Chapter 1
Structural Mechanics

Introduction

There are many different types of structures all around us. Each structure has a specific purpose or function. Some structures are simple, while others are complex; however there are two basic principles of composing structures.

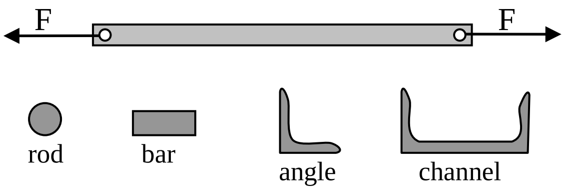
- They must be capable of carrying the loads that they are designed for without collapsing.
- They must support the various parts of the external load in the correct relative position.

A structure refers to a system with connected parts used to support a load. Some examples related to civil engineering are buildings, bridges and towers. However, these structures are very complex for analyze and design. At first, we will consider simple examples of structures and parts of structures like beams, trusses, frames etc. It is important for a structural engineer to recognize the various type of elements composing a structures and to be able to classify them as to there form and function. We will introduce some of these aspects.

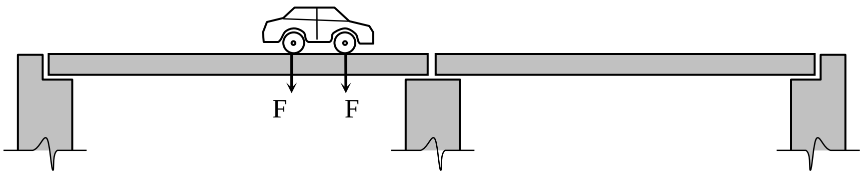
Structural elements:

Some of most common structural elements are as follow:

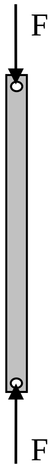
✦ **Tie rods** – structural members subjected to a tensile force. Due to the nature of the load, these elements are rather slender and are often chosen from rods, bars, angels, or channels.



✦ **beams** – straight horizontal members are used generally to carry vertical loads.



Beams may be designed from several of element and materials – concrete, metal etc. with rectangular or other cross section.



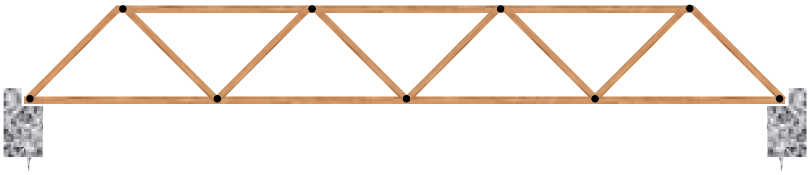
✦ **columns –**members are generally vertical and resist axial compressive loads.

Columns are elements similar to the tie rods but they carry vertical loads.

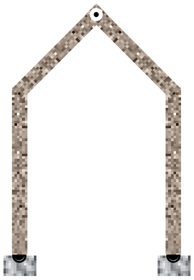
Type of structures

1. **Frame structures:** trusses, three-hinged frame, frames,

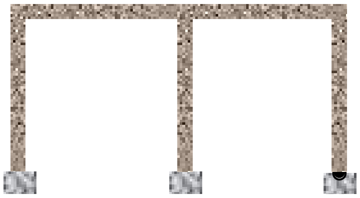
✦ **trusses**: they are composed of slender rods usually arrenged trintriangular fashion. Trusses are suitible for constructions with large span when the depth is not an important criterion for desing. Plane trusses are composed of members that lie in the same plane and are frequantly used for bridge and roof support.



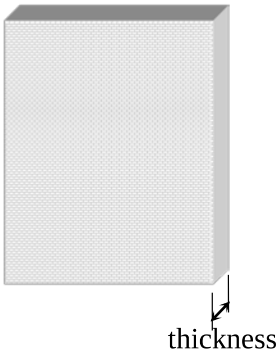
✦ **three-hinged frame**: this structure is simple determinate frame used generally for base element for complicated frame structures.

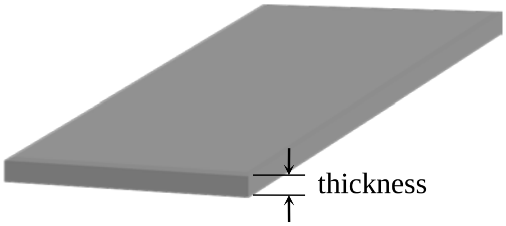


✦ **frames:** they are often used in buildings and are composed of beams and columns which are with hinge or rigid connections. These structures are usually indeterminate and the load causes generally bending of its members.

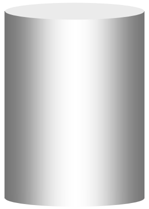


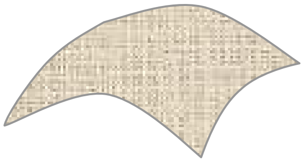
✦ **plane structures:** plates, walls and etc. These structures have two significant dimensions and one small called thickness. The theory of elasticity is capable to analyze such structures.





- ✦ **surface structures:** shells and etc. These structures can be made from flexible or rigid material and has a three-dimensional shape like a cylinder hyperbolic paraboloid etc. The analysis of these structures is also aim of theory of elasticity.

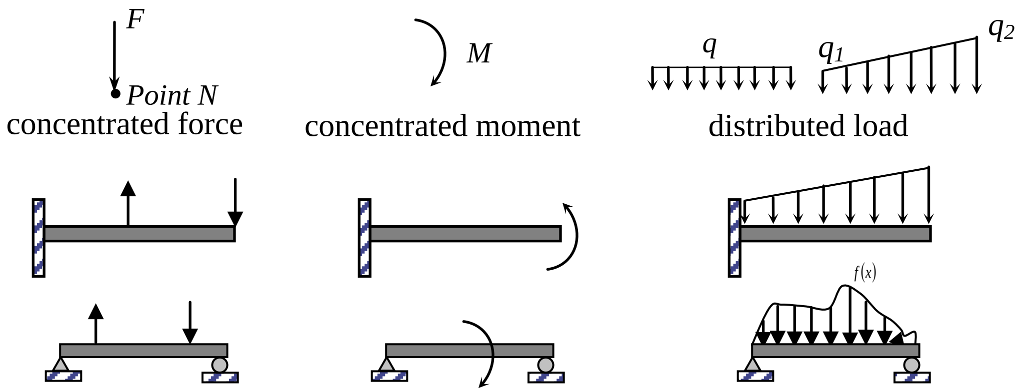




Loads:

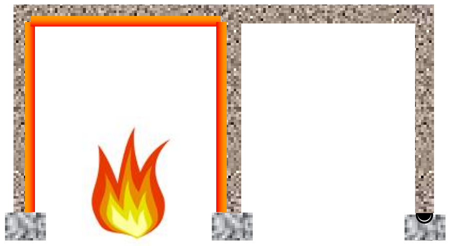
In statical structural analysis of frame structures we define statical (dead) load. We distinguish types of loads:

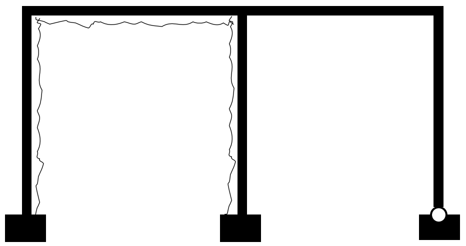
- ✦ **force load:** concentrated force or moment, distributed load.



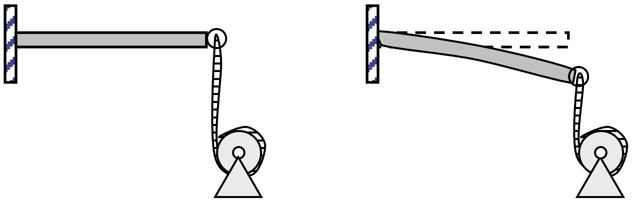
concentrated force concentrated moment distributed load

- ✦ **temperature load:** load caused by fire.





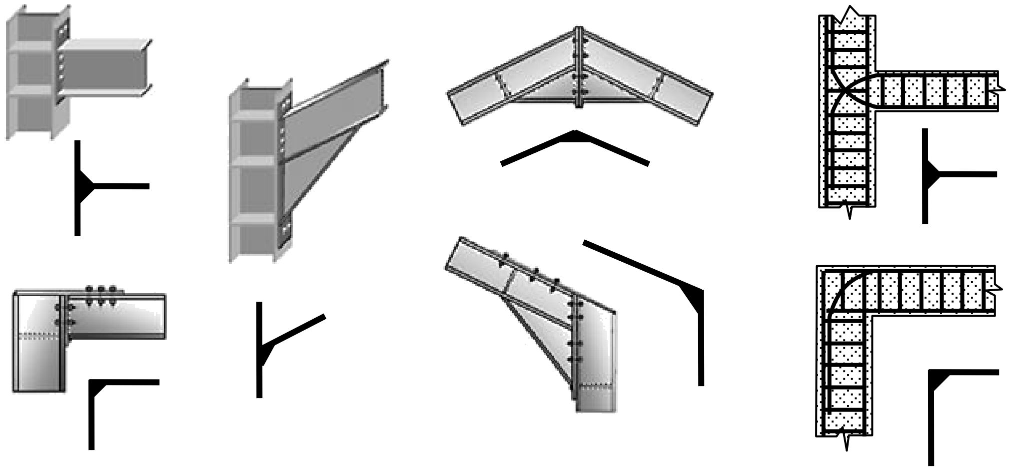
✦ **displacement load:** load displacement is caused from displacement of some point or points of the structure.



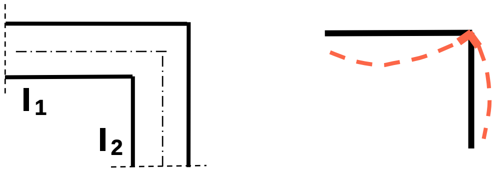
Idealized structures:

Idealized structure is needed to the engineer to perform a practical force analysis of the whole frame and its member. This is the reason in this section to show different member connections and supports and there idealizations. If one know these models may compose idealized model of each real structure after all perform the analysis and design.

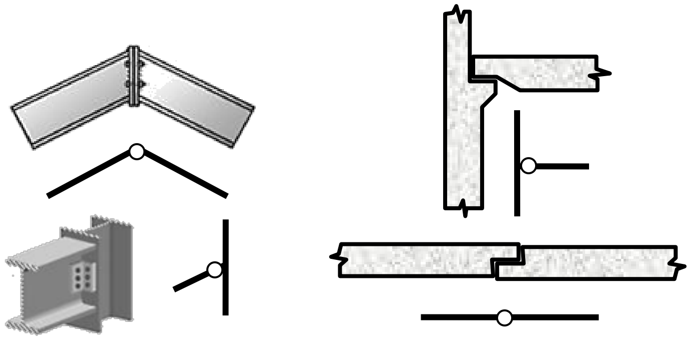
✦ **rigid (fixed) connections:** this connection carry moment, shear and axial forces between different members. In addition, in this case all members including in such a connection have one and the same rotation and displacements – the nodal rotation and displacements. Typical rigid connections between members in metal and in reinforced concrete constructions and there idealized models are shown in the following figure:



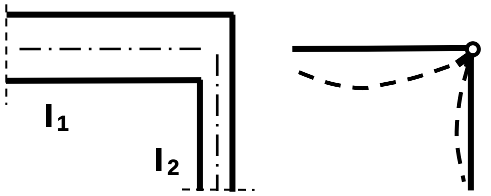
Rigid connection, $I_1 = I_2$



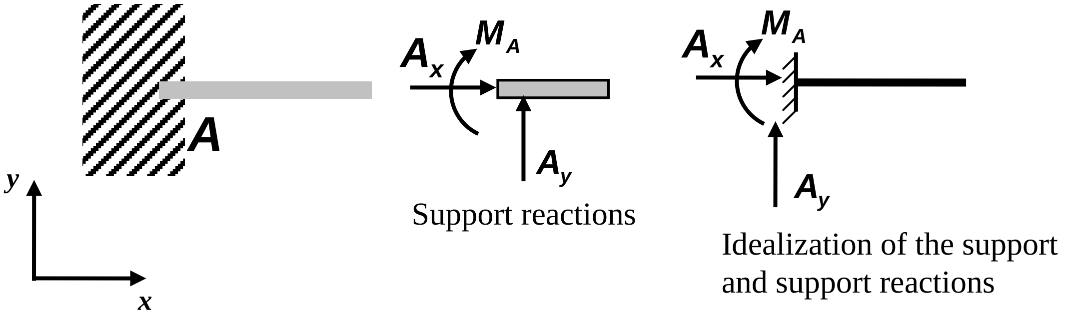
✦ **hinged (pin) connections:** this connection carry shear and axial forces but not moment between different members. Hinged connection allow to the jointed members to have different rotations but the same displacements. Typical hinged connections between members in metal and in reinforced concrete constructions and there idealized models are shown in the next figure:



hinged connection, $I_2 << I_1$



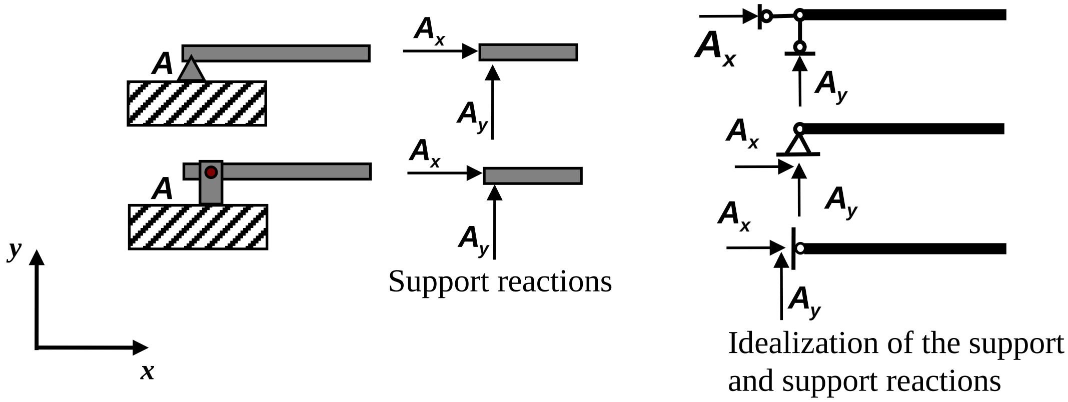
✦ **fixed support**: this support carry moment, shear and axial forces between different members. This kind of support doesn’t allow any displacements of the support point. So if the displacement along the x axis is u, the displacement along y axis is y and the rotation is called φ then we can say that: $u_A = 0$; $v_A = 0$ and $\varphi_A = 0$.



Support reactions

Idealization of the support and support reactions

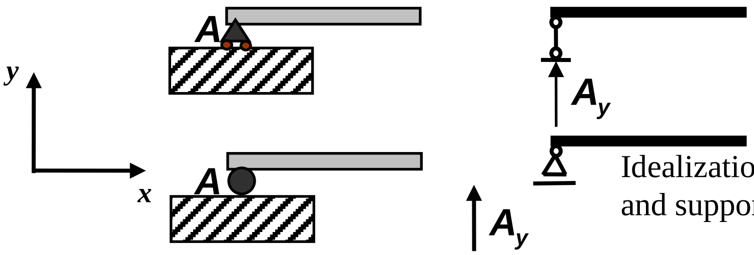
✦ **hinged (pin) support:** this support carry shear and axial forces but not moment between different members. The hinged support allows rotation of the support point but the two displacement are equal zero or: **$u_A = 0$; $v_A = 0$** and **$\varphi_A \neq 0$.**



Support reactions

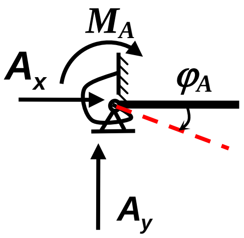
Idealization of the support and support reactions

✦ **roller support:** this support carry only shear forces between jointed members. The roller support allows rotation and one displacement of the support point: $\boldsymbol{u_A \neq 0}$; $\boldsymbol{v_A = 0}$ **and** $\boldsymbol{\varphi_A \neq 0}$**.**

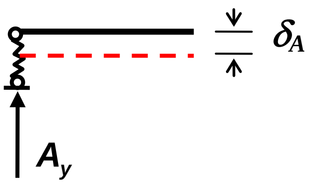


Idealization of the support and support reactions

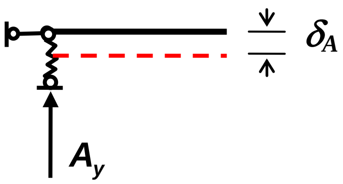
✦ **spring supports:** These supports are like the previous but with the difference that they are not ideally rigid but with some real stiffness. The spring has a stiffness constant ***c*** equals to the force caused by displacement ***d*** = 1.



Fixed spring support
$\boldsymbol{M_A = c.\ \varphi_A}$

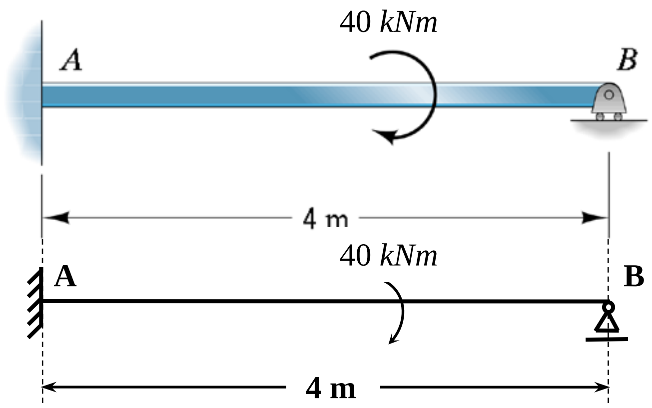


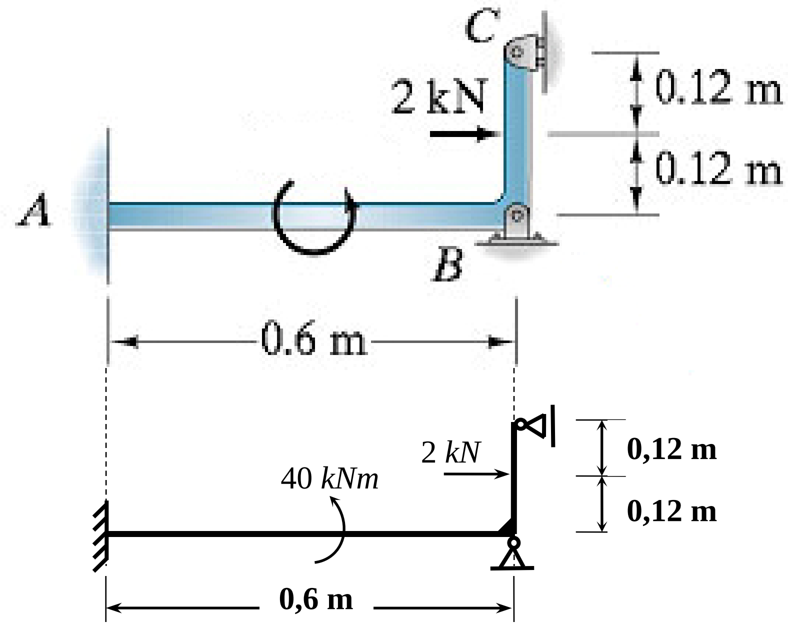
Roller spring support
$\boldsymbol{A_y = c.\ \delta_A}$



Pin (hinged) spring support
$\boldsymbol{A_y = c.\ \delta_A}$

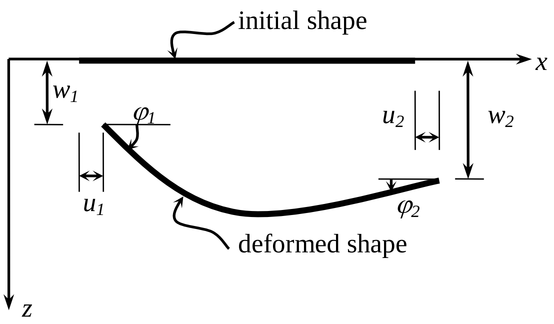
✦ **structure idealization**: The main idea of this idealization is to made a mathematical model of the real construction to be convenient for analysis and calculation. After we know the idealization of different joints and supports, we will take care about whole structure idealization. To make this we follow the middle axis of the elements of the structure. In the following figure are shown some real and idealized structures:

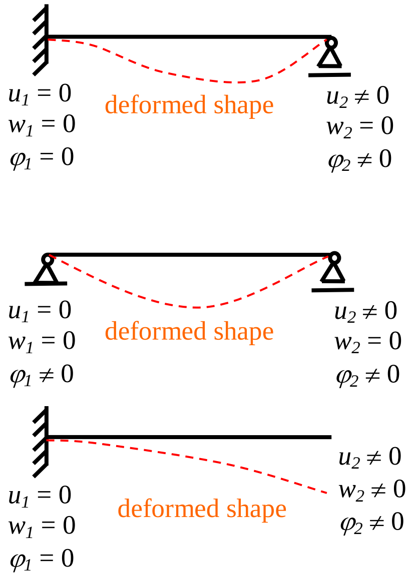




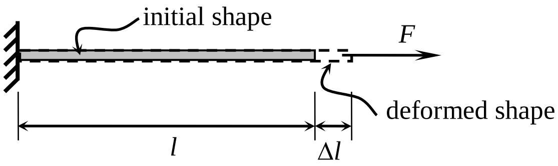
Principles and preconditions:

✦ **displacements:** Every two dimensional deformable element has three degrees of freedom (two displacements and one rotation) of each its end node. With using different support links, we control these degrees of freedom so the elements cannot move on the limited direction or it moves with controlled value. These limitations are called boundary conditions. On the following figure are shown the degrees of freedom and some boundary conditions for elements:



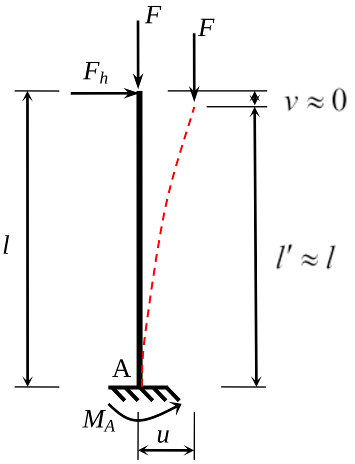


✦ **deformation:** deformation or strain is the change in the metric properties of a continuous body (element) caused by some load. A change in the metric properties means that the element changes its length and shape when displaced to a curve in the final position – the deformed shape.



$$\varepsilon = \frac{\Delta l}{l} - \text{elongation (deformation) of the element}$$

✦ **preconditions about displacements and deformations:** we presume that the displacements are small according the dimensions of the element and deformations are small according the unit. These preconditions allow us to write equilibrium conditions for the initial shape of the structure and also to neglect the small displacement of the structure.



Equilibrium condition for deformed shape:

$$\sum M_A = o : M_A = F.u + F_h.(l - v)$$

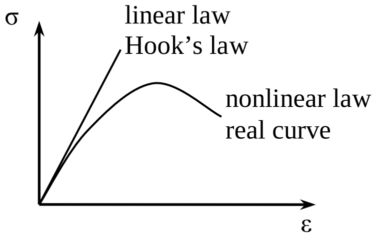
Equilibrium condition for deformed shape if $v \approx 0$:

$$\sum M_A = o : M_A = F.u + F_h.l$$

Equilibrium condition for the initial shape and $v \approx 0$ (small displacement and small deformations):

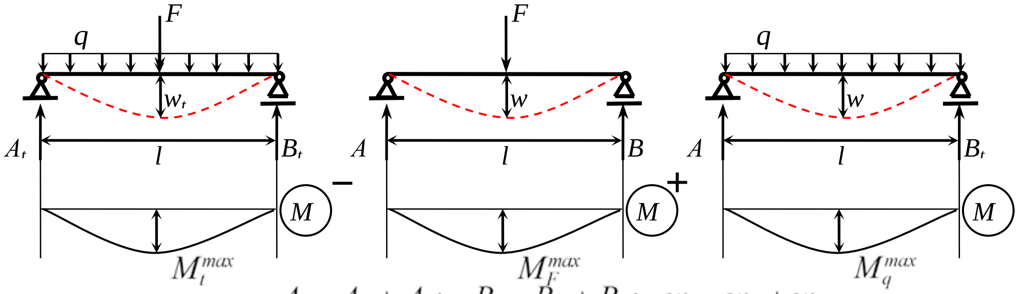
$$\sum M_A = o : M_A = F_h.l$$

✦ **precondition about the material:** we suppose that the connection between stress and strain is linear so the Hook's law is valid. This is acceptable because of presumption of small deformation.



$$\sigma = E.\varepsilon - \text{Hook's law}$$

✦ **principal of superposition:** The previous two preconditions allow us to use the principle of superposition. It may be stated as follow: The total displacement or internal forces at a point in a structure subjected to several external loadings can be determinate by adding together the displacements or internal forces caused by each of the external loads acting separately.



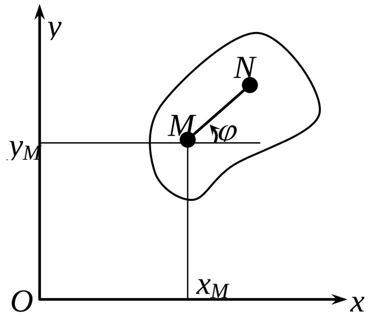
$$A_t = A_F + A_q; \quad B_t = B_F + B_q; \quad w_t = w_F + w_q$$

$$M_t = M_F + M_q; \; M_t^{max} = M_F^{max} + M_q^{max}$$

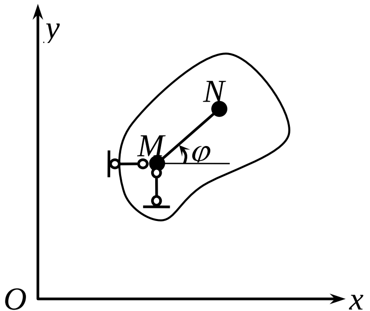
Chapter 2
Kinematical analysis of structures

Determination of degrees of freedom:

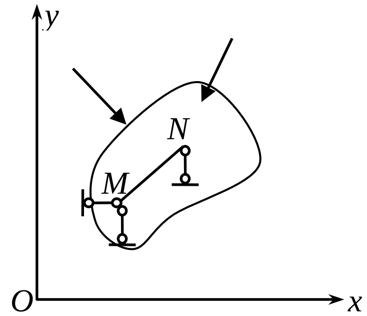
We know that each body situated in one plane has three degrees of freedom – three independent parameters determining its movement. By using support links we limit this movement possibility. So if we put on three special arranged support links at a body than it will be stable – without any movement possibility. In this way the body is able to carry different loads and we call it structure. Then as a response of the load in the support links appears support reactions we can determine. Structure with exact number of links is called determinate structure. If this body has less then tree links then some movement will be possible. Such a body is called mechanism. If we put on more then three support links on the body than it is indeterminate structure.



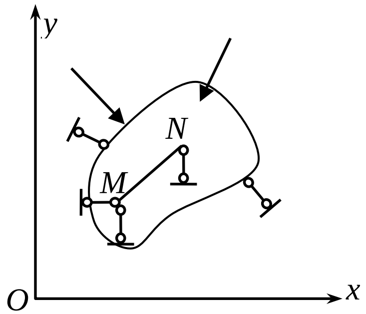
Plane body's degree of freedom (movement possibility): x_M, y_M *and* φ



One degree of freedom mechanism. Possible movement is **T**



Determinate structure (no movement possibility) able to carry some load.



Indeterminate structure (no movement possibility) able to carry load.

In case we have no one body but several numbers, the degrees of freedom is depending of the body's connection and supports. The way to calculate of the degrees of freedom in such complicated structure is following:

If there are no one element closed loops:

$$w = 3d - 2k - a\,;$$

where:

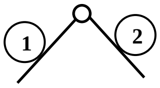
w is degree of freedom (mobility);

d is number of bodies (elements);

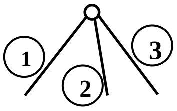
k is number of one-degree-of-freedom kinematic pin joints;

a is number of support links.

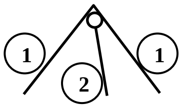
The numbers of *k* is calculated by the formulae: $k = d - 1$, where *d* is number of the connected at the pin joint elements.



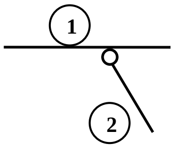
$d = 2$
$k = 1$



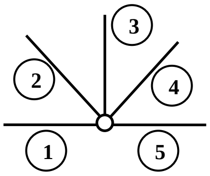
$d = 3$
$k = 2$



$d = 2$
$k = 1$

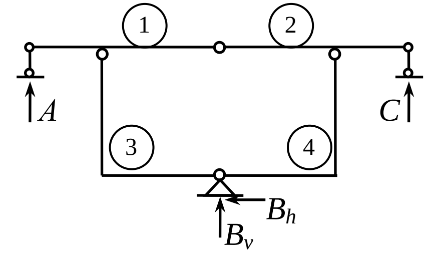


$d = 2$
$k = 1$



$d = 5$
$k = 4$

We show some examples for determination of degree of freedom.
In the following example we have closed loop but composed by two elements.

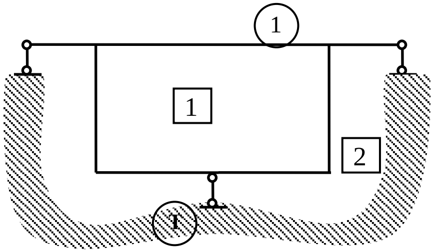


$d = 4$; $k = 4$; $a = 4$;
$w = 3.4 - 2.4 - 4 = 12 - 12 = 0$

If there are one-element-closed-loops:

$$w = -(3m - k)$$

where:
m is number of the closed loops (including the basic disk – ground (terra));
k is number of one-degree-of-freedom kinematic pin joints.

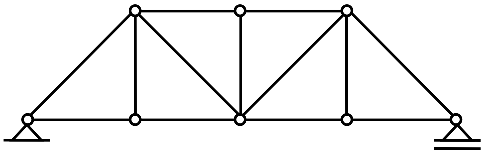


$m = 2$; $k = 3$
$w = -(3.2 - 3) = -3$

In the case of the rod structures (trusses) we may use the next formulae:

$$w = 2k' - d - a$$

where:
d is number of elements;
k' is number of hinges;
a is number of support links.

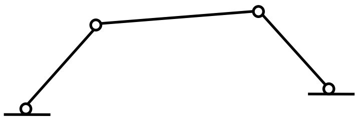


$d = 13$; $k = 8$; a = 3
$w = 2.8 - 13 - 3 = 0$

And finally in the case of chains we have:

$$w = d - 2$$

where:
d is number of elements;



$d = 3$; $w = 3 - 2 = +1$

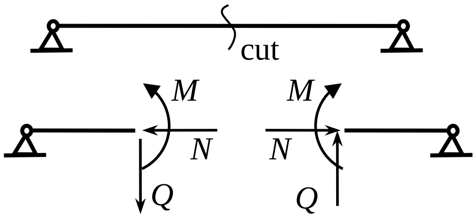
On previous examples we saw that the number of degrees of freedom *w* may be positive, negative or zero. So we distinguish three different cases for *w*:

$w > 0$ - the system is mechanism. In the case of mechanism we don't have a structure carrying any load;
$w = 0$ - determinate structure. We have a structure and it is possible to analyze it with only equilibrium conditions.
$w < 0$ - indeterminate structure. We have a structure and it is possible to analyze it with equilibrium conditions and additional equations.

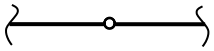
In this first stage we will analyze only determinate structures namely structures with $w = 0$.

Basic kinematical elements and links:

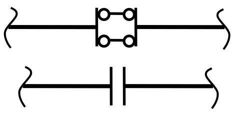
It's known that if we cut the body of the beam par example, in the cut sections there are three body force shown on the figure:



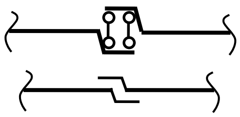
It is useful for some structures to construct different type of connection between the disks like pin joint we has shown and some other displayed on the next figure. These types of connection are called releases.



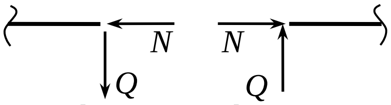
pin joint (hinge)
***M* – release**



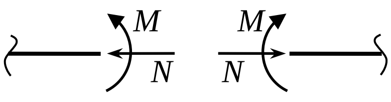
***Q* – release**



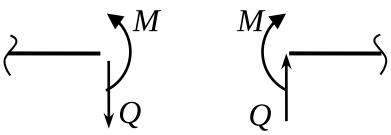
***N* – release**



In this section there is no bending moment - *M*



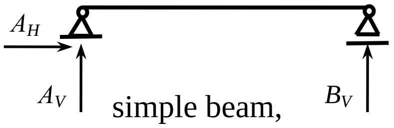
In this section there is no shear force - *Q*



In this section there is no axial force - *N*

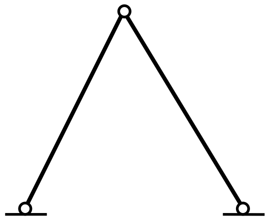
- ✦ **basic kinematical elements** – cantilever beam, simple beam and dyad. They are simple, stable determinate structure and we use them for composing complicated systems.

The dyad is stable only if the three joints are not lying on one and the same line. If they are then the dyad is unstable and we call it “singular dyad”.

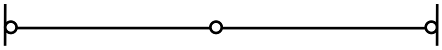


simple beam,
$w = 0$; stable system.

cantilever beam,
$w = 0$; stable system.



dyad,
$w = 0$; stable system.



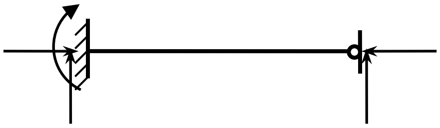
singular dyad, $w = 0$; **unstable** system.

We may distinguish three types of basic kinematical links (elements) for composing structures:

- **kinematical links Type 1 –** this link carry only axial load if there is no transverse load.

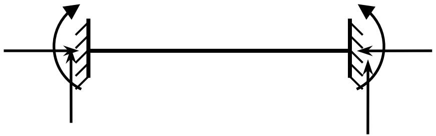
$w = -1$; indeterminate system.

- **kinematical links Type 2 –** this link carry axial and bending load and has one fixed and one pin support.



$w = -2$; indeterminate system.

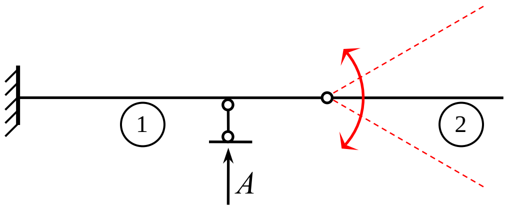
- **kinematical links Type 3 –** this link carry axial and bending load and has two fixed support.



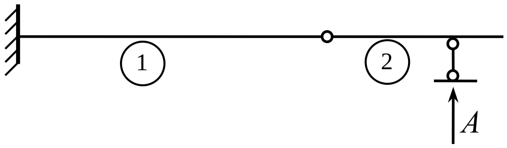
$w = -3$; indeterminate system.

Kinematical analysis of determinate structures:

By using basic kinematical elements, links and chains we may compose different complicated structures. With the upper formulas we control if the composed structures are determinate or not. But it is possible construction to be determinate and to be mechanism at the same time. This phenomenon we call kinematical instability and such a system – mechanism. So the upper formulas give us information only for the number of the links but not for the kinematical stability. That is why we need a kinematical analysis. In the fowling example we show this phenomenon.

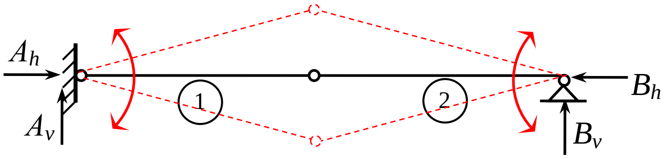


$d = 2$; $k = 1$; a = 4
$w = 3.2 - 2.1 - 4 = 1$

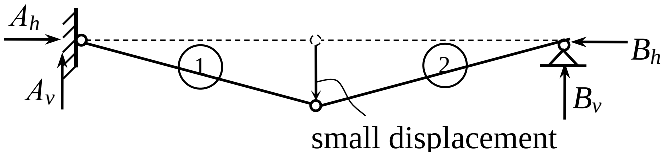


$d = 2$; $k = 1$; a = 4
$w = 3.2 - 2.1 - 4 = 0$ – the system is stable.

Another possibility of this phenomenon is instantly unstable system. Instantly unstable because after some displacement the system came stable but therefore not good for design.

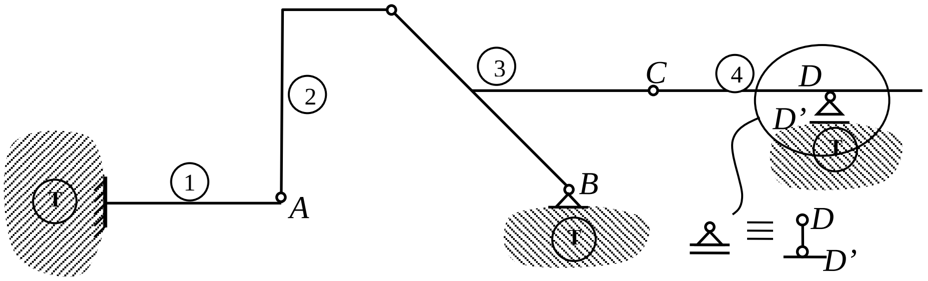


$d = 2$; $k = 1$; a = 4
$w = 3.2 - 2.1 - 4 = 0$ – but the system is instantly unstable.



After some displacement the system is stable but not good for design. This system is called “singular dyad”.

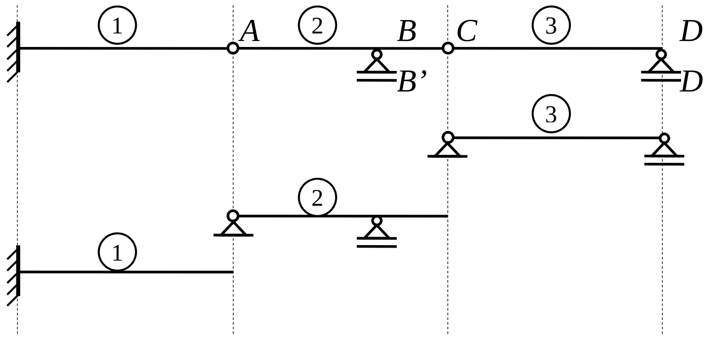
The kinematical analysis consist a way of composing the complicated structure. If we use only stable basic elements like a cantilever beam, a simple beam or a dyad the result should be stable structure.



The kinematical analysis of this structure is the next: At first we have only the earth (terra). After that we construct the cantilever beam (element 1). It is stable structure. Point *A* already exists. On the next step we construct the dyad 2.3. This dyad is based at point *A* on the cantilever beam and at point *B* on the earth. The dyad is also stable structure. Last step is composing of the simple beam 4. It is based on the dyad at point *C* and on the earth at point *D*. This explanation of the kinematical analysis we write in the fowling way:

$$KA:[T+1(w=0)+2.3(w=0)+4.DD'(w=0)](w=0)$$

Another example - compound beam:



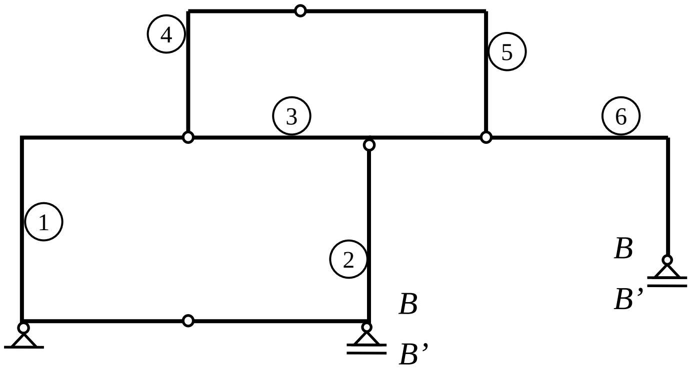
$$KA : [T + 1(w = 0) + 2.BB'(w = 0) + 3.DD'(w = 0)](w = 0)$$

The compound beam is composed by one cantilever beam and two simple beams, all of them lying on one line. Beam 2 is based on beam 1 and on the roller support *BB’*. Beam 3 is based on beam 2 and the roller support *DD’*. Beam 1 is supported only on the earth so this beam we call primary beam and the two other we call secondary beams.

Depending on kinematical analysis we classify structures on two types:

- **Type I:** structures composed only by using other stable structures: cantilever beam, simple beam or dyad.
- **Type II:** system which consist chains and links.

Previous examples were of type I. Now we will shall some examples of structures type II:



$$KA : [T + 1.2.BB'(w = +1) + 3(w = -1) + 4.5(w = 0) + 6.BB'(w = 0)](w = 0)$$

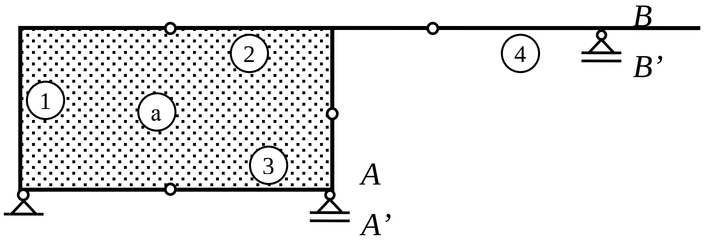
This system consist chain and link. So it is not sure if the system is stable or not. It is necessary to be made additional verification.

There are four typical ways for composing different structures:

- **way I:** compose structures only by using cantilever beams, simple beams and dyads starting of the earth (base disk).

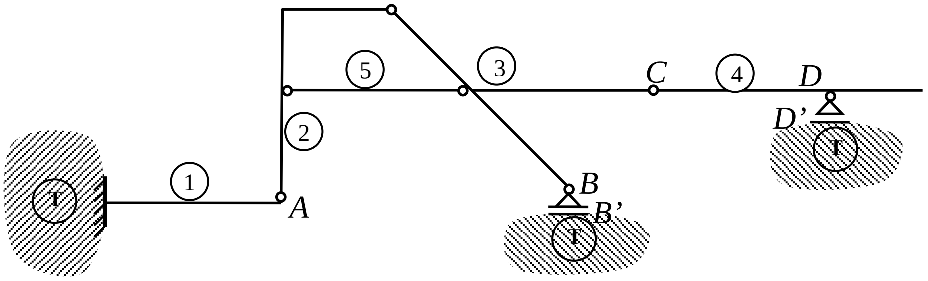
Using this way we have structure type I and we are sure it is stable structure.

- **way II:** compose structures only by using simple beams and dyads but using one of the disks for base element. As a result we have composed a stable close loop. After that we base it on the earth. The kinematical analysis in this case has two stages. The first composing the closed loop and the second composing earth based structure.



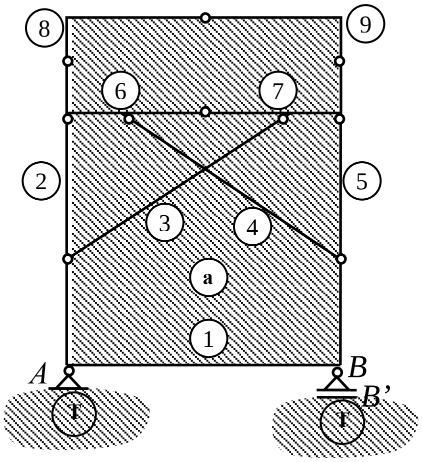
First stage: $KA:\ a = [1 + 2.3(w = 0)](w = 0)$
Second stage: $[T + a.AA'(w = 0) + 4.BB'(w = 0)](w = 0)$

✦ **way III:** compose structures by using cantilever, simple beams, dyads and chains and links. In this case we use terra for a base of the structure. As a result it is not sure if the structure is stable or not. It is necessary to make additional analysis.



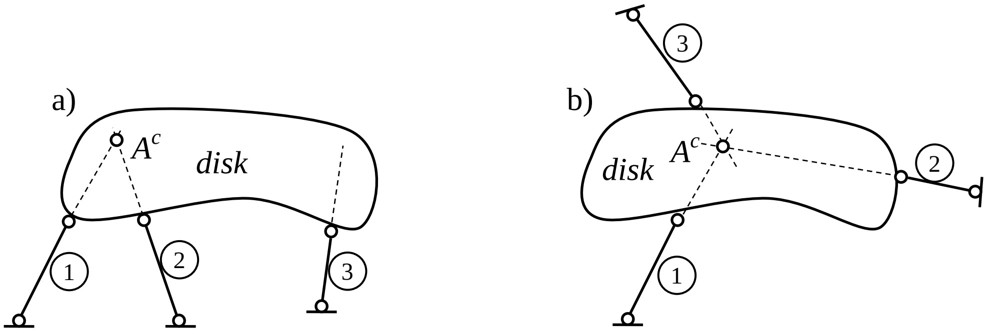
$$KA: [T + 1(w = 0) + 2.3.BB'(w = +1) + 5(w = -1) + 4.DD'(w = 0)](w = 0)$$

✦ **way IV:** compose structures by chains and links. In this case we use some of the disks for a base element. The kinematical analysis is in two stages. As a result it is not sure if the structure is stable or not. It is necessary to make additional analysis.



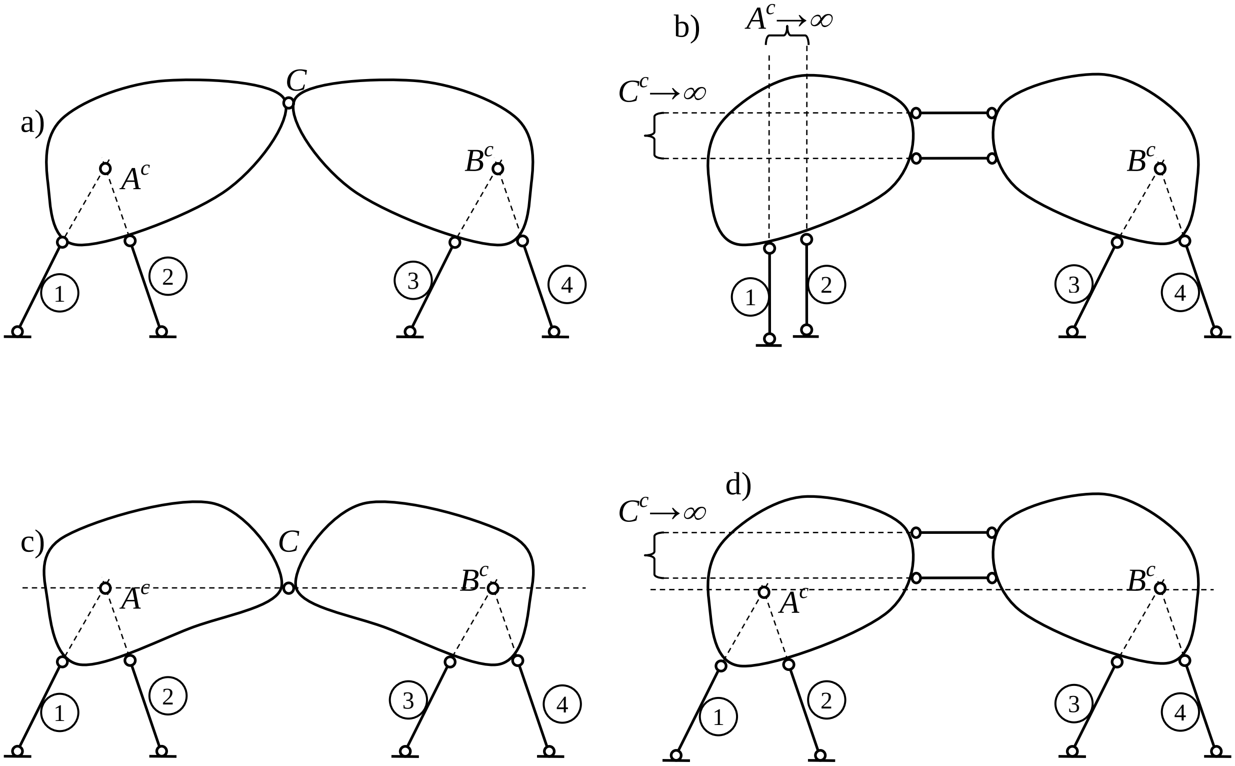
First stage: $KA:\ a = [1 + 2.6.7.3(w = +2) + 4(w = -1) + 5(w = -1) + 8.9(w = 0)](w = 0)$
Second stage: $[T + a.BB'(w = 0)](w = 0)$

Let us consider in details a way of support of a simple beam and a dyad. The simple beam is composed by one disk and three support links like it's shown:

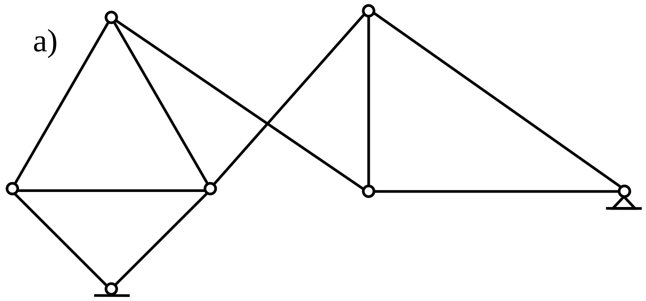


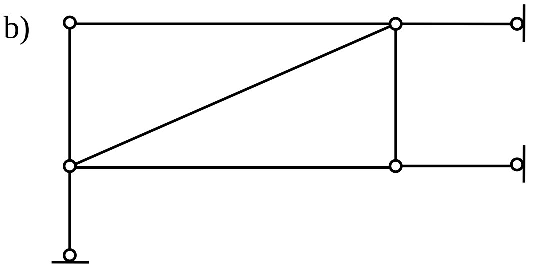
In case *a)* the direction of support links *1* and *2* intersect in common hinge A^c as a rotating point but support link *3* obstruct this rotation so the structure is stable. In contrary in case *b)* the direction of the three support links intersect at on and the same point – the common hinge A^c. As a result the rotation is possible and the structure is unstable.

As we know the dyad is composed by two disks connected by a hinge (common or not) and supported by two fixed support (one fixed support is composed by two intersected support links).



In case *a)* the dyad has two fixed supports at the common hinges A^c and B^c and one real hinge *C* between the disks. The three hinges A^c, B^c and C^c are not lying at one line, that is why the dyad is a stable one. The same situation is in case *b)* but the common hinges A^c and C^c are at infinity. In cases *c)* and *d)* the three hinges are lying at one line and that is the reason the dyad is unstable. The difference is that in case *d)* the common hinge C^c is at the infinity but as is known all horizontal lines intersect at the horizontal infinity and all vertical lines intersect at the vertical infinity.
If we know enough about principles of structural composing and common hinges we may answer the question “Is the construction stable or not?” very easy some time as in the following cases:



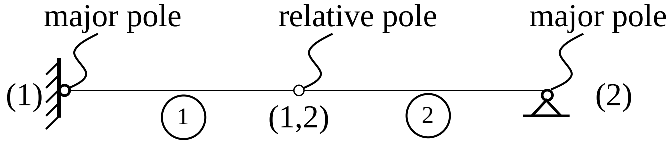


At the first glance at the pictures one may say that these are very complicated structures but after that it should be clear that at figure *a)* we have three-hinged beam and at figure *b)* it is a simple beam. At figure *a)* the two triangles are close-loops and have a sense of one disk each of them. These two disks are connected with two links crossed at meddle where is the common hinge. The left triangle – disk is supported by two links crossed at one point. This point is actually fixed support for the disk. The right disk is supported directly by fixed support. At figure *b)* the two triangles compose one close-loop disk which is supported by one link (in the left) equal to roller support and one fixed support at the horizontal infinity composed by the two horizontal links. So at the result we have simple beam at figure *b)*. As well as we know that these structures are simples we can be sure that they are stable if they agree with upper rules. This way for analyze structures is very convenient in most cases but there are some situation in which it isn't possible to use it. That is the reason to perform common method for analyzing complicated structures for there determination and stability.

Common method for kinematical analysis of determinate structures:

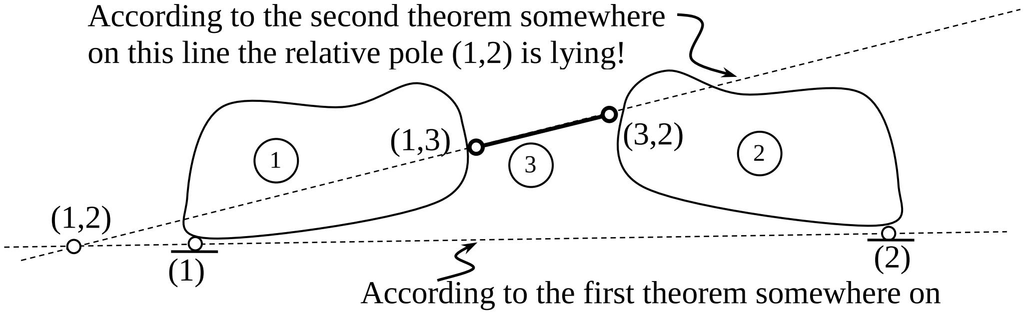
Before we present the common method we should explain some kinematical theorems and determinations.

- **Major pole of rotation:** This is the pole around which rigid body rotates. Fixed supports are usually major poles.
- **Relative pole of rotation:** This is a point around which two rigid bodies relatively rotates. Middle and common hinges are usually relative poles of rotation.
- **First Theorem:** If we have a mechanism of two connected bodies so they have major poles each of them and one relative pole. Allays these three poles (the two major and one relative) are lying at one line.



- **Second Theorem:** If we have a mechanism of three connected bodies there relative poles are lying at one line.

According to the second theorem somewhere on this line the relative pole (1,2) is lying!



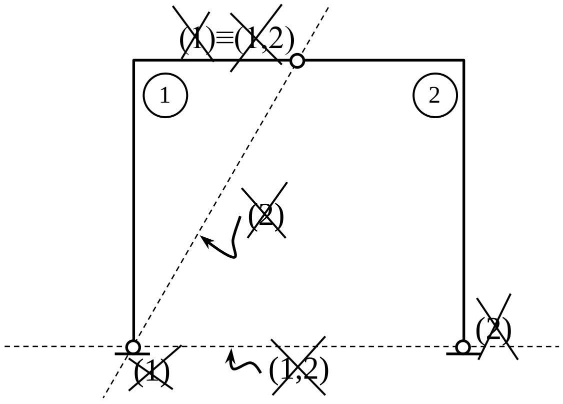
According to the first theorem somewhere on this line the relative pole (1,2) is lying!

As it's shown at the figure using these two theorems we may find the position of some pole which is not obvious at first. Additionally we may write these conditions in the following provisionally way:

$$\left|\begin{matrix}(1)+(2)=(1,2)\\(1,3)+(3,2)=(1,2)\end{matrix}\right\}\Rightarrow(1,2)$$

This "equation" should be red as: The major poles (1) and (2) of the disks 1 and 2 determine a line on which the relative pole (1,2) should lie. The relative poles (1,3) and (3,2) determine a line on which the relative pole (1,2) should lie. The cross point of the two lines determine the exact position of the relative pole (1,2).

- **First Additional Theorem:** If one major pole appears at two points at the same time then the pole do not exist. When one major pole don't exist then the corresponding body do not moves.
- **Second Additional Theorem:** If one relative pole appears at two points at the same time then the pole do not exist. When one relative pole doesn't exist then the corresponding bodies do not relatively moves. They move like one and the same disk (body).



In this example the major pole (2) should lie at a line determinate by poles (1) and (1,2) but at the same time it is known the exact position of this pole (at the right fixed support). Hence the major pole (2) appears at two different positions so do not exist. As a result the disk 2 doesn't moves. Owing to this the relative pole (1,2) becomes major pole (1). But this pole already exists so the disk 1 doesn't moves too.

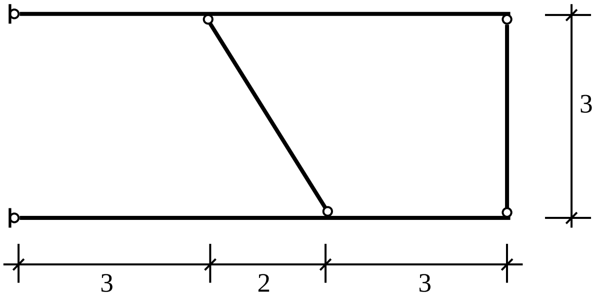
In the other hands the relative pole (1,2) should lie at a line determine by poles (1) and (2) but its exact position is known by the middle hinge. As a result pole (1,2) do not exist. In consequence disks 1 and 2 moves like one disk. But this one disk has two major poles at the two fixed supports. Hence this disk hasn't a major pole. Thus the disk doesn't moves.

Actually the shown system is dyad and as we know is stable. That is why poles of movement do not exist. But this is very good example to show haw the additional theorems works. This example also shows that the theorems of kinematical mechanism can be used as a source for verification of structural stability. Thus follows the common method for kinematical analysis.

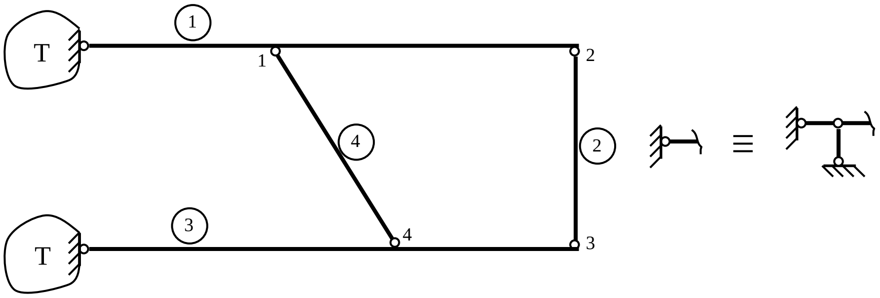
First we make Determination of degrees of freedom. If the structure is statically determinate we continue with the next step: kinematical analysis – the way of composition. If the structure is composed by using firs or second way of composition then it is known the structure is stable and more verification isn't needed. If it is used third or fourth way then verification for stability is needed. If it is possible we may identify the structure as an elementary one – simple beam or three-hinged frame – dyad like it was shown previously. If not we continue with the common method of verification.

- **Common method for kinematical analysis.**
  1) We remove the last link composed according to the kinematical analysis. The two ends points of the link determine a line we call it ***a-b***.
  2) The removed link has connected two other disks. We compose the plan of the poles and find the relative pole to these disks.
  3) If this relative pole is lying at the line ***a-b*** then the determinate structure is instantly unstable one. If the relative pole isn't lying at the line ***a-b*** then the system is stable and we may determine reactions and internal forces caused by some loads.

Example 1: Make full kinematical analysis of the structure.



1. Determination of degrees of freedom: $w = 3d - 2k - a$.



$$d = 4; k = 4; a = 4 \quad \Rightarrow \quad w = 3.4 - 2.4 - 4 = 12 - 12 = 0$$

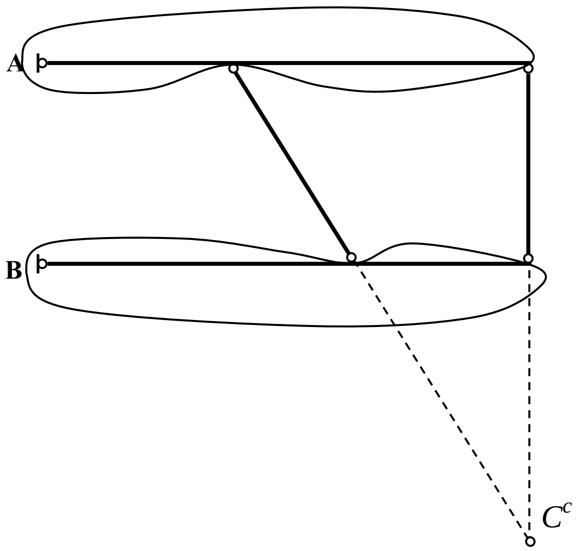
$w = 0 \;\Rightarrow$ The system is statically determinate.

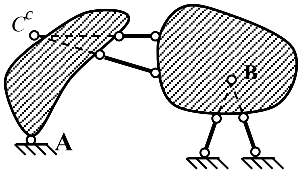
2. Kinematical analysis of the structure:

$$\left[T + 1.2.3(w = +1) + 4(w = -1)\right](w = 0)$$

The system is composed by using chains and links (way IV); therefore it is second type structure. It is necessary to make verification for kinematical stability.

3. Identification of the structure as an "Elementary system with common hinges":





This system can be considered as an elementary one. It is “three-hinged frame” type, with two real support links, called *A* and *B* and a common medial hinge C^c. The three hinges are not lying at one and the same line, so the system is stable.

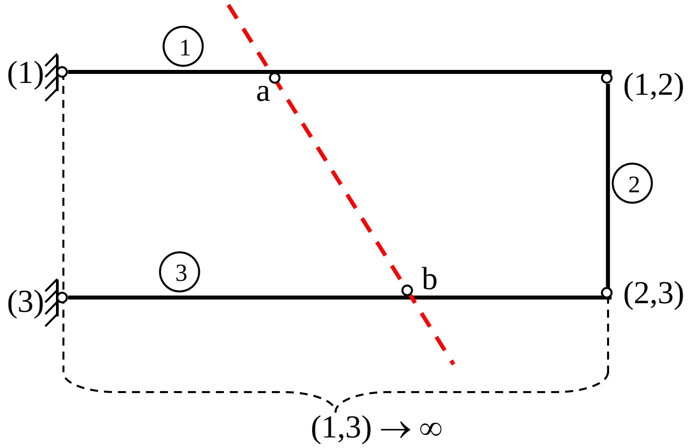
It is not needed but we will show the common method of verification.

4. Verification for kinematical stability by the common method.

   1) At first we remove the last link according to the kinematical analysis – link 4.

$$\left[T + 1.2.3(w = +1) + 4(w = -1)\right](w = 0)$$

   2) We compose the plan of the poles. Searching for relative pole *(1, 3)* – the relative pole of disks, which were connected by the removed link:

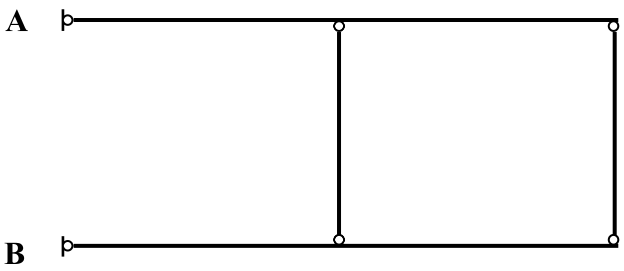


$$\left.\begin{array}{l}(1) + (3) = (1,3) \\ (1,2) + (2,3) = (1,3)\end{array}\right\} (1,3) \rightarrow \infty$$

The serched relative pole *(1, 3)* is at the vertical infinity.

The relative pole *(1, 3)* is not lying at the straight line *a-b*, so the system is stable.

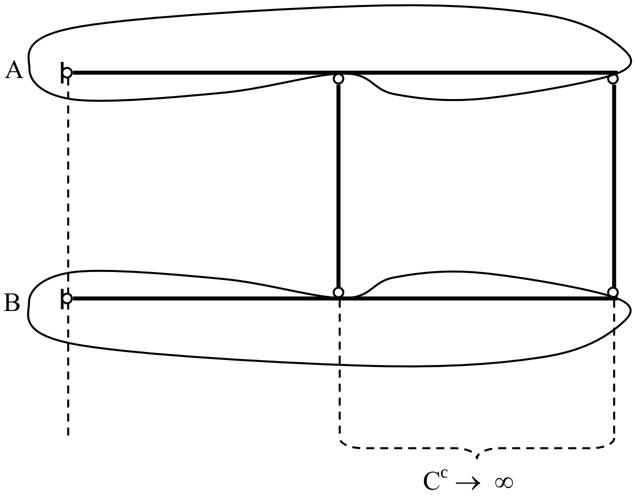
Example 2: If the two connecting disks of the upper system are vertical, it is transferred into an instantly unstable system.

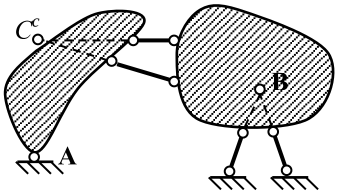


The kinematical analysis is the same as the previous one.

$$\left[T + 1.2.3(w = +1) + 4(w = -1)\right](w = 0)$$

1. Identification of the structure as an "Elementary system with common hinges":





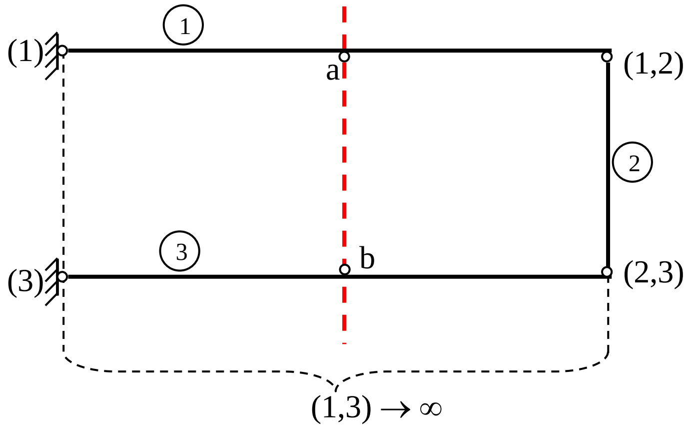
This system can be identify once again as an elementary system with common hinges and "three – hinged system" type with two real and one common hinge. In this case the medial hinge C^c is on the vertical infinity, so three hinges are lying at one line (all parallel lines are crossed in one point to infinity). If the three hinges (*A*, *B* and C^c) are lying at one straight line, then the system is an kinematically unstable.

2. Verification for kinematical stability by the common method.

   a. Remove the last link according to the kinematical analysis – link 4.

$$\left[T + 1.2.3(w = +1) + 4(w = -1)\right](w = 0)$$

   b. Compose the plan of the poles searching for relative pole *(1, 3)* – the relative pole of disks, which were connected by the removed link:

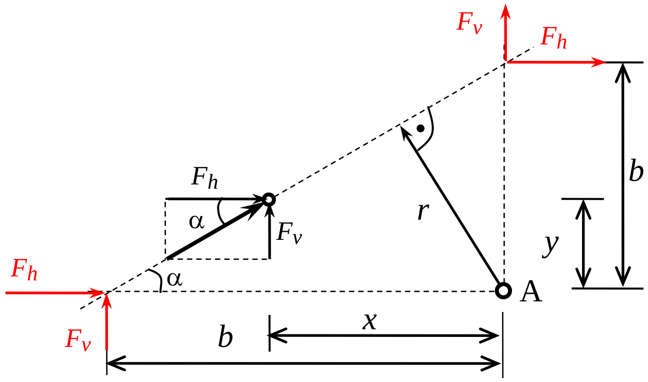


$$\left.\begin{array}{l}(1) \;\;+\;\; (3) \;\;= (1,3)\\(1,2) + (2,3) = (1,3)\end{array}\right\}\;(1,3) \to \infty$$

The serched relative pole *(1, 3)* is at the vertical infinity again.
In this case however, the relative pole *(1, 3)* **is lying** at the straight line *a-b*, so the system is **instantly unstable**.

Chapter 3
Analysis of elementary structures.

In this chapter, we will consider the procedure of analysis of elementary structures like a simple beam, cantilever beam, dyad (three-hinged frame) and compound beam. For this reason, first of all remind the definition of a force and a moment of force to some point. From the physics, it is known that the force is a vector, which has a sign, direction, and value and application point. The force is a representation of some load, which causes damages (deformations and displacements) of the body on which act. If the force acts at arbitrary direction, we may decompose it at the two mane directions – horizontal and vertical as it's shown on a figure. The action of the decomposed force is the same as this of the whole one.



The force components are:

$$F_h = F\cos\alpha$$
$$F_v = F\sin\alpha$$

Also for the force components we have the following dependencies:

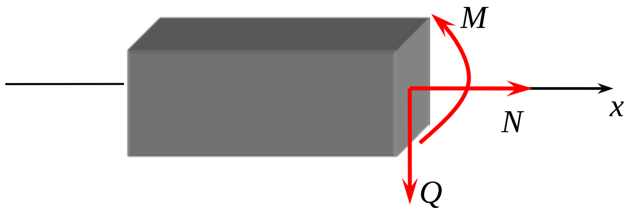
$$\frac{F_h}{F_v} = \frac{a}{b} \Rightarrow F_h = F_v\frac{a}{b} \quad and \quad F_v = F_h\frac{b}{a}$$

The moment of the force related to the point A may be calculated by following different ways According to the figure:

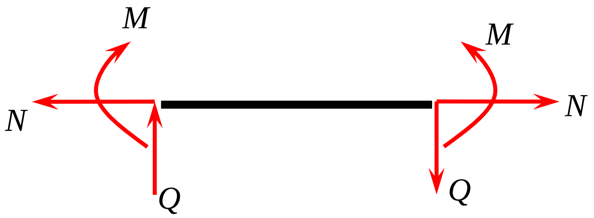
$$M_A = F.r;$$
$$M_A = F_h.b;$$
$$M_A = F_v.a;$$
$$M_A = F_h.y + F_v.x$$

The frequently used ways are the first and the last ones. In the second and the third cases is used the translated forces along the directrix of the force.

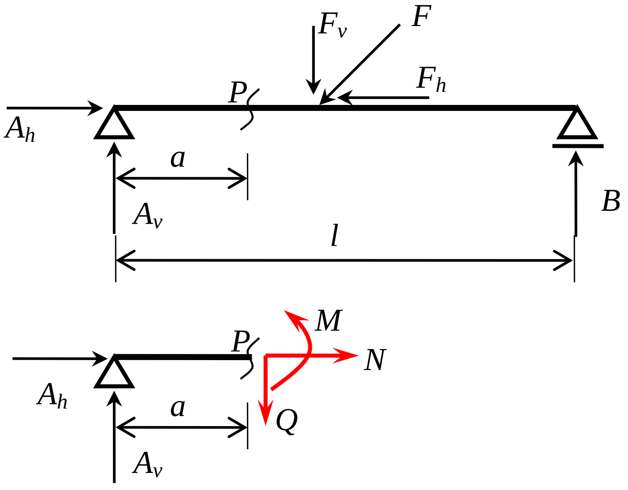
Furthermore when we talk about internal forces in a beam element then the forces and the moments are integral (a reduction) of the stresses acting in the center of the cross section of the beam. When the beam is under planar load the as an internal forces we have two forces: axial and transversal and one moment. Their positive positions are shown at the next figure:



In our next explanations, we will show it in the simple way as following:



From the physics is known that all actions have counteractions. Therefore, if we know the moment and the resultant forces of the load we may say the values of the internal forces of the beam element, because they are equal. Actually if we know the support, reactions and loads we just needs to compose the three equilibrium equations for the cross sectional point and will find the values of the internal forces as it is shown at the next figure:



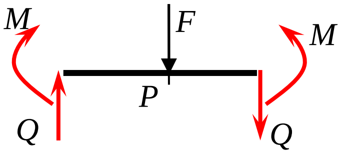
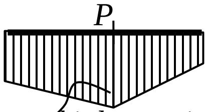
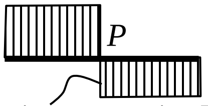
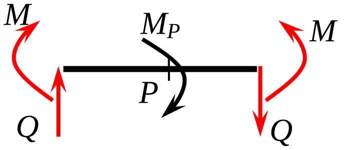
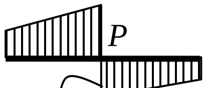
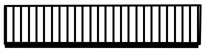
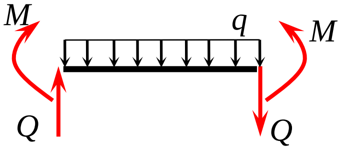
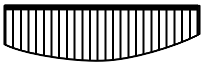
$$\sum H = 0: \qquad A_h + N = 0 \rightarrow N = -A_h$$
$$\sum V = 0: \qquad A_v - Q = 0 \rightarrow Q = A_v$$
$$\sum M_P = 0: \qquad A_v.a - M = 0 \rightarrow M = A_v.a$$

Along with we always may determine the internal forces at each character point of the beam. In addition, if we know some rules we may compose the internal force diagram. Some of these rules are as follow:

1) At force load point the internal moment diagram has a kink and a shear force diagram has a jump;
2) If some section of the beam hasn't any load then the internal moment diagram is linear and shear force diagram is a constant;

3) If some section of the beam is under distributed load then the internal moment diagram is parabolic of second degree and shear force diagram is linear;

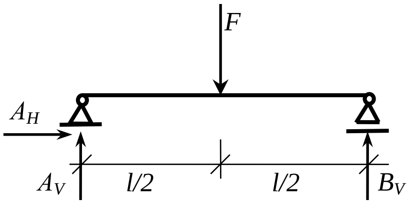
Next table illustrates these and some other rules briefly:

Load	Moment Diagram	Shear Diagram
	 kink at point P	 jump at point P
	 jump at point P	 constant
	 parabolic function	 linear function

Now it will be illustrated procedure of analysis of some elementary structures:

1. **Support reactions.** Before the member is “cut” or sectioned it is necessary to determine the member’s support reactions so that the equilibrium equations are written for the whole member.
2. **Internal force diagrams:** The members should be sectioned or “cut” at specific points and equilibrium equations should be written for the separate part for determining internal forces and to compose the internal forces diagram.

Example 1: Simple beam under point force load.



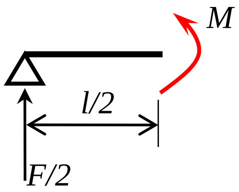
1. **Support reactions.**

$$\sum H = 0: \qquad A_h = 0$$

$$\sum M_A = 0: \qquad B_v.l - F.\frac{l}{2} = 0 \rightarrow B_v = \frac{F}{2}$$

$$\sum M_B = 0: \qquad A_v.l - F.\frac{l}{2} = 0 \rightarrow A_v = \frac{F}{2}$$

2. **Internal force determination.** The beam has two sections and one specific point at the middle of the beam to find the moment and shear values.

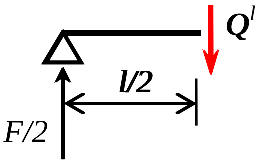


$$\sum M = 0:\ M = A_v.\frac{l}{2} = \frac{F}{2}.\frac{l}{2} = \frac{F.l}{4}$$

The internal moment is positive because the lower bands are bended. The moment diagram should be plotted at the bended side of the member. The moment diagram is linear with a kink under the load point.

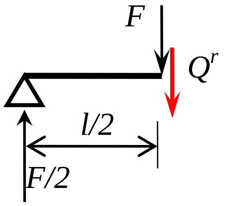
About the internal shear force, we have two specific points at the middle of the beam, one next before the force end the second next after the load force. The reason of this is to determine the jump in the shear force diagram as it is illustrated in the up table. The shear force diagram is a constant with a jump under the load point.

next before the load force:



$$\sum V = 0:\ Q^l = A_v = \frac{F}{2};$$

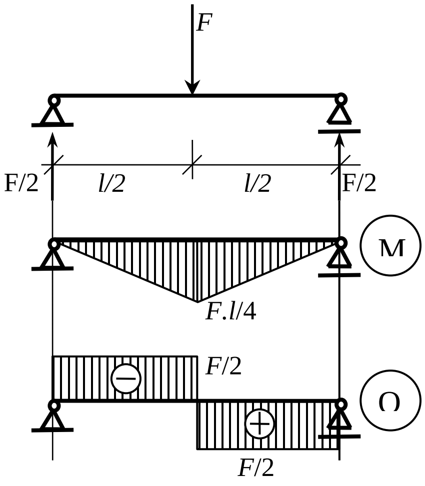
next after the load force:



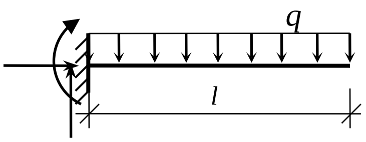
$$\sum V = 0:\ Q^r = A_v - F = -\frac{F}{2}$$

The normal force is zero because there isn't any horizontal load.

3. **Internal force diagram.**

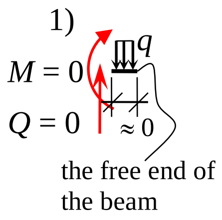


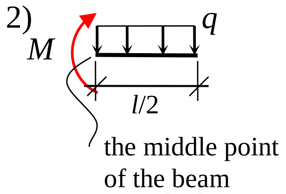
Example 2: Cantilever beam under distributed load.



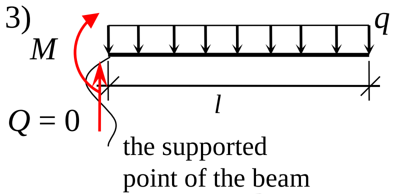
In this case, it is not necessary to determine the support reactions because the right side of the body is free of supports and it is possible to cut and to separate the right-hand side part of the member and to determine the internal forces. The distributed load loads the cantilever beam so that the internal moment diagram should be parabolic function. There are needed three values for plotting the diagram. The first one can be the free end of the beam, the second one is at the middle of the beam and the last one is at the support.

The shear force diagram is linear and it is sufficient to determine its value at two points – at the free end and at the supported end of the beam.





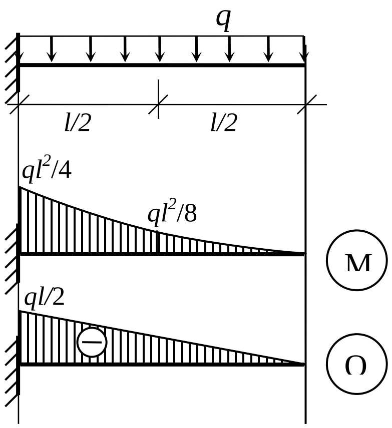
$$\sum M = 0 : M + q.\frac{l}{2}.\frac{l}{4} = 0 : M = -\frac{ql^2}{8}$$



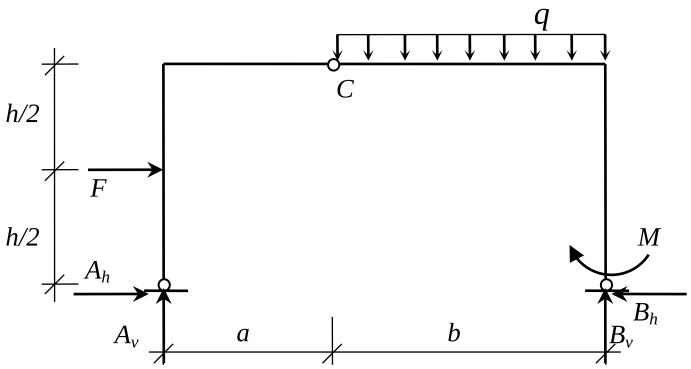
$$\sum M = 0 : M + q.l.\frac{l}{2} = 0 : M = -\frac{ql^2}{2}$$

$$\sum V = 0 : Q - q.l = 0 : Q = q.l$$

1. **Internal force diagram.**



Example 3: Three-hinged frame.



1. **Support reactions.**

The three-hinged frame has two support reactions at each pin support. It is possible to write only three equilibrium equations to the whole frame but there are four reactions. It is necessary to find one more equation. The best equation in this case is a moment equation about the middle hinge. The reason of this is that we know the moment at the hinge point is zero. When we write this equation, we take one part of the frame – the right one or the left. If we take the right one we use the left part to verify the results. If we take the left one, we use the right for verification. This is illustrated bellow.

Equilibrium equations for the whole frame:

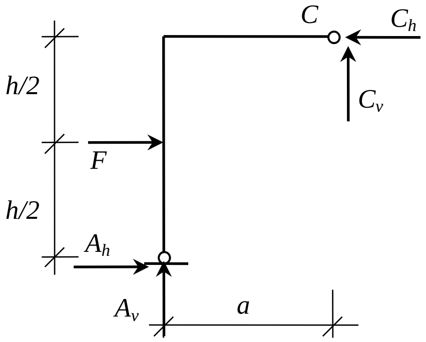
$$\sum M_A = 0 : F.\frac{h}{2} + M + q.l.\left(a + \frac{b}{2}\right) - B_v.(a+b) = 0 \rightarrow B_v = \cdots$$

$$\sum M_B = 0 : F.\frac{h}{2} + M - q.l.\frac{b}{2} + A_v.(a+b) = 0 \rightarrow A_v = \cdots$$

$$\sum H = 0 : A_h + F - B_h = 0 \rightarrow B_h = A_h + F$$

The case of frames with pin supports on one level is easier because from the first two equations we can find directly the values of the vertical support reactions.

Equilibrium equation for the middle hinge of the frame (the right-hand side):



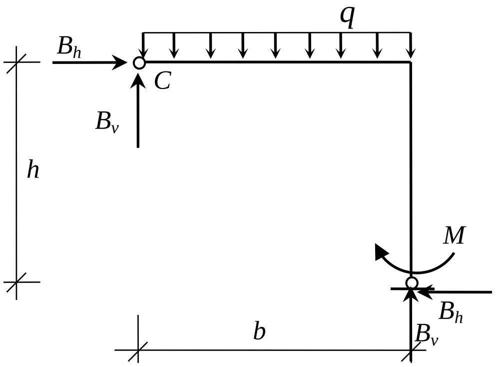
$$\sum M_C^l = 0 : -F.\frac{h}{2} + A_v..a - A_h.h = 0 \rightarrow A_h = \cdots$$

$$B_h = A_h + F \rightarrow B_h = \cdots$$

Verification of the results:

Equilibrium equation for the middle hinge of the frame (the left-hand side):

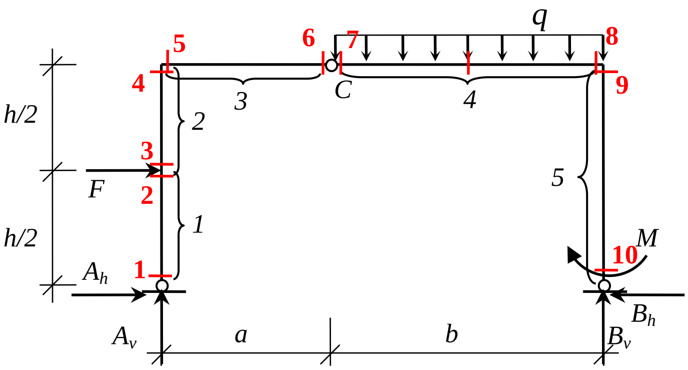
$$\sum M_C^r \overset{?}{=} 0 : M + q.l.\frac{b}{2} - B_v.b + B_h.h \overset{?}{=} 0$$



Equilibrium equation for the whole frame:

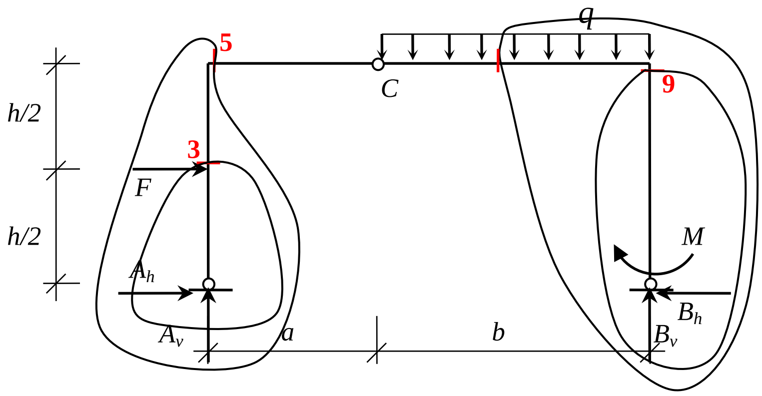
$$\sum V \overset{?}{=} 0 : A_v - q.l + B_v \overset{?}{=} 0$$

As we know the correct values of the support reactions, we may compose the internal forces diagrams. In the present frame, we have 5 segments (shown on the figure bellow) for which we should determine these diagrams. Consequently, we have at least 10 specific points for the shear force and 7 for the moment because we know the moment value at the hinges is zero. The major specific points are shown on red at the scheme bellow.

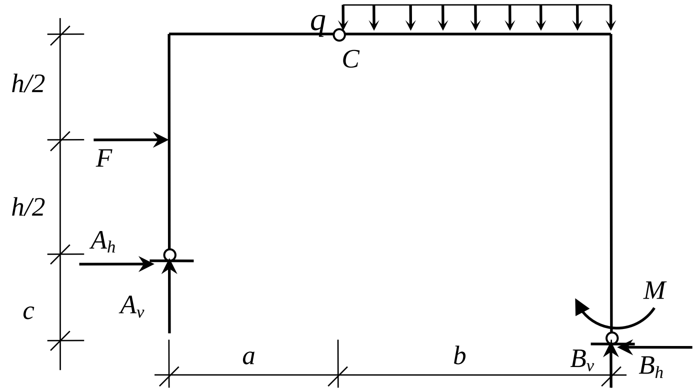


Actually, we may compose the internal moment diagram using only 3 specific points and using this diagram we compose the shear and normal forces diagrams but if we know all characteristics of the internal forces which will be explained in the next chapter.

Composing the diagram, we cut part of the frame and separate it. After that, we calculate the moment the shear and normal forces at the specific point. The way of separating of the frame for some of the specific points is shown at the figure bellow.



More details about diagram composing will be shown in later. Now will be demonstrated the difference when the pin support of the frame are at different level. The mane procedure of solution is the same as previous with only one different aspect, namely that we cannot find directly the support reactions. In this situation we have system of linear equations.



One way for determining the reactions is as follow:

$$\sum M_A = 0: F.\frac{h}{2} + M + q.l.\left(a + \frac{b}{2}\right) - B_v.(a+b) + B_h.c = 0$$

$$\sum M_B = 0: F.\frac{h}{2} + M - q.l.\frac{b}{2} + A_v.(a+b) + A_h.c = 0$$

$$\sum H = 0: A_h + F - B_h = 0$$

$$\sum M_c^l = 0: \ -F.\frac{h}{2} + A_v.a - A_h.h = 0$$

Verifications:

$$\sum M_C^r \overset{?}{=} 0: M + q.l.\frac{b}{2} - B_v.b + B_h.h \overset{?}{=} 0$$

$$\sum V \overset{?}{=} 0: A_v - q.l + B_v \overset{?}{=} 0$$

Another way is similar as next:

$$\sum M_A = 0: F.\frac{h}{2} + M + q.l.\left(a + \frac{b}{2}\right) - B_v.(a+b) + B_h.c = 0$$

$$\sum M_B = 0: F.\frac{h}{2} + M - q.l.\frac{b}{2} + A_v.(a+b) + A_h.c = 0$$

$$\sum H = 0: A_h + F - B_h = 0$$

$$\sum M_C^r = 0: M + q.l.\frac{b}{2} - B_v.b + B_h.h = 0$$

Verifications:

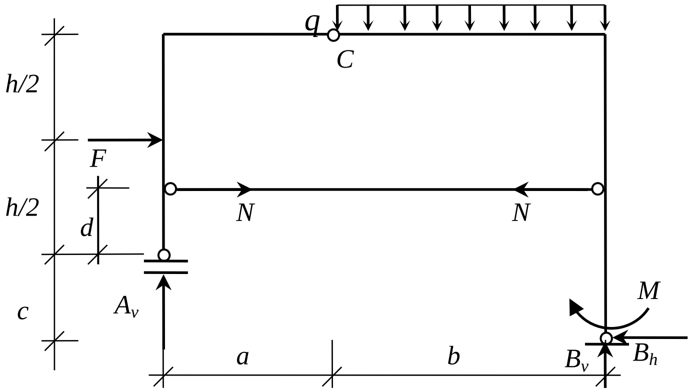
$$\sum M_c^l \overset{?}{=} 0: \ -F.\frac{h}{2} + A_v.a - A_h.h \overset{?}{=} 0$$

$$\sum V \overset{?}{=} 0: A_v - q.l + B_v \overset{?}{=} 0$$

The difference is only at the fourth equation and the first one at the verifications. Every four equilibrium equations are useful for determination of the support reactions and at least one equation verifying results. Therefore, these two variants are not only the possible. **Nevertheless, one should**

be always careful with a partial equations and one should know that only 3 equations can be written for the whole system! The composing of the internal forces diagrams is the same.

The next modification of the three-hinged frame is the tied three-hinged frame. We will illustrate its' solution.



One of the supports of this frame is roller and another is pin. As addition is added a link of type I with only one internal force – the normal one. That is why the three equations for the whole system are enough for reactions determination. Whit using the partial equation we determine the normal force at the link.

$$\sum H = 0: F - B_h = 0 \to B_h = \cdots$$

$$\sum M_A = 0: F.\frac{h}{2} + M + q.l.\left(a + \frac{b}{2}\right) - B_v.(a+b) + B_h.c = 0 \to B_v = \cdots$$

$$\sum M_B = 0: F.\frac{h}{2} + M - q.l.\frac{b}{2} + A_v.(a+b) = 0 \to A_v = \cdots$$

$$\sum M_c^l = 0: -F.\frac{h}{2} + A_v.a - N.(h-d) = 0 \to N = \cdots$$

Verifications:

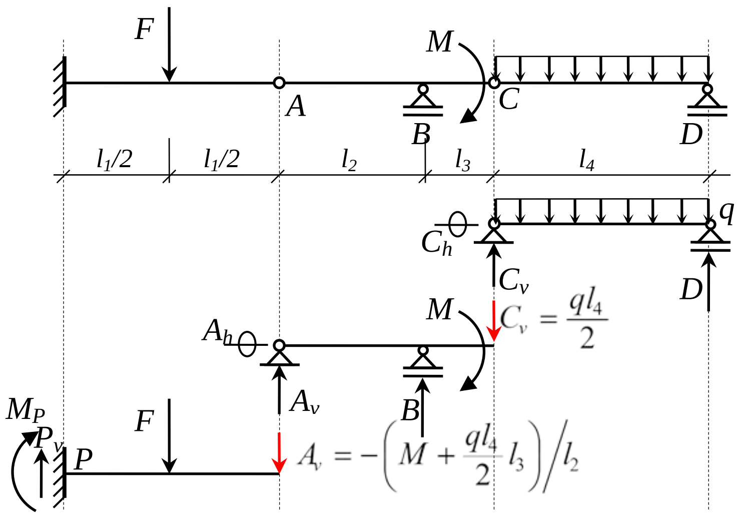
$$\sum M_C^r \overset{?}{=} 0: M + q.l.\frac{b}{2} - B_v.b + N.(h-d) \overset{?}{=} 0$$

$$\sum V \overset{?}{=} 0: A_v - q.l + B_v \overset{?}{=} 0$$

The composing of the internal forces diagrams is the same.

Example 4: Compound beam:

As it is known from the kinematical analysis, basic and secondary beams compose the compound beam. The secondary beams transfer the loads to the basics ones. That is why we first analyze the secondary beams and with their support reactions we load the basic beam.



1. **Support reactions.**

Beam CD – simple beam:

$$\sum H = 0: \qquad C_h = 0$$

$$\sum M_D = 0: \qquad C_v.l_4 - ql_4.\frac{l_4}{2} = 0 \rightarrow C_v = \frac{ql_4}{2}$$

$$\sum M_C = 0: \qquad D_v.l_4 - ql_4.\frac{l_4}{2} = 0 \rightarrow D_v = \frac{ql_4}{2}$$

Beam AB – simple beam with overhangs:

$$\sum H = 0: \qquad A_h = 0$$

$$\sum M_B = 0: \qquad A_v.l_2 + M + \frac{ql_4}{2} l_3 = 0 \rightarrow A_v = -\left(M + \frac{ql_4}{2} l_3\right) \Big/ l_2$$

$$\sum M_A = 0: \qquad \frac{ql_4}{2}(l_3 + l_2) + M - B.l_2 = 0 \rightarrow B = \cdots$$

Beam PA – cantilever beam:

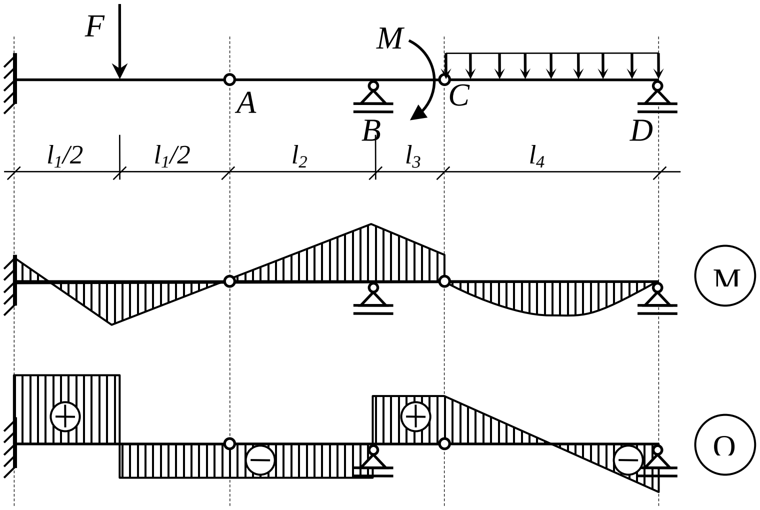
$$\sum H = 0: \qquad P_h = 0$$

$$\sum M_P = 0: \qquad A_v.l_1 + F.\frac{l_1}{2} + M_P = 0 \rightarrow M_P = \cdots$$

$$\sum V = 0: \qquad P_v - F - A_v = 0 \rightarrow P_v = \cdots$$

2. **Internal force diagram.**

The diagram composing makes for every beam separately and after that, we join them for the whole compound beam. Here we show only the shape of the final diagrams because the exact values are not so important at this moment.


F
M
A
B
C
D
$l_1/2$
$l_1/2$
l_2
l_3
l_4
M
Q

Chapter 4
Shear and moment functions.
Analysis of structures type I.

As it is known from the "Strength of materials", there is a connection between the internal moment and the internal shear force at a beam element. As a basis of this connection, we will discus some characteristics of the moment and shear diagram. As we say already if one know this characteristics may compose these diagrams without any problems and with a few calculations and as most important on can check if the composed diagrams are correct or not.

Connections between distributed loads, shear and moment functions:

$$\frac{dQ}{dx} = -q(x)$$

$$\left.\begin{array}{r}\text{slope of shear}\\ \text{diagram}\end{array}\right\} = \left\{\begin{array}{l}\text{intensity of}\\ \text{distributed load}\end{array}\right.$$

$$\frac{dM}{dx} = Q(x)$$

$$\left.\begin{array}{r}\text{slope of moment}\\ \text{diagram}\end{array}\right\} = \left\{\text{shear function}\right.$$

The first equation states that the slope of the shear diagram at a point is equal to the intensity of the distributed load at the point. Likewise, the second equation states that the slope of the moment diagram is equal to the shear at the point. These equations can be integrated from one point to another between concentrated point or couples and as a result we have as follow:

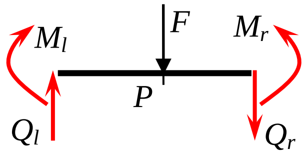
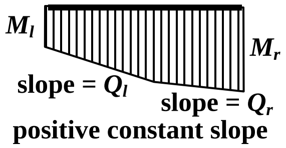
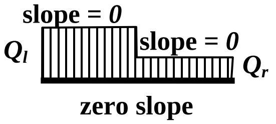
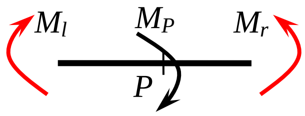
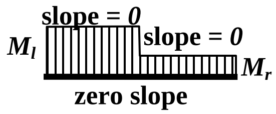
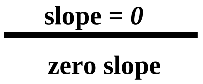
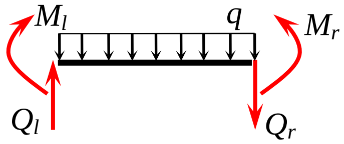
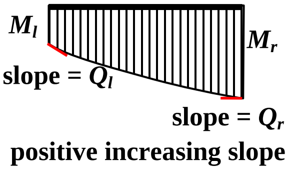
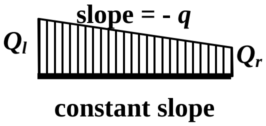
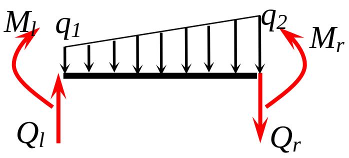
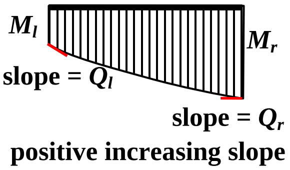
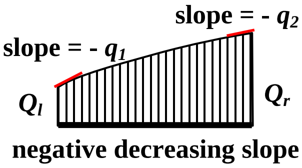
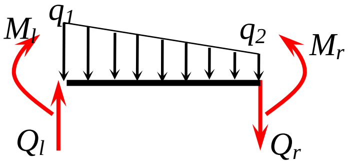
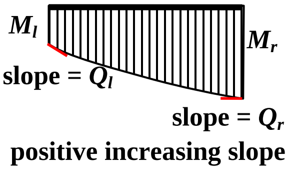
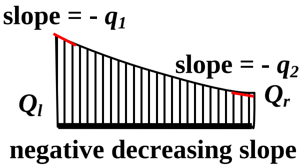
$$\Delta Q = -\int q(x)dx$$

$$\left.\begin{array}{r}\text{change in}\\ \text{shear}\end{array}\right\} = \left\{\begin{array}{l}\text{area under distributed}\\ \text{loading diagram}\end{array}\right.$$

$$M = \int Q(x)dx$$

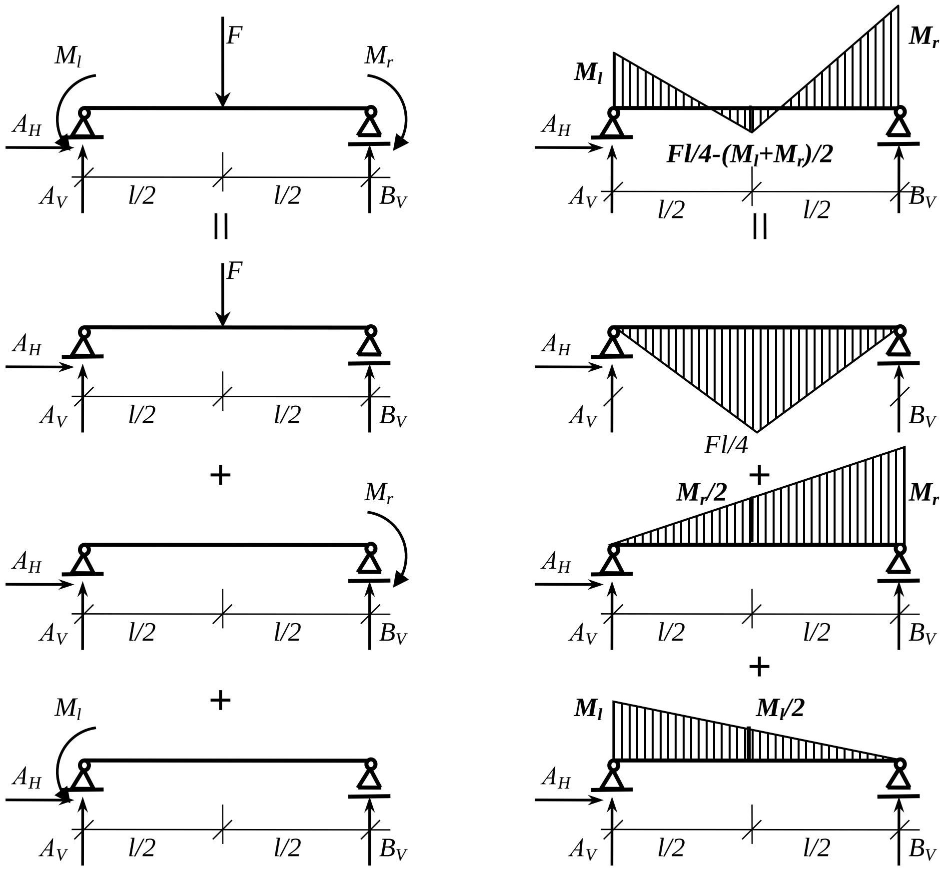
$$\left.\begin{array}{r}\text{change in}\\ \text{moment}\end{array}\right\} = \left\{\begin{array}{l}\text{area under}\\ \text{shear function}\end{array}\right.$$

As it is noted the first equation stats that the change in the shear between any two points on a beam equals the area under the distributed loading diagram between the points. Likewise, the next equation states that the change in the moment between the two points equals the area under the shear diagram between the points. If the area under the load and shear diagrams are easy to compute then these equations can be used for determining the numerical values of the shear and the moment at a various points along a beam except points with a concentrated force or moment. The next table illustrates the application of these equations for some common loadings cases. The slope at various points is indicated. Each of these results should be studied carefully so that one becomes fully aware of how shear and moment diagrams can be constructed based on knowing the variation of the slope from the load and the shear diagram respectively.

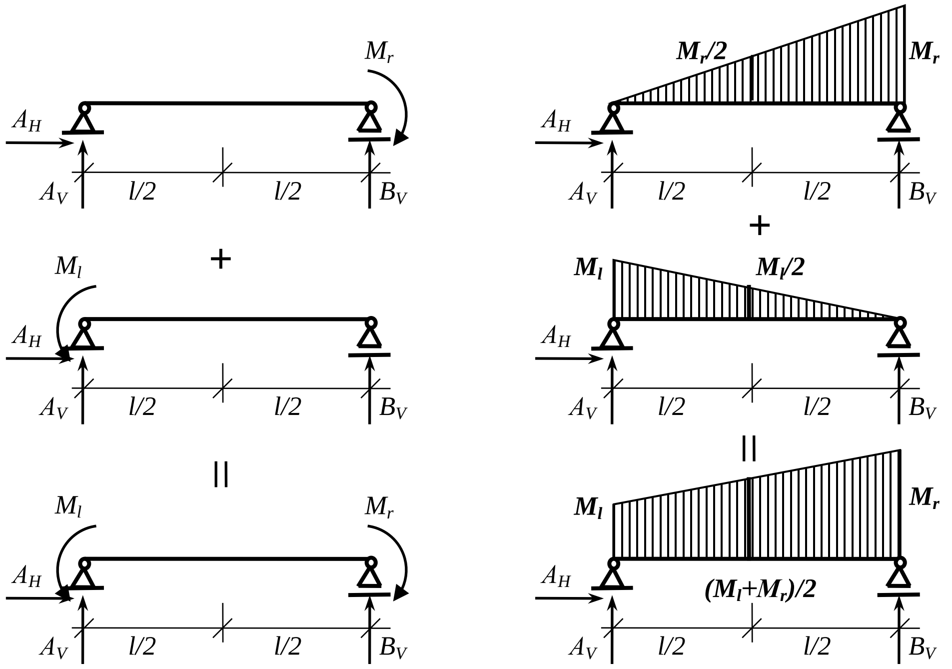
Load	Moment Diagram	Shear Diagram
		
		
		
		
		

As an addition helping information we will explain the usage of a method of superposition for composing the diagrams. We already notice above that one system loaded by more then one loads can by solved for every one of then separately and the result is a sum of all separate solutions. The next figure illustrates shortly the principle of superposition:

In this case we have simply supported beam loaded by a point force at the middle and two moments at the two ends of the beam. The moment diagram from these loads is shown at the right top of the figure and the separates loads and diagrams are shown bellow.

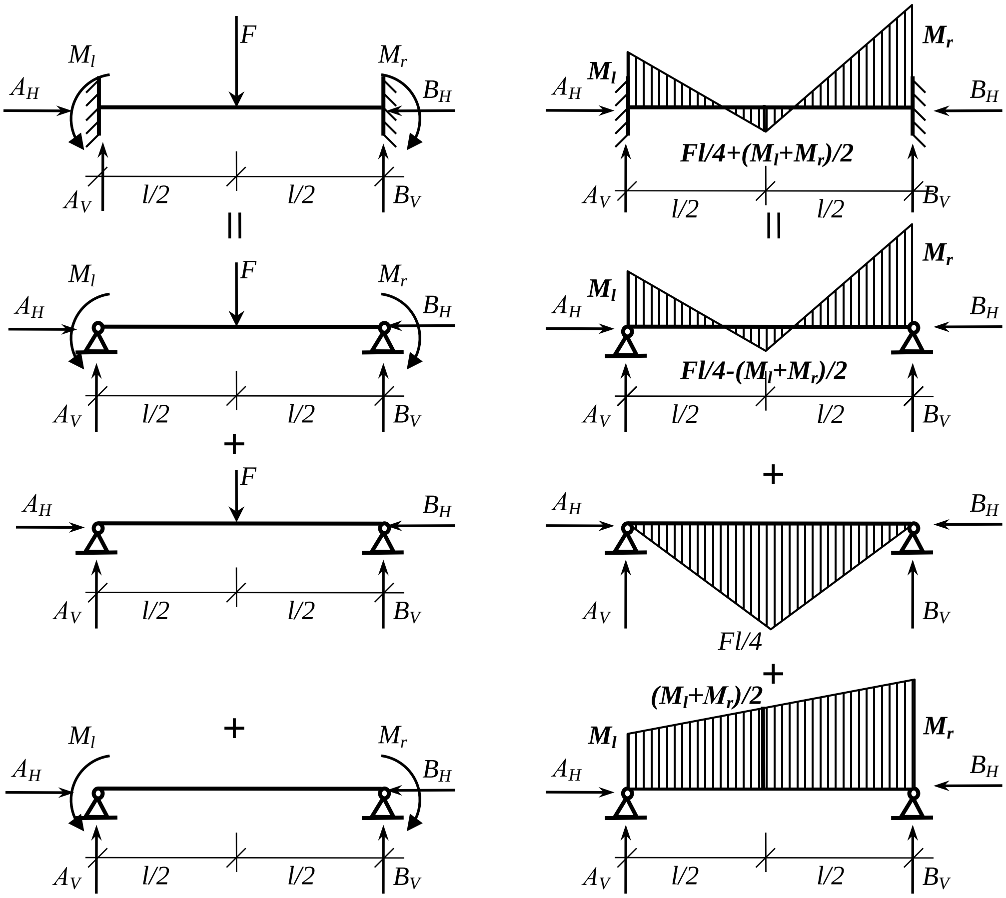


As one can see the two moments, acting at the two ends of the beam can be considered together and if we know their values, it is very easy to compose the diagram. It is always a trapezium as it is shown bellow and the middle value of the diagram equals to the middle value of the trapezium. Therefore, if we know that for the above simple beam is only needed to add the diagram from the point load to the trapezium and the summary diagram will be computed.



Now we will continue with another aspect of the usage of the superposition principle. Let us consider a beam element type III as it is shown at the next picture. Let us consider we know all support reactions caused by the point load for example. Let us now compose the moment diagram in the beam element.

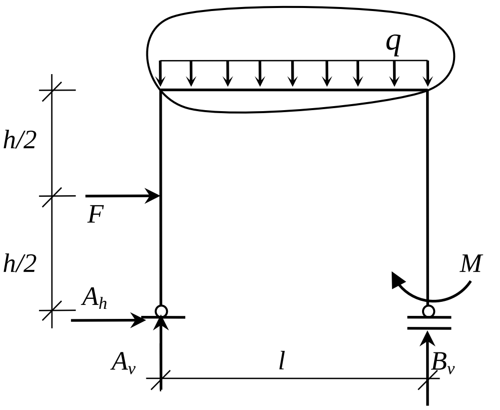
Here all support reactions are known so the force system of the beam element produces zero force and moment resultant or in other words the system is in equilibrium state. The moment diagram is shown right to the beam element. Note that the same diagram will be produced by simple beam loaded by the same load system (including support reactions), because the load system is at equilibrium state.. In this case, the two horizontal reactions are included but they are equal and are not significant for the solution.

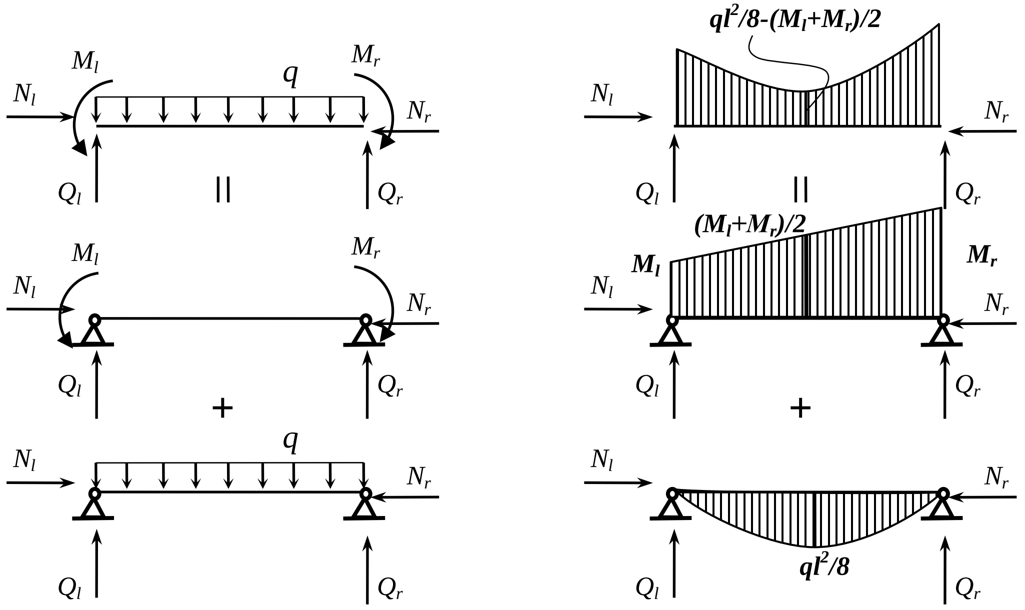


As a result, we consider a simple beam with known moments at the ends and a point load at the middle as in the previous example. Therefore, for composing the summary moment diagram it is enough to sum the middle value of the trapezium (received by the two moments) and the middle value of the moment diagram in the simple beam loaded by the point load at the middle point. Note that nowhere we use the vertical and horizontal support reactions. Only the two ends moment as support reactions are used.

The moment diagram produced by these two ends moment we call “reference diagram” and we us it as a benchmark. The diagram caused by the external load (the point load in this example) we call “additional diagram”. Superposing the reference diagram with the additional one, we receive the summary moment diagram. Note that the additional diagram is always at a simple beam. The next example illustrates this again.

Let us consider a part of the frame loaded by the distributed load. Notice that if we separate this part it will be the same as a beam element type III as a previous one. Therefore, the summary moment diagram can be produced as a summation of a reference and additional diagram if we know the two ends moment of the frame part.





Note that in this case again the internal horizontal and vertical forces are not included in the solution. Therefore, it is enough to know the two ends moment of the frame part.

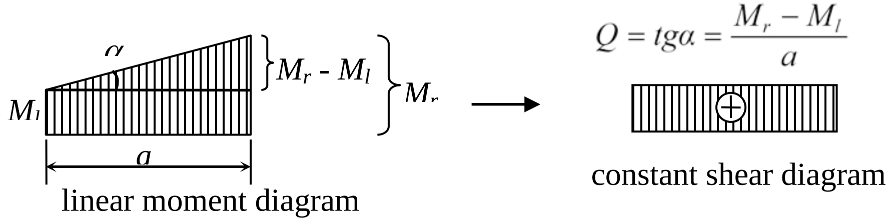
This usage of the superposition principle is the most powerful and faster method for composing the moment diagram. That is why we will use it further.

The next step is the composing the shear force diagram with a faster method. This is very easy if we use the connections between the moment and the shear force. We already explained this connection and now we will illustrate its usage.

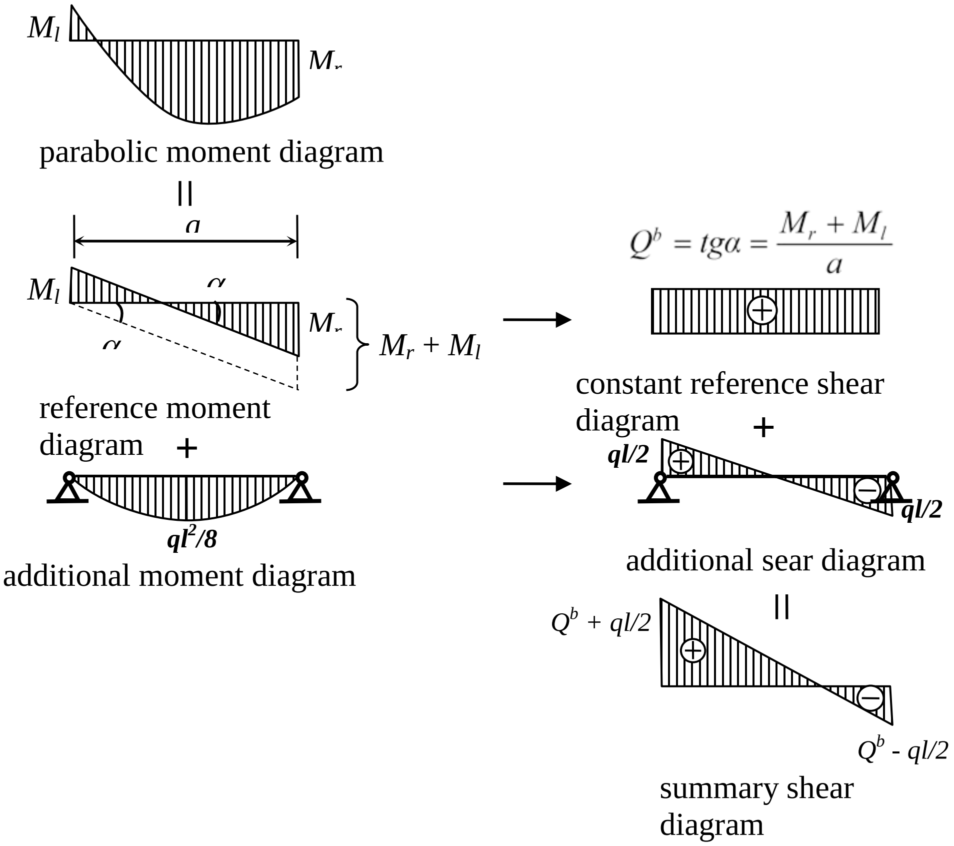
As two important rules, we will mention the next:

- The shear force is takes from the moment diagram as its’ tangent at a point (the shear force is equal to the slope of a moment at a point).
- If the moment diagram is a rising function then the shear is positive and if the moment is decreasing function then the shear is negative.

In most cases, the moment function is linear so the tangent is very easy to find and the shear function is a constant:



If the moment diagram is parabolic function then we decompose it to a reference – linear and additional diagram and compose the shear force diagram using the same idea. The reference shear diagrams takes from the reference moment diagram as a tangent and the additional shear diagram is a diagram at a simple beam:

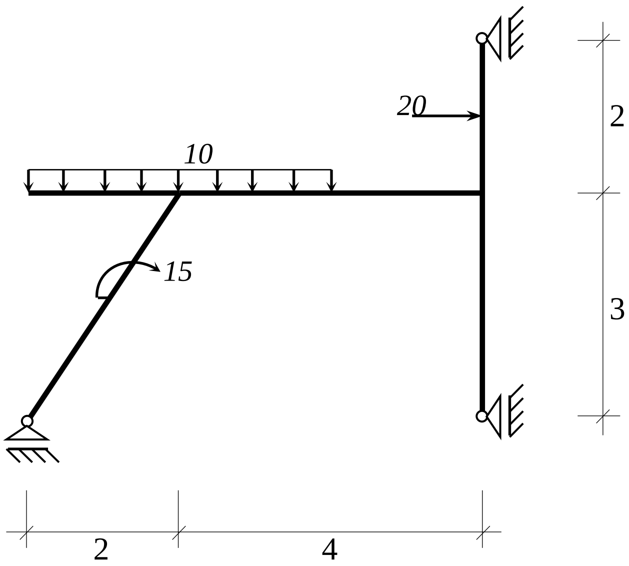


The normal force diagram composes using the support reactions and shear forces at the corners nodes.

Analysis of structures type I.

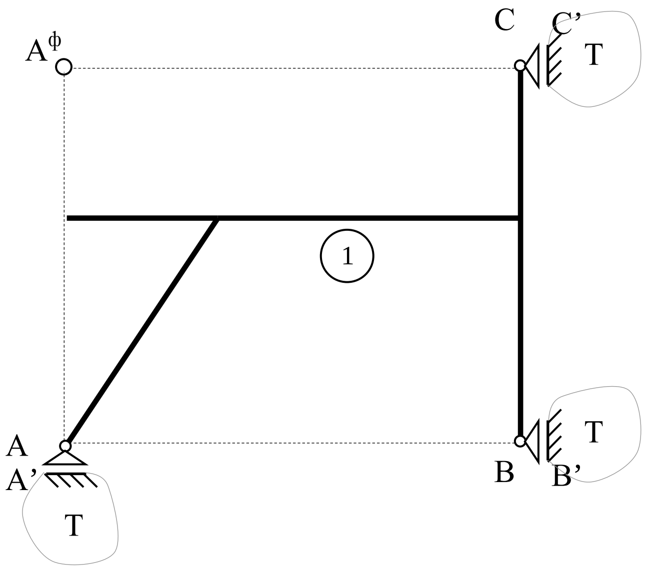
As we already know haw to compose diagrams faster and easier we are ready to analyze fully different complicated structures. First, we should make the kinematical analysis of the structure. The kinematical analysis consist a way of composing the complicated structure. If we use only stable basic elements like a cantilever beam, a simple beam or a dyad the result should be stable structure. If the structure is statically determinate and stable we may analyze it in way opposite to its composing. After that for the decomposed structure, we calculate the support reactions and summaryly using decomposed structure, we produce the internal forces diagrams.

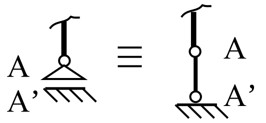
Example 1:

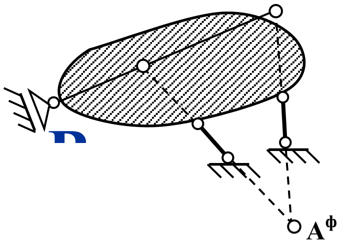


1. Kinematical analysis:

$$\left[T + AA'.1.BB'(n = +1) + CC'(n = -1)\right](n = 0)$$





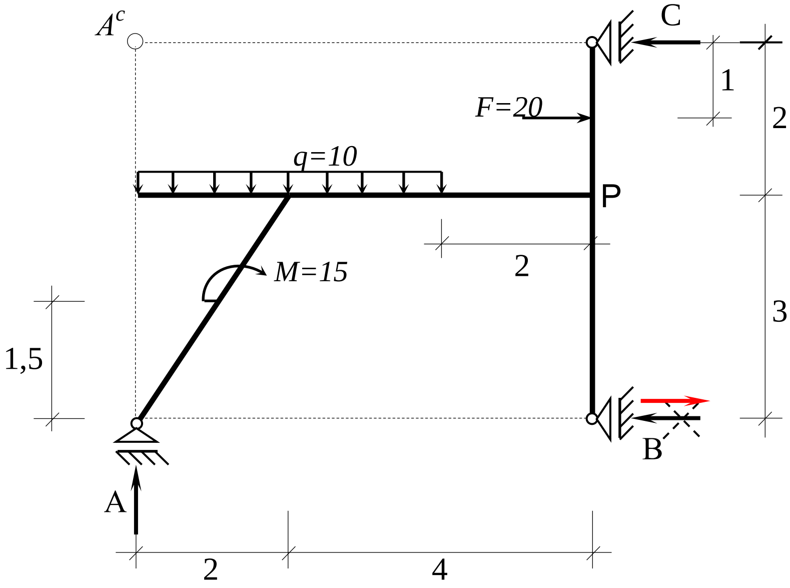


The system statically determinate and is composed with using of chains and links, so it is second type. Verification of instantaneously unstable is necessary. The system can be identified like elementary one “simple beam” type. The directions of the three support links are not crossing at one point therefore, the system is stable.

2. Support reactions.

This system is elementary one that is why it is not necessary to decompose it. We may compute the support reactions and internal forces diagrams directly.

Lets equilibrium equations be written for such a points if possible for which only a single unknown participate.



$$\sum M_{A^c} = 0: \quad 15+10.4.2+B.5-20.1=0 \quad \rightarrow \quad B=-15$$

$$\sum M_{A} = 0: \quad 15+10.4.2-C.5+20.4=0 \quad \rightarrow \quad C=35$$

$$\sum V = 0: \quad A-10.4=0 \quad \rightarrow \quad A=40$$

Verifications :

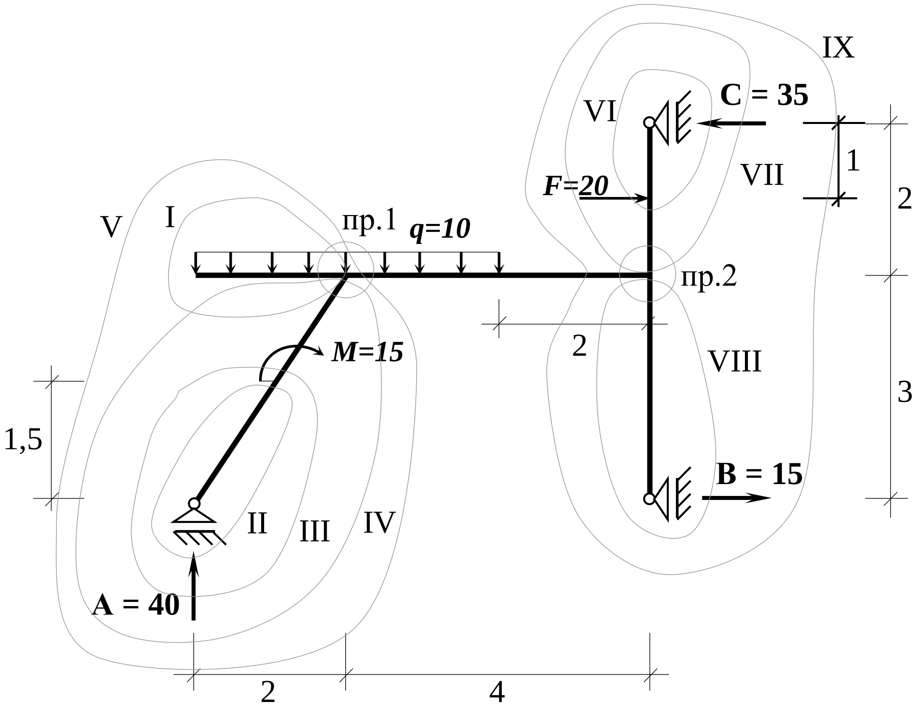
$$\sum H = 0: \quad 20+15-35\overset{?}{=}0 \quad \rightarrow 35-35=0$$

$$\sum M_{P} = 0: \quad 15-10.4.4-35.2+20.1-15.3+40.6\overset{?}{=}0 \quad \rightarrow 275-275=0$$

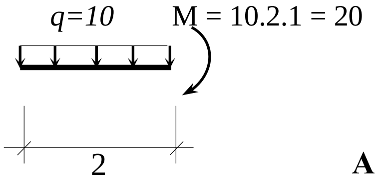
For verification of support reactions we use usually at minimum one moment equation. As advise use such a point for wich participate maximum number of already determinate supports reactions.

3. Internal forces diagrams.

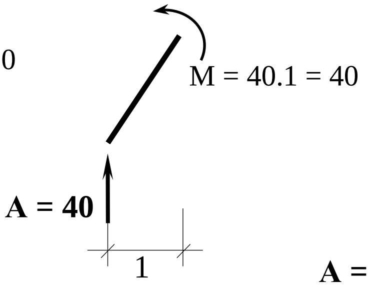
Moment diagram is organized by sections at characteristic points and the already explained properties of the diagram. In this system, the locations of the sections and their sequence are shown on the next figure. The separated sections are also shown. The shear diagram is produced using the moment diagram according to their connections and the normal force diagram is composed by the shear force diagram and supports reactions. At last we verifications are made using the corner nodes force equilibrium.



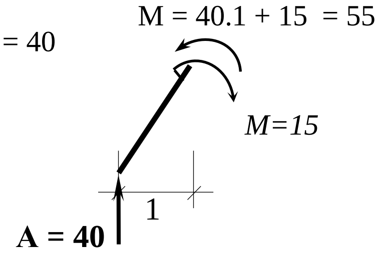
I – I



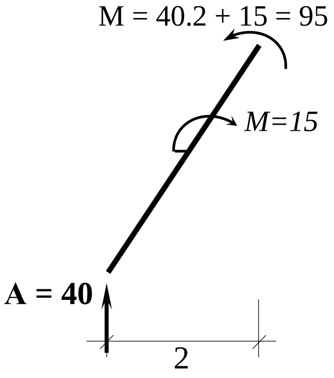
II – II



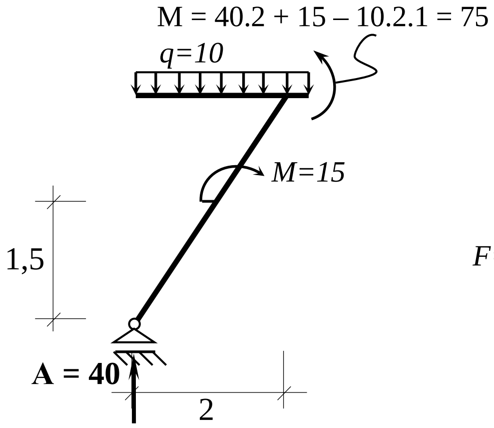
III – III



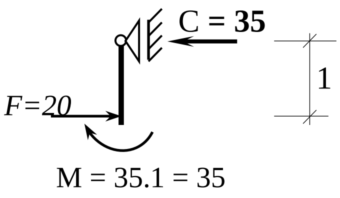
IV – IV



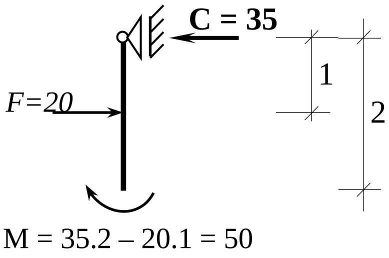
V – V



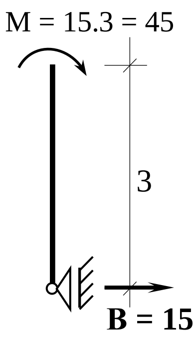
VI – VI



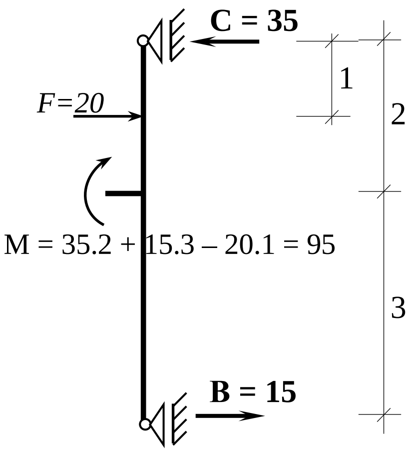
VII – VII



VIII – VIII

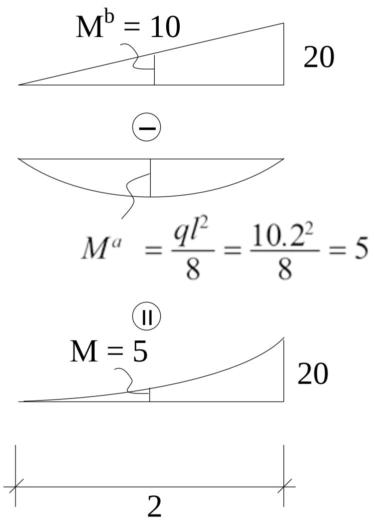


IX – IX

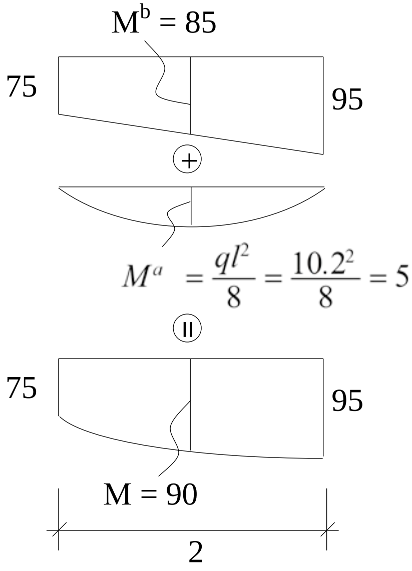


Determination of the moment and shear diagrams for the parts loaded by the distributed loads.

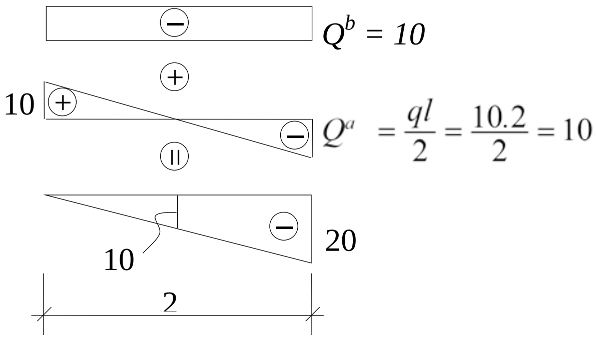
for the cantilever part:



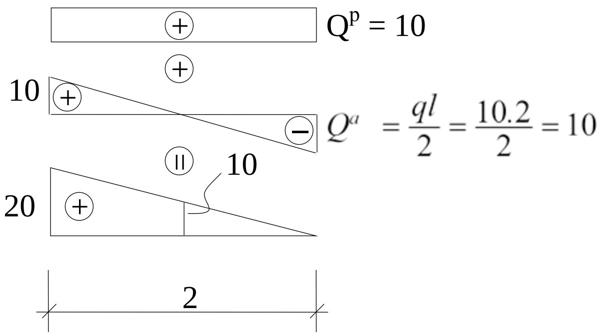
for the internal part:

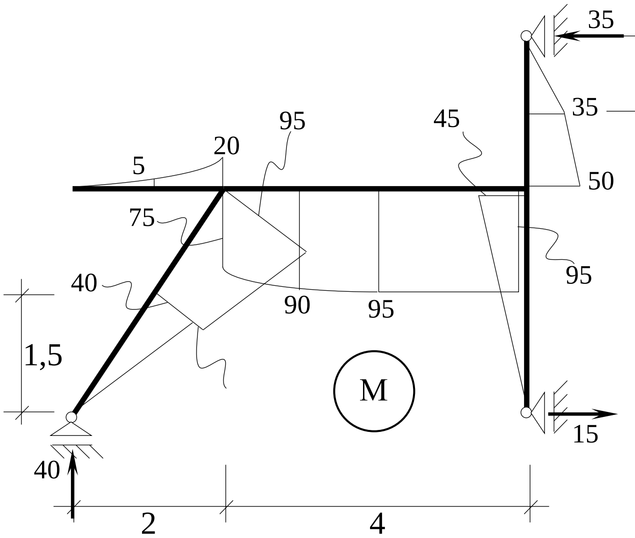


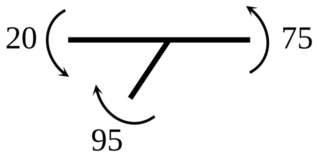
for the cantilever part:



for the internal part:

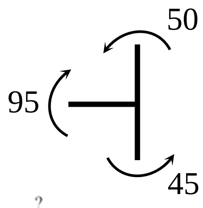






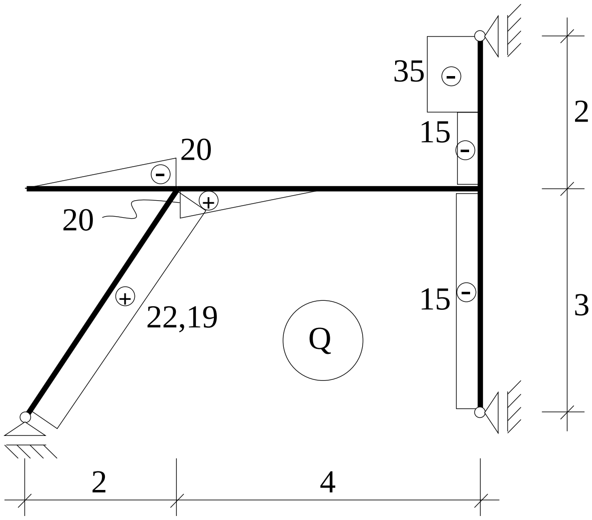
$$\sum M \overset{?}{=} 0: \qquad 95 - 75 - 20 \overset{?}{=} 0$$

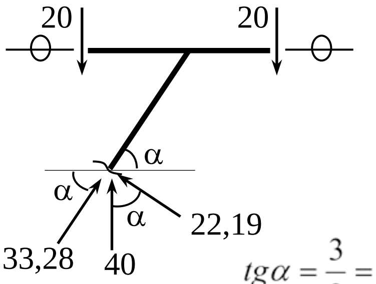
$$95 - 95 = 0$$



$$\sum M \overset{?}{=} 0: \qquad 95 - 50 - 45 \overset{?}{=} 0$$

$$95 - 95 = 0$$





$$tg\alpha = \frac{3}{2} = 1{,}5$$

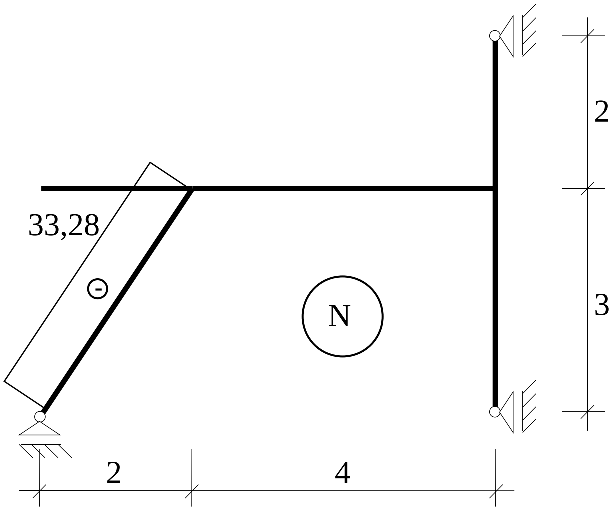
$$\alpha = 56{,}31^\circ$$

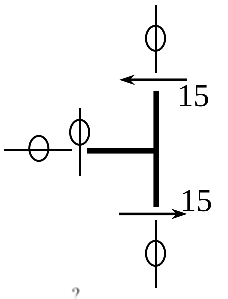
$$\sum H \overset{?}{=} 0: \qquad 33{,}28\cos\alpha - 22{,}19\sin\alpha \overset{?}{=} 0$$

$$18{,}460 - 18{,}463 = 0$$

$$\sum V \overset{?}{=} 0: \qquad 40 - 20 - 20 \overset{?}{=} 0$$

$$40 - 40 = 0$$

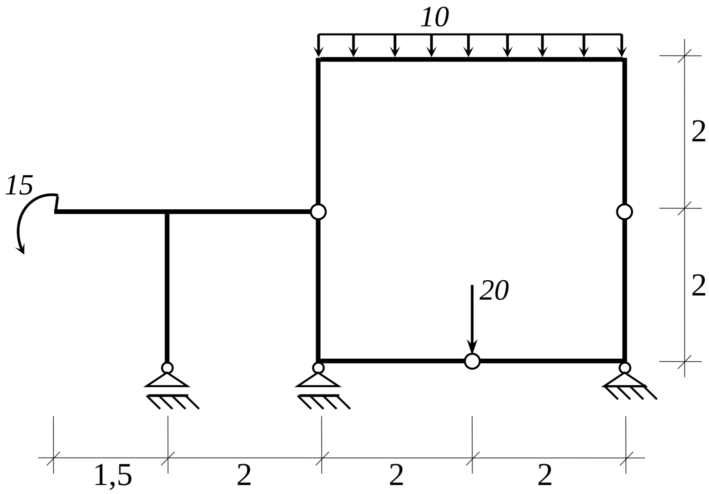




$$\sum H \overset{?}{=} 0: \qquad 15 - 15 = 0$$

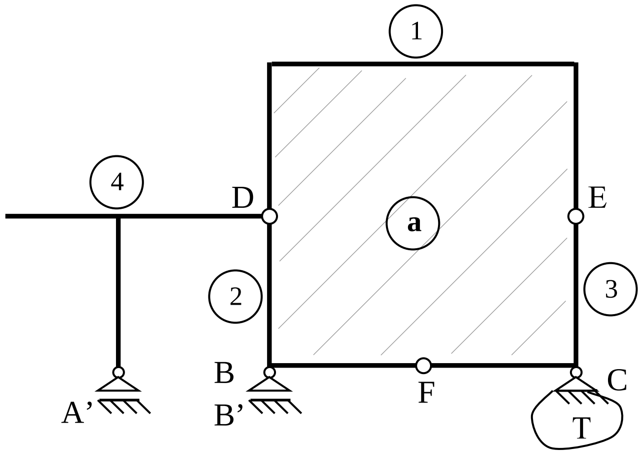
$$\sum V \overset{?}{=} 0: \qquad 0 = 0$$

Example 2:



1. Kinematical analysis:

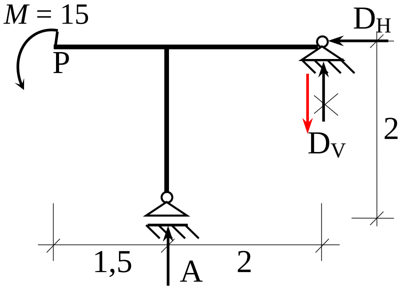
$$a = \left[1 + 2.3(n = 0)\right](n = 0)$$



$$\left[T + a.BB'(n = 0) + 4.AA'(n = 0)\right](n = 0)$$

This system is statically determine an composed by using the second way of kinematical composing. Therefore, the structure is cinematically stable and should be analyzed in order opposite of the composing order. That is why we will analyze first *disk 4* after that the common *disk a*, next the *dyad 2.3* and in the end *disk 1*.

2. Support reactions. *disk 4:*



$$\sum M_D = 0: \quad 15 - A.2 = 0: \quad A = 7,5$$

$$\sum M_G = 0: \quad 15 + D_V.2 = 0: D_V = -7,5$$

$$\sum H = 0: \quad D_H = 0$$

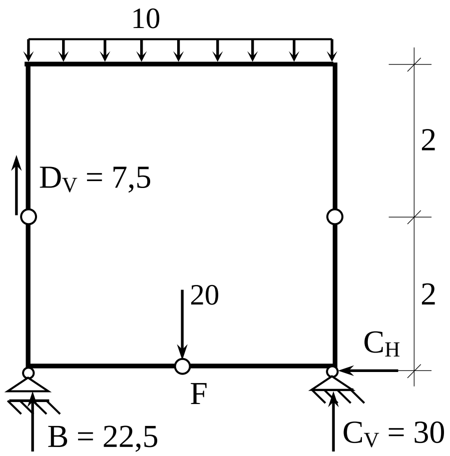
verification:

$$\sum V = 0: \quad 7,5 - 7,5 = 0$$

$$\sum M_P = 0: \quad 7,5.1,5 - 7,5.3,5 + 15 \overset{?}{=} 0$$

$$26,25 - 26,25 = 0$$

disk a:



$$\sum M_C = 0: \quad 20.2 + 10.4.2 - 7,5.4 - B.4 = 0 : B = 22,5$$

$$\sum M_B = 0: \quad 10.4.2 + 20.2 - C_V.4 = 0 : \quad C_V = 30$$

$$\sum H = 0: \quad C_H = 0$$

Verifications :

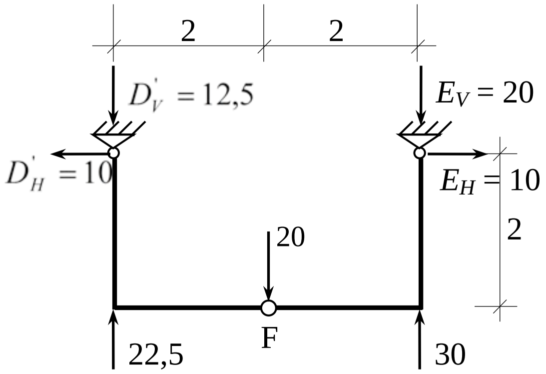
$$\sum V = 0: \quad 10.4 + 7,5 + 20 - 30 - 37,5 \overset{?}{=} 0$$

$$67,5 - 67,5 = 0$$

$$\sum M_F = 0: \quad 7,5.2 + 30.2 - 37,5.2 \overset{?}{=} 0$$

$$75 - 75 = 0$$

dyad 2.3:



$$\sum M_E = 0: \quad 22,5.4 - 20.2 - D_V^{'}.4 = 0 : \; D_V^{'} = 12,5$$

$$\sum M_D = 0: \quad 30.4 - 20.2 - E_V.4 = 0 : \quad E_V = 20$$

$$\sum M_F^{\text{д.ч.}} = 0: \quad 20.2 - 30.2 + E_H.2 = 0 : \quad E_H = 10$$

$$\sum H = 0: \quad D_H^{'} = 10$$

Verifications :

$$\sum V = 0: \quad 32,5 + 40 - 20 - 30 - 22,5 \overset{?}{=} 0$$

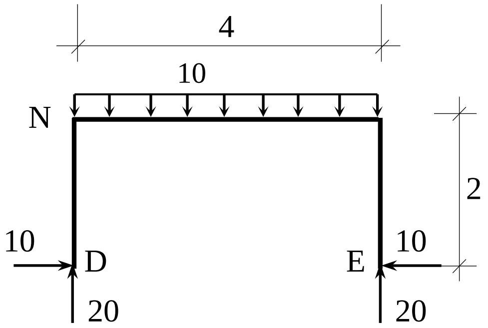
$$72,5 - 72,5 = 0$$

$$\sum M_F = 0: \quad 32,5.2 + 10.2 + 30.2 - 22,5.2 -$$

$$-10.2 - 40.2 \overset{?}{=} 0$$

$$145 - 145 = 0$$

disk 1: In this disk there is any unknown that is why its’ equilibrium is only for verifications:



Verifications :

$$\sum V = 0: \quad 20 + 20 - 10.4 \overset{?}{=} 0 :$$

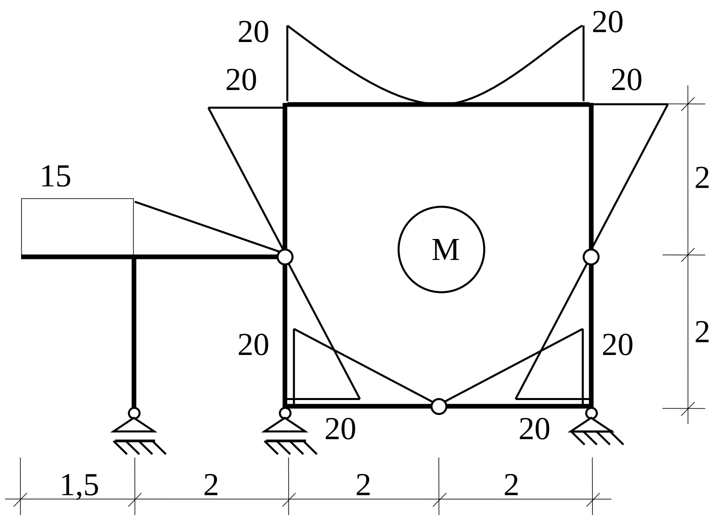
$$40 - 40 = 0$$

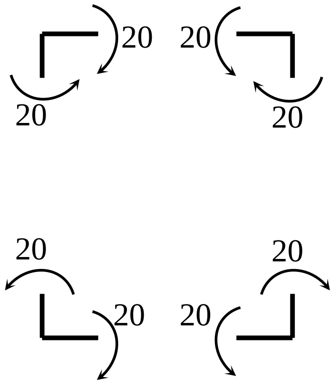
$$\sum H = 0: \quad 10 - 10 = 0$$

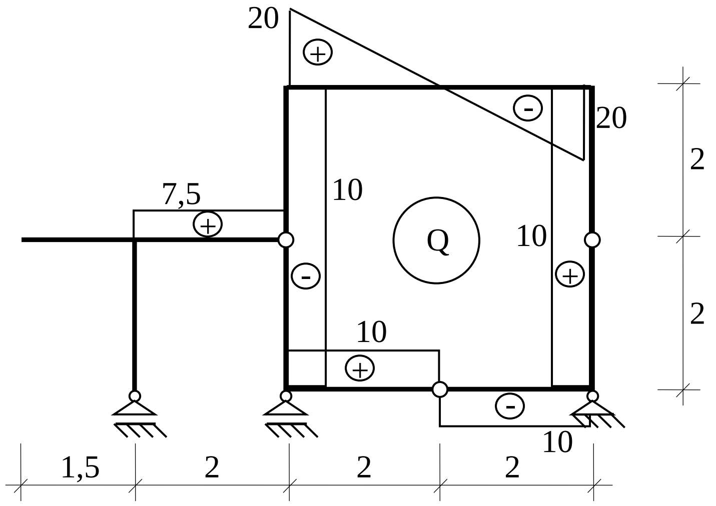
$$\sum M_N = 0: \quad 10.2 - 10.2 + 20.4 - 10.4.2 \overset{?}{=} 0$$

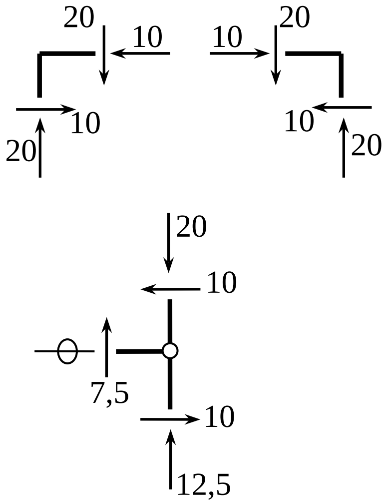
$$80 - 80 = 0$$

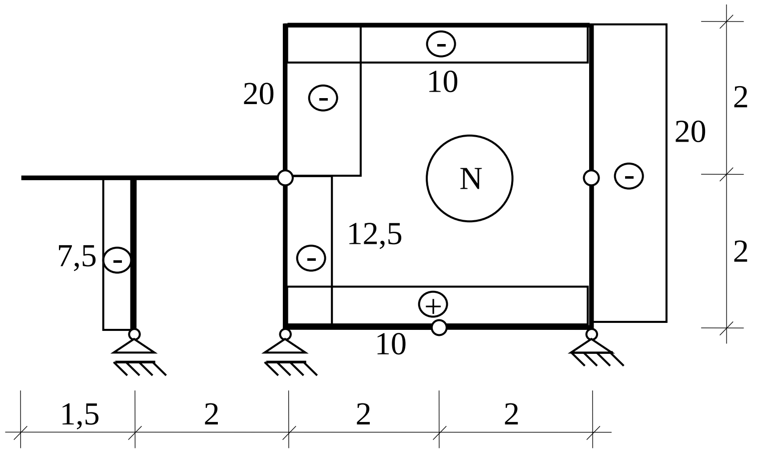
3. Internal forces diagrams.

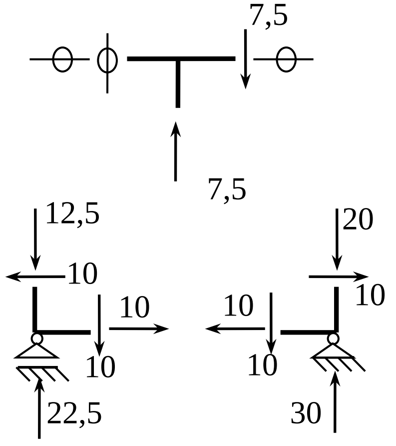




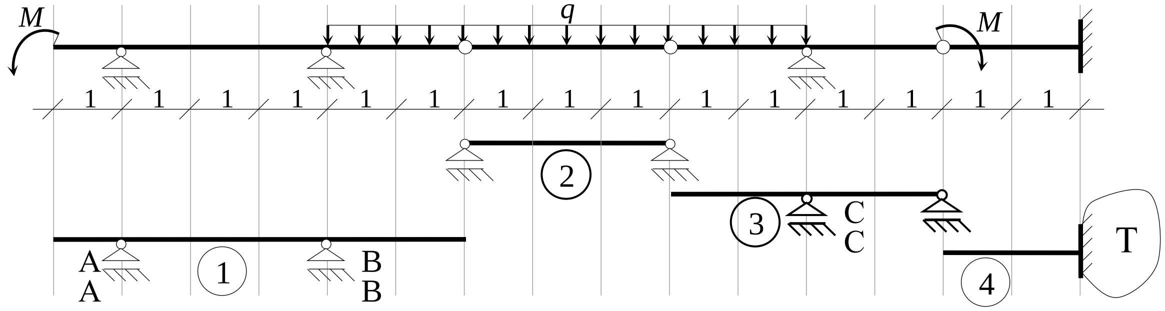






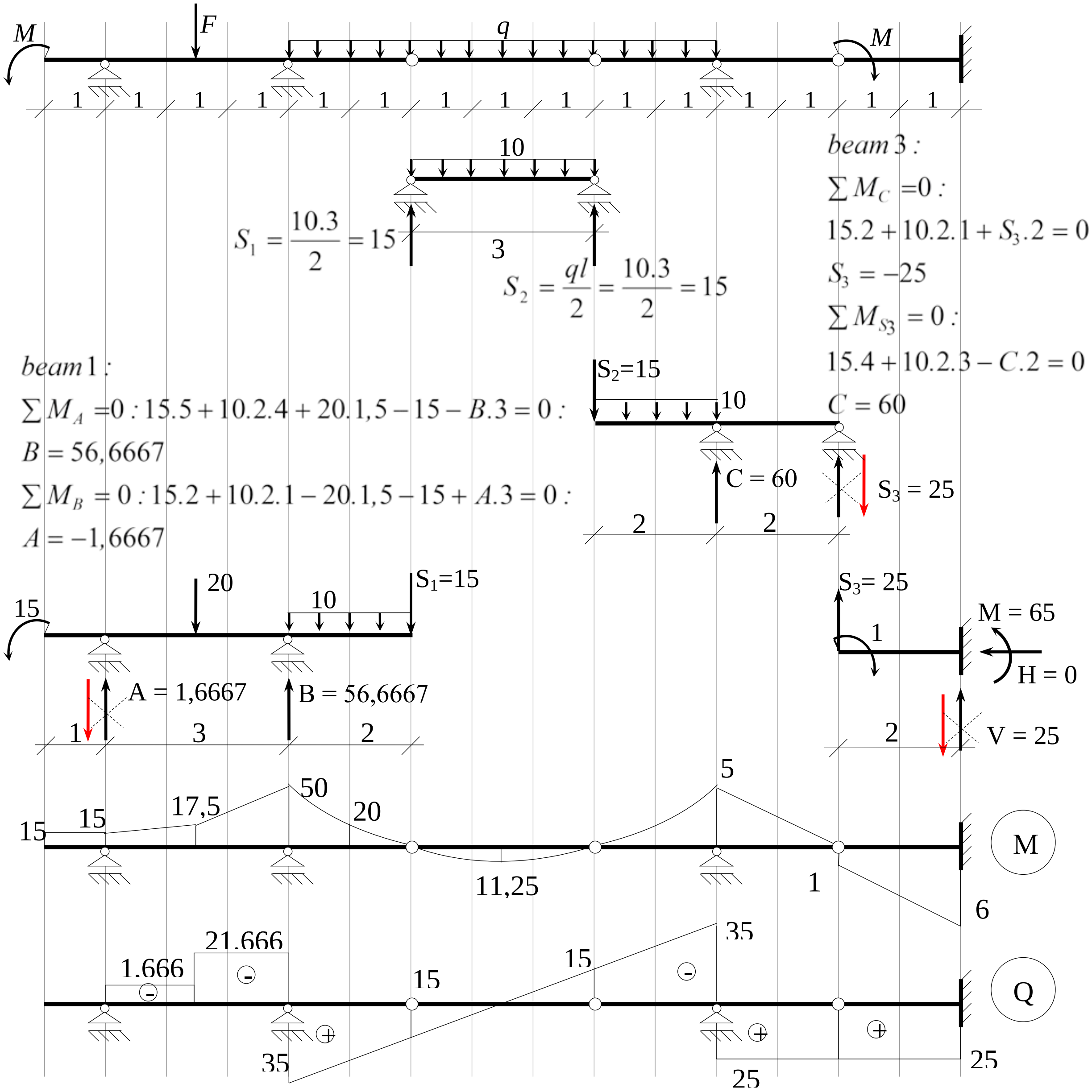


Example 3:



1. Kinematical analysis:

The separation on parts of the compound beam into basic and secondary beams is shown on the figure above. The system has two basic and two secondary beams. The basic beams are *1* and the cantilever *4*. Beam *3* and beam *2* are secondary. The whole solution is shown bellow.



Chapter 5
Principle of virtual work for rigid body. Energy methods.

Beams may be analyzed using the equations of static equilibrium and the method of sections, as illustrated. Alternatively, the principle of virtual work may be utilized to provide a simple and convenient solution. In this chapter we will illustrate for analysis of simple structures. At first we will explain the principle of virtual work for rigid bodies.

Principle of virtual work:

The principle of virtual work may be defined as follows: Consider a structure in equilibrium under a system of applied forces is subjected to a system of displacements compatible with the external restraints and the geometry of the structure. The total work done by the applied forces during these external displacements equals the work done by the internal forces, corresponding to the applied forces, during the internal deformations corresponding to the external displacements.

Or more simple: When a rigid body that is in equilibrium is subject to virtual compatible displacements, the total virtual work of all external forces is zero; and conversely, if the total virtual work of all external forces acting on a rigid body is zero then the body is in equilibrium.

The expression “virtual work” signifies that the work done is the product of a real loading system and imaginary displacements or an imaginary loading system and real displacements.

Consider a system of disks, in static equilibrium state,that is why the resutant force roar all disks is zero.http://en.wikipedia.org/wiki/Virtual_work - cite_note-Torby1984-0#cite_note-Torby1984-0

$$\sum_i F_i^R = 0$$

i – is a number of disks.

Summing the work produced by the force on each disk that acts through an arbitrary virtual displacement, δr_i, of the system leads to an expression for the virtual work that must be zero since the resutant force is zero:

$$\delta W = \sum_i F_i^R . \delta r_i = 0$$

In this equation the exprecion “the resultant force” means a total force – force and moment resultant. The same equation in more details:

$$\delta W = \sum_j F_j . \delta \overline{d}_j + \sum_j M_j . \delta \varphi_j = 0$$

here:

j is the number of forces or moments;

$\delta \overline{d}$ is the projected displacement;

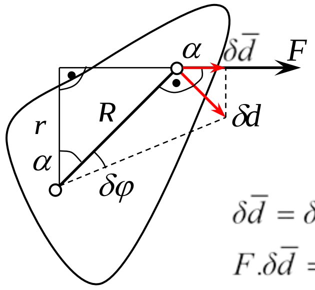
F is the force working on the projected displacement;

$\delta \varphi$ is rotation;

M is the moment working on the rotation.

The original vector equation could be recovered by using virtual work equation. That work expression must hold for arbitrary virtual displacements.

If arbitrary virtual displacements are assumed to be in directions that are orthogonal to the constraint forces, the constraint forces do no work. Such displacements are said to be consistent with the constraints.



δd – full virtual displacement of the application force point;

$\delta\overline{d}$ – projected virtual displacement of the application force point;

$\delta\varphi$ –virtual rotation of the rigid body;

r – ray of the force refer to point of rotation;

$$\delta\overline{d} = \delta d.\cos\alpha$$

$$F.\delta\overline{d} = F.\delta d.\cos\alpha$$

$$\delta d = R.\delta\varphi$$

$$F.\delta\overline{d} = F.R.\delta\varphi.\cos\alpha$$

$$R.\cos\alpha = r$$

$$F.\delta\overline{d} = F.r.\delta\varphi$$

$$\delta W = F.\delta\overline{d} = F.r.\delta\varphi$$

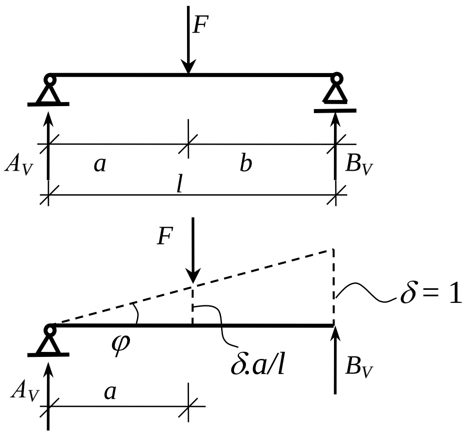
Usage of the principle of virtual work for determination of reactions or internal forces:

Procedure of analysis:

1) Remove the link, which holds the searching reaction or internal force;
2) Impose a virtual displacement to the system;
3) Write a virtual work equation and determine the searching reaction or internal force.

When impose the virtual displacement (rotation) we use the plan of the poles (if it is necessary) and express all rotations and displacement as a function of the virtual one, therefore we should have only one unknown parameter – the virtual displacement.

Example 1: Determination of the support reaction and internal forces for a simply supported beam loaded by point force.

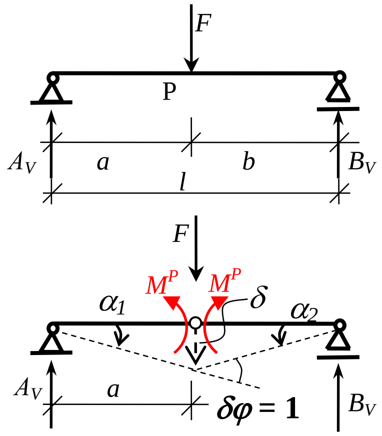


It is possible to use only forces or moments for writing the virtual work expression.

Using projected displacement: $\delta W = \sum F.d = F.\left(-\frac{\delta a}{l}\right) + B_v.\delta = 0 \rightarrow B_v = \frac{F.\frac{\delta.a}{l}}{\delta} = \frac{Fa}{l}$

Using the rotation φ: $\delta W = \sum M.\varphi = B_v.l.\varphi - F.a.\varphi = 0 \rightarrow B_v = \frac{F.a.\varphi}{l.\varphi} = \frac{Fa}{l}$

Similarly, as shown at next figure, the bending moment produced at point P by the applied load may be determinate by cutting the beam (removing the link) at P and imposing a unit virtual relative rotation of $\delta\varphi = 1$. Evaluating internal and external work done gives:



$$\alpha_1 = \theta$$

$$tg\alpha_2 = \frac{\delta}{b}; \quad tg\alpha_1 = \frac{\delta}{a} \quad \rightarrow \delta = a.tg\alpha_1$$

$$tg\alpha_2 \cong \alpha_2;\ tg\alpha_1 \cong \alpha_1 = \theta \Rightarrow \delta = a.\theta$$

$$\alpha_2 = \frac{\theta.a}{b}$$

$$\delta W = \delta W_{ext} + \delta W_{int} = 0 \rightarrow \delta W_{ext} = -\delta W_{int}$$

$$\delta W_{ext} = F.\delta = F.a.\theta$$

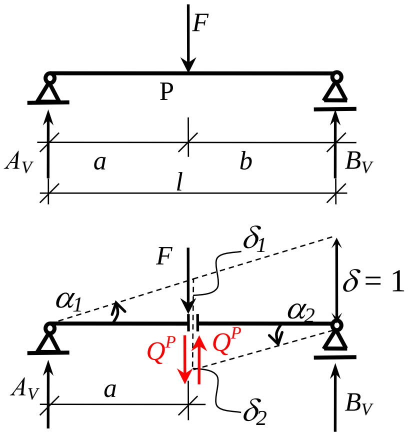
$$\delta W_{int} = -M^P.\alpha_1 - M^P.\alpha_2 = -M^P\theta - M^P\frac{a\theta}{b} = -M^P\theta\left(1+\frac{a}{b}\right) = -M^P\theta\frac{l}{b}$$

$$F.a.\theta = M^P\theta\frac{l}{b}$$

$$Fa = M^P\frac{l}{b} \rightarrow M^P = F\frac{ab}{l}$$

When the direction of the force and the dispacements are at the same then the work done is positive when they are oposite the work done is negative. When the direction of the moment and the rotation are at the same then the work done is positive when they are oposite the work done is negative.

If we want to evaluate the shere force at point P we should cut the beam (removing the link) at P and imposing a unit virtual relative displacement of $\delta = 1$.



$$\frac{\delta}{l} = \frac{\delta_1}{a} \Rightarrow \delta_1 = \frac{a}{l} \rightarrow \frac{\delta_1}{a} = \alpha_1 = \theta$$

$$\frac{\delta}{l} = \frac{\delta_2}{b} \Rightarrow \delta_2 = \frac{b}{l}$$

$$\alpha_1 = \alpha_2 = \theta$$

$$\frac{\delta_1}{a} = \alpha_1 = \theta \rightarrow \delta_1 = a.\theta$$

$$\frac{\delta_2}{b} = \alpha_2 = \theta \rightarrow \delta_2 = b.\theta$$

At the first solution we will use the virtual displacement $\delta = 1$ as a parameter in the equation. The external force we will impose at the left side of the beam and will receive the right side internal shear force. This solution is done bellow:

$$\delta W_{ext} = -\delta W_{int}$$

$$\delta W_{ext} = -F.\delta_1 = -F\frac{a}{l}$$

$$\delta W_{int} = -Q^P\delta_1 - Q^P\delta_2 = -Q^P\frac{a}{l} - Q^P\frac{b}{l} = -Q^P\left(\frac{a}{l}+\frac{b}{l}\right) = -Q^P$$

$$-F\frac{a}{l} = Q^P \rightarrow Q^P = -F\frac{a}{l}$$

At the second solution we will use the virtual displacement $\theta = 1$ as a parameter. The external force we will impose at the right side of the beam and will receive the left side internal shear force. In this case we use the fact that the two lines of the displaced form of the beam are parallel. This solution is done bellow:

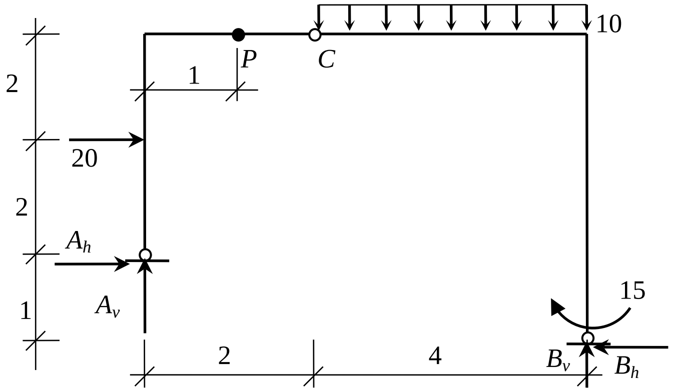
$$\delta W_{ext} = -\delta W_{int}$$

$$\delta W_{ext} = F.\delta_2 = F.b.\theta$$

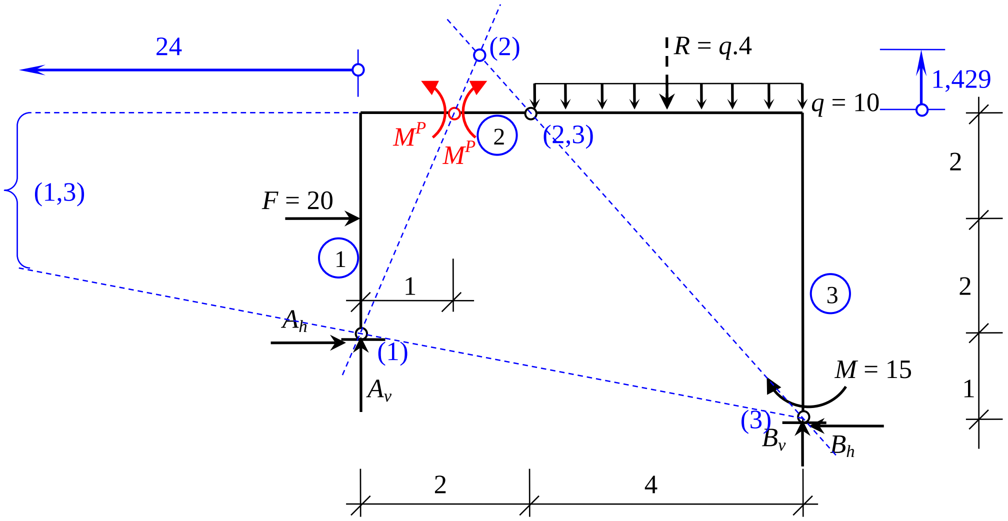
$$\delta W_{int} = -Q^P\delta_1 - Q^P\delta_2 = -Q^P.a.\theta - Q^P b.\theta = -Q^P(a+b)\theta = -Q^P l\theta$$

$$F.b.\theta = Q^P l\theta \rightarrow Q^P = F\frac{b}{l}$$

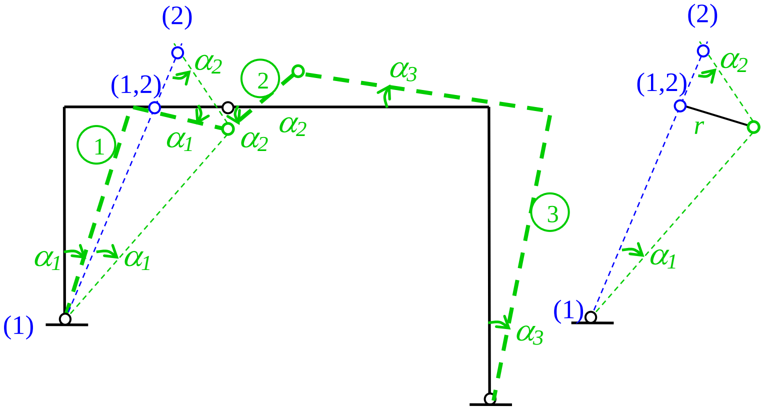
Example 2: Determination of the internal moment for a three-hinged frame using the plane of the poles.



First we sould compose the plan of poles. It is shown at the next picture with the blue lines and all nessacery dimensions. How it mades we already discused.



The next step is to determine how disks rotates when impose the relative rotation at point P now it is point (1,2). The displaced shape of the frame is shown with the green lines at the figure bellow:



Using showed geometry is determinate the rotation of the disk *2*.

$$\frac{r}{(1)(1,2)} = \alpha_1 \rightarrow r = (1)(1,2).\alpha_1$$

$$\frac{r}{(2)(1,2)} = \alpha_2 \rightarrow r = (2)(1,2).\alpha_2$$

$$\Rightarrow (1)(1,2).\alpha_1 = (2)(1,2).\alpha_2 \rightarrow \alpha_2 = \frac{(1)(1,2)}{(2)(1,2)}.\alpha_1$$

This formulae may be used for every two disks and can be written generally for any disks *n* and *m* as follow:

$$\alpha_m = \frac{(n)(n,m)}{(m)(n,m)}.\alpha_n$$

here:

$(n)(n,m)$ is the dimension of the segment between the mane pole (*n*) and the relative pole (*m*,*n*);

$(m)(n,m)$ is the dimension of the segment between the mane pole (*m*) and the relative pole (*m*,*n*);

α_m is the rotation of disk *m*;

α_n is the rotation of disk *n*.

Using this we determine all rotations at function of one of them as follow:

$$\alpha_1 = \theta$$

$$\alpha_2 = \frac{(1)(1,2)}{(2)(1,2)}.\alpha_1 = \frac{4,123}{0,9817} = 4,2.\theta$$

$$\alpha_3 = \frac{(1)(1,3)}{(3)(1,3)}.\alpha_1 = \frac{24}{30} = 0,8.\theta$$

Verification :

$$\alpha_3 = \frac{(2)(2,3)}{(3)(2,3)}.\alpha_2 = \frac{1,2197}{6,4031}.4,2\theta = 0,8.\theta$$

If we are not sure about the directions of the disk rotation, it is easily to choose a positive direction of rotation. For example, let choose the clockwise direction for positive, then each force, which rotates at such a direction related to the mane pole of rotation, will have a positive work done. Opposite, each force, which rotates at anticlockwise a direction related to the mane pole of rotation, will have a negative work done. If the moment rotate at clockwise direction will have a positive work done and opposite if moment rotate at anticlockwise a direction will have a negative work done.

After all, we may compute the internal moment at point P for the three-hinged beam as follow:

$$\delta W = F.2.\alpha_1 - M^P.\alpha_1 + M^P.\alpha_2 - R.2.\alpha_3 + M\alpha_3 = 0$$

$$F.2.\theta - M^P.\theta + M^P.4,2\theta - R.2.0,8.\theta + 15.0,8\theta = 0$$

$$3,2.M^P = -20.2 + 40.2.0,8 - 15.0,8$$

$$M^P = 3,75$$

Chapter 6

Influence line for statically determinate structures.

In the previous chapter, we have explained a technique for analyzing structures for dead (fixed) load. If the load is moving, not fixed, then the moment end shear internal forces should be analyzed by using influence line. Influence lines have important application for the design of structures that should resist different live loads. Here we will discuss how to draw the influence line for a statically determinate structures and its application for determination of the absolute maximum live shear and moment in a members.

An Influence line represents the variation of the reaction, shear, moment, or deflection at a specific point in a member as a concentrated force moves over the member. Once this line is constructed, one can tell where the moving load should be placed on the structure so that it creates the greatest influence for the internal forces of the specific point. Furthermore, the magnitude of the associated reaction, shear, moment, or deflection at the point can then be calculated from the ordinates of the influence-line diagram.

Although the procedure for constructing an influence line is rather basic and one should clearly be aware of the difference between constructing an influence line and constructing a shear or moment diagram. The influence lines present the effect of the moving load only at a specified point on a member, whereas shear and moment diagrams represent the effect of a fixed loads at all points along the member.

Influence lines composition:

For constructing influence lines, we should know the following important notes:

- The influence lines are composed by straight lines for statically determinate structures;
- For constructing the influence lines, we use a moving concentrated vertical force at a dimensionless magnitude of unity;
- The way where the force moves we will call the “moving path”;
- We draw the influence lines at a basic line, not at the structure axis;
- The positive values of the influence line we draw at a bottom side of the basic line;
- An ordinate (the value) of some point at an influence-line diagram correspond to the position of the unit load.

We will discus the two methods of determination of a influence line: the static method and the cinematic one. First, we will present the static method by its variations.

Static method for constructing influence lines:

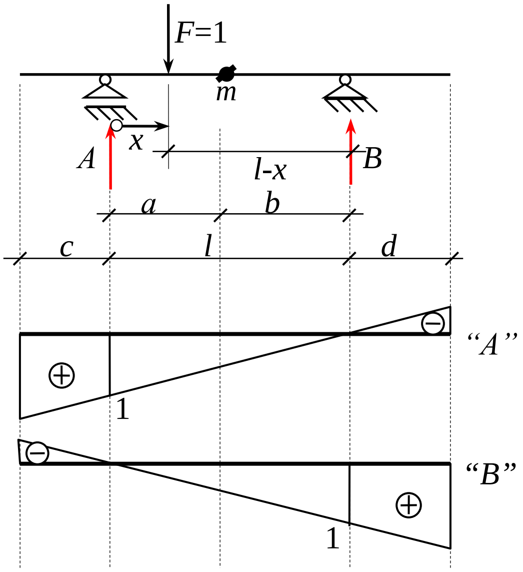
- Usage of an influence-lines function:

An influence line can be constructed by placing the unit load at a variable position *x* on the member and then computing the value of the internal force or reaction at a specific point as a function of the unit load position. In this way, the functions of the different influence line straight segment can be computed and plotted.

Example 1:

Composing of influence line for the support reactions:

We place the unit force at a point x and compute the support reactions using equilibrium equations. So we have a function of the support reaction in dependence of the unit force position. After that, we calculate the support reactions value for some value of x variable corresponding to character position of the unit load. Finally using this value and knowing the influence line is straight one we draw the support reaction influence-line diagram.



Computing of the sup port reactions :

$$\sum M_B = 0:\ A.l - F.(l-x) = 0 \rightarrow A = \frac{l-x}{l}:$$

$$x = 0 \rightarrow A = 1$$

$$x = l \rightarrow A = 0$$

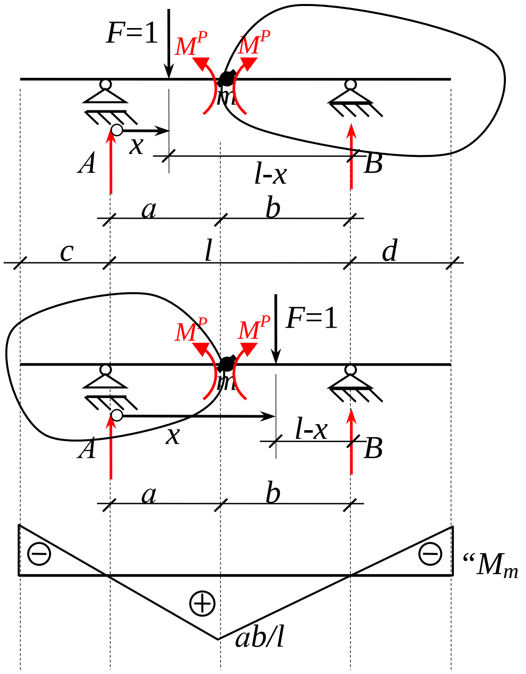
$$\sum M_A = 0:\ B.l - F.x = 0 \rightarrow B = \frac{x}{l}:$$

$$x = 0 \rightarrow B = 0$$

$$x = l \rightarrow B = 1$$

Composing of influence line for the shear and moment at a specific point:

When composing influence line for the internal forces at a specified point we should be careful with choosing the variable *x*. Once at left of the specified point therefore at the other side.



The unit load at left side :

$\sum M_m = 0 : \; B.b - M^P = 0$

$M^P = B.b = \frac{x}{l} b$

$x = 0 : M^P = 0$

$x = a : M^P = \frac{ab}{l}$

The unit load at rightt side :

$\sum M_m = 0 : \; A.a - M^P = 0$

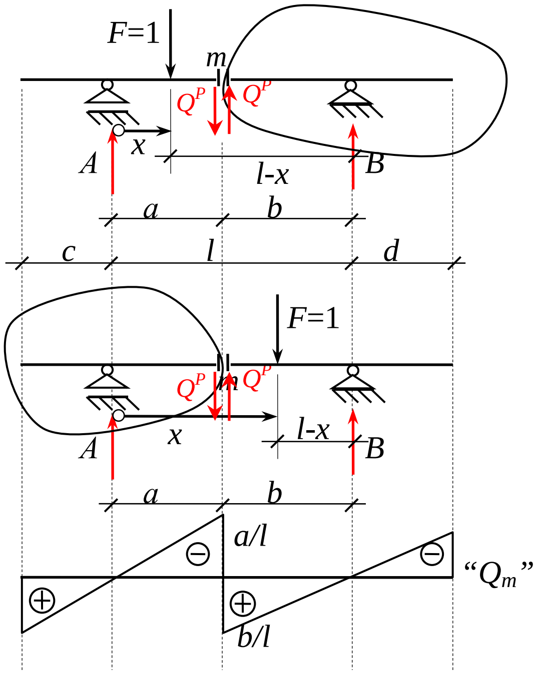
$M^P = A.a = \frac{l - x}{l} a$

$x = a : M^P = \frac{ab}{l}$

$x = l : M^P = 0$

When the force is at left, it is easier to write the equilibrium equation for the right side because of the load absence. In this case the variable *x* changes from *0* to *a* because in other case in the equilibrium equation we should include the unit load.

Otherwise, when the force is at right, we write the equilibrium equation for the left side so the variable *x* changes from *a* to *l* because in other case in the equilibrium equation we should include the unit load. In this way, we compose the influence line for the internal moment for the specified point *m*.



The unit load at left side :

$\sum V = 0 : \; B + Q^P = 0$

$Q^P = -B = -\frac{x}{l}$

$x = 0 : Q^P = 0$

$x = a : Q_P^l = -\frac{a}{l}$

The unit load at rightt side :

$\sum V = 0 : \; A - Q^P = 0$

$Q^P = A = \frac{l - x}{l}$

$x = a : Q_P^r = \frac{b}{l}$

$x = l : Q_P^r = 0$

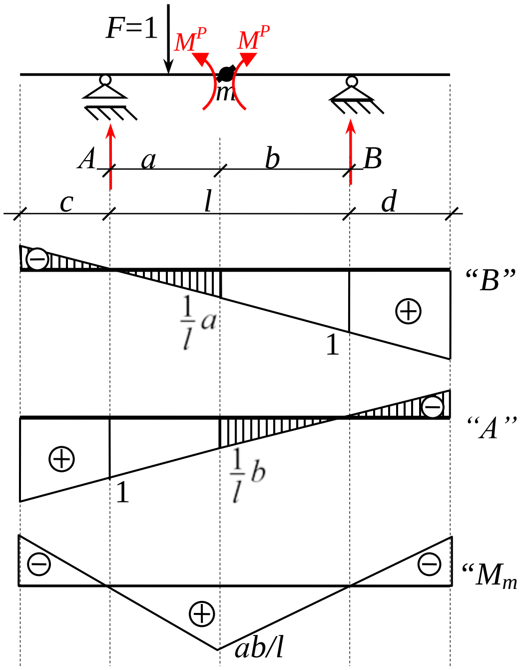
On the contrary of the moment influence line where is one and the same value at the two sides of the point *m* for the shear force influence line one can see there is a jump at *m*. The reason of this is that the load is a unit vertical force. That is why the jump should be with unit value. Indeed, we have:

$$Q_P^l + Q_P^r = \frac{a}{l} + \frac{b}{l} = \frac{a+b}{l} = 1$$

- Usage of other influence lines:

The same result may be achieved if we use already determinate influence lines to construct another one. As an example, we will show the composition of the moment and shear influence line for point m at the previous simple beam.

The main idea is as follow: Write the equilibrium equation for the searched reaction, shear, moment, or deflection and put a quotation marks to all variables which have influence lines. After that, perform all summations and multiplications of the equilibrium equation for the influence lines. As a result, we will have the needed influence line. Performing these operations, we should take care about the unit force position and if it is necessary, we should write two equilibrium equations – for right and left sides.



The unit load at left side :

$\sum M_m = 0: \; B.b - M^P = 0$

$M^P = "B".b$

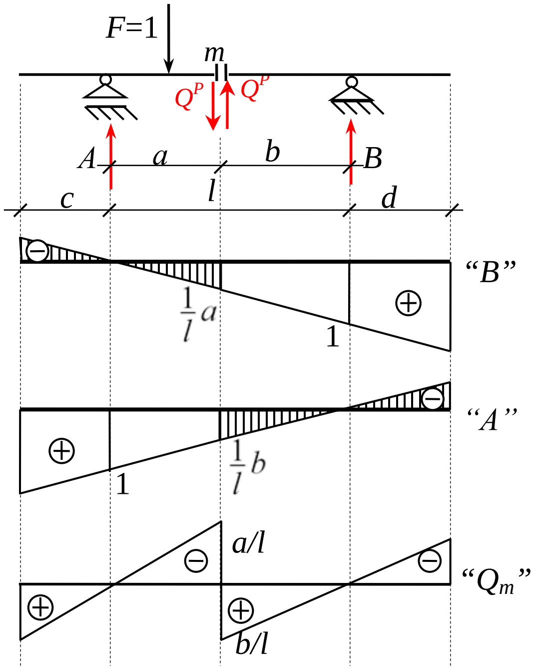
The unit load at rightt side :

$\sum M_m = 0: \; A.a - M^P = 0$

$M^P = "A".a$

When the unit load is at left side of the beam then we use equilibrium equation for the right side of the beam for to neglect the force. As a result the point *m* moment depends of the *B* reaction but we should use the left part of the *B* influence line because the unite load is there. The ordinate of the influence lines corresponds to the unit load position. Otherwise, when the force is at right of point *m* we perform equilibrium equation for the left side of the beam. Then the M_m depends of the *A* reaction but we should use the right part of the ”*A*” the unite load is there.

In this manner we construct the *m* point shear influence line:



The unit load at left side :

$$\sum V = 0: \; B + Q^P = 0$$

$$Q_P^r = -"B"$$

The unit load at rightt side :

$$\sum V = 0: \; A - Q^P = 0$$

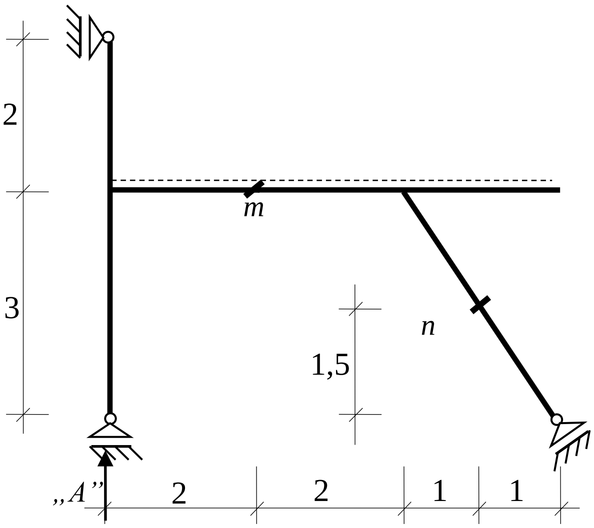
$$Q_P^l = "A"$$

- ➢ Usage of character position for the unit load:

The main idea is to move the unit point load at fixed the character points one-by-one, and using equilibrium equations for the reaction, shear, moment or the deflection to determine a character ordinates of the searching influence line. This method is useful because of the fact that the influence lines for the statically determine systems are combination of the straight segments. The number of straight lines corresponds to the number of the disks lay upon the moving path. If the specified point is at the moving path, then the corresponding disk divides into two separated disks.

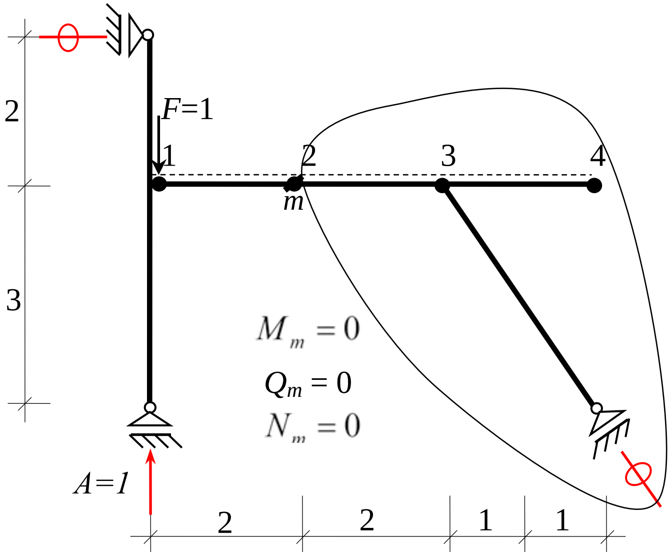
All character points that we must stop the force put on are the beginning of the moving path, upon supports, at a specified point, above hinges (joints) or other apparatuses and at the end of the moving path.

Example 2:

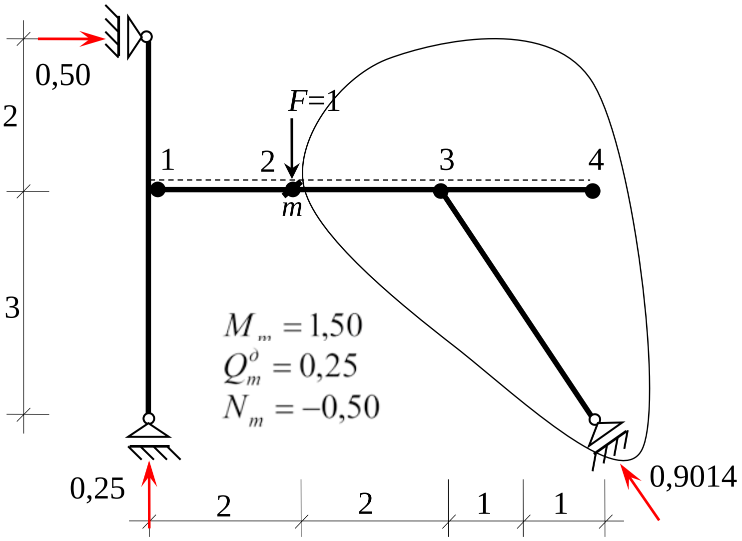


Composing of influence line for the M_m:

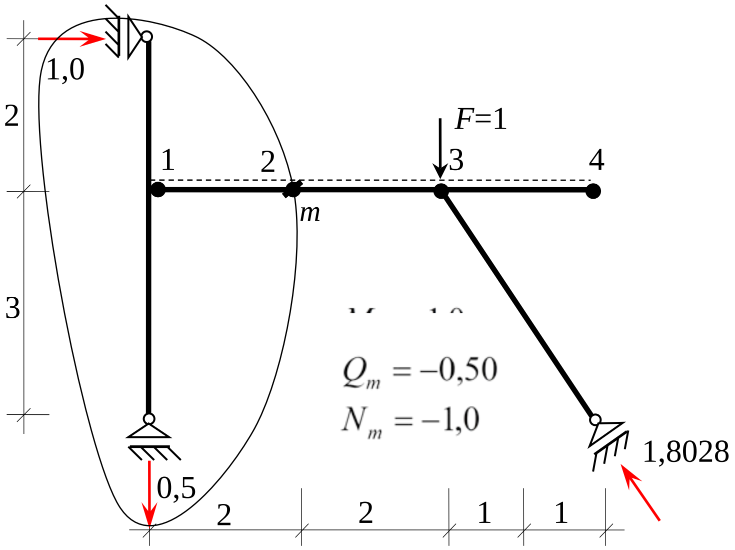
The influence line M_m consists two straight segments, because the specified point m is lying on the moving path. For this reason, it is necessary to put the force on three points at least. We fix the unit load at the characteristic point 1 and determine the internal force M_m.

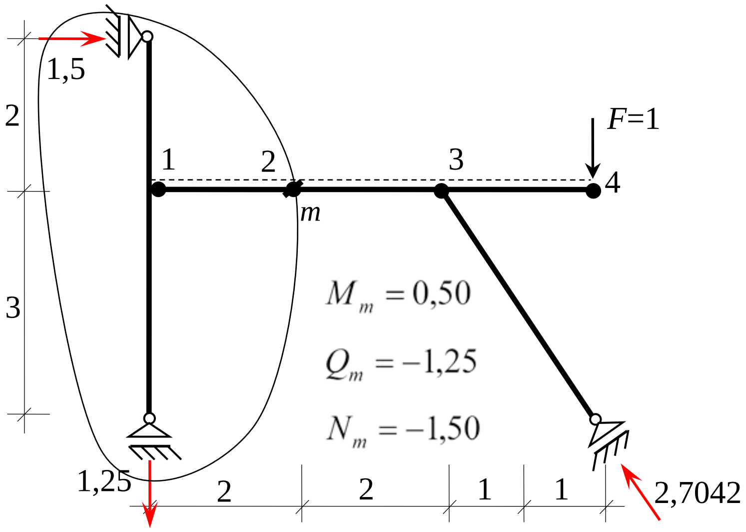


After that we move the unit load at the characteristic point 2 and again determine the internal M_m. When the force is infinitely next to left or right of the specified point m, the result is identically for the moment.

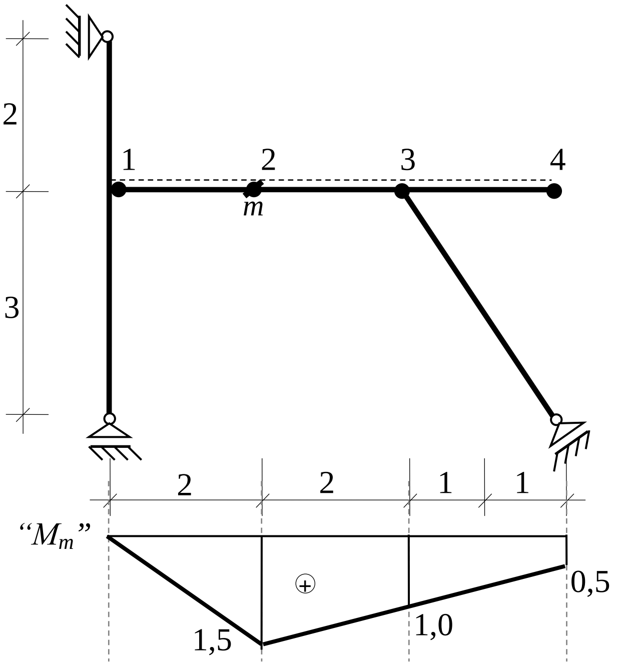


We fix the unit force at characteristic points 3 and 4, and determine internal force M_m.





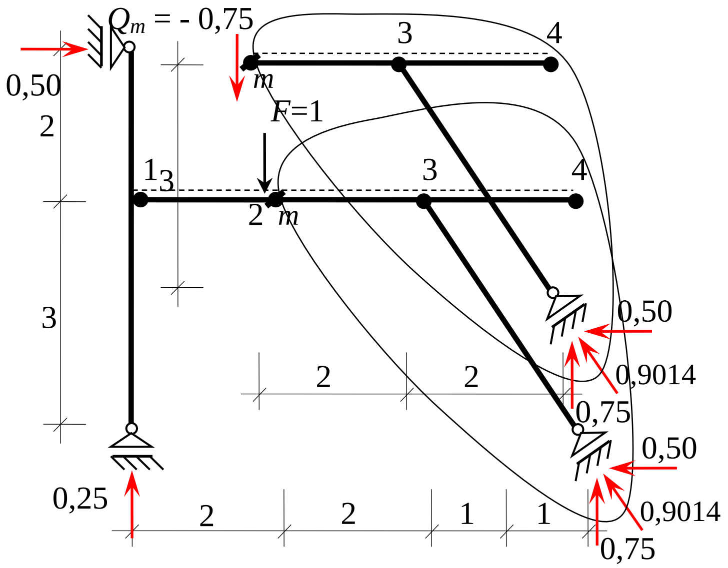
Using the obtained results we may compose the influence line and don’t forget that the obtained value for the moment when the unit load is at the characteristic point n is the ordinate of the influence line at the characteristic point n. The ordinates are connected with straight lines. we can use as verification the fact that all ordinates are lying on a straight line,. Thus, point 4 is an extra one, so the corresponding ordinate is used as verification.



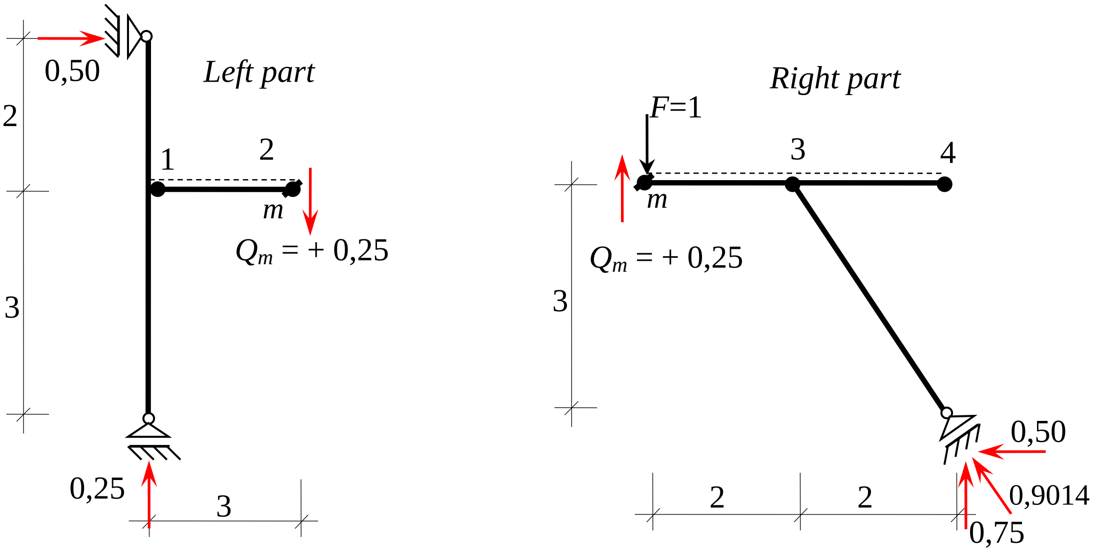
Determine influence line „Q_m”:

This solution is similar as this for “M_m” influence line. When the force is at the characteristic points 1, 3, and 4 it is possible to use the solutions made before, but we will calculate internal force Q_m , instead of M_m. That is why the solutions are done and shown on the upper figures and the internal force Q_m is read next to M_m. More complicate is the calculation of the sear force when the unit load is at the point 2, actually the specified point for which we compute the influence lines. We already described the mane idea of this solution in the previous example. Now we will use this idea.

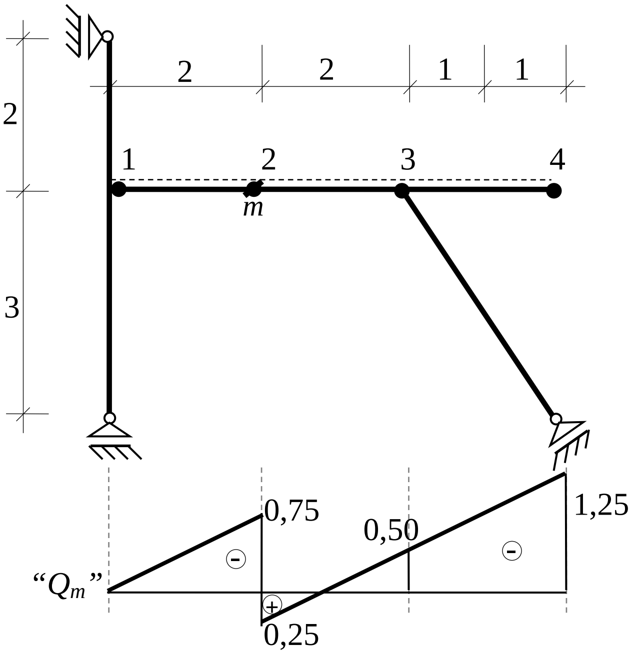
First, the unit force is next left to the section. Support reactions are the same as solution for M_m. Using vertical forces equilibrium equation with right cut part of the system, we determine the value and the sign of the internal force Q_m.



The next situation is when the force runs upon the section and now it is right next of the specified force. reactions do not change. The value of internal force Q_m can be determine by cutting left and right parts of the frame. Here we show both sections of the frame, which causes a clear results.



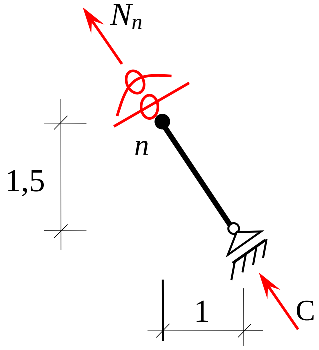
It is clear that presented frame-parts have the same value of the internal force, independent of the chosen part of the frame and the result is sure. That is why we prefer that part of the internal force, which will be less for calculations. The important detail is that the received internal force value must be fixed as an ordinate of influence line exactly under the position of the unit force. Here we mean that ordinate *+0,25* is next right of the specified point *m*. It is seen from the obtained influence line that is in section *m* has a jump equal to unit. The result is obvious because there is an external unit force in section *m*.



Determination of the influence line "N_m":

The determination of the influence line for the normal force can be done by the analogical way with the above lines. In this situation, point *2* is not singularity point because the external force is vertical and does not cause a any jump for the normal force. The values of the normal force in any position of the unit load are shown next to the others at the previous solutions.

Other method for calculating this influence line is using another one. In this case is more convenient to cut out the down part of the structure near support *C*. In that way the solution is independent of the place of the unit load.



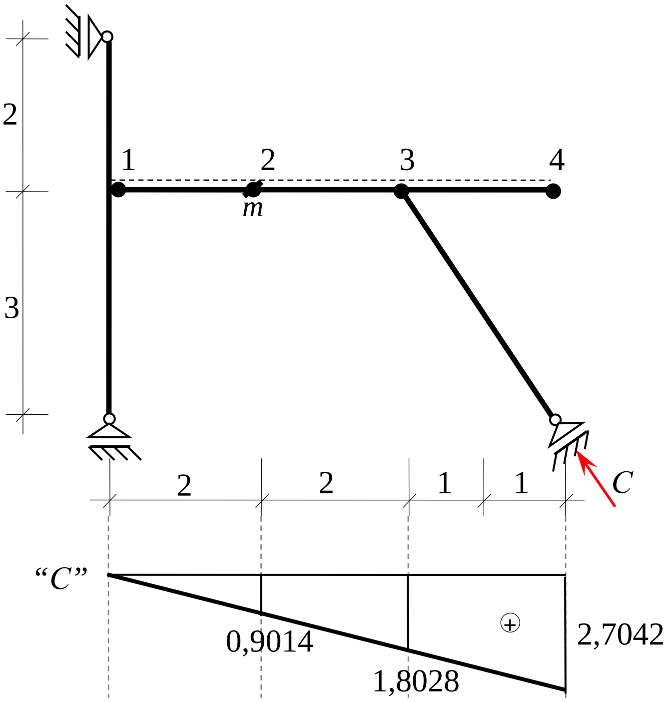
From this section, it is clear that the moment and shear force at point *n* are always zero therefore those influence lines are zero. We can compute the normal force by the supporting reaction *C* with the following expression:

$$N_n = -C$$

Therefore, if we know influence line of the support reaction, we can find the influence line of the normal force as follow:

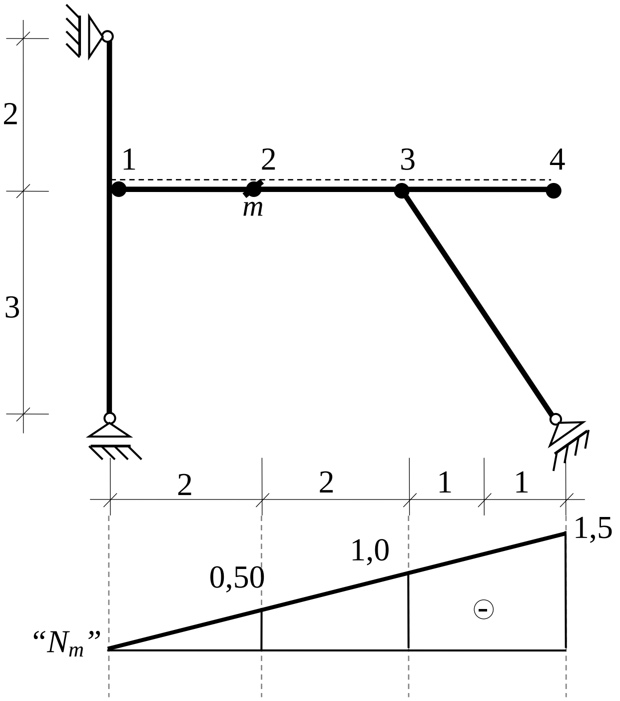
$$"N_n" = -"C"$$

For composing the support reaction influence line we use the same way as for the moment and the shear for the specified point *m*. We choose positive direction of the reaction and calculate its value for the different positions of the unit load. Actually, they are already computed and it is only necessary to draw the influence line.



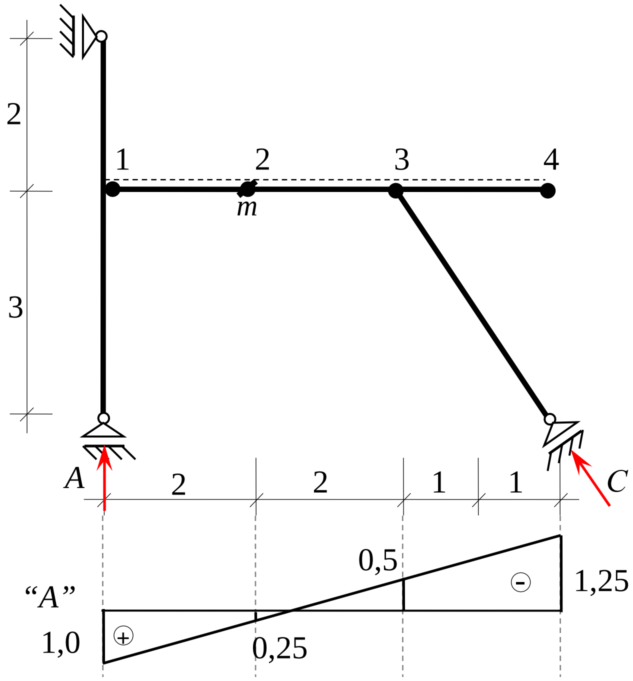
For the normal force influence line, we have:

$$"N_n" = -"C"$$



Determination of the influence line for the support reaction "*A*":

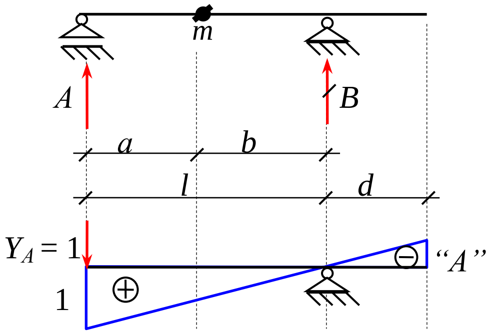
The influence line of supporting reaction „*A*" is determine like the support reaction line„*C*", that is why we will show only the result:



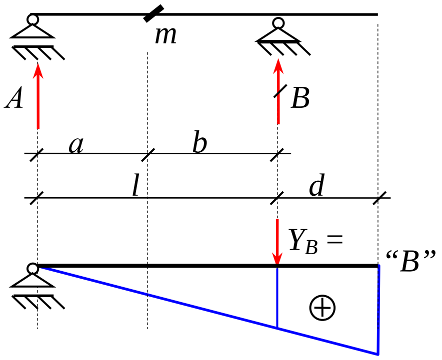
Kinematical method for constructing influence lines:
(Qualitative influence lines using the Müller Breslau principle**)**

Kinematical method is very easy and suitable technique for determining influence lines. In the international literature this technique is known as qualitative influence lines using the Müller Breslau principle. The Müller Breslau Principle (1886) states that: *the ordinate value of an influence line for any function on any structure is proportional to the ordinates of the deflected shape that is obtained by removing the restraint corresponding to the function from the structure and introducing a corresponding unit displacement as the function makes the negative work.*

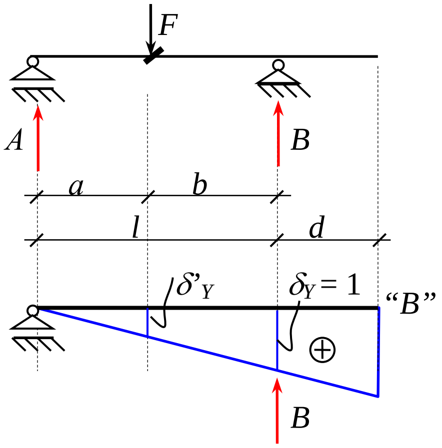
For example, to obtain the influence line for the support reaction at *A* for the simple beam shown in next figure, above, remove the support corresponding to the reaction and apply a unit displacement in the direction of Y_A. The resulting deflected shape will be proportional to the true influence line for this reaction. i.e., for the support reaction at *A*. The deflected shape due to a unit displacement at *A* is shown below. Notice that the deflected shape is linear, i.e., the beam rotates as a rigid body without any curvature. This is true only for statically determinate systems.



Similarly, to construct the influence line for the support reaction *B*, we remove the support at *B* and apply a vertical unit displacement δ_Y. The resulting deflected shape is the qualitative influence line for the support reaction.



The proof of the Breslau principle can be established by using a virtual work principle. Recall that the work is the product of either a linear displacement and a force in the direction of that displacement or a rotational displacement and moment in the direction of that rotation. If the rigid body (beam) is in equilibrium, the sum of all forces and moments must be equal to zero. Consequently, if the body is given a virtual displacement the work done by all these forces and moments must also be equal to zero. Consider, for example, the previous simple beam and the influence line for support reaction "*B*". The beam is subjected to a unit load placed at an arbitrary point along its length. If the beam is given a virtual vertical displacement δ_Y at support *B* then only the support reaction and the unit force do virtual work. In this case, the support reaction does the negative work and the unit load does the positive. The support reaction *A* doesn't a work because its point don't moves. Since the beam is in equilibrium the virtual work sum must be zero:



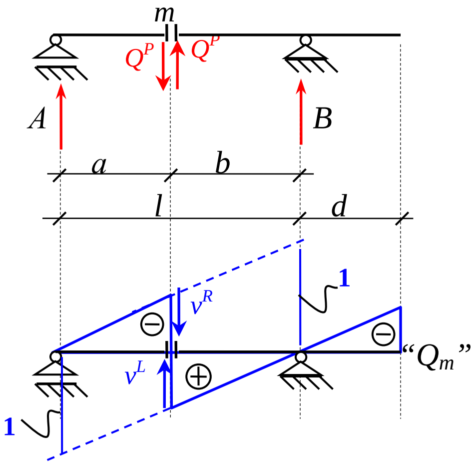
$$F.\delta'_Y - B.\delta_Y = 0$$
$$B.\delta_Y = F.\delta'_Y$$

Since the load force is a unit and the virtual displacement is unit than we have:

$$B = \delta'_Y$$

In other words, the value of *B* is equals to the vertical displacement at the position of the unit load, it shows of the displaced shape represents the influence line for the support reaction *B*.

Similarly if we want to construct the influence line for the shear force at specified point m we should remove the restrain corresponding to the shear force and to subject the unit relative vertical displacement at such direction the work done by the shear to be negative. The deflected shape is shown bellow:



The sum of the right and the left parts of the displacement is equal to unit:

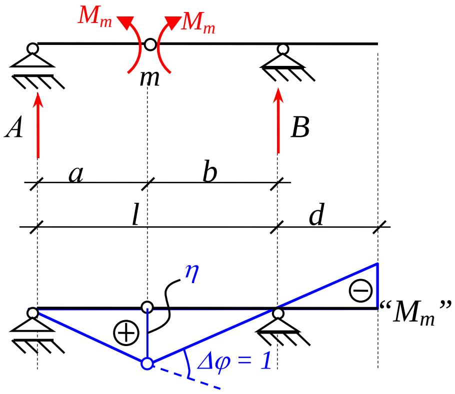
$$v^L + v^R = 1$$

Using the geometry, we compute the two parts of the displacements:

$$\frac{v^L}{a} = \frac{1}{l} \to v^L = \frac{a}{l}$$
$$\frac{v^R}{b} = \frac{1}{l} \to v^R = \frac{b}{l}$$

As a result, we have the "Q_m" influence line.

To obtain a qualitative influence line for the bending moment at a section, remove the moment restraint at the section, but maintain axial and shear force resistance. The moment resistance is eliminated by inserting a hinge in the structure at the section location. Apply equal and opposite introduce a unit relative rotation between the two tangents of the deflected shape at the hinge. The corresponding elastic curve for the beam, under these conditions, is the influence line for the bending moment at the section. The resulting influence line is shown below.



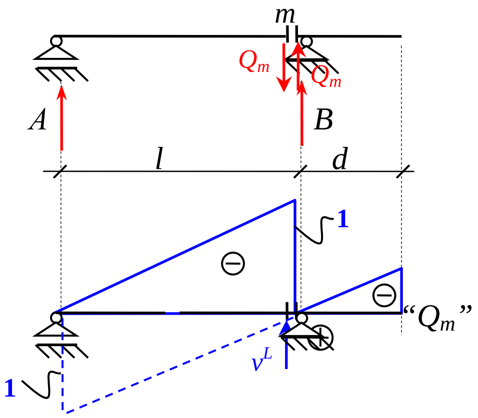
One can compute the value of the ordinate at the specified point using geometry.

$$\alpha_1 + \alpha_2 = \Delta\varphi = 1$$

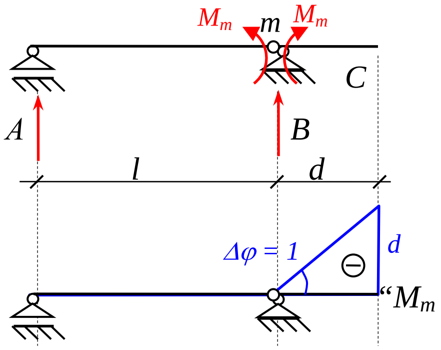
$$\alpha_1 + \alpha_2 = \frac{\eta}{a} + \frac{\eta}{b} = \frac{l}{ab}\eta = 1 \to \eta = \frac{ab}{l}$$

Let see the influence line for the specified shape just next to the support *B* at left.

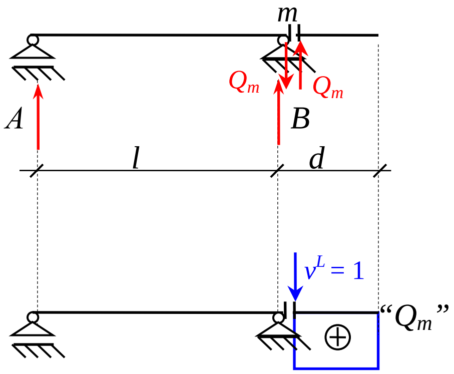
The shear just to the left side of support *B* can be constructed using the ideas explained above. Simply imagine that section s in the previous example is moved just to the left of *B*. By doing this, the magnitude of the positive shear decreases until it reaches zero, while the negative shear increases to 1.



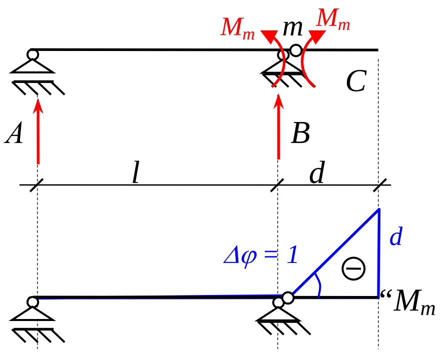
The qualitative influence line for the bending moment at B is obtained by introducing a hinge at support *B* and applying a moment that introduces a unit relative rotation. Notice that no deflection occurs between supports *A* and *B* since neither of the supports were removed. Therefore, the only portion that will rotate is part *BC* as shown below.



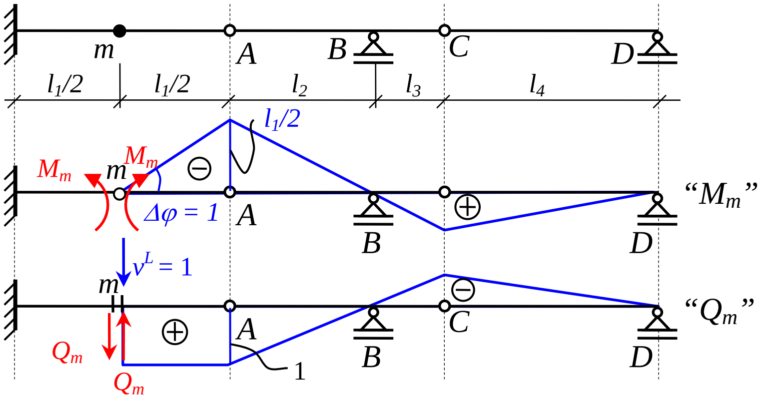
In the figure bellow are shown the influence line when the section moves next to the support *B* but at right.

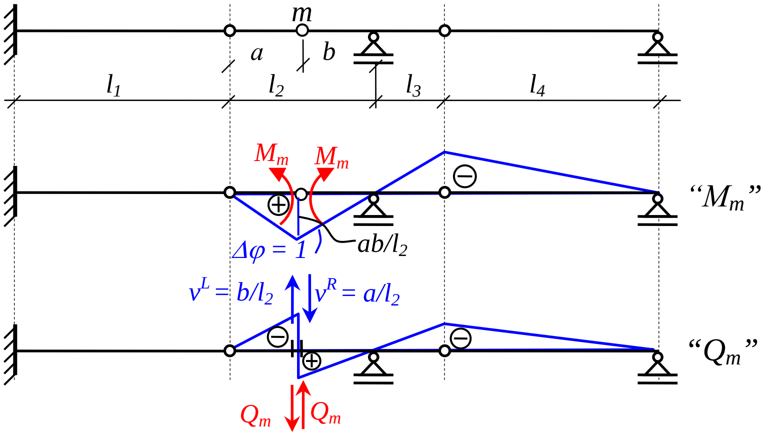


Notice that no deflection occurs between *A* and *B*, since neither of those supports were removed and hence the deflections at *A* and *B* must remain zero.



For the moment influence line there is no difference if the section is at left or at right to the support. Using this idea we will show some influence lines for the compound beam:





When we use the kinematical way for constructing influence lines for simple structures it is easy to determine the displaced shape after removing the resistant link but when we use it for more complicated structures it is not so easy. That is why it is necessary to use the plane of poles to determine the displaced shape.

For this technique we have the next work sequence:

- remove the restrains, corresponding to the function for which we construct the influence line;
- subject a unit displacement so that a negative work be done;
- the structure changes to a mechanism with one degree of freedom and we construct its plan of poles;
- determine the displaced shape of the disk on the moving path;
- the vertical displacements give a shape of the influence line;

- the values we can compute using the static method.

For the vertical displacements one should know following marks:

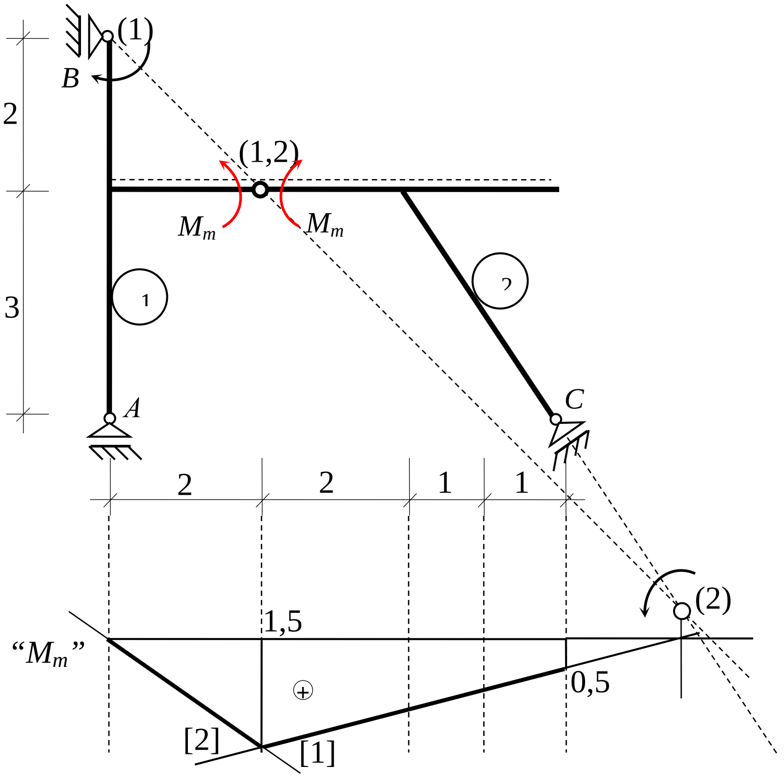
- under the main pole there is no a vertical displacement because this point is like a common pin support. For this reason the influence line have a zero value at this point.
- under the relative pole there is a kink for the vertical displacement shape because this point is a relative rotation point. For this reason the influence line have a kink at this point.

when the relative pole is at the infinity the corresponding disks moves parallel. For this reason at the influence line the corresponding lines are parallel.

This procedure is presented on the next example. We will determine the influence lines to the frame we already done using the static way.

Influence line „M_m”:

In section *m* remove the moment and put a hinge. Thus the structure is divided into two disks. Hence the influence line will consist two straight lines. After that we construct the plane of poles.



$$\begin{aligned}&(1)+(1{,}2)=(2)\\&CC'=(2)\end{aligned}$$

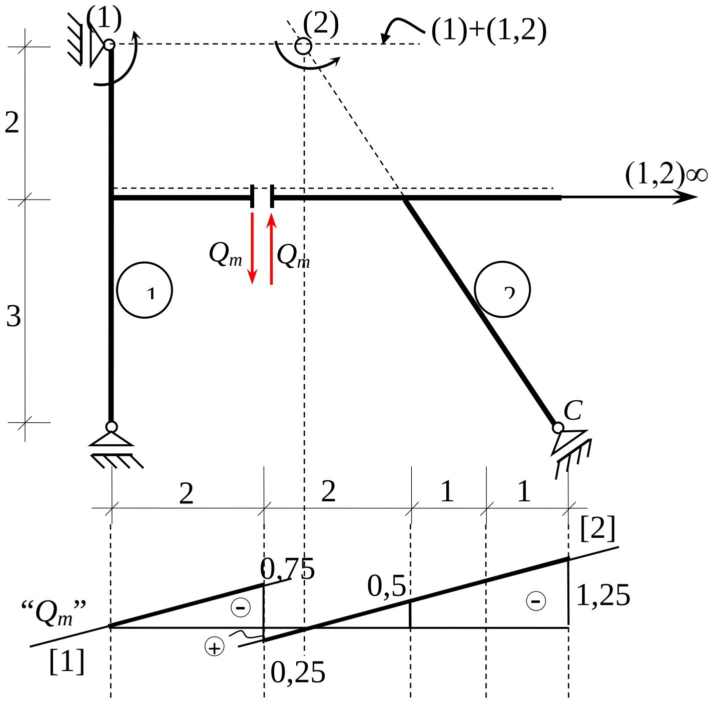
Moment M_m, which is at disk 1 rotates anticlockwise direction so, disk 1 must rotates clockwise direction for M_m do negative work. Since disk 1 rotates clockwise direction, then disk 2 should rotate anticlockwise direction. According the rule, under main pole in the influence line has a zero under pole(1) (marked as [1] in the drawing) and will be rotated to clockwise direction. The straight line correspond to disk 2 – [2] has zero under pole (2) and it is connected to the straight line, corresponding to disk 1 under relative pole (1, 2). Notice that the projection of pole 2 is out of the geometry of the frame.

Influence line „Q_m”:

In section *m* remove the shear and put a *Q*-release. Thus, the structure is divided into two disks. Hence, the influence line will consist two straight lines. After that, we construct the plane of poles.

$$\left| \begin{array}{l} (1)+(1,2)=(2) \\ CC'=(2) \end{array} \right.$$

It is obvious that pole (1,2) is at the horizontal infinity. For finding pole (2) is needed to connect pole (1) to pole (1,2) at infinity. It is practically done when draft a horizontal straight line across (1). The shear force Q_m corresponding to disk 1, rotates the disk around pole (1) clockwise direction, therefore Q_m will done negative work if disk 1 turns anticlockwise. As disk 1 turns anticlockwise, thus disk 2 turns anticlockwise too, because the relative pole (1, 2) is at infinity. Therefore, the influence lines consist two parallel lines according to the rules mentioned before.

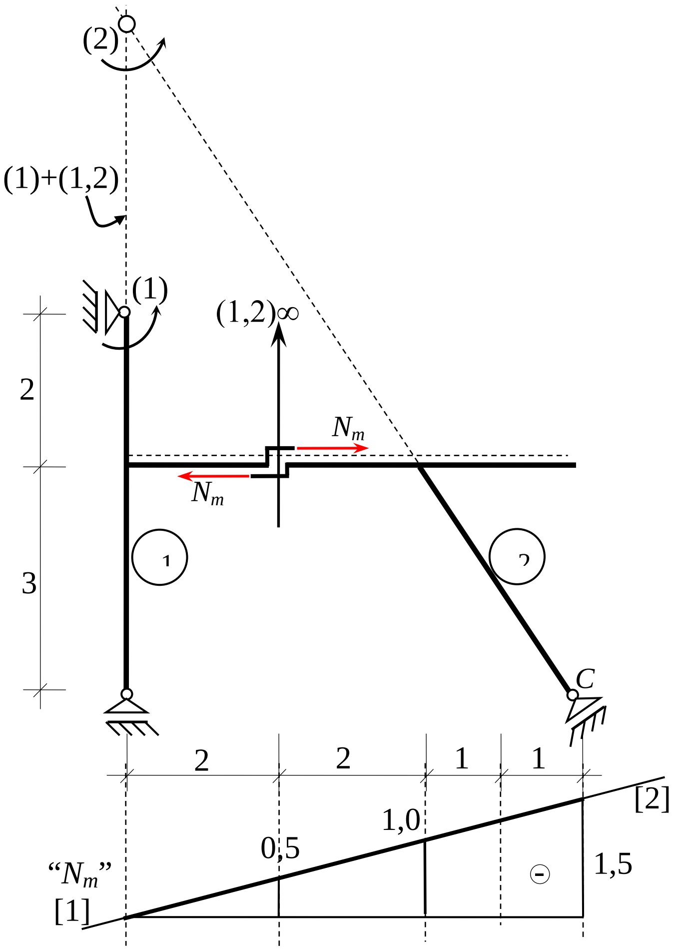


Influence line „N_m”:

In section *m* remove the normal force and put a *N*-release. Thus, the structure is divided into two disks. Hence, the influence line will consist two straight lines. After that, we construct the plane of poles.

$$\left| \begin{array}{l} (1)+(1,2)=(2) \\ CC'=(2) \end{array} \right.$$

Pole (1,2) is at the vertical infinity. For finding pole (2) is needed to connect pole (1) to pole (1,2) at infinity. It is practically done when draft a verical straight line across (1). The normal force N_m corresponding to disk 1, rotates the disk around pole (1) anticlockwise direction, therefore Q_m will done negative work if disk 1 turns anticlockwise. As disk 1 turns clockwise, thus disk 2 turns clockwise too, because the relative pole (1, 2) is at infinity. Therefore, the influence lines consist two parallel lines.

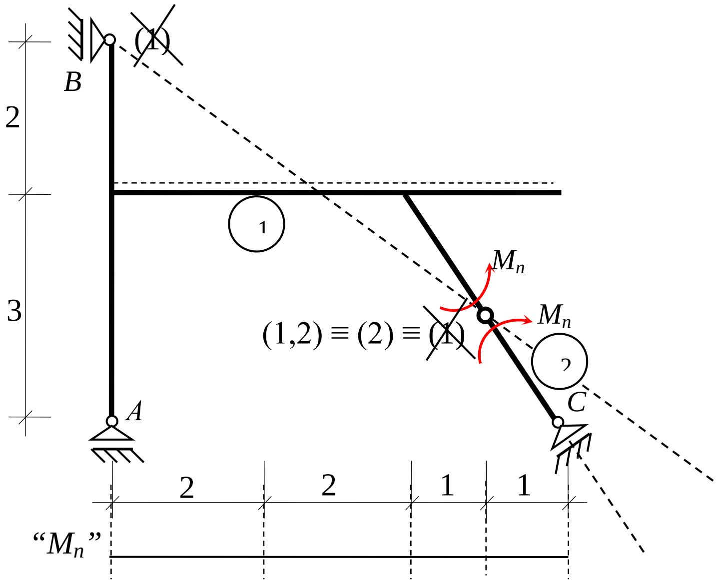


Influence line „M_n":

In section *n* remove the moment and put a hinge. Thus the structure is divided into two disks. Hence the influence line will consist two straight lines. After that we construct the plane of poles.

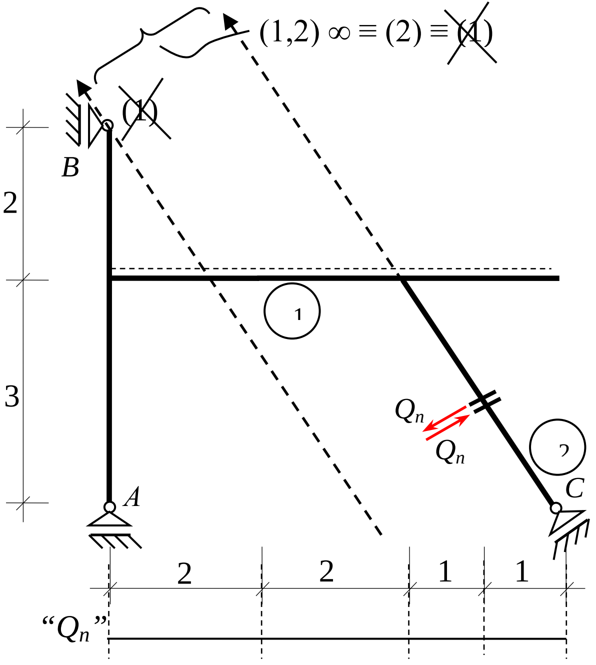
$$\left|\begin{array}{l}(1)+(1,2)=(2)\\ CC'=(2)\end{array}\right.$$

In this case pole 2 coincides to pole (1, 2), therefore pole (1) must be located at the same point, but pole (1) is stand in other place. As we already know when some pole is located at two points that means it does not exist. When a main pole does not exist its then the disk is stable (does not moves). The disk 1 is on the moving path and haven't a vertical displacement, therefore corresponding influence line is zero.



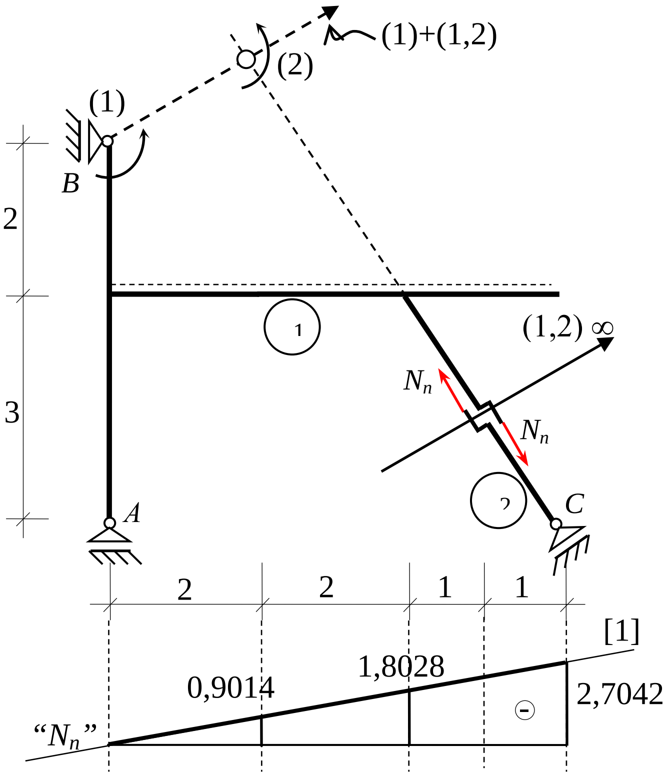
Influence line „Q_n":

Here we have again the same situation; the disk 1 is on the moving path and haven't a vertical displacement, therefore corresponding influence line is zero.

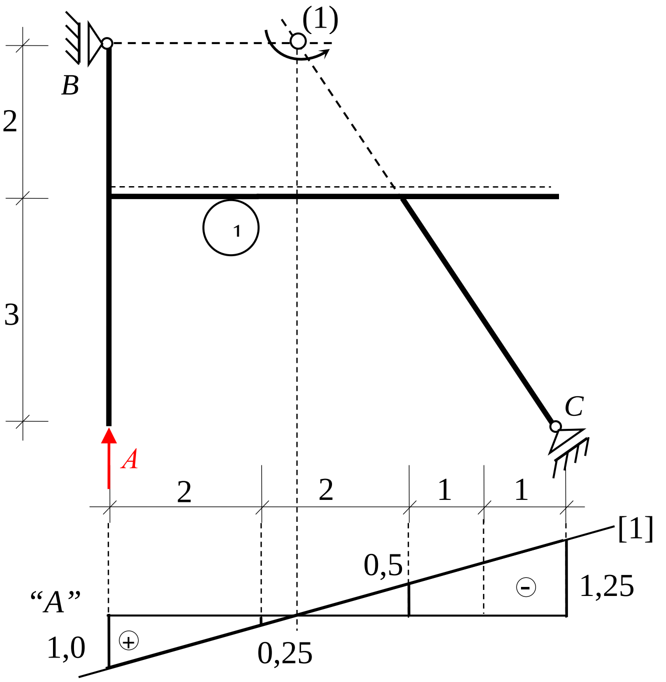


The solution of last two influence lines is clear, therefore only the final result is shown below.

Determine influence line „N_n”:



Determine influence line„A”:

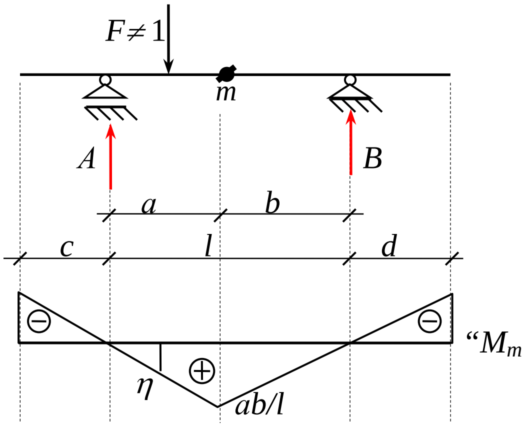


Calculating of the internal forces by using influence lines.

By using the influence lines one can calculate the values of the internal forces for a specified point caused by some external loads. This can be done very easy using the idea of the influence lines and the principle of the superposition.

As it is known each ordinates means the value of the function for which is composed the influence line when the unit force is up of the ordinates place. Therefore, if the force is not unit but F then the ordinate of the influence line (constructed using force F) will present the value of the function caused by force F so we may calculate the function caused by the force F.

$$S_k = F.\eta$$

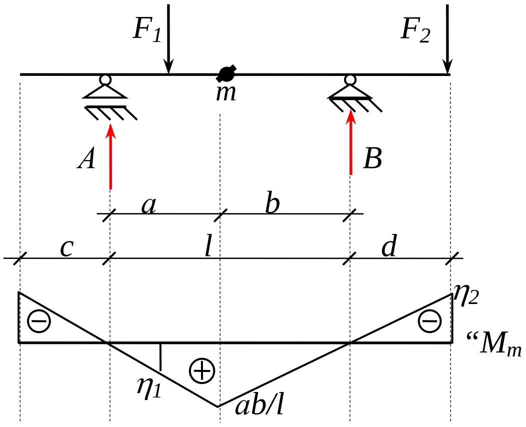


$$M_m = F.\eta$$

If there are more then one force one can use the superposition principle, then the function value will be as follow:

$$S_k = \sum F_i.\eta_i$$

If the force load is directed down then one must take the sign of the ordinate from the influence line.

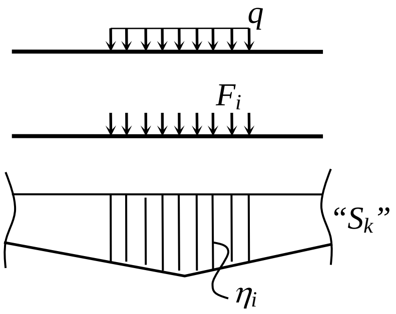


$$M_m = F_1.\eta_1 + F_2.\eta_2$$

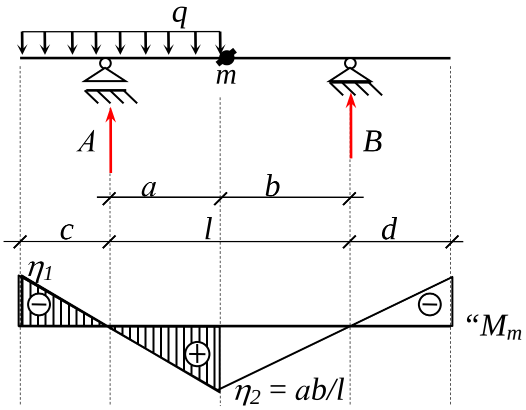
If the beam is subjected by the distributed load one can compute the function at a specified point by the formulae:

$$S_k = \sum q_i . \Omega_i$$

where the Ω_i is the area of the influence line under the *i*-th distributed load.
Why is used the area it is easy to understand if present the distributed load as a many forces at infinity small distance between each other.



Now one can use the previous formulae for the applied force and as the ordinates are very close to each other so the sum of them equals to the area. For the sing of the multiplier one must take the sing of the area of the influence line.

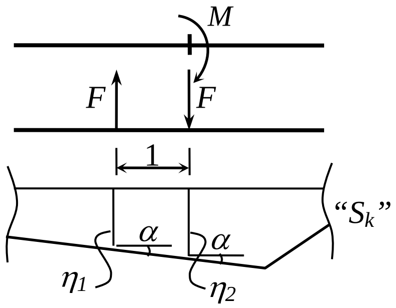


$$M_m = q.\left(-\frac{\eta_1 .c}{2}\right) + q.\frac{\eta_2 .a}{2}$$

Similarly, if the beam is under the concentrate moment load then the value of the function can be computed as follow:

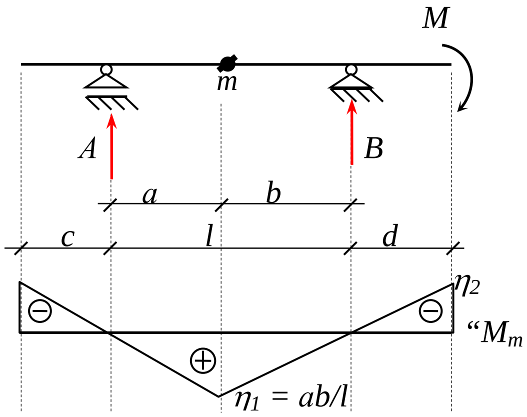
$$S_k = \sum M_i tg\alpha_i$$

This is easy to understanding if present the moment as a couple of forces whit the arm of unity. Then as we know the value of the forces is the same one can write:



$$S_k = \sum F_i.\eta_i = -F.\eta_1 + F.\eta_1 = F\left(\frac{\eta_2 - \eta_1}{1}\right) = F.1.tg\alpha = M.tg\alpha$$

When the moment load and the angle rotate at the same direction then the sing of the multiplier is positive. The direction of rotation determines as the reference line rotates to the influence line.



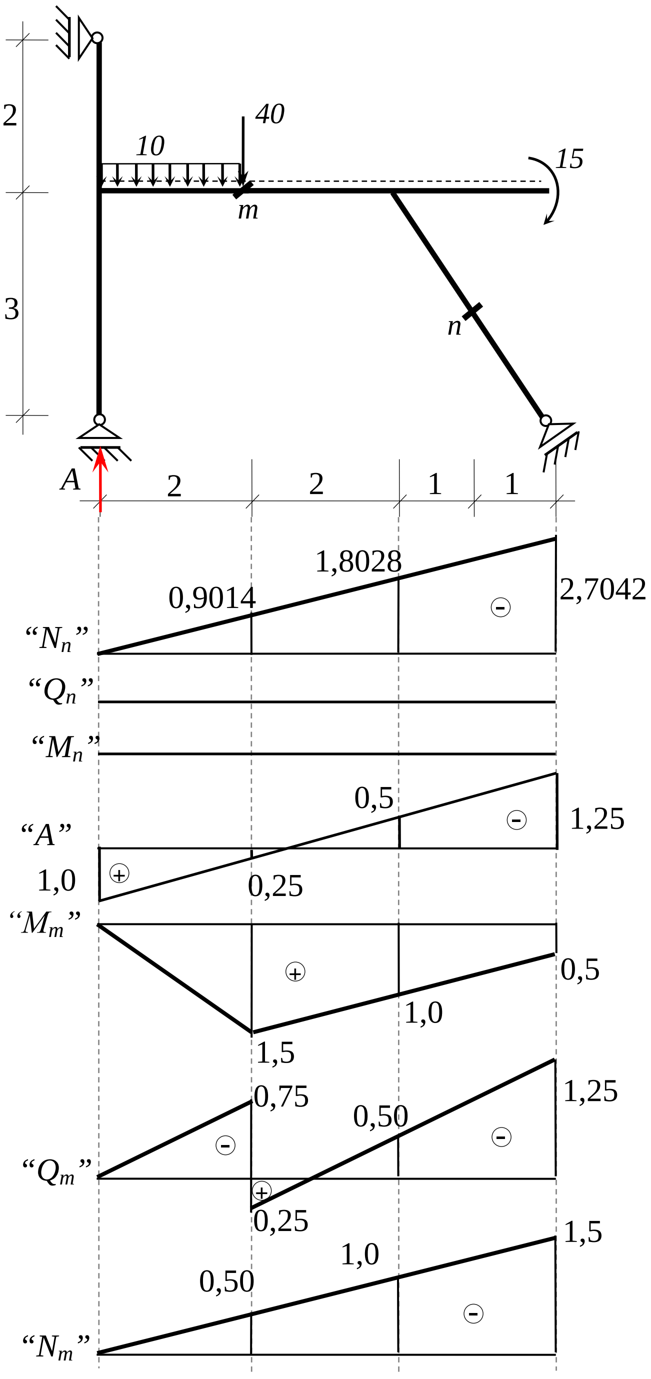
$$M_m = -M.\frac{\eta_1 + \eta_2}{b + d}$$

If a beam (frame) is subjected at the same time to forces, moments and a distributed loads then the value of the function for which the influence line is constructed on can compute as follow:

$$S_k = \sum F_i.\eta_i + \sum q_i.\Omega_i + \sum M_i tg\alpha_i$$

In the following example is presented this formula for the frame which influence lines where already composed.

If one already construct the internal force diagram can use the influence lines for verification of the obtained result.



$$N_n = -40.0,9014 - 10.\frac{0,9014.2}{2} - 15.\frac{2,7042 - 1,8028}{2} = 51,8305\,kN$$

$Q_n = 0$; $\qquad M_n = 0$

$$A = 40.0,25 + 10.\frac{(1,0 + 0,25).2}{2} - 15.\frac{1,25 - 0,5}{2} = 16,875\,kN$$

$$M_m = 40.1,5 + 10.\frac{1,5.2}{2} - 15.\frac{1,0 - 0,5}{2} = 71,25\,kN$$

$$Q_m^{\partial} = -40.0,75 - 10.\frac{0,75.2}{2} - 15.\frac{1,25 - 0,5}{2} = -43,125\,kN$$

$$Q_m^л = +40.0{,}25 - 10.\frac{0{,}75.2}{2} - 15.\frac{1{,}25 - 0{,}5}{2} = 3{,}125\,kN$$

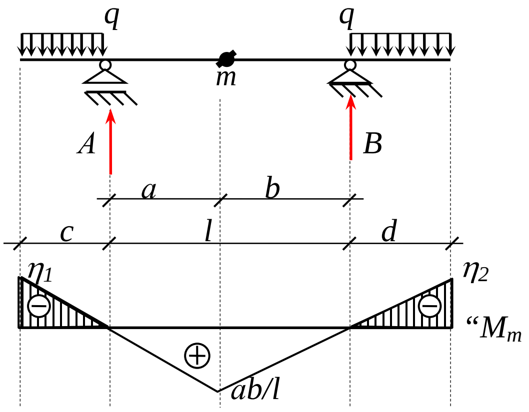
$$N_m = -40.0{,}5 - 10.\frac{0{,}5.2}{2} - 15.\frac{1{,}5 - 1{,}0}{2} = -28{,}75\,kN$$

Maximum value of the function

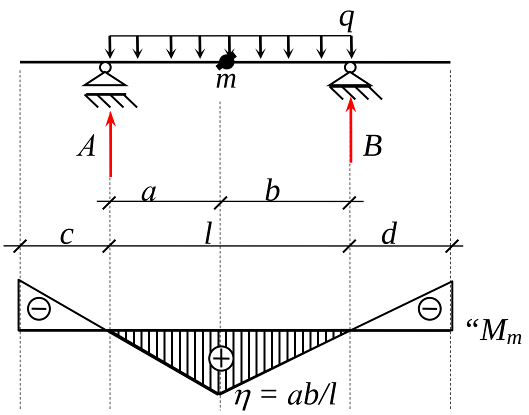
Using this idea one can determine the absolute maximum of the function for which the influence line is constructed from the distributed load passing through the beam (train). If the distributed load has a specific position at the beam then at the point for which the influence line is the function will have a maximum value. There are two questions:

1. What is the right position of the distributed load to produce the maximum effect?
2. What will happens if the influence line is constructed for the changing section – section at distance *x*?

The answer of the first question is as follow: If the influence line has negative and positive parts we must place the distributed load up to the whole positive part of the influence line to produce a maximum positive value of the function. Similarly if we place the distributed load up to the whole negative part of the influence line it will produce a maximum negative value of the function.

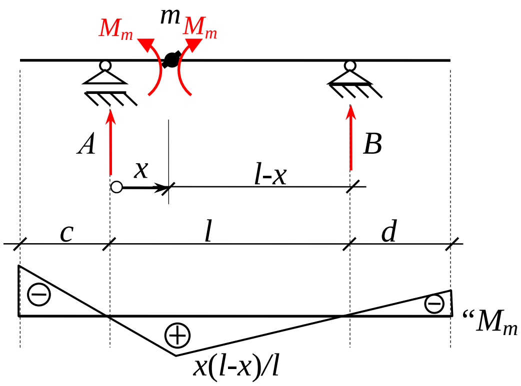


The maximum negative value of the M_m is: $M_m = q.\left(-\frac{\eta_1.c}{2}\right) + q.\left(-\frac{\eta_2.d}{2}\right) = -q\left(\frac{\eta_1.c}{2} + \frac{\eta_2.d}{2}\right)$

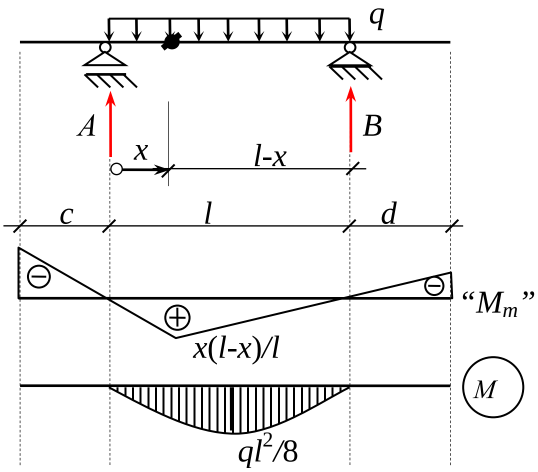


The maximum positive value of the M_m is: $M_m = +q.\frac{\eta.l}{2}$

The answer of the second question is as follow: If we construct an influence line for a section at distance x from the support, for example, we will have influence line for a function for each section of the beam span. The influence line will be a function of the distance x.



As we already know if now we have an external load we may compute the value of the function at section at distance x caused by the external load. However, this value will be valid for each section of the beam spam because the influence line is valid for each one section. Therefore, this value also will be a function of the position of the section – function of x. Thus if we draw this function we will have a diagram of the function caused of the external load.



$$M_m = q.\Omega = q.\frac{x.(l-x)}{l}.\frac{l}{2} = q.\frac{x.(l-x)}{2}$$

$$x = 0 :\rightarrow M = 0$$

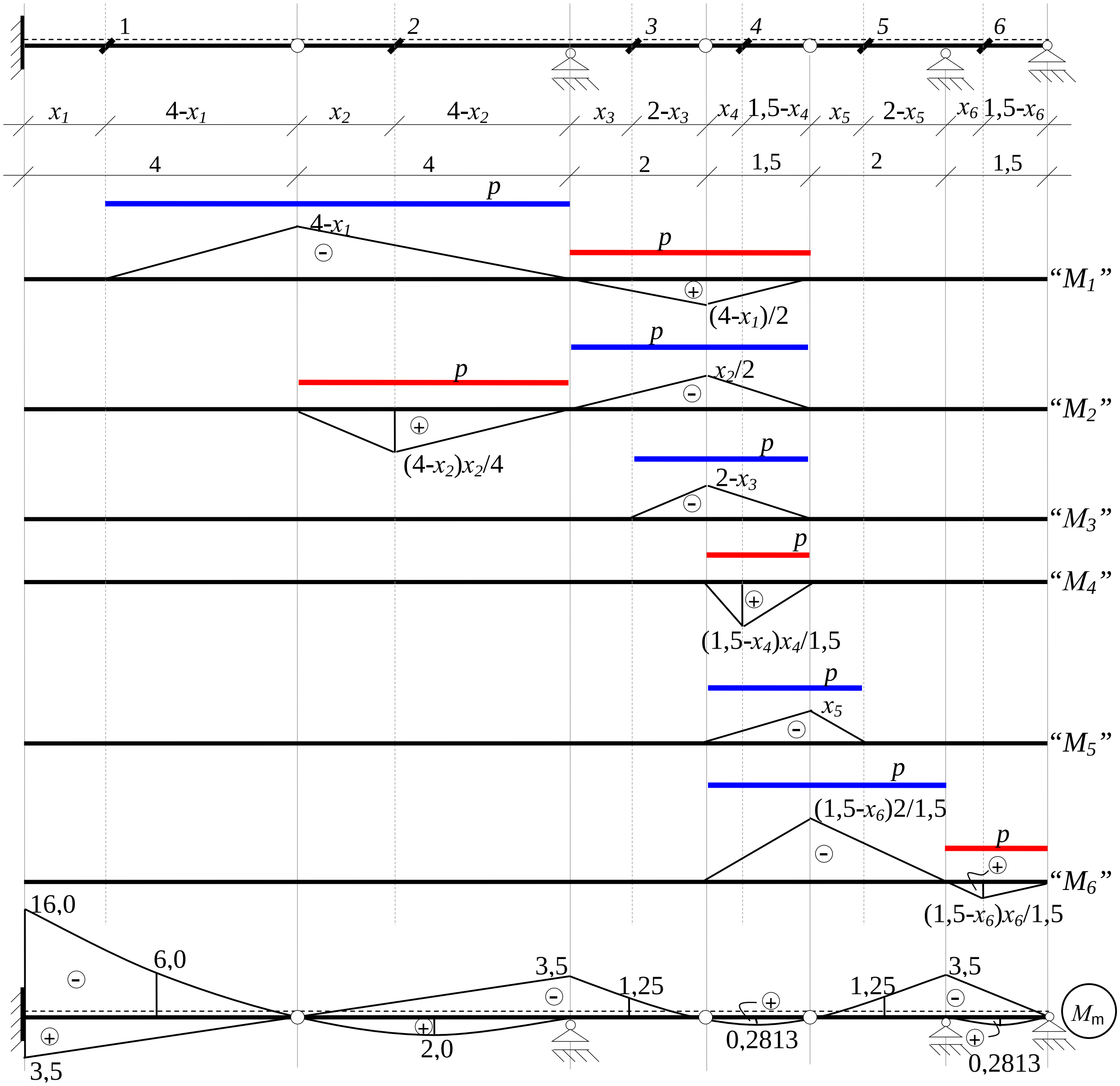
$$x = l/2 :\rightarrow M = q.\frac{\frac{l}{2}.\frac{l}{2}}{2} = \frac{ql^2}{8}$$

$$x = l :\rightarrow M = 0$$

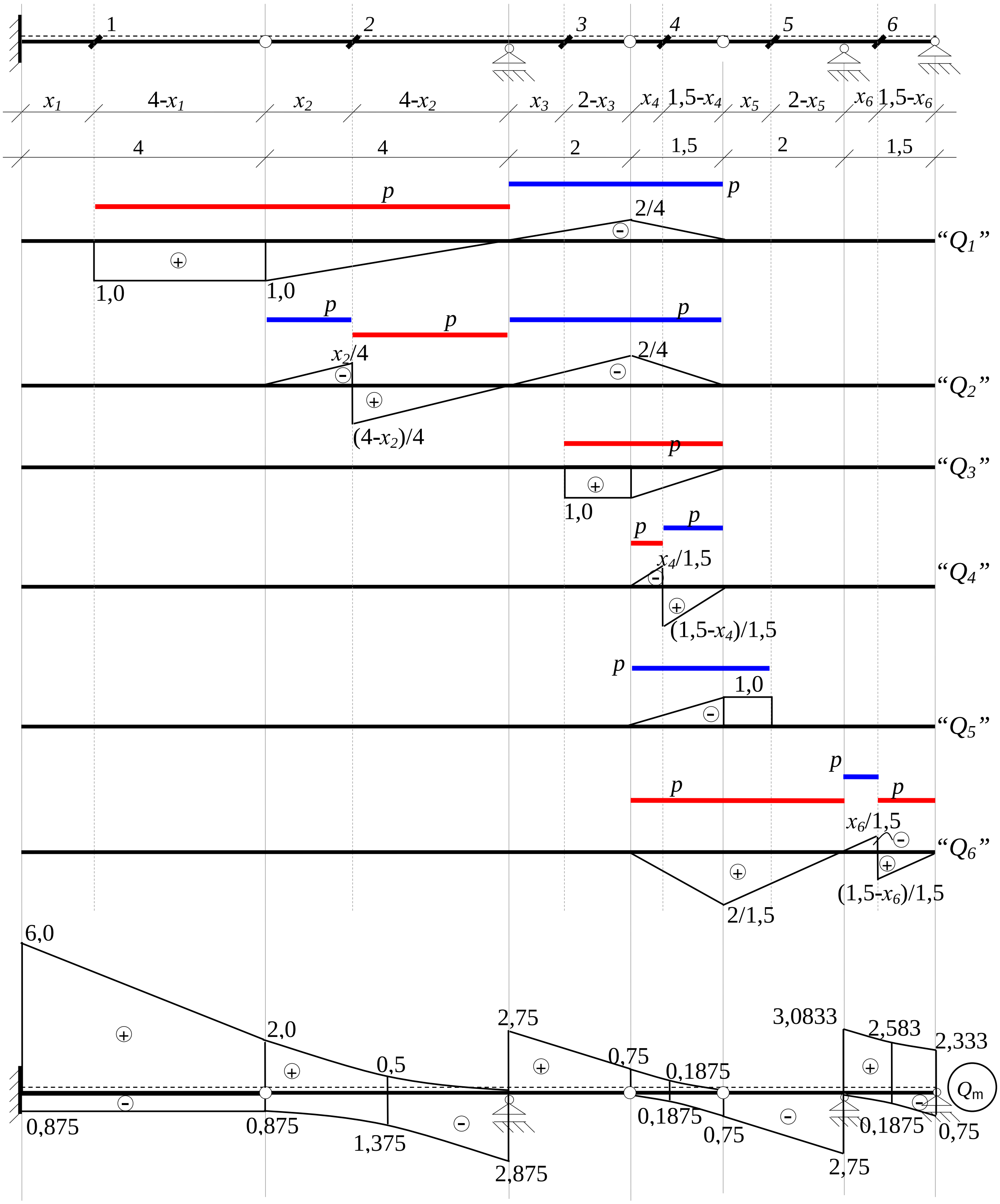
Using this idea one can compose an extreme diagram for the moment, for example, if the external load is placed at the specified place as it was described above.

This extreme diagram is useful for the bridge beams where is very important to know the maximum negative and maximum positive internal forces neglecting load place. For rich this diagram first must be composed the influence lines for each span of the bridge (compound) beam.

Next example presents this idea for a compound beam.



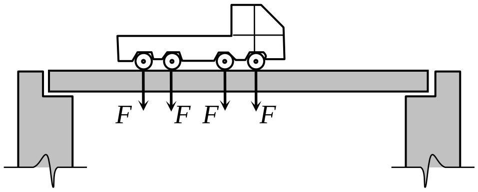
part		Functions	l	$x = 0$	$x = l/2$	$x = l$
1	**M–**	$[(4-x_1)(4-x_1+4)/2].p=(4-x_1)(8-x_1)/2$	4	16	6	0
	M+	$[((4-x_1)/2)(3{,}5/2)].p=0{,}875(4-x_1)$		3,5	1,75	0
2	**M–**	$(x_2/2)(3{,}5/2).p=0{,}875x_2$	4	0	1,75	3,5
	M+	$[((4-x_2)x_2/4)(4/2)].p=(4-x_2)x_2/2$		0	2	0
3	**M–**	$[(2-x_3)(2-x_3+1{,}5)/2].p=(2-x_3)(3{,}5-x_3)/2$	2	3,5	1,25	0
4	**M+**	$[((1{,}5-x_4)x_4/1{,}5)(1{,}5/2)].p=(1{,}5-x_4)x_4/2$	1,5	0	0,2815	0
5	**M–**	$[(1{,}5+x_5)x_5/2].p=x_5(1{,}5+x_5)/2$	2	0	1,25	3,5
6	**M–**	$[((1{,}5-x_6)2/1{,}5).((2+1{,}5)/2)].p=2{,}3333.(1{,}5-x_6)$	1,5	3,5	1,75	0
	M+	$[((1{,}5-x_6)x_6/1{,}5)(1{,}5/2)].p=(1{,}5-x_6)x_6/2$		0	0,2815	0

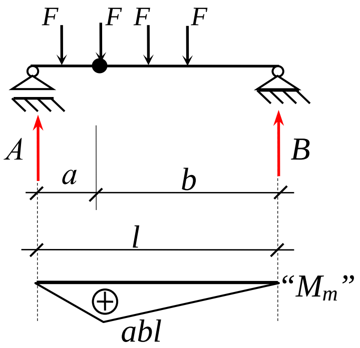

1
2
3
4
5
6
x_1
4-x_1
x_2
4-x_2
x_3
2-x_3
x_4
1,5-x_4
x_5
2-x_5
x_6
1,5-x_6
4
4
2
1,5
2
1,5
p
p
2/4
“Q_1”
1,0
1,0
p
p
p
x_2/4
2/4
“Q_2”
(4-x_2)/4
p
“Q_3”
1,0
p
p
x_4/1,5
“Q_4”
(1,5-x_4)/1,5
p
1,0
“Q_5”
p
p
p
x_6/1,5
“Q_6”
(1,5-x_6)/1,5
2/1,5
6,0
2,0
0,5
2,75
0,75
0,1875
3,0833
2,583
2,333
Q_m
0,875
0,875
1,375
2,875
0,1875
0,75
2,75
0,1875
0,75

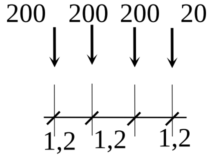
part		Functions	*l*	*x* = 0	*x* = *l*/2	*x* = *l*
1	**Q−**	$[(2/4)(3{,}5/2)].p = 0{,}875$	4	0,875	0,875	0,875
	Q+	$[1(4 - x_1) + 1.4/2].p = (4 - x_1) + 2$		6	4	2
2	**Q−**	$[x_2 x_2/(4.2) + 4.3{,}5/(2.2)].p = x_2^2/8 + 3{,}5$	4	0,875	1,375	2,875
	Q+	$[(4 - x_2)(4 - x_2)/(4.2)].p = (4 - x_2)^2/8$		2	0,5	0
3	**Q+**	$[1(2 - x_3) + 1.1{,}5/2].p = (2 - x_3) + 0{,}75$	2	2,75	1,75	0,75
4	**Q+**	$[x_4 x_4/(2.1{,}5)].p = x_4^2/3$	1,5	0	0,1875	0,75
	Q−	$[(1{,}5 - x_2)(1{,}5 - x_2)/(1{,}5.2)].p = (1{,}5 - x_2)^2/3$		0,75	0,1875	0
5	**Q−**	$[1.x_5 + (1.1{,}5)/2].p = x_5 + 0{,}75$	2	0,75	1,75	2,75
6	**Q−**	$[x_6 x_6/(2.1{,}5)].p = x_6^2/3$	1,5	0	0,1875	0,75
	Q+	$[(1{,}5 - x_6)(1{,}5 - x_6)/(1{,}5.2) + (2.3{,}5)/(1{,}5.2)].p =$ $= (1{,}5 - x_6)^2/3 + 2{,}3333$		3,0833	2,583	2,333

Absolute maximum live moment in bridges.

It is necessary to exam the situation when the motor truck pass through the bridge, then at the construction appears some maximum internal forces. There are different position of this vehicle at which the internal forces are very high values. The influence lines are useful for examine the internal forces in dependence of the position of the vehicle. For this examining is used a standart vehicle as shown at the next figure:







The absolute maximum of the internal force searches trying different position of the four forces carrying the following points:

- one of the forces should be up to the maximum value of the influence line;
- if it is possible the all four should be at the positive (or negative) part of the influence line;
- if it is not possible then try only one of the forces be at the part with the opposite sing with very low ordinate value.

Chapter 7

Deflections using energy methods.

It is very useful to determine the deflections and displacements in a frame or beam by using the energy methods in the structural mechanic. For doing this on can understand the meaning of the deflection, virtual work and the energy theorems. In this chapter we will explain the principles of virtual work and how to determine displacements end deflections using these principles.

External work and strain energy:

Before developing any energy method we will explain the external work and the strain energy (the internal work) done by a force and a moment.

The external work done by a force:

When a force F undergoes a displacement *dx* in the same direction as the force, the work done is: $dW_{ext} = F\left(x\right)dx$. If the total displacement is *x*, the work becomes:

$$dW_{ext} = F\left(x\right)dx \qquad \textbf{7. 1}$$

Consider now the effect caused by an axial force applied to the end of a bar. As the magnitude of the force increase from zero to some value $\overline{F}$ the final elongation of the bar becomes Δ. As in the statical analysis, the material has a linear elastic response (The Hook's low is valid), then the force value at some moment will be as follow:

$$F(x) = \frac{\overline{F}}{\mathit{\Delta}}x \qquad \textbf{7. 2}$$

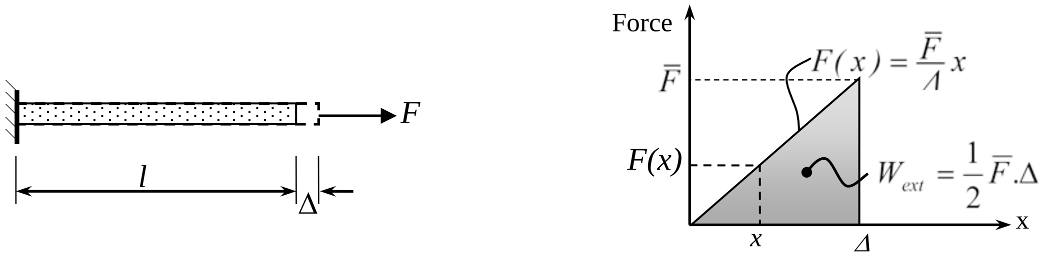


Figure 7 - 1

The work done by this force, we will be defined as:

$$W_{ext} = \int_0^{\Delta} F(x)dx; \qquad F(x) = \frac{\overline{F}}{\Delta}.x$$

$$W_{ext} = \int_0^{\Delta} \frac{\overline{F}}{\Delta}.x.dx = \frac{\overline{F}}{\Delta}\int_0^{\Delta}.x.dx = \frac{\overline{F}}{\Delta}.\frac{x^2}{2}\bigg|_0^{\Delta} = \frac{1}{2}\overline{F}.\Delta \qquad \textbf{7. 3}$$

$$W_{ext} = \frac{1}{2}\overline{F}.\Delta$$

If there are several external forces, the work done will be:

$$W_{ext} = \frac{1}{2}\sum_{i=1}^{n} \overline{F}_i.\Delta_i \qquad \textbf{7. 4}$$

Where $\overline{F_i}$ is the final value of the *i*-th force and Δ_i is the corresponding displacement.

Deformation caused by the internal normal (axial) force:

The normal force causes the normal deformations at a differentially small part of the beam. This deformation is defined as follows:

From the geometry, we have connection between the elongation, deformation and the length of the differentially small element. From the definition of the stresses, we have connection between the normal stress and the normal force. From the Hooke's law we have connection between the normal stress and the deformation. As we use all these connections together we will obtain for the elongation the following expression:

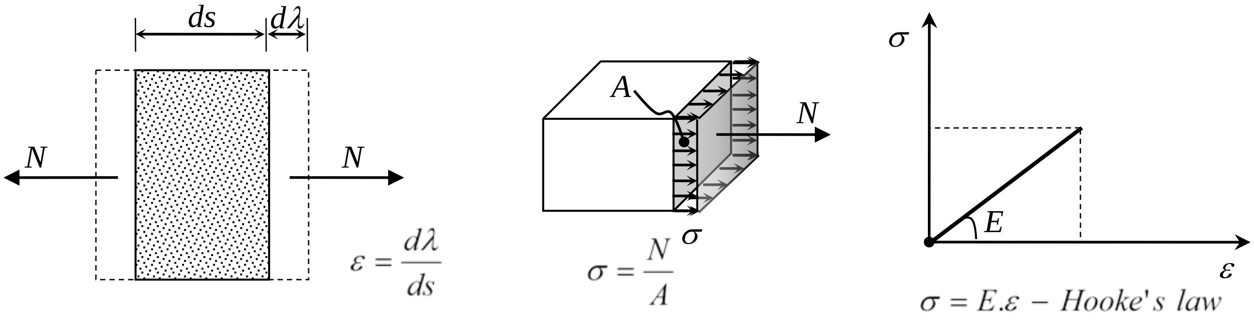


Figure 7 - 2

$$\frac{N}{A} = E.\varepsilon = E.\frac{d\lambda}{ds} \qquad \rightarrow \qquad d\lambda = \frac{N}{EA}ds \qquad \textbf{7. 5}$$

Deformation caused by the internal moment:

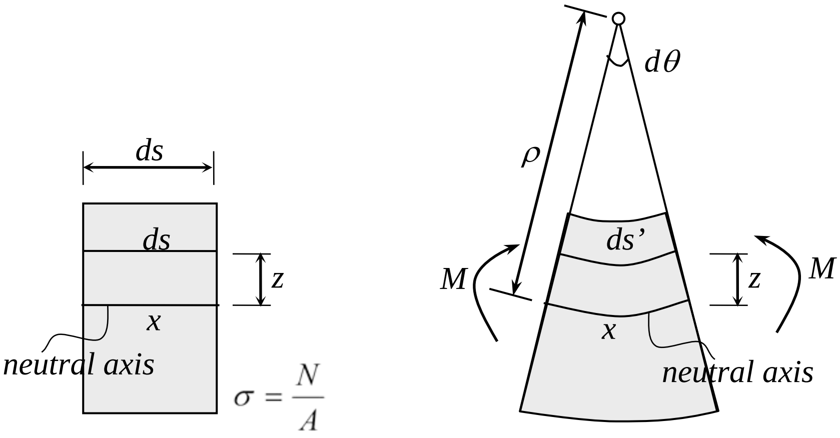


Figure 7 - 3

The strain in arc *ds* located at a position *z* from the neutral axis *x* is:

$$\varepsilon = \frac{ds' - ds}{ds} \qquad \textbf{7. 6}$$

However from the geometry $ds = dx = \rho.d\theta$ and $ds' = (\rho - z).d\theta$ so:

$$\varepsilon = \frac{(\rho - z)d\theta - \rho.d\theta}{\rho.d\theta} = -\frac{z}{\rho} \qquad 7.7$$

Or:

$$\frac{1}{\rho} = -\frac{\varepsilon}{z} \qquad \mathbf{7.8}$$

Where: $\frac{1}{\rho}$ is the curvature.

Using Hooke’s law and stress definition for bending, we obtain:

$$\varepsilon = \frac{\sigma}{E}; \qquad \sigma = -\frac{M.z}{I} \qquad \Rightarrow \qquad \varepsilon = -\frac{M.z}{EI} \qquad \mathbf{7.9}$$

And finally:

$$\frac{1}{\rho} = \kappa = \frac{M}{EI} \qquad \mathbf{7.10}$$

Where: *EI* is the flexural rigidity; *I* is the moment of inertia computed about the neutral axis.

$$dx = \rho.d\theta$$
$$d\theta = \frac{M}{EI}ds \qquad \mathbf{7.11}$$

Deformation caused by the shear force:

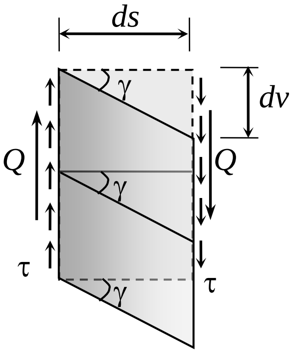


Figure 7 - 4

$$dv = \gamma.ds; \qquad \gamma = \frac{\tau}{G}; \qquad G = \frac{2(1+\nu)}{E}; \qquad \tau = \frac{Q}{A}.\kappa = \frac{Q}{A_Q} \qquad \mathbf{7.12}$$

κ - coefficient depending on the cross section area.

$$\boldsymbol{dv = \frac{Q}{G.A_Q}ds} \qquad \mathbf{7.13}$$

Strain energy of the body:

If the material is linear elastic and isotropic then the strain energy caused the axial force will be expressed as follows:

$$\boldsymbol{W_{int}^{N} = -\frac{1}{2}N.d\lambda = -\frac{1}{2}\frac{N^2}{EA}ds} \qquad \mathbf{7.14}$$

Similarly for the strain energy caused by the internal moment:

$$W_{int}^{M} = -\frac{1}{2}M.d\theta = -\frac{1}{2}\frac{M^2}{EI}ds \qquad 7.15$$

And for the shear force:

$$W_{int}^{Q} = -\frac{1}{2}Q.dv = -\frac{1}{2}\frac{Q^2}{GA_Q}ds \qquad 7.16$$

The internal work (the strain energy) for the differentially small element is:

$$W_{int} = -\frac{1}{2}\left(\frac{M^2}{EI}+\frac{N^2}{EA}+\frac{Q^2}{GA_Q}\right)ds \qquad 7.17$$

And for the whole element is:

$$W_{int} = -\frac{1}{2}\left(\sum\int\frac{M^2}{EI}ds+\sum\int\frac{N^2}{EA}ds+\sum\int\frac{Q^2}{GA_Q}ds\right) \qquad 7.18$$

Work expression for element under external load:

If the body is in equilibrium then the internal and external work will be equal whit reversed sing or:

$$W_{int} + W_{ext} = 0 \qquad 7.19$$

Or

$$\frac{1}{2}\sum F_i.\Delta_i - \frac{1}{2}\left(\sum\int\frac{M^2}{EI}ds+\sum\int\frac{N^2}{EA}ds+\sum\int\frac{Q^2}{GA_Q}ds\right) = 0 \qquad 7.20$$

And finally:

$$\sum F_i.\Delta_i = \left(\sum\int\frac{M^2}{EI}ds+\sum\int\frac{N^2}{EA}ds+\sum\int\frac{Q^2}{GA_Q}ds\right) \qquad 7.21$$

Deformation caused by temperature load;

The temperature induce deflections on element and these deflections induce internal forces – moments and axial forces.

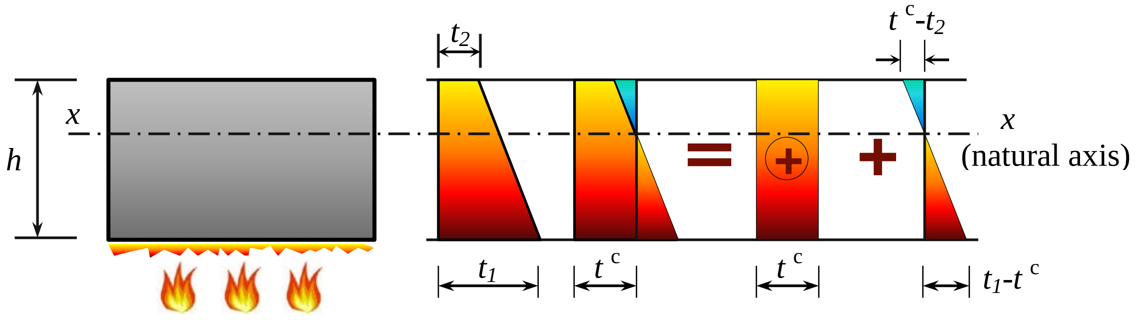


Figure 7 - 5

When the element is subjected on the temperature load et one of its sides the temperature is higher than the other. This difference cause internal forces in the element. Temperature distribution along the element is given on Figure 7 -5. If *x*-*x* is the natural axis, it divide the temperature distribution at two parts. The first is a constant temperature distribution and the second is temperature difference whit a zero value at the natural axis (Figure 7 - 5). The constant temperature distributions cause an elongation of the element and as a consequence an internal normal force. The temperature difference cause a deflection of the element and as a a consequence an internal moment.

Elongation and internal force

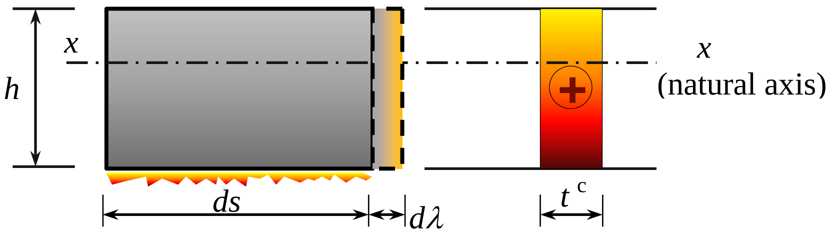


Figure 7 - 6

$$d\lambda = \varepsilon_t . ds \qquad 7.22$$

From the physics is known that:

$$\varepsilon_t = \alpha_t . t^c \qquad 7.23$$

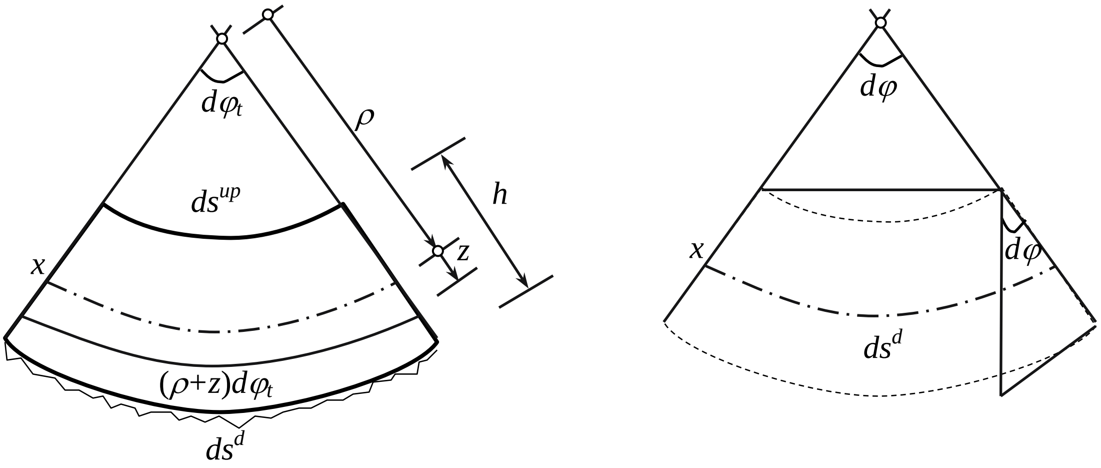
Where, α_t is the coefficient of the thermal extension and t_c is the temperature at the cross section:

$$tc = \frac{t_1 + t_2}{2}$$ - for rectangular cross section.

It follows:

$$d\lambda = \alpha_t . t^c . ds \qquad 7.24$$

Deflection from the temperature differences:



The extensions of the upper and down bars are as follow:

$$ds^{up} = \alpha_t(t^c - t_2)ds \tag{7. 25}$$

$$ds^d = \alpha_t(t_1 - t^c)ds \tag{7. 26}$$

The angle of the deflection is:

$$d\varphi_t = \frac{ds^d - ds^{up}}{h} = \frac{\alpha_t(t_1 - t^c) - \alpha_t(t^c - t_2)}{h}ds \tag{7. 27}$$

$$d\varphi_t = \frac{\alpha_t(t_1 - t_2)}{h}ds = \frac{\alpha_t \Delta t}{h}ds \tag{7. 28}$$

Internal work at an element caused by the internal forces from the temperature load:

$$W_{int}^t = -\frac{1}{2}N.\lambda_t - \frac{1}{2}M.d\varphi_t \tag{7. 29}$$

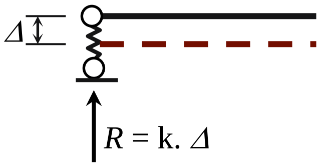
$$W_{int}^t = -\frac{1}{2}\left(N.\alpha_t.t^c + M.\alpha_t\frac{\Delta t}{h}\right) \tag{7. 30}$$

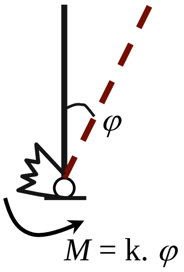
For more than one element;

$$W_{int}^t = -\frac{1}{2}\left(\sum\int\frac{M.\alpha_t.\Delta t}{h}ds + \sum\int N.\alpha_t.t^c\,ds\right) \tag{7. 31}$$

Internal work of the support reaction at a spring supports:

If the system has a springs supports then there have a displacements and the support reactions done internal work.





Were *k* is the rigidity of the spring;
Δ (*φ*) is the displacement at the spring support.

If *S* is generally the support reaction (moment or force) and *Δ* is the displacement (linear or rotational) at the support we can write:

$$S = \mathrm{k}.\Delta \tag{7. 32}$$

Or

$$\Delta = \frac{S}{k} \tag{7. 33}$$

And the work done by a reaction will be:

$$W_{int}^{sp} = -\frac{1}{2}\mathbf{S}.\Delta = -\frac{1}{2}\frac{S^2}{k} \tag{7. 34}$$

If there are *n* spring supports, the work be:

$$W_{int}^{sp} = -\frac{1}{2}\sum_1^n S.\Delta = -\frac{1}{2}\sum_1^n \frac{S^2}{k} \qquad 7.\ 35$$

If there is a system including *n* elements under external and temperature loads and spring supports then if the system is at equilibrium then the work expression will be:

$$\sum F.\Delta = \sum\int\frac{M^2}{EI}ds + \sum\int\frac{N^2}{EA}ds + \sum\int\frac{Q^2}{GA_Q}ds + \sum\int\frac{M.\alpha_t.\Delta t}{h}ds + \sum\int N.\alpha_t.t^c\,ds + \sum_1^n\frac{S^2}{k} \qquad 7.\ 36$$

Principle of virtual work:

As we already mention the principle of virtual work may be defined as follows: Consider a structure in equilibrium with a system of applied forces is subjected to a system of virtual displacements compatible with the external restraints and the geometry of the structure. The total work done by the applied forces during these external displacements equals the work done by the internal forces, corresponding to the applied forces, during the internal deformations corresponding to the external displacements.

The expression “virtual work” signifies that the work done is the product of a real loading system and imaginary displacements or an imaginary loading system and real displacements.

$$\delta W_{ext} + \delta W_{int} = 0 \qquad 7.\ 37$$

If we impose a virtual displacement at a deformable system there will appear a virtual deflections of the body as follow:

$$\overline{\kappa} = \frac{\overline{M}}{EI} \qquad \overline{d\lambda} = \frac{\overline{N}}{EA}ds \qquad \overline{dv} = \frac{\bar{Q}}{G.A_Q}ds \qquad 7.\ 38$$

$$\overline{d\lambda} = \alpha_t.t^c.ds \qquad \overline{d\varphi_t} = \frac{\alpha_t\Delta t}{h}ds \qquad \overline{\Delta} = \frac{\bar{S}}{k} \qquad 7.\ 39$$

The work done by the real internal forces with the virtual deflections will be:

$$\delta W_{int} = \frac{1}{2}\left(\sum\int\frac{\overline{M}M}{EI}ds + \sum\int\frac{\overline{N}N}{EA}ds + \sum\int\frac{\bar{Q}Q}{GA_Q}ds + \sum\int\frac{M.\alpha_t.\Delta t}{h}ds + \sum\int N.\alpha_t.t^c\,ds + \sum_1^n\frac{\bar{S}S}{k}\right) \qquad 7.\ 40$$

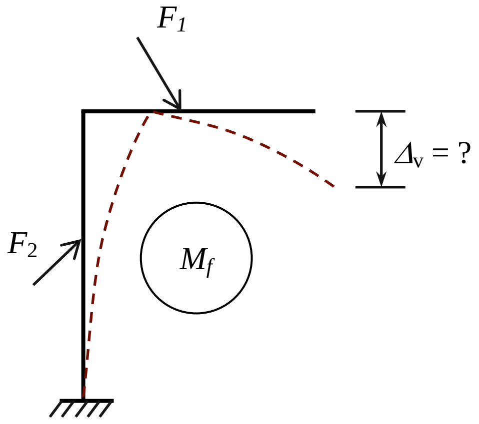
The work done by the real external forces with the virtual displacements will be:

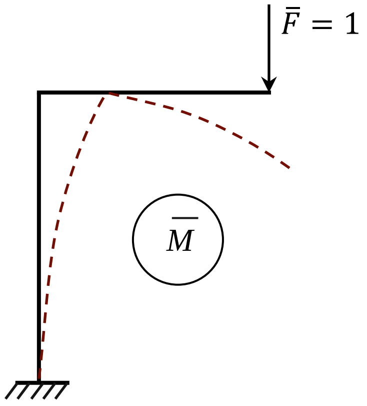
$$\delta W_{ext} = -\frac{1}{2}\sum F.\overline{\Delta} \qquad 7.\ 41$$

So, principle of the virtual work gives following expression:

$$\sum F.\overline{\Delta} = \sum\int\frac{\overline{M}M}{EI}ds + \sum\int\frac{\overline{N}N}{EA}ds + \sum\int\frac{\bar{Q}Q}{GA_Q}ds + \sum\int\frac{\overline{M}.\alpha_t.\Delta t}{h}ds + \sum\int \overline{N}.\alpha_t.t^c\,ds + \sum_1^n\frac{\bar{S}S}{k} \qquad 7.\ 42$$

This expression gives as a possibility to obtain the value of displacement at some point of the structure. If a unit force is imposed at the point and toke this system of forces and displacement as a virtual then the right side of the expression *7.42* will be equal to the displacement at the point we are trying to find.





$$\bar{F}.\Delta_v = \Delta_v = \sum\int\frac{\bar{M}M_f}{EI}ds + \sum\int\frac{\bar{N}N_f}{EA}ds + \sum\int\frac{\bar{Q}Q_f}{GA_Q}ds$$

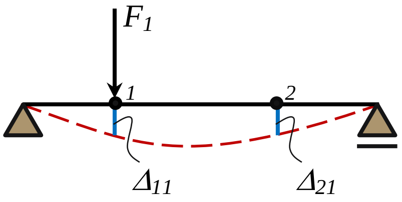
Energy theorems:

Betti's theorem, (discovered by Enrico Betti in 1872),

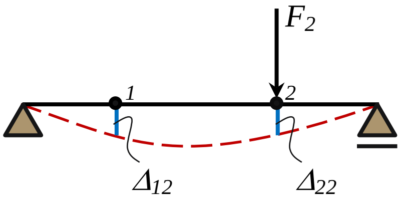
States that for a linear elastic structure subject to two sets of forces $\{P_i\}$ $i = 1,...,m$ and $\{Q_j\}$, $j = 1,2,...,n$, the work done by the set P through the displacements produced by the set Q is equal to the work done by the set Q through the displacements produced by the set P.

Example:

Stage 1



Stage 2



If stage 1 is real and stage 2 is virtual then we have:

$$W_{ext} = F_1.\Delta_{12} = W_{int}$$
$$F_1\Delta_{12} = \sum\int\frac{M_1M_2}{EI}ds$$

If stage 2 is real and stage 1 is virtual then we have:

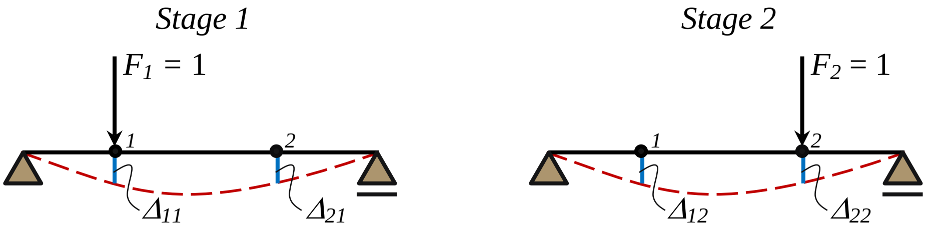
$$W_{ext} = F_2.\Delta_{21} = W_{int}$$
$$F_2\Delta_{21} = \sum\int\frac{M_2M_1}{EI}ds$$

It follows that:

$F_1\Delta_{12} = F_2\Delta_{21}$ – Betti's theorem

Maxwell’s theorem of reciprocal displacements:

States that a displacement at point I on a structure produced by a unit force at point k is equal to the displacement at point k when the unit force is acting at point i.
This theorem follows from the Betti’s theorem if the forces F_1 and F_2 have a unit value.



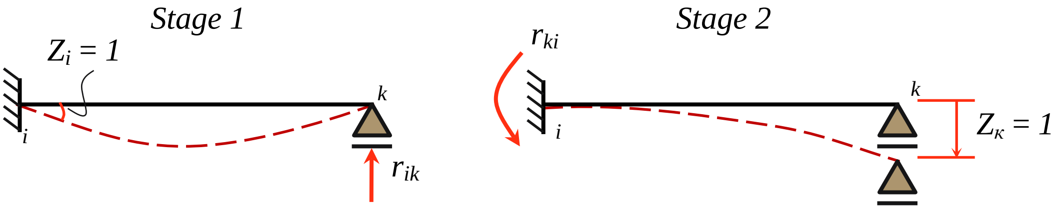
$$1.\Delta_{12} = 1.\Delta_{21}$$

or

$$\Delta_{12} = \Delta_{21}$$

Theorem of the reciprocal reactions:

States that the reaction at point *i* produced by unit displacement at point *k* is equal of the reaction at point *k* caused by unit displacement at point *i*.
This theorem follows from the Betti’s theorem if the displacements Z_i and Z_k have a unit value.



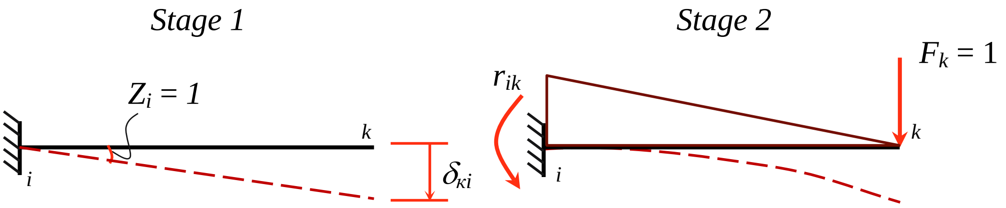
$$r_{ik}Z_i = r_{ki}Z_k$$
$$r_{ik}.1 = r_{ki}.1$$

or

$$r_{ik} = r_{ki}$$

Theorem of the reciprocal reactions and displacements:

States that the reaction at point *i* produced by unit displacement at point *k* is equal to the displacement at point *k* caused by unit force at point *I* with inversed sing.
This theorem follows from the Betti’s theorem if the displacements Z_i and F_k have a unit value.



$$\delta_{ki}F_k = -r_{ik}Z_i$$
$$\delta_{ki}.1 = -r_{ik}.1$$

or

$$\delta_{ki} = -r_{ik}$$

Chapter 8

Displacements at statically determinate structures.

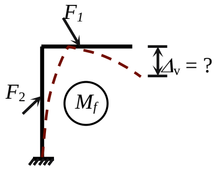
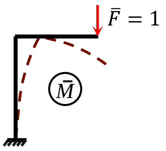
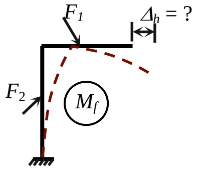
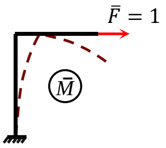
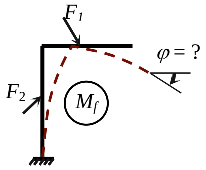
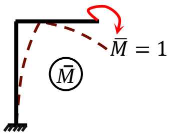
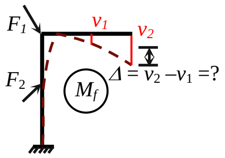
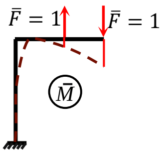
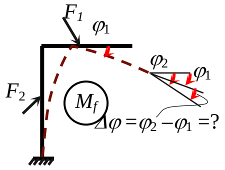
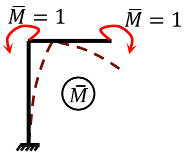
A principle of virtual work can be used for determining of the displacement of some point in the statically determinate frame or beam. It was shown at a previous chapter how the deflections are determinate and how the virtual work is used for calculating displacement of specific point.

According to principle of the virtual work for a system subjected to an external force and temperature load and including springs supports the displacement at a specific point calculates using the next expression:

$$\Delta = \sum\int \frac{\bar{M}M}{EI}ds + \sum\int \frac{\bar{N}N}{EA}ds + \sum\int \frac{\bar{Q}Q}{GA_Q}ds + \sum\int \frac{\bar{M}.\alpha_t.\Delta t}{h}ds + \sum\int \bar{N}.\alpha_t.t^c\,ds + \sum_1^n \frac{\bar{S}S}{k}$$

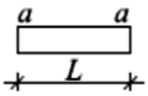
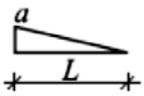
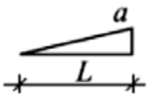
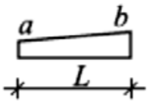
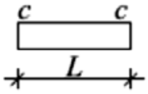
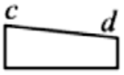
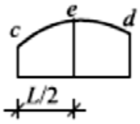
Where $\bar{M}, \bar{Q}, \bar{N}$, and $\bar{S}$ are internal and spring forces caused by a virtual unit load at the specific point for which we are calculating the displacement.

This virtual load correspond to the displacement. It is shown at the next table:

N	Description	Displacement	Virtual load
1	If the displacement is linear vertical then the unit load is a vertical force.		
2	If the displacement is linear horizontal then the unit load is a horizontal force.		
3	If the displacement is rotation then the unit load is a moment.		
4	If the displacement is linear relative then the unit load is a couple of forces		
5	If the displacement is relative rotation then the unit load is a couple of moments.		

Before to show some examples for calculating displacements is needed to make some comments about how to calculate the integrals at a formula 8.1. This formula means that we should find an integral of multiplication of two functions. The first is the function of the diagram from the external load and the second is the function of the diagram from the virtual load. The first function can be arbitrary – from constant to a parabolic by third degree. The second according to the virtual load is at maximum linear function. From this reason, these integrals are not so difficult for numerical calculation and it is done for all cases at a table. Such a table is shown below:

$$I = \int_0^l f_1(x) f_1(x) dx$$

$f_1(x)$ $f_2(x)$					
1		acL	$\frac{acL}{2}$	$\frac{acL}{2}$	$\frac{c(a+b)L}{2}$
2		$\frac{acL}{2}$	$\frac{acL}{3}$	$\frac{acL}{6}$	$\frac{c(2a+b)L}{6}$
3		$\frac{acL}{2}$	$\frac{acL}{6}$	$\frac{acL}{3}$	$\frac{c(a+2b)L}{6}$
4		$\frac{a(c+d)L}{2}$	$\frac{a(2c+d)L}{6}$	$\frac{a(c+2d)L}{6}$	$\frac{[ac+(a+b)(c+d)+bd]L}{6}$
5		$\frac{a(c+4e+d)L}{6}$	$\frac{a(c+2e)L}{6}$	$\frac{a(d+2e)L}{6}$	$\frac{[ac+2e(a+b)+bd]L}{6}$

Where $f_1(x)$ is the function from the virtual load and $f_2(x)$ is the function from the external load. The integrals are calculated for different elements or parts of elements where the functions are steady. The integral calculates from the beginning of the element (or part of the element) to its end.

Calculation of displacements at statically determinate structures from external load.

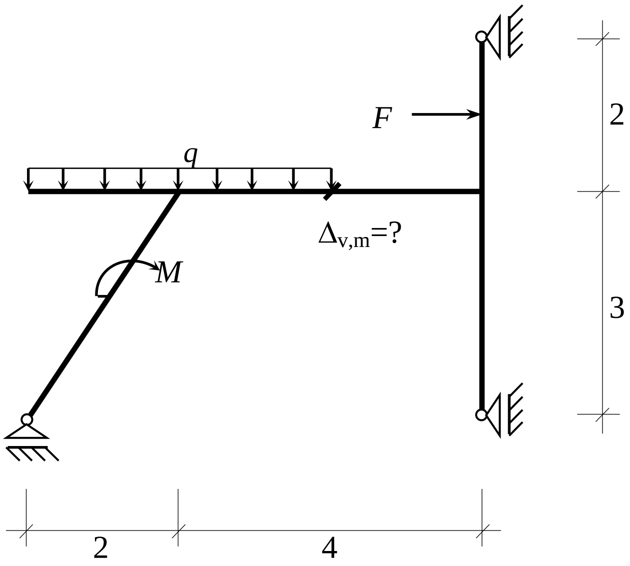
If there is a frame without springs supports and loaded with an external force load then the displacement calculates as follows:

$$\Delta = \sum \int \frac{\bar{M} M_f}{EI} ds + \sum \int \frac{\bar{N} N_f}{EA} ds + \sum \int \frac{\bar{Q} Q_f}{GA_Q} ds$$

Usually major influence of the displacement values has the internal moment and the influence of the normal and shear forces is negligible. That is why (when we make hand calculations) we usually ignore the second and the third part of the integral. Sometimes we include the integral of the normal forces if the structure has bars working of tension and compression but in this case it isn’t so obligate. Only when the structure is a truss then of course the only way to calculate the displacement is to use integral of the normal forces. The other two integrals are zero.

Now we will show an example of calculation of displacement at a frame under force load using only the integral of the moments and calculating integrals using tables.

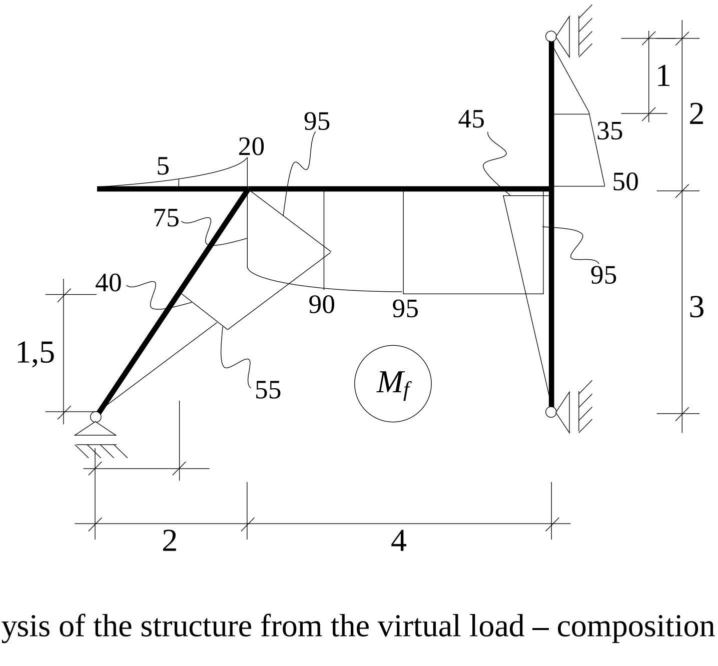
Calculation of vertical displacement of point *m* done from the external load:



Data:	***beams(0.25/0.315)**:*	***column(0.25/0.25)**:*
$F = 20\ kN$	$I_b = 0.000651\ m^4$	$I_{col} = 0.0003255\ m^4$
$M = 15\ kNm$	$A_b = 0.0788\ m^2$	$A_{col} = 0.0625\ m^2$
$q = 10\ kN/m'$		
$E = 2.4.10^7\ kN/m^2$	$\mathbf{I_{гр} = 2.I_{кол}}$	$\alpha_t = 1.10^{-5}$

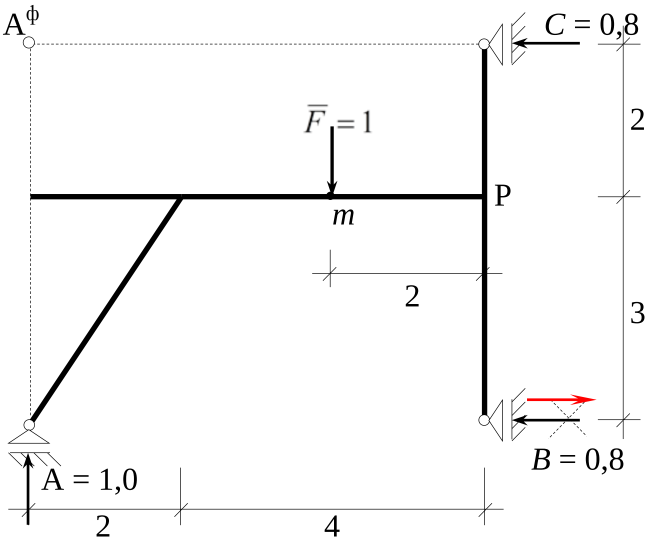
- **Step 1:** Analysis of the structure from the external load – composition of the M_f diagram:

The result is:

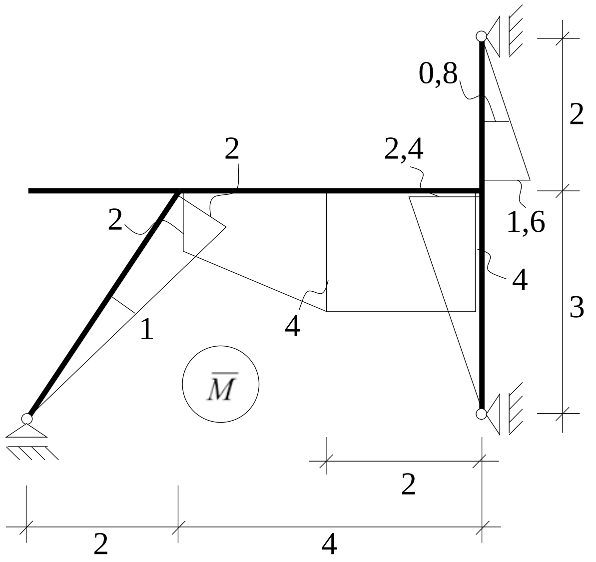


- **Step 2:** Analysis of the structure from the virtual load – composition of the $\overline{M}$ diagram.

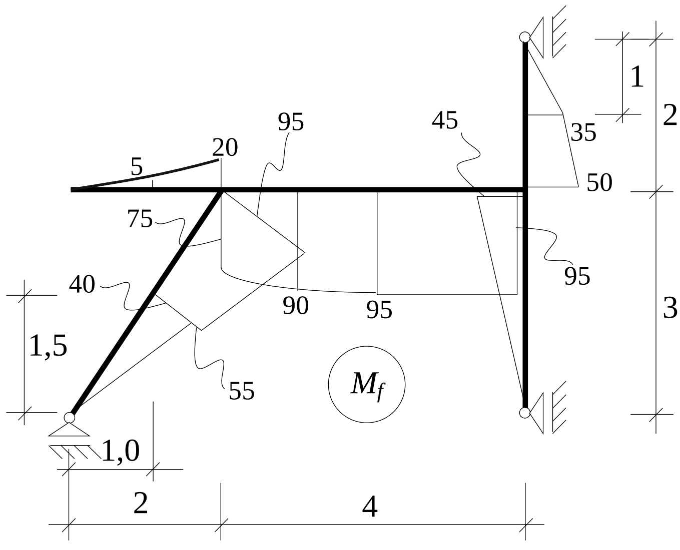
The virtual load is a vertical force at point *m*:

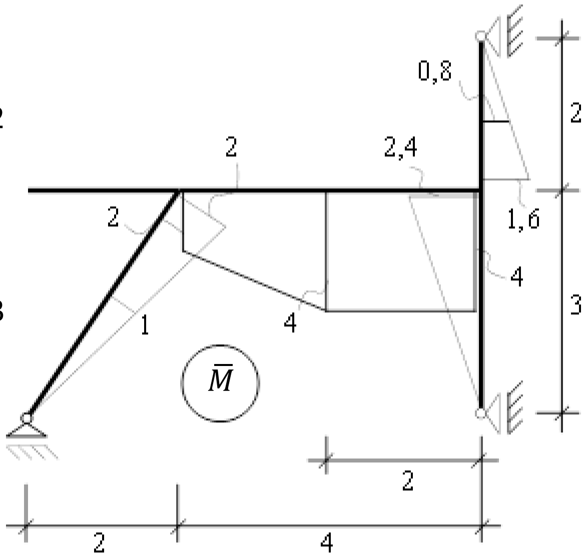


And the diagram is:



- **Step 3:** Calculation of the displacement.

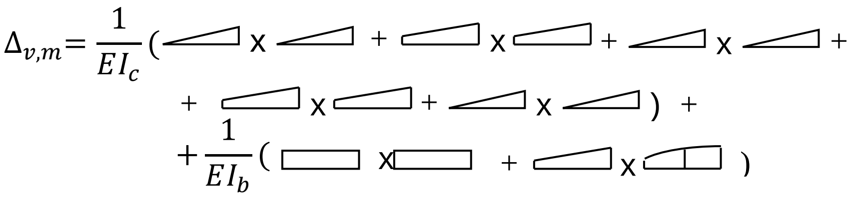




One should be careful because the moment of inertia of the beams and columns are different. We will write first all columns parts and after that beams parts.

$$\Delta_{v,m}= \sum \int \frac{\boldsymbol{\bar{M}M_f}}{\boldsymbol{EI}}\boldsymbol{ds}$$

The different parts of the calculation can be illustrated by following way:



And the calculation is:

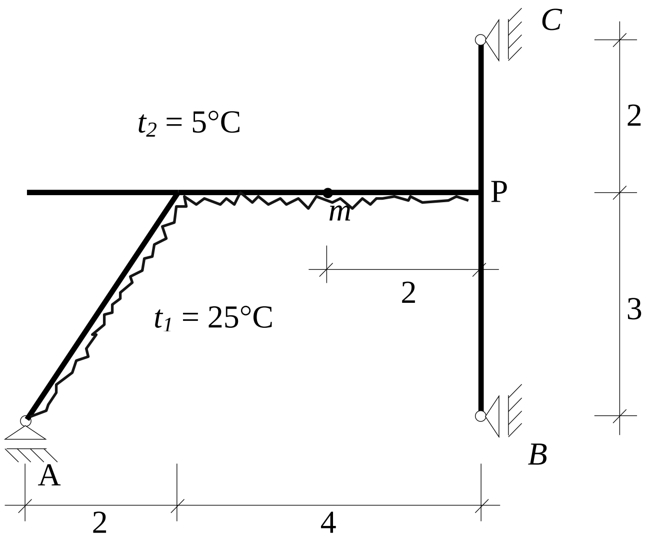
$$EI_c\Delta_{v,m} = \frac{1}{7812}\left[\frac{1}{3}1.40.1{,}803+\frac{1}{6}\left(1.55+(1+2)(55+95)+2.95\right).1{,}803+\frac{1}{3}2{,}4.45.3\right.$$

$$\left.+\frac{1}{6}\left(1{,}6.50+(1{,}6+0{,}8)(50+35)+0{,}8.35\right).1+\frac{1}{3}0{,}8.35.1\right]+$$

$$+\frac{1}{15624}\left[\frac{1}{6}\left(2.75+2.90.(2+4)+4.95\right).2+4.95.2\right]$$

$$\Delta_{v,m} = 0{,}1345m = 13{,}45cm$$

Calculation of vertical displacement of point *m* done from the temperature load:



If the same system is subjected to a temperature load as it is shown at the figure then there will appears displacements. It is very important to know that:

At a statically determinate systems subjected to a temperature load will appears only displacements without any internal forces (without any moments, shear and axial forces).

That is why we cannot draw a moment diagram at a determinate structure from a temperature load but we can calculate the displacement at some point at the structure.

To calculate this displacement we need to analyze the structure from the virtual unit force (or moment) and to know the initial data for the temperature and the material of the structure – the coefficient of the thermal expansion.

In this case, the columns and the beams of the structures are rectangular cross-sections so the natural axis is at the haft of the high. The constant temperature and temperature different ire:

$$t^c = \frac{t_1 + t_2}{2} = 15℃$$

and

$$\Delta t = t_1 - t_2 = 20℃$$

The displacement at the point calculates using the next expression:

$$\Delta_t = \sum \int \frac{\boldsymbol{\overline{M}.\alpha_t.\Delta t}}{\boldsymbol{h}} \boldsymbol{ds} + \sum \int \boldsymbol{\overline{N}.\alpha_t.t^c\, ds}$$

In this case in not needed to use the tables of the numerical calculation of the integrals because the temperature difference, the constant temperature, the high of the elements and the coefficient of the thermal expansion are constants and we can write:

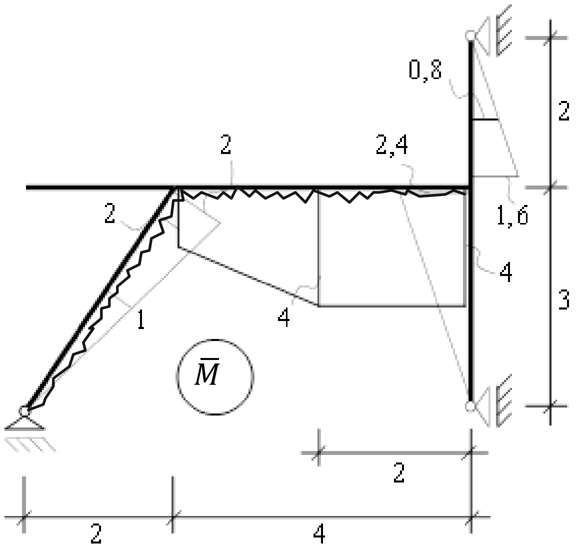
$$\Delta_t = \sum \frac{\boldsymbol{\alpha_t.\Delta t}}{\boldsymbol{h}} \int \boldsymbol{\overline{M}\, ds} + \sum \boldsymbol{\alpha_t.t^c} \int \boldsymbol{\overline{N}\, ds}$$

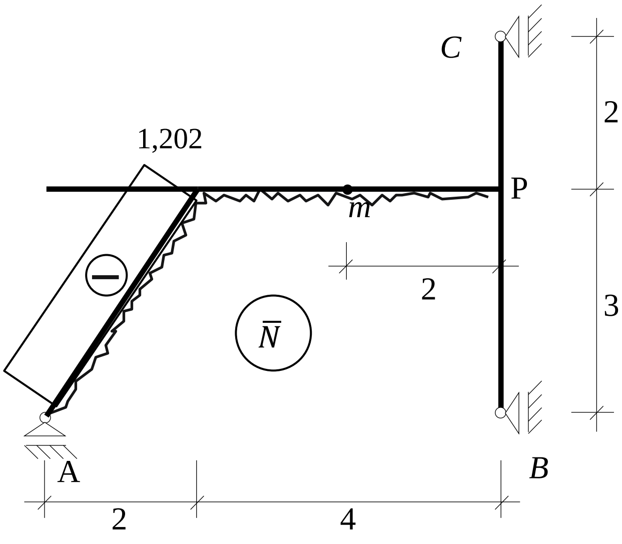
It follows that the integrals are equal to the area of the diagrams.
It is important to know following facts:

1. This formula uses only for the elements which are subjected to a temperature load. Because for the other elements Δt and t^c are zero and as one can see at the upper expression the integrals become zero.
2. When the moment diagram from the virtual load is from the same side of the elements where the higher temperature is then the sing of the integral is positive (plus). In the other case is negative (minus). That is because we draw the moment diagram from the extended bars of the element and such a moment forms the same deflection as the higher temperature.
3. When the normal force from the virtual load is positive then the integral is also positive and if the normal force is negative the integral is also negative. This is because the positive normal force cause an elongation of the element the same as the positive constant temperature.

Now we can show the procedure of the calculation:

- **Step 1:** Analysis of the structure from the virtual load – composition of the $\overline{M}$ and $\overline{N}$ diagrams.





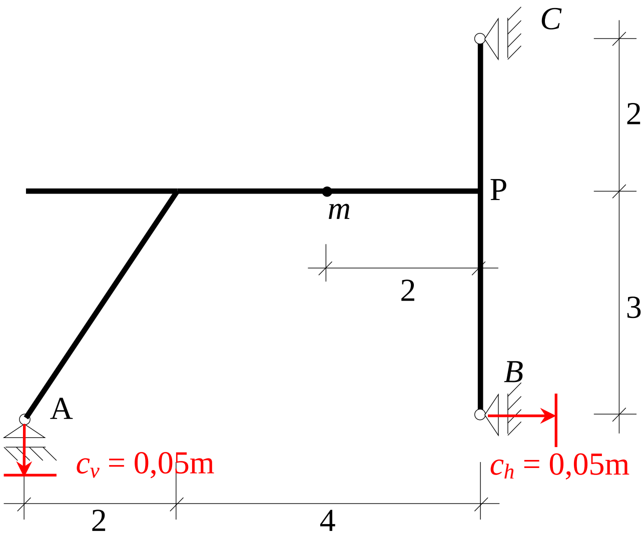
- **Step 2:** Calculation of the displacement.

$$\Delta_{m,t} = \sum \frac{\alpha_t.\Delta t}{h} \int \bar{M}\,ds + \sum \alpha_t.t^c \int \bar{N}\,ds =$$

$$= \frac{1.10^{-5}.20}{0,25}.\frac{2.3,605}{2} + \frac{1.10^{-5}.20}{0,315}\left(\frac{(2+4).2}{2} + 4.2\right) + 1.10^{-5}.15.(-1,202.3,605)$$

$$\Delta_{m,t} = 2,884.10^{-3} + 8,8889.10^{-3} - 0,65.10^{-3} = 11,123.10^{-3}m = 1,1123cm$$

Calculation of vertical displacement of point *m* done from the support settlement:



If the same system is subjected to a support-settlement load as it is shown at the figure then there will appears displacements. It is very important to know that:

At a statically determinate systems subjected to a support-settlement load will appears only displacements without any internal forces (without any moments, shear and axial forces).

That is why we cannot draw a moment diagram at a determinate structure from a support-settlement load but we can calculate the displacement at some point at the structure.

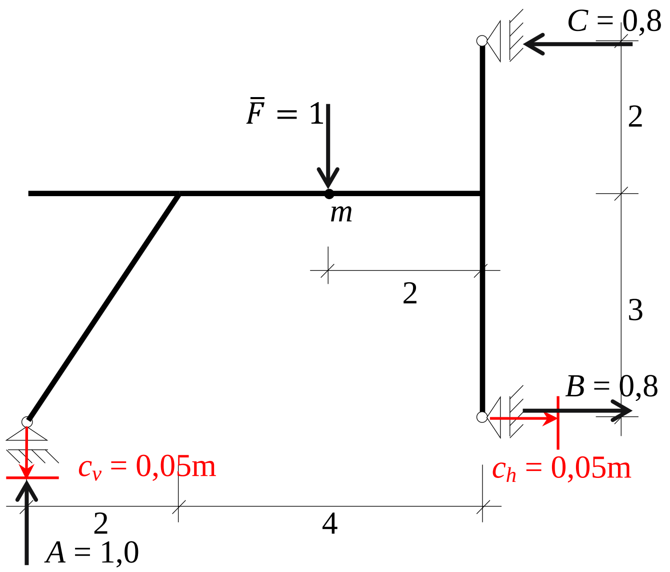
To calculate this displacement we need to analyze the structure from the virtual unit force (or moment) and to know the initial data for the support-settlement. The displacement at the point calculates analogically of the displacement at a system including springs supports. In this case again the internal work is equal to the multiplication of the support reaction by the displacement of the support using the next expression:

$$\boldsymbol{\Delta_c} = \sum_{\boldsymbol{1}}^{\boldsymbol{n}} \mathrm{R.c}$$

Where *R* is the support reaction and *c* is the displacement at the support. When the support reaction and the displacements are at one and the same direction then the multiplication (the work) is positive in the other case is negative.

Now we can show the procedure of the calculation:

- **Step 1:** Analysis of the structure from the virtual load – we need only the support reactions.



- **Step 2: Calculation of the displacement.**

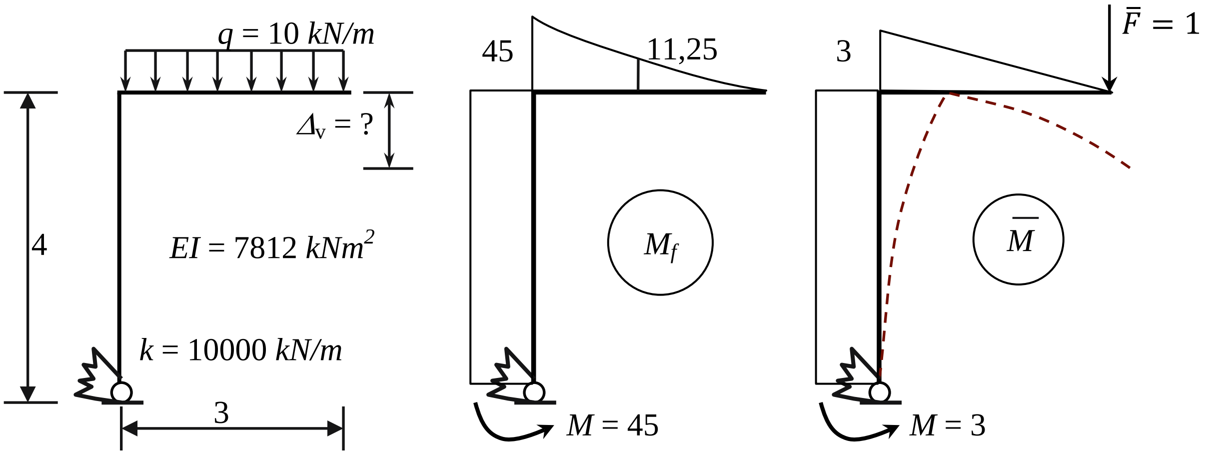
$$\boldsymbol{\Delta_{m,c}} = \sum_{\boldsymbol{1}}^{\boldsymbol{n}} \mathrm{R.c} = -1{,}0.0{,}05 + 0{,}8.0{,}05 = 0{,}09m = 9cm$$

The same result we can rich using geometrical solution. This makes composing the displaced shape of the structure. Such a solution we will illustrate in the next chapter.

Calculation of vertical displacement of point *m* done from the external load for structures including springs:

Important to know: When one determinate system includes a spring support it is not a mechanism. It is a normal determinate structure but whit a displacement at a support. This support displacement is limited because the spring has stiffness. The support reaction at a spring can be obtained in the same way as a ideally rigid support. The difference is only at the displacements. So, the support reactions and the diagrams are as at the structure whit ideally rigid supports.

On the next figure are shown the moment diagrams from the external an virtual load of the structure including spring support:



The vertical displacement one can obtain using the following expression:

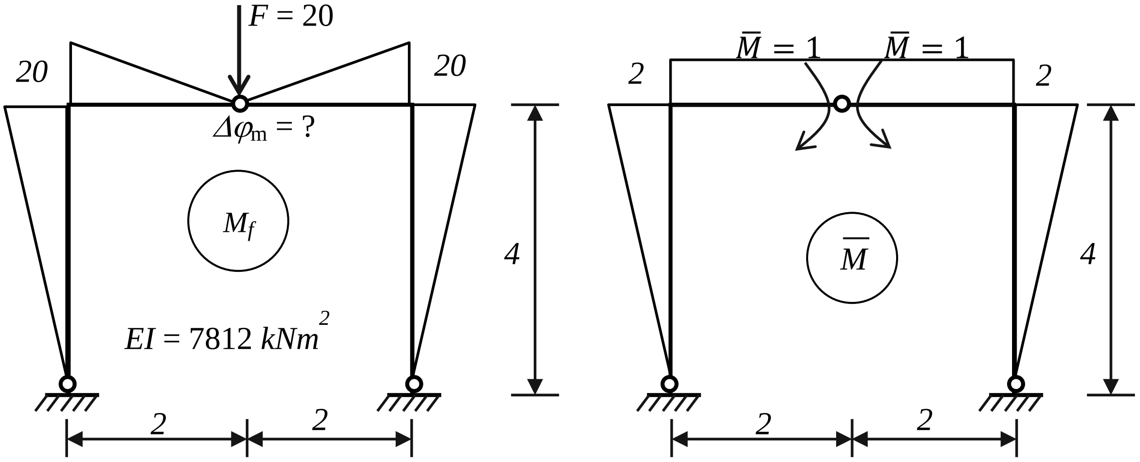
$$\Delta_v = \sum \int \frac{\bar{M}M}{EI} ds + \sum_{1}^{n} \frac{\bar{S}S}{k}$$

$$\Delta_v = \frac{1}{7812}\left(\frac{1}{6}3.(45 + 4.11,25 + 0).3 + 3.45.4\right) + \frac{45.3}{10000}$$

$$\Delta_v = 0.01728 + 0,0135 = 0.03078m = 3,078cm$$

Calculation of relative rotation of point *m* done from the external load:

Important to know: When we need to find a relative displacement (or rotation) there is only one difference from the situation whit a single displacement (or rotation). This difference is only of the virtual load. As it was mentioned before, in this case the virtual load is a couple (forces or moments). On the next example is shown the solution for the relative rotation at a middle hinge of the three-hinged frame from the external load:



The relative rotation displacement can be obtained using the following expression:

$$\Delta_v = \sum \int \frac{\bar{M}M}{EI} ds$$

$$\Delta_v = \frac{1}{7812}\left(\frac{1}{3}2.20.4 + \frac{1}{2}2.20.2\right).2$$

$$\Delta_v = 0.023895rad$$

Chapter 9

Method of forces (Force method) for analysis of statically indeterminate structures.

On the beginning of this course we saw that the number of degrees of freedom way be calculated using the next formula:

$$w = 3d - 2k - a;$$

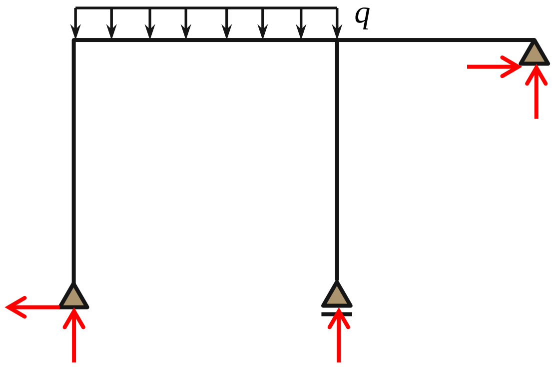
where:
w is degree of freedom (mobility);
d is number of bodies (elements);
k is number of one-degree-of-freedom kinematic pin joints;
a is number of support links.

As a result w may be positive, negative or zero. Therefore, we distinguish three different cases for w:

$w > 0$ - the system is mechanism. In the case of mechanism, we don't have a structure carrying any load;
$w = 0$ - determinate structure. We have a structure and it is possible to analyze it with only equilibrium conditions.
$w < 0$ - indeterminate structure. We have a structure and it is possible to analyze it with equilibrium conditions and additional equations.

In this first stage we will analyze indeterminate structures namely structures with $w < 0$.

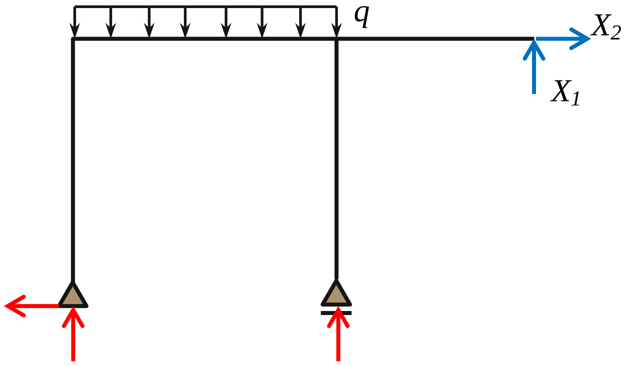
When we have, indeterminate structures we need additional equations for find the unknown reactions. On the next figure, we show two times indeterminate structure.



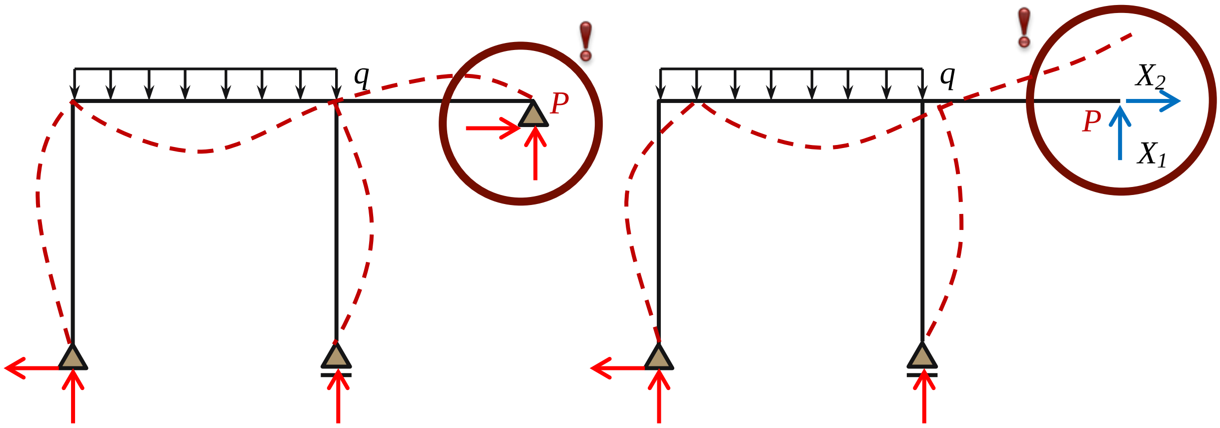
$$w = 3.1 - 0 - 5 = -2$$

If we have two times indeterminate structure, we need 2 additional equations to find reactions because for such a structure we may write only 3 equilibrium equations but we have 5 support reactions. In the force method (or flexibility method), the additional equations are the compatibility equations. Now we will explain the main idea of this method.

For example let take the same system but as a determinate structure. If we know the values of the two unknowns, the two systems will be equivalents for all internal forces.



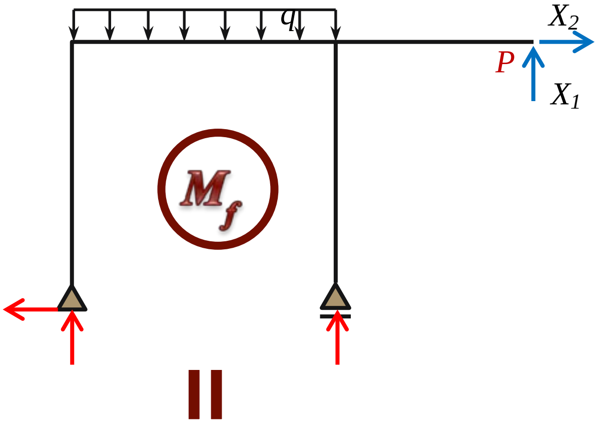
So, if we can compose the moment diagram for this structure it will be the same as for the indeterminate.
But let try to imagine what will be the deformed shape in the two cases.

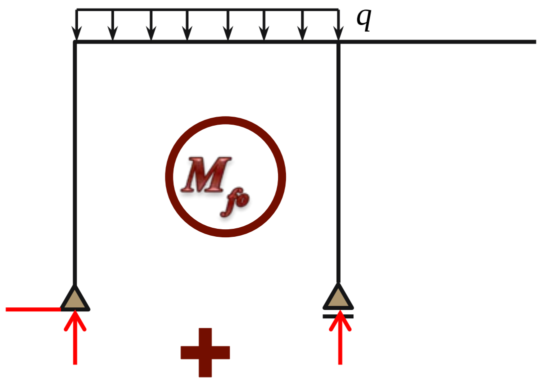


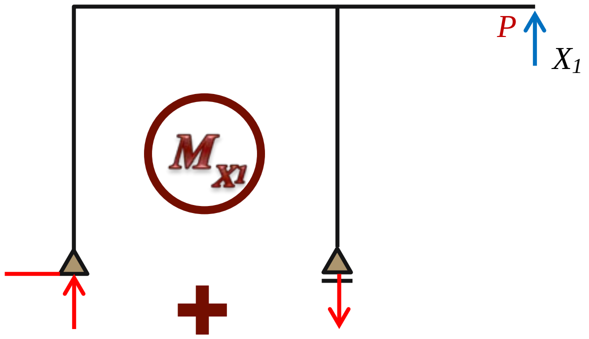
The deformed shapes are different because in the indeterminate structure, there is a fixed support at point *P* but in the determinate structure, it is not. The result is that in the determinate structure at point *P* there are displacements but in the indeterminate, they are zero. The equivalence of the internal forces of the two systems is not enough. Therefore, if we want equivalence of the systems we need to find the displacement at point *P* of the determinate system and to put them to be zero.

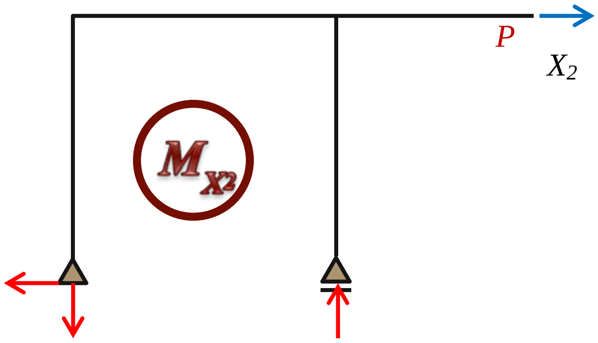
That will be the additional equations – **the equations of the displacement consistency or compatibility equations**.

The question is: How to find this displacements as we don't know the extra forces X_1 and X_2?
To answer of this question we will use the principle of superposition:





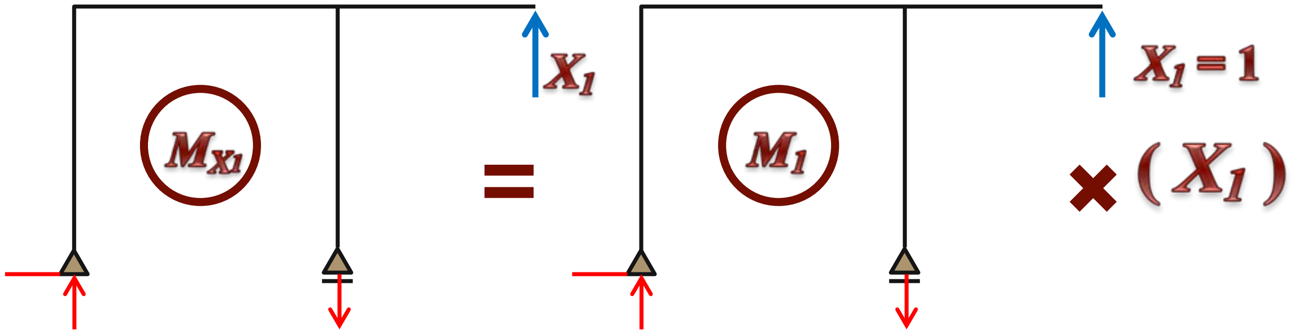




Or

$$M_f = M_{f_0} + M_{X_1} + M_{X_2}$$

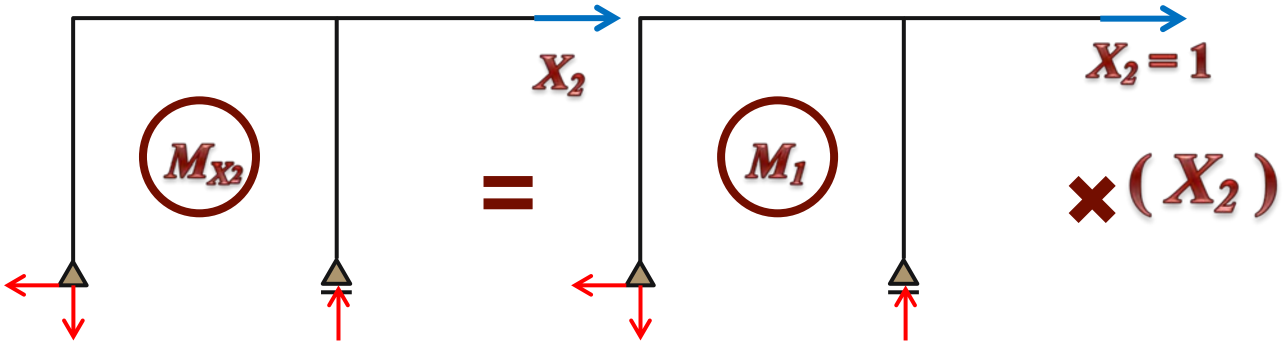
According to the principle of superposition, the determinate structure subjected to an external load and extra forces loads can be separated by three situations: the first is the system subjected to the external load; the second is the system subjected by X_1 and the last one is the system subjected by X_2. On the other side as we know already if the force with value X1 subjects the system one may compose the internal moment diagram M_{X1}. However, one may compose the diagram from the unit force and to multiply it by the value X_1 and the result will be the same.



Or:

$$M_{X_1} = M_1 \times X_1$$

Same situation we have from the X_2 force.



Or:

$$M_{X_2} = M_2 \times X_2$$

We may use the last two results at the previous one to write the next:

$$M_f = M_{f_0} + M_1 \times X_1 + M_2 \times X_2$$

Where:

M_f – is internal moment diagram obtained by the external load at the statically **indeterminate system**;

M_{f_0}– is internal moment diagram obtained by the external load at the statically **determinate system**;

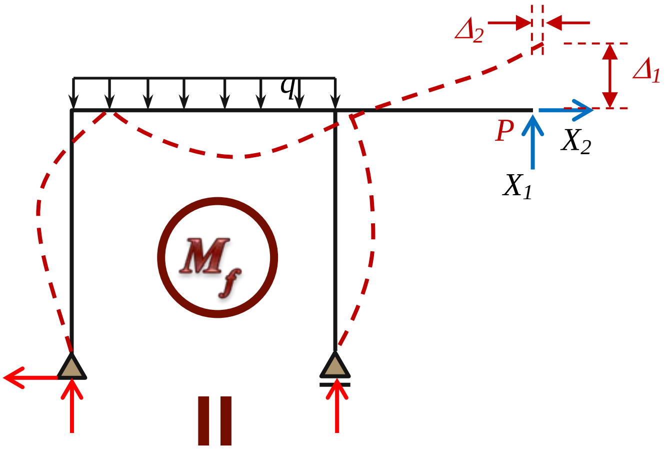
M_1– is internal moment diagram obtained by the unit value of the X_1 force at the statically **determinate system**;

M_2– is internal moment diagram obtained by the unit value of the X_2 force at the statically **determinate system**;

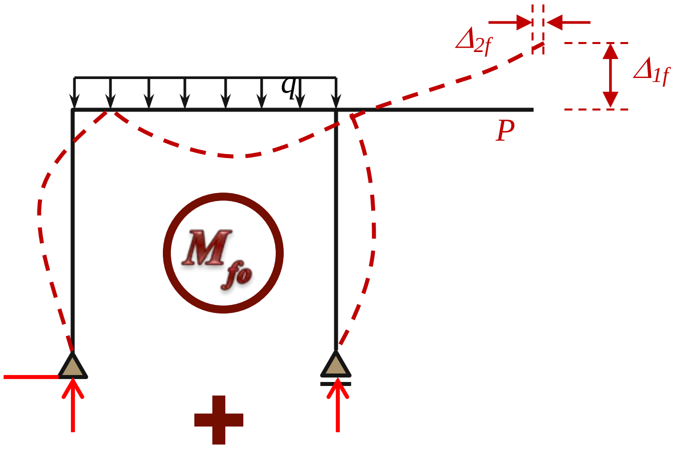
X_1 and X_2 are the unknown extra forces.

Important to know: This idea is valid and may be used for everything – displacements, deformations, support reactions and shear and axial forces. All internal forces, displacement and support reactions at the indeterminate system may be determinate using determinate system. We call this determinate system – **primary system**!

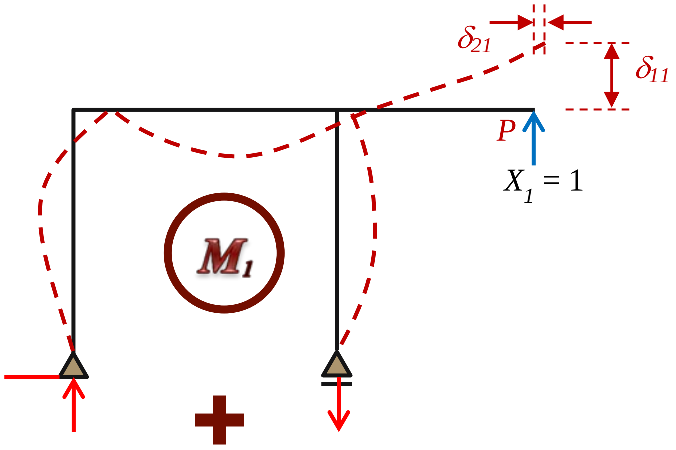
This is important to know because we will use it for the displacement at the point P (in presented case) to find the displacements at the point and to put them zero value. As the primary system is determinate, we can obtain the displacements at a point P from every load – external, $X_1 = 1$ and $X_2 = 1$.



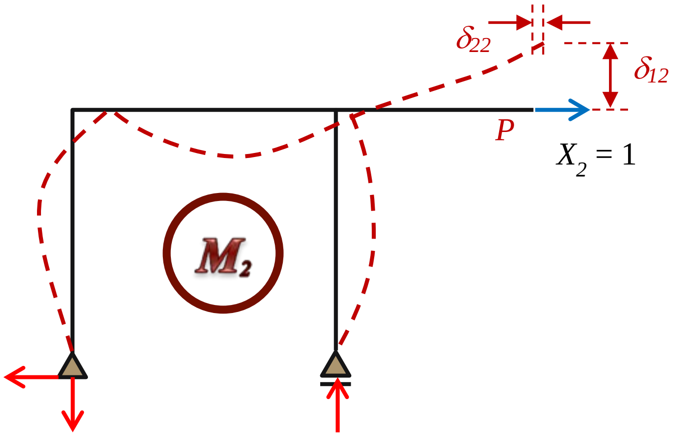
$\boldsymbol{\Delta_1}$ – is the displacements at X_1 application point (point P) at X_1 direction from all loads – external load and unknown values of X_1 and X_2 in the primary system.
$\boldsymbol{\Delta_2}$ – is the displacements at X_2 application point (point P) at X_2 direction from all loads – external load and unknown values of X_1 and X_2 in the primary system.
And they are equals to:



$\boldsymbol{\Delta_{1f}}$ – is the displacements at X_1 application point (point P) at X_1 direction from external load only in the primary system.
$\boldsymbol{\Delta_{2f}}$ – is the displacements at X_2 application point (point P) at X_2 direction from external load only in the primary system.



$\boldsymbol{\delta_{11}}$ – is the displacements at X_1 application point (point P) at X_1 direction from $X_1 = 1$ only in the primary system.
$\boldsymbol{\delta_{21}}$ – is the displacements at X_2 application point (point P) at X_2 direction from $X_1 = 1$ only in the primary system.



$\boldsymbol{\delta_{12}}$ – is the displacements at X_1 application point (point *P*) at X_1 direction from $\boldsymbol{X_2 = 1}$ only in the primary system.

$\boldsymbol{\delta_{21}}$ – is the displacements at X_2 application point (point *P*) at X_2 direction from $\boldsymbol{X_2 = 1}$ only in the primary system.

Or:

$$\Delta_1 = \Delta_{1f} + \delta_{11}.X_1 + \delta_{12}.X_2$$
$$\Delta_2 = \Delta_{2f} + \delta_{21}.X_1 + \delta_{22}.X_2$$

But according the displacement consistency we must have:

$$\Delta_1 = 0$$
$$\Delta_2 = 0$$

Therefore we have:

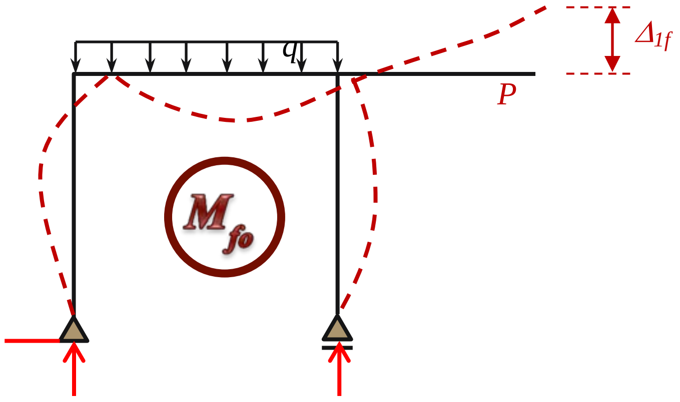
$$\left|\begin{array}{l}\delta_{11}.X_1 + \delta_{12}.X_2 + \Delta_{1f} = 0\\ \delta_{21}.X_1 + \delta_{22}.X_2 + \Delta_{2f} = 0\end{array}\right.$$

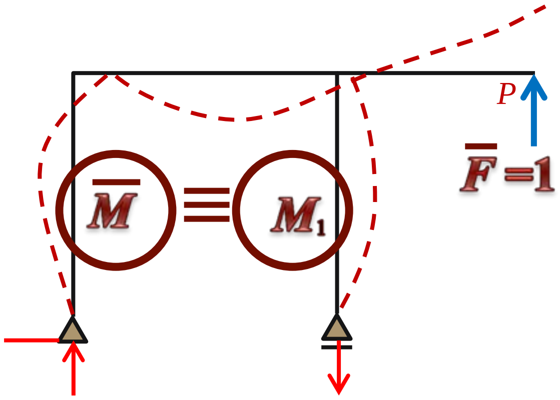
Finally, we have two additional equations and we may obtain the values of the two unknowns – the extra forces $\boldsymbol{X_2}$; $\boldsymbol{X_2}$**.**

Next step to solve is haw to obtain all these displacements. This is already solved problem at the previous chapter. To find the displacement at some point we need to analyze the system from a virtual unit load.

1. Displacement from the external load:

To find the Δ_{1f} displacement we need a virtual vertical unit force:



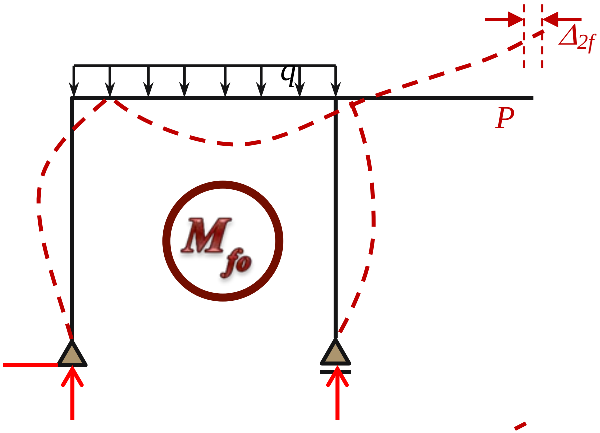


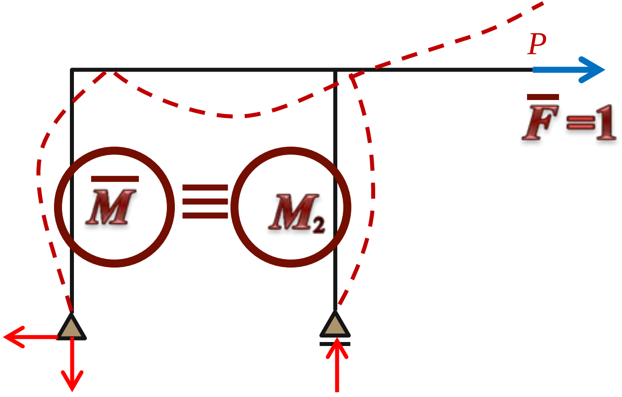
The result from this virtual vertical force is the same as this from the unit value of the X_1 so we may use the diagram from $X_1 = 1$ that we already have instead of the $\overline{M}$ diagram.

And the displacement is:

$$\Delta_{1f} = \sum \int \frac{\bar{M} M_{fo}}{EI} ds = \sum \int \frac{M_1 M_{fo}}{EI} ds$$

To find the Δ_{2f} displacement we need a virtual horizontal unit force:



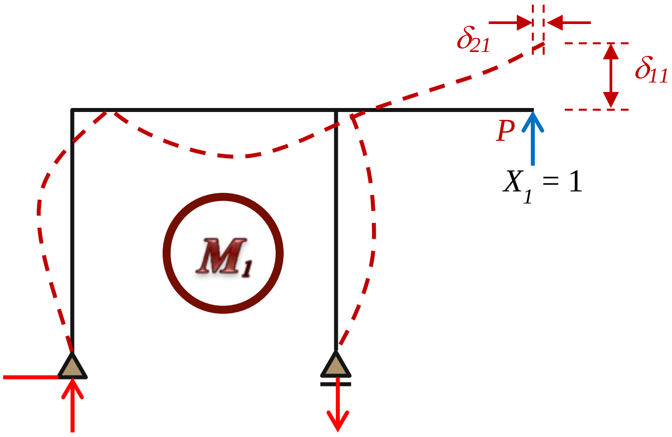


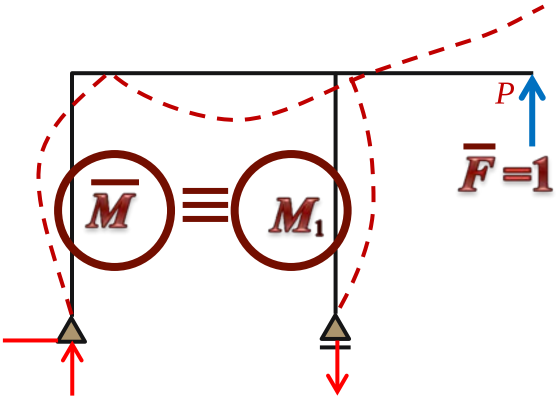
The result from this horizontal force is the same as this from the unit value of the X_2 so we may use the diagram from $X_2 = 1$ that we already have instead of the $\bar{M}$ diagram.
And the displacement is:

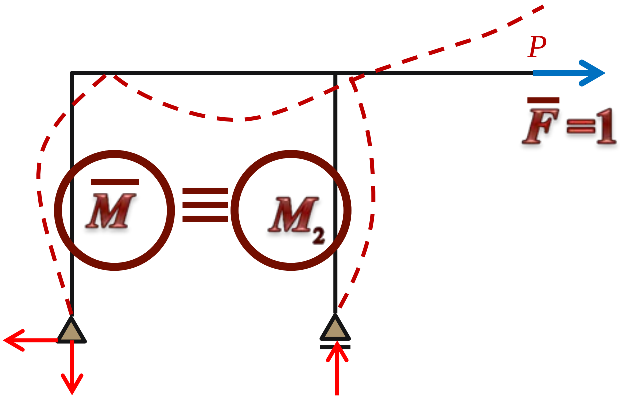
$$\Delta_{2f} = \sum \int \frac{\bar{M} M_{fo}}{EI} ds = \sum \int \frac{M_2 M_{fo}}{EI} ds$$

2. Displacement from the unit values of the extra-forces $X_1 = 1$ and $X_2 = 1$:

To find the δ_{11} displacement we need a virtual vertical unit force and for δ_{21} displacement we need a virtual horizontal unit force:





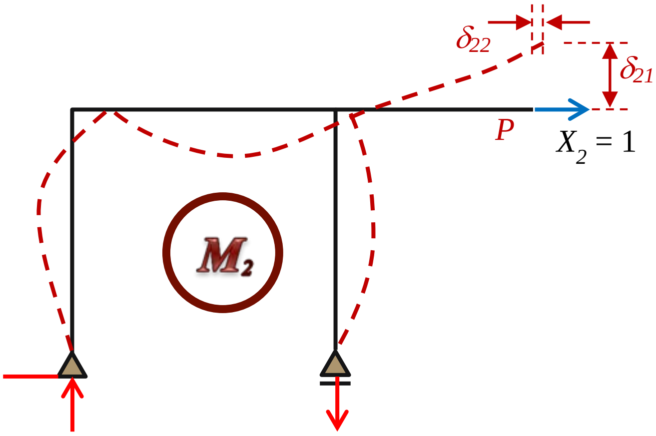


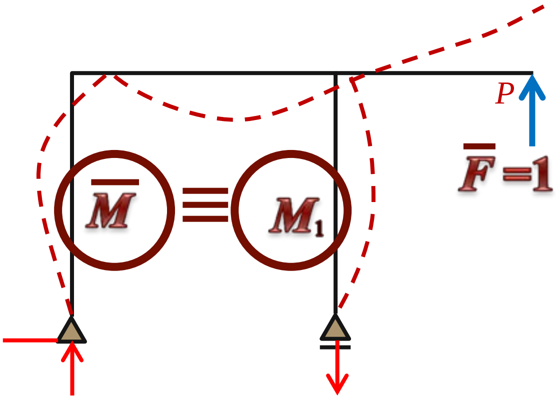
The result from these virtual forces is the same as this from the unit value of the X_1 and X_2 so we may use M_1 and M_2 diagrams instead of $\bar{M}$ diagram.
And the displacements are:

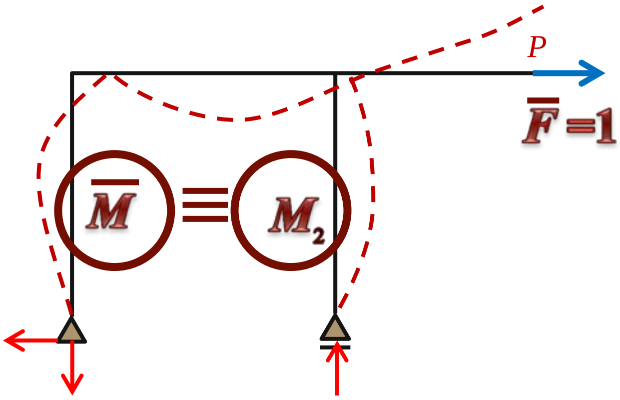
$$\delta_{11}=\sum\int\frac{\bar{M}M_1}{EI}ds=\sum\int\frac{M_1M_1}{EI}ds=\sum\int\frac{M_1^2}{EI}ds$$

$$\delta_{21}=\sum\int\frac{\bar{M}M_1}{EI}ds=\sum\int\frac{M_2M_1}{EI}ds$$

To find the δ_{21} displacement we need a virtual vertical unit force and for δ_{22} displacement we need a virtual horizontal unit force:







The result from these virtual forces is the same as this from the unit value of the X_1 and X_2 so we may use M_1 and M_2 diagrams instead of $\bar{M}$ diagram.
And the displacements are:

$$\delta_{21}=\sum\int\frac{\bar{M}M_2}{EI}ds=\sum\int\frac{M_1M_2}{EI}ds$$

$$\delta_{22} = \sum \int \frac{\bar{\boldsymbol{M}} M_2}{\boldsymbol{EI}} \boldsymbol{ds} = \sum \int \frac{M_2 M_2}{\boldsymbol{EI}} \boldsymbol{ds} = \sum \int \frac{M_2^2}{\boldsymbol{EI}} \boldsymbol{ds}$$

Important to know:

1. As it is obvious from the expressions $\delta_{12} = \delta_{21}$ and it is not necessary no calculate two of them.
2. When we analyze indeterminate systems, it is not necessary to compose $\bar{M}$ diagrams because we already know that they are same as M_1 and M_2.
3. The presented idea is valid not only for two-times indeterminate structures but for *n*-times indeterminate also. The procedure is the same. The compatibility equations for *n*-times indeterminate structure are:

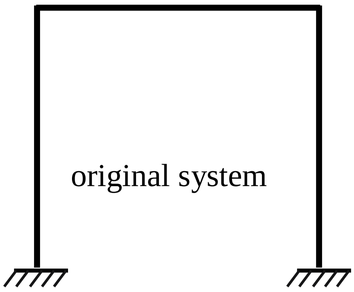
$$\left|\begin{array}{l} \delta_{11}.X_1 + \delta_{12}.X_2 + \delta_{13}.X_3 + \cdots + \delta_{1n}.X_n + \Delta_{1f} = 0 \\ \delta_{21}.X_1 + \delta_{22}.X_2 + \delta_{23}.X_3 + \cdots + \delta_{2n}.X_n + \Delta_{2f} = 0 \\ \delta_{31}.X_1 + \delta_{32}.X_2 + \delta_{33}.X_3 + \cdots + \delta_{3n}.X_n + \Delta_{3f} = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \delta_{n1}.X_1 + \delta_{n2}.X_2 + \delta_{n3}.X_3 + \cdots + \delta_{nn}.X_n + \Delta_{nf} = 0 \end{array}\right.$$

Choice of a primary system.

The choice of a primary system is very important because for only one indeterminate system there are many different variants of a primary system. One should choose easier, most appropriate and correct primary system.

How to compose a primary system: We compose a primary system removing links from the indeterminate structure. We remove so many links as is the number of the degree of freedom *w*. We may remove as external links and internal. When we already have some primary system we must do the kinematical analysis of it to prove that it is stable primary system.

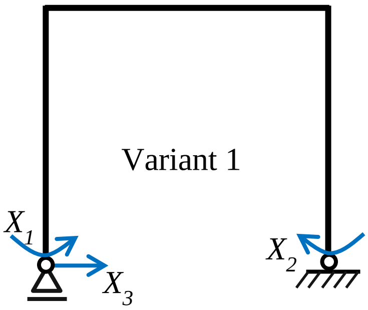
Example1: For presented indeterminate structure, compose different primary systems.



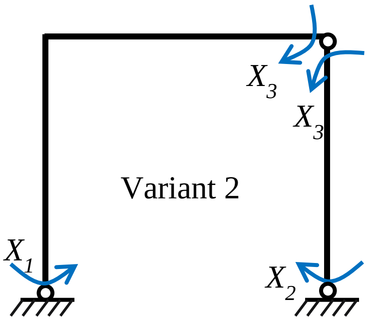
First, we should calculate how many times tis structure is indeterminate:

$$w = 3.d - 2.k - a = 3.1 - 2.0 - 6 = -3$$

We need to remove three links. First, we will remove only external links.

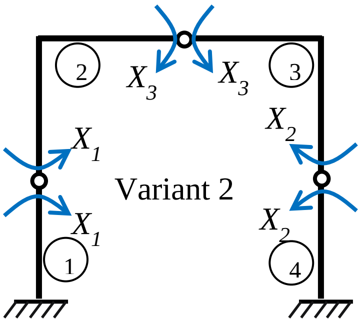


Variant 1 of the primary system is a simple beam. It is a stable structure.
At the next variant we will remove not only external links but internal.



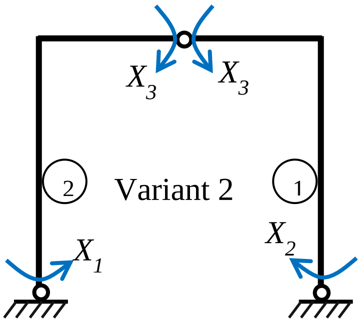
Variant 2 of the primary system is a simple beam. It is a stable structure.
Important to know: When internal links have been removed the unknowns are always couple!

At a Variant 3 we will remove only internal links.



$$KA: [1 + 4 + 2.3](w = 0)$$

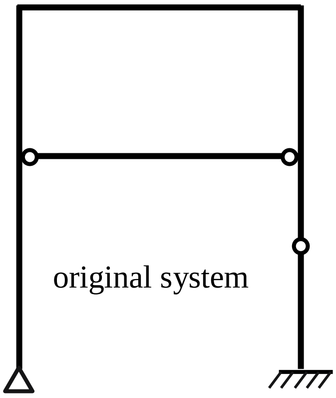
Variant 3 of the primary system is a structure type I and is a stable structure.



$$KA: [1.2](w = 0)$$

Variant 3 of the primary system is a three-hinged frame and is a stable structure.
It is possible to compose many other variant of a primary structures but it is not necessary.
Important to know: The primary system should be **type I** (if it is possible). When it is type I we are sure the system is stable.

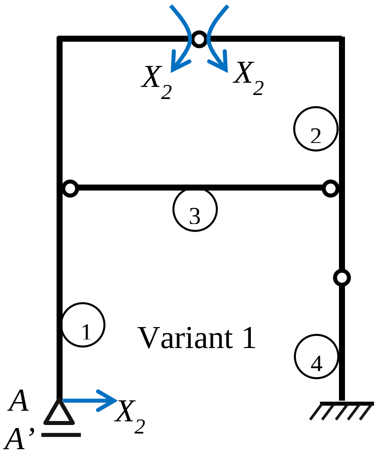
Example2: For presented indeterminate structure, compose different primary systems.



First, we should calculate how many times tis structure is indeterminate:

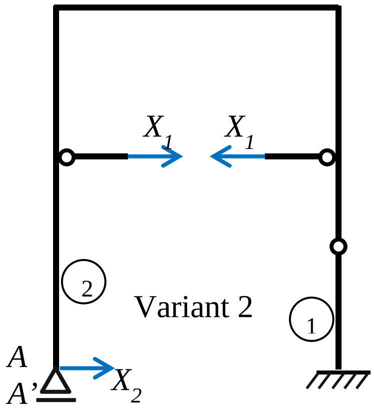
$$w = 3.d - 2.k - a = 3.3 - 2.3 - 5 = -2$$

We need to remove two links.



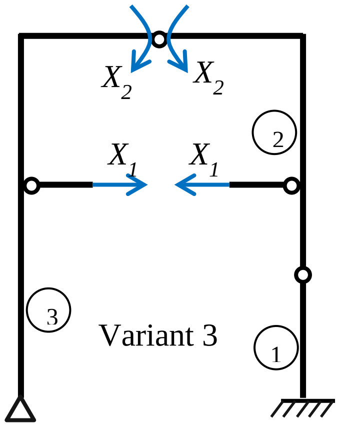
$$KA: [4 + 2.1. AA'(w = +1) + 3(w = -1)](w = 0)$$

Variant 1 of the primary system is a fixed beam and a tied three-hinged frame. It is a stable structure.



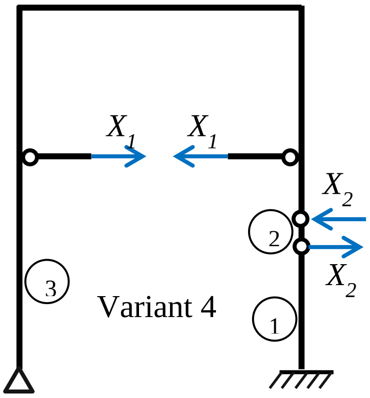
$$KA: [1 + 2. AA'](w = 0)$$

Variant 2 of the primary system is a fixed beam and a simple beam. It is a stable structure.



$$KA: [1 + 2.3](w = 0)$$

Variant 3 of the primary system is a fixed beam and a three-hinged frame. It is a stable structure.



$$KA: [1 + 2.3](w = 0)$$

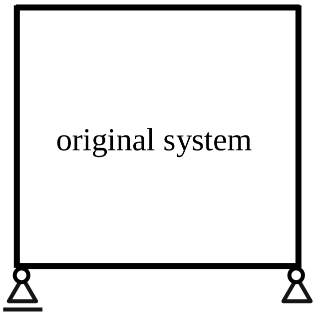
Variant 4 of the primary system is a fixed beam and a simple beam. It is a stable structure.

How to compose a primary system: The primary system should be easier because we analyze it many times from many lad cases. When the primary system is simple, we obtain moment diagrams easy and fast. Diagrams are simples and one can compose them without mistakes. When diagrams are easy the next calculations (multiplications of the diagrams) are also easy. For the example1 Variant 1 is simple and suitable and can be used properly. Variant 4 is also appropriate because is symmetrical.

Important to know: When the original structure is symmetrical is recommended the primary system to be symmetrical too.

For the example2 Variant 2 is simpler and appropriate because it is. Variant 4 is a simple beam too but it is difficult to think off such a primary system.

Example3:



Tis system is composed by 1 disk supported as a simple beam but the disk is closed. This is a closed-loop system. The degrees of freedom calculate as follows:

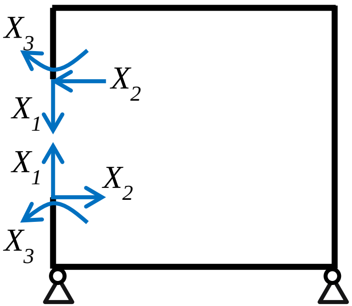
$$w = -(3m - k)$$

m is number of the closed loops (including the basic disk – ground)
k is number of one-degree-of-freedom kinematic pin joints.
For the present example is:

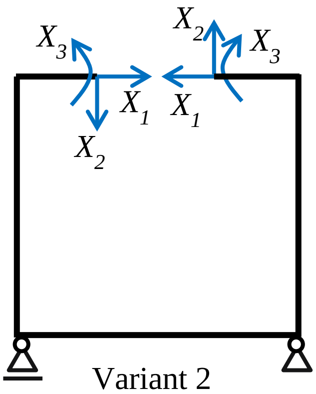
$$w = -(3.2 - 3) = -3$$

It is easy to see this result if cut the close-loop. This system is supported as a simple beam and one can calculates the support reactions. But for to calculate internal forces diagrams we need to cut at two places so we cannot compose the diagram without obtain first the extra-forces. When we cut it three internal forces appears and so the system is three times indeterminate. Such systems are **inner indeterminate**.

Actually cutting the loop, we compose the primary system.



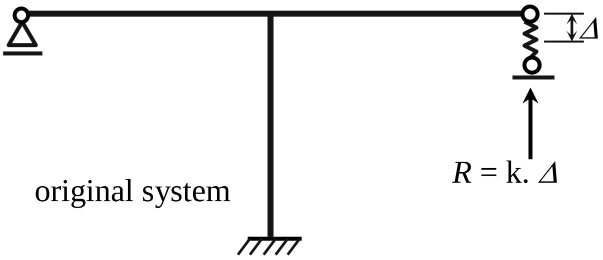
Variant 1



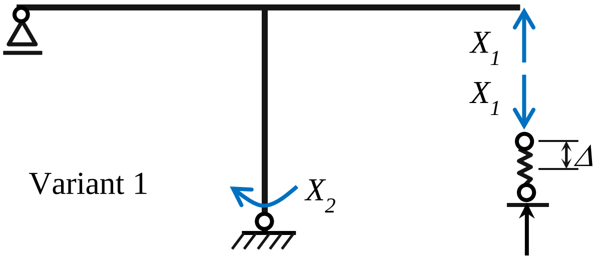
Variant 2

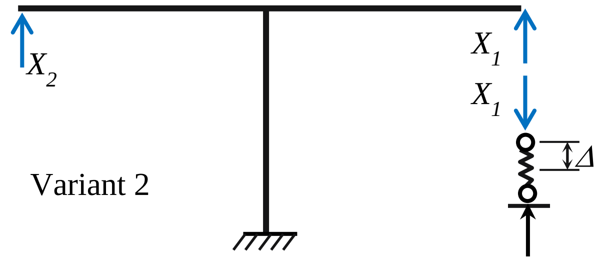
Since the original structure is symmetrical as we already mentioned the primary system should be symmetrical too. That is why Variant 2 is appropriate for this example.
Example 2 also includes close-loop and composing primary system we cut it without mention it.

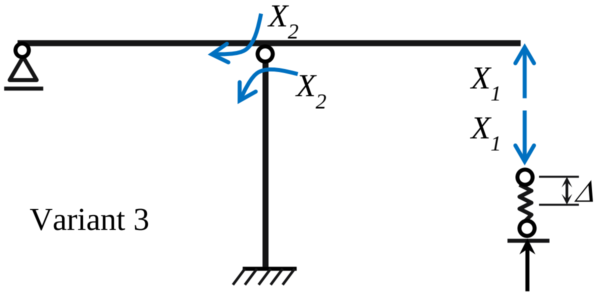
Example4:



$$w = 3.d - 2.k - a = 3.1 - 2.0 - 5 = -2$$

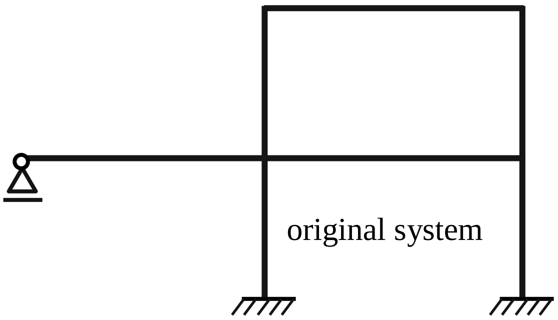






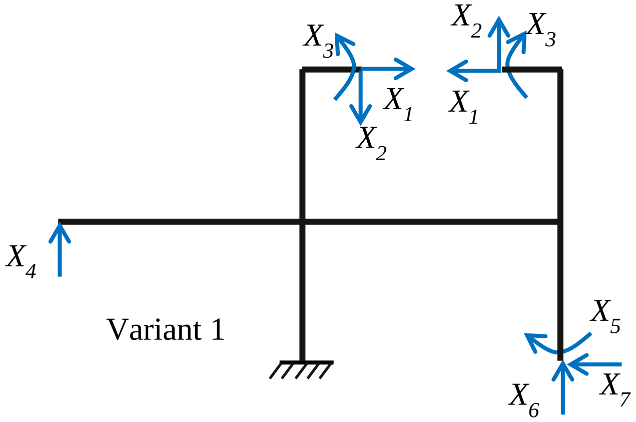
When the system include a spring support is better to cut the spring to compose the primary system.
For tis example the easier end the proper variant is Variant2.

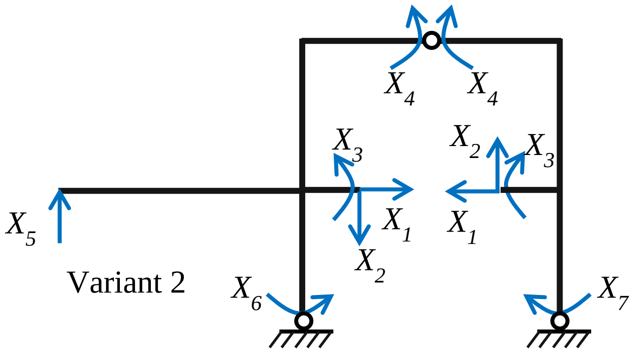
Example5:



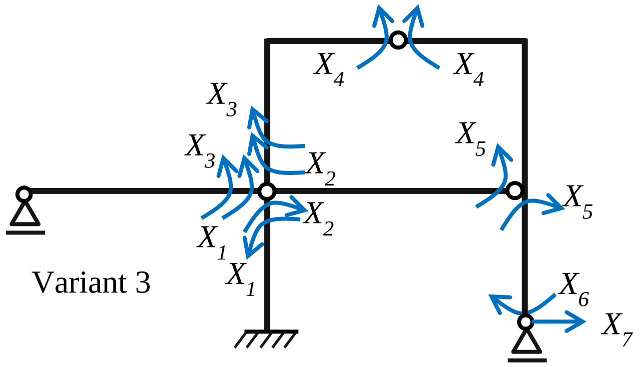
$$w = -(3.2 - 1) = -3$$

This system is a complicated one. It has 7 extra-forces and also has a closed-loop. It is not only inner indeterminate but external too.





Variant 2



Variant 3

At the Variant 1we removed 4 external links and cut off the closed-loop. As a result, the primary system is a fixed beam. At the Variant 2we do the same but in this case primary system is a three-hinged frame. Variant 3 is very complicate. In this case, we didn't cut the close loop and only include a lot of hinges. As a result, we have a structure type I so it is stable structure but is it the appropriate variant?

Important to know: When we put a hinge in the primary system we should be careful about the number of the one-degree-of-freedom kinematic pin joints *k*. This number tell as haw many internal moments we should put in the primary system.

At a Variant3 the left hinge is with *k* = 3 so there are three internal moments we removed and we should put them as unknown in the primary system. The other two hinges are with k = 1 that is why we put only one unknown internal moment.

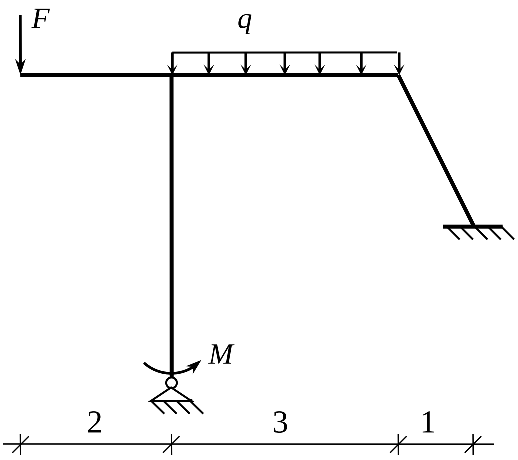
After all this analysis what is the better primary system? The preferable primary system is Variant2 because it is partly symmetrical but Variant1 is also proper if one prefer it.

Full example of analysis of statically indeterminate frame:

Analysis of statically indeterminate frame loaded by external load:

Data:

$F = 20\ kN$	***beams (0.25/0.315):***	***columns (0.25/0.25):***
$M = 15\ kNm$	$I_b = 0.000651\ m^4$	$I_{col} = 0.0003255\ m^4$
$q = 10\ kN/m'$	$A_b = 0.0788\ m^2$	$A_{col} = 0.0625\ m^2$
	$\boldsymbol{I_b = 2.I_{col.}}$	$E = 2.4.10^7\ kN/m^2$
	$t_1 = 0^o$	$\alpha_t = 0{,}00001$
	$t_2 = 25$	$c^h = c^v = 0{,}05\ m$
		$\varphi = 0{,}05\ rad$



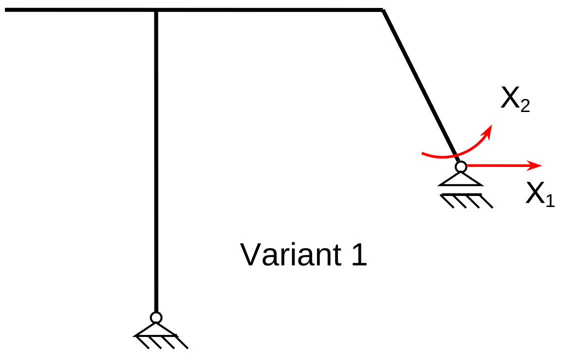
1. **Degree of indeterminacy:**

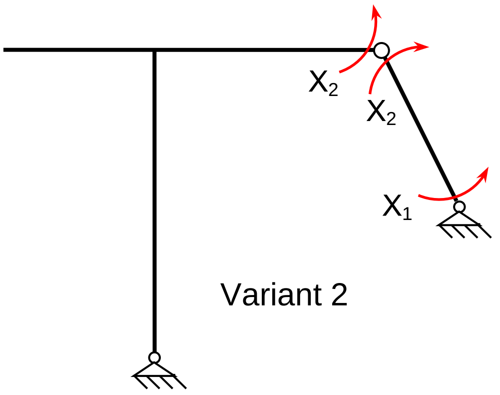
$n = 3d - 2k - a; \qquad d = 1;\ k = 0;\ a = 3+2 = 5$

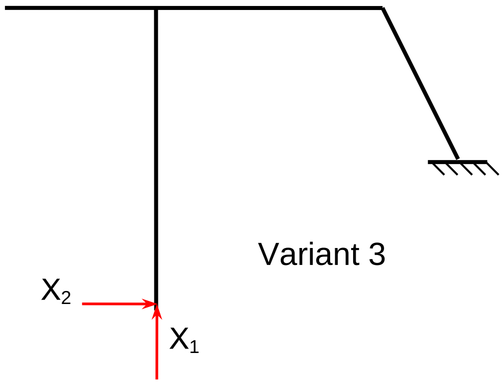
$n = 3.1 - 5 = -2 \Rightarrow$ The system is two times statically indeterminate

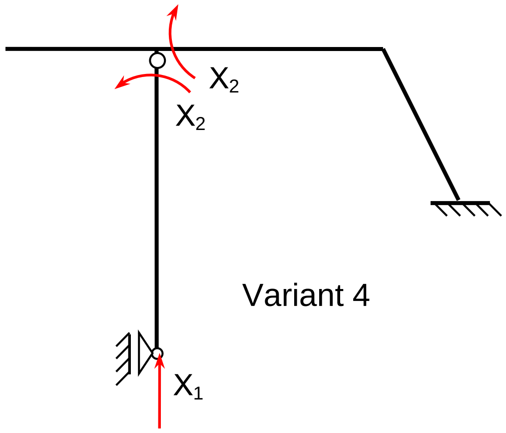
2. **Chose of primary system:**

The primary system is composed by removing 2 links. The primary system should be statically determinate, a stable.



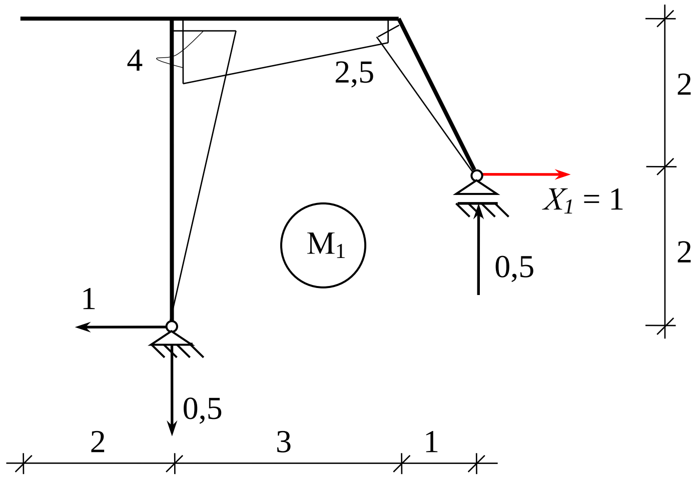




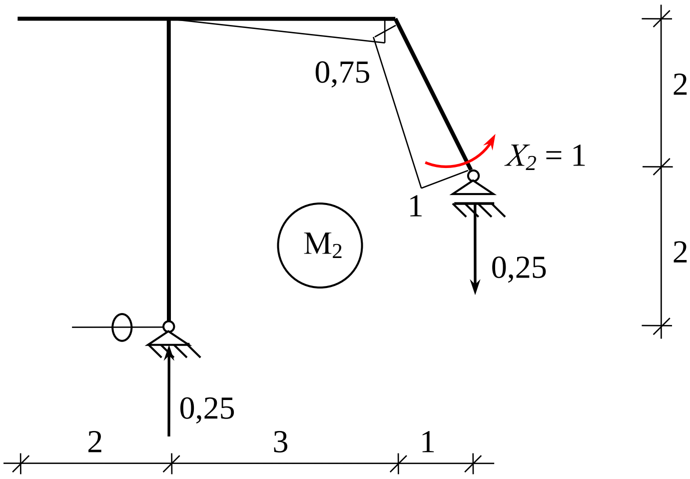


The primary system will be analyzed several times from different loads. That is why the primarily structure should be simpler for to obtain diagrams easy. In this case will be used Variant 1.

1. Unit diagrams:



The M_1 diagram is a diagram in determinate primary system from the unit value of the unknown force X_1.



The M_2 diagram is a diagram in the determinate primary system from the unit value of the unknown moment X_2.

2. Flexibility coefficients:

δ_{ij} – the displacement of the application point of the unknown X_i upon its direction, caused by unknown $X_j = 1$. Values of δ_{ij} calculate using Maxwell-Moor's integrals.

$EI_{col} = 7812; \quad EI_b = 15624$

$$\delta_{11} = \sum\int \frac{M_1^2}{EI} ds = \frac{1}{7812}\left(\frac{1}{3}4^2.4 + \frac{1}{3}2,5^2.2,2361\right) + \frac{1}{15624}\left(\frac{1}{6}\left[4^2 + \left(4+2,5^2\right)^2 + 2,5^2\right].3\right) = 53,913.10^{-4}$$

$$\delta_{12} = \sum\int \frac{M_1 M_2}{EI} ds = \frac{1}{15624}\left(\frac{1}{6}0,75.\left(2.2,5+4\right).3\right) + \frac{1}{7812}\left(\frac{1}{6}2,5.\left(2.0,75+1\right).2,2361\right) = 5,14167.10^{-4}$$

$$\delta_{21} = \sum\int \frac{M_2 M_1}{EI} ds = \delta_{12} = 5,14167.10^{-4}$$

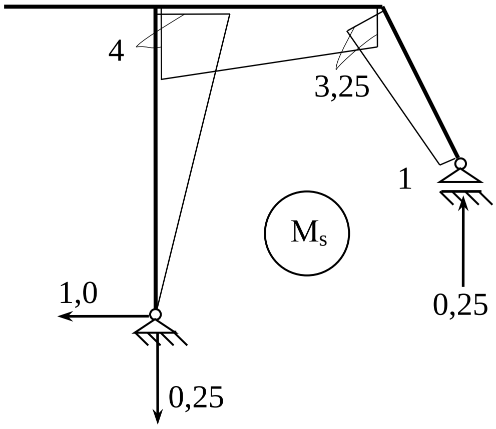
$$\delta_{22} = \sum\int \frac{M_2^2}{EI} ds = \frac{1}{15624}\left(\frac{1}{3}0,75^2.3\right) + \frac{1}{7812}\left(\frac{1}{6}\left[0,75^2 + \left(0,75+1\right)^2 + 1^2\right].2,2361\right) = 2,5666.10^{-4}$$

3. Verification of the flexibility coefficients caused by the unit value of the unknowns:

$$\sum \delta_{ij} \overset{?}{=} \sum \int \frac{M_s^2}{EI} ds$$

For this verification is needed to compose a summary moment diagram M$_s$. It is composed by adding the values of the unit diagrams at each specific points.

$$M_s = M_1 + M_2$$

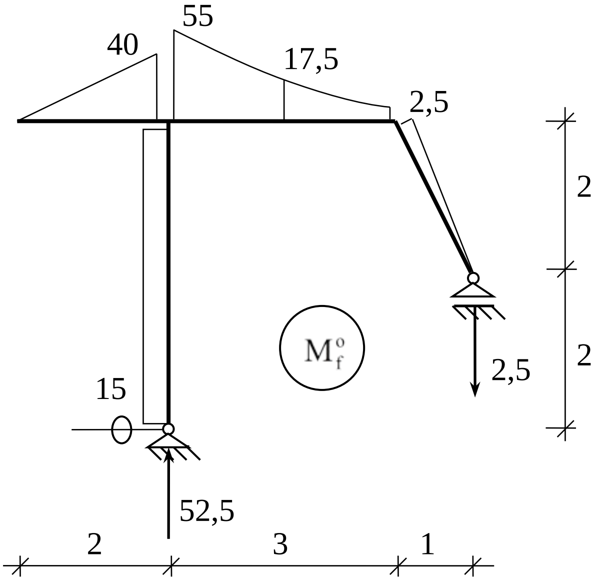


$$\sum\int \frac{M_s^2}{EI} ds = \frac{1}{15624}\left(\frac{1}{6}\left[4^2+(4+3,25)^2+3,25^2\right].3\right)+$$
$$+\frac{1}{7812}\left(\frac{1}{3}4^2.4+\frac{1}{6}\left[1^2+(1+3,25)^2+3,25^2\right].2,2361\right)=66,763.10^{-4}$$

$$\sum\delta_{ij} = 53,913.10^{-4}+2.5,142.10^{-4}+2,5666.10^{-4}=66,763.10^{-4} \qquad \Rightarrow \qquad \sum\delta_{ij}=\sum\int\frac{M_s^2}{EI}ds$$

During coefficients summering we should remember that $\delta_{12} = \delta_{21}$ and it should be included the two coefficients.

4. Diagram of the primary system from external load:



5. Flexibility coefficients from the external load:

$$\Delta_{1f} = \sum\int \frac{M_1 M_f^o}{EI} ds = \frac{1}{7812}\left(-\frac{1}{2}4.15.4 - \frac{1}{3}2{,}5.2{,}5.2{,}5.2{,}2361\right) +$$

$$+\frac{1}{15624}\left(-\frac{1}{6}[55.4 + 2.17{,}5.(4+2{,}5) + 2{,}5.2{,}5].3\right) = -304{,}782.10^{-4}$$

$$\Delta_{2f} = \sum\int \frac{M_2 M_f^o}{EI} ds = \frac{1}{15624}\left(-\frac{1}{6}0{,}75.(2{,}5 + 2.17{,}5).3\right) + \frac{1}{7812}\left(-\frac{1}{6}2{,}5(2.0{,}75+1).2{,}2361\right) = -11{,}983.10^{-4}$$

6. Verification of the coefficients from the external load:

$$\sum\Delta_{if} \overset{?}{=} \sum\int \frac{M_s M_f^o}{EI} ds$$

$$\sum\int \frac{M_s M_f^o}{EI} ds = \frac{1}{7812}\left(-\frac{1}{2}15.4.4 - \frac{1}{6}2{,}5(2.3{,}25+1).2{,}2361\right) +$$

$$+\frac{1}{15624}\left(-\frac{1}{6}[55.4 + 2.17{,}5.(4+3{,}25) + 3{,}25.2{,}5].3\right) = -316{,}765.10^{-4}$$

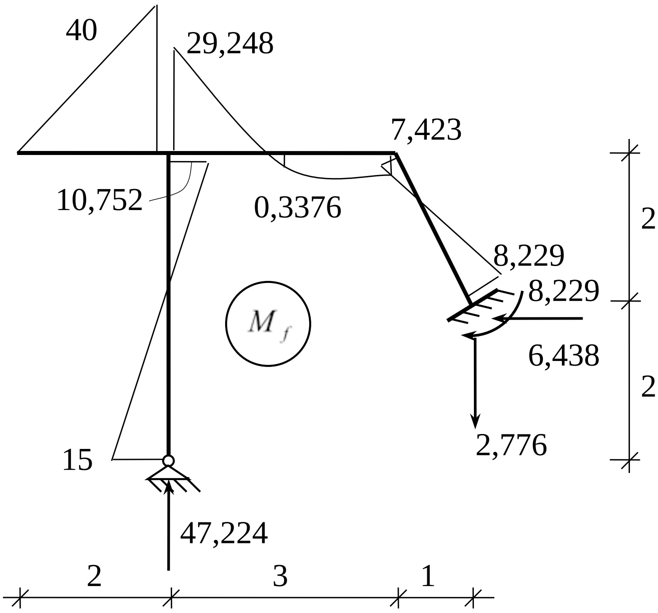
$$\sum\Delta_{if} = -304{,}782.10^{-4} - 11{,}983.10^{-4} = -316{,}765 \Rightarrow \qquad \sum\Delta_{if} = \sum\int \frac{M_s M_f^o}{EI} ds$$

7. Compatibility equations:

$$\left| \begin{array}{l} 53{,}913X_1 + 5{,}142X_2 - 304{,}782 = 0 \quad \rightarrow \qquad X_1 = 6{,}438 \\ \;\; 5{,}142X_1 + 2{,}5666X_2 - 11{,}983 = 0 \rightarrow \; X_2 = -8{,}229 \end{array} \right.$$

8. Final diagrams:

It is composed by using next connection at each specific point: $M_f = M_f^o + M_1.X_1 + M_2.X_2$



9. Compatibility verification:

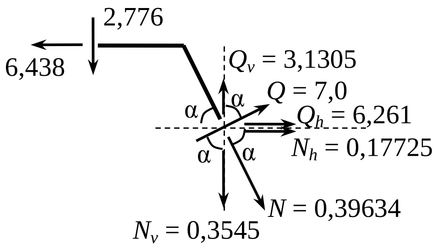
$$\sum\int\frac{M_s M_f}{EI}ds\overset{?}{=}0$$

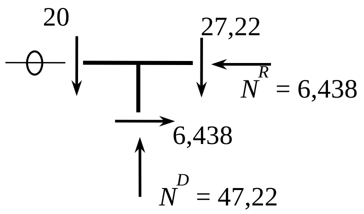
$$\sum\int\frac{M_s M_f}{EI}ds=\frac{1}{7812}\left(\frac{1}{6}4.(2.10,752-15).4+\frac{1}{6}[3,25.7,423+(3,25+1)(7,423-8,229)-8,229.1].2,2361\right)+$$

$$+\frac{1}{15624}\left(\frac{1}{6}[-29,248.4+2.0,3376(4+3,25)+3,25.7,423].3\right)=22,202.10^{-4}-28,153.10^{-4}+5,9485.10^{-4}\overset{?}{=}0$$

$$28,1505.10^{-4}+28,153.10^{-4}=0 \qquad 0,0087\%-error$$

If the consistency condition is satisfied, we can continue with composing the shear and axial force diagrams. The shear force diagram is composed by using connection between the moment and the shear. The normal force diagram is composed using joints equilibrium.





$$\alpha=\arctan\left(\frac{2}{1}\right)=63,4349^\circ$$

$$Q_h=Q\sin\alpha=6,261$$

$$Q_v=Q\cos\alpha=3,1305$$

$$\sum V=0:2,776+N_v-6,261=0\rightarrow N_v=0,3545$$

$$N=\frac{N_v}{\sin\alpha}=0,39634$$

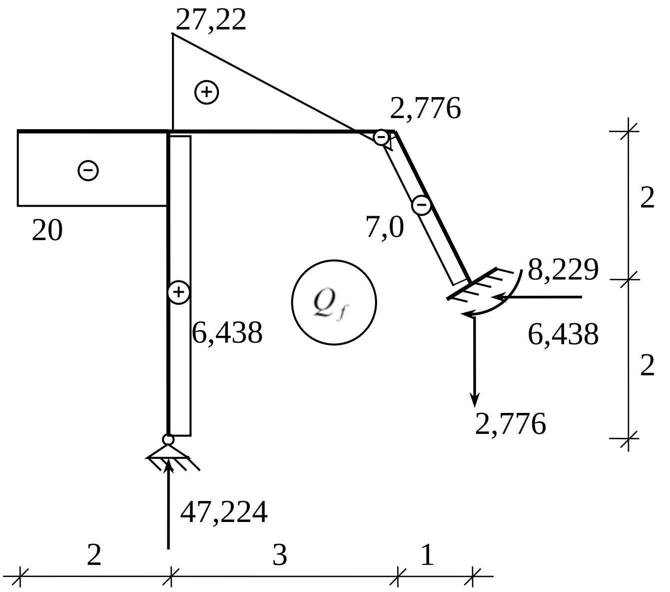
$$N_h=N\cos\alpha=0,17725$$

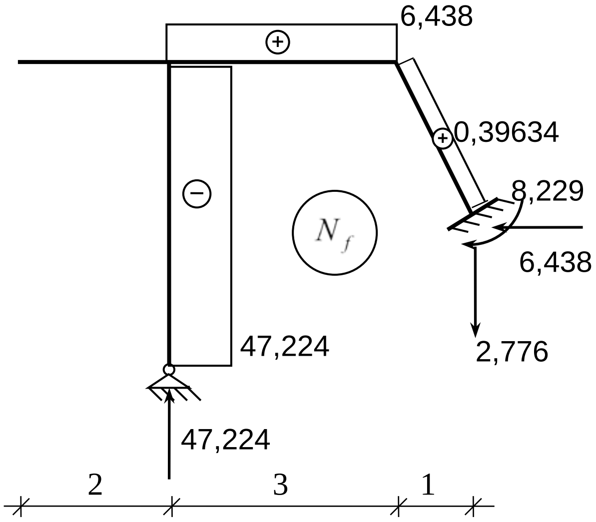
$$check:\sum H=0:\ 0,17725+6,261-6,438=0$$

$$6,43825-6,438=0\rightarrow 0,004\%\ error$$

$$\sum V=0:20+27,22-N^D=0\rightarrow N^D=47,22$$

$$\sum H=0:6,438+0-N^R=0\rightarrow N^R=6,438$$





Analysis of statically indeterminate frame loaded by temperature load:

The analysis of indeterminate frames from the temperature load is the same as this from the external load whit only one difference. The final diagram made using the following expression:

$$M_t = +M_1.X_{1t} + M_2.X_{2t}$$

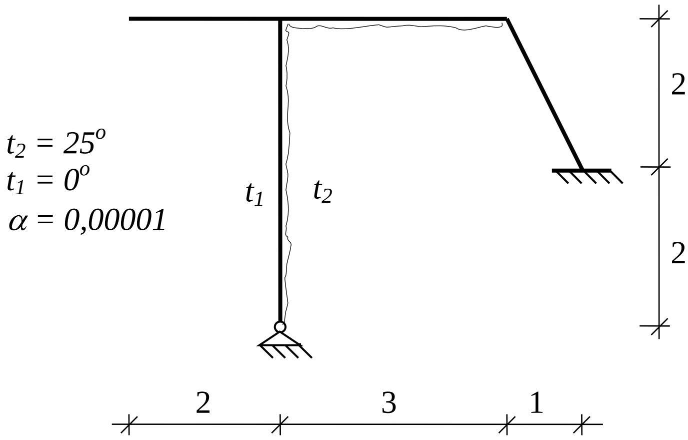
The reason of this result is that at a determinate structure (the primary system) there is no diagram from the temperature load. Therefore, to compose the final diagrams we need only to calculate new values of the unknowns. To calculate them is necessary to determine the displacement of the unknown's application points caused by the temperature load – Δ_{1t}, Δ_{2t}. Then the compatibility equations will be:

$$\begin{vmatrix} \delta_{11}.X_{1t} + \delta_{12}.X_{2t} + \Delta_{1t} = 0 \\ \delta_{21}.X_{1t} + \delta_{22}.X_{2t} + \Delta_{2t} = 0 \end{vmatrix}$$

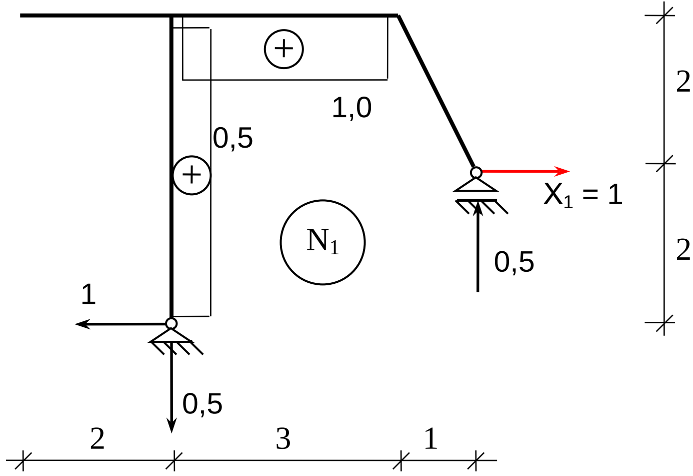
The displacements Δ_{1t} and Δ_{2t} are displacements at a determinate system so the way of there calculation is already explained and it is:

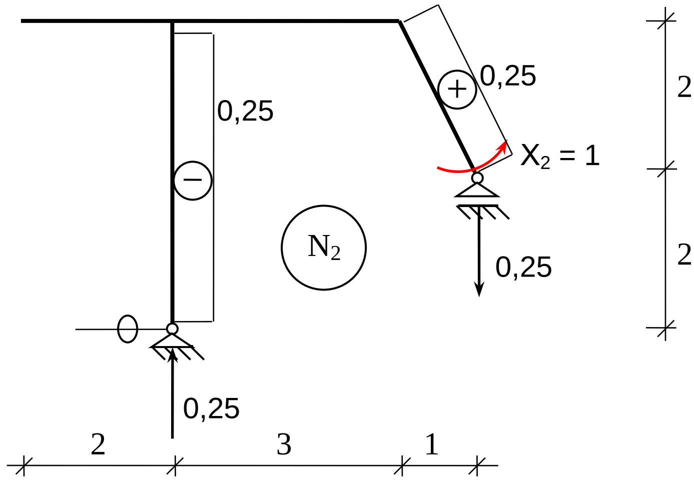
$$\Delta_{it} = \sum \frac{\boldsymbol{\alpha_t.\Delta t}}{\boldsymbol{h}} \int M_i \, \boldsymbol{ds} + \sum \boldsymbol{\alpha_t.t^c} \int N_i \, \boldsymbol{ds}$$

For the present example, the solution from the temperature load is as follows:



This solution needs to compose the normal forces diagram from the unit loads.





For the rectangular cross-section, t_c is the middle temperature: $t_c = \frac{t_1 + t_2}{2} = \frac{0+25}{2} = 12{,}5°;\ \Delta t = 25°$

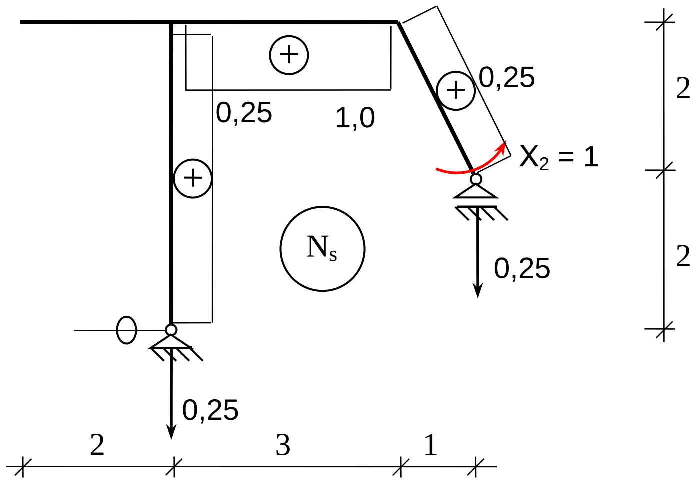
Calculation of the coefficients:

$$\Delta_{1t} = 0{,}00001.\left[\frac{25}{0{,}25}\frac{1}{2}4.4 + \frac{25}{0{,}315}\frac{4+2{,}5}{2}3 + 12{,}5(0{,}5.4 + 1.3)\right] = 163{,}631.10^{-4}$$

$$\Delta_{2t} = 0{,}00001.\left[\frac{25}{0{,}315}0{,}75\frac{3}{2} - 12{,}5.0{,}25.4\right] = 7{,}6786.10^{-4}$$

Verification of the coefficients: $\sum \Delta_{it} = \sum \frac{\alpha \Delta t}{h} \int M_s ds + \sum \alpha . t_c \int N_s ds$

Composing the summary diagram N_s



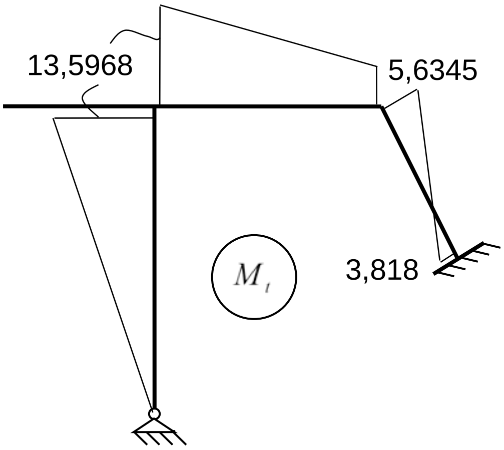
$\sum\Delta_{it} = 171{,}31.10^{-4}$

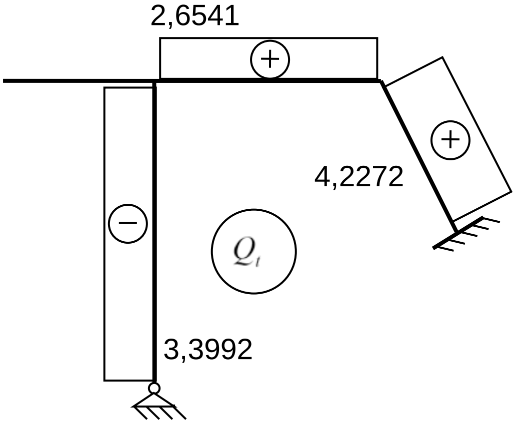
$$\alpha.\left[\sum\frac{\Delta t}{h}\int M_S ds + \sum t_c \int N_S ds\right] = 1.10^{-5}\left[\frac{25}{0{,}25}\frac{1}{2}4.4 + \frac{25}{0{,}315}\frac{4+3{,}25}{2}3 + 12{,}5\left(0{,}25.4 + 1.3\right)\right] = 171{,}31.10^{-4}$$

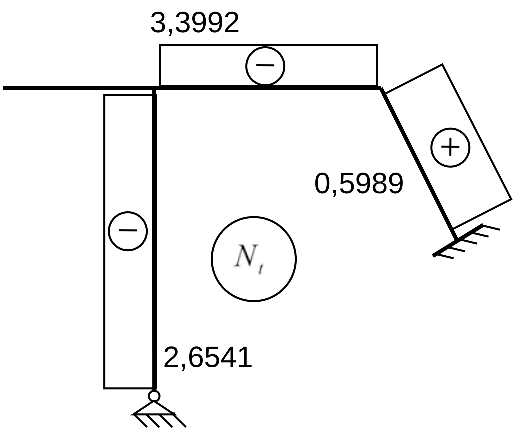
Compatibility equations for temperature load:

$$\left|\begin{array}{ll} 53{,}913X_{1t} + 5{,}142X_{2t} + 163{,}631 = 0 \rightarrow & X_{1t} = -3{,}3992 \\ 5{,}142X_{1t} + 2{,}5666X_{2t} + 7{,}6786 = 0 \rightarrow & X_{2t} = 3{,}818 \end{array}\right.$$

Final diagrams from the temperature loads: $M_t = M_1.X_{1t} + M_2.X_{2t}$







Compatibility verification:

$$\sum \frac{M_1 M_t}{EI} ds = \Delta_{1t}$$

$$\sum \frac{M_1 M_t}{EI} ds = \frac{1}{7812}\left(-\frac{1}{3}4.13,5968.4++\frac{1}{6}2,5.[2.(-5,6345)+3,818].2,2361\right)-$$

$$-\frac{1}{15624}\left(\frac{1}{6}[4.13,597+(13,597+5,6345).(4+2.5)+2,5.5,6345].3\right)=163,36.10^{-4}=\Delta_{1t}=163,361.10^{-4}$$

Analysis of statically indeterminate frame loaded by support settlement load:

The analysis of indeterminate frames from the support settlement load is the same as this from the external load whit only one difference. The final diagram made using the following expression:

$$M_c = +M_1.X_{1c} + M_2.X_{2c}$$

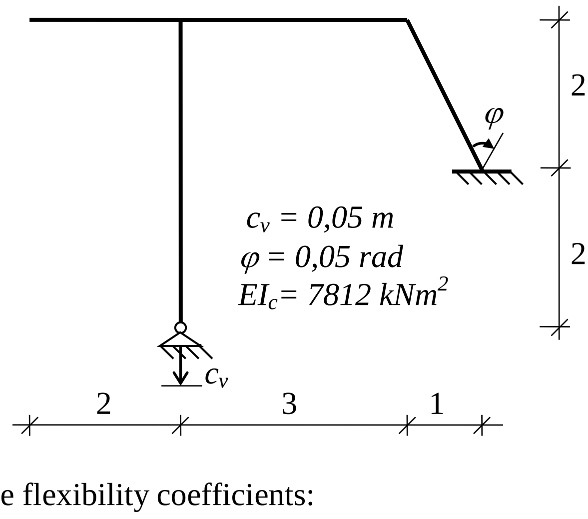
The reason of this result is that at a determinate structure (the primary system) there is no diagram from the support settlement. Therefore, to compose the final diagrams we need only to calculate new values of the unknowns. To calculate them is necessary to determine the displacement of the unknown's application points caused by the support settlement – Δ_{1c}, Δ_{2c}. Then the compatibility equations will be:

$$\begin{vmatrix} \delta_{11}.X_{1c} + \delta_{12}.X_{2c} + \Delta_{1c} = 0 \\ \delta_{21}.X_{1c} + \delta_{22}.X_{2c} + \Delta_{2c} = 0 \end{vmatrix}$$

The displacements Δ_{1c} and Δ_{2c} are displacements at a determinate system so the way of their calculation is already explained and it is:

$$\boldsymbol{\Delta_{ic}} = \sum_{\mathbf{1}}^{\boldsymbol{n}} \mathrm{R}_i.c$$

For the present example, the solution from the temperature load is as follows:



Determination of the flexibility coefficients:
To obtain these coefficients we need reactions at the unit diagrams and use the formula:

$$\Delta_{ic} = -\sum R_i.c_i$$

The sum of the right side of the equation represents the work of the support reactions at the unit diagrams through the support settlements.

$$\Delta_{1c} = -(0{,}05.0{,}5+0) = -250.10^{-4}$$

$$\Delta_{2c} = -(-0{,}05.0{,}25-0{,}05.1) = 625.10^{-4}$$

Verification of the coefficients: $\sum\Delta_{ic} \overset{!}{=} -\sum R_{s,i}.c_i$;

$R_{s,i}$ – support reaction of the summary moment diagrams – M_S **.**

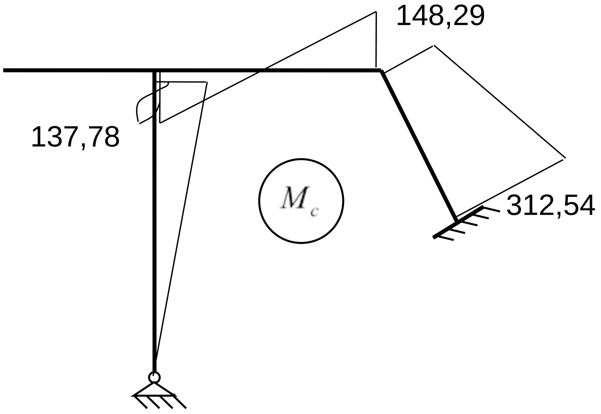
$$-\sum R_{s,i}.c_i = -(0{,}05.0{,}25-0{,}05.1) = 375.10^{-4}$$

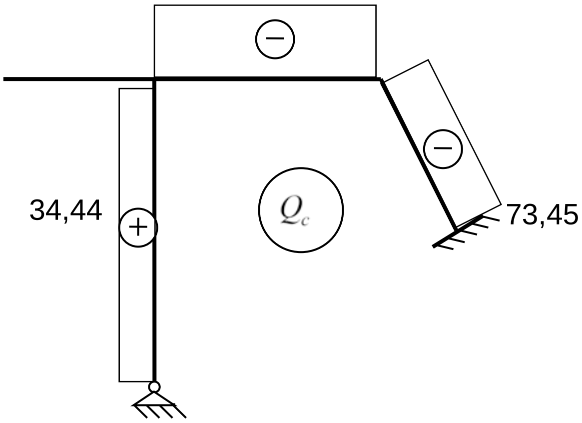
$$\sum\Delta_{ic} = 625.10^{-4} - 250.10^{-4} = 375.10^{-4}$$

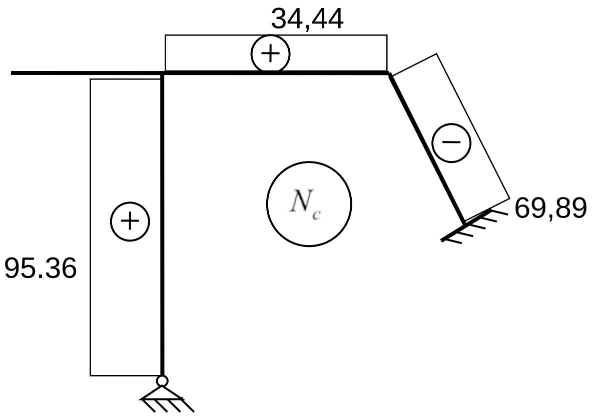
Compatibility equations for the support settlements analysis:

$$\left| \begin{array}{lll} 53{,}913X_1 + 5{,}142X_2 - 250 = 0 & \rightarrow & X_1 = 34{,}44 \\ 5{,}142X_1 + 2{,}5666X_2 + 625 = 0 & \rightarrow & X_2 = -312{,}51 \end{array} \right.$$

Final diagrams from support settlements:







Compatibility verification:

$$\sum \frac{M_1 M_c}{EI} ds = -\Delta_{1c}$$

$$\sum \frac{M_1 M_c}{EI} ds = \frac{1}{7812}\left(\frac{1}{3}4.137{,}78.4 - \frac{1}{6}2{,}5.[2148{,}29 + 312{,}54].2{,}2361\right) +$$

$$+\frac{1}{15624}\left(\frac{1}{6}[4.137{,}78 + (137{,}78 - 148{,}29).(4 + 2.5) - 2{,}5.148{,}29].3\right) = 250.10^{-4} = -\Delta_{1c} = -\left(-250.10^{-4}\right)$$

Symmetrical indeterminate frames.

If one indeterminate system is symmetrical is better to use this property when analyze such a system. In this section, we will show this property of the system and how to use it.

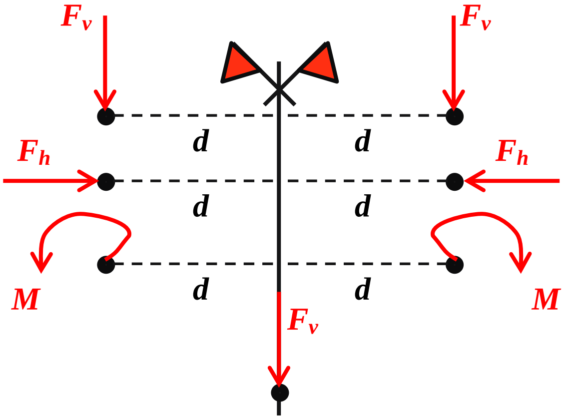
One system is symmetrical if it has the following properties:

1. the geometry is symmetrical;
2. the supports are symmetrical;
3. the cross-sections (*A* and *I*) are symmetrical;
4. the physical data (*E*-module) is symmetrical.

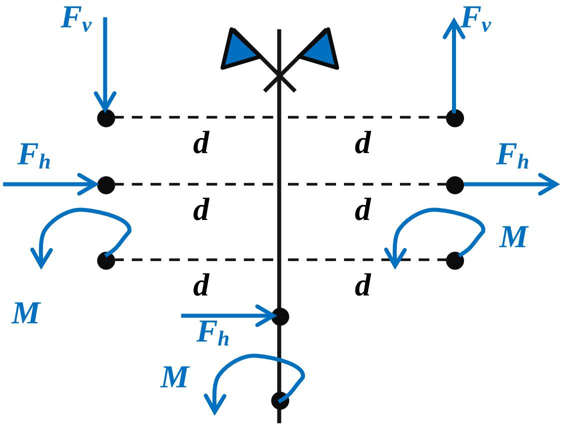
The external load can be symmetrical, antisymmetrical or common.

First of all we will show when one force or moment is symmetrical and when it is antisymmetrical.

- symmetrical forces (moments):



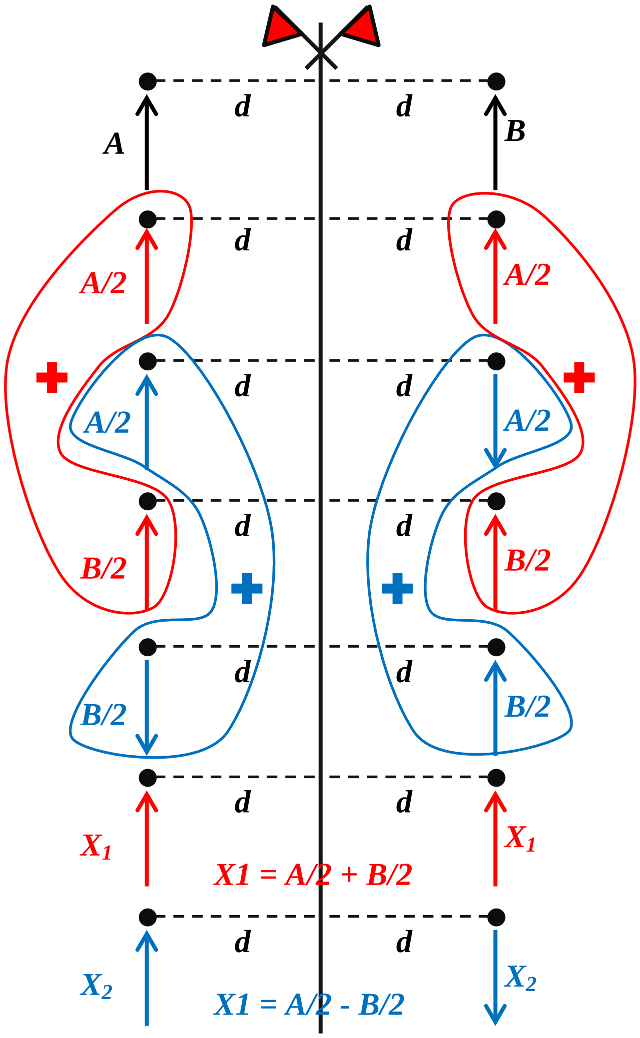
1. Two vertical forces are symmetrical when the application points of the forces are at one and the same distance from the axe of symmetry and the forces at the two sides of the symmetrical axe have one and the same direction.
2. Two horizontal forces are symmetrical when the application points of the forces are at one and the same distance from the axe of symmetry and the forces at the two sides of the symmetrical axe have inverse directions.
3. Two moments are symmetrical when the application points of the moments are at one and the same distance from the axe of symmetry and the moments at the two sides of the symmetrical axe have inverse directions.
4. One vertical force is symmetrical when the application point is on the axe of symmetry and it doesn't matter of its direction.



1. Two vertical forces are antisymmetrical when the application points of the forces are at one and the same distance from the axe of symmetry and the forces at the two sides of the symmetrical axe have inverse direction.
2. Two horizontal forces are antisymmetrical when the application points of the forces are at one and the same distance from the axe of symmetry and the forces at the two sides of the symmetrical axe have one and the same directions.
3. Two moments are antisymmetrical when the application points of the moments are at one and the same distance from the axe of symmetry and the moments at the two sides of the symmetrical axe have one and the same directions.
4. One horizontal force is antisymmetrical when the application point is on the axe of symmetry and it doesn't matter of it's direction.
5. One moment is antisymmetrical when the application point is on the axe of symmetry and it doesn't matter of it's direction.

Important to know: Internal normal force and internal moment on the axe of symmetry are symmetrical forces. Internal shear force on the axe of symmetry is antisymmetrical force.

Now will show how to transfer two common forces to a symmetrical and antisymmetrical:



Each force can be transfer at two parts one symmetrical and one antisymmetrical. In this way the two forces *A* and *B* are transferred to a two couples – symmetrical and antisymmetrical. After that if one take a sum of the symmetrical couples of the two forces will obtain one symmetrical force (couple) X_1. If one take a sum of the antisymmetrical couples of the two forces will obtain one antisymmetrical force (couple) X_2.

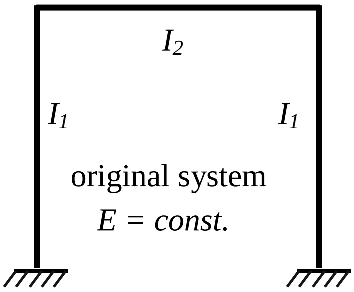
And inversely if one have directly values of the two couples a symmetrical couple X_1 and antisymetrical couple X_2 one can obtain the values of the common forces A and B as follows:

$$A = X_1/2 + X_2/2 \text{ and } B=X_1/2\ \ X_2/2.$$

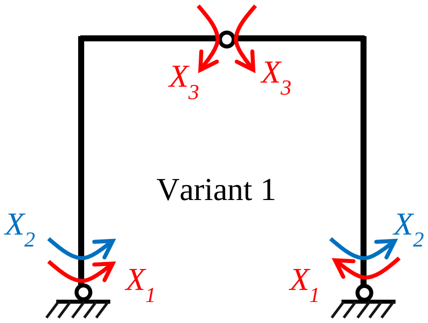
This idea is common not only for forces but for moments and displacements olso.

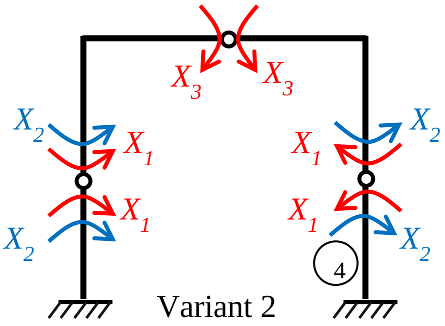
Actually in the structural analysis we use the last way of the decomposition. For the force method we use directly symmetrical and antisymmetrical couples X_1 and X_2 as unknowns and in the end if it is necessary we obtain the real forces A and B.

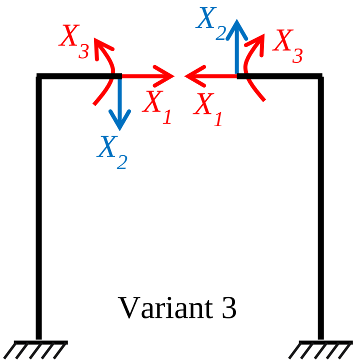
To be possible to use a symmetry in the force method not only indeterminate system should be symmetrical but the primary system must be too and to use couple of unknowns. Now will show some examples of a symmetrical indeterminate system and different symmetrical primary systems:



This structure is symmetrical because the geometry, supports, cross-sections and the physical data are symmetrical. So, the primary system should be symmetrical.







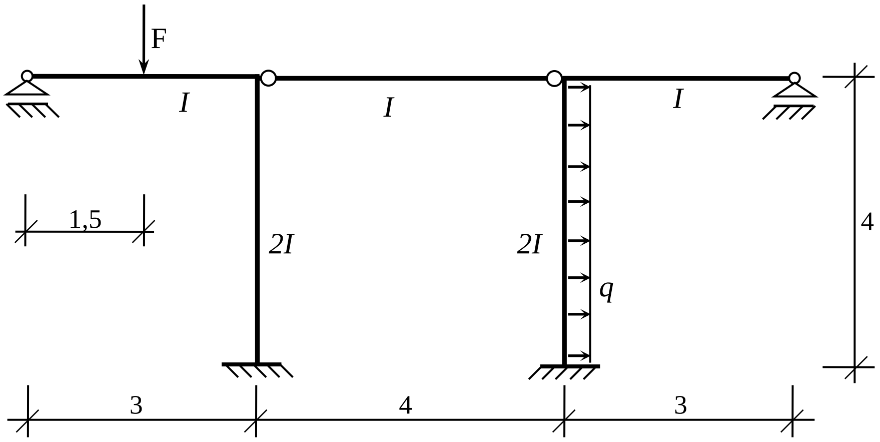
At the first variant the two supports moments are presented by couples of symmetrical antisymmetrical unknowns. X_1 is the symmetrical couple, X_2 is the antisymmetrical couple. X_3 is the internal moment for the frame, that is why it is a couple and because it is at the axe of symmetry, it is a symmetrical unknown. X_1 and X_3 are symmetrical and X_2 is antisymmetrical.

At a variant *2* all unknowns are internal moments but X_3 is on the axe of symmetry and it is a symmetrical moment. It is not necessary to transfer it to couples. Therefore, the X_1 and X_2 are symmetrical and antisymmetrical couples and they present the transferred couples of the internal moments in the left and in the right columns.

At the last variant, the frame is cut at the axe of symmetry so we directly have symmetrical and antisymmetrical unknowns. X_1 and X_3 are symmetrical and X_2 is antisymmetrical.

Important to know: The most important effect of using symmetry is at the multiplication of the diagrams for calculation of the displacements. When the unknown is symmetrical, the internal moment diagram is also symmetrical. When the unknown is antisymmetrical, the internal moment diagram is also antisymmetrical. When symmetrical and antisymetrical diagrams are multiplied the result is zero. Follows the displacement is zero.

Full example of analysis of a symmetrical frame using this property from the external force load.



$F = 40\ kN$

$q = 16\ kN/m'$

$I_b = 0{,}5I_{col}$

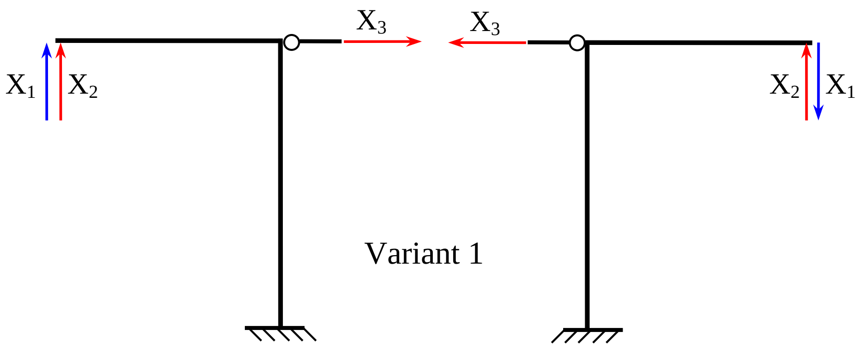
$A_b = 1{,}5A_{col}$

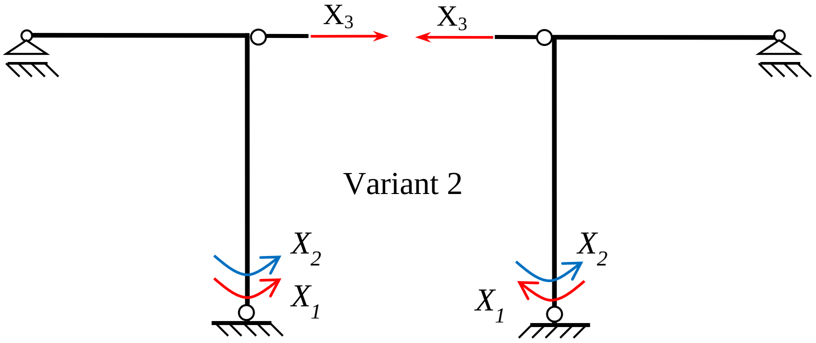
$A_c = A_b$

$I_c/A_c = 0{,}01$

On tis example in addition to symmetry we will use not only moments but normal forces to calculate displacements. The reason of this is that as part of the frame there is a rod. We are not obligate to use normal forces but it is better to do it. For the present example, there is not essential difference in the analysis with and without taking into account normal forces but we will show the analysis including normal forces.

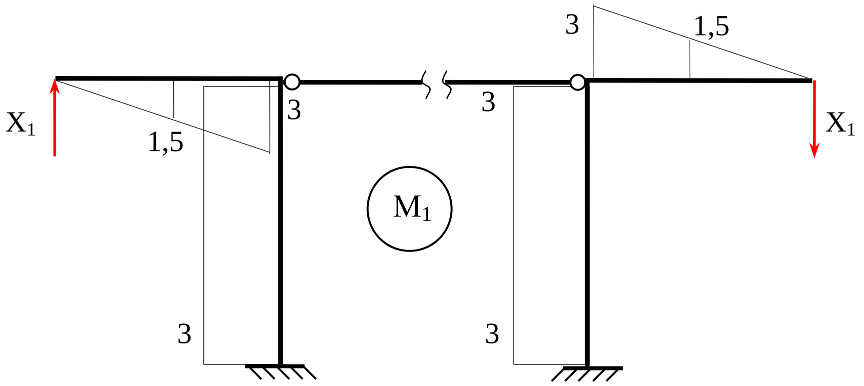
1. **Primary system:**

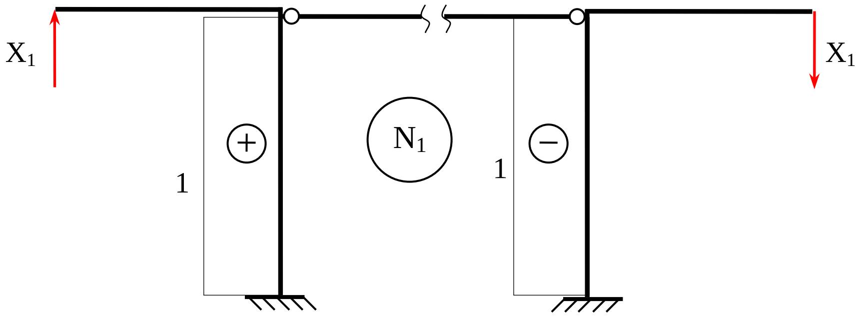


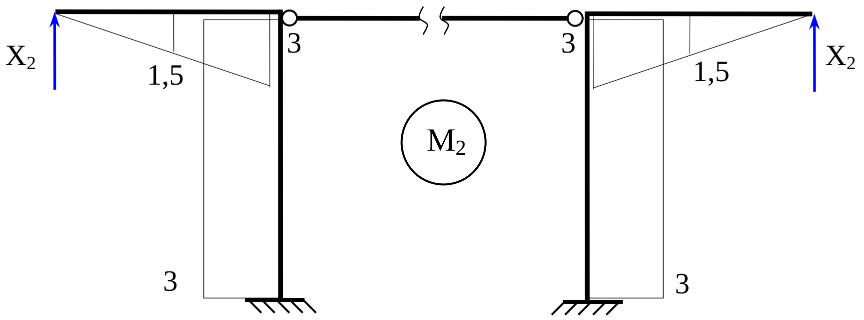


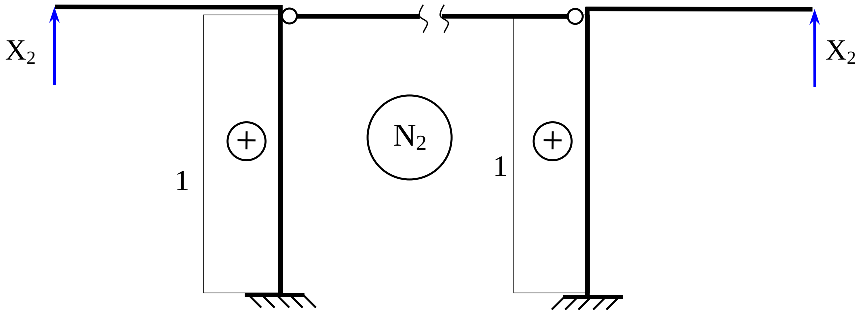
At the two variants, X_1 and X_3 are symmetrical and X_2 is antisymmetrical. Variant 1 will be chosen.

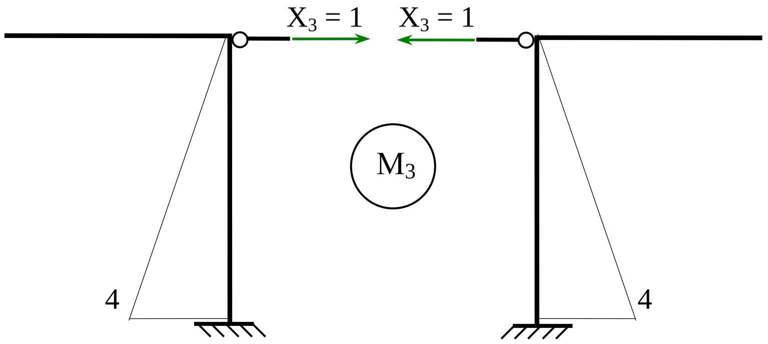
2. Unit diagrams:

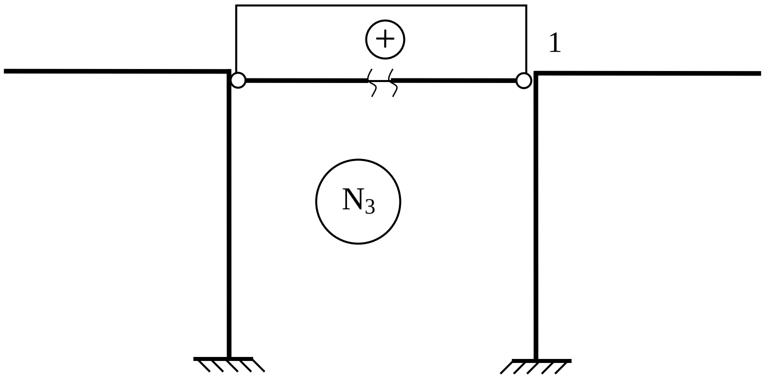






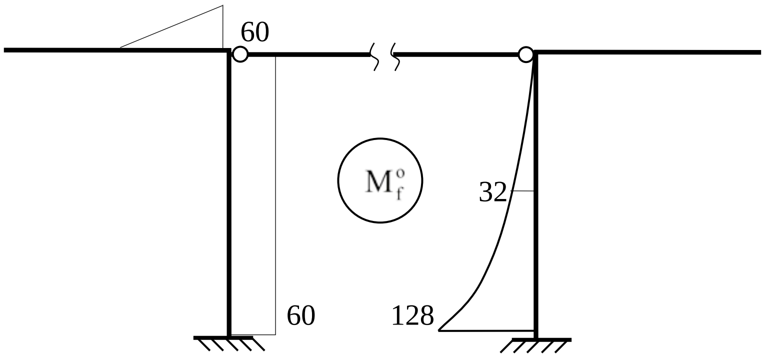


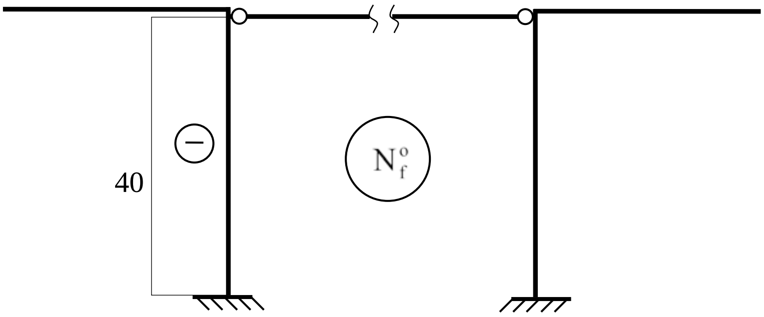




The diagrams M_1, N_1, M_3, N_3 are symmetrical. The diagrams M_2 and N_3 are antisymmetrical.

3. Diagrams from the external load at a primary system:





4. Calculation of the displacements:

$$EI = 2{,}4.10^7.3{,}255.10^4 = 7812$$
$$2EI = 2.2{,}4.10^7.6{,}512.10^4 = 15625$$
$$EA = 2{,}4.10^7.0{,}0625 = 1500000$$
$$1{,}5EA = 1{,}5.2{,}4.10^7.0{,}0625 = 2250000$$

$$\delta_{11} = \sum\int\frac{M_1^2}{EI}ds' + \sum\frac{N_1^2}{EA}ds = \frac{1}{3}\frac{3^2.3.2}{7812} + \frac{3^2.4.2}{15625} + \frac{1.4.2}{2250000} = 2{,}304.10^{-3} + 4{,}608.10^{-3} + 3{,}556.10^{-6} = 6{,}916.10^{-3}$$

$$\delta_{12} = \sum\int\frac{M_1M_2}{EI}ds' + \sum\frac{N_1N_2}{EA}ds = 0$$

$$\delta_{13} = \sum\int\frac{M_1M_3}{EI}ds' + \sum\frac{N_1N_3}{EA}ds = 0$$

$$\delta_{22} = \sum\int\frac{M_1^2}{EI}ds' + \sum\frac{N_1^2}{EA}ds = \frac{1}{3}\frac{3^2.3.2}{7812} + \frac{3^2.4.2}{15625} + \frac{1.4.2}{2250000} = 2{,}304.10^{-3} + 4{,}608.10^{-3} + 3{,}556.10^{-6} = 6{,}916.10^{-3}$$

$$\delta_{23} = \sum\int\frac{M_2M_3}{EI}ds' + \sum\frac{N_2N_3}{EA}ds = \frac{1}{2}\frac{4.3.4.2}{15625} + 0 = 3{,}072.10^{-3}$$

$$\delta_{33} = \sum\int\frac{M_3^2}{EI}ds' + \sum\frac{N_2^2}{EA}ds = \frac{1}{3}\frac{4^2.4.2}{15625} + \frac{1.4}{1500000} = 2{,}731.10^{-3} + 2{,}6667.10^{-6} = 2{,}7337.10^{-3}$$

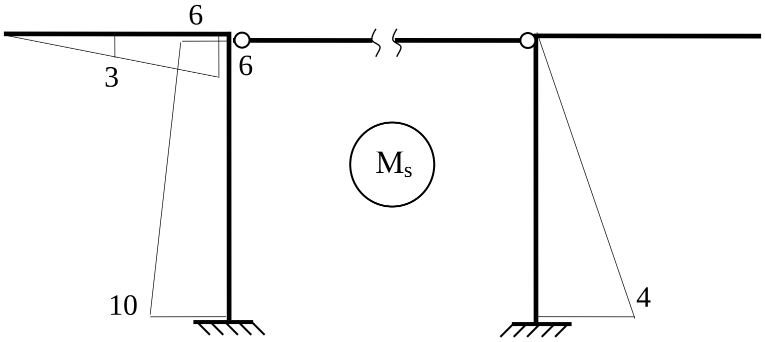
$$\Delta_{1F} = \sum\int\frac{M_1M_f^o}{EI}ds' + \sum\frac{N_1N_f^o}{EA}ds = -\frac{1}{6}\frac{60.(2.3+1{,}5).1{,}5}{7812} - \frac{3.60.4}{15625} + \frac{1}{3}\frac{3.128.4}{15625} -$$
$$-\frac{1.40.4}{2250000} = -14{,}4.10^{-3} - 46{,}08.10^{-3} + 32{,}77.10^{-3} - 7{,}1111.10^{-5} = -27{,}78.10^{-3}$$

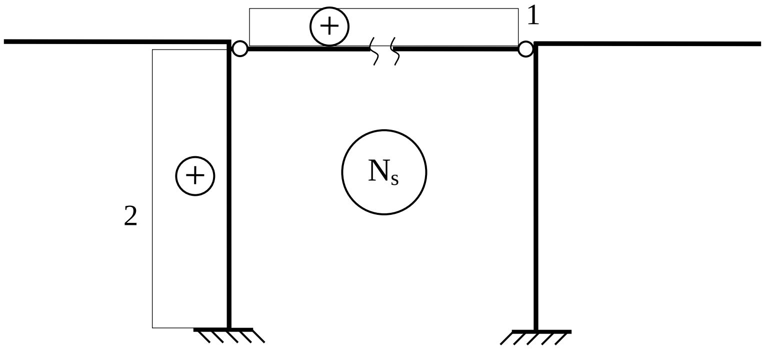
$$\Delta_{2F} = \sum\int\frac{M_2M_f^o}{EI}ds' + \sum\frac{N_2N_f^o}{EA}ds' = -\frac{1}{6}\frac{60.(2.3+1{,}5).1{,}5}{7812} - \frac{3.60.4}{15625} - \frac{1}{3}\frac{3.128.4}{15625} -$$
$$-\frac{1.40.4}{2250000} = -14{,}4.10^{-3} - 46{,}08.10^{-3} - 32{,}77.10^{-3} - 7{,}1111.10^{-5} = -93{,}32.-10^{-3}$$

$$\Delta_{3F} = \sum\int \frac{M_3 M_f^o}{EI} ds' + \sum \frac{N_1 N_f^o}{EA} ds = -\frac{1}{2}\frac{4.60.4}{15625} - \frac{1}{4}\frac{4.128.4}{15625} + 0 = -30{,}72.10^{-3} - 32{,}77.10^{-3} = -63{,}49.10^{-3}$$

5. Verification of the coefficients:

$$\sum \delta_{ij} = \sum\int \frac{M_s^2}{EI} ds' + \sum \frac{N_s^2}{EA} ds$$

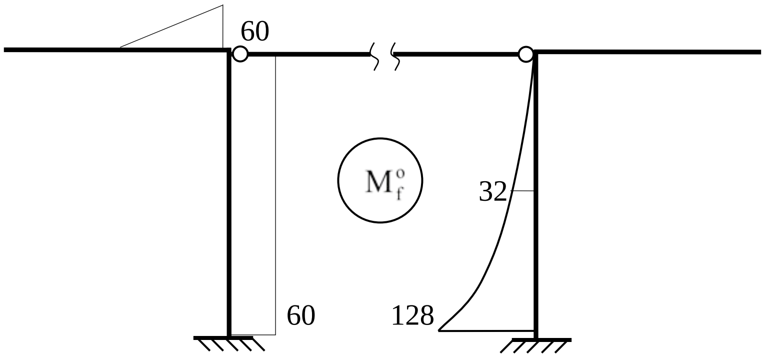


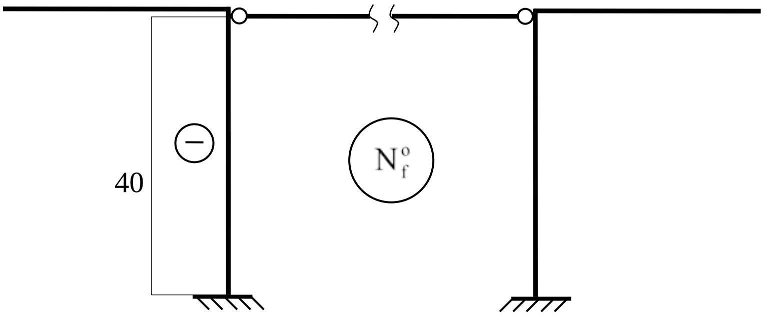


$$\sum\int \frac{M_s^2}{EI} ds' + \sum \frac{N_s^2}{EA} ds = \frac{1}{3}\frac{6^2.3}{7812} + \frac{1}{6}\left[6^2 + (6+10)^2 + 10^2\right]\frac{4}{15625} + \frac{1}{3}\frac{4^2.4}{15625} +$$

$$+ \frac{2^2.4}{2250000} + \frac{1^2.4}{1500000} = 4{,}608.10^{-3} + 16{,}73.10^{-3} + 1{,}365.10^{-3} + 7{,}111.10^{-6} + 2{,}667.10^{-6} = 22{,}71.10^{-3}$$

$$\sum \delta_{ij} = 6{,}916.10^{-3} + 6{,}916.10^{-3} + 3{,}072.10^{-3} + 3{,}072.10^{-3} + 2{,}7337.10^{-3} = 22{,}709.10^{-3}$$





$$\sum\int\frac{M_s M_f^o}{EI}ds' + \sum\frac{N_s N_f^o}{EA}ds' = -\frac{1}{6}\frac{60.(2.6+3).1,5}{7812} - \frac{1}{2}\frac{60.(6+10).4}{15625} - \frac{1}{6}\frac{4.(128+2.32).4}{15625} -$$

$$-\frac{2.40.4}{2250000} = -28{,}801.10^{-3} - 122{,}88.10^{-3} - 32{,}768.10^{-3} - 7{,}111.10^{-5} = -184{,}52.10^{-3}$$

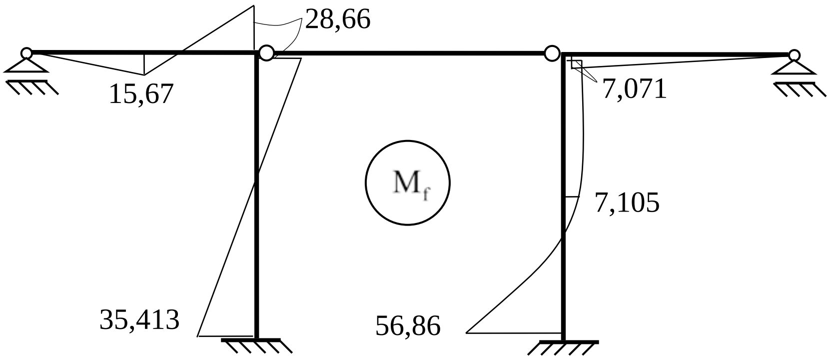
$$\sum\Delta_{ij} = -27{,}78.10^{-3} - 93{,}32.-10^{-3} - 63{,}49.10^{-3} = 184{,}59.10^{-3}$$

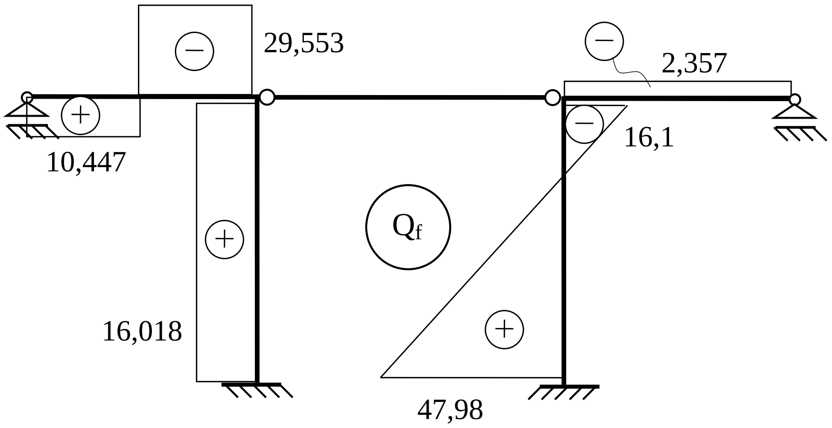
6. Compatibility equations:

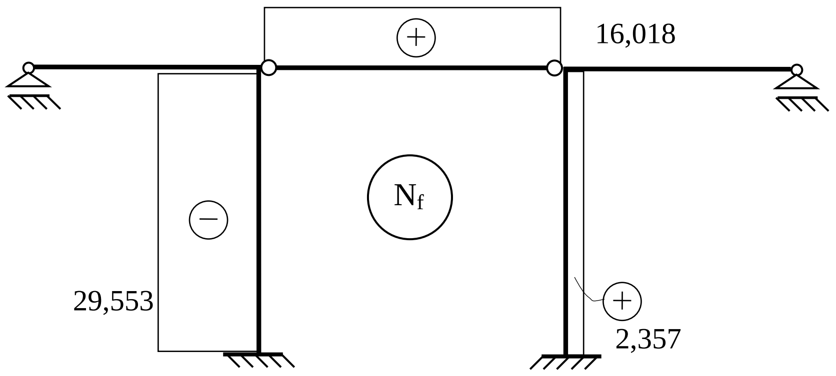
$$\left|\begin{array}{c} 6{,}916.10^{-3}X_1 - 27{,}78.10^{-3} = 0 \\ 6{,}916.10^{-3}X_2 + 3{,}072.10^{-3}X_3 - 93{,}32.10^{-3} = 0 \\ 3{,}072.10^{-3}X_2 + 2{,}7337.10^{-3}.X_3 - 63{,}49.10^{-3} = 0 \end{array}\right| : 10^{-3}$$

$$\left|\begin{array}{ll} 6{,}916X_1 - 27{,}78 = 0 & \rightarrow X_1 = 4{,}017 \\ 6{,}916X_2 + 3{,}072X_3 - 93{,}32 = 0 & \rightarrow X_2 = 6{,}344 \\ 3{,}072X_2 + 2{,}7337X_3 - 63{,}49 = 0 & \rightarrow X_3 = 16{,}096 \end{array}\right.$$

7. Final diagrams:







8. Compatibility verification:

$$\sum\int\frac{M_sM_f}{EI}ds'+\sum\frac{N_sN_f}{EA}ds'\overset{?}{=}0$$

$$\sum\int\frac{M_sM_f}{EI}ds'+\sum\frac{N_sN_f}{EA}ds'=\frac{1}{3}\frac{3.15,67.1,5}{7812}+\frac{1}{6}\left[3.15,67+(3+6)(15,67-28,66)-6.28,66\right].\frac{1,5}{7812}+$$

$$+\frac{1}{6}\left[10.35,413+(10+6)(35,413-28,66)-6.28,66\right].\frac{4}{15625}+$$

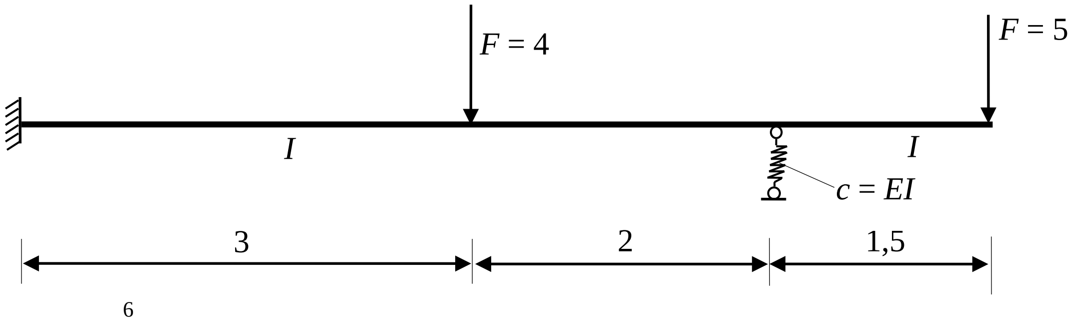
$$+\frac{1}{6}4.(-56,86+2.7,105).\frac{4}{15625}-\frac{2.29,553.4}{2250000}+\frac{1.16,018.4}{1500000}\overset{?}{=}0$$

$$3,01.10^{-3}-7,74.10^{-3}+12,38.10^{-3}-7,28.10^{-3}-1,05.10^{-4}+4,27.10^{-5}\overset{?}{=}0$$

$$15,43.10^{-3}-15,125.10^{-3}=0$$

Indeterminate frames containing springs.

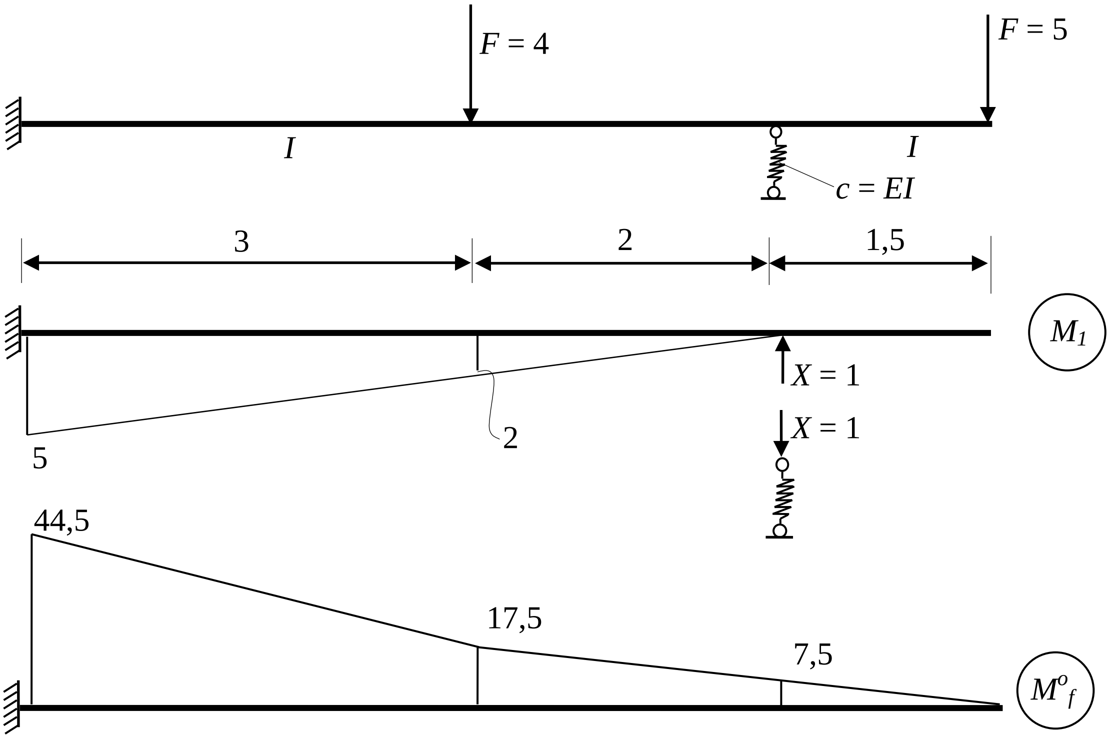
As we already mentioned when an indeterminate frame includes spring best way to compose a primary system is to cut the spring. Therefore, the solution has the same idea. Here is presented full example of the analysis of system containing spring support:



$E = 2.10^6$

$I = 0,000675\ m^4$ $\quad c = EI = 1350$ $\quad d = 1/EI = 7,407.10^{-4}$

When we take such a primary system the unknown is the spring reaction. When we load the system by X_1 don't forget to load the spring whit the same load and to include its influence at the displacements.



It is suitable to cut the spring because in this situation there is no reaction in the spring from the external load. Only from the unknown X_1 (the unknown at the spring) it appears the spring influence. In the concrete case at δ_{11}.

$$\delta_{11} = \sum\int\frac{M_1^2}{\boldsymbol{EI}}\boldsymbol{ds} + \sum_{\boldsymbol{1}}^{\boldsymbol{n}}\frac{\boldsymbol{S_1^2}}{\boldsymbol{c}}$$

$$\delta_{11} = \frac{1}{3}.\frac{5^2.5}{EI} + \frac{1^2}{EI} = \frac{42,6667}{1350} = 0,03160$$

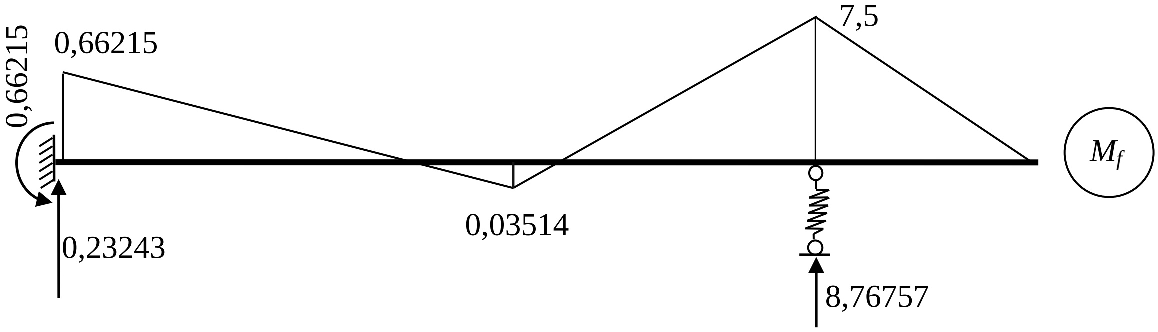
$$\Delta_{1f} = \sum\int\frac{M_1 M_{fo}}{\boldsymbol{EI}}\boldsymbol{ds} + \sum_{\boldsymbol{1}}^{\boldsymbol{n}}\frac{S_1 . S_f^0}{\boldsymbol{c}}$$

$$\Delta_{1f} = \frac{1}{1350}\left(\frac{-1}{6}.3.[5.44,5+(5+2)(44,5+17,5)+2.17,5]-\frac{1}{6}.2.2.[2.17,5+7,5]\right)+0=-\frac{374,083}{1350}$$

$$\Delta_{1f} = -0,2771$$

$$X_1 = -\frac{\Delta_{1f}}{\delta_{11}} = \frac{0,2771}{0,03160} = 8,76757$$

$$M_f = M_{f_0} + M_{X_1}$$



Chapter 10

Displacements at statically indeterminate structures.

The displacement at statically indeterminate structures calculates in the same way as in the determinate systems.

According to principle of the virtual work for a system subjected to an external force and temperature load and including springs supports the displacement at a specific point calculates using the next expression:

$$\Delta_f^n = \sum\int\frac{\bar{M}^n M_f^n}{EI}ds + \sum\int\frac{\bar{N}^n N_f^n}{EA}ds + \sum\int\frac{\bar{Q}^n Q_f^n}{GA_Q}ds + \sum\int\frac{\bar{M}^n.\alpha_t.\Delta t}{h}ds + \sum\int\bar{N}^n.\alpha_t.t^c\,ds + \sum_1^n\frac{\bar{S}^n S_f^n}{k}$$

Where $\bar{M}^n, \bar{Q}^n, \bar{N}^n,$ and $\bar{S}^n$ are internal and spring forces caused by a virtual unit load in the indeterminate frame at a specific point for which we are calculating the displacement. $M_f^n, Q_f^n, N_f^n,$ andS_f^n are internal and spring forces caused by external load in the indeterminate frame.

In the previous chapter, it was shown haw to analyze indeterminate frames. e necessary to compose primary system and to analyze it many times from a different loads. I it is needed to compute a displacement at some point in indeterminate system we need to analyze the system ones again but from some virtual load. Moreover, diagrams in indeterminate structures are complicate and it is possible to make a lot of mistake of their multiplication. Conclusion is that calculation of displacement in indeterminate frame is a complicated problem.

However, it is possible one of the solutions to be in determinate system. Now will be shown this idea and how to use it. To be easier let take only external load internal moments integral in the expression of the displacements. The main idea for all other parts is the same and it is not needed to use it now. So, we have:

$$\Delta_f^n = \sum\int\frac{\bar{M}^n M_f^n}{EI}ds$$

Now let think that indeterminate frame is analyzed using force method. Let also make assumption that the frame is two times indeterminate. If the frame is more times indeterminate the expressions are the same but longer. The external load internal moments diagram we can compose using following expression:

$$M_f^n = M_f^o + M_1.X_1 + +M_2.X_2$$

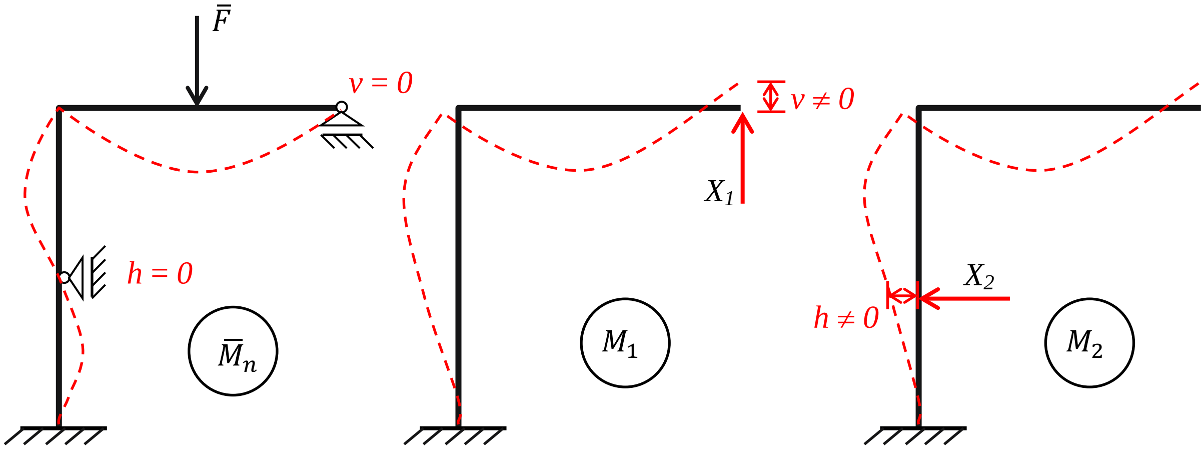
Let substitute this expression in the expression for the displacement:

$$\begin{aligned}\Delta_f^n &= \sum\int\frac{1}{EI}\left(M_f^o + M_1.X_1 + +M_2.X_2\right)\bar{M}^n ds\\ &= \sum\int\frac{1}{EI}M_f^o\bar{M}^n ds + \sum\int\frac{1}{EI}(M_1.X_1 + +M_2.X_2)\bar{M}^n ds\\ &= \sum\int\frac{1}{EI}M_f^o\bar{M}^n ds + \sum\int\frac{1}{EI}M_1.X_1\bar{M}^n ds + \sum\int\frac{1}{EI}M_2.X_2\bar{M}^n ds\\ &= \sum\int\frac{M_f^o\bar{M}^n}{EI}ds + \sum X_1\int\frac{M_1.\bar{M}^n}{EI}ds + \sum X_2\int\frac{M_2.\bar{M}^n}{EI}ds\end{aligned}$$

The question is what are the last two integrals?

$$\sum X_1 \int \frac{M_1.\overline{\boldsymbol{M}}^n}{\boldsymbol{EI}}\boldsymbol{ds} \;\; ; \;\; \sum X_2 \int \frac{M_2.\overline{\boldsymbol{M}}^n}{\boldsymbol{EI}}\boldsymbol{ds}.$$

$\overline{\boldsymbol{M}}^n$ is the moment diagram in the indeterminate frame from a virtual load.
M_1 and M_2 are the moment diagrams unspecified primary system from a unit load.
The integrals present the displacements of the X_1 and X_2 application points from a virtual load.



If the indeterminate frame is on equilibrium than the displacements of the X_1 and X_2 application points should be zero!

The conclusion is that:

$$\sum X_1 \int \frac{M_1.\overline{\boldsymbol{M}}^n}{\boldsymbol{EI}}\boldsymbol{ds} = 0;$$
$$\sum X_2 \int \frac{M_2.\overline{\boldsymbol{M}}^n}{\boldsymbol{EI}}\boldsymbol{ds} = 0.$$

Finally the result about the displacement in indeterminate frame is:

$$\Delta_{\boldsymbol{f}}^{\boldsymbol{n}} = \sum \int \frac{M_f^o \overline{\boldsymbol{M}}^n}{\boldsymbol{EI}}\boldsymbol{ds}$$

The diagram from the external load can be determinate in arbitrary determinate frame.

On the same way we can prof that the other one diagram can be composed in arbitrary determinate frame, or:

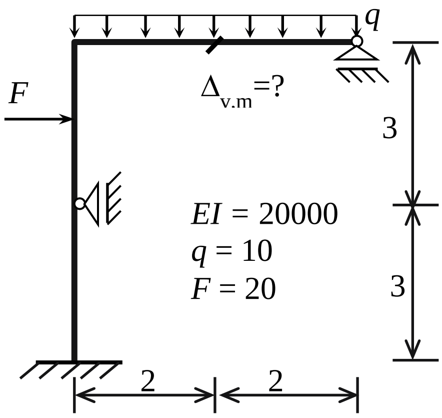
$$\Delta_{\boldsymbol{f}}^{\boldsymbol{n}} = \sum \int \frac{M_f^n \overline{\boldsymbol{M}}^o}{\boldsymbol{EI}}\boldsymbol{ds}$$

As conclusion, we may say: The displacement of some point in indeterminate system can be calculated using internal forces diagram from external and virtual load in indeterminate structure. Using internal forces diagram from external load in indeterminate structure and virtual load in determinate system. Using internal forces diagram from virtual load in indeterminate structure and internal forces diagram from external load in determinate system.

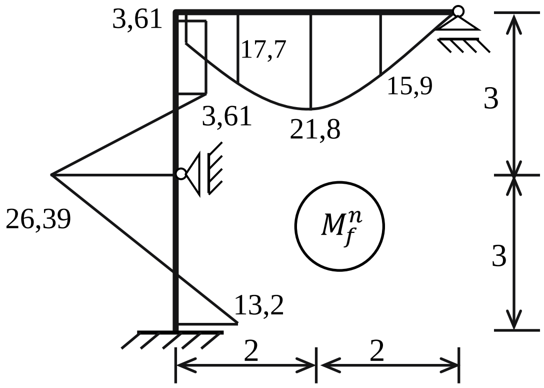
$$\Delta_{\boldsymbol{f}}^{\boldsymbol{n}} = \sum \int \frac{\overline{\boldsymbol{M}}^n \boldsymbol{M}_{\boldsymbol{f}}^{\boldsymbol{n}}}{\boldsymbol{EI}}\boldsymbol{ds} = \sum \int \frac{M_f^o \overline{\boldsymbol{M}}^n}{\boldsymbol{EI}}\boldsymbol{ds} = \sum \int \frac{M_f^n \overline{\boldsymbol{M}}^o}{\boldsymbol{EI}}\boldsymbol{ds}$$

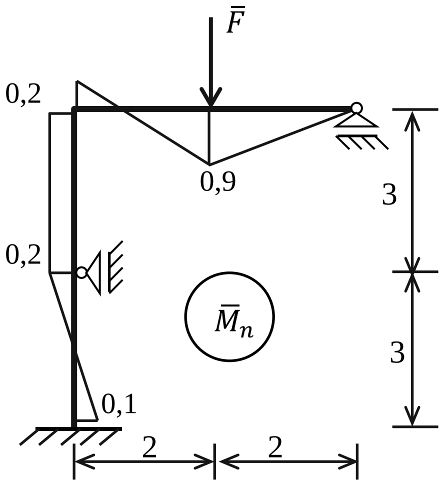
Example:

Calculation of displacement in indeterminate frame from external load:



The internal moment diagram from the external load in indeterminate structure and the internal moment diagram the virtual load in indeterminate structure are:





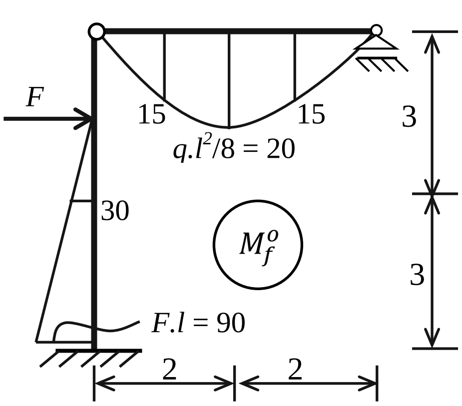
The calculated displacement using two indeterminate frames is:

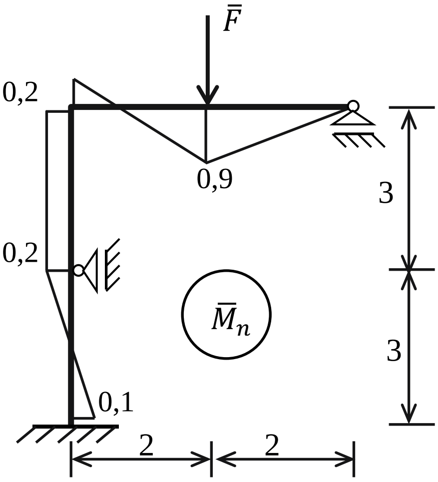
$$\Delta_f^{\boldsymbol{n}} = \sum \int \frac{\boldsymbol{\bar{M}^n M_f^n}}{\boldsymbol{EI}} \boldsymbol{ds}$$

$$= \frac{1}{20000}\left[\frac{1}{6}(13{,}2.0{,}1 + (13{,}2 - 26{,}39)(0{,}1 - 0{,}2) + 26{,}39.0{,}2).3 \right.$$
$$+ \frac{1}{2}0{,}2(26{,}39 - 3{,}61).1{,}5 - 3{,}61.0{,}2.1{,}5$$
$$\left. + \frac{1}{6}(-3{,}61.0{,}2 + 2.17{,}7.(-0{,}2 + 0{,}9) + 21{,}8.0{,}9).2 + \frac{1}{6}0{,}9(21{,}8 + 2.15{,}9).2\right]$$
$$= \frac{1}{20000}(3{,}9585 + 3{,}417 - 1{,}083 + 14{,}5593 + 16{,}8) = \frac{36{,}93}{20000}$$

$$\Delta_f^{\boldsymbol{n}} = \sum \int \frac{\boldsymbol{\bar{M}^n M_f^n}}{\boldsymbol{EI}} \boldsymbol{ds} = 1{,}847.10^{-3}m = 0{,}185cm.$$

The moment diagram from the external load in determinate structure and the moment diagram the virtual load in indeterminate structure are:
The determinate frame is arbitrary and is better to be such system to compose diagrams easier.





The calculated displacement in this case is:

$$\Delta_f^{\boldsymbol{n}} = \sum \int \frac{M_f^o \bar{\boldsymbol{M}}^n}{\boldsymbol{EI}} \boldsymbol{ds}$$

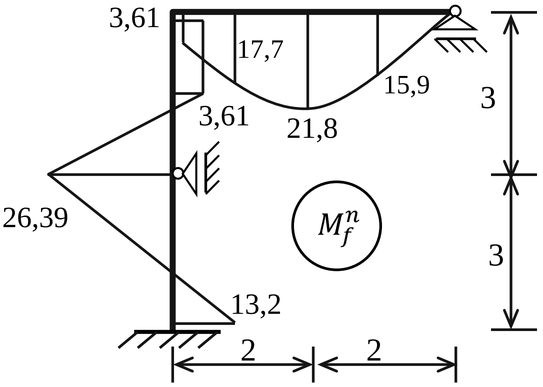
$$= \frac{1}{20000}\left[\frac{1}{6}(-0{,}1.90 + (90+30)(-0{,}1+0{,}2) + 30.0{,}2).3 + \frac{1}{2}30.0{,}2.1{,}5\right.$$

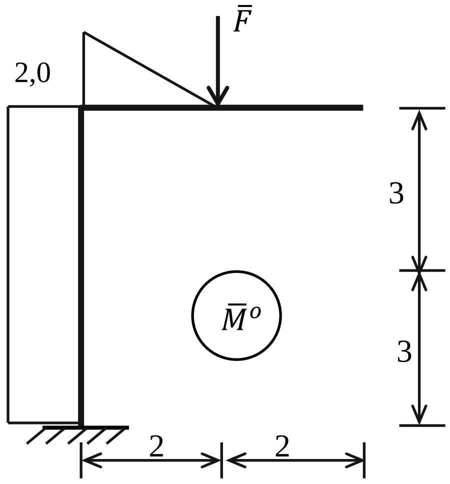
$$\left. + \frac{1}{6}(-0{,}2.0 + 2.15(-0{,}2+0{,}9) + 0{,}9.20).2 + \frac{1}{6}0{,}9(20+2.15).2\right]$$

$$= \frac{1}{20000}(4{,}5 + 4{,}5 + 13 + 15) = \frac{37}{20000}$$

$$\Delta_f^{\boldsymbol{n}} = \sum \int \frac{M_f^o \bar{\boldsymbol{M}}^n}{\boldsymbol{EI}} \boldsymbol{ds} = 1{,}85.10^{-3} m = 0{,}185 cm.$$

The moment diagram from the external load in indeterminate structure and the moment diagram the virtual load in determinate structure are:
The determinate frame is arbitrary and is better to be such system to compose diagrams easier.





Presented determinate frame is not easier because in this case the multiplication of the diagrams is not so easy but we will use it to show that it is no difference if use different determinate systems. Better determinate system is the previous one because the diagram will be only on the simple beam and when multiply will have only two parts for multiplication.

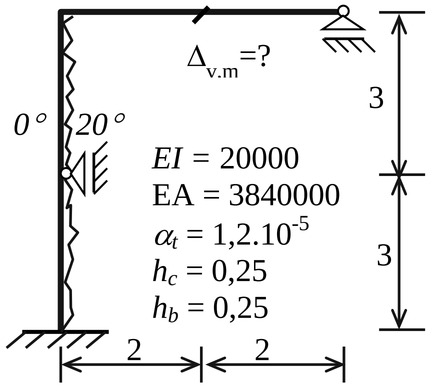
The calculated displacement in this case is:

$$\Delta_{\boldsymbol{f}}^{\boldsymbol{n}} = \sum \int \frac{M_f^n \boldsymbol{\bar{M}^o}}{\boldsymbol{EI}} \boldsymbol{ds}$$

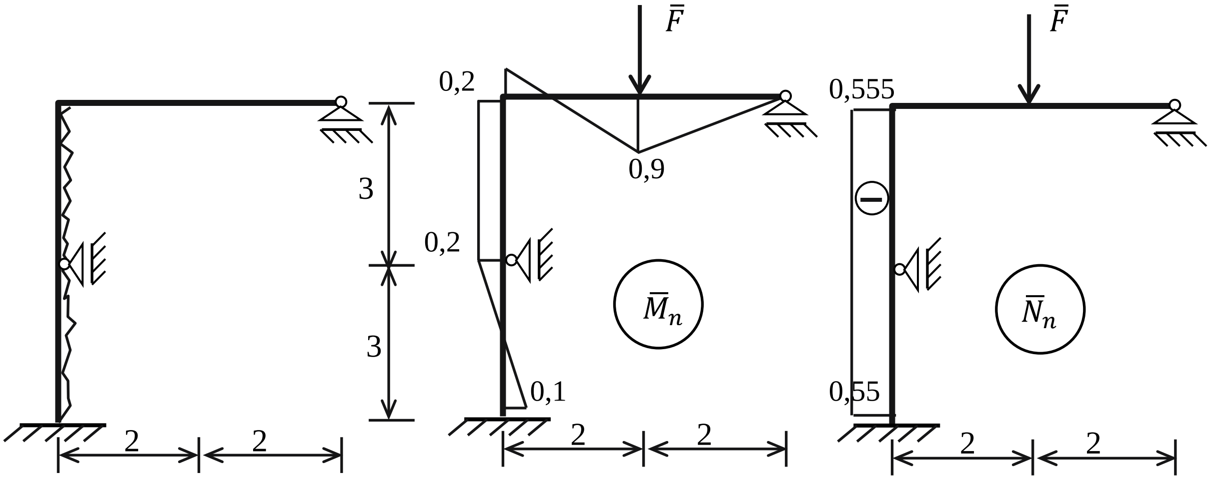
$$= \frac{1}{20000}\left[-\frac{1}{6}2(3,61 + 2.17,7).2 - 2.3,61.1,5 + \frac{1}{2}2(26,39 - 3,61).1,5 + \frac{1}{2}2(26,39 - 13,2).3\right] = \frac{1}{20000}(-26 - 10,83 + 34,17 + 39,57) = \frac{36,91}{20000}$$

$$\Delta_{\boldsymbol{f}}^{\boldsymbol{n}} = \sum \int \frac{M_f^n \boldsymbol{\bar{M}^o}}{\boldsymbol{EI}} \boldsymbol{ds} = 1,8455.10^{-3}m = 0,185cm.$$

Calculation of displacement in indeterminate frame from temperature load:



If we don’t have solution of the indeterminate frame from the temperature load it is better to perform analysis of the frame from the virtual load. In this case the diagrams should be composed for the indeterminate frame. To calculate displacement from temperature load we need not only internal moment diagram but normal forces too. The diagrams for the virtual load in indeterminate frame are as follows:



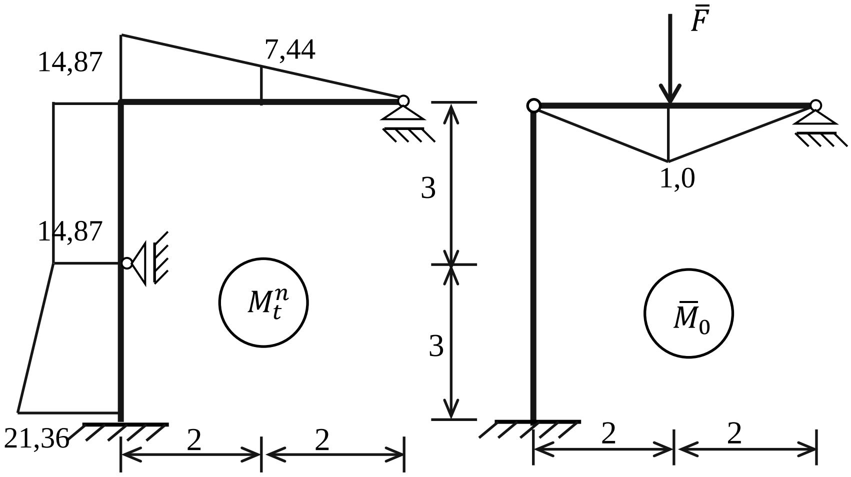
$$\Delta_{\boldsymbol{t}}^{\boldsymbol{n}} = \sum \int \frac{\boldsymbol{\bar{M}^n.\alpha_t.\Delta t}}{\boldsymbol{h}} \boldsymbol{ds} + \sum \int \boldsymbol{\bar{N}^n.\alpha_t.t^c\, ds}$$

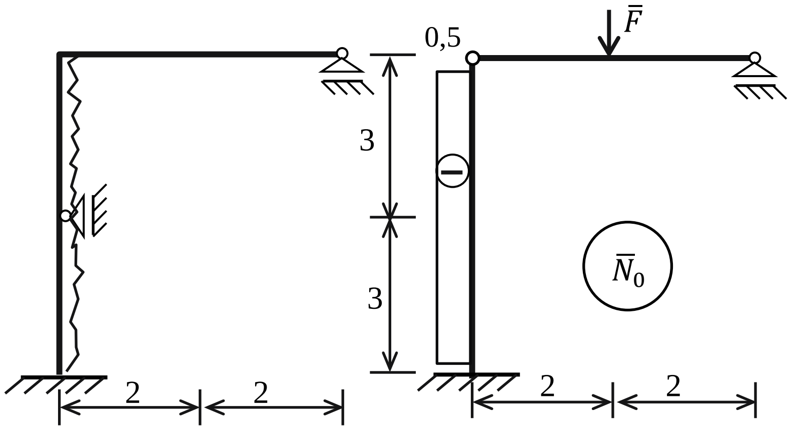
$$\Delta_{\boldsymbol{t}}^{\boldsymbol{n}} = \frac{1,2.10^{-5}.20}{0,25}\left(-0,2.3 + \frac{0,1 - 0,2}{2}3\right) - 1,2.10^{-5}.10.0,55.6 = -7,2.10^{-4} - 3,96.10^{-4}$$

$$\Delta_{\boldsymbol{t}}^{\boldsymbol{n}} = 1,116.10^{-3}m = 0,112cm.$$

Other way to calculate this displacement is to use the diagram at indeterminate frame from temperature load. In this case the solution from the virtual load can be at determinate structure and the displacement is:

$$\Delta_t^{\boldsymbol{n}} = \sum \int \frac{M_t^n \boldsymbol{\bar{M}^o}}{\boldsymbol{EI}} \boldsymbol{ds} + \sum \int \frac{\boldsymbol{\bar{M}}^0 . \boldsymbol{\alpha_t} . \boldsymbol{\Delta t}}{\boldsymbol{h}} \boldsymbol{ds} + \sum \int \boldsymbol{\bar{N}}^0 . \boldsymbol{\alpha_t} . \boldsymbol{t^c}\, \boldsymbol{ds}$$



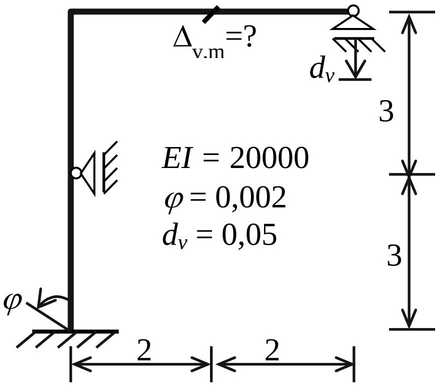


$$\Delta_{\mathrm{t}}^{\mathrm{n}} = \frac{1}{20000}\left[-\frac{1}{6}1(14{,}87 + 2.7{,}44).2 - \frac{1}{3}1.7{,}44.2\right] - 0{,}5.6.10.1{,}2.10^{-5}$$

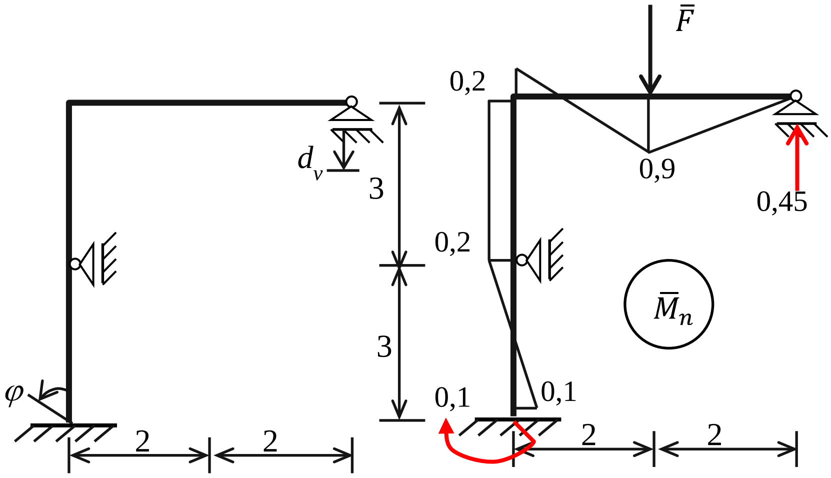
$$= -\frac{14{,}877}{20000} \pm 3{,}6.10^{-4} = -7{,}4385..10^{-4} + 3{,}6.10^{-4}$$

$$\Delta_{\mathrm{t}}^{\mathrm{n}} = -1{,}104.10^{-3} m = 0{,}1104 cm$$

Calculation of displacement in indeterminate frame from support settlement load:



If we don’t have solution of the indeterminate frame from the support settlement load it is better to perform analysis of the frame from the virtual load. In this case the diagrams should be composed for the indeterminate frame. To calculate displacement from support settlement load we need not only internal moment diagram but reactions too. The diagrams for the virtual load in indeterminate frame are as follows:



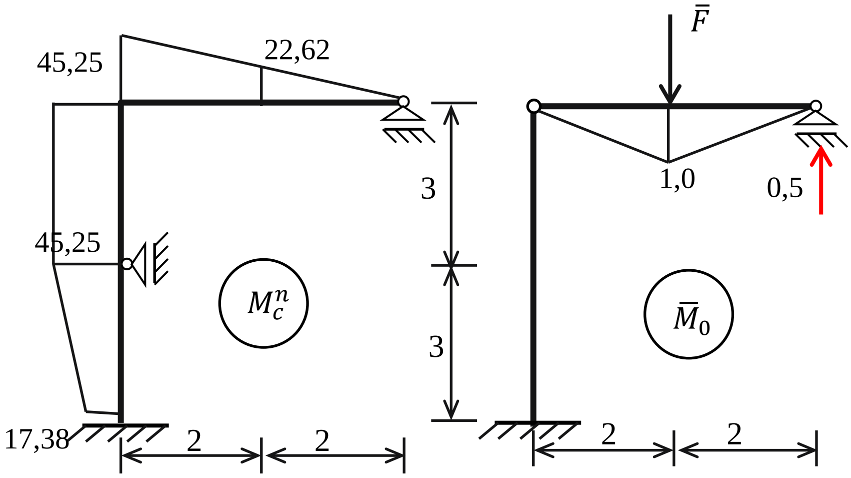
$$\Delta_c^{\boldsymbol{n}} = -\sum \bar{\boldsymbol{R}}^n . d$$

where *d* is the support settlement.

$$\Delta_c^{\boldsymbol{n}} = -\sum \bar{\boldsymbol{R}}^n . d = -(-0{,}1.0{,}002 - 0{,}05.0{,}45) = 0{,}0227m = 2{,}27cm$$

Other way to calculate this displacement is to use the diagram at indeterminate frame from support settlement load. In this case the solution from the virtual load can be at determinate structure and the displacement is:

$$\Delta_t^{\boldsymbol{n}} = \sum \int \frac{M_c^n \bar{\boldsymbol{M}}^o}{\boldsymbol{EI}} \boldsymbol{ds} - \sum \bar{\boldsymbol{R}}^0 . d$$



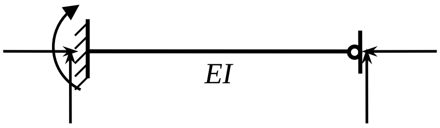
$$\Delta_c^{\boldsymbol{n}} = \sum \int \frac{M_c^n \bar{\boldsymbol{M}}^o}{\boldsymbol{EI}} \boldsymbol{ds} - \sum \bar{\boldsymbol{R}}^0 . d$$
$$= \frac{1}{20000}\left[-\frac{1}{6}1(22{,}62.2 + 45{,}25).2 - \frac{1}{3}1.22{,}62.2\right] - 0{,}5.(-0.05)$$
$$= \frac{-30{,}163 - 15{,}08}{20000} + 0{,}025 = -2{,}262.10^{-3} + 0{,}025$$
$$\Delta_c^{\boldsymbol{n}} = 0{,}0227m = 2{,}27cm$$

Chapter 11

Analysis of simple indeterminate structures using force method.

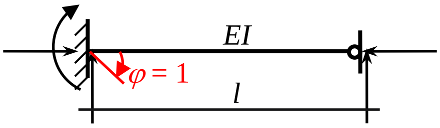
In this chapter, we will present the analysis of so-called kinematical links Type II and Type III. We already have presented them at Chapter 2 as basic elements for composition of complicate structures. Actually, they are indeterminate structure so they need additional analysis. As we already familiar to the force method, we may use it on this analysis.

Analysis of kinematical link Type II

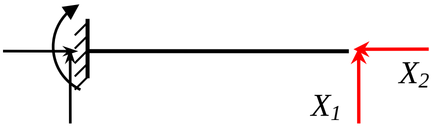


Two times indeterminate system.

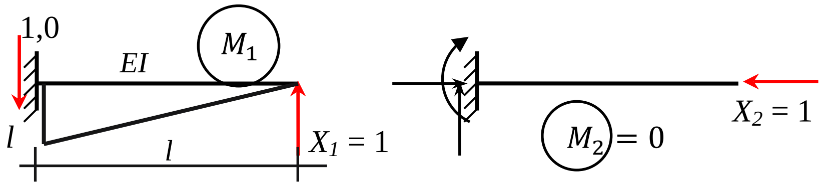
As a external load we will take different situation of a support settlements. Firstly will be a rotation at the fixed support with a unit value.



As a primary system, we will choose a fixed beam.



Unit diagrams and displacements:



$$\delta_{11} = \sum \int \frac{M_1^2}{EI} ds = \frac{l^3}{3EI}$$

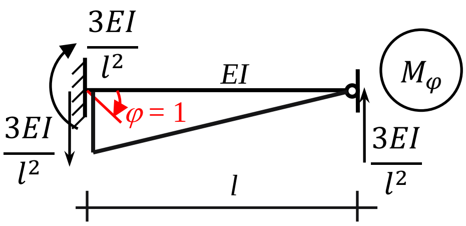
$$\delta_{12} = \sum \int \frac{M_1 M_2}{EI} ds = \delta_{22} = \sum \int \frac{M_2^2}{EI} ds = 0$$

$$\Delta_{1f} = -\sum R_1 . \varphi = -1. l = -l$$

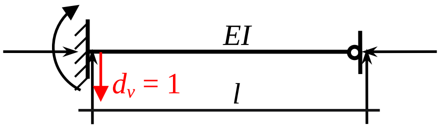
$$\Delta_{2f} = -\sum R_2 . \varphi = 0$$

$$X_1 = -\frac{\Delta_{1f}}{\delta_{11}} = \frac{l}{\frac{l^3}{3EI}} = \frac{3EI}{l^2}$$

$$X_2 = 0$$



Therefore, we have a solution of the link type II from unit rotation of the fixed support. Let do the same but from vertical unit displacement of the fixed support.



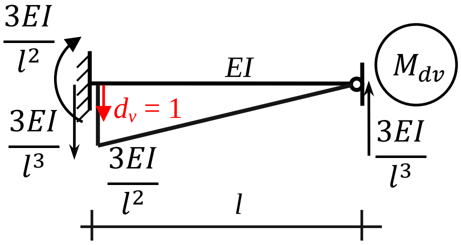
The primary system, unit diagrams and the displacement from the unit forces are the same. Only the displacement from the external load will changes and it is:

$$\Delta_{1f} = -\sum R_1 . d_v = -1.1 = -1$$

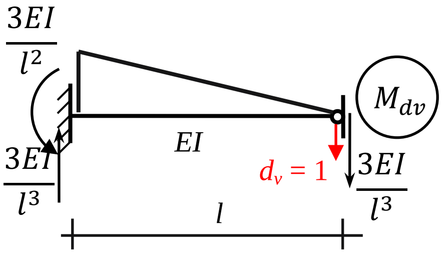
$$\Delta_{2f} = -\sum R_2 . d_v = 0$$

$$X_1 = -\frac{\Delta_{1f}}{\delta_{11}} = \frac{1}{\frac{l^3}{3EI}} = \frac{3EI}{l^3}$$

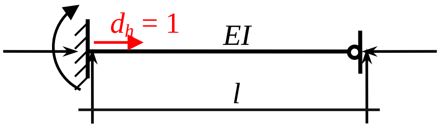
$$X_2 = 0$$



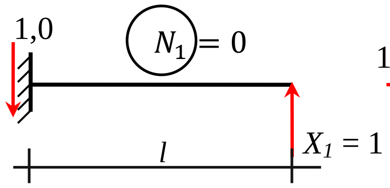
If the vertical displacement is on the pin support the result will be the same but with a reverse sing.

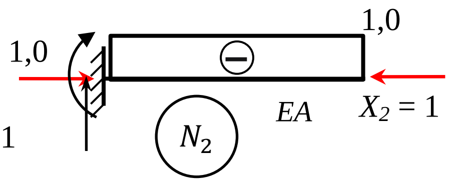


Let do the same but from horizontal unit displacement of the fixed support.



In this case we need only normal forces. The primary system is the same but the displacements are different:





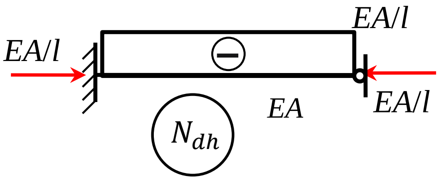
$$\delta_{11} = \delta_{12} = 0$$

$$\delta_{22} = \sum \int \frac{N_1^2}{EA} ds = \frac{1.l}{EA}$$

$$\Delta_{1f} = -\sum R_1.d_h = 0$$

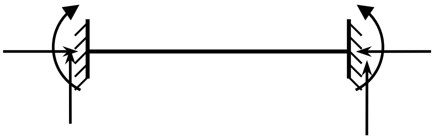
$$\Delta_{2f} = -\sum R_2.d_h = -1$$

$$X_2 = -\frac{\Delta_{2f}}{\delta_{22}} = \frac{1}{\frac{l}{EA}} = \frac{EA}{l}$$



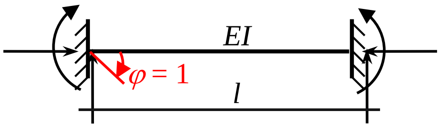
The same result will be if the horizontal displacement is at the other support.

Analysis of kinematical link Type III

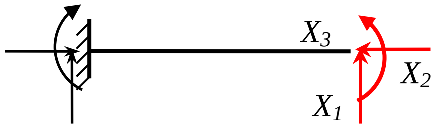


Three times indeterminate.

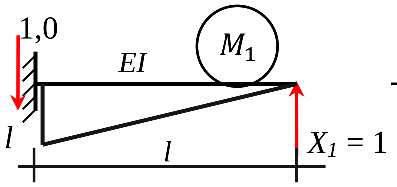
As a external load we will take different situation of a support settlements. Firstly will be a rotation at the left fixed support with a unit value.

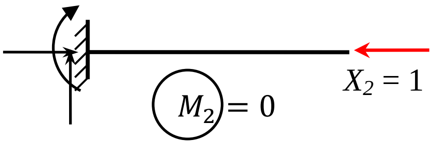


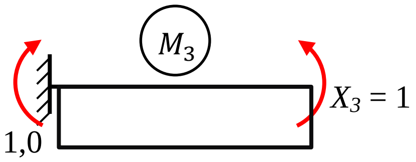
As a primary system, we will choose a fixed beam.



Unit diagrams and displacements:







$$\delta_{11}=\sum\int\frac{M_1^2}{EI}ds=\frac{l^3}{3EI}$$

$$\delta_{12}=\sum\int\frac{M_1M_2}{EI}ds=\delta_{23}=\sum\int\frac{M_2M_3}{EI}ds=\delta_{22}=\sum\int\frac{M_2^2}{EI}ds=0$$

$$\delta_{13}=\sum\int\frac{M_1M_3}{EI}ds=\frac{1.l^2}{2EI}$$

$$\delta_{33}=\sum\int\frac{M_3^2}{EI}ds=\frac{l}{EI}$$

$$\Delta_{1f}=-\sum R_1.\varphi=-1.l=-l$$

$$\Delta_{2f}=-\sum R_2.\varphi=0$$

$$\Delta_{3f}=-\sum R_3.\varphi=-1.1=-1$$

$$\left|\begin{array}{l}\delta_{11}X_1+\delta_{21}X_2+\delta_{13}X_3+\Delta_{1f}=0\\ \delta_{21}X_1+\delta_{22}X_2+\delta_{23}X_3+\Delta_{2f}=0\\ \delta_{31}X_1+\delta_{32}X_2+\delta_{33}X_3+\Delta_{3f}=0\end{array}\right.$$

$$\left|\begin{array}{l}\frac{l^3}{3EI}X_1+\frac{l^2}{2EI}X_3-l=0\\ \frac{l^2}{2EI}X_1+\frac{l}{EI}X_3-1=0\end{array}\right.$$

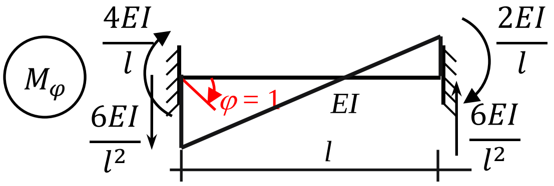
$$D=\frac{l^3}{3EI}\frac{l}{EI}-\frac{l^2}{2EI}\frac{l^2}{2EI}=\frac{4.l^4}{12EI}-\frac{3.l^4}{12EI}=\frac{l^4}{12EI^2}$$

$$\frac{12EI^2}{l^4}\begin{bmatrix}\frac{l}{EI} & -\frac{l^2}{2EI}\\ -\frac{l^2}{2EI} & \frac{l^3}{3EI}\end{bmatrix}\begin{Bmatrix}X_1\\X_3\end{Bmatrix}=\begin{Bmatrix}l\\1\end{Bmatrix}$$

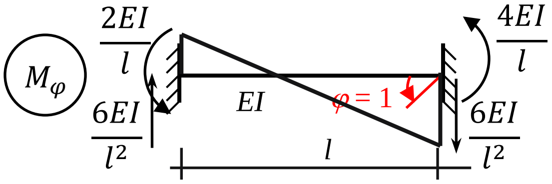
$$X_1=\frac{12EI^2}{l^4}\left(\frac{l}{EI}l-\frac{l^2}{2EI}1\right)=\frac{12EI^2}{l^4}\left(\frac{2l^2}{2EI}-\frac{l^2}{2EI}\right)=\frac{12EI^2}{l^4}\frac{l^2}{2EI}=\frac{6EI}{l^2}$$

$$X_3=\frac{12EI^2}{l^4}\left(-\frac{l^2}{2EI}l+\frac{l^3}{3EI}1\right)=\frac{12EI^2}{l^4}\left(-\frac{3l^3}{6EI}+\frac{2l^3}{6EI}\right)=-\frac{12EI^2}{l^4}\frac{l^3}{6EI}=\frac{2EI}{l}$$

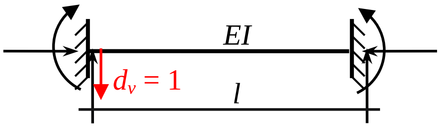
$$X_2=0$$



If the rotation is on the other support, the result will be the same but mirror.



Therefore, we have a solution of the link type III from unit rotation of the fixed supports. Let do the same but from vertical unit displacement of the left fixed support.



The primary system, unit diagrams and the displacement from the unit forces are the same. Only the displacement from the external load will changes and it is:

$$\Delta_{1f} = -\sum R_1 . d_v = -1.1 = -1$$

$$\Delta_{2f} = -\sum R_2 . d_v = 0$$

$$\Delta_{2f} = -\sum R_3 . d_v = 0$$

$$\left| \begin{array}{l} \delta_{11}X_1 + \delta_{21}X_2 + \delta_{13}X_3 + \Delta_{1f} = 0 \\ \delta_{21}X_1 + \delta_{22}X_2 + \delta_{23}X_3 + \Delta_{2f} = 0 \\ \delta_{31}X_1 + \delta_{32}X_2 + \delta_{33}X_3 + \Delta_{3f} = 0 \end{array} \right.$$

$$\left| \begin{array}{l} \frac{l^3}{3EI}X_1 + \frac{l^2}{2EI}X_3 - 1 = 0 \\ \frac{l^2}{2EI}X_1 + \frac{l}{EI}X_3 + 0 = 0 \end{array} \right.$$

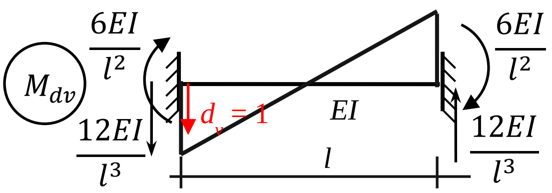
$$D = \frac{l^4}{12EI^2}$$

$$\frac{12EI^2}{l^4} \begin{bmatrix} \frac{l}{EI} & -\frac{l^2}{2EI} \\ -\frac{l^2}{2EI} & \frac{l^3}{3EI} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

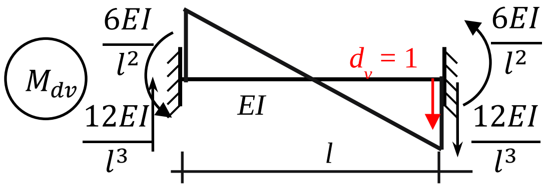
$$X_1 = \frac{12EI^2}{l^4}\left(\frac{l}{EI}1 - \frac{l^2}{2EI}.0\right) = \frac{12EI^2}{l^4}\frac{l}{EI} = \frac{12EI}{l^3}$$

$$X_3 = \frac{12EI^2}{l^4}\left(-\frac{l^2}{2EI}1 + \frac{l^3}{3EI}.0\right) = -\frac{12EI^2}{l^4}\frac{l^2}{2EI} = \frac{6EI}{l^2}$$

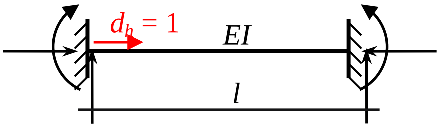
$$X_2 = 0$$



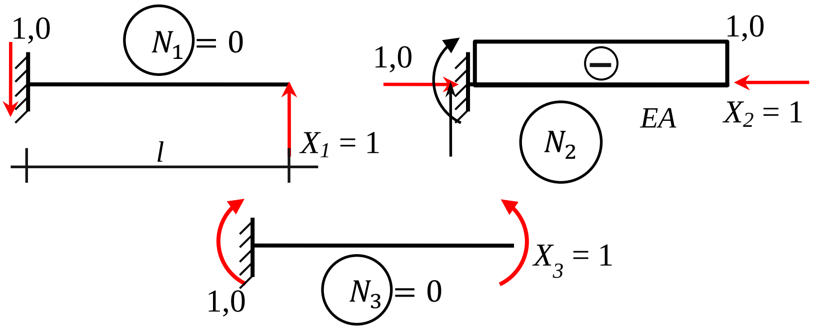
If the vertical displacement is on the other support, the result will be the same but mirror.



Let do the same but from horizontal unit displacement of the left fixed support.



In this case we need only normal forces. The primary system is the same but the displacements are different:



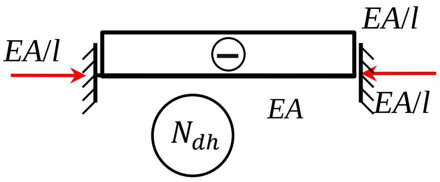
$$\delta_{22} = \sum \int \frac{N_1^2}{EA} ds = \frac{1.l}{EA}$$

$$\delta_{11} = \delta_{33} = \delta_{13} = \delta_{12} = \delta_{23} = 0$$

$$\Delta_{1f} = \Delta_{3f} = -\sum R_1.d_h = 0$$

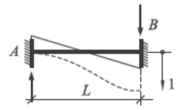
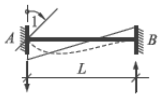
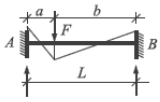
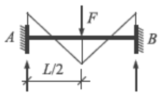
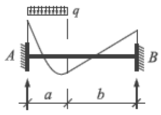
$$\Delta_{2f} = -\sum R_2.d_h = -1$$

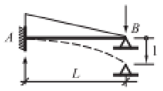
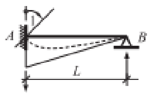
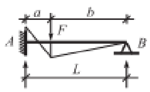
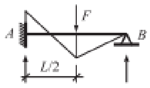
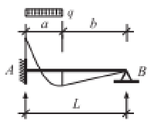
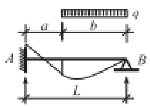
$$X_2 = -\frac{\Delta_{2f}}{\delta_{22}} = \frac{1}{\frac{l}{EA}} = \frac{EA}{l}$$



The same result will be if the horizontal displacement is at the other support.

In this way, we already have obtained results about links type II and III from all possible support settlements. Such solutions we may obtain from different external load too. All these results are made once for all and are arranged at a table can be used directly. Such a table is shown below:

		M_A	M_B	A	B
1		$\frac{6EI}{L^2}=\frac{6i}{L}$	$\frac{6EI}{L^2}=\frac{6i}{L}$	$\frac{12EI}{L^3}=\frac{12i}{L^2}$	A
2		$\frac{4EI}{L}=4i$	$\frac{2EI}{L}=2i$	$\frac{6EI}{L^2}=\frac{6i}{L}$	A
3		$\frac{Fab^2}{L^2}$	$\frac{Fa^2b}{L^2}$	$\frac{Fb^2}{L^2}\left(1+\frac{2a}{L}\right)$	$F-A$
4		$\frac{FL}{8}$	$\frac{FL}{8}$	$\frac{F}{2}$	A
5		$\frac{qa^2}{12}\left(6-\frac{8a}{L}+\frac{3a^2}{L^2}\right)$	$\frac{qa^2}{12}\left(\frac{4a}{L}-\frac{3a^2}{L^2}\right)$	$\frac{qa}{2}\left(2-\frac{2a^2}{L^2}+\frac{a^3}{L^3}\right)$	$qa-A$

		M_A	A	B
11		$\frac{3EI}{L^2}=\frac{3i}{L}$	$\frac{3EI}{L^3}=\frac{3i}{L^2}$	A
12		$\frac{3EI}{L}=3i$	$\frac{3EI}{L^2}=\frac{3i}{L}$	A
13		$\frac{Fab}{2L}\left(1+\frac{b}{L}\right)$	$\frac{Fb}{2L}\left(3-\frac{b^2}{L^2}\right)$	$F-A$
14		$\frac{3FL}{16}$	$\frac{11F}{16}$	$\frac{5F}{16}$
15		$\frac{qa^2}{8}\left(2-\frac{a}{L}\right)^2$	$\frac{qa}{8}\left(8-\frac{4a^2}{L^2}+\frac{a^3}{L^3}\right)$	$qa-A$
16		$\frac{qb^2}{8}\left(2-\frac{b^2}{L^2}\right)$	$\frac{qb}{8}\left(\frac{6b}{L}-\frac{b^3}{L^3}\right)$	$qb-A$

In these tables is used label ***i*** for the linear stiffness and it is:

$$i=\frac{EI}{l}$$

Chapter 12

Displacement method for analysis of statically indeterminate structures.

The displacement method is one other method for analyzing indeterminate structures. Some time it is suitable not to search extra-forces but node displacements.

The displacement method is used to calculate the response of statically indeterminate structures to loads and/or imposed deformations. The method is based on calculating unknown rotations and displacements at the joints of frames based on conditions of equilibrium at the joints.

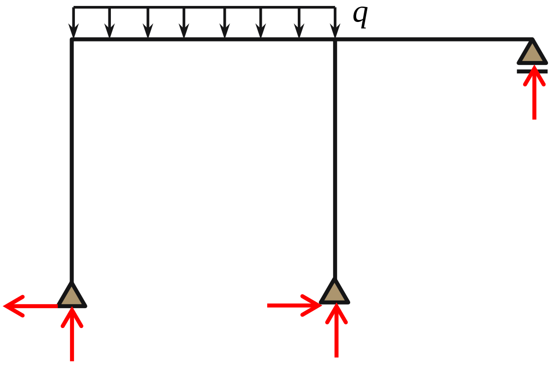
The force method is a method for calculating the response of statically indeterminate structures by which the unknowns are force quantities (the redundant forces X_1, X_2, ..., X_n) and the equations used to solve for the unknowns are based on geometrical conditions (compatibility conditions at the location of each extra-force).

It is possible to consider an analogous method for calculating the response of statically indeterminate structures in which the unknowns are displacement quantities and the equations used to solve for the unknowns are based on statical conditions (equilibrium conditions). This method will be referred to as the classical displacement method.

The procedure of enveloping the theory.

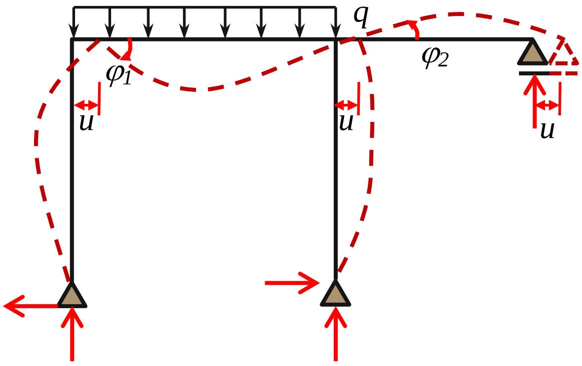
The procedure of enveloping the theory is logically same as in the force method. We use again the principle of superposition and multiplication of unit diagrams by extra values. The main difference as we have mentioned is that now we will think about displacement not for forces and we will add links not to remove them.

Let consider the next example and its deformation:



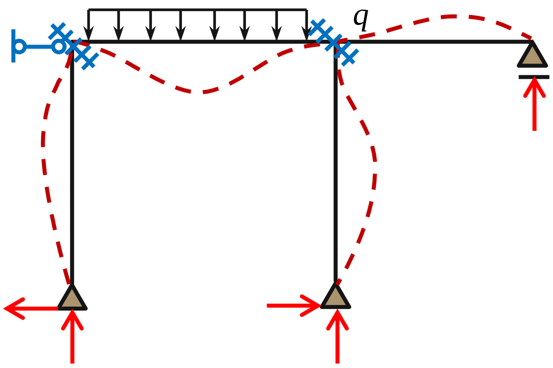
$$w = 3.1 - 0 - 5 = -2$$

The present structure is two times indeterminate according force method. What about its joint displacements:

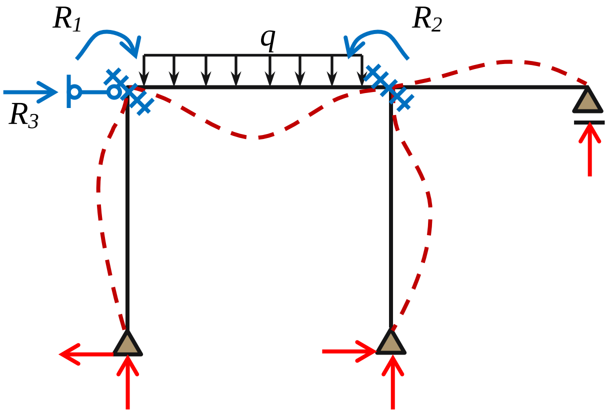


The two joints of the frame are rotated and the beam is displaced horizontally. Main question is: Is it possible to determine these joint displacement and rotations? If it is possible we will be able to determine full deformation of the frame.

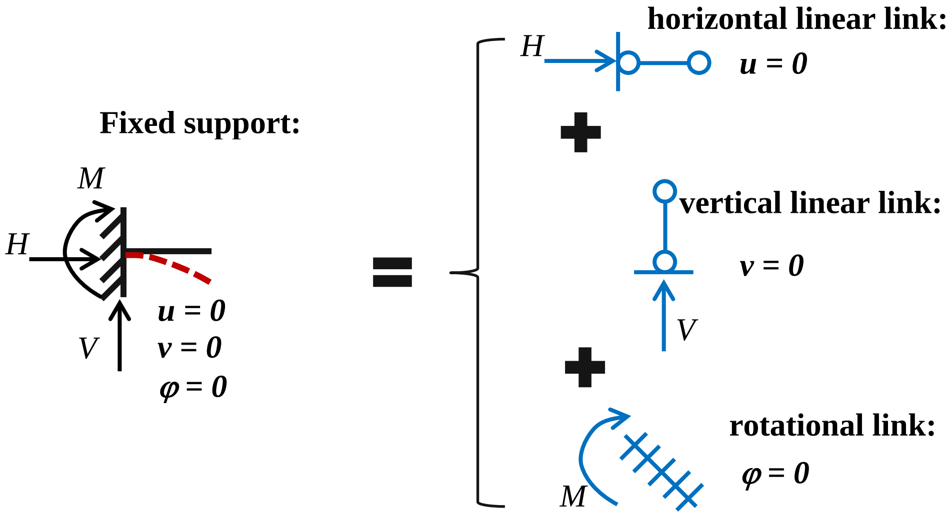
What will happens if we add some links to stop the joints displacements and rotations. We should add two rotational links and one linear horizontal.



In this case the nodes of the frame are fully fixed end the separate parts of the frame have only local and independent deformations but no displacements. Of this reason the local deformation is fully determinate (it is known by Strength of materials). In addition, we have displacement-controlled system so, we can give a unit values of the joint displacement and rotations. We will give them unit value because we don't know the real one (φ_1 = Z_1 = 1; φ_2 = Z_2 = 1; u = Z_3 = 1). Actually, what we search for? As we add links on the frame, they stop the displacements but as a result, it appears additional reactions on them.



Furthermore, we should add some information about rotational link. Let consider the consistency of the fixed support.



The fixed support includes two linear links and one rotational link. One should remember that the rotational link can exist separately (independently) as the linear links.

As a result when we put on the frame one rotational link we stop only the rotation of the joint and add only a moment as a reaction.

Return on the indeterminate frame and the displacement method. In the original frame there is no any link on the nodes and consequently any joint reactions. Logically if the two frames should be equivalent these new reactions should be zero. And this is the equilibrium equation from which we will find the values of the joint displacements and reactions.

$$R_1 = 0;\ R_2 = 0;\ R_3 = 0$$

$$\begin{aligned} R_1 &= R_{1f} + r_{11}.Z_1 + r_{12}.Z_2 + r_{13}.Z_3 = 0 \\ R_2 &= R_{2f} + r_{21}.Z_1 + r_{22}.Z_2 + r_{23}.Z_3 = 0 \\ R_3 &= R_{3f} + r_{31}.Z_1 + r_{32}.Z_2 + r_{33}.Z_3 = 0 \end{aligned}$$

where:

$r_{ij}.Z_j$ is the reaction on link ***i*** from the real value of the rotation (displacement) φ_j (u).

r_{ij} is is the reaction on link ***i*** from the unit value of the rotation(displacement) φ_j (u) = Z_j = 1.

R_{if} is the reaction on link ***i*** from the external load.

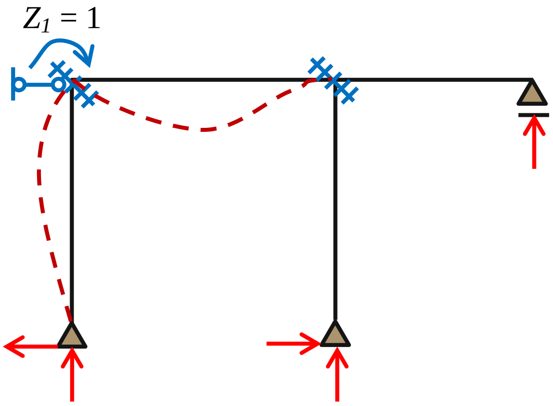
From the obtained equilibrium equations we will calculate the real values of the nodes rotations and displacements:

$$\left| \begin{array}{l} r_{11}.Z_1 + r_{12}.Z_2 + r_{13}.Z_3 + R_{1f} = 0 \\ r_{21}.Z_1 + r_{22}.Z_2 + r_{23}.Z_3 + R_{2f} = 0 \\ r_{31}.Z_1 + r_{32}.Z_2 + r_{33}.Z_3 + R_{3f} = 0 \end{array} \right.$$

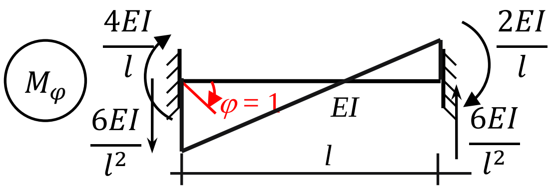
The question is how to compose internal moment diagrams from the unit displacements of the displacement and how to calculate the additional reactions at the additional links. Let see the situation $Z_1 = 1$ – the rotation of the left node of the frame.

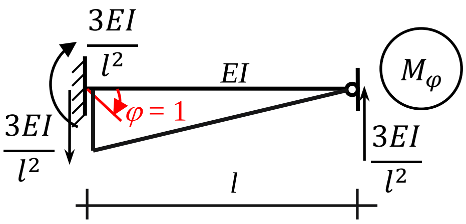
Composing of unit diagrams.

When we have added rotational and linear links then the nodes of the frames are fully fixed – there is no displacements and rotations. If we impose rotation of the left joint only connected to it members of the frame will deform. The other will be undeforming.

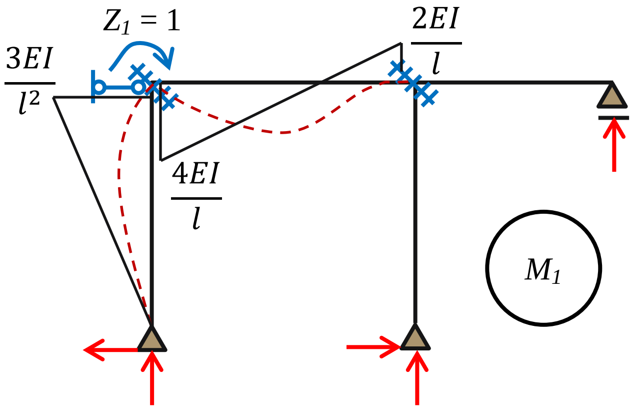


In addition, the two deformed members are independent to each other. In other words, the hall frame is separated to independent members. In this way, the problem to analyze frame from the rotation at the left node transforms to a problem to analyze the different members. This problem we already analyzed in previous chapter.

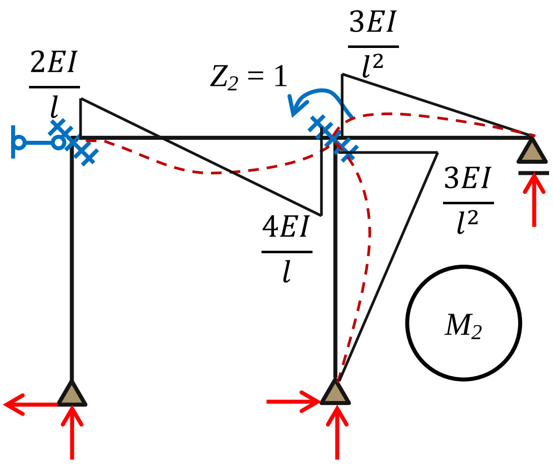




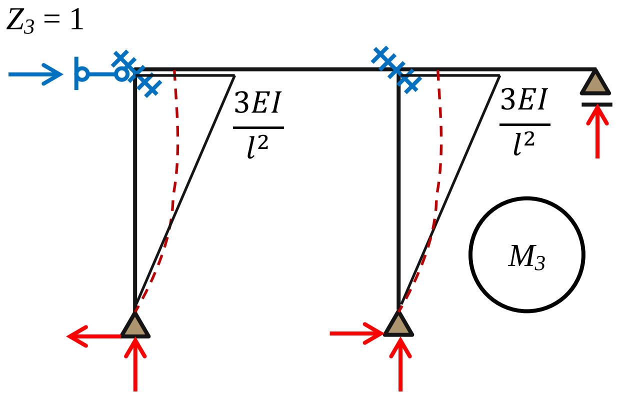
So the internal moment diagram we will compose as composing diagrams from the different parts. It is enough only to think carefully how to rotate the diagrams.



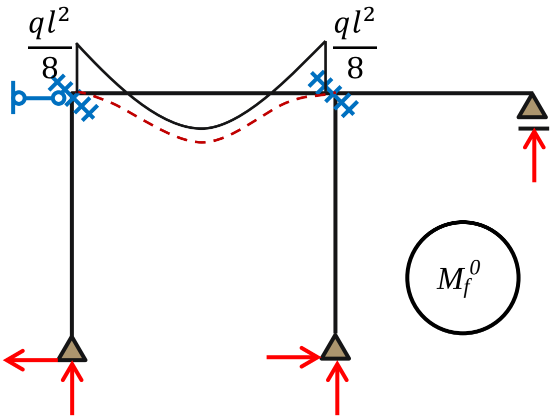
The best way to draw the diagrams correctly is to look at the deformed shapes of the different parts and to draw the diagrams on the tensile side of the member.
The internal moment diagrams from the unit rotation of the right node of the frame will be:



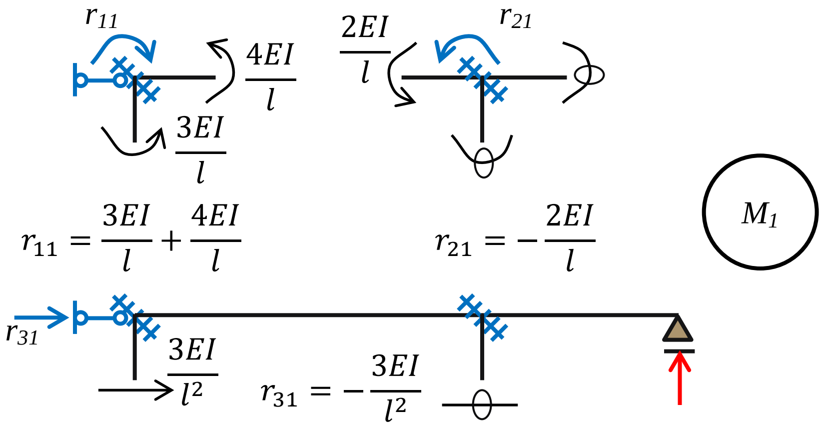
The internal moment diagrams from the unit displacement will be:



The internal moment diagrams from the external load will be:



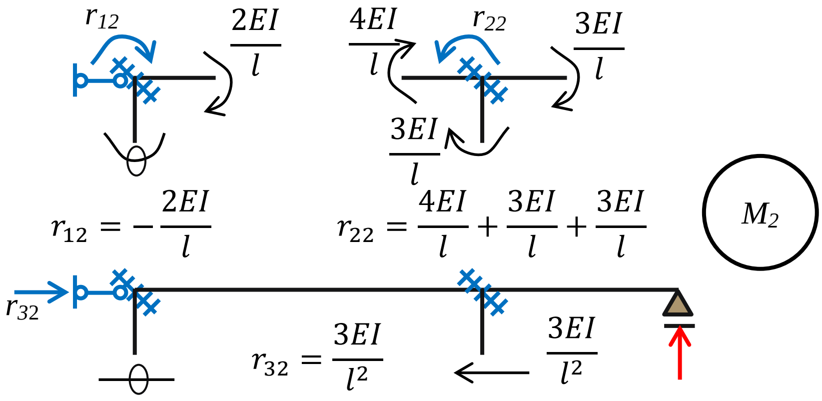
The reactions r_{ij} and R_{if} we may find very easy using joint equilibrium. It follows:



$$r_{11} = \frac{3EI}{l} + \frac{4EI}{l}$$

$$r_{21} = -\frac{2EI}{l}$$

$$r_{31} = -\frac{3EI}{l^2}$$

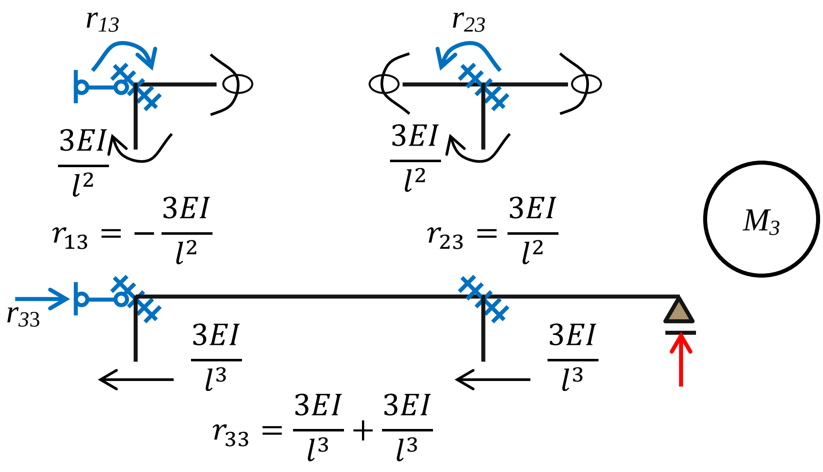


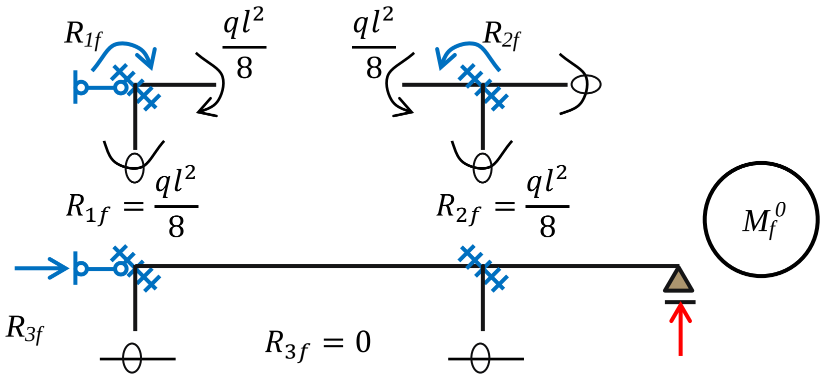
$$r_{12} = -\frac{2EI}{l}$$

$$r_{22} = \frac{4EI}{l} + \frac{3EI}{l} + \frac{3EI}{l}$$

$$r_{32} = \frac{3EI}{l^2}$$

It is important to understand that even parametric values of the moments are equal the numerical values are different because the cross sections, length and E-modulus can be different.





Coefficient verifications:

Again we have symmetry for the coefficients $r_{ij} = r_{ji}$ and positive matrix $r_{ii} > 0$. This can be used as verification. Other way to check the coefficient is as follows:

$$r_{ij} = \sum \int \frac{M_1 M_2}{\boldsymbol{EI}} \boldsymbol{ds}$$

$$R_{if} = -\sum \int \frac{M_i \overline{M}_f^o}{\boldsymbol{EI}} \boldsymbol{ds}$$

where $\overline{M}_f^o$ is the internal moment diagram from the external load at arbitrary statically determinate structure. This expressions are based on a principle of virtual work.

The final diagrams compose using next expression:

$$M_f = M_{f_0} + M_1.Z_1 + M_2.Z_2 + M_3.Z_3$$

As we already have the internal moment diagram we can obtain the shear and normal forces diagrams. And in the end to find reactions and as final check is the equilibrium of the system.

Presented idea is valid not only for two-times indeterminate structures but for *n*-times indeterminate also. The procedure is same. The equilibrium equations for *n*-times indeterminate structure are:

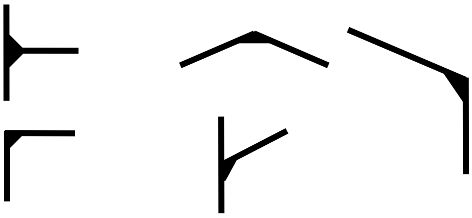
$$\left| \begin{array}{l} r_{11}.Z_1 + r_{12}.Z_2 + r_{13}.Z_3 + \cdots + r_{1n}.Z_n + R_{1f} = 0 \\ r_{21}.Z_1 + r_{22}.Z_2 + r_{23}.Z_3 + \cdots + r_{2n}.Z_n + R_{2f} = 0 \\ r_{31}.Z_1 + r_{32}.Z_2 + r_{33}.Z_3 + \cdots + r_{3n}.Z_n + R_{3f} = 0 \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ r_{n1}.Z_1 + r_{n2}.Z_2 + r_{n3}.Z_3 + \cdots + r_{nn}.Z_n + R_{nf} = 0 \end{array} \right.$$

Primary system and full example.

As a main difference from the force method is that in the displacement, method is only one possible and correct primary system. The primary system in the displacement method develops on two stages. First is adding rotational links and second is adding linear links. Here will be explained the two stages of composing primary system.

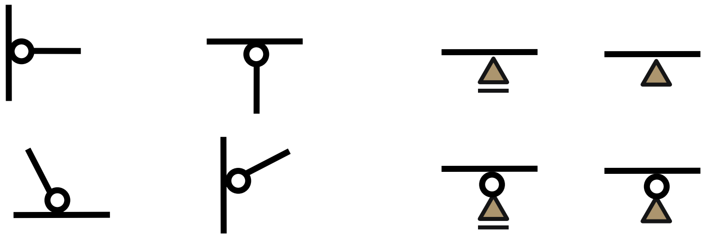
Stage I: Rotational links.

We place rotational links at every **rigid joint** of the frame! What is a rigid connection have been explained at the first chapter. Here we will remember only the idealized schemes of them.



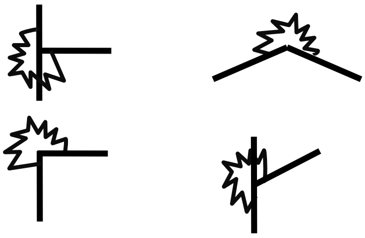
Rigid connections. Rigid joints of frames.

Other places where we place rotational links are the **partial hinges** and **partial supports**.



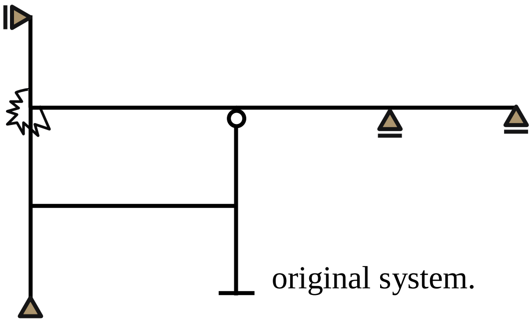
Partial hinges, partial supports.

Final places where we place rotational links are the **rotational springs**.

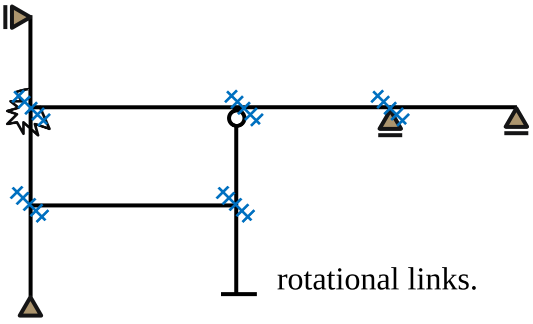


rotational springs.

Example:



original system.

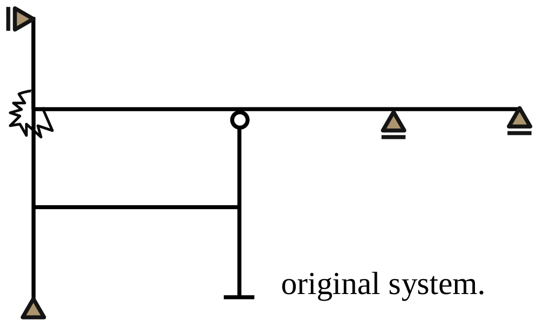


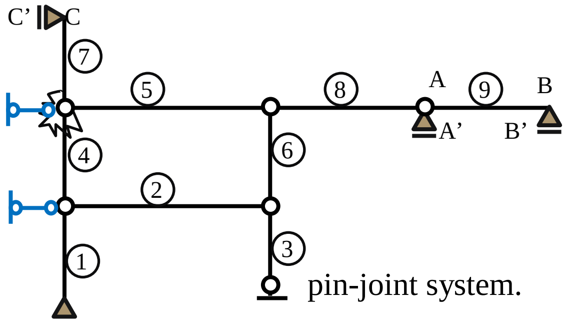
rotational links.

Stage II: Linear links.

The second stage of composing primary system is to determine the number and places of the linear links. For this reason we compose the hinged-joint system placing at each joint of the structure a hinge. We do the kineatical analysis of the hinged-joint system. It should be composed by dyads. If it is not we add linear links to make it composed by dyads (composed by not singular dyads). Adding this links actually we determine their places and numbers.

Example:


original system.

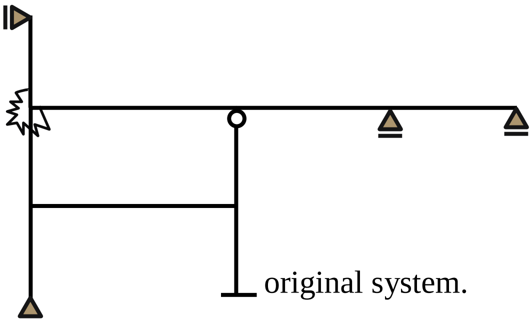


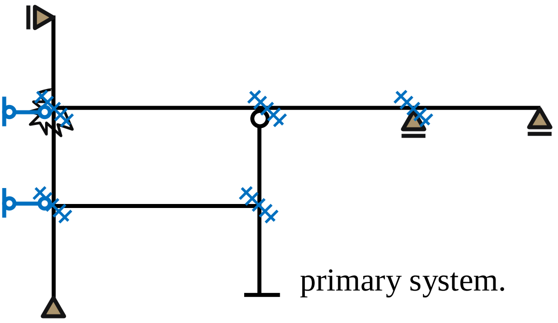
pin-joint system.

$$KA: [G + 1.2.3(w = +1) + 4.5.6(w = +1) + 8.AA' + 9.BB' + 7.CC'](w = +2)$$

In the present pin-joint system there are 2 chains. To remove their degrees of freedom we should add two linear links. The first chain is 1.2.3 and has a horizontal degree of freedom so, we will add a horizontal linear link. Situation with the second chain is same so, as a result for this system we will have two horizontal links.

Including the rotational and linear links to the original system we obtain the primary system on the displacement method. The number of these links is the number of the kinematical indeterminacy of the system.


original system.

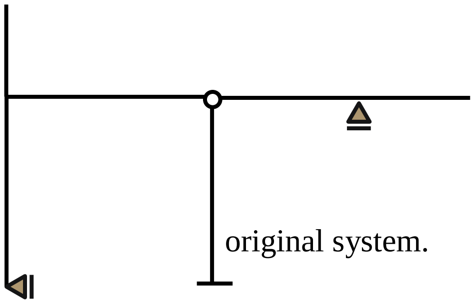

primary system.

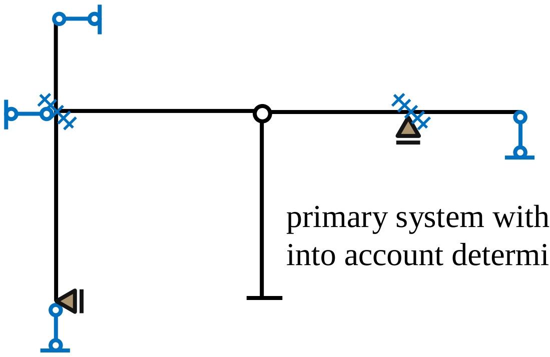
The present example is 7 times cinematically indeterminate system.

System including determinate parts.

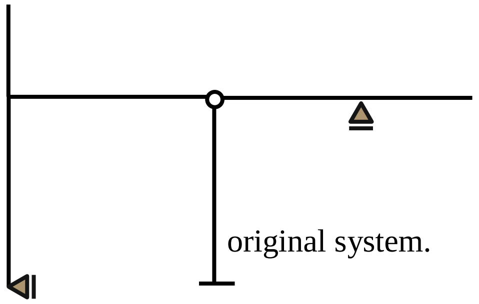
When in the original system includes determinate parts it is better to cut them off because it decrease the degree of the kinematical indeterminacy. Also, it is good to know that some time the linear links are vertical not only horizontal.

Example:

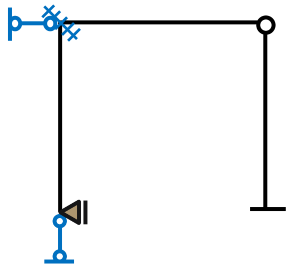

original system.


primary system without taking into account determinate parts.

This system includes two determinate parts. First one is the cantilever part up of the frame and the second one is the simple beam on right of the frame. When we taking into account this parts the system is 6 times cinematically indeterminate. When we cut them off the system is 2 times cinematically indeterminate.

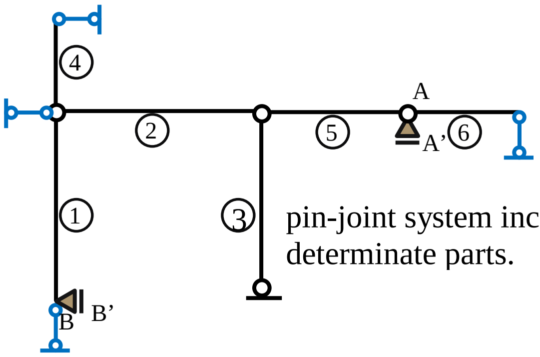


original system.

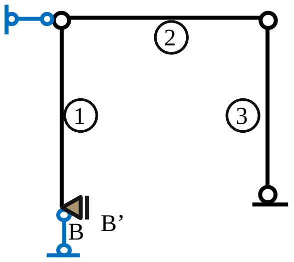


primary system with taking into account determinate parts.

On the next figure is shown the pin-jont system of the two situations and their kinematical analysis.



pin-joint system including determinate parts.



pin-joint system excluding determinate parts.

For the pin-joint system including determinate parts:

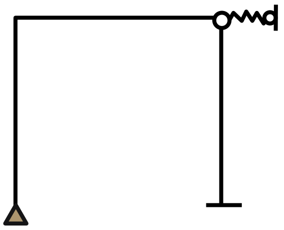
$$KA: [G + 1.2.3. BB'(w = +2) + 4(w = +1) + 5. AA' + 6(w = +1)](w = +4)$$

For the pin-joint system excluding determinate parts:

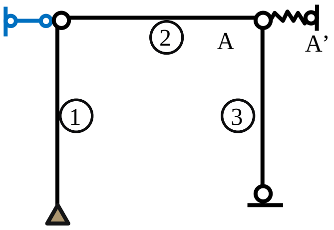
$$KA: [G + 1.2.3. BB'(w = +2)](w = +2)$$

System including linear springs.

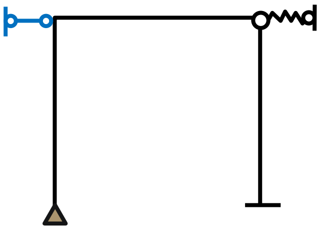
When in the original system includes linear springs it is necessary to place a linear link at the spring because there is a displacement even the system don't need linear links according the pin-joint system analysis.



original system including linear spring.



pin-joint system including linear spring.



primary system including linear spring.

Kinematical analysis of the pin-joint system including linear spring:

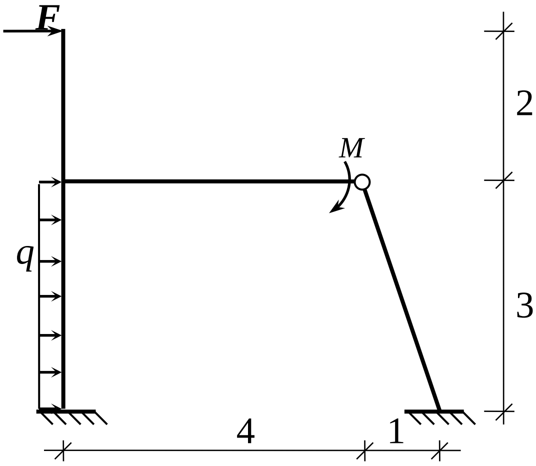
$$KA: [G + 3.AA' + 1.2](w = 0)$$

Full example of analysis of cinematically indeterminate frame:

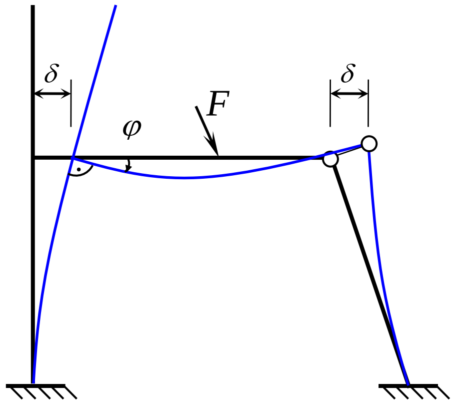
Analysis of cinematically indeterminate frame loaded by external load:

Main Data:

F = 20 kN	***beams (0.25/0.315):***	***columns (0.25/0.25):***
M = 15 kNm	$I_b = 0.000651\ m^4$	$I_{col} = 0.0003255\ m^4$
q = 10 kN/m'	$A_b = 0.0788\ m^2$	$A_{col} = 0.0625\ m^2$
	$\boldsymbol{I_b = 2.I_{col.}}$	$E = 2.4.10^7\ kN/m^2$



Kinematical indeterminacy:



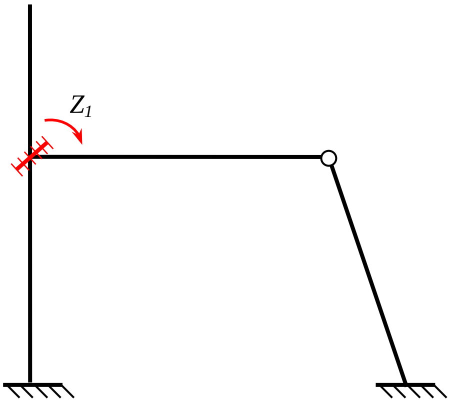
Each construction has a deformations caused by some arbitrarily load. In this case we know the joint slope (rotation) φ and displacement δ, so the deformation of the system will be completely determined because the law of the deflection of the elements is known. That is why the system is 2 times kinematical indeterminate.

1. **Choice of primary system according displacement method:**

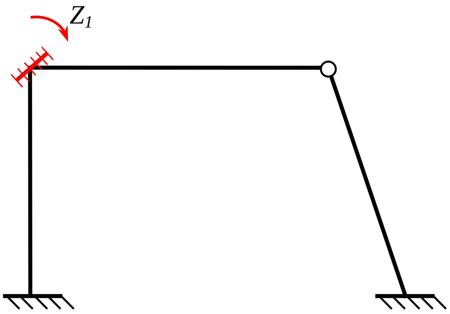
It is very important to remember that in the force method are possible several correct primary systems, but in displacement the correct primary system is **only one**.

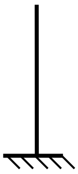
The primary system will be composed in two stages.

First stage: To place rotational links. We place rotational links at the rigid joint. We use them to control the joint rotation (slope).



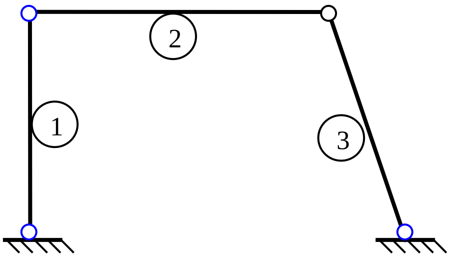
The unknown parameter in this case the rotation (slope) of the joint and it’s label is Z_1. The cantilever part of the system is statically determinate part and may be removed.



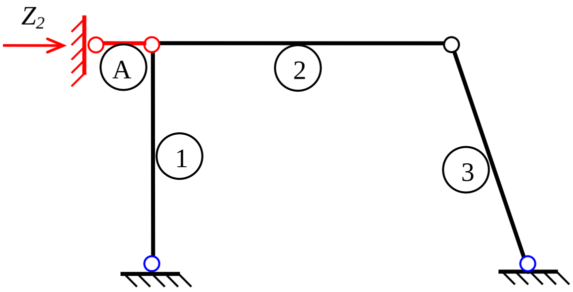


determinate
part(piece)

The second stage: The second stage of composing primary system is to determine the number and places of the linear links. For this reason we compose the hinged-joint system placing hinges at each joint of the structure.

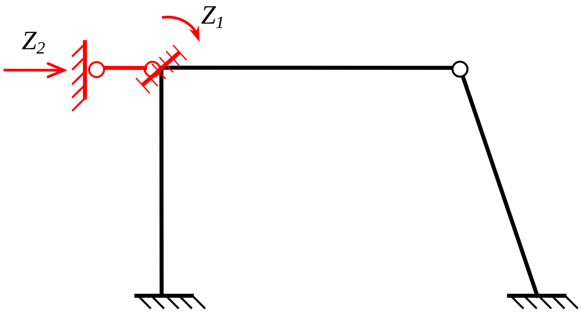


KA: [G+1.2.3(w=+1)](w=+1)



KA: [G+1.A +2.3](w=0)

After that we make a kinematical analysis of the hinged-joint system. It should be composed by dyads. If it is not we add linear links to make it composed by dyads (composed by not singular dyads).
As we determine the number and positions of the linear links we obtain the primary system:



The original system is two times cinematically indeterminate. We analyze the primary system from unit values of the unknowns – displacements and rotations of points.

2. **Unite diagrams:**

For to make calculations easier we will make some reductions. As we know the value of is:

$$i = \frac{EI}{l}$$

If we divide this equation by EI_{column} (or EI_{beam}) the linear stiffness will become:

$$\frac{1}{EI_c} i_i = \frac{EI_i}{l_i} \frac{1}{EI_c} = \frac{1}{l_i \frac{I_c}{I_i}}; \qquad l'_i = l_i \frac{I_c}{I_i}; \qquad \frac{1}{EI_c} i_i = \frac{1}{l'_i}$$

where EI_c is the stiffness of the column(EI_{column}) or of the beam (EI_{beam}) and EI_i is the stiffness of the *i*-th member of the frame.
i_i and l_i are the linear stiffness and the length of the *i*-th member of the frame.
After this transformation all stiffness coefficients r_{ij} (reactions from the unit displacement or rotations) will be reduced by $1/EI_c$:

$$\frac{1}{EI_c} r_{ij}$$

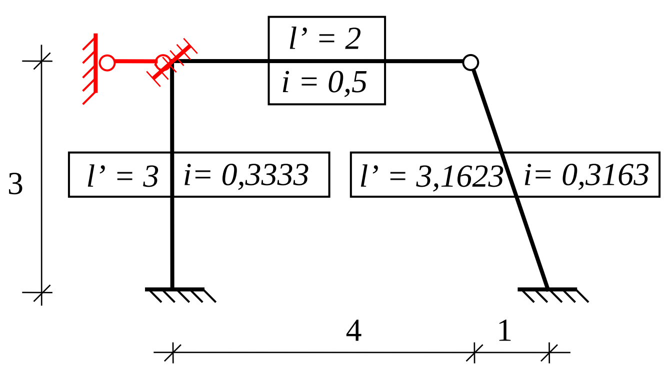
The coefficients from the external load are real R_{if}. In this way for the equilibrium equations in matrix form we will have:

$$\frac{1}{EI_c}[r]\{Z\} + \{R\} = 0$$

$$\{Z\} = -\frac{\{R\}}{\frac{1}{EI_c}[r]} = -\frac{\{R\}}{[r]}(EI_c)$$

As a result, we obtained not real values of the unknowns but multiplied by EI_c. If we need their real values, we should divide the obtained value by EI_c. All the time we have not real values of the stiffness coefficients but their $1/EI_c$ reduction values.

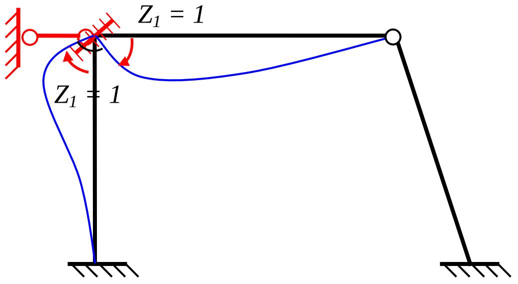
This transformation is not obligate but is better for to work with suitable numbers. The present example is done using this idea.

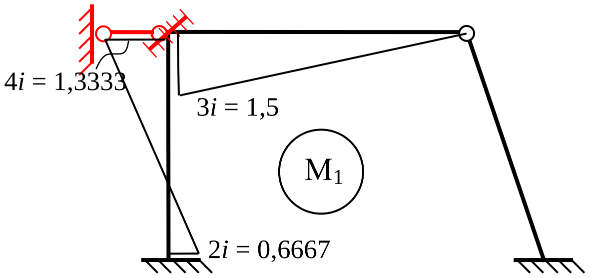


$I_c = I_{col.}$
$I_c / I_{b.} = 0{,}5$

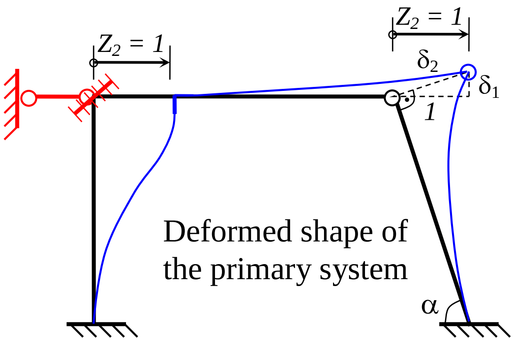
$l' = l \frac{I_c}{I}$
$i = \frac{1}{l'}$

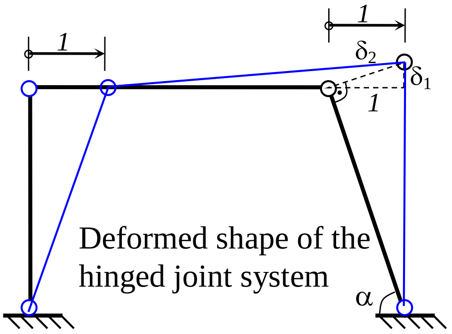
Deformed shapes from the unit rotation of the joint:

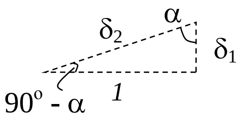




Deformed shape from unit horizontal displacement. First, we obtain the displaced shape of the hinged-joint system after that for the primary system and in the end, we obtain the diagram.



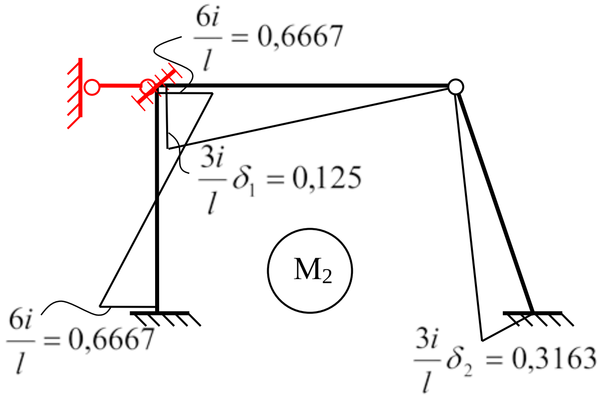




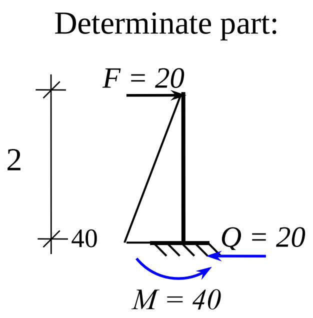
$tg\alpha = \frac{3}{1} = 3 \quad \rightarrow \quad \alpha = 71{,}565^{\circ}$

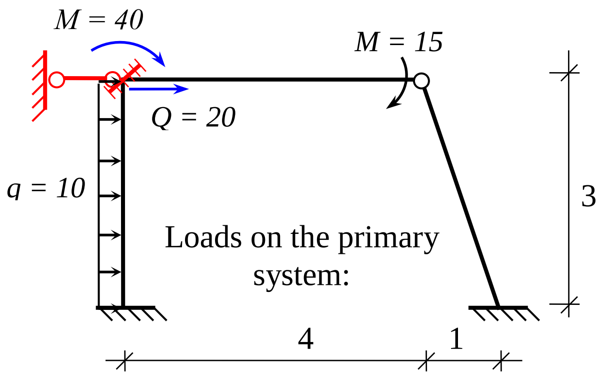
$\delta_1 = \frac{1}{tg\alpha} \qquad \delta_2 = \frac{1}{sin\alpha}$

$\delta_1 = 0{,}3333 \qquad \delta_2 = 1{,}0541$

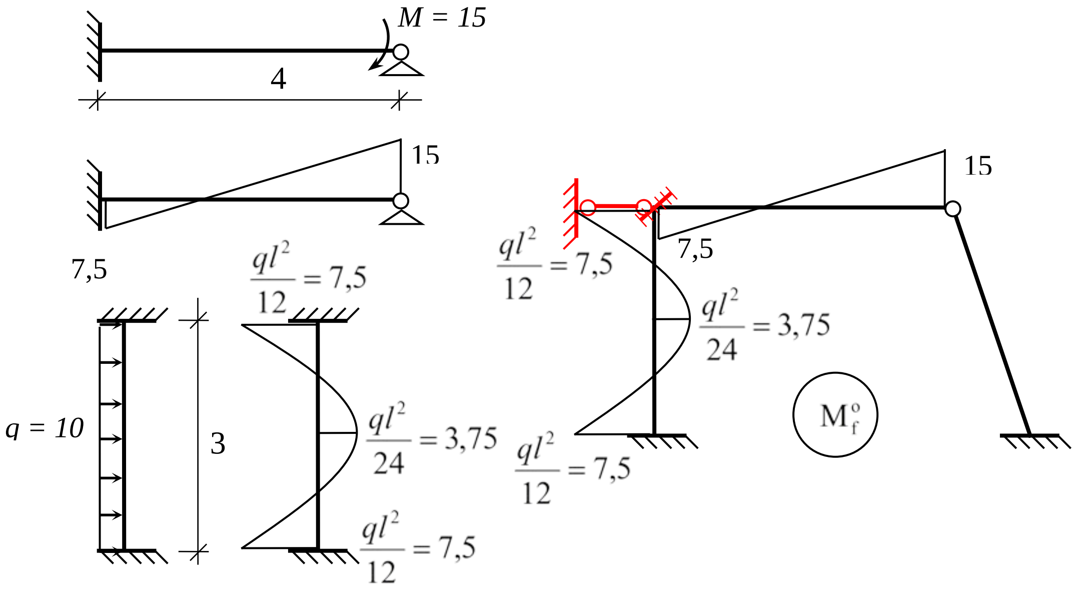


3. Diagram from the external load in the primary system:





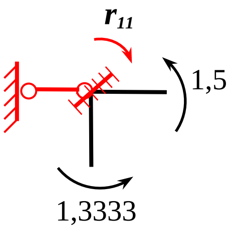
In advance, we analyze the determinate part of the structure and its reactions are used as actions of the primary system. In this situation, the actions are in the joint with a linear and a rotational link, so they do not cause any internally moment forces in the elements.



4. Reaction in the links:

The reactive forces are introduced by the same sing as the unknowns. After that the signs become positive or negative because of equilibrium equation.

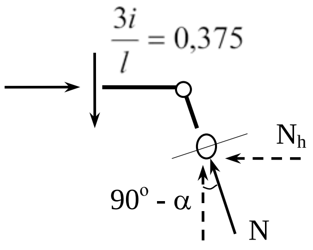
From $\mathbf{M_1}$



$\sum M = 0:$

$1{,}5 + 1{,}3333 - r_{11} = 0$

$r_{11} = 2{,}8333$

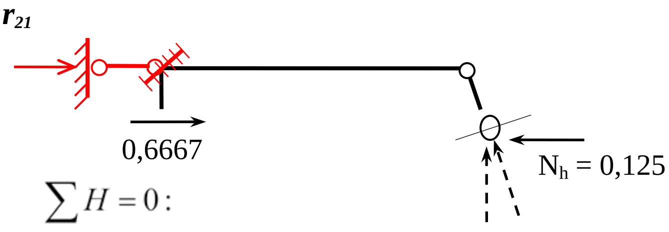


$\sum V = 0:$

$N\sin\alpha - 0{,}375 = 0$

$N = 0{,}3953$

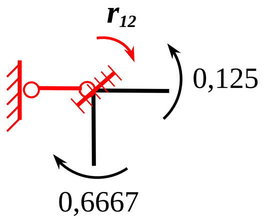
$N_h = N\cos\alpha = 0{,}125$



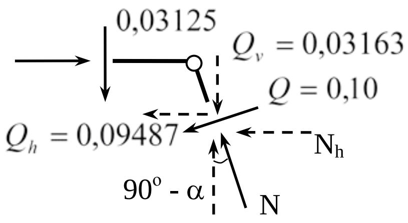
$\sum H = 0:$

$0{,}6667 - 0{,}125 + r_{21} = 0 \quad \rightarrow \quad r_{21} = -0{,}54167$

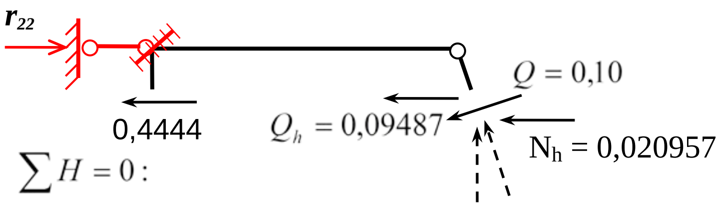
From $\mathbf{M_2}$



$$\sum M = 0:$$
$$r_{12} - 0{,}125 + 0{,}6667 = 0$$
$$r_{12} = -0{,}54167$$

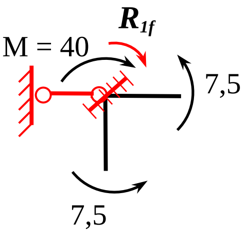


$$\sum V = 0:$$
$$0{,}03125 + 0{,}03163 - N \sin\alpha = 0$$
$$N = 0{,}066274$$
$$N_h = N\cos\alpha = 0{,}020957$$

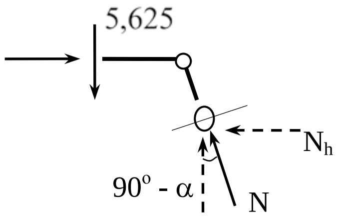


$$\sum H = 0:$$
$$r_{22} - 0{,}4444 - 0{,}09487 - 0{,}020957 = 0$$
$$r_{22} = 0{,}56027$$

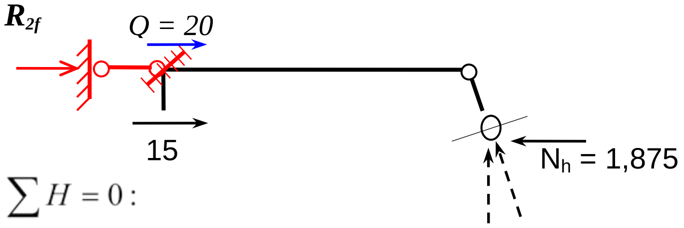
From M_f^o



$$\sum M = 0:$$
$$R_{1f} - 7{,}5 - 7{,}5 + 40 = 0$$
$$R_{1f} = -25$$



$$\sum V = 0:$$
$$5{,}625 - N\sin\alpha = 0$$
$$N = 5{,}9293$$
$$N_h = N\cos\alpha = 1{,}875$$



$$\sum H = 0:$$
$$R_{2f} + 15 - 1{,}875 + 20 = 0 \rightarrow R_{2f} = -33{,}125$$

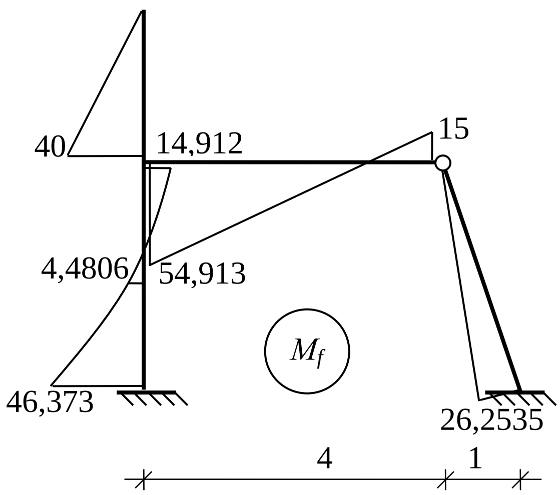
R_{if} – is the reactions of the additional links from the external load.

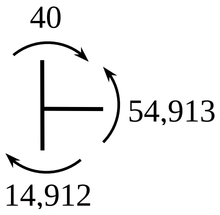
4. Solution of the system of the equations:

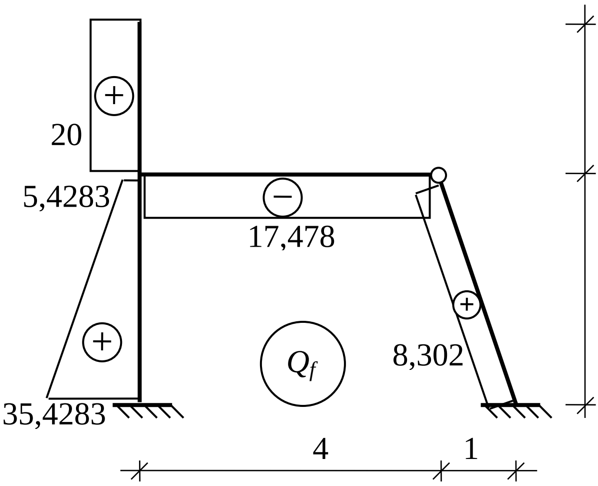
$$\left| \begin{array}{lll} 2{,}8333Z_1 - 0{,}54167Z_2 - 25 = 0 & \rightarrow & Z_1 = 24{,}692 \\ -0{,}54167Z_1 + 0{,}56022Z_2 - 33{,}125 = 0 & \rightarrow & Z_2 = 83{,}003 \end{array} \right.$$

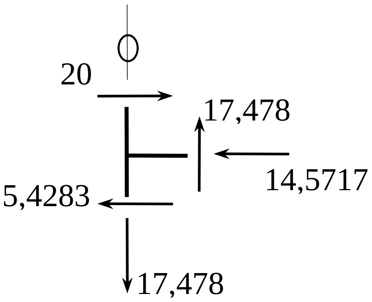
5. Final diagrams:

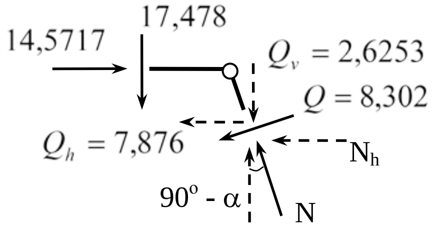
The final diagram is composed for each characteristic point and it is calculated by the next expression: $M_f = M_f^o + M_1.Z_1 + M_2.Z_2$











$$\sum V = 0:$$

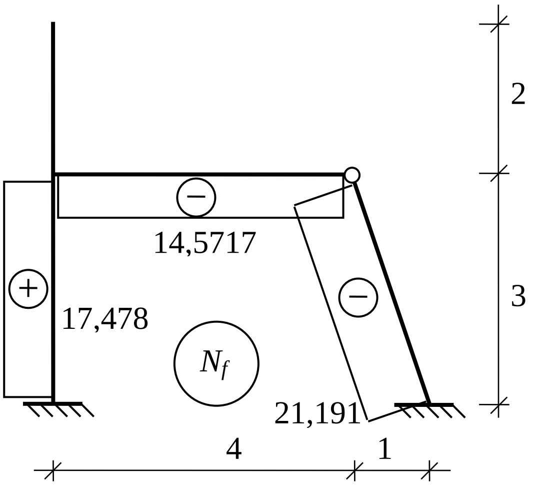
$$17,478 + 2,6253 - N\sin\alpha = 0$$

$$N = 21,191$$

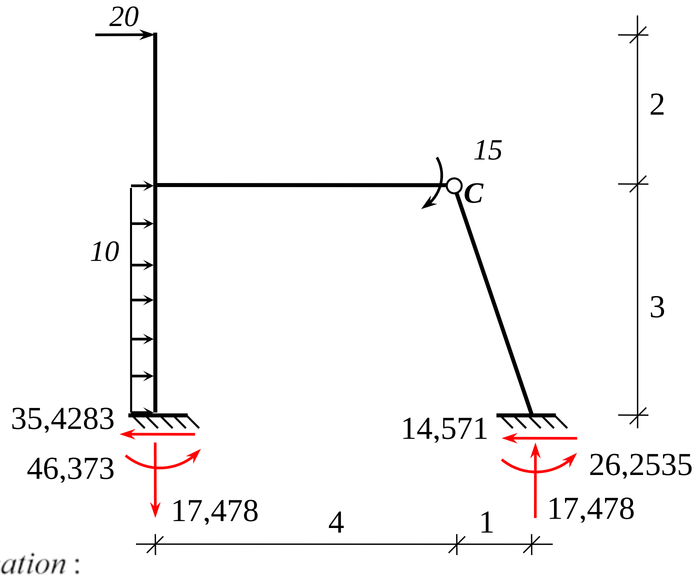
$$N_h = N\cos\alpha = 6,701$$

Проверка: $\sum H \overset{?}{=} 0$: $14,5717 - 7,876 - 6,701 \overset{?}{=} 0$

$14,5717 - 14,577 = 0 \quad \Rightarrow$ 0,037% *грешка*



6. Equilibrium verification:



Verification :

$$\sum M_c \overset{?}{=} 0:$$

$$15+20.2+35{,}4283.3+14{,}571.3-17{,}478.4-46{,}373-26{,}2535-17{,}478.1-10.3.1{,}5\overset{?}{=}0$$

$$204{,}998-205{,}02=0 \quad \Rightarrow \qquad 0{,}01\%\ \textit{error}$$

$$\sum H \overset{?}{=} 0: \qquad 20+10.3-35{,}4283-14{,}571\overset{?}{=}0: \qquad 50-49{,}9993=0$$

$$\sum V \overset{?}{=} 0: \qquad 17{,}478-17{,}478=0$$

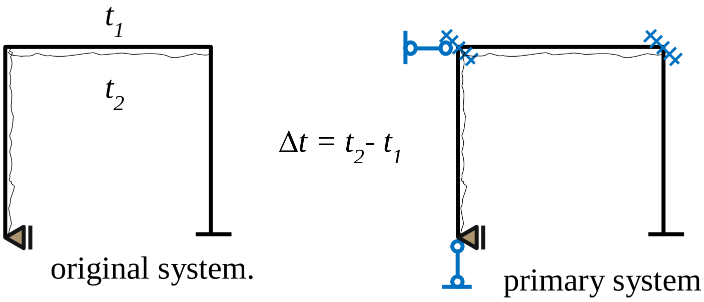
Analysis from temperature load, support settlement.

The analysis of the structures from the temperature load has two stages. We should remember from Chapter 8 the influence of the angular and axial deformations. Angular deformation cause only internal moments and axial deformations cause only normal forces. In the case of cinematically indeterminate structures under temperature load, we can distinguish two stages of deformations. As it is known the temperature load divides to a load from temperature difference and load from constant load.
Finally the using displacement method we divide the analysis of a primary system to a two parts and it follows the internal moment diagram in the primary system has two parts.

$$M_t^o = M_{t,\Delta t} + M_{t,tc}$$

Internal moment diagrams from temperature difference.

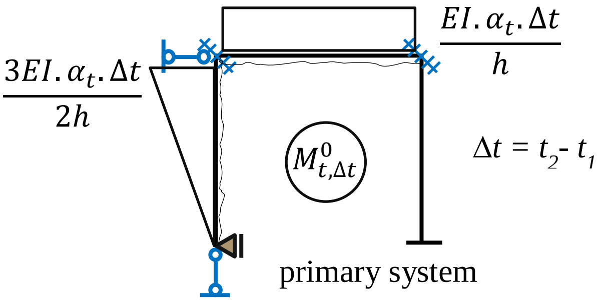
The nodes of the primary system have no possibility to move or rotate so the temperature difference will cause only deformations of the elements.



original system.

primary system

As a result, there is diagram only at the loaded elements. The moment values for every member is obtained using force method as the solutions from the unit displacements. The result is:



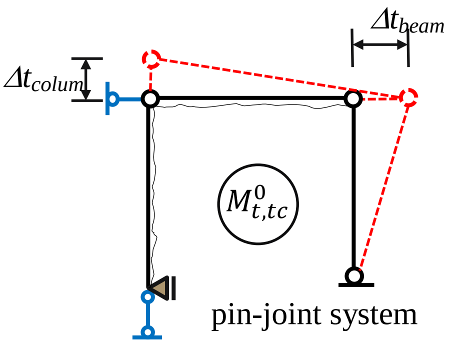
primary system

Internal moment diagrams from constant temperature.

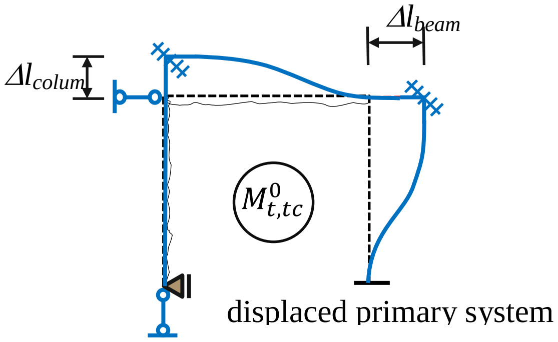
The constant temperature cause only linear elongation. There is no deflections so it cannot be a reason of appearing moment diagram but it is a reason of nodal displacement. Element elongation is:

$$\Delta l = \alpha_t.t_c.l$$
$$t_c = \frac{t_2 + t_1}{2}$$

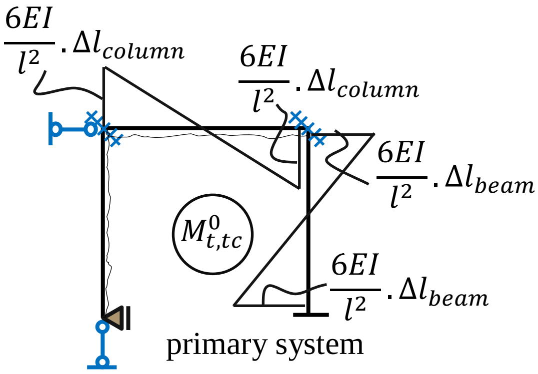
As we know the elongations of the elements we may compose the deformed shape of the system but first we will compose the displaced shape of the pin-joint system. When compose the displaced pin-joint system we should be careful for the linear links. They stops the displacements at some direction and the displacements is possible only at one directions.



After we have the displacement pin-joint system we know the new places of the frame nodes and we may compose the deformed shape of the primary system.



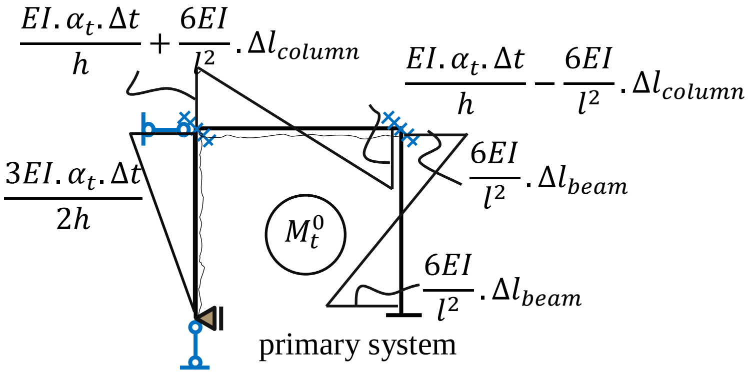
One can see that the elongation by itself don't cause internal moments but the displacements of the joints at the primary system cause the deflections of the elements. Actually these deflections cause internal moments in the primary system. For the present example the elongation of the left column causes deflections at the beam and the elongation of the beam causes deflections at the right column. As a result internal moments diagram will appears at the beam element and at the right column. The values of the moments are as moments from unit displacement of one of the joints. The difference is that the displacement is not unit but equal to Δl_{beam} (or Δl_{column}). The result is:



As we already have the two parts of the internal moments we can compose full moment diagram in the primary system from the temperature load.

$$M_t^0 = M_{t,\Delta t}^0 + M_{t,tc}^0$$

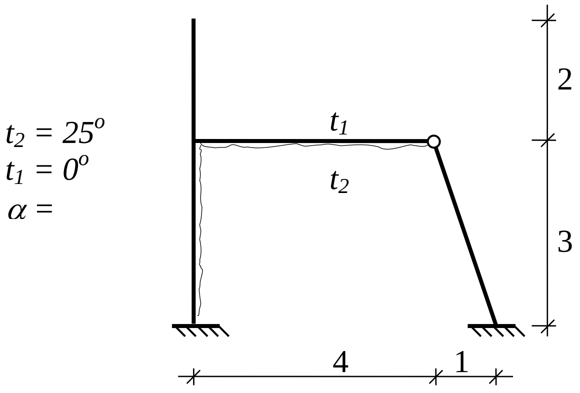
For the present example this diagram is as follows:



Once we have this diagram we may calculate the reactions on the added links from the exteran temperature load R_{it} and to compose the equilibrium equations to calculate unknowns Z_{it}.

Next is presented a numerical example.

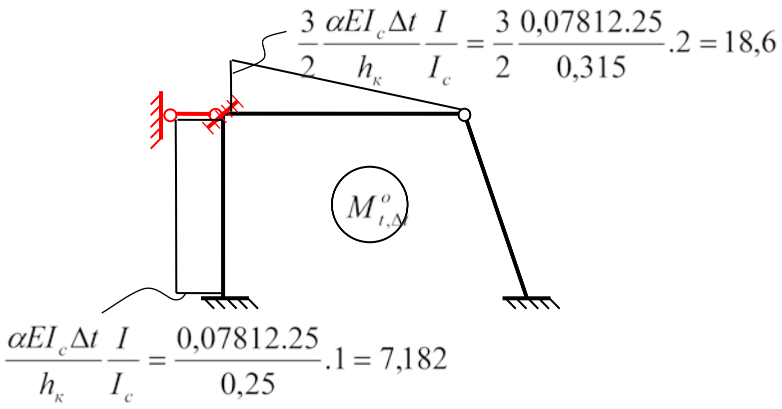
Example:



The diagram from temperature load in the primary system is composed in two stages.
First stage – In this stage is composed the moment diagram in each member separately from the temperature difference Δt.

$$EI_c = 7812 \rightarrow \alpha EI_c = 0,07812; \quad t_{cp} = \frac{t_2+t_1}{2} = \frac{25+0}{2} = 12,5^\circ; \quad \Delta t = t_2 - t_1 = 25^\circ$$

1. **Diagram in the primary system:**



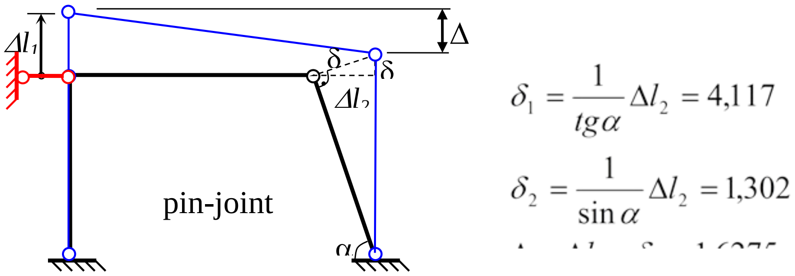
Second stage: Diagram from the constant temperature.

The elements elongations are:

$$\Delta l_1 = \alpha EI_c t_{cp} .l_1 = 0{,}07812.12{,}5.3 = 2{,}9295$$

$$\Delta l_2 = \alpha EI_c t_{cp} .l_2 = 0{,}07812.12{,}5.4 = 3{,}906$$

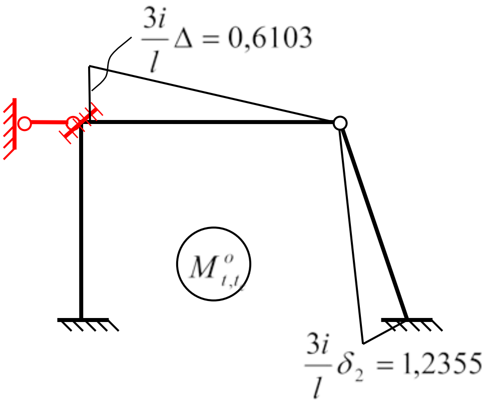
The displaced pin-jointed system is:



$$\delta_1 = \frac{1}{tg\alpha}\Delta l_2 = 4{,}117$$

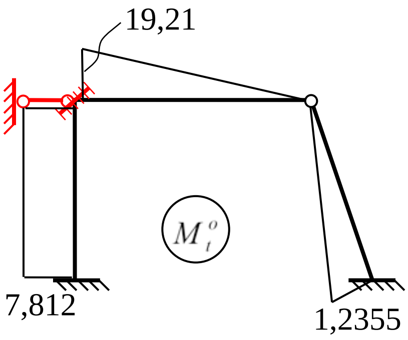
$$\delta_2 = \frac{1}{\sin\alpha}\Delta l_2 = 1{,}302$$

Diagram caused by the displaced frame nodes is:



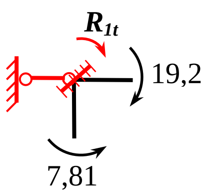
Follows full diagram from the temperature load at the primary system:

$$M_t^0 = M_{t,\Delta t}^0 + M_{t,tc}^0$$

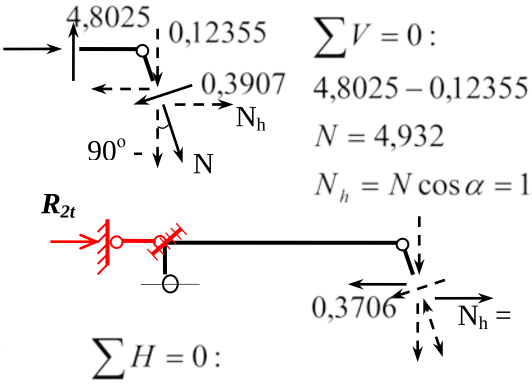


2. Computing reactions at the added links:

From M_t^o



$$\sum M = 0:$$
$$R_{1t} - 7{,}812 + 19{,}21 = 0$$
$$R_{1t} = -11{,}398$$

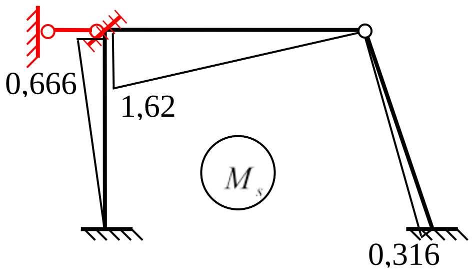


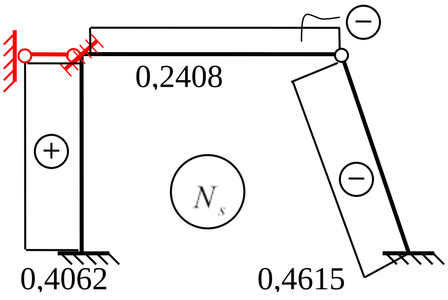
$$\sum V = 0:$$
$$4{,}8025 - 0{,}12355 - N\sin\alpha = 0$$
$$N = 4{,}932$$
$$N_h = N\cos\alpha = 1{,}5596$$

$$\sum H = 0:$$
$$R_{2t} + 1{,}5596 - 0{,}3706 = 0 \rightarrow R_{2t} = -1{,}189$$

Verification:

$$\sum R_{it} = -EJ_c \sum \int M_s \frac{\alpha \Delta t}{h} ds + EJ_c \sum \int N_s \alpha . t_{cp} ds = -\alpha EJ_c \left[\sum \frac{\Delta t}{h} \int M_s ds + \sum t_{cp} \int N_s ds \right]$$





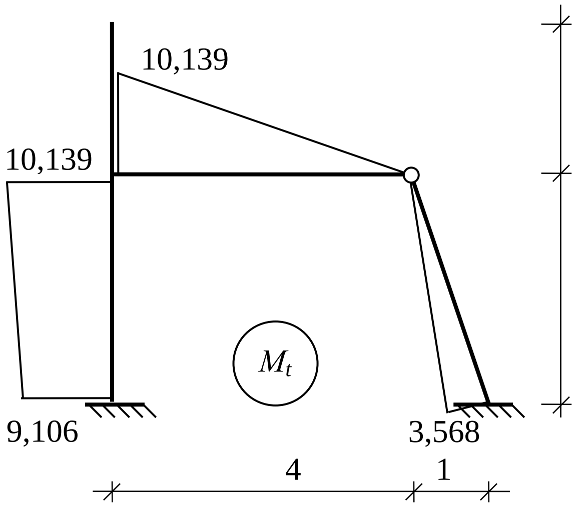
$$-\alpha EJ_c \left[\sum \frac{\Delta t}{h} \int M_s ds + \sum t_{cp} \int N_s ds \right] =$$
$$= -0{,}07812. \left[\frac{25}{0{,}25} \left(-\frac{0{,}66667.3}{2} \right) + \frac{25}{0{,}315} \frac{1{,}625.4}{2} + 12{,}5.0{,}40625.3 - 12{,}5.0{,}240827.4 \right] =$$
$$= -0{,}07812. [-100 + 257{,}94 + 15{,}234 - 12{,}041] = -0{,}07812.161{,}133 = -12{,}587$$
$$\sum R_{it} = -12{,}587 \quad \rightarrow \quad -12{,}587 \overset{!}{=} -12{,}587$$

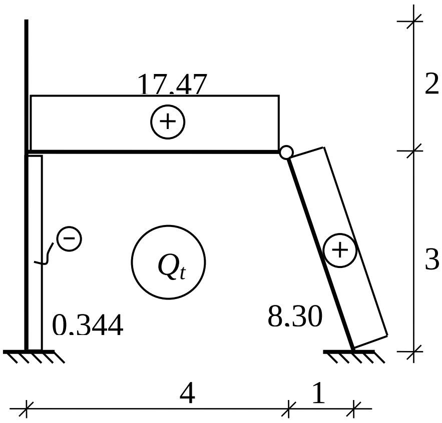
3. Equilibrium equations.

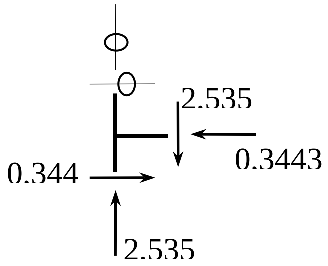
$$\left| \begin{array}{ll} 2{,}8333Z_1 - 0{,}54167Z_2 - 11{,}398 = 0 \quad \rightarrow & Z_1 = 5{,}433 \\ -0{,}54167Z_1 + 0{,}56022Z_2 - 1{,}189 = 0 \rightarrow & Z_2 = 7{,}375 \end{array} \right.$$

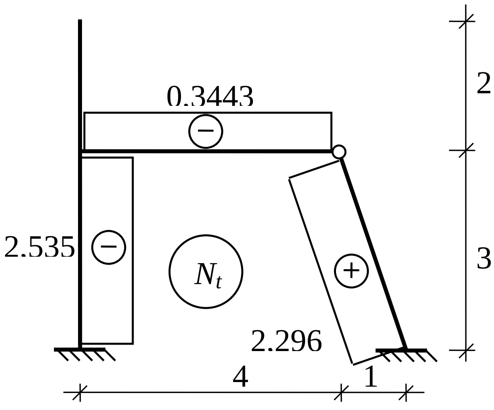
1. Final diagram.

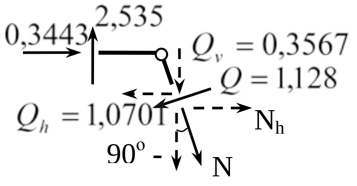
$$M_t = M_t + M_1.Z_1 + M_2.Z_2$$











$\sum V = 0:$

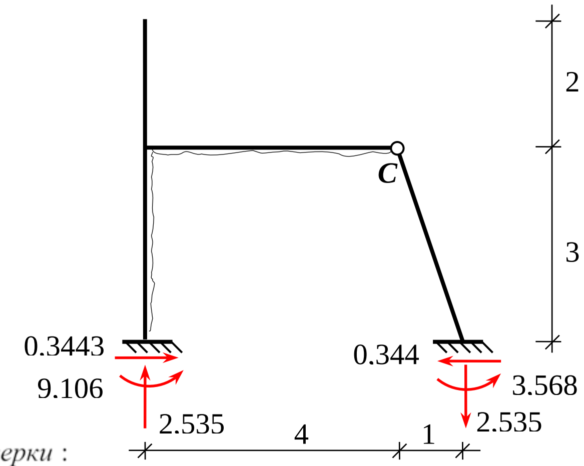
$0,3567 - 2,535 + N \sin\alpha = 0$

$N = 2,296$

$N_h = N \cos\alpha = 0,726$

Проверка: $\sum H \overset{?}{=} 0$: $0,3443 + 0,726 - 1,0701 \overset{?}{=} 0$

$1,0703 - 1,0701 = 0 \quad \Rightarrow \quad$ 0,019% *грешка*



Проверки :

$$\sum M_c \overset{?}{=} 0: \quad 9{,}106 + 3{,}568 + 0{,}3443.3 - 0{,}3443.3 - 2{,}535.4 - 2{,}535.1 \overset{?}{=} 0$$

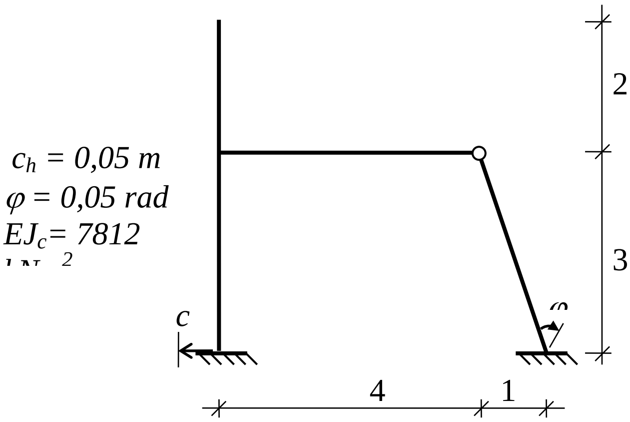
$$12{,}674 - 12{,}675 = 0 \quad \Rightarrow \quad 0{,}008\%\textit{грешка}$$

$$\sum H \overset{?}{=} 0: \quad 0{,}3443 - 0{,}3443 = 0; \qquad \sum V \overset{?}{=} 0: \quad 2{,}535 - 2{,}535 = 0$$

Analysis from the support settlement.

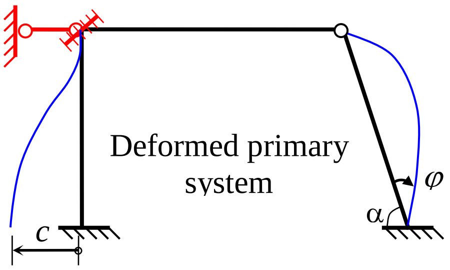
This analysis is not so different from the standard one. The support settlement cause of the primary system node displacements which cause element displacements. This deflections cause internal moments at the deflected elements. The idea is the same as the idea of the diagrams from the constant temperature load. Actually, we should be careful with the displaced pin-joint system (if it is necessary) and as a result the deflected primary system. The last one will show as which elements have displaced nodes and with what value so we will be able to compose the internal moment diagram. Of course one shouldn't forgot that the displacements causing moments diagrams in the elements are not equal to unit but are equal to the support settlements.

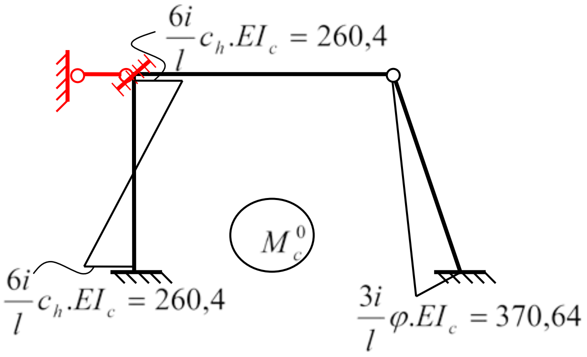
Here will be present only the numerical example:



1. Deformed primary system:

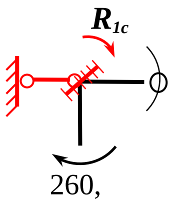
The present example is calculated with a reductions of the unit reactions that is why moment diagram at the primary system from the support settlement is multiplied by EI_c.





2. Computing reactions at the added links:

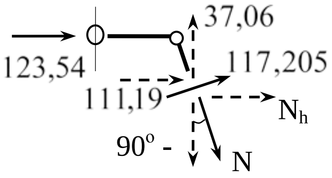
From



$\sum M = 0:$

$R_{1c} + 260,4 = 0$

$R_{1c} = -260,4$

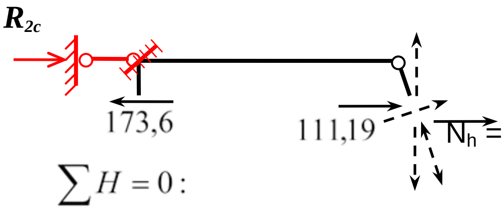


$\sum V = 0:$

$37,06 - N\sin\alpha = 0$

$N = 39,065$

$N_h = N\cos\alpha = 12,35$



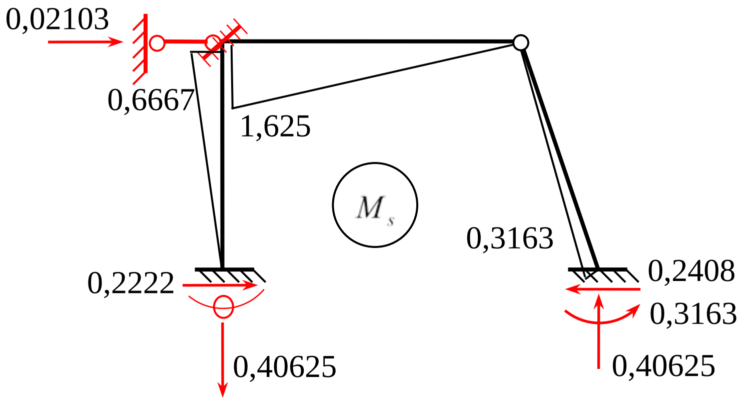
$\sum H = 0:$

$R_{2c} + 111,19 + 12,35 - 173,6 = 0 \rightarrow R_{2c} = 50,06$

Verification:

$$\sum R_{ic} = +EJ_c \sum R_s.c_i$$

To do this verification it is need to determine the support reactions in the summary diagram from unit values of the unknowns.



$$\sum R_{ic} = -260{,}4 + 50{,}06 = -210{,}34$$

The corresponding reactions of this diagram are multiplied to the values of the support settlements:

$$EJ_c \sum R_i.c_i = 7812[0{,}2222.(-0{,}05) + 0{,}3163.(-0{,}05)] = -210{,}338$$

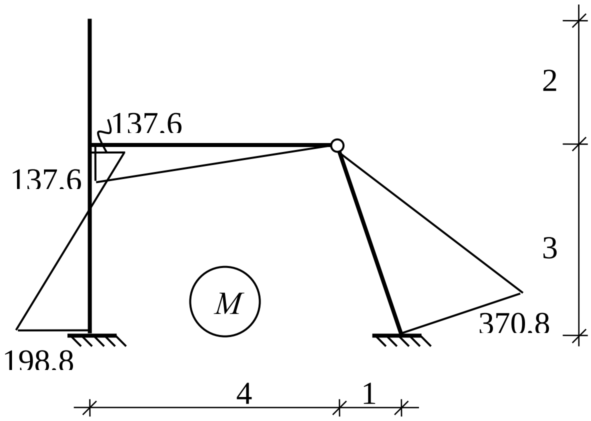
$$\sum R_{ic} = +EJ_c \sum R_i.c_i \quad \rightarrow \quad -210{,}34 \overset{!}{=} -210{,}338$$

3. **Equillibrium equations:**

$$\left| \begin{array}{ll} 2{,}8333Z_1 - 0{,}54167Z_2 - 260{,}4 = 0 \quad \rightarrow & Z_1 = 91{,}79 \\ -0{,}54167Z_1 + 0{,}56022Z_2 + 50{,}06 = 0 \rightarrow & Z_2 = -0{,}606 \end{array} \right.$$

4. Final diagrams from the support settlements:

$$M_t = M_1.Z_1 + M_2.Z_2 + M_c^o$$



Finale verifications , normal and shear diagrams are not shown here but they are all right.

Analysis of structures including elements with infinite high system.

One should know that this elements are actually rigid body elements. As a result they cannot deform. They only rotates or moves without deformations – without changing their shapes.

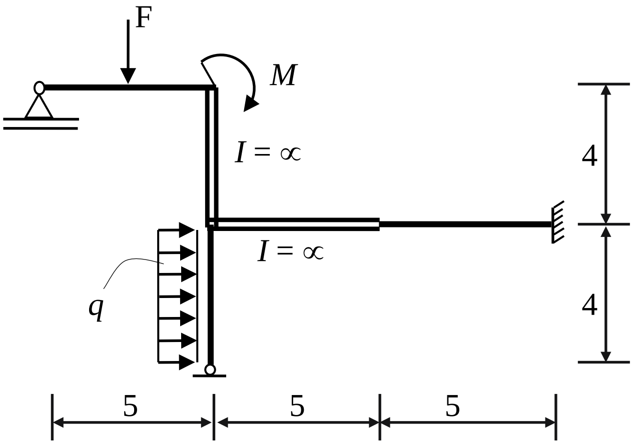
This fact must be taken into account when composing the primary system. Also when such an element rotates it rotate and the flexible elements. In the end of the normal elements will appears not only displacements but rotation too.

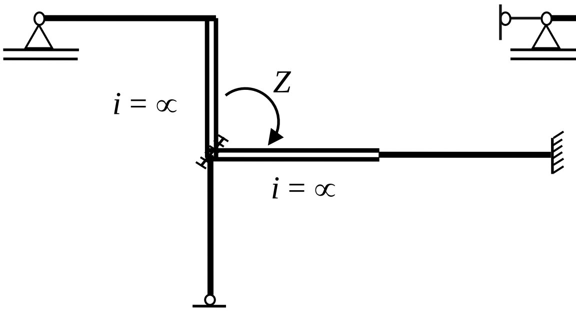
Also, it is important to understand that the flexible elements are rigidity connected at the rigid element but the rigid element is connected at the flexible element with a joint connection!

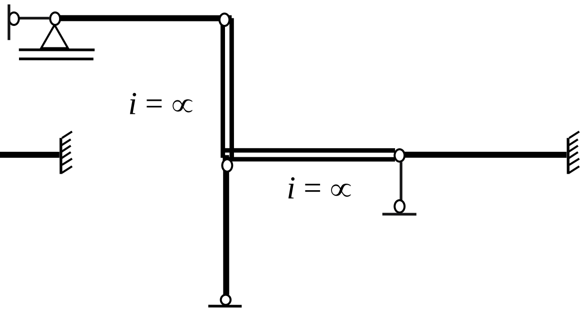
This will be shown at the next example.

Example:

$F = 200$, $M = 10$, $q = 40$, $EI = 50\ 000\ kNm^2$





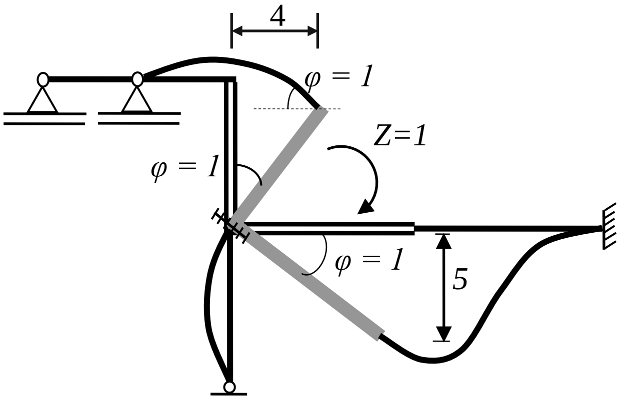


If we see the pin-joint system of the present example, we will need two linear links. But as the system has a rigid element so the two linear displacements are dependent one to the other. Actually, they are dependent to the rotation of the rigid element so this structure has only one independent displacement parameter – the rotation of the rigid body. That is why this system is one time cinematically indeterminate system. If we know this rotation we will know the displacements of the other nodes of the system.

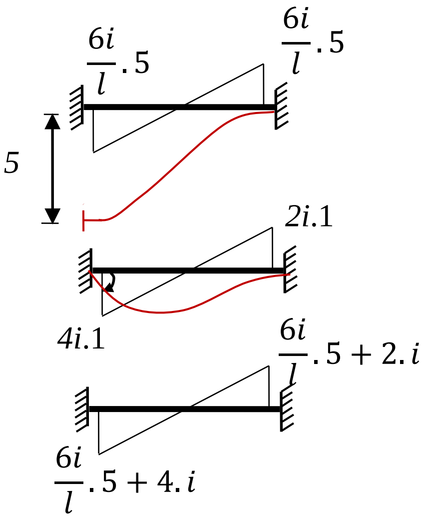
Next stage is to compose the displaced shape of the primary system from the unit rotation of the rigid body.

When rigid body rotates to a unit angle the point at a distance 5 metre will have 5 metre displacement also the rigid connection between the rigid body and the flexible one cause additional rotation at the flexible element.

As a result the displaced shape of the primary system is shown bellow:

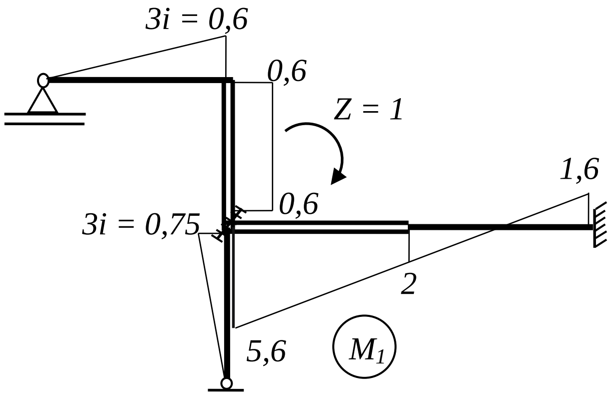


To compose the moment diagram for some elements we should be careful if at the element has only displacement or displacement plus rotation. At the present example the right beam element has displacement and rotation and the other flexible elements deflect only because of the rotation of the rigid body. The diagram of the right beam element is composed by two parts and is presented on next figure.

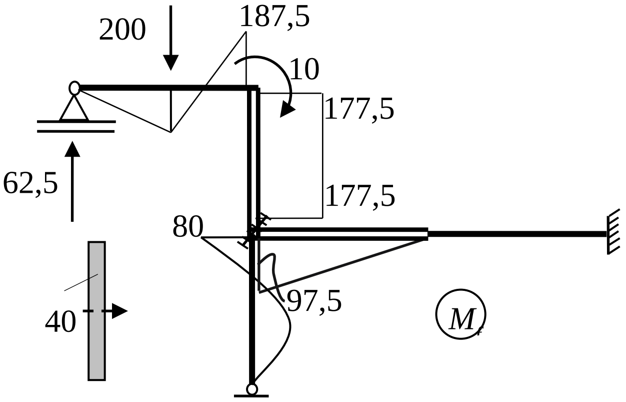


At the connections between rigid body and flexible elements the moment diagram composes after equilibrium of the joint.

In this why the unit moment diagram is presented on next figure:



The diagram from the external load is composed in the standard way only taking into account the connections between rigid and flexible body.

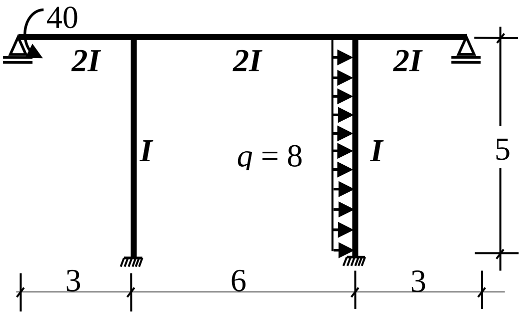


Next solution continues standard.

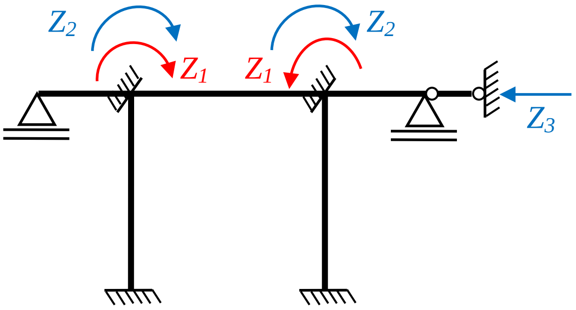
Analysis of symmetrical structures.

The idea of the symmetrical system is the same as this in the force method. We use again the couple of unknowns but in the displacement method, the unknowns are nodes displacements. That is why we will show example directly.

Example:

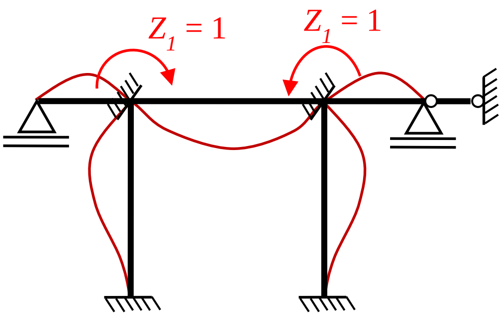


The primary system and the coupled unknowns are shown bellow:

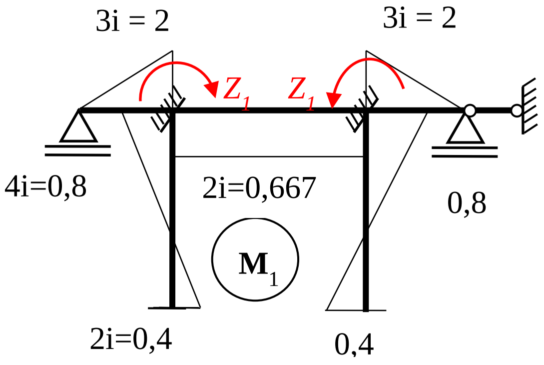


Primary system

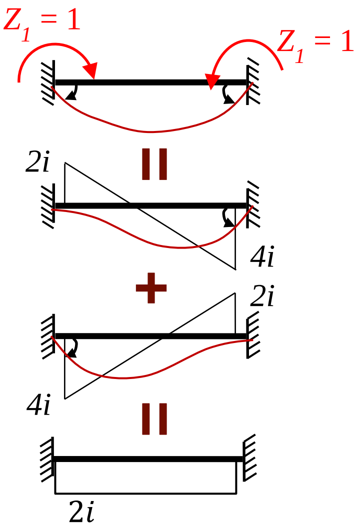
As one can see there are one symmetrical and two antisymmetrical unknowns. First unknown is a couple and symmetrical, second is again couple but antisymmetrical and the last one is not couple (it is only one force) and is antisymmetrical. When compose diagrams from the coupled unknowns one should be careful because at some members there are rotations at the two ends of the element. To compose the diagram in this case one should use the principle of the superposition shown below. In middle beam element there is double rotation from the unit value of the first unknown so, the diagram will be composed in two parts using superposition.



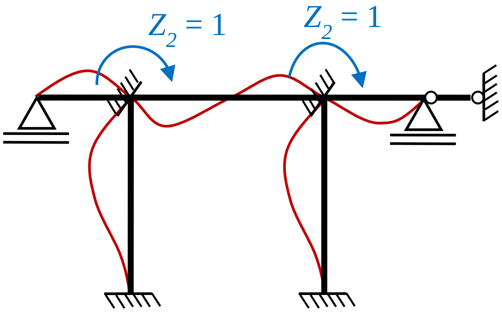
Deformed shape



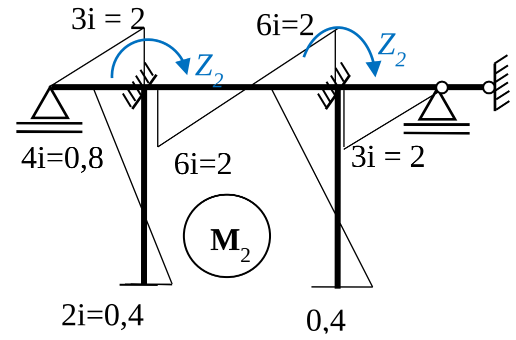
Unit diagram



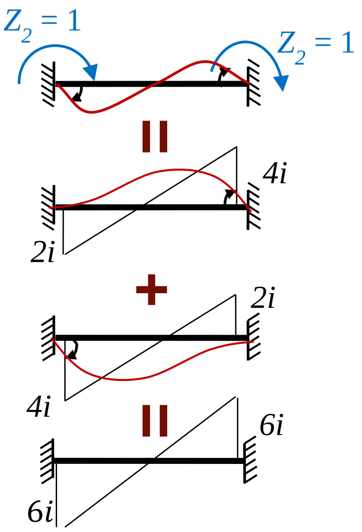
From unit value of the second unknown there is again doubled rotation in the middle beam element but in this case at different direction. The diagram composes in same way, only the result is different.



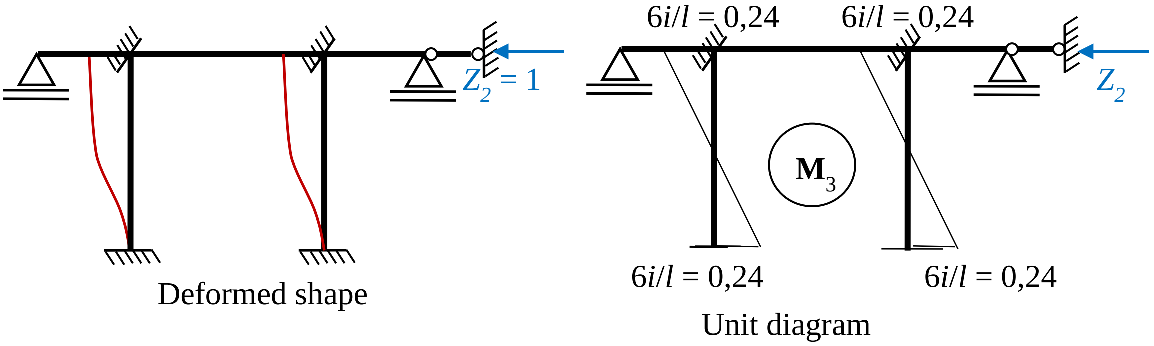
Deformed shape



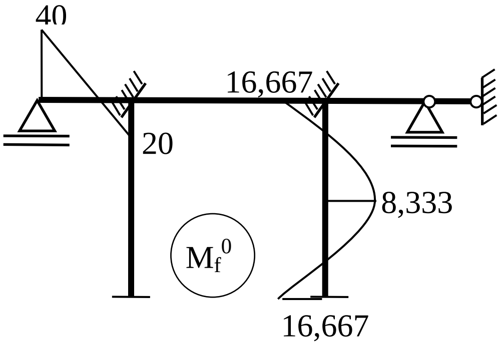
Unit diagram



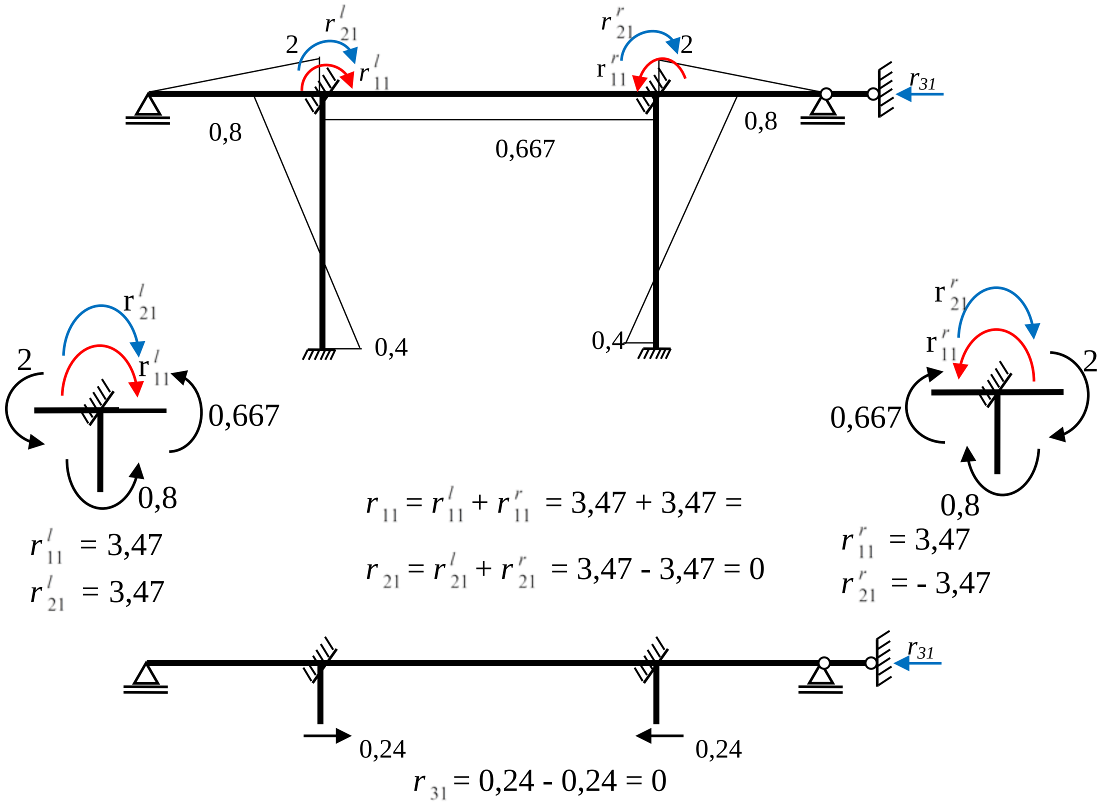
From the unit value of the last unknown there is no such effects. The unit diagram composes simply. The result is shown bellow.



Moment diagram from the external load is nonsymmetrical and is composed standartly.



Next stage of the solution is to calculate reactions at the additional links from the previous loads. In the case of couples unknowns the reactions are also coupled – left and right part. The full reaction which we should use in the equilibrium equation is sum of the two parts. This summation is shown bellow only for the first unit diagram. For the other diagrams the results obtain analogically.



$r^l_{11} = 3,47$

$r^l_{21} = 3,47$

$r_{11} = r^l_{11} + r^r_{11} = 3,47 + 3,47 =$

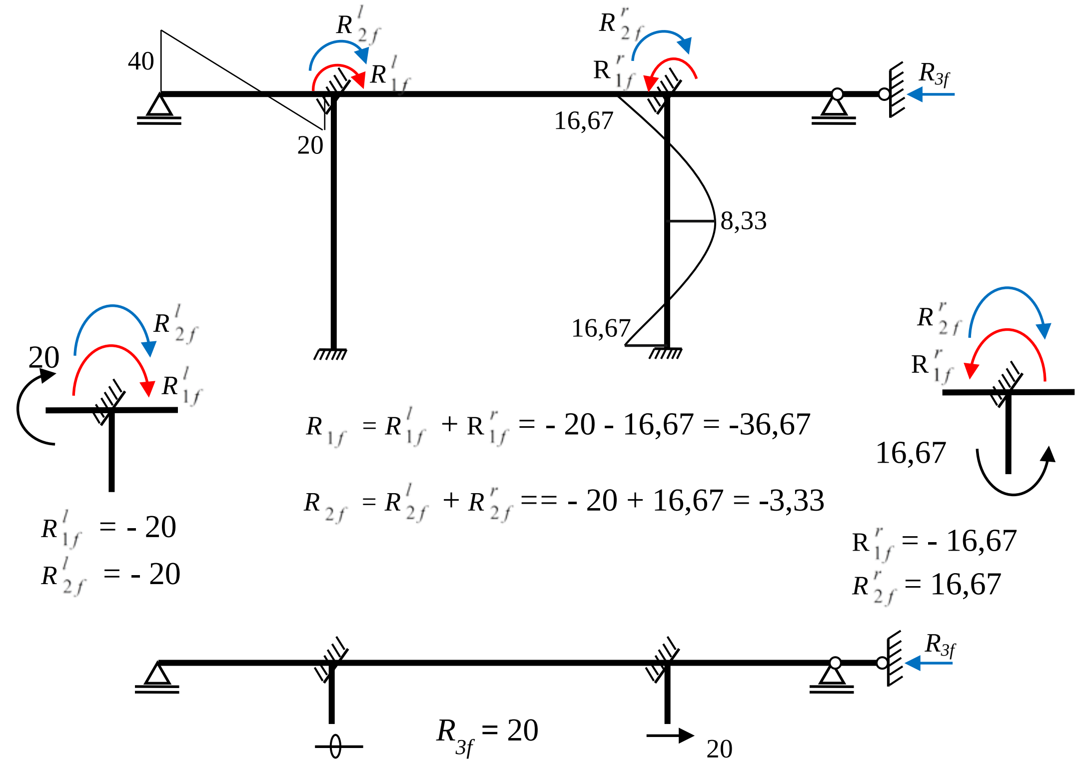
$r_{21} = r^l_{21} + r^r_{21} = 3,47 - 3,47 = 0$

$r^r_{11} = 3,47$

$r^r_{21} = - 3,47$

$r_{31} = 0,24 - 0,24 = 0$

As one can see the antisymmetrical reactions from the symmetrical load are zero and analogically the symmetrical reactions from the antisymmetrical load are zero. The internal moment diagram from the external load is non-symmetrical but the reactions are again coupled. Their calculation is shown bellow.



$R_{1f} = R^l_{1f} + R^r_{1f} = - 20 - 16,67 = -36,67$

$R_{2f} = R^l_{2f} + R^r_{2f} == - 20 + 16,67 = -3,33$

$R^l_{1f} = - 20$

$R^l_{2f} = - 20$

$R^r_{1f} = - 16,67$

$R^r_{2f} = 16,67$

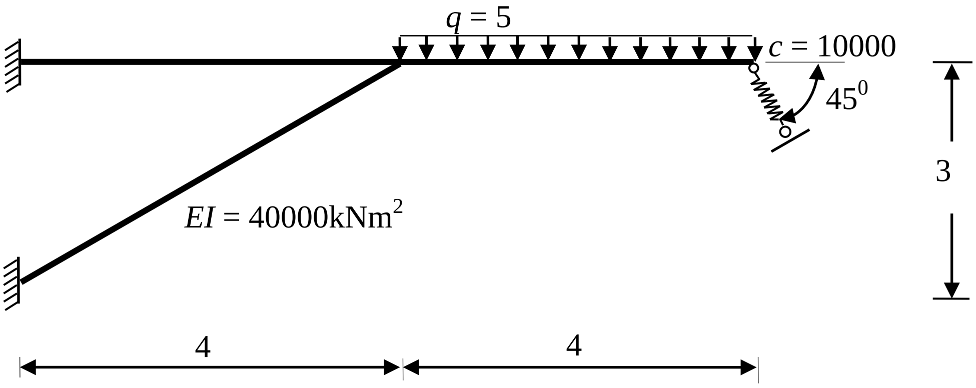
$R_{3f} = 20$

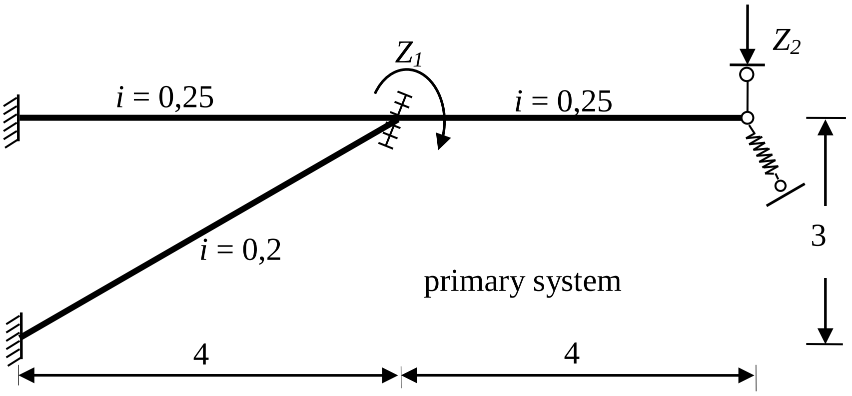
As we have all coefficients the solution continuous as standard one.

Analysis of structures including springs.

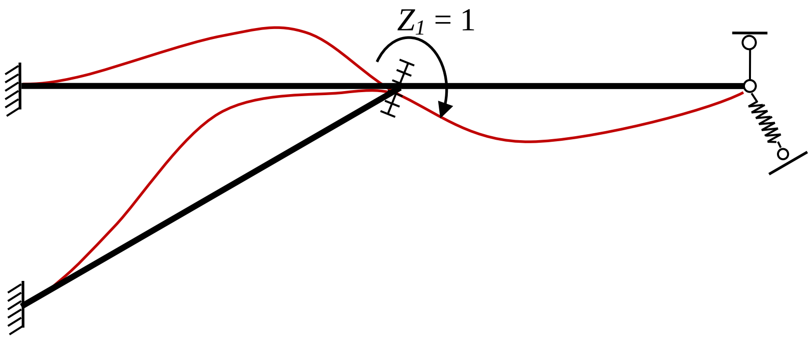
The analysis of structures including springs is generally standard with only deferens we have mentioned one should add a linear link at the spring if the spring is linear and rotational link if the spring is rotational. We do this to control the displacement (rotation) of the spring. At the next example is shown a structure with a linear spring. Analysis of a structure including rotational spring is analogical.

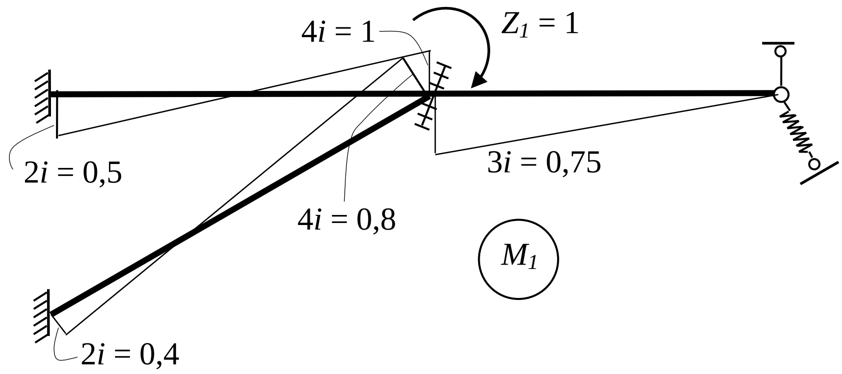
Example:



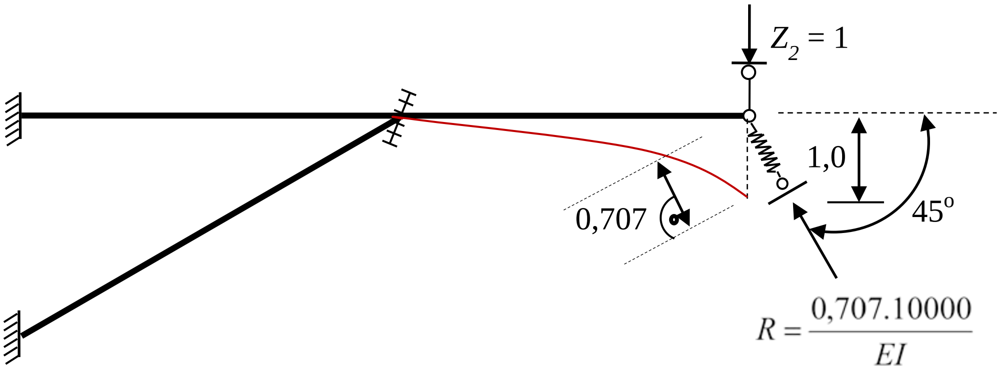


As it is shown it is not necessary the linear link to be in the same direction as the spring. The deformation and the diagram from the unit value of the first unknown is standard. The spring cannot deform because of the linear link.

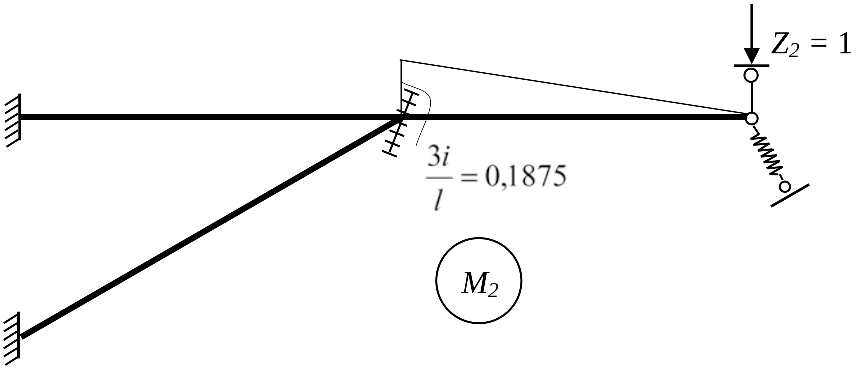




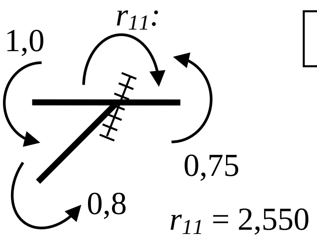
The deformation from the unit values of the second unknown is shown on the next figure:



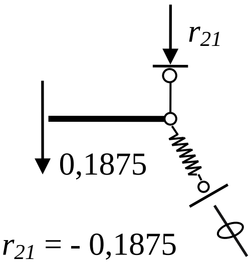
As we do unit vertical displacement (Z_2 = 1) at the spring appears displacement equal to 0,707 because of the angle 45°. This displacement at the spring evoke reaction at the spring equal to the displacement multiplied by the stiffness of the spring. In our case we have divided this reaction by *EI* because we produce solution using reduced values of the reaction (we have shown already). And the moment diagram is:

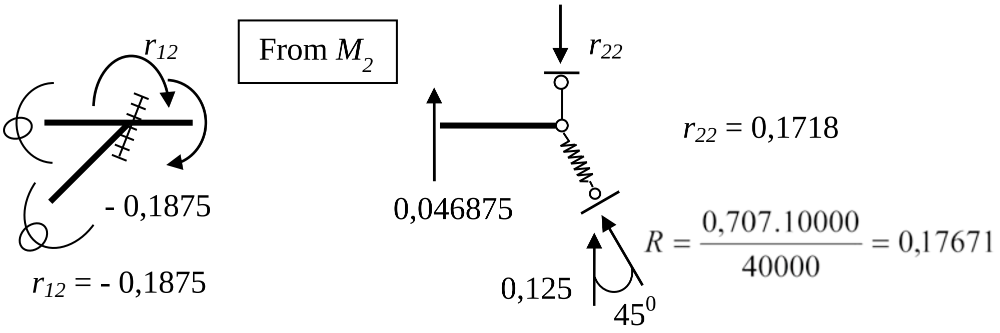


It is important to understand that the reaction at the spring is always zero only in the case of unit displacement of the linear link is different of zero. Reactions at the added links are:

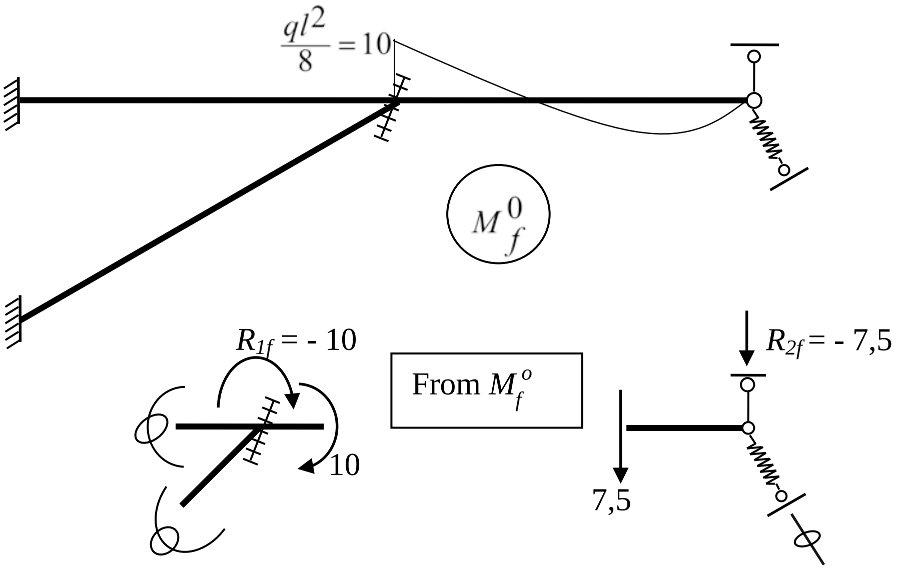


From M_1





The internal moment diagram and reactions from the external load are:



As we have all coefficients the solution continuous as standard one.

Literature:

1. Методично ръководство за решаване на задачи по строителна статика, Т. Карамански, Р. Рангелов, издателство „Техника“ 1971г.;
2. Structural Analysis, R.C. Hibeller, Prentice Hall 2006;
3. Structural Analysis in theory and practice, A. Williams, International Codes Council 2009.