Instructor Solutions Manual

to accompany

University Physics

Second Edition

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***UNIVERSITY PHYSICS*, Second Edition**

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Conceptual Questions

1.17. (a) In Europe, gas consumption is in L/100 km. In the US, fuel efficiency is in miles/gallon. Let's relate these two: 1 mile = 1.609 km, 1 gal = 3.785 L.

$$\frac{1\text{ mile}}{\text{gal}}=\frac{1.609\text{ km}}{3.785\text{ L}}=\frac{1.609}{3.785}\left(\frac{1}{100}\right)(100)\frac{\text{km}}{\text{L}}=(0.00425)\left(\frac{1}{\text{L/100 km}}\right)=\frac{1}{235.24\text{ L/100 km}}$$

Therefore, 1 mile/gal is the reciprocal of 235.2 L/100 km.

(b) Gas consumption is $\frac{12.2\text{ L}}{100\text{ km}}$. Using $\frac{1\text{ L}}{100\text{ km}}=\frac{1}{235.24\text{ miles/gal}}$ from part (a),

$$\frac{12.2\text{ L}}{100\text{ km}}=12.2\left(\frac{1\text{ L}}{100\text{ km}}\right)=12.2\left(\frac{1}{235.24\text{ miles/gal}}\right)=\frac{1}{19.282\text{ miles/gal}}.$$

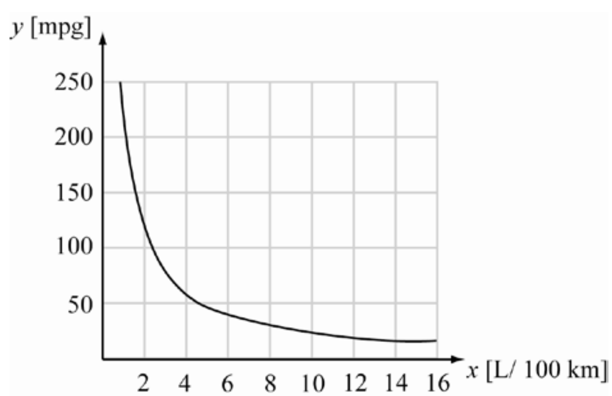
Therefore, a car that consumes 12.2 L/100 km of gasoline has a fuel efficiency of 19.3 miles/gal.

(c) If the fuel efficiency of the car is 27.4 miles per gallon, then

$$\frac{27.4\text{ miles}}{\text{gal}}=\frac{27.4}{235.24\text{ L/100 km}}=\frac{1}{8.59\text{ L/100 km}}.$$

Therefore, 27.4 miles/gal is equivalent to 8.59 L/100 km.

(d)



1.18. A vector is described by a set of components in a given coordinate system, where the components are the projections of the vector onto each coordinate axis. Therefore, on a two-dimensional sheet of paper there are two coordinates and thus, the vector is described by two components. In the real three-dimensional world, there are three coordinates and a vector is described by three components. A four-dimensional world would be described by four coordinates, and a vector would be described by four components.

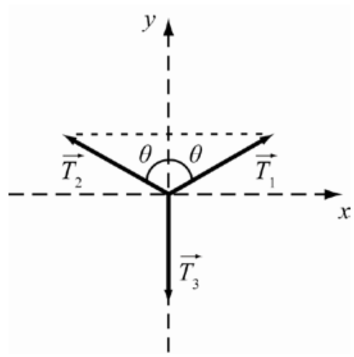
1.19. A vector contains information about the distance between two points (the magnitude of the vector). In contrast to a scalar, it also contains information direction. In many cases knowing a direction can be as important as knowing a magnitude.

1.20. In order to add vectors in magnitude-direction form, each vector is expressed in terms of component vectors which lie along the coordinate axes. The corresponding components of each vector are added to obtain the components of the resultant vector. The resultant vector can then be expressed in magnitude-direction form by computing its magnitude and direction.

1.21. The advantage to using scientific notation is two-fold: Scientific notation is more compact (thus saving space and writing), and it also gives a more intuitive way of dealing with significant figures since you can only write the necessary significant figures and extraneous zeroes are kept in the exponent of the base.

1.22. The SI system of units is the preferred system of measurement due to its ease of use and clarity. The SI system is a metric system generally based on multiples of 10, and consisting of a set of standard measurement units to describe the physical world. In science, it is paramount to communicate results in the clearest and most widely understood manner. Since the SI system is internationally recognized, and its definitions are unambiguous, it is used by scientists around the world, including those in the United States.

1.23. It is possible to add three equal-length vectors and obtain a vector sum of zero. The vector components of the three vectors must all add to zero. Consider the following arrangement with $|T_1| = |T_2| = |T_3|$:



The horizontal components of T_1 and T_2 cancel out, so the sum $T_1 + T_2$ is a vertical vector whose magnitude is $T\cos\theta + T\cos\theta = 2T\cos\theta$. The vector sum $T_1 + T_2 + T_3$ is zero if

$$2T\cos\theta - T = 0$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = 60^\circ$$

Therefore it is possible for three equal-length vectors to sum to zero.

1.24. Mass is not a vector quantity. It is a scalar quantity since it does not make sense to associate a direction with mass.

1.25. The volume of a sphere is given by $V = (4/3)\pi r^3$. Doubling the volume gives $2V = 2(4/3)\pi r^3 = (4/3)\pi(2^{3/3})r^3 = (4/3)\pi(2^{1/3}r)^3$. Now, since the distance between the flies is the diameter of the sphere, $d = 2r$, and doubling the volume increases the radius by a factor of $2^{1/3}$, the distance between the flies is then increased to $2(2^{1/3}r) = 2^{1/3}(2r) = 2^{1/3}d$. Therefore, the distance is increased by a factor of $2^{1/3}$.

1.26. The volume of a cube of side r is $V_\text{c} = r^3$, and the volume of a sphere of radius r is $V_\text{sp} = (4/3)\pi r^3$. The ratio of the volumes is:

$$\frac{V_\text{c}}{V_\text{sp}} = \frac{r^3}{\frac{4}{3}\pi r^3} = \frac{3}{4\pi}.$$

The ratio of the volumes is independent of the value of *r*.

1.27. The surface area of a sphere is given by $4\pi r^2$. A cube of side length s has a surface area of $6s^2$. To determine s set the two surface areas equal:

$$6s^2 = 4\pi r^2 \Rightarrow s = \sqrt{\frac{4\pi r^2}{6}} = r\sqrt{\frac{2\pi}{3}}.$$

1.28. The mass of Sun is $2 \cdot 10^{30}$ kg, the number of stars in the Milky Way is about $100 \cdot 10^9 = 10^{11}$, the number of galaxies in the Universe is about $100 \cdot 10^9 = 10^{11}$, and the mass of an H-atom is $2 \cdot 10^{-27}$ kg.

(a) The total mass of the Universe is roughly equal to the number of galaxies in the Universe multiplied by the number of stars in a galaxy and the mass of the average star:

$$M_{\text{universe}} = (10^{11})(10^{11})(2 \cdot 10^{30}) = 2 \cdot 10^{(11+11+30)} \text{ kg} = 2 \cdot 10^{52} \text{ kg}.$$

(b) $n_{\text{hydrogen}} \approx \dfrac{M_{\text{universe}}}{M_{\text{hydrogen}}} = \dfrac{2 \cdot 10^{52} \text{ kg}}{2 \cdot 10^{-27} \text{ kg}} = 10^{79}$ atoms

1.29. The volume of 1 teaspoon is about $4.93 \cdot 10^{-3}$ L, and the volume of water in the oceans is about $1.35 \cdot 10^{21}$ L.

$$\frac{1.35 \cdot 10^{21} \text{ L}}{4.93 \cdot 10^{-3} \text{ L/tsp}} = 2.74 \cdot 10^{23} \text{ tsp}$$

There are about $2.74 \cdot 10^{23}$ teaspoons of water in the Earth's oceans.

1.30. The average arm-span of an adult human is $d = 2$ m. Therefore, with arms fully extended, a person takes up a circular area of $\pi r^2 = \pi (d/2)^2 = \pi (1 \text{ m})^2 = \pi \text{ m}^2$. Since there are approximately $6.5 \cdot 10^9$ humans, the amount of land area required for all humans to stand without being able to touch each other is $6.5 \cdot 10^9 \text{ m}^2 (\pi) = 6.5 \cdot 10^9 \text{ m}^2 (3.14) = 2.0 \cdot 10^{10} \text{ m}^2$. The area of the United States is about $3.5 \cdot 10^6$ square miles or $9.1 \cdot 10^{12} \text{ m}^2$. In the United States there is almost five hundred times the amount of land necessary for all of the population of Earth to stand without touching each other.

1.31. The diameter of a gold atom is about $2.6 \cdot 10^{-10}$ m. The circumference of the neck of an adult is roughly 0.40 m. The number of gold atoms necessary to link to make a necklace is given by:

$$n = \frac{\text{circumference of neck}}{\text{diameter of atom}} = \frac{4.0 \cdot 10^{-1} \text{ m}}{2.6 \cdot 10^{-10} \text{ m/atom}} = 1.5 \cdot 10^9 \text{ atoms}.$$

The Earth has a circumference at the equator of about $4.008 \cdot 10^7$ m. The number of gold atoms necessary to link to make a chain that encircles the Earth is given by:

$$N = \frac{\text{circumference of Earth}}{\text{diameter of a gold atom}} = \frac{4.008 \cdot 10^7 \text{ m}}{2.6 \cdot 10^{-10} \text{ m}} = 1.5 \cdot 10^{17} \text{ atoms}.$$

Since one mole of substance is equivalent to about $6.022 \cdot 10^{23}$ atoms, the necklace of gold atoms has $\left(1.5 \cdot 10^9 \text{ atoms}\right) / \left(6.022 \cdot 10^{23} \text{ atoms/mol}\right) = 2.5 \cdot 10^{-15}$ moles of gold. The gold chain has $\left(1.5 \cdot 10^{17} \text{ atoms}\right) / \left(6.022 \cdot 10^{23} \text{ atoms/mol}\right) = 2.5 \cdot 10^{-7}$ moles of gold.

1.32. The average dairy cow has a mass of about $1.0 \cdot 10^3$ kg. Estimate the cow's average density to be that of water, $\rho = 1000. \text{ kg/m}^3$.

$$\text{volume} = \frac{\text{mass}}{\rho} = \frac{1.0 \cdot 10^3 \text{ kg}}{1000. \text{ kg/m}^3} = 1.0 \text{ m}^3$$

Relate this to the volume of a sphere to obtain the radius.

$$\text{volume} = \frac{4}{3}\pi r^3 \Rightarrow r = \left[\frac{3V}{4\pi}\right]^{1/3} = \left[\frac{3\left(1.0 \text{ m}^3\right)}{4\pi}\right]^{1/3} \approx 0.62 \text{ m}$$

A cow can be roughly approximated by a sphere with a radius of 0.62 m.

1.33. The mass of a head can be estimated first approximating its volume. A rough approximation to the shape of a head is a cylinder. To obtain the volume from the circumference, recall that the circumference is $C = 2\pi r$, which gives a radius of $r = C/2\pi$. The volume is then:

$$V = \left(\pi r^2\right)h = \pi\left(\frac{C}{2\pi}\right)^2 h = \frac{C^2 h}{4\pi}.$$

The circumference of a head is about 55 cm = 0.55 m, and its height is about 20 cm = 0.20 m. These values can be used in the volume equation:

$$V = \frac{(0.55 \text{ m})^2}{4\pi}(0.20 \text{ m}) = 4.8\cdot 10^{-3} \text{ m}^3.$$

Assuming that the density of the head is about the same as the density of water, the mass of a head can then be estimated as follows:

$$\text{mass} = \text{density}\cdot\text{volume} = \left(1.0\cdot 10^3 \text{ kg/m}^3\right)\left(4.8\cdot 10^{-3} \text{ m}^3\right) = 4.8 \text{ kg}.$$

1.34. The average adult human head is roughly a cylinder 15 cm in diameter and 20. cm in height. Assume about 1/3 of the surface area of the head is covered by hair.

$$A_{\text{hair}} = \frac{1}{3}\left(A_{\text{cylinder}}\right) = \frac{1}{3}\left(2\pi r^2 + 2\pi r h\right) = \frac{2\pi}{3}\left(r^2 + rh\right) = \frac{2\pi}{3}\left[(7.5 \text{ cm})^2 + (7.5 \text{ cm})(20. \text{ cm})\right]$$
$$\approx 4.32\cdot 10^2 \text{ cm}^2$$

On average, the density of hair on the scalp is $\rho_{\text{hair}} = 2.3\cdot 10^2 \text{ hairs/cm}^2$. Therefore, you have $A_{\text{hair}} \times \rho_{\text{hair}}$ hairs on your head.

$$A_{\text{hair}}\rho_{\text{hair}} = \left(4.32\cdot 10^2 \text{ cm}^2\right)\left(2.3\cdot 10^2 \text{ hairs/cm}^2\right) = 9.9\cdot 10^4 \text{ hairs.}$$

Exercises

1.35. (a) Three (b) Four (c) One (d) Six (e) One (f) Two (g) Three

1.36. **THINK:** The known quantities are: $F_1 = 2.0031$ N and $F_2 = 3.12$ N. Both F_1 and F_2 are in the same direction, and act on the same object. The total force acting on the object is F_{total}.

SKETCH:

RESEARCH: Forces that act in the same direction are summed, $F_{\text{total}} = \sum F_i$.

SIMPLIFY: $F_{\text{total}} = \sum F_i = F_1 + F_2$

CALCULATE: $F_{\text{total}} = 2.0031 \text{ N} + 3.12 \text{ N} = 5.1231 \text{ N}$

ROUND: When adding (or subtracting), the precision of the result is limited by the least precise value used in the calculation. F_1 is precise to four places after the decimal and F_2 is precise to only two places after the decimal, so the result should be precise to two places after the decimal: $F_{\text{total}} = 5.12$ N.

DOUBLE-CHECK: This result is reasonable as it is greater than each of the individual forces acting on the object.

1.37. The result should have the same number of decimal places as the number with the fewest of them. Therefore, the result is 2.0600 + 3.163 + 1.12 = 6.34.

1.38. In a product of values, the result should have as many significant figures as the value with the smallest number of significant figures. The value for x only has two significant figures, so $w = (1.1\cdot 10^3)(2.48\cdot 10^{-2})(6.000) = 1.6\cdot 10^2$.

1.39. Write "one ten-millionth of a centimeter" in scientific notation. One millionth is $1/10^6 = 1\cdot 10^{-6}$. Therefore, one ten-millionth is $1/\left[10\cdot 10^6\right] = 1/10^7 = 1\cdot 10^{-7}$ cm.

1.40. 153,000,000 = $1.53\cdot 10^8$

1.41. There are 12 inches in a foot and 5280 feet in a mile. Therefore there are 63,360 inch/mile. 30.7484 miles · 63,360 inch / mile = 1948218.624 inches. Rounding to six significant figures and expressing the answer in scientific notation gives $1.94822\cdot 10^6$ inches.

1.42. (a) kilo (b) centi (c) milli

1.43. $1 \text{ km} = 1 \text{ km}\left(\frac{1000 \text{ m}}{1 \text{ km}}\right)\left(\frac{1000 \text{ mm}}{1 \text{ m}}\right) = 1{,}000{,}000 \text{ mm} = 1\cdot 10^6 \text{ mm}$

1.44. 1 hectare = 100 ares, and 1 are = 100 m^2, so:

$$1 \text{ km}^2 = 1 \text{ km}^2\left(\frac{(1000)^2 \text{ m}^2}{1 \text{ km}^2}\right)\left(\frac{1 \text{ are}}{100 \text{ m}^2}\right)\left(\frac{1 \text{ hectare}}{100 \text{ ares}}\right) = 100 \text{ hectares.}$$

1.45. 1 milliPascal

1.46. **THINK:** The known quantities are the masses of the four sugar cubes. Crushing the sugar cubes doesn't change the mass. Their masses, written in standard SI units, using scientific notation are $m_1 = 2.53\cdot 10^{-2}$ kg, $m_2 = 2.47\cdot 10^{-2}$ kg, $m_3 = 2.60\cdot 10^{-2}$ kg and $m_4 = 2.58\cdot 10^{-2}$ kg.

SKETCH: A sketch is not needed to solve this problem.

RESEARCH:

(a) The total mass equals the sum of the individual masses: $M_{\text{total}} = \sum_{j=1}^{4} m_j$.

(b) The average mass is the sum of the individual masses, divided by the total number of masses:

$$M_{\text{average}} = \frac{m_1 + m_2 + m_3 + m_4}{4}.$$

SIMPLIFY:

(a) $M_{\text{total}} = m_1 + m_2 + m_3 + m_4$

(b) $M_{\text{average}} = \frac{M_{\text{total}}}{4}$

CALCULATE:

(a)
$$\begin{aligned} M_{\text{total}} &= 2.53\cdot 10^{-2} \text{ kg} + 2.47\cdot 10^{-2} \text{ kg} + 2.60\cdot 10^{-2} \text{ kg} + 2.58\cdot 10^{-2} \text{ kg} \\ &= 10.18\cdot 10^{-2} \text{ kg} \\ &= 1.018\cdot 10^{-1} \text{ kg} \end{aligned}$$

(b) $M_{\text{average}} = \frac{10.18\cdot 10^{-2} \text{ kg}}{4} = 2.545\cdot 10^{-2} \text{ kg}$

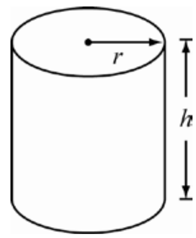
ROUND:

(a) Rounding to three significant figures, $M_{\text{total}} = 1.02\cdot 10^{-1}$ kg.

(b) Rounding to three significant figures, $M_{\text{average}} = 2.55\cdot 10^{-2}$ kg.

DOUBLE-CHECK: There are four sugar cubes weighing between $2.53\cdot 10^{-2}$ kg and $2.60\cdot 10^{-2}$ kg, so it is reasonable that their total mass is $M_{\text{total}} = 1.02\cdot 10^{-1}$ kg and their average mass is $2.55\cdot 10^{-2}$ kg.

1.47. **THINK:** The cylinder has height h = 20.5 cm and radius r = 11.9 cm.
SKETCH:



RESEARCH: The surface area of a cylinder is $A = 2\pi rh + 2\pi r^2$.

SIMPLIFY: $A = 2\pi r(h+r)$

CALCULATE: $A = 2\pi(11.9\text{ cm})(20.5\text{ cm} + 11.9\text{ cm}) = 2422.545\text{ cm}^2$

ROUND: Three significant figures: $A = 2.42\cdot 10^3\text{ cm}^2$.

DOUBLE-CHECK: The units of area are a measure of distance squared so the answer is reasonable.

1.48. **THINK:** When you step on the bathroom scale, your mass and gravity exert a force on the scale and the scale displays your weight. The given quantity is your mass m_1 = 125.4 lbs. Pounds can be converted to SI units using the conversion 1 lb = 0.4536 kg. Let your mass in kilograms be m_2.
SKETCH: A sketch is not needed to solve this problem.

RESEARCH: $m_2 = m_1\left(\dfrac{0.4536\text{ kg}}{1\text{ lb}}\right)$

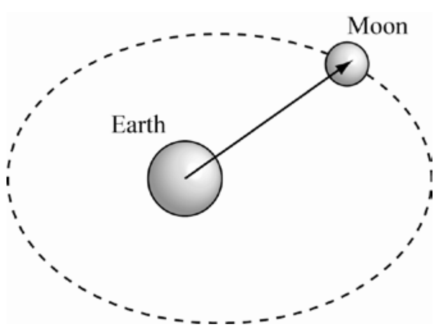
SIMPLIFY: It is not necessary to simplify.

CALCULATE: $m_2 = 125.4\text{ lbs}\left(\dfrac{0.4536\text{ kg}}{\text{lb}}\right) = 56.88144\text{ kg}$

ROUND: The given quantity and conversion factor contain four significant figures, so the result must be rounded to 56.88 kg.

DOUBLE-CHECK: The SI units of mass are kg, so the units are correct.

1.49. **THINK:** The orbital distance from the center of the Moon to the center of the Earth ranges from 356,000 km to 407,000 km. Recall the conversion factor 1 mile = 1.609344 kilometer.
SKETCH:



RESEARCH: Let d_1 be a distance in kilometers, and d_2 the equivalent distance in miles. The formula to convert from kilometers to miles is $d_2 = d_1 / 1.609344$.

SIMPLIFY: It is not necessary to simplify.

CALCULATE: $356{,}000\text{ km}\left(\dfrac{1\text{ mile}}{1.609344\text{ km}}\right)=221208.144\text{ miles}$

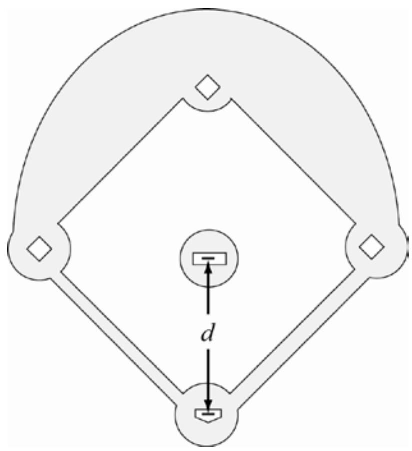
$$407{,}000\text{ km}\left(\frac{1\text{ mile}}{1.609344\text{ km}}\right)=252898.0752\text{ miles}$$

ROUND: The given quantities have three significant figures, so the calculated values must be rounded to 221,000 miles and 253,000 miles respectively.

DOUBLE-CHECK: A kilometer is roughly 2/3 of a mile, and the answers are roughly 2/3 of the given values, so the conversions appear correct.

1.50. **THINK:** It is a distance d = 60 feet, 6 inches from the pitcher's mound to home plate. Recall the conversion factors: 1 foot = 12 inches, 1 inch = 2.54 cm, 100 cm = 1 m.

SKETCH:



RESEARCH: If the distance is x in meters and y in feet, then using the conversion factor c, $x=cy$.

$$c=1\text{ foot}\left(\frac{12\text{ inches}}{1\text{ foot}}\right)\left(\frac{2.54\text{ cm}}{1\text{ inch}}\right)\left(\frac{1\text{ m}}{100\text{ cm}}\right)/\text{ foot}$$

SIMPLIFY: $c=0.3048$ meters/foot

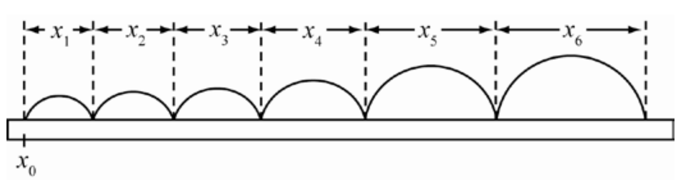
CALCULATE: 60 feet plus 6 inches = 60.5 feet. Then, converting 60.5 feet to meters: $d=60.5\text{ ft}\left(\dfrac{0.3048\text{ m}}{1\text{ ft}}\right)=18.440\text{ m}.$

ROUND: Rounding to three significant figures, the distance is 18.4 m.

DOUBLE-CHECK: The answer is a reasonable distance for a pitcher to throw the ball.

1.51. **THINK:** The given quantities, written in scientific notation and in units of meters, are: the starting position, $x_o=7\cdot 10^{-3}$ m and the lengths of the flea's successive hops, $x_1=3.2\cdot 10^{-2}$ m, $x_2=6.5\cdot 10^{-2}$ m, $x_3=8.3\cdot 10^{-2}$ m, $x_4=10.0\cdot 10^{-2}$ m, $x_5=11.5\cdot 10^{-2}$ m and $x_6=15.5\cdot 10^{-2}$ m. The flea makes six jumps in total.

SKETCH:



RESEARCH: The total distance jumped is $x_{\text{total}}=\sum_{n=1}^{6}x_n$. The average distance covered in a single hop is:

$$x_{\text{avg}}=\frac{1}{6}\sum_{n=1}^{6}x_n.$$

SIMPLIFY: $x_{\text{total}} = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$, $x_{\text{avg}} = \dfrac{x_{\text{total}}}{6}$

CALCULATE: $x_{\text{total}} = (3.2 \text{ m} + 6.5 \text{ m} + 8.3 \text{ m} + 10.0 \text{ m} + 11.5 \text{ m} + 15.5 \text{ m}) \cdot 10^{-2} = 55.0 \cdot 10^{-2} \text{ m}$

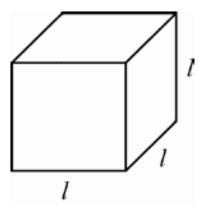
$$x_{\text{avg}} = \frac{55.0 \cdot 10^{-2} \text{ m}}{6} = 9.16666 \cdot 10^{-2} \text{ m}$$

ROUND: Each of the hopping distances is measured to 1 mm precision. Therefore the total distance should also only be quoted to 1 mm precision: $x_{\text{total}} = 55.0 \cdot 10^{-2}$ m. Rounding the average distance to the right number of significant digits, however, requires a few more words. As a general rule of thumb the average distance should be quoted to the same precision as the least precision of the individual distances, if there are only a few measurements contributing to the average. This is the case here, and so we state $x_{\text{avg}} = 9.17 \cdot 10^{-2}$ m. However, suppose we had 10,000 measurements contributing to an average. Surely we could then specify the average to a higher precision. The rule of thumb is that we can add one additional significant digit for every order of magnitude of the number of independent measurements contributing to an average. You see that the answer to this problem is yet another indication that specifying the correct number of significant figures can be complicated and sometimes outright tricky!

DOUBLE-CHECK: The flea made 6 hops, ranging from $3.2 \cdot 10^{-2}$ m to $15.5 \cdot 10^{-2}$ m, so the total distance covered is reasonable. The average distance per hop falls in the range between $3.2 \cdot 10^{-2}$ m and $1.55 \cdot 10^{-1}$ m, which is what is expected.

1.52. **THINK:** The question says that 1 cm^3 of water has a mass of 1 g, that $1 \text{ mL} = 1 \text{ cm}^3$, and that 1 metric ton is 1000 kg.

SKETCH:



RESEARCH: For the first part of the question, use the conversion equation:

$$1 \text{ L} = 1 \text{ L}\left(\frac{1000 \text{ mL}}{1 \text{ L}}\right)\left(\frac{1 \text{ cm}^3}{1 \text{ mL}}\right)\left(\frac{1 \text{ g}}{1 \text{ cm}^3}\right)\left(\frac{1 \text{ kg}}{1000 \text{ g}}\right).$$

For the second part of the question, use:

$$1 \text{ metric ton} = 1 \text{ metric ton}\left(\frac{1000 \text{ kg}}{1 \text{ metric ton}}\right)\left(\frac{1000 \text{ g}}{1 \text{ kg}}\right)\left(\frac{1 \text{ cm}^3}{1 \text{ g}}\right).$$

For the last part, recall that the volume of a cube is $V = l^3$.

SIMPLIFY: Re-arranging the formula for the volume of the cubical tank to solve for the length gives $l = \sqrt[3]{V_c}$.

CALCULATE: $1 \text{ L} = 1 \text{ L}\left(\dfrac{1000 \text{ mL}}{1 \text{ L}}\right)\left(\dfrac{1 \text{ cm}^3}{1 \text{ mL}}\right)\left(\dfrac{1 \text{ g}}{1 \text{ cm}^3}\right)\left(\dfrac{1 \text{ kg}}{1000 \text{ g}}\right) = 1 \text{ kg}$

$$1 \text{ metric ton} = 1 \text{ metric ton}\left(\frac{1000 \text{ kg}}{1 \text{ metric ton}}\right)\left(\frac{1000 \text{ g}}{1 \text{ kg}}\right)\left(\frac{1 \text{ cm}^3}{1 \text{ g}}\right) = 1000000 \text{ cm}^3$$

$l = \sqrt[3]{1{,}000{,}000} = 100 \text{ cm} = 1 \text{ m}$

ROUND: No rounding is necessary.

DOUBLE-CHECK: In each calculation the units work out correctly, so the answers are reasonable.

1.53. **THINK:** The given quantity is the speed limit, which is 45 miles per hour. The question asks for the speed limit in millifurlongs per microfortnight. The conversions 1 furlong = 1/8 mile, and 1 fortnight = 2 weeks are given in the question.

SKETCH: A sketch is not needed.

RESEARCH:

$$\frac{1\text{ mile}}{1\text{ hour}} = \frac{1\text{ mile}}{1\text{ hour}}\left(\frac{8\text{ furlongs}}{1\text{ mile}}\right)\left(\frac{10^3\text{ millifurlongs}}{1\text{ furlong}}\right)\left(\frac{24\text{ hours}}{1\text{ day}}\right)\left(\frac{14\text{ days}}{1\text{ fortnight}}\right)\left(\frac{1\text{ fortnight}}{10^6\text{ microfortnights}}\right)$$

SIMPLIFY: $\frac{1\text{ mile}}{1\text{ hour}} = 2.688\frac{\text{millifurlongs}}{\text{microfortnight}}$

CALCULATE: $\frac{45\text{ miles}}{\text{hour}} = 45\left(2.688\frac{\text{millifurlongs}}{\text{microfortnight}}\right) = 120.96\frac{\text{millifurlongs}}{\text{microfortnight}}$

ROUND: Because the given quantity contains two significant figures, the result must be rounded to remain consistent. A speed of 45 miles per hour is equivalent to a speed of 120 millifurlongs/microfortnight.

DOUBLE-CHECK: The conversion factor works out to be roughly 3 millifurlongs per microfortnight to each mile per hour, so the answer is reasonable.

1.54. **THINK:** The density of water is $\rho = 1000.\text{ kg/m}^3$. Determine if a pint of water weighs a pound. Remember that 1.00 kg = 2.21 lbs and 1.00 fluid ounce = 29.6 mL.

SKETCH: A sketch is not needed.

RESEARCH: 1 pint = 16 fluid ounces, mass = density · volume

SIMPLIFY: $1\text{ pint} = 1\text{ pint}\left(\frac{16\text{ fl. oz}}{1\text{ pint}}\right)\left(\frac{29.6\text{ mL}}{1.00\text{ fl. oz}}\right)\left(\frac{1\text{ cm}^3}{1\text{ mL}}\right)\left(\frac{1\text{ m}^3}{(100)^3\text{ cm}^3}\right) = 4.736\cdot 10^{-4}\text{ m}^3$

CALCULATE: $m = \left(1000.\text{ kg/m}^3\right)(4.736\cdot 10^{-4}\text{ m}^3) = 0.4736\text{ kg}$

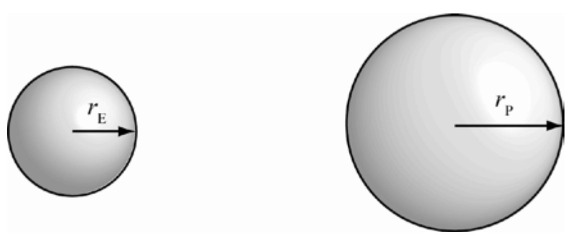
In pounds m is equal to $0.4736\text{ kg}\left(\frac{2.21\text{ lbs}}{1.00\text{ kg}}\right) = 1.046656\text{ lbs}$.

ROUND: Rounding to three significant figures, the weight is 1.05 lbs.

DOUBLE-CHECK: A pint is still a common measure for beverages, such as beer. A beer is relatively light and mainly comprised of water, so the answer is reasonable.

1.55. **THINK:** The radius of a planet, r_{p}, is 8.7 times greater than the Earth's radius, r_{E}. Determine how many times bigger the surface area of the planet is compared to the Earth's. Assume the planets are perfect spheres.

SKETCH:



RESEARCH: The surface area of a sphere is $A = 4\pi r^2$, so $A_{\text{E}} = 4\pi r_{\text{E}}^2$, and $A_{\text{p}} = 4\pi r_{\text{p}}^2$, and $r_p = 8.7 r_E$.

SIMPLIFY: $A_{\text{p}} = 4\pi\left(8.7 r_{\text{E}}\right)^2$

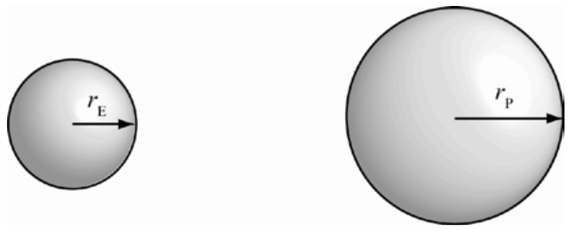
CALCULATE: $A_{\text{p}} = (75.69)4\pi r_{\text{E}}^2$, and $A_{\text{E}} = 4\pi r_{\text{E}}^2$. By comparison, $A_{\text{p}} = 75.69 A_{\text{E}}$.

ROUND: Rounding to two significant figures, the surface area of the planet is 76 times the surface area of Earth.

DOUBLE-CHECK: Since the area is proportional to the radius squared, it is expected that the surface area of the planet will be much larger than the surface area of the Earth, since its radius is 8.7 times Earth's radius.

1.56. **THINK:** The radius of the planet r_p is 5.8 times larger than the Earth's radius r_E. Assume the planets are perfect spheres.

SKETCH:



RESEARCH: The volume of a sphere is given by $V=(4/3)\pi r^3$. The volume of the planet is $V_p=(4/3)\pi r_p^{\ 3}$. The volume of the Earth is $V_E=(4/3)\pi r_E^{\ 3}$. $r_P=5.8r_E$.

SIMPLIFY: $V_p=(4/3)\pi(5.8r_E)^3$

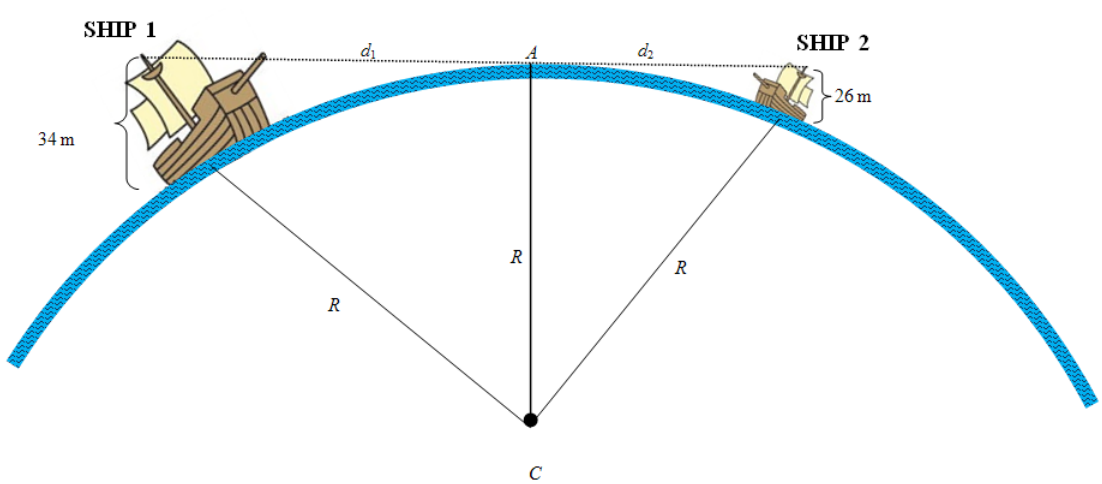
CALCULATE: $V_p=(4/3)\pi(5.8r_E)^3=195.112r_E^{\ 3}(4/3)\pi$. Recall, $V_E=(4/3)\pi r_E^{\ 3}$. Comparing the expressions, $V_p=195.112V_E$.

ROUND: To two significant figures, so $V_p=2.0\cdot 10^2 V_E$.

DOUBLE-CHECK: The volume of the planet is about 200 times the volume of the Earth. The volume of a sphere is proportional to the radius cubed, it is reasonable to get a much larger volume for the planet compared to the Earth's volume.

1.57. **THINK:** It is necessary to take the height of both masts and the curvature of the Earth into account when calculating the distance at which they can see one another. If the ships are at the maximum distance at which the sailors can see one another, then the line between the first sailor and the second sailor will be tangent to the surface of the earth.

SKETCH: Since it is necessary to take the curvature of the earth into account when solving this problem, the sketch will not be to scale. The radius of the Earth is labeled R and the center of the earth is labeled C. The farthest point on the horizon that can be seen by both sailors, which is also the point at which the line of sight between them is tangent to the Earth, is labeled A. The distance from the first sailor to point A is d_1 and the distance from the second sailor to point A is d_2.



RESEARCH: Because the line of sight between the sailors is tangent to the earth, it is perpendicular to the radius of the earth at point *A*. This means that the triangle formed by the first sailor, point *A*, and the center of the earth (point *C*) is a right triangle. The second sailor, point *A*, and point *C* also form a right triangle. Examining the figure, we can use the Pythagorean Theorem to find equations relating d_1 and d_2 to *R*: $R^2 + d_1^2 = (R+34)^2$ and $R^2 + d_2^2 = (R+26)^2$. The total distance will be the sum $d_1 + d_2$.

SIMPLIFY: First find expressions for the distances d_1 and d_2 and then use those to find the sum. The equation for d_1 gives:

$$R^2 + d_1^2 - R^2 = (R+34)^2 - R^2 \Rightarrow$$
$$d_1^2 = \left(R^2 + 2\cdot 34R + 34^2\right) - R^2 \Rightarrow$$
$$d_1 = \sqrt{2\cdot 34R + 34^2}$$

Similar calculations are used to find d_2:

$$R^2 + d_2^2 - R^2 = (R+26)^2 - R^2 \Rightarrow$$
$$d_2^2 = \left(R^2 + 2\cdot 26R + 26^2\right) - R^2 \Rightarrow$$
$$d_2 = \sqrt{2\cdot 26R + 26^2}$$

Combine to get an expression for the total distance between the ships: $d_1 + d_2 = \sqrt{2\cdot 34R + 34^2} + \sqrt{2\cdot 26R + 26^2}$.

CALCULATE: The radius of the earth is given in Solved Problem 1.2 as 6.37×10^6 m . Using this gives a final answer of:

$$d_1 + d_2 = \sqrt{2\cdot 34\text{ m}\cdot\left(6.37\times 10^6 34\text{ m}\right) + (34\text{ m})^2} + \sqrt{2\cdot 26\text{ m}\cdot\left(6.37\times 10^6\text{ m}\right) + (26\text{ m})^2}$$
$$= 39{,}012.54259\text{ m}.$$

ROUND: The radius of the earth used in this problem is known to three significant figures. However, the heights of the masts of the two ships are given to two significant figures. So, the final answer should have two significant figures: 3.9×10^4 m.

DOUBLE-CHECK: The maximum distance between the ships is a distance, so the units of meters seem correct. The calculated maximum distance at which the two sailors can see one another is 39 km. Calculating

$$d_1 = \sqrt{2\cdot 26\text{ m}\cdot\left(6.37\times 10^6\text{ m}\right) + (26\text{ m})^2} = 21\text{ km}$$

and

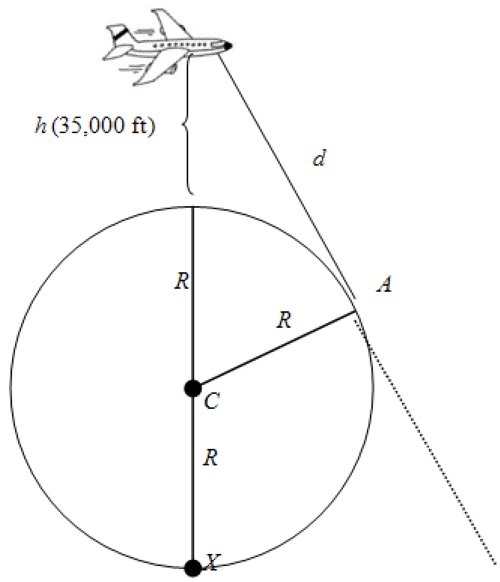
$$d_2 = \sqrt{2\cdot 26\text{ m}\cdot\left(6.37\times 10^6\text{ m}\right) + (26\text{ m})^2} = 18\text{ km}$$

confirms that the sailor from ship 1, sitting at the top of a slightly taller mast, can see about 3 km further than his companion. These distances seem reasonable: an average person looking out over the ocean sees about 4.7 km, and the view from 413 m atop the Willis Tower was calculated to be 72.5 km. Since the masts are significantly taller than a person but much shorter than the height of the Willis Tower, the final answer of 39 km seems reasonable.

An alternate way to calculate this would have been to use the secant-tangent theorem, which states that the square of the distance from the sailor to the horizon equals the product the height of the mast times the sum of the height of the mast and the diameter of the earth: $(d_1)^2 = (2R+34)\cdot 34$ and $(d_2)^2 = (2R+26)\cdot 26$. Using this formula confirms the answer:

$$\begin{aligned} d_1 + d_2 &= \sqrt{(2R+34)\cdot 34} + \sqrt{(2R+26)\cdot 26} \\ &= \sqrt{\left(2\left(6.37\times 10^6 \text{ m}\right)+34 \text{ m}\right)\cdot 34 \text{ m}} + \sqrt{\left(2\left(6.37\times 10^6 \text{ m}\right)+26 \text{ m}\right)\cdot 26 \text{ m}} \\ &= 39 \text{ km} \end{aligned}$$

1.58. **THINK:** The altitude of the jet liner is given in feet, so it will be necessary to convert to meters before calculating the answer. The horizon is the furthest point that can be seen in perfect weather conditions. Since we don't know where and when the plane is flying, we will approximate the Earth as a perfect sphere.
SKETCH: Since the radius of the earth is important here, the sketch will not be to scale. Point *A* is a furthest point on the horizon that can be seen from the plane, and *C* marks the center of the Earth, and *R* indicates the radius of the Earth. Point *X* is on the surface of the Earth directly opposite from where the plane is flying.



RESEARCH: The line of sight from the plane to the furthest point on the horizon (point *A*) is tangential to the Earth at point *A*. So, it must be perpendicular to the radius of the Earth at point *A*. This means that the plane, point *A*, and point *C* form a right triangle. The Pythagorean Theorem states that $d^2 + R^2 = (R+h)^2$. To find the distance *d*, it is necessary to use consistent units. Since the radius of the Earth (*R*) is given in meters, it is easiest to convert the height *h* from feet to meters using the fact that 1 m = 3.281 ft.

SIMPLIFY: First convert the height of the plane from feet to meters, multiplying *h* by $\frac{1 \text{ m}}{3.281 \text{ ft}}$. Then, solve the expression $d^2 + R^2 = (R+h)^2$ for *d*, the distance we want to find:

$$\begin{aligned} d^2 + R^2 - R^2 &= (R+h)^2 - R^2 \Rightarrow \\ d^2 &= \left(R^2 + 2Rh + h^2\right) - R^2 \Rightarrow \\ d &= \sqrt{2Rh + h^2} \end{aligned}$$

CALCULATE: The radius of the Earth $R = 6.37\cdot 10^6$ m and the plane is flying $h = 35{,}000 \text{ ft} \cdot \frac{1 \text{ m}}{3.281 \text{ ft}}$ above the ground. Using these numbers, the distance to the horizon is

$$d = \sqrt{2\left(6.37\times 10^6 \text{ m}\right)\left(35{,}000 \text{ ft}\frac{1 \text{ m}}{3.281 \text{ ft}}\right) + \left(35{,}000 \text{ ft}\frac{1 \text{ m}}{3.281 \text{ ft}}\right)^2} = 368{,}805.4813 \text{ m}.$$

ROUND: Though it is ambiguous, the height of the jetliner above the ground is known to at least two significant figures. The radius of the Earth is known to three significant figures and the conversion from feet to meters uses four significant figures. So, the answer is known to two significant digits. This gives a final distance of $3.7 \cdot 10^5$ m or 370 km.

DOUBLE-CHECK: The answer is given in units of meters or kilometers. Since the distance to the horizon is a length, the units are correct. 370 km is the approximate distance between Los Angeles and Las Vegas. Indeed, in an airplane at cruising altitude, it is just possible to see the Los Angeles coast as you fly over Las Vegas, so this answer seems reasonable. It is also possible to check the answer by working backwards. The secant-tangent theorem states that the square of the distance d equals the product of the height of the plane over the Earth h and the distance from the jetliner to point X on the other side of the Earth. Use this to find the height of the plane in terms of the distance to the horizon and the radius of the earth:

$$d^2 = h(h+2R) = h^2 + 2Rh \Rightarrow$$
$$0 = h^2 + 2Rh - d^2 \Rightarrow$$
$$h = \frac{-2R \pm \sqrt{(2R)^2 + 4d^2}}{2} = -R \pm \sqrt{R^2 + d^2}$$
$$= -\left(6.37 \times 10^6 \text{ m}\right) + \sqrt{\left(6.37 \times 10^6 \text{ m}\right)^2 + \left(3.7 \times 10^5 \text{ m}\right)^2}$$
$$= 10736.63 \text{ m}$$

Converting this back to feet and rounding to 2 significant figures gives confirmation that the answer was correct: $10736.63459 \text{ m} \cdot \frac{3.281 \text{ ft}}{1 \text{ m}} = 35{,}000 \text{ ft}.$

1.59. **THINK:** The given quantity is 1.56 barrels of oil. Calculate how many cubic inches are in 1.56 barrels. 1 barrel of oil = 42 gallons = (42 gal)(231 cu. in./gal) = 9702 cubic inches.

SKETCH: A sketch is not needed.

RESEARCH: If a volume V_1 is given in barrels then the equivalent volume V_2 in cubic inches is given by the formula $V_2 = V_1 \frac{9702 \text{ cu. in.}}{1 \text{ barrel}}$

SIMPLIFY: Not applicable.

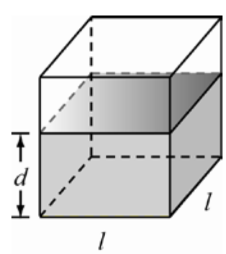
CALCULATE: $1.56 \text{ barrels}\left(\frac{9702 \text{ cu. in}}{1 \text{ barrel}}\right) = 15135.12 \text{ cu. in.}$

ROUND: The value given in the question has three significant figures, so the final answer is that 1.56 barrels is equivalent to $1.51 \cdot 10^4$ cubic inches.

DOUBLE-CHECK: Barrels are not commonly used units. However, since the proper conversion factor of 9702 cubic inches per barrel was used, the answer is accurate.

1.60. **THINK:** The car's gas tank has the shape of a rectangular box with a square base whose sides measure $l = 62$ cm. The tank has only 1.5 L remaining. The question asks for the depth, d of the gas remaining in the tank. The car is on level ground, so that d is constant.

SKETCH:

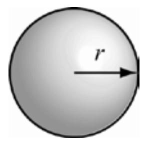


RESEARCH: $A_{\text{tank}} = l^2$. The volume of gas remaining is $V_{\text{gas}} = A_{\text{tank}} \times d$. Convert the volume 1.5 L to 1500 cm^3 by using $1 \text{ mL} = 1 \text{ cm}^3$.
SIMPLIFY: $d = V_{\text{gas}} / A_{\text{tank}}$, but $A_{\text{tank}} = l^2$, so substitute this into the expression for *d*: $d = V_{\text{gas}} / l^2$.

CALCULATE: $d = \dfrac{1500 \text{ cm}^3}{(62 \text{ cm})^2} = 0.390218 \text{ cm}$

ROUND: To two significant figures $d = 0.39$ cm.
DOUBLE-CHECK: The car's gas tank will hold 52 L but only has 1.5 L remaining. The sides of the gas tank are 62 cm and because the gas tank is almost empty, there should be a small depth of gas in the bottom of the tank, so the answer is reasonable.

1.61. **THINK:** The formula for the volume of a sphere is given by $V_{\text{sphere}} = (4/3)\pi r^3$. The formula for density is given by $\rho = m/V$. Refer to Appendix B in the text book and express the answers in SI units using scientific notation.
SKETCH:



RESEARCH: The radius of the Sun is $r_{\text{S}} = 6.96 \cdot 10^8$ m, the mass of the Sun is $m_{\text{S}} = 1.99 \cdot 10^{30}$ kg, the radius of the Earth is $r_{\text{E}} = 6.37 \cdot 10^6$ m, and the mass of the Earth is $m_{\text{E}} = 5.98 \cdot 10^{24}$ kg.
SIMPLIFY: Not applicable.
CALCULATE:

(a) $V_{\text{S}} = \dfrac{4}{3}\pi r_{\text{S}}^3 = \dfrac{4}{3}\pi (6.96 \cdot 10^8)^3 = 1.412265 \cdot 10^{27} \text{ m}^3$

(b) $V_{\text{E}} = \dfrac{4}{3}\pi r_{\text{E}}^3 = \dfrac{4}{3}\pi (6.37 \cdot 10^6)^3 = 1.082696 \cdot 10^{21} \text{ m}^3$

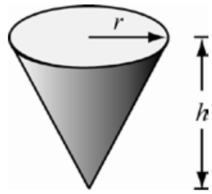
(c) $\rho_{\text{S}} = \dfrac{m_{\text{S}}}{V_{\text{S}}} = \dfrac{1.99 \cdot 10^{30}}{1.412265 \cdot 10^{27}} = 1.40908 \cdot 10^3 \text{ kg/m}^3$

(d) $\rho_{\text{E}} = \dfrac{m_{\text{E}}}{V_{\text{E}}} = \dfrac{5.98 \cdot 10^{24}}{1.082696 \cdot 10^{21}} = 5.523249 \cdot 10^3 \text{ kg/m}^3$

ROUND: The given values have three significant figures, so the calculated values should be rounded as:
(a) $V_{\text{S}} = 1.41 \cdot 10^{27} \text{ m}^3$
(b) $V_{\text{E}} = 1.08 \cdot 10^{21} \text{ m}^3$
(c) $\rho_{\text{S}} = 1.41 \cdot 10^3 \text{ kg/m}^3$
(d) $\rho_{\text{E}} = 5.52 \cdot 10^3 \text{ kg/m}^3$
DOUBLE-CHECK: The radius of the Sun is two orders of magnitude larger than the radius of the Earth. Because the volume of a sphere is proportional to the radius cubed, the volume of the Sun should be $(10^2)^3$ or 10^6 larger than the volume of the Earth, so the calculated volumes are reasonable. Because density depends on mass and volume, and the Sun is roughly 10^6 times larger and more massive than the Earth, it is not surprising that the density of the Sun is on the same order of magnitude as the density of the Earth (e.g. $10^6 / 10^6 = 1$). Earth is primarily solid, but the Sun is gaseous, therefore it is reasonable that the Earth is denser than the Sun.

1.62. **THINK:** The tank is in the shape of an inverted cone with height $h =$ 2.5 m and radius r = 0.75 m. Water is poured into the tank at a rate of w = 15 L/s. Calculate how long it will take to fill the tank. Recall the conversion $1\text{ L} = 1000\text{ cm}^3$.

SKETCH:



RESEARCH: The volume of a cone is $V_{\text{cone}} = \frac{1}{3}\pi r^2 h$. The rate water enters the cone is $w = \frac{V_{\text{water}}}{t}$ where t is time. When the cone is full, $V_{\text{cone}} = V_{\text{water}}$, therefore $\frac{1}{3}\pi r^2 h = wt$.

SIMPLIFY: $t = \frac{\pi r^2 h}{3w}$

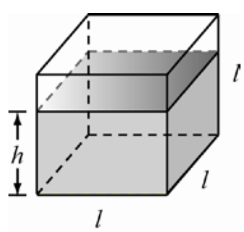
CALCULATE: $t = \frac{\pi(75\text{ cm})^2(250\text{ cm})}{3\left(\frac{15000\text{ cm}^3}{\text{s}}\right)} = 98.1748\text{ s}$

ROUND: To two significant figures, t = 98 s.

DOUBLE-CHECK: The calculation resulted in the correct units, so the answer is reasonable.

1.63. **THINK:** The rate of water flow is 15 L/s, the tank is cubical, and the top surface of the water rises by 1.5 cm/s. Let h be the height of the water in the tank.

SKETCH:



RESEARCH: The change in the volume of the water, ΔV_{water}, is $15\text{ L/s} = 15000\text{ cm}^3\text{/s}$. The change in the height of the water is $\Delta h = 1.5\text{ cm/s}$. An equation to find the side length of the tank is $\Delta V_{\text{water}} = l^2 \Delta h$.

SIMPLIFY: $l = \sqrt{\frac{\Delta V_{\text{water}}}{\Delta h}}$

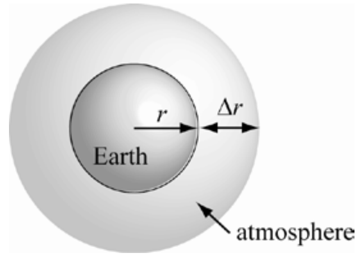
CALCULATE: $l = \sqrt{\left(\frac{15000\text{ cm}^3\text{ / s}}{1.5\text{ cm/s}}\right)} = 100.\text{ cm}$

ROUND: $l = 1.0 \cdot 10^2\text{ cm}$

DOUBLE-CHECK: The flow rate of 15 L/s is quite fast, but the level of the water is rising by only 1.5 cm/s, so it is reasonable that the tank is relatively large.

1.64. **THINK:** The atmosphere has an effective weight of 15 pounds per square inch. By computing the surface area of the Earth, it will be easy to compute the mass of the atmosphere. Then, since the atmosphere is assumed to have a uniform density of 1.275 kg/m^3, the mass can be converted to a volume. The volume of the atmosphere is the difference of two spheres, whose radii are the radius of the Earth, r_{E}, and the radius of the Earth plus the thickness of the atmosphere, Δr. The result will be a cubic equation with one real root which can be approximated to give the thickness of the atmosphere.

SKETCH:



RESEARCH: Recall the conversions 1 inch = 0.0254 m and 1 kg = 2.205 lbs. The radius of the Earth is about 6378 km. The surface area of the Earth is $A_\text{E} = 4\pi r_\text{E}^{\;2}$. The mass of the atmosphere is $m_\text{A} = A_\text{E}\left(15 \text{ lb/sq in}\right)$. The volume of the atmosphere can be computed using the ratio $V_\text{A} = m_\text{A} / \rho_\text{A}$, where ρ_A is the density of the atmosphere. This volume is the difference of the two spheres, as shown in the sketch. The volume of the Earth (without its atmosphere) is $V_\text{E} = (4/3)\pi r_\text{E}^{\;3}$, and the volume of the Earth and atmosphere is $V_\text{EA} = (4/3)\pi\left(r_\text{E} + \Delta r\right)^3$. A second method of computing the volume of the atmosphere is $V_\text{A} = V_\text{EA} - V_\text{E}$. Set the two values of V_A equal and solve for *r*.

SIMPLIFY: The first expression for the volume of the atmosphere is

$$V_\text{A} = \frac{m_\text{A}}{\rho_\text{A}} = \frac{4\pi r_\text{E}^{\;2}}{\rho_\text{A}}\left(\frac{15 \text{ lb}}{1 \text{ square inch}}\right).$$

The second expression is $V_\text{A} = (4\pi/3)\left((r_\text{E} + \Delta r)^3 - r_\text{E}^3\right)$. Setting these expressions equal to each other gives an equation to solve for Δr.

CALCULATE: $r_\text{E} = 6378 \text{ km}\left(\frac{1000 \text{ m}}{1 \text{ km}}\right)\left(\frac{1 \text{ in}}{0.0254 \text{ m}}\right) = 2.511 \cdot 10^8 \text{ in}$

$$\rho_\text{A} = 1.275 \frac{\text{kg}}{\text{m}^3}\left(\frac{0.0254 \text{ m}}{1 \text{ in}}\right)^3 = 2.089 \cdot 10^{-5} \frac{\text{kg}}{\text{in}^3}$$

Substituting into the first equation for V_A gives:

$$V_\text{A} = \frac{4\pi\left(2.511 \cdot 10^8 \text{ in}\right)^2}{2.089 \cdot 10^{-5} \text{kg/in}^3}\left(\frac{15 \text{ lb}}{1 \text{ square inch}}\right)\left(\frac{1 \text{ kg}}{2.205 \text{ lb}}\right) = 2.580 \cdot 10^{23} \text{ inch}^3.$$

The second equation for V_A becomes:

$$V_\text{A} = \frac{4\pi}{3}\left((2.511 \cdot 10^8 \text{ in} + \Delta r)^3 - \left(2.511 \cdot 10^8 \text{ in}\right)^3\right) = 4.1888\Delta r^3 + \left(3.155 \cdot 10^9\right)\Delta r^2 + \left(7.923 \cdot 10^{17}\right)\Delta r.$$

Setting the two equations for V_A equal results in the equation:

$$4.1888\Delta r^3 + \left(3.155 \cdot 10^9\right)\Delta r^2 + \left(7.923 \cdot 10^{17}\right)\Delta r = 2.580 \cdot 10^{23},$$

a cubic equation in Δr. This equation can be solved by a number of methods. A graphical estimate is sufficient. It has one real root, and that is at approximately

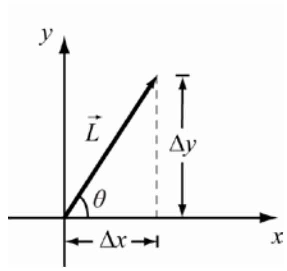
$$\Delta r = 325300 \text{ in} = 325300 \text{ in } \frac{0.0254 \text{ m}}{1 \text{ in}} = 8263 \text{ m}.$$

ROUND: The least precise value given in the question had two significant figures, so the answer should be rounded to 8300 m.

DOUBLE-CHECK: The result has units of distance, which is what is expected. What may not be expected is that our result is not as big as the height of the tallest mountain on Earth, Mt. Everest, which has a height of 8.8 km. We can obtain a simple approximation of our result by realizing that our calculated value for Δr is small compared to the radius of Earth. This means that the surface of a sphere with radius $R_\text{E} + \Delta r$ and one with radius R_E are not very different, allowing us to write an approximation to our result as $\Delta r \approx V_A / \left(4\pi r_E^2\right) = \left(2.580 \cdot 10^{23} \text{ inch}^3\right) / \left(4\pi \left(2.511 \cdot 10^8 \text{ inch}\right)^2\right) = 3.256 \cdot 10^5 \text{ inch} = 8.3 \text{ km}.$

1.65. **THINK:** Let $\vec{L}$ be the position vector. Then $\left|\vec{L}\right| = 40.0$ m and $\theta = 57.0°$ (above x-axis).

SKETCH:



RESEARCH: From trigonometry, $\sin\theta = \Delta y / \left|\vec{L}\right|$ and $\cos\theta = \Delta x / \left|\vec{L}\right|$. The length of the vector $\vec{L}$ is given by the formula $\left|\vec{L}\right| = \sqrt{\Delta x^2 + \Delta y^2}$.

SIMPLIFY: $\Delta x = \left|\vec{L}\right|\cos\theta$, $\Delta y = \left|\vec{L}\right|\sin\theta$

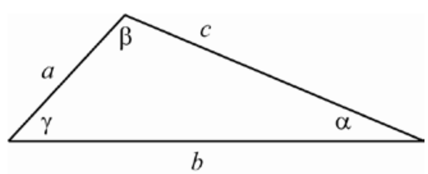
CALCULATE: $\Delta x = \left(40.0 \text{ m}\right)\cos(57.0°) = 21.786 \text{ m}$, $\Delta y = \left(40.0 \text{ m}\right)\sin(57.0°) = 33.547 \text{ m}$

ROUND: $\Delta x = 21.8$ m and $\Delta y = 33.5$ m.

DOUBLE-CHECK: $\left|\vec{L}\right| = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{(21.8 \text{ m})^2 + (33.5 \text{ m})^2} \approx 40.0$ m, to three significant figures.

1.66. **THINK:** $a = 6.6$ cm, $b = 13.7$ cm, and $c = 9.2$ cm are the given quantities.

SKETCH:



RESEARCH: Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\gamma$

SIMPLIFY:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$
$$2ab\cos\gamma = a^2 + b^2 - c^2$$
$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\gamma = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

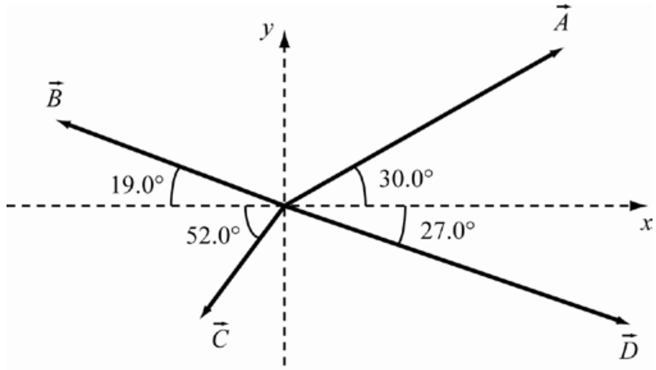
CALCULATE: $\gamma = \cos^{-1}\left[\frac{(6.6 \text{ cm})^2 + (13.7 \text{ cm})^2 - (9.2 \text{ cm})^2}{2(6.6 \text{ cm})(13.7 \text{ cm})}\right] = 35.83399°$

ROUND: $\gamma = 36°$

DOUBLE-CHECK: The angle γ in the figure is less than $45°$, so the answer is reasonable.

1.67. **THINK:** The lengths of the vectors are given as $\left|\vec{A}\right| = 75.0$, $\left|\vec{B}\right| = 60.0$, $\left|\vec{C}\right| = 25.0$ and $\left|\vec{D}\right| = 90.0$. The question asks for the vectors to be written in terms of unit vectors. Remember, when dealing with vectors, the x- and y-components must be treated separately.

SKETCH:



RESEARCH: The formula for a vector in terms of unit vectors is $\vec{V} = V_x\hat{x} + V_y\hat{y}$. Since $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, $\sin\theta = \frac{A_y}{|\vec{A}|}$ and $\cos\theta = \frac{A_x}{|\vec{A}|}$.

$\theta_A = 30.0°$, $\theta_B = 19.0° = 161.0°$ (with respect to the positive x-axis),

$\theta_C = 52.0° = 232.0°$ (with respect to the positive x-axis),

$\theta_D = 27.0° = 333.0°$ (with respect to the positive x-axis).

SIMPLIFY: $A_x = |\vec{A}|\cos\theta_A$, $A_y = |\vec{A}|\sin\theta_A$, $B_x = |\vec{B}|\cos\theta_B$, $B_y = |\vec{B}|\sin\theta_B$, $C_x = |\vec{C}|\cos\theta_C$, $C_y = |\vec{C}|\sin\theta_C$, $D_x = |\vec{D}|\cos\theta_D$, and $D_y = |\vec{D}|\sin\theta_D$.

CALCULATE: $A_x = 75.0\cos 30.0° = 64.9519\hat{x}$, $A_y = 75.0\sin 30.0° = 37.5\hat{y}$

$B_x = 60.0\cos 161.0° = -56.73\hat{x}$, $B_y = 60.0\sin 161.0° = 19.534\hat{y}$

$C_x = 25.0\cos 232.0° = -15.3915\hat{x}$, $C_y = 25.0\sin 232.0° = -19.70027\hat{y}$

$D_x = 90.0\cos 333.0° = 80.19058\hat{x}$, $D_y = 90.0\sin 333.0° = -40.859144\hat{y}$

ROUND: The given values had three significant figures so the answers must be rounded to:

$\vec{A} = 65.0\hat{x} + 37.5\hat{y}$, $\vec{B} = -56.7\hat{x} + 19.5\hat{y}$, $\vec{C} = -15.4\hat{x} - 19.7\hat{y}$, $\vec{D} = 80.2\hat{x} - 40.9\hat{y}$.

DOUBLE-CHECK: Comparing the calculated components to the figure provided shows that this answer is reasonable.

1.68. **THINK:** Use the components in Question 1.65 to find the sum of the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$. Also, calculate the magnitude and direction of $\vec{A} - \vec{B} + \vec{D}$. Remember, when dealing with vectors the x and y components must be treated separately. Treat the values given in the question as accurate to the nearest decimal, and hence as having two significant figures.

SKETCH: Not applicable.

RESEARCH:

(a) The resultant vector is $\vec{V} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$.

(b) The magnitude of a vector is $|\vec{V}| = \sqrt{(V_x)^2 + (V_y)^2}$. The direction of the vector $\vec{V}$ is $\theta_V = \tan^{-1}\left(V_y / V_x\right)$.

SIMPLIFY:

(a) $\vec{A} + \vec{B} + \vec{C} + \vec{D} = (A_x + B_x + C_x + D_x)\hat{x} + (A_y + B_y + C_y + D_y)\hat{y}$

(b) $|\vec{V}| = |\vec{A} - \vec{B} + \vec{D}| = \sqrt{(A_x - B_x + D_x)^2 + (A_y - B_y + D_y)^2}$

$$\theta_V = \tan^{-1}\left(\frac{A_y - B_y + D_y}{A_x - B_x + D_x}\right)$$

CALCULATE:

(a) $\vec{A}+\vec{B}+\vec{C}+\vec{D}=(65.0-56.7-15.4+80.2)\hat{x}+(37.5+19.5-19.7-40.9)\hat{y}$

$$=73.1\hat{x}-3.6\hat{y}$$

(b) $\left|\vec{A}-\vec{B}+\vec{D}\right|=\sqrt{(65.0-(-56.7)+80.2)^2+(37.5-19.7-40.9)^2}=203.217$

$$\theta_V=\tan^{-1}\left[\frac{37.5-19.7.-40.9}{65.0-(-56.7)+80.2}\right]=-6.5270°$$

ROUND:

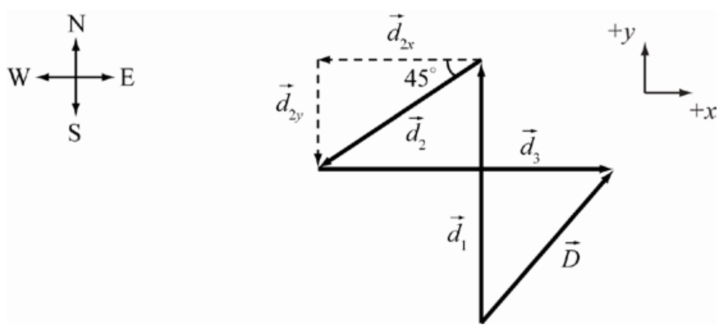
(a) Not necessary.

(b) The given magnitudes have three significant figures, so $\left|\vec{A}-\vec{B}+\vec{D}\right|=203$, at $-6.53°$ (below the x-axis).

DOUBLE-CHECK: The length of the resulting vector is less than the sum of the lengths of the component vectors. Since the vector points into the fourth quadrant, the angle made with the x-axis should be negative, as it is.

1.69. **THINK:** The problem involves adding vectors, therefore break the vectors up into their components and add the components. SW is exactly $45°$ south of W. $\vec{d}_1$ = 4.47 km N, $\vec{d}_2=2.49$ km SW, $\vec{d}_3$ = 3.59 km E.

SKETCH:



RESEARCH: Use $\vec{D}$ = $\vec{d}_1+\vec{d}_2+\vec{d}_3$ = $D_x\hat{x}+D_y\hat{y}$, and recall the formula for the length of $\vec{D}$: $\left|\vec{D}\right|$ = $\sqrt{D_x^{\ 2}+D_y^{\ 2}}$. Decompose each summand vector into components $\vec{d}_i$ = $d_{ix}\ \hat{x}$ + $d_{iy}\ \hat{y}$, with summand vectors: $\vec{d}_1=d_1\hat{y},\ \vec{d}_2=d_{2x}\hat{x}+d_{2y}\hat{y}=-d_2\sin(45°)\hat{x}-d_2\cos(45°)\hat{y},\ \vec{d}_3=d_3\hat{x}.$

SIMPLIFY: Therefore, $\vec{D}=\vec{d}_1+\vec{d}_2+\vec{d}_3=(d_3-d_2\sin(45°))+(d_1-d_2\cos(45°))$ and $\left|\vec{D}\right|=\sqrt{(d_3-d_2\sin(45°))^2+(d_1-d_2\cos(45°))^2}$.

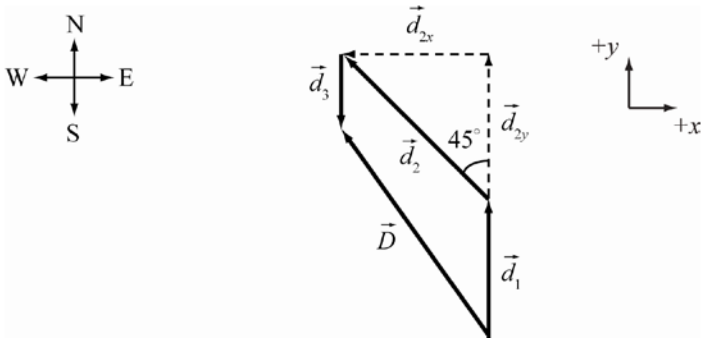
CALCULATE: $\left|\vec{D}\right|=\sqrt{(3.59-2.49\cos(45°))^2+(4.47-2.49\sin(45°))^2}$ = 3.269 km

ROUND: $\left|\vec{D}\right|=3.27$ km

DOUBLE-CHECK: Given that all vectors are of the same order of magnitude, the distance from origin to final position is less than d_1, as is evident from the sketch. This means that the calculated answer is reasonable.

1.70. **THINK:** The problem involves adding vectors, therefore break the vectors up into their components and add the components. NW is exactly $45°$ north of west. $\vec{d}_1$ = 20 paces N, $\vec{d}_2=30$ paces NW, $\vec{d}_3$ = 10 paces S. Paces are counted to the nearest integer, so treat the number of paces as being precise.

SKETCH:



RESEARCH:, Use $\vec{D} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3 = D_x \hat{x} + D_y \hat{y}$, and recall the formula for the length of $\vec{D}$: $\left|\vec{D}\right| = \sqrt{D_x^{\ 2} + D_y^{\ 2}}$. Decompose each summand vector into components $\vec{d}_i = d_{ix}\hat{x} + d_{iy}\hat{y}$, with summand vectors: $\vec{d}_1 = d_1\hat{y}$, $\vec{d}_2 = d_{2x}\hat{x} + d_{2y}\hat{y} = -d_2 \sin(45^\circ)\hat{x} - d_2\cos(45^\circ)\hat{y}$, $\vec{d}_3 = -d_3\hat{y}$

SIMPLIFY: $\vec{D} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3 = -d_2 \sin(45°)\hat{x} + (d_1 - d_3 + d_2\cos(45°))\hat{y}$ and

$\left|\vec{D}\right| = \sqrt{(-d_2\sin(45°))^2 + (d_1 - d_3 + d_2\cos(45°))^2}$.

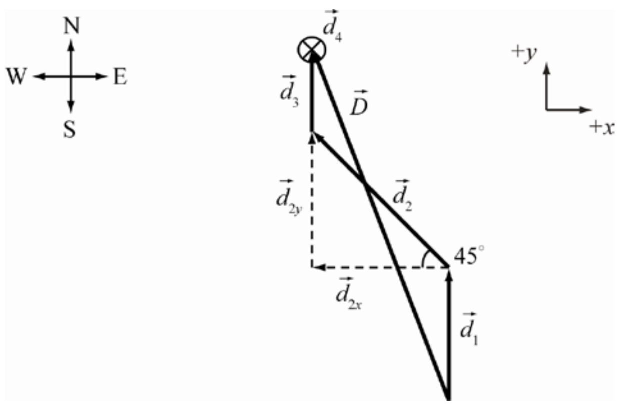
CALCULATE: $\left|\vec{D}\right| = \sqrt{(-30\sin(45°))^2 + (20 - 10 + 30\cos(45°))^2} = 37.739$ paces

ROUND: 38 paces

DOUBLE-CHECK: Given that $d_1 > d_3$, the calculated answer makes sense since the distance D should be greater than d_2.

1.71. **THINK:** The problem involves adding vectors, therefore break the vectors up into their components and add the components. NW is 45° north of west. $\vec{d}_1$ = 20 paces N, $\vec{d}_2$ = 30 paces NW, $\vec{d}_3$ = 12 paces N, $\vec{d}_4$ = 3 paces into ground ($\vec{d}_4$ implies 3 dimensions). Paces are counted to the nearest integer, so treat the number of paces as being precise.

SKETCH:



RESEARCH: $\vec{D} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4$, $\vec{d}_i = d_{ix}\hat{x} + d_{iy}\hat{y} + d_{iz}\hat{z}$, $\left|\vec{D}\right| = \sqrt{D_x^{\ 2} + D_y^{\ 2} + D_z^{\ 2}}$, $\vec{d}_1 = d_1\hat{y}$,

$\vec{d}_2 = -d_{2x}\hat{x} + d_{2y}\hat{y} = -d_2\cos(45°)\hat{x} + d_2\sin(45°)\hat{y}$, $\vec{d}_3 = d_3\hat{y}$, and $\vec{d}_4 = -d_4\hat{z}$.

SIMPLIFY: $\vec{D} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3 + \vec{d}_4 = -d_2\cos(45°)\hat{x} + (d_1 + d_3 + d_2\sin(45°))\hat{y} - d_4\hat{z}$ and

$\left|\vec{D}\right| = \sqrt{(-d_2\cos(45°))^2 + (d_1 + d_3 + d_2\sin(45°))^2 + (-d_4)^2}$.

CALCULATE: $\vec{D} = -30\frac{\sqrt{2}}{2}\hat{x} + \left(20 + 12 + 30\frac{\sqrt{2}}{2}\right)\hat{y} - 3\hat{z}$

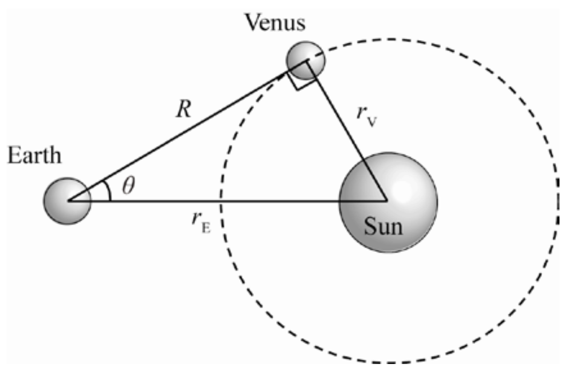
$\left|\vec{D}\right| = \sqrt{(-21.213)^2 + (53.213)^2 + (-3)^2} = 57.36$ paces

ROUND: $\vec{D} = -15\sqrt{2}\hat{x} + \left(32 + 15\sqrt{2}\right)\hat{y} - 3\hat{z}$ and round the number of paces to the nearest integer: $\left|\vec{D}\right| = 57$ paces.

DOUBLE-CHECK: Distance should be less than the sum of the magnitudes of each vector, which is 65. Therefore, the calculated answer is reasonable.

1.72. **THINK:** Consider the Sun to be the centre of a circle with the distance from the Sun to Venus, as the radius. Earth is located a distance $r_{\text{E}} = 1.5 \cdot 10^{11}$ m from the Sun, so that the three bodies make a triangle and the vector from Earth to the Sun is at ̊0 The vector pointing from Earth to Venus inte rsects Venus' orbit one or two times, depending on the angle Venus makes with the Earth. This angle is at a maximum when the vector intersects the orbit only once, while all other angles cause the vector to intersect twice. If the vector only intersects the circle once, then that vector is tangential to the circle and therefore is perpendicular to the radius vector of the orbit. This means the three bodies make a right triangle with r_{E} as the hypotenuse. Simple trigonometry can then be used to solve for the angle and distance.

SKETCH:



RESEARCH: $r_{\text{E}}^2 = r_{\text{V}}^2 + R^2$, $r_{\text{E}} \sin\theta = r_{\text{V}}$

SIMPLIFY: $R = \sqrt{r_{\text{E}}^2 - r_{\text{V}}^2}$, $\theta = \sin^{-1}\left(r_{\text{V}} / r_{\text{E}}\right)$

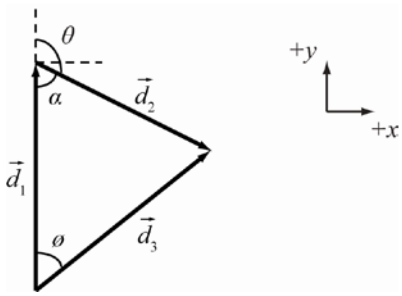
CALCULATE: $R = \sqrt{(1.5 \cdot 10^{11})^2 - (1.1 \cdot 10^{11})^2} = 1.0198 \cdot 10^{11}$ m , $\theta = \sin^{-1}\left(\frac{1.1 \cdot 10^{11}}{1.5 \cdot 10^{11}}\right) = 47.17°$

ROUND: $R = 1.0 \cdot 10^{11}$ m, $\theta = 47°$

DOUBLE-CHECK: If it had been assumed that $\theta = \tan^{-1}\left(r_{\text{V}} / r_{\text{E}}\right)$ when the E-to-S-to-V angle was $90°$, then $\tan\theta$ would be about $36°$. Therefore the maximum angle should be greater than this, so the answer is reasonable.

1.73. **THINK:** All angles and directions of vectors are unknown. All that is known are the distances walked, d_1 = 550 m and d_2 = 178 m, and the distance d_3 = 432 m that the friend is now away from you. Since the distances are the sides of a triangle, use the cosine law to determine the internal (and then external) angles. Also, since $d_3 < d_1$, he must have turned back towards you partially, i.e. he turned more than $90°$, but less than $180°$.

SKETCH:



RESEARCH: $d_2^{\,2} = d_1^{\,2} + d_3^{\,2} - 2d_1d_3\cos\phi,\;\; d_3^{\,2} = d_1^{\,2} + d_2^{\,2} - 2d_1d_2\cos\alpha,\;\; \theta + \alpha = 180°$

SIMPLIFY: $2d_1d_3\cos\phi = d_1^{\,2} + d_3^{\,2} - d_2^{\,2} \;\Rightarrow\; \phi = \cos^{-1}\left(\dfrac{d_1^{\,2} + d_3^{\,2} - d_2^{\,2}}{2d_1d_3}\right)$

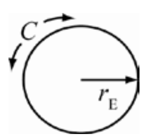
Likewise, $\alpha = \cos^{-1}\left(\dfrac{d_1^{\,2} + d_2^{\,2} - d_3^{\,2}}{2d_1d_2}\right)$.

CALCULATE: $\phi = \cos^{-1}\left(\dfrac{(550\text{ m})^2 + (432\text{ m})^2 - (178\text{ m})^2}{2(550\text{ m})(432\text{ m})}\right) = 15.714°$

$\alpha = \cos^{-1}\left(\dfrac{(550\text{ m})^2 + (178\text{ m})^2 - (432\text{ m})^2}{2(550\text{ m})(178\text{ m})}\right) = 41.095°,\;\; \theta = 180° - 41.095° = 138.905°$

ROUND: Since $d_1 = 550\text{ m}$ has two significant figures (which is the fewest) the answers should be rounded to two significant figures. This means: $\phi = 16°$, $\alpha = 41°$ and then $\theta = 139°$. The two possibilities are that the friend turned to the right or the left (a right turn is shown in the diagram). The direction the friend turned doesn't matter, he turns by the same amount regardless of which direction it was.
DOUBLE-CHECK: The friend turned through an angle of 140 degrees. The angle between the initial departure and the final location is 16 degrees. These are both reasonable angles.

1.74. **THINK:** Assume that the Earth is a perfect sphere with radius, r_{E} = 6378 km, and treat the circumference of Earth as the circumference of a circle.
SKETCH:

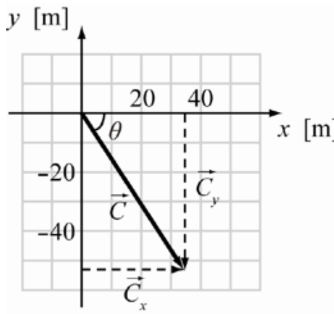


RESEARCH: The circumference of a circle is given by $C = 2\pi r$.
SIMPLIFY: $C = 2\pi r_{\text{E}}$
CALCULATE: $C = 2\pi\left(6378\text{ km}\right) = 40074\text{ km}$
ROUND: The instructions from the question say to round to three significant figures: $C = 4.01\cdot 10^4$ km.
DOUBLE-CHECK: Assuming a hot air balloon has an average velocity of 20 km/h, then it would take about 80 days to travel, hence the phrase around the world in 80 days.

1.75. $4{,}308{,}229 \approx 4\cdot 10^6$; $44 \approx 4\cdot 10^1, \left(4\cdot 10^6\right)\left(4\cdot 10^1\right) = 16\cdot 10^7 = 2\cdot 10^8$

1.76. $3\hat{x} + 6\hat{y} - 10\hat{z} + \vec{C} = -7\hat{x} + 14\hat{y}, \qquad \vec{C} = (-7\hat{x} - 3\hat{x}) + (14\hat{y} - 6\hat{y}) + 10\hat{z} = -10\hat{x} + 8\hat{y} + 10\hat{z}$

1.77. **THINK:** An angle is measured counter-clockwise from the positive x-axis (0°). $\vec{C} = (34.6 \text{ m}, -53.5 \text{ m})$. It is also possible to measure clockwise from the positive x-axis and consider the measure to be negative.
SKETCH:



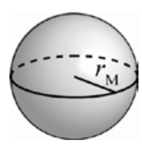
RESEARCH: $\left|\vec{C}\right| = \sqrt{{C_x}^2 + {C_y}^2}$, $\tan\theta = \left(\dfrac{C_y}{C_x}\right)$

SIMPLIFY: $\theta = \tan^{-1}\left(\dfrac{C_y}{C_x}\right)$

CALCULATE: $\left|\vec{C}\right| = \sqrt{(34.6 \text{ m})^2 + (-53.5 \text{ m})^2} = 63.713 \text{ m}$, $\theta = \tan^{-1}\left(\dfrac{-53.5 \text{ m}}{34.6 \text{ m}}\right) = -57.108^\circ$

ROUND: $\vec{C} = 63.7 \text{ m}$, $\theta = -57.1^\circ$ or 303° (equivalent angles).
DOUBLE-CHECK: The magnitude is greater than each component but less than the sum of the components and the angle is also in the correct quadrant. The answer is reasonable.

1.78. **THINK:** Assume Mars is a sphere whose radius is $r_{\text{M}} = 3.39 \cdot 10^6$ m.
SKETCH:



RESEARCH: $C = 2\pi r$, $A = 4\pi r^2$, $V = \dfrac{4}{3}\pi r^3$

SIMPLIFY: $C = 2\pi r_{\text{M}}$, $A = 4\pi {r_{\text{M}}}^2$, $V = \dfrac{4}{3}\pi {r_{\text{M}}}^3$

CALCULATE: $C = 2\pi(3.39 \cdot 10^6 \text{ m}) = 2.12999 \cdot 10^7 \text{ m}$

$A = 4\pi(3.39 \cdot 10^6 \text{ m})^2 = 1.44414 \cdot 10^{14} \text{ m}^2$

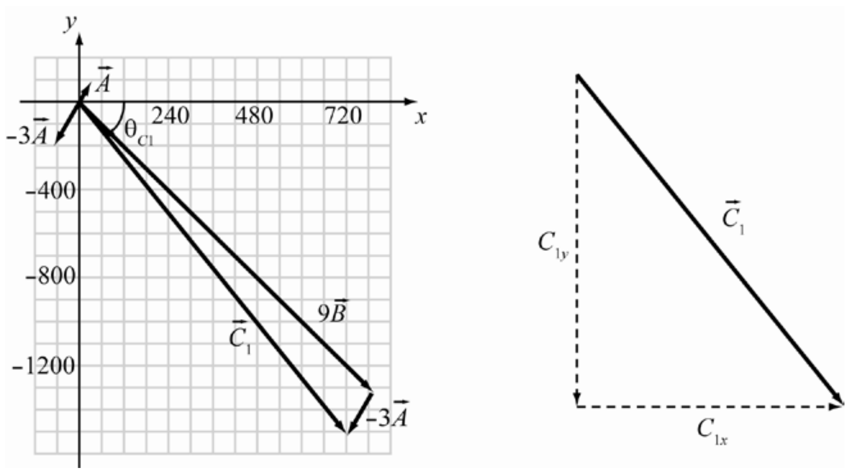
$V = \dfrac{4}{3}\pi(3.39 \cdot 10^6 \text{ m})^3 = 1.63188 \cdot 10^{20} \text{ m}^3$

ROUND: $C = 2.13 \cdot 10^7$ m, $A = 1.44 \cdot 10^{14} \text{ m}^2$, $V = 1.63 \cdot 10^{20} \text{ m}^3$
DOUBLE-CHECK: The units are correct and the orders of magnitude are reasonable.

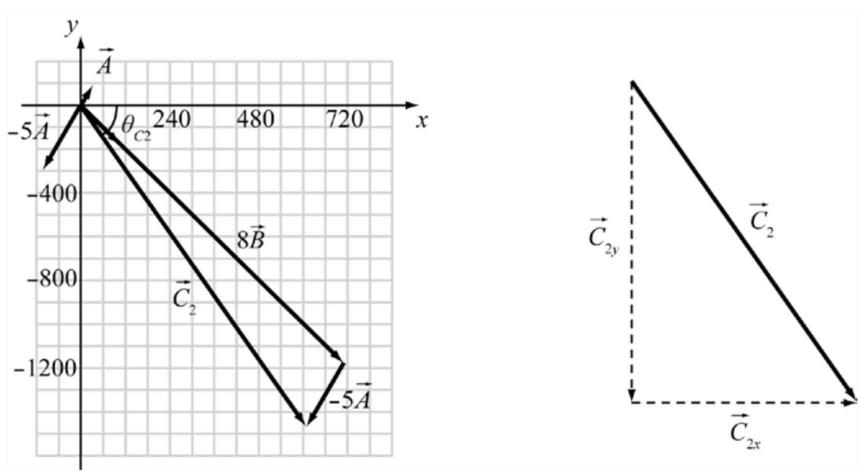
1.79. **THINK:** Sum the components of both vectors and find the magnitude and the angle from the positive x-axis of the resultant vector. $\vec{A}=(23.0,59.0)$ and $\vec{B}=(90.0,-150.0)$.

SKETCH:

(a)



(b)



RESEARCH: $\vec{C}=(C_x,C_y)$, $C_i=nA_i+mB_i$, $\left|\vec{C}\right|=\sqrt{C_x^{\ 2}+C_y^{\ 2}}$, $\tan\theta_C=\dfrac{C_y}{C_x}$

SIMPLIFY:

(a) Since $n=-3$ and $m=9$, $C_x=-3A_x+9B_x$ and $C_y=-3A_y+9B_y$. Also, $\theta_C=\tan^{-1}\left(C_y/C_x\right)$.

(b) Since $n=-5$ and $m=8$, $C_x=-5A_x+8B_x$ and $C_y=-5A_y+8B_y$. Also, $\theta_C=\tan^{-1}\left(C_y/C_x\right)$.

CALCULATE:

(a) $C_x=-3(23.0)+9(90.0)=741.0,\ \ C_y=-3(59.0)+9(-150)=-1527.0$

$\vec{A}=(A_x,A_y)=(-30.0\text{ m},-50.0\text{ m})$

(b) $=(30.0\text{ m},50.0\text{ m})$. $\left|\vec{C}\right|=\sqrt{(605.0)^2+(-1495.0)^2}=1612.78$

$$\theta_C=\tan^{-1}\left(\frac{-1495.0}{605.0}\right)=-67.97°$$

ROUND:

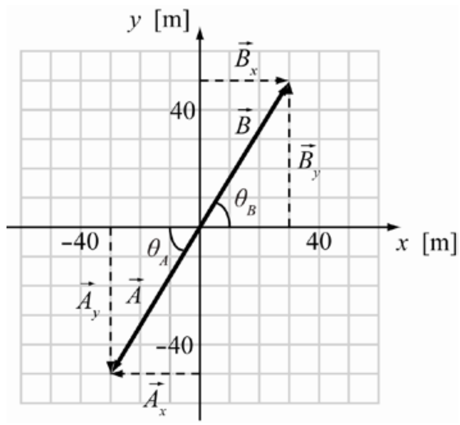
(a) $\vec{C}=1.70\cdot 10^3$ at $-64.1°$ or $296°$

(b) $\vec{C} = 1.61 \cdot 10^3$ at $-68.0°$ or $292°$

DOUBLE-CHECK: Each magnitude is greater than the components but less than the sum of the components and the angles place the vectors in the proper quadrants. The calculated answers are reasonable.

1.80. **THINK:** The vectors are $\vec{A} = (A_x, A_y) = (-30.0 \text{ m}, -50.0 \text{ m})$ and $\vec{B} = (B_x, B_y) = (30.0 \text{ m}, 50.0 \text{ m})$. Find the magnitude and angle with respect to the positive x-axis for each.

SKETCH:



RESEARCH: $\vec{C} = \sqrt{C_x^{\ 2} + C_y^{\ 2}}$, $\tan\theta_C = \dfrac{C_y}{C_x}$

SIMPLIFY: $\vec{A} = \sqrt{A_x^{\ 2} + A_y^{\ 2}}$, $\vec{B} = \sqrt{B_x^{\ 2} + B_y^{\ 2}}$, $\theta_A = \tan^{-1}\left(\dfrac{A_y}{A_x}\right)$, $\theta_B = \tan^{-1}\left(\dfrac{B_y}{B_x}\right)$

CALCULATE: $\left|\vec{A}\right| = \sqrt{(-30.0 \text{ m})^2 + (-50.0 \text{ m})^2} = 58.3095 \text{ m}$, $\left|\vec{B}\right| = \sqrt{(30.0 \text{ m})^2 + (50.0 \text{ m})^2} = 58.3095 \text{ m}$

$$\theta_A = \tan^{-1}\left(\frac{-50.0 \text{ m}}{-30.0 \text{ m}}\right) = 59.036° \Rightarrow 180° + 59.036° = 239.036°$$

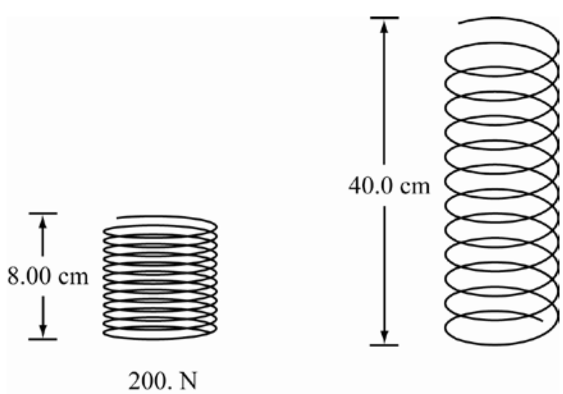
$$\theta_B = \tan^{-1}\left(\frac{50.0 \text{ m}}{30.0 \text{ m}}\right) = 59.036°$$

ROUND: $\vec{A} = 58.3$ m at 239° or $-121°$, and $\vec{B} = 58.3$ m at 59.0°.

DOUBLE-CHECK: Each magnitude is greater than the components of the vector but less than the sum of the components and the angles place the vectors in the proper quadrants.

1.81. **THINK:** A variable is proportional to some other variable by a constant. This means the ratio of one variable to another is a constant. Therefore, both ratios are equal. $F_1 = 200.$ N, $x_1 = 8.00$ cm and $x_2 = 40.0$ cm.

SKETCH:



RESEARCH: $\frac{F_1}{x_1}=\frac{F_2}{x_2}$

SIMPLIFY: $F_2=\frac{F_1x_2}{x_1}$

CALCULATE: $F_2=\frac{(200.\text{ N})(40.0\text{ cm})}{8.00\text{ cm}}=1000.0\text{ N}$

ROUND: $F_2=1.00\cdot 10^3\text{ N}$

DOUBLE-CHECK: The ratio of distance to force remains 1:25 for the two distances. The answers are reasonable.

1.82. **THINK:** When a variable is proportional to another, it is equal to the other variable multiplied by a constant. Call the constant "*a*".

SKETCH: A sketch is not needed to solve this problem.

RESEARCH: $d=at^2$

SIMPLIFY: $d_0=at_0^{\;2},\ d_0'=a(3t_0)^2$

CALCULATE: $d_0'=9at_0^{\;2}\ =\ 9d_0$

ROUND: The distance increases by a factor of 9.

DOUBLE-CHECK: Acceleration is a quadratic relationship between distance and time. It makes sense for the amount of time to increase by a factor larger than 3.

1.83. **THINK:** Consider the $90°$ turns to be precise turns at right angles.

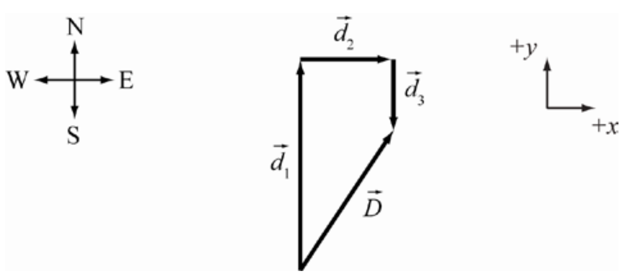
(a) The pilot initially flies N, then heads E, then finally heads S. Determine the vector $\vec{D}$ that points from the origin to the final point and find its magnitude. The vectors are $\vec{d}_1=155.3\text{ miles N}$, $\vec{d}_2=62.5\text{ miles E}$ and $\vec{d}_3=47.5\text{ miles S}$.

(b) Now that the vector pointing to the final destination has been computed, $\vec{D}=d_2\hat{x}+(d_1-d_3)\hat{y}=(62.5\text{ miles})\hat{x}+(107.8\text{ miles})\hat{y}$, determine the angle the vector makes with the origin. The angle the pilot needs to travel is then 180° from this angle.

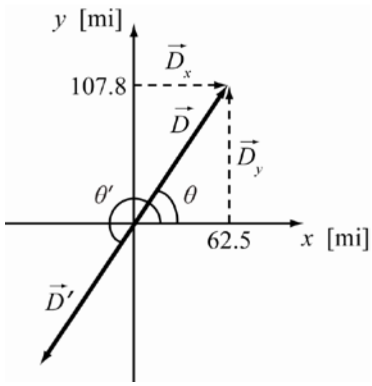
(c) Before the pilot turns S, he is farthest from the origin. This is because when he starts heading S, he is negating the distance travelled N. The only vectors of interest are $\vec{d}_1=155.3\text{ miles N}$ and $\vec{d}_2=62.5\text{ miles E}$.

SKETCH:

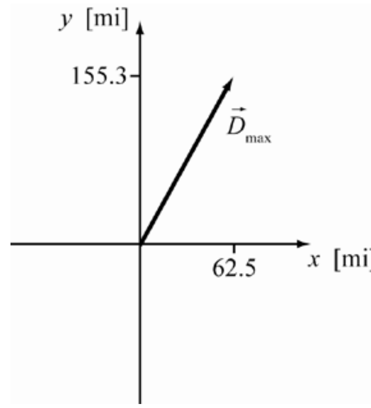
(a)



(b)



(c)



RESEARCH:

(a) $\vec{D} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3 = D_x\hat{x} + D_y\hat{y},\ \ \vec{d}_i = d_{ix}\hat{x} + d_{iy}\hat{y},\ \ \left|\vec{D}\right| = \sqrt{D_x^{\ 2} + D_y^{\ 2}}$

(b) $\tan\theta = \dfrac{D_y}{D_x},\ \ \theta' = \theta \pm 180°$

(c) $\vec{D}_{\max} = \vec{d}_1 + \vec{d}_2,\ \ \vec{d}_i = d_{ix}\hat{x} + d_{iy}\hat{y},\ \ \left|\vec{D}_{\max}\right| = \sqrt{D_x^{\ 2} + D_y^{\ 2}}$

SIMPLIFY:

(a) $\vec{d}_1 = d_1\hat{y},\ \ \vec{d}_2 = d_2\hat{x},\ \ \vec{d}_3 = -d_3\hat{y}$

Therefore, $\vec{D} = d_2\hat{x} + (d_1 - d_3)\hat{y}$ and $\left|\vec{D}\right| = \sqrt{d_2^2 + (d_1 - d_3)^2}$.

(b) $\theta = \tan^{-1}\left(\dfrac{D_y}{D_x}\right)$ and $\theta' = \tan^{-1}\left(\dfrac{D_y}{D_x}\right) \pm 180°$

(c) $\vec{d_1} = d_1\hat{y},\ \ \vec{d_2} = d_2\hat{x},\ \ \vec{D}_{\max} = d_2\hat{x} + d_1\hat{y} \Rightarrow \left|\vec{D}_{\max}\right| = \sqrt{d_2^2 + d_1^2}$

CALCULATE:

(a) $\left|\vec{D}\right| = \sqrt{(62.5 \text{ miles})^2 + (155.3 \text{ miles} - 47.5 \text{ miles})^2}$

$= 124.608 \text{ miles}$

(b) $\theta' = \tan^{-1}\left(\dfrac{107.8 \text{ miles}}{62.5 \text{ miles}}\right) \pm 180°$

$= 59.896° \pm 180° = 239.896°$ or $-120.104°$

(c) $\left|\vec{D}_{\max}\right| = \sqrt{(62.5 \text{ miles})^2 + (155.3 \text{ miles})^2} = 167.405 \text{ miles}$

ROUND:

(a) $\left|\vec{D}\right| = 125$ miles

(b) $\theta' = 240.°$ or $-120.°$ (from positive x-axis or E)

(c) $\left|\vec{D}_{\max}\right| = 167$ miles

DOUBLE-CHECK:

(a) The total distance is less than the distance travelled north, which is expected since the pilot eventually turns around and heads south.

(b) The pilot is clearly NE of the origin and the angle to return must be SW.

(c) This distance is greater than the distance which included the pilot travelling S, as it should be.

1.84. **THINK:**

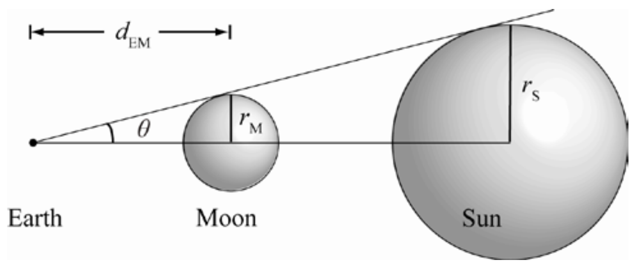
(a) If an observer sees the Moon fully cover the Sun, then light rays from the outer edge of the Sun are blocked by the outer edge of the Moon. This means a line pointing to the outer edge of the Moon also points to the outer edge of the Sun. This in turn means that the lines share a common angle. The radii of the Moon and Sun are, respectively, $r_{\text{M}} = 1.74 \cdot 10^6$ m and $r_{\text{S}} = 6.96 \cdot 10^8$ m. The distance from the Moon to the Earth is $d_{\text{EM}} = 3.84 \cdot 10^8$ m.

(b) In part (a), the origin of the light ray is assumed to be the centre of the Earth. In fact, the observer is on the surface of the Earth, $r_{\text{E}} = 6378$ km. This difference in observer position should then be related to the actual distance to the Moon. The observed Earth to Moon distance remains the same, $d_{\text{EM}} = 3.84 \cdot 10^8$ m, while the actual distance is the observed distance minus the radius of the Earth.

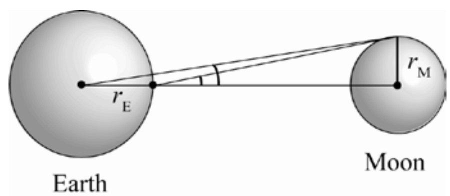
(c) Given the relative error of 1.69% between the actual and observed distance to the Moon, there should be the same relative error in the difference between the observed and actual distance to the Sun. $d_{\text{ES}}(\text{observed}) = 1.54 \cdot 10^{11}$ m.

SKETCH:

(a)



(b)



(c) Not applicable.

RESEARCH:

(a) $\tan\theta = \left(\dfrac{\text{opposite}}{\text{adjacent}}\right)$

(b) relative error $= \dfrac{d_{\text{EM}}(\text{observed}) - d_{\text{EM}}(\text{actual})}{d_{\text{EM}}(\text{actual})}$

(c) $d_{\text{ES}}(\text{actual}) = (1 - \text{relative error})d_{\text{ES}}(\text{observed})$

SIMPLIFY:

(a) $\tan\theta = \left(\dfrac{r_{\text{M}}}{d_{\text{EM}}}\right) = \left(\dfrac{r_{\text{S}}}{d_{\text{ES}}}\right) \Rightarrow d_{\text{ES}} = \dfrac{r_{\text{S}} d_{\text{EM}}}{r_{\text{M}}}$

(b)
$$\text{relative error} = \frac{d_{\text{EM}}(\text{observed}) - (d_{\text{EM}}(\text{observed}) - r_{\text{E}})}{d_{\text{EM}}(\text{observed}) - r_{\text{E}}} = \frac{r_{\text{E}}}{d_{\text{EM}}(\text{observed}) - r_{\text{E}}}$$

(c)
$$d_{\text{ES}}(\text{actual}) = (1 - 0.0169)d_{\text{ES}}(\text{observed}) = 0.9831 d_{\text{ES}}(\text{observed})$$

CALCULATE:

(a) $d_{\text{ES}} = \dfrac{(6.96\cdot 10^8\text{ m})(3.84\cdot 10^8\text{ m})}{(1.74\cdot 10^6\text{ m})} = 1.536\cdot 10^{11}\text{ m}$

(b) relative error = $\dfrac{6378000\text{ m}}{3.84\cdot 10^8\text{ m} - 6378000} = 0.01689$

(c) $d_{\text{ES}}(\text{actual}) = 0.9831(1.54\cdot 10^{11}\text{ m}) = 1.513\cdot 10^{11}\text{ m}$

ROUND:

(a) $d_{\text{ES}} = 1.54\cdot 10^{11}\text{ m}$

(b) relative error $= 1.69\%$

(c) $d_{\text{ES}}(\text{actual}) = 1.51\cdot 10^{11}\text{ m}$

DOUBLE-CHECK:

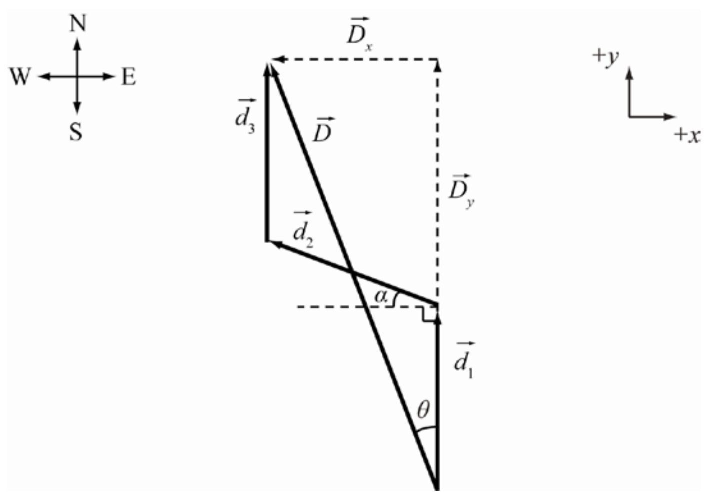
(a) The distance from the Earth to the Sun is about 300 times the distance from the Earth to the Moon, so the answer is reasonable.

(b) The radius of Earth is small compared to the distance from the Earth to the Moon, so the error calculated is small.

(c) The relative error is small so there should be a small difference between the actual and the observed distance from the Earth to the Sun.

1.85. **THINK:** The problem involves adding vectors. Break the vectors into components and sum the components. The vectors are: $\vec{d}_1 = 1.50\text{ km due N}$, $\vec{d}_2 = 1.50\text{ km } 20.0°\text{ N of W}$ and $\vec{d}_3 = 1.50\text{ km due N}$. Find the length of the resultant, and the angle it makes with the vertical. Let $\alpha = 20.0°$.

SKETCH:



RESEARCH: $\vec{D} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3,\ \vec{d}_i = d_{ix}\hat{x} + d_{iy}\hat{y},\ \left|\vec{D}\right| = \sqrt{D_x^{\ 2} + D_y^{\ 2}},\ \tan\theta = \left(\dfrac{D_x}{D_y}\right)$

SIMPLIFY: $\vec{d}_1 = d_1\hat{y},\ \vec{d}_2 = -d_{2x}\hat{x} + d_{2y}\hat{y} = -d_2(\cos\alpha)\hat{x} + d_2(\sin\alpha)\hat{y},\ \vec{d}_3 = d_3\hat{y}$

$$\left|\vec{D}\right| = \sqrt{(-d_2\cos\alpha)^2 + (d_1 + d_3 + d_2\sin\alpha)^2}$$

$$\theta = \tan^{-1}\left(\frac{D_x}{D_y}\right)$$

CALCULATE: $\left|\vec{D}\right| = \sqrt{(-1.50\cos(20.0°)\text{ km})^2 + (3.00\text{ km} + 1.50\sin(20.0°)\text{ km})^2}$

$$= \sqrt{1.9868\text{ km}^2 + 12.3414\text{ km}^2} = 3.7852\text{ km}$$

$$\theta = \tan^{-1}\left(\frac{-1.4095\text{ km}}{3.5130\text{ km}}\right) = -21.862°$$

ROUND: $\left|\vec{D}\right| = 3.79$ km at 21.9° W of N

DOUBLE-CHECK: The only directions travelled were N or NW, so the final direction should be in the NW region.

1.86. **THINK:** If the number of molecules is proportional to the volume, then the ratio of volumes should be the same as the ratio of the molecules. 1 mol = $6.02\cdot 10^{23}$ molecules, volume of mol = 22.4 L and the volume of one breath is 0.500 L. Only 80.0% of the volume of one breath is nitrogen.

SKETCH: Not applicable.

RESEARCH: $V_{\text{Nitrogen}} = 0.800 V_{\text{breath}}$, $\dfrac{V_{\text{Nitrogen}}}{V_{\text{mol}}} = \dfrac{\#\text{ molecules in one breath}}{\#\text{ molecules in a mol}} = \dfrac{N_{\text{breath}}}{N_{\text{mol}}}$

SIMPLIFY: $N_{\text{breath}} = \dfrac{V_{\text{Nitrogen}}(N_{\text{mol}})}{V_{\text{mol}}} = \dfrac{0.800 V_{\text{Breath}}(N_{\text{mol}})}{V_{\text{mol}}}$

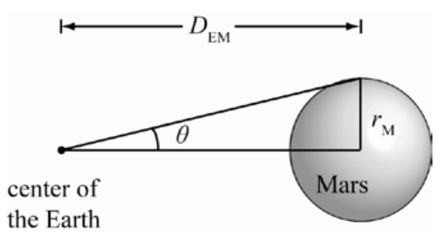
CALCULATE: $N_{\text{breath}} = \dfrac{0.800(0.500\text{ L})(6.02\cdot 10^{23}\text{ molecules})}{(22.4\text{ L})} = 1.07500\cdot 10^{22}$ molecules

ROUND: $N_{\text{breath}} = 1.08\cdot 10^{22}$ molecules

DOUBLE-CHECK: Since the volume of one breath is about 50 times smaller than the volume of one mole of gas, the number of nitrogen molecules in one breath should be about 50 times smaller than the number of molecules in a mole.

1.87. **THINK:** 24.9 seconds of arc represents the angle subtended by a circle with diameter = $2r_{\text{M}}$ located a distance D_{EM} from Earth. This value must be converted to radians. The diameter of Mars is $2r_{\text{M}}$ = 6784 km.

SKETCH:



RESEARCH: The angular size is related to the angle θ shown in the sketch by $\theta_{\text{angular size}} = 2\theta$. From the sketch, we can see that

$$\tan\theta = \frac{r_{\text{M}}}{D_{\text{EM}}}.$$

Because Mars is a long distance from the Earth, even at closet approach, we can make the approximation $\tan\theta \approx \theta$.

SIMPLIFY: Putting our equations together gives us $\theta_{\text{angular size}} = 2\theta = \dfrac{2r_{\text{M}}}{D_{\text{EM}}}$.

CALCULATE: We first convert the observed angular size from seconds of arc to radians

$$24.9\text{ arc seconds}\left(\frac{1°}{3600\text{ arc seconds}}\right)\left(\frac{2\pi\text{ radians}}{360°}\right) = 1.207\cdot 10^{-4}\text{ radians.}$$

The distance is then

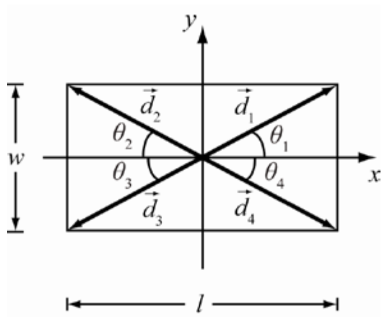
$$D_{\text{EM}} = \frac{2r_{\text{M}}}{\theta_{\text{angular size}}} = \frac{6784\text{ km}}{(1.207\cdot 10^{-4}\text{ radians})} = 5.6205\cdot 10^{7}\text{ km.}$$

ROUND: We specify our answer to three significant figures, $D_{\text{EM}} = 5.62\cdot 10^{7}$ km.

DOUBLE-CHECK: The mean distance from Earth to Mars is about $7\cdot 10^7$ km. Because the distance calculated is for a close approach and the distance is less than the mean distance, the answer is reasonable.

1.88. **THINK:** If the quarterback is in the exact centre of a rectangular field, then each corner should be the same distance from the centre. Only the angle changes for each corner. The width of the field is 53 1/3 yards and the length is 100. yards. Since the question states that the length is exactly 100 yards, the precision of the final answer will be limited by the width.

SKETCH:



RESEARCH: $\vec{d}_i = d_{ix}\hat{x} + d_{iy}\hat{y},\ \left|\vec{d}_i\right| = \sqrt{{d_{ix}}^2 + {d_{iy}}^2},\ \tan\theta_i = \dfrac{d_{iy}}{d_{ix}}$

SIMPLIFY: $\left|\vec{d}_1\right| = \left|\vec{d}_2\right| = \left|\vec{d}_3\right| = \left|\vec{d}_4\right| = \sqrt{\left(\dfrac{w}{2}\right)^2 + \left(\dfrac{l}{2}\right)^2} = \left|\vec{d}\right|,\ \theta_1 = \tan^{-1}\left[\dfrac{\left(\frac{w}{2}\right)}{\left(\frac{l}{2}\right)}\right] = \tan^{-1}\left(\dfrac{w}{l}\right)$

CALCULATE:

(a) $\left|\vec{d}\right| = \sqrt{\left(\dfrac{53\ 1/3 \text{ yards}}{2}\right)^2 + \left(\dfrac{100 \text{ yards}}{2}\right)^2} = 56.667 \text{ yards},\ \theta_1 = \tan^{-1}\left(\dfrac{53\ 1/3}{100}\right) = 28.072°$

(b) $\theta_2 = 180° - \theta_1 = 180° - 28.072° = 151.928°$

$\theta_3 = 180° + \theta_1 = 180° + 28.072° = 208.072°$

$\theta_4 = 360° - \theta_1 = 360° - 28.072° = 331.928°$

ROUND:

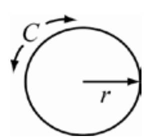
(a) $\vec{d}_1 = 56.7$ yards at 28.1°

(b) $\vec{d}_2 = 56.7$ yards at 152°, $\vec{d}_3 = 56.7$ yards at 208°, $\vec{d}_4 = 56.7$ yards at 332°

DOUBLE-CHECK: $\vec{d}_1$ & $\vec{d}_3$ and $\vec{d}_2$ & $\vec{d}_4$ are 180° apart. This is expected when throwing at opposite corners of the field. The answers are reasonable.

1.89. **THINK:** Assume the Cornell Electron Storage Ring is a perfect circle with a circumference of $C = 768.4$ m. Recall the exact conversion $1 \text{ m} = (100/2.54)$ inches.

SKETCH:



RESEARCH: $C = 2\pi r,\ d = 2r$

SIMPLIFY: $d = \dfrac{C}{\pi}\left(\dfrac{100 \text{ in}}{2.54 \text{ m}}\right)$

CALCULATE: $d = \dfrac{(768.4 \text{ m})}{\pi}\left(\dfrac{100 \text{ in}}{2.54 \text{ m}}\right) = 9629.5007 \text{ inches}$

ROUND: $d = 9630.$ inches

DOUBLE-CHECK: There are 12 inches in a foot and 5280 feet in a mile. Therefore there are 63,360 inch/mile. Our answer for the Cornell ring is thus about 1/6th of a mile, which seems the right order of magnitude.

1.90. **THINK:** 4% of the 0.5 L for each exhalation is composed of carbon dioxide. Assume 1 mole ($6.02\cdot 10^{23}$ molecules) has a volume of 22.4 L. The particular numbers are actually not that important. The only important thing is that they have the right order of magnitude. So it also could be 0.3 or 0.6 L that we exhale in each breath, which are also numbers you can find in the literature; and some sources quote 5% CO_2 in the air that we breathe out.
SKETCH: Not applicable.
RESEARCH: How many times do we breathe per day? You can count the number of breaths you take in a minutes, and that number is around 15. This means that you breath around 800 to 1,000 times per hour and around 20,000 to 25,000 times per day.

$$V_{\text{CO}_2} = 0.04V_{\text{breath}}, \quad \frac{V_{\text{CO}_2}}{V_{\text{mol}}} = \frac{\#\text{ molecules in one breath}}{\#\text{ molecules in a mol}} = \frac{CO_{2_{breath}}}{CO_{2_{mol}}}$$

SIMPLIFY: $CO_{2_{\text{breath}}} = \dfrac{V_{\text{CO}_2}}{V_{\text{mol}}}(CO_{2_{mol}}) = \dfrac{0.04V_{\text{breath}}}{V_{\text{mol}}}(CO_{2_{mol}})$

CALCULATE: $CO_{2_{\text{breath}}} = \dfrac{0.04(0.5\text{ L})}{22.4\text{ L}}(6.02\cdot 10^{23}\text{ molecules}) = 5.375\cdot 10^{20}\text{ molecules}$

(a)
$$\begin{aligned} CO_{2_{day}} &= \#\text{ molecules exhaled in a day} \\ &= \left(2.5\cdot 10^4\right)CO_{2_{\text{breath}}} \\ &= \left(2.5\cdot 10^4\right)\left(5.375\cdot 10^{20}\text{ molecules}\right) \\ &= 1.34375\cdot 10^{25}\text{ molecules} \end{aligned}$$

(b) $$m_{\text{CO}_2} = \frac{1.34375\cdot 10^{25}\text{ molecules}}{1\text{ day}}\left(\frac{365\text{ days}}{1\text{ year}}\right)\left(\frac{1\text{ mole}}{6.02\cdot 10^{23}\text{ molecules}}\right)\left(\frac{44\text{ g}}{1\text{ mole}}\right) = 3.58482\cdot 10^2\text{ kg/year}$$

ROUND: In this case we only estimate order of magnitudes. And so it makes no sense to give more than one significant digit. We can therefore state our answer as

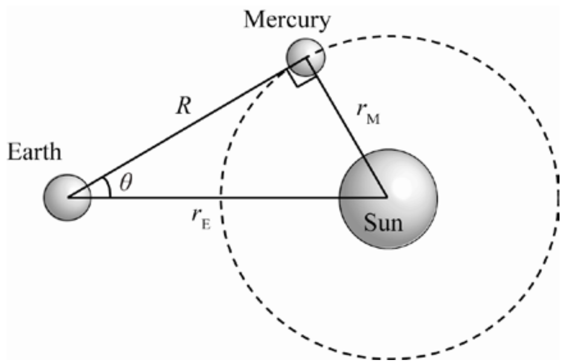
(a) $CO_{2_{day}} = 10^{25}$ molecules

(b) $m_{\text{CO}_2} = 300$ to 400 kg/year

DOUBLE-CHECK: Does it makes sense that we breathe out around 300 to 400 kg of CO_2 in a year, which implies that we breathe out approximately 1 kg of CO_2 in a day. Where does this materials come from? The oxygen comes from the air we breathe in. So the carbon has to be part of what we eat each day. Since ~1/4 of the mass of a CO_2 molecule resides in the carbon, this means that we have to eat at least ~1/2 of a pound of carbon each day. Since carbon, hydrogen, and oxygen are the main components of our food, and since we eat several pounds of food per day, this seems in the right ballpark.

1.91. **THINK:** Consider the Sun to be at the centre of a circle with Mercury on its circumference. This gives $r_{\text{M}} = 4.6\cdot 10^{10}\text{ m}$ as the radius of the circle. Earth is located a distance $r_{\text{E}} = 1.5\cdot 10^{11}\text{ m}$ from the Sun so that the three bodies form a triangle. The vector from Earth to the Sun is at $0°$. The vector from Earth to Mercury intersects Mercury's orbit once when Mercury is at a maximum angular separation from the Sun in the sky. This tangential vector is perpendicular to the radius vector of Mercury's orbit. The three bodies form a right angle triangle with r_{E} as the hypotenuse. Trigonometry can be used to solve for the angle and distance.

SKETCH:



RESEARCH: $r_E^{\ 2} = r_M^{\ 2} + R^2, \ \ r_E \sin\theta = r_M$

SIMPLIFY: $R = \sqrt{r_E^{\ 2} - r_M^{\ 2}}, \ \ \theta = \sin^{-1}\left(\frac{r_M}{r_E}\right)$

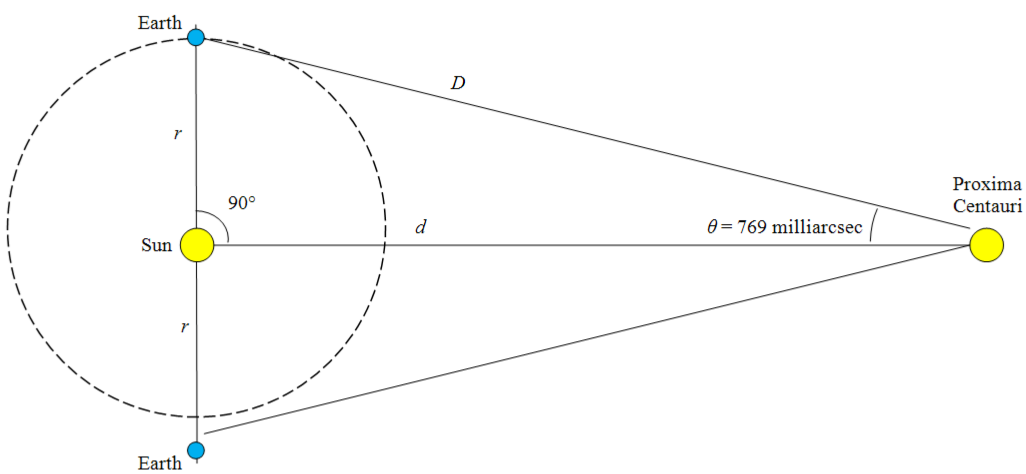
CALCULATE: $R = \sqrt{(1.5\cdot 10^{11})^2 - (4.6\cdot 10^{10})^2} = 1.4277\cdot 10^{11} \text{ m}, \ \ \theta = \sin^{-1}\left(\frac{4.6\cdot 10^{10}}{1.5\cdot 10^{11}}\right) = 17.858°$

ROUND: $R = 1.4\cdot 10^{11}$ m, $\theta = 18°$

DOUBLE-CHECK: If it had been assumed that the maximum angular separation occurred when the Earth to Sun to Mercury angle was $90°$, $\theta = \tan^{-1}(r_M / r_E)$ would be about 17°. The maximum angle should be greater than this and it is, so the answer is reasonable.

1.92. **THINK:** This question asks about the distance to Proxima Centauri, which can be calculated using parallax. To do so, it will be necessary to know the radius of Earth's orbit. It will also be necessary to convert from milliarcseconds to degrees or radians. Then, geometry should be used to find the distance.

SKETCH: Because of the distances involved, the diagram will not be to scale. The earth is shown at two locations, ½ year apart in its revolution around the Sun. The radius of Earth's orbit is labeled *r* and the distance to Proxima Centauri is labeled *d*.



RESEARCH: The goal is to find *d*, the distance between the Sun and Proxima Centauri. Note that the Earth at either of the positions shown, the Sun, and Proxima Centauri form right triangles. The textbook lists the mean orbital radius of Earth as 1.496×10^8 km. The final answer needs to be in light-years, so it will be necessary to convert from km to light-years at some point using the fact that 1 light-year = 9.461 ×

10^{12} km. Knowing the parallax and the radius of the Earth's orbit, it is then possible to use trigonometry to find the distance d from the Sun to Proxima Centauri: $\tan\theta = \frac{r}{d}$.

SIMPLIFY: Using algebra to find the distance d in terms of r and θ gives $d = \frac{r}{\tan\theta}$. It is more difficult to convert from milliarcseconds to a more familiar unit of angle measure, the degree. Since there are 60 arcseconds to the arcminute, and 60 arcminutes make one degree, the conversion will look like this:

$$\text{angle in milliarcseconds}\cdot\frac{10^{-3}\text{ arcseconds}}{1\text{ milliarcsecond}}\cdot\frac{1\text{ arcminute}}{60\text{ arcseconds}}\cdot\frac{1\text{ degree}}{60\text{ arcminutes}} = \text{angle in degrees.}$$

CALCULATE: It is important to perform this calculation with the computer or calculator in degree (not radian) mode. Using the textbook value for the radius of the earth $r = 1.496 \times 10^8$ km and the given value for the parallax of 769 milliarcsec gives:

$$\begin{aligned} d &= \frac{r}{\tan\theta} \\ &= \frac{1.496\times10^8\text{ km}}{\tan\left(769\text{ milliarcsec}\cdot\frac{10^{-3}\text{ arcsec}}{1\text{ milliarcsec}}\cdot\frac{1\text{ arcminute}}{60\text{ arcsec}}\cdot\frac{1\text{ degree}}{60\text{ arcminutes}}\right)}\cdot\frac{1\text{ light-year}}{9.461\times10^{12}\text{ km}} \\ &= 4.241244841\text{ light-years} \end{aligned}$$

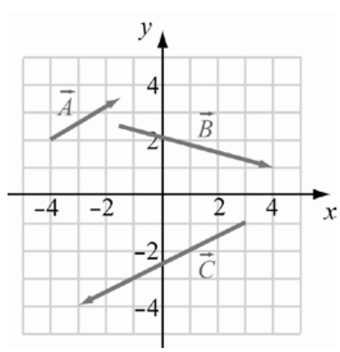
ROUND: The parallax has three significant figures. The radius of the earth is given to four, and all of the conversions are exact except light-years to km, which is given to four. So the final answer should have three figures. This gives a total distance of 4.24 light-years.

DOUBLE-CHECK: A distance to the Proxima Centauri of 4.24 light-years means that it takes light about 4¼ years to travel between the Sun and Proxima Centauri. Knowing what we do of astronomical scales, this makes sense.

Multi-Version Exercises

1.93. **THINK:** The lengths of the x and y components of the vectors can be read from the provided figure. Remember to decompose the vectors in terms of their x and y components.

SKETCH:



RESEARCH: A vector can be written as $\vec{V} = V_x\hat{x} + V_y\hat{y}$, where $V_x = x_f - x_i$ and $V_y = y_f - y_i$.

SIMPLIFY: Not applicable.

CALCULATE: $\vec{A} = (-1.5-(-4))\hat{x} + (3.5-2)\hat{y} = 2.5\hat{x} + 1.5\hat{y}$, $\vec{B} = (4-(-1.5))\hat{x} + (1-2.5)\hat{y} = 5.5\hat{x} - 1.5\hat{y}$

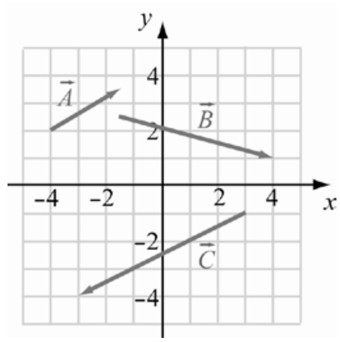
$\vec{C} = (-3-3)\hat{x} - (4-(-1))\hat{y} = -6\hat{x} - 3\hat{y}$

ROUND: Not applicable.

DOUBLE-CHECK: Comparing the signs of the x- and y-components of the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ to the provided figure, the calculated components all point in the correct directions. The answer is therefore reasonable.

1.94. **THINK:** The question asks for the length and direction of the three vectors. The x and y components of the vectors can be read from the provided figure. Remember when dealing with vectors, the components must be treated separately.

SKETCH:



RESEARCH: The length of a vector is given by the formula $\vec{L} = \sqrt{x^2 + y^2}$. The direction of a vector (with respect to the x-axis) is given by $\tan\theta = y / x$.

SIMPLIFY: $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

CALCULATE: $\left|\vec{A}\right| = \sqrt{(2.5)^2 + (1.5)^2} = 2.9,\ \theta_A = \tan^{-1}\left(\frac{1.5}{2.5}\right) = 30.9638°$

$$\left|\vec{B}\right| = \sqrt{(5.5)^2 + (-1.5)^2} = 5.700877,\ \theta_B = \tan^{-1}\left(\frac{-1.5}{5.5}\right) = -15.2551°$$

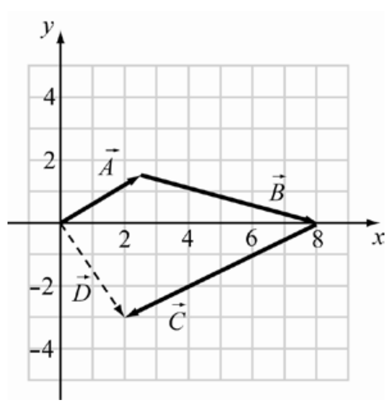
$$\left|\vec{C}\right| = \sqrt{(-6)^2 + (-3)^2} = 6.7082,$$

$$\theta_C = \tan^{-1}\left(\frac{-3}{-6}\right) = 26.565° = 180° + 26.565° = 206.565°$$

ROUND: The figure can reasonably be read to two significant digits, so the rounded values are $\left|\vec{A}\right| = 2.9$ $\theta_A = 31°$, $\left|\vec{B}\right| = 5.7$, $\theta_B = -15°$, $\left|\vec{C}\right| = 6.7$, and $\theta_C = 210°$.

DOUBLE-CHECK: Comparing the graphical values to the calculated values, the calculated values are reasonable.

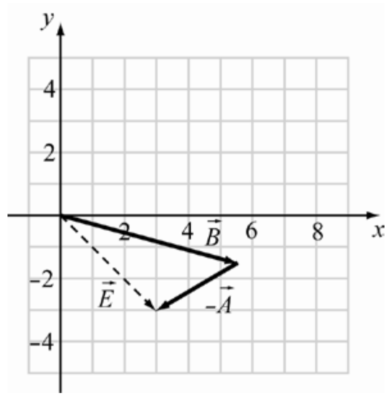
1.95. Vectors add tip to tail, $\vec{A} + \vec{B} + \vec{C} = \vec{D}$.



By inspecting the image, it is clear that $\vec{D} = (2, -3)$.

1.96. **THINK:** To subtract two vectors, reverse the direction of the vector being subtracted, and treat the operation as a sum. Denote the difference as $\vec{E} = \vec{B} - \vec{A}$.

SKETCH:



RESEARCH: $\vec{E} = \vec{B} - \vec{A} = \vec{B} + \left(-\vec{A}\right)$

SIMPLIFY: No simplification is necessary.

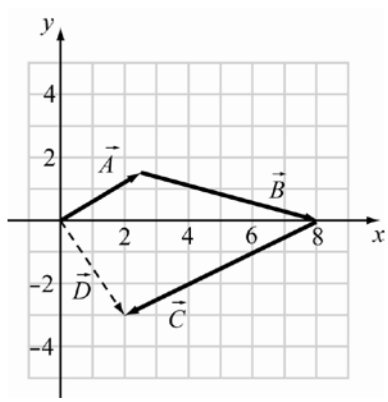
CALCULATE: By inspection, $\vec{E} = (3,-3)$.

ROUND: No rounding is necessary.

DOUBLE-CHECK: The resultant vector $\vec{E}$ points from the origin to the fourth quadrant, so its x-component should be positive and its y-component should be negative. This gives some support to the reasonableness of the answer.

1.97. **THINK:** When adding vectors, you must add the components separately.

SKETCH:



RESEARCH: $\vec{D} = \vec{A} + \vec{B} + \vec{C}$

SIMPLIFY: $\vec{D} = (A_x + B_x + C_x)\hat{x} + (A_y + B_y + C_y)\hat{y}$

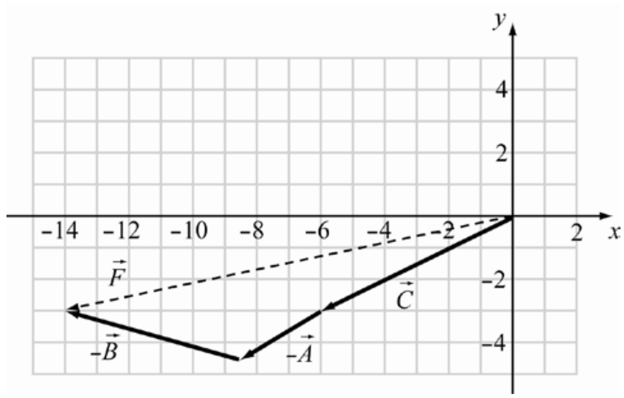
CALCULATE: $\vec{D} = (2.5 + 5.5 - 6)\hat{x} + (1.5 - 1.5 - 3)\hat{y} = 2\hat{x} - 3\hat{y}$

ROUND: The answers are precise, so no rounding is necessary.

DOUBLE-CHECK: The calculation seems consistent with the provided figure.

1.98. **THINK:** When subtracting vectors, you must subtract the x and y components separately.

SKETCH:



RESEARCH: $F_x = C_x - A_x - B_x$ and $F_y = C_y - A_y - B_y$. The length is computed using $\left|\vec{F}\right| = \sqrt{F_x^2 + F_y^2}$ with $F = F_x\hat{x} + F_y\hat{y}$.

SIMPLIFY: $\left|\vec{F}\right| = \sqrt{(C_x - A_x - B_x)^2 + (C_y - A_y - B_y)^2}$

CALCULATE: $\left|\vec{F}\right| = \sqrt{\left((-6.0) - 2.5 - 5.5\right)^2 + \left((-3.0) - (1.5) - (-1.5)\right)^2}$

$= \sqrt{205}$

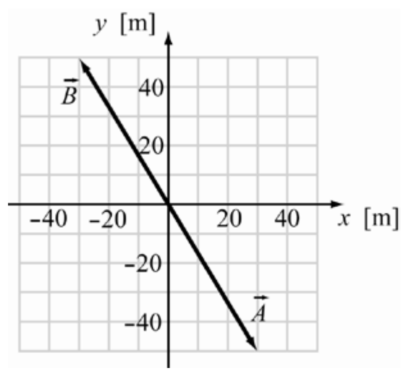
$= 14.318$

ROUND: To two significant figures, the length of $\vec{F}$ is 14.

DOUBLE-CHECK: The size of $\left|\vec{F}\right|$ is reasonable.

1.99. **THINK:** The two vectors are $\vec{A} = (A_x, A_y) = (30.0 \text{ m}, -50.0 \text{ m})$ and $\vec{B} = (B_x, B_y) = (-30.0 \text{ m}, 50.0 \text{ m})$. Sketch and find the magnitudes.

SKETCH:



RESEARCH: The length of a vector $\vec{C} = C_x\hat{x} + C_y\hat{y}$ is $\left|\vec{C}\right| = \sqrt{C_x^{\ 2} + C_y^{\ 2}}$.

SIMPLIFY: $\left|\vec{A}\right| = \sqrt{A_x^{\ 2} + A_y^{\ 2}}$, $\left|\vec{B}\right| = \sqrt{B_x^{\ 2} + B_y^{\ 2}}$

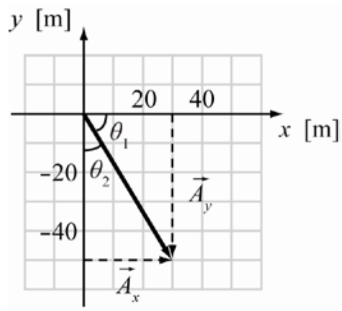
CALCULATE: $\left|\vec{A}\right| = \sqrt{(30)^2 + (-50)^2} = 58.3095$ m, $\left|\vec{B}\right| = \sqrt{(-30)^2 + (50)^2} = 58.3095$ m

ROUND: $\left|\vec{A}\right| = 58.3$ m, $\left|\vec{B}\right| = 58.3$ m

DOUBLE-CHECK: The calculated magnitudes are larger than the lengths of the component vectors, and are less than the sum of the lengths of the component vectors. Also, the vectors are opposites, so they should have the same length.

1.100. **THINK:** Use trigonometry to find the angles as indicated in the sketch below. $\vec{A} = (A_x, A_y) = (30.0 \text{ m}, -50.0 \text{ m})$.

SKETCH:



RESEARCH: $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

SIMPLIFY: $\tan\theta_1 = (A_y / A_x) \Rightarrow \theta_1 = \tan^{-1}(A_y / A_x)$, $\tan\theta_2 = (A_x / A_y) \Rightarrow \theta_2 = \tan^{-1}(A_x / A_y)$

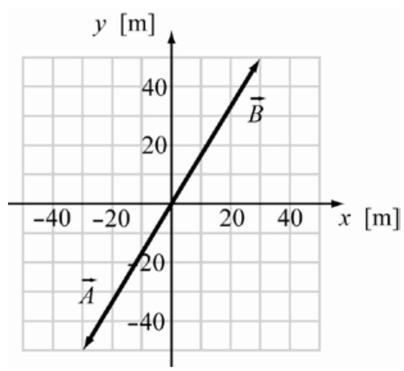
CALCULATE: $\theta_1 = \tan^{-1}(-50/30) = -59.036°$, $\theta_2 = \tan^{-1}(30/-50) = -30.963°$

ROUND: Drop the signs of the angles and just use their size: $\theta_1 = 59.0°$, $\theta_2 = 31.0°$.

DOUBLE-CHECK: The two angles add up to 90˚, which they should. The answers are reasonable.

1.101. **THINK:** The two vectors are $\vec{A} = (A_x, A_y) = (-30.0 \text{ m}, -50.0 \text{ m})$ and $\vec{B} = (B_x, B_y) = (30.0 \text{ m}, 50.0 \text{ m})$. Sketch and find the magnitudes.

SKETCH:



RESEARCH: $|\vec{C}| = \sqrt{C_x^2 + C_y^2}$

SIMPLIFY: $|\vec{A}| = \sqrt{A_x^2 + A_y^2}$, $|\vec{B}| = \sqrt{B_x^2 + B_y^2}$

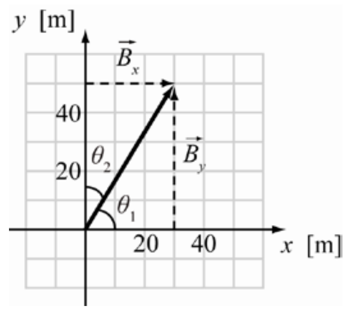
CALCULATE: $|\vec{A}| = \sqrt{(-30.0 \text{ m})^2 + (-50.0 \text{ m})^2} = 58.3095 \text{ m}$, $|\vec{B}| = \sqrt{(30.0 \text{ m})^2 + (50.0 \text{ m})^2} = 58.3095 \text{ m}$

ROUND: $|\vec{A}| = 58.3 \text{ m}$, $|\vec{B}| = 58.3 \text{ m}$

DOUBLE-CHECK: The magnitudes are bigger than individual components, but not bigger than the sum of the components. Therefore, the answers are reasonable.

1.102. **THINK:** Using trigonometry find the angles indicated in the diagram below. The vector $\vec{B} = (B_x, B_y)$ $= (30.0 \text{ m}, 50.0 \text{ m})$.

SKETCH:



RESEARCH: $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

SIMPLIFY: $\theta = \tan^{-1}\left(\dfrac{\text{opposite}}{\text{adjacent}}\right)$, $\theta_1 = \tan^{-1}\left(\dfrac{B_y}{B_x}\right)$, $\theta_2 = \tan^{-1}\left(\dfrac{B_x}{B_y}\right)$

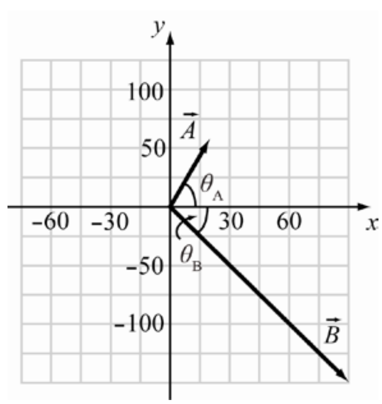
CALCULATE: $\theta_1 = \tan^{-1}\left(\dfrac{50.0 \text{ m}}{30.0 \text{ m}}\right) = 59.036°$, $\theta_2 = \tan^{-1}\left(\dfrac{30.0 \text{ m}}{50.0 \text{ m}}\right) = 30.963°$

ROUND: $\theta_1 = 59.0°$, $\theta_2 = 31.0°$

DOUBLE-CHECK: The angles sum to 90°, which is expected from the sketch. Therefore, the answers are reasonable.

1.103. **THINK:** The two vectors are $\vec{A} = (23.0, 59.0)$ and $\vec{B} = (90.0, -150.0)$. Find the magnitude and angle with respect to the positive x-axis.

SKETCH:



RESEARCH: For any vector $\vec{C} = C_x\hat{x} + C_y\hat{y}$, the magnitude is given by the formula $\left|\vec{C}\right| = \sqrt{C_x^{\ 2} + C_y^{\ 2}}$, and the angle θ_C made with the x-axis is such that $\tan\theta_C = \dfrac{C_y}{C_x}$.

SIMPLIFY: $\left|\vec{A}\right| = \sqrt{A_x^{\ 2} + A_y^{\ 2}}$, $\left|\vec{B}\right| = \sqrt{B_x^{\ 2} + B_y^{\ 2}}$, $\theta_A = \tan^{-1}\left(\dfrac{A_y}{A_x}\right)$, $\theta_B = \tan^{-1}\left(\dfrac{B_y}{B_x}\right)$

CALCULATE: $\left|\vec{A}\right| = \sqrt{(23.0)^2 + (59.0)^2} = 63.3246$, $\left|\vec{B}\right| = \sqrt{(90.0)^2 + (-150.0)^2} = 174.9286$

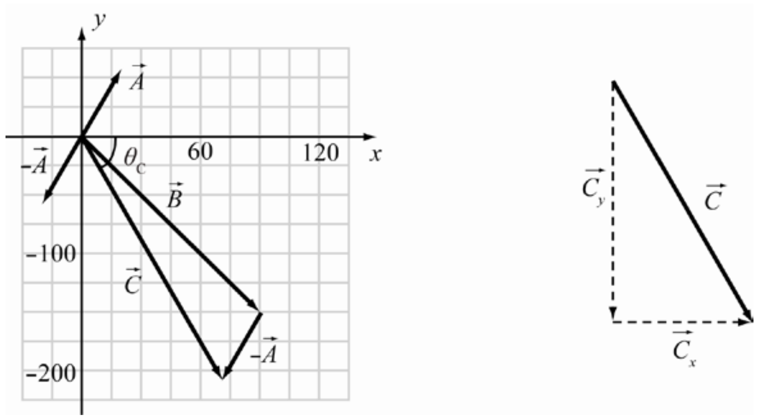
$$\theta_A = \tan^{-1}\left(\frac{59.0}{23.0}\right) = 68.7026°, \ \theta_B = \tan^{-1}\left(\frac{-150.0}{90.0}\right) = -59.0362°$$

ROUND: Three significant figures: $\vec{A} = 63.3$ at $68.7°$, $\vec{B} = 175$ at $-59.0°$ or $301.0°$.

DOUBLE-CHECK: Each magnitude is greater than the components but less than the sum of the components, and the angles place the vectors in the proper quadrants.

1.104. **THINK:** Add the components of the vectors. Find the magnitude and the angle from the positive x-axis of the resultant vector. $\vec{A} = (23.0, 59.0)$ and $\vec{B} = (90.0, -150.0)$.

SKETCH:



RESEARCH: $\vec{C} = (C_x, C_y)$, $C_i = nA_i + mB_i$ with $n = -1$ and $m = +1$, $\left|\vec{C}\right| = \sqrt{C_x^{\ 2} + C_y^{\ 2}}$, $\tan\theta_C = \dfrac{C_y}{C_x}$.

SIMPLIFY: $C_x = -A_x + B_x$, $C_y = -A_y + B_y$, $\theta_C = \tan^{-1}\left(\dfrac{C_y}{C_x}\right)$

CALCULATE: $C_x = -23.0 + 90.0 = 67.0$, $C_y = -59.0 + (-150) = -209.0$,

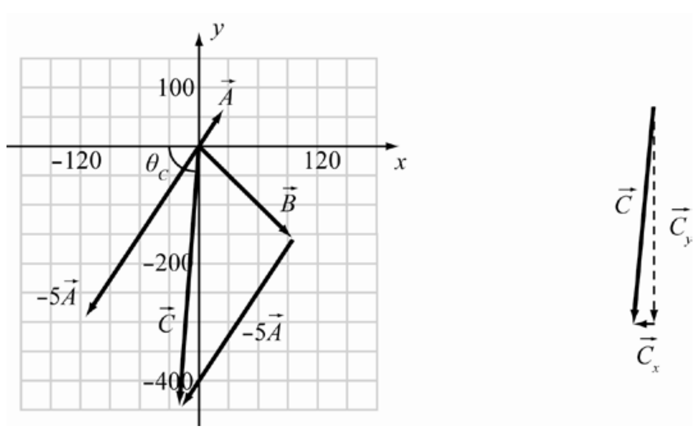
$\left|\vec{C}\right| = \sqrt{(67.0)^2 + (-209.0)^2} = 219.477$, and $\theta_C = \tan^{-1}\left(\dfrac{-209.0}{67.0}\right) = -72.225°$.

ROUND: $\vec{C} = 219$ at $-72.2°$ or $288°$

DOUBLE-CHECK: The magnitude is greater than each component but less than the sum of the components and the angle is also in the correct quadrant. The answer is reasonable.

1.105. **THINK:** Add the components of the vectors (with applicable multiplication of each vector). Find the magnitude and the angle from the positive x-axis of the resultant vector. $\vec{A} = (23.0, 59.0)$ and $\vec{B} = (90.0, -150.0)$.

SKETCH:



RESEARCH: $\vec{C} = (C_x, C_y)$, $C_i = nA_i + mB_i$ with $n = -5$ and $m = +1$, $\left|\vec{C}\right| = \sqrt{C_x^{\ 2} + C_y^{\ 2}}$, $\tan\theta_C = \dfrac{C_y}{C_x}$.

SIMPLIFY: $C_x = -5A_x + B_x$, $C_y = -5A_y + B_y$, $\left|\vec{C}\right| = \sqrt{C_x^{\ 2} + C_y^{\ 2}}$, $\theta_C = \tan^{-1}\left(\frac{C_y}{C_x}\right)$

CALCULATE: $C_x = -5(23.0) + 90.0 = -25.0$, $C_y = -5(59.0) + (-150) = -445.0$

$$\left|\vec{C}\right| = \sqrt{(-25.0)^2 + (-445.0)^2} = 445.702$$

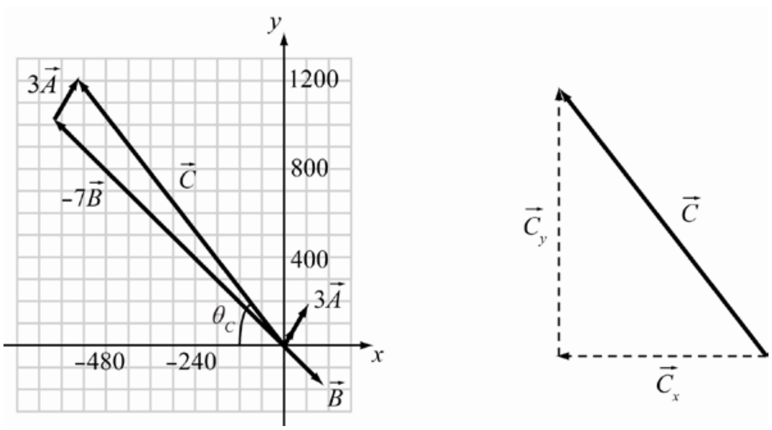
$$\theta_C = \tan^{-1}\left(\frac{-445.0}{-25.0}\right) = 86.785° \Rightarrow 180° + 86.785° = 266.785°$$

ROUND: $\vec{C} = 446$ at $267°$ or $-93.2°$

DOUBLE-CHECK: The magnitude is greater than each component but less than the sum of the components and the angle is also in the correct quadrant. The answer is reasonable.

1.106. **THINK:** Add the components of the vectors (with applicable multiplication of each vector). Find the magnitude and the angle from the positive x-axis of the resultant vector. $\vec{A} = (23.0, 59.0)$ and $\vec{B} = (90.0, -150.0)$.

SKETCH:



RESEARCH: $\vec{C} = (C_x, C_y)$, $C_i = nA_i + mB_i$ with $n = 3$ and $m = -7$, $\left|\vec{C}\right| = \sqrt{C_x^{\ 2} + C_y^{\ 2}}$, $\tan\theta_C = \frac{C_y}{C_x}$.

SIMPLIFY: $C_x = 3A_x - 7B_x$, $C_y = 3A_y - 7B_y$, $\theta_C = \tan^{-1}\left(\frac{C_y}{C_x}\right)$

CALCULATE: $C_x = 3(23.0) - 7(90.0) = -561.0$, $C_y = 3(59.0) - 7(-150) = 1227.0$

$$\left|\vec{C}\right| = \sqrt{(-561.0)^2 + (1227.0)^2} = 1349.17$$

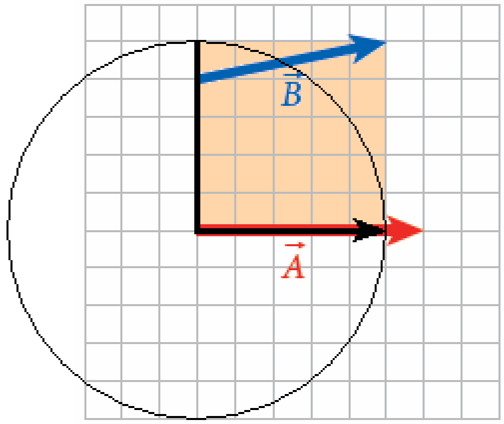
$$\theta_C = \tan^{-1}\left(\frac{1227.0}{-561.0}\right) = -65.43° \Rightarrow 180° - 65.43° = 114.57°$$

ROUND: $\left|\vec{C}\right| = 1.35 \cdot 10^3$ at $115°$

DOUBLE-CHECK: The magnitude is greater than each component but less than the sum of the components and the angle is also in the correct quadrant.

1.107. **THINK:** The scalar product of two vectors equals the length of the two vectors times the cosine of the angle between them. Geometrically, think of the absolute value of the scalar product as the length of the projection of vector $\vec{B}$ onto vector $\vec{A}$ times the length of vector $\vec{A}$, or the area of a rectangle with one side the length of vector $\vec{A}$ and the other side the length of the projection of vector $\vec{B}$ onto vector $\vec{A}$. Algebraically, use the formula $\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y$ to find the scalar product from the components, which can be read from the graphs.

SKETCH: Using the geometric interpretation, sketch the projection of vector $\vec{B}$ onto vector $\vec{A}$ and then draw the corresponding rectangular area, for instance for case (e):



Note, however, that this method of finding the scalar product is cumbersome and does not readily produce exact results. The algebraic approach is much more efficient.

RESEARCH: Use the formula $\vec{A}\bullet\vec{B}=A_xB_x+A_yB_y$ to find the scalar product from the components, which can be read from the graphs. $(A_x,A_y)=(6,0)$ in all cases. In (a), $(B_x,B_y)=(1,5)$; in (b), $(B_x,B_y)=(0,3)$; in (c), $(B_x,B_y)=(2,2)$; in (d), $(B_x,B_y)=(-6,0)$; in (e), $(B_x,B_y)=(5,1)$; and in (f), $(B_x,B_y)=(1,4)$.

SIMPLIFY: Using the formula $\vec{A}\bullet\vec{B}=A_xB_x+A_yB_y$, find that in part (a), $\vec{A}\bullet\vec{B}=6\cdot 1+0\cdot 5$. In part (b), $\vec{A}\bullet\vec{B}=6\cdot 0+0\cdot 3$. In part (c), $\vec{A}\bullet\vec{B}=6\cdot 2+0\cdot 2$. In part (d), $\vec{A}\bullet\vec{B}=6\cdot -6+0\cdot 0$. In part (e), $\vec{A}\bullet\vec{B}=6\cdot 5+0\cdot 1$. Finally, in part (f), $\vec{A}\bullet\vec{B}=6\cdot 1+0\cdot 4$.

CALCULATE: Performing the multiplication and addition as shown above, the scalar product in (a) is 6 units, in (b) it is 0 units, and in part (c) the scalar product is 12 units. In parts (d), (e), and (f), the scalar products are –36 units, 30 units, and 6 units, respectively. The one with the largest absolute value is case (d), $|-36|=36$.

ROUND: Rounding is not required in this problem.

DOUBLE-CHECK: Double-check by looking at the rectangles with sides the length of vector $\vec{A}$ and the length of the projection of vector $\vec{B}$ onto vector $\vec{A}$. The results agree with what was calculated using the formula.

1.108. When the scalar products are evaluated as described in the preceding solution, the one with the smallest absolute value is case (b), where $|\vec{A}\bullet\vec{B}|=0$. It is characteristic of the scalar product that it comes out zero for perpendicular vectors, and zero is of course the smallest possible absolute value.

1.109. The vector product of two non-parallel vectors $\vec{A}$ and $\vec{B}$ that lie in the *xy*-plane is a vector in the *z*-direction. As given by Eq. (1.32), the magnitude of that vector is $|A_xB_y-A_yB_x|$.

(a) $|A_xB_y-A_yB_x|=|(6)(5)-(0)(1)|=30$

(b) $|A_xB_y-A_yB_x|=|(6)(3)-(0)(0)|=18$

(c) $|A_xB_y-A_yB_x|=|(6)(2)-(0)(2)|=12$

(d) $|A_xB_y-A_yB_x|=|(6)(0)-(0)(-6)|=0$

(e) $\left|A_xB_y - A_yB_x\right| = \left|(6)(1)-(0)(5)\right| = 6$

(f) $\left|A_xB_y - A_yB_x\right| = \left|(6)(4)-(0)(1)\right| = 24$

The largest absolute value of a vector product is case (a).

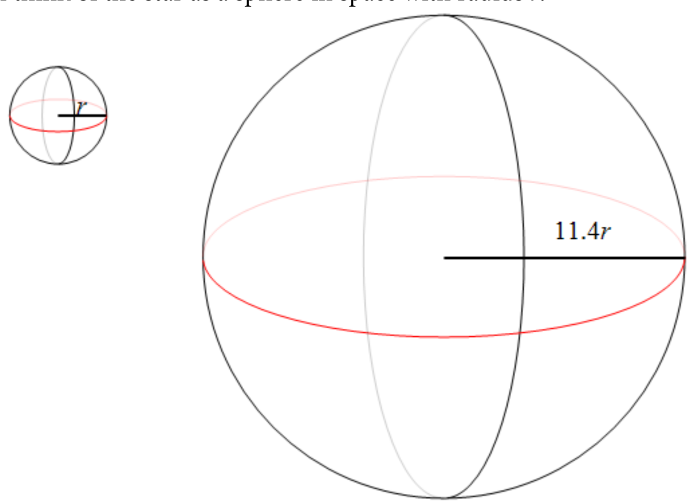
1.110. When the vector products are evaluated as described in Solution 1.109, the one with the smallest absolute value is case (d). It is characteristic of the vector product that it comes out zero for parallel or antiparallel vectors, and zero is of course the smallest possible absolute value.

1.111. Taking the absolute value and ranking in order from least to greatest, we find that $0 < 6 = 6 < 12 < 30 < 36$. This gives us the ordering from least to greatest of the absolute value of the scalar product in parts b, a = f, c, e, and d.

1.112. Ranking the absolute values found in Solution 1.109 in order from least to greatest, we find that $0 < 6 < 12 < 18 < 24 < 30$. This gives us the ordering from least to greatest of the absolute value of the vector product in parts d, e, c, b, f, and a.

1.113. **THINK:** We are given the change in the star's radius. So, if we can express the surface area, circumference, and volume in terms of the radius, we can find by what factors these change as the radius changes.

SKETCH: We can think of the star as a sphere in space with radius *r*.



RESEARCH: We can use the formulas for volume and surface area of a sphere given in Appendix A. We find that the volume of the sphere on the left is $\frac{4}{3}\pi r^3$ and its surface area is $4\pi r^2$. Similarly, the volume of the sphere on the right is $\frac{4}{3}\pi(11.4r)^3$ and its surface area is $4\pi(11.4r)^2$. The circumference of a sphere is the same as the circumference of a great circle around it (shown in red in the sketch). Finding the radius of the circle will give us the radius of the sphere. Using this method, we find that the circumference of the sphere on the left is $2\pi r$, while the sphere on the right has a circumference of $2\pi(11.4r)$.

SIMPLIFY: We use algebra to find the volume, surface area, and circumference of the larger sphere in terms of the volume, surface area, and circumference of the smaller sphere. (a) We find the surface area of the sphere on the right is $4\pi(11.4r)^2 = 4\pi\left(11.4^2r^2\right) = 11.4^2\cdot 4\pi r^2$ (b) The circumference of the larger sphere is $2\pi(11.4r) = 11.4\cdot(2\pi r)$. (c) The volume of the larger sphere is $\frac{4}{3}\pi(11.4r)^3 = \frac{4}{3}\pi\left(11.4^3r^3\right) = \left(11.4^3\right)\frac{4}{3}\pi r^3$.

CALCULATE: Since we don't know the star's original radius, we take the ratio of the new value divided by the old value to get the factor by which the surface area, volume, and circumference have increased. In part (a), we find that the ratio of the new surface area to the original surface area is $\frac{11.4^2 \cdot 4\pi r^2}{4\pi r^2} = 11.4^2 \frac{4\pi r^2}{4\pi r^2} = 11.4^2 \cdot 1 = 129.96$. (b) Similarly, we can divide the new circumference by the original one to get $\frac{11.4 \cdot (2\pi r)}{2\pi r} = 11.4 \frac{2\pi r}{2\pi r} = 11.4 \cdot 1 = 11.4$. (c) The new volume divided by the original volume is $\frac{\left(11.4^3\right)\frac{4}{3}\pi r^3}{\frac{4}{3}\pi r^3} = \left(11.4^3\right)\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi r^3} = 11.4^3 \cdot 1 = 1481.544$.

ROUND: For all of these calculations, we round to three significant figures. This gives us that (a) the surface area has increased by a factor of130., (b) the circumference has increased by a factor of 11.4, and (c) the volume has increased by a factor of 1.48×10^3.

DOUBLE-CHECK: Think about what these values represent. The circumference is a one-dimensional quantity, with units such as km, which is proportional to *r*. The surface area is a two-dimensional quantity with units such as km^2, and is proportional to r^2. The volume is a three-dimensional quantity with units such as km^3 and is proportional to r^3. So it makes sense that, when we increase the radius by a given amount (11.4 in this case), the circumference increases in proportion to that amount, while the surface area increases by that amount squared, and the volume increases by the cube of that amount.

1.114. The circumference is directly proportional to the radius.

(a) The surface area is proportional to the square of the radius and therefore to the square of the circumference. It will increase by a factor of $12.5^2 = 156$.

(b) The radius is directly proportional to the circumference. It will increase by a factor of 12.5.

(c) The volume is proportional to the cube of the radius and therefore to the cube of the circumference. It will increase by a factor of $12.5^3 = 1950$.

1.115. The volume is proportional to the cube of the radius, so if the volume increases by a factor of 872, then the radius increases by a factor of $\sqrt[3]{872} = 9.553712362$.

(a) The surface area is proportional to the square of the radius. It will increase by a factor of $9.553712362^2 = 91.3$.

(b) The circumference is directly proportional to the radius. It will increase by a factor of 9.55.

(c) The diameter is directly proportional to the radius. It will increase by a factor of 9.55.

1.116. (a) The volume is proportional to the cube of the radius, so if the volume increases by a factor of 274, then the radius increases by a factor of $\sqrt{274} = 16.55294536 = 16.6$.

(b) The volume is proportional to the cube of the radius. It will increase by a factor of $16.55294536^3 = 4535.507028 = 4540$.

(c) The density is *inversely* proportional to the volume. It will decrease by a factor of $4535.507028^{-1} = 2.20 \cdot 10^{-4}$.

Chapter 2: Motion in a Straight Line

Concept Checks

2.1. d **2.2.** b **2.3.** b **2.4.** c **2.5.** a) 3 b) 1 c) 4 d) 2 **2.6.** c **2.7.** d **2.8.** c **2.9.** d

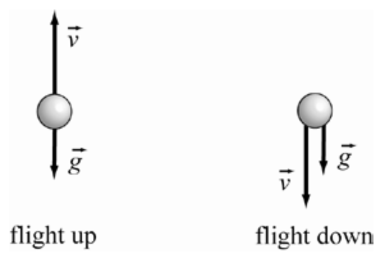
Multiple-Choice Questions

2.1. e **2.2.** c **2.3.** c **2.4.** b **2.5.** e **2.6.** a **2.7.** d **2.8.** c **2.9.** a **2.10.** b **2.11.** b **2.12.** d **2.13.** c **2.14.** d **2.15.** a **2.16.** c

Conceptual Questions

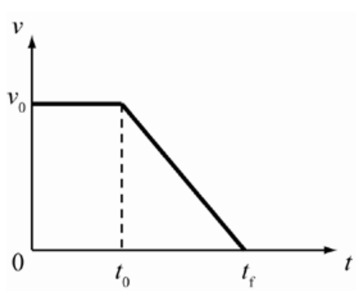
2.17. Velocity and speed are defined differently. The magnitude of average velocity and average speed are the same only when the direction of movement does not change. If the direction changes during movement, it is known that the net displacement is smaller than the net distance. Using the definition of average velocity and speed, it can be said that the magnitude of average velocity is less than the average speed when the direction changes during movement. Here, only Christine changes direction during her movement. Therefore, only Christine has a magnitude of average velocity which is smaller than her average speed.

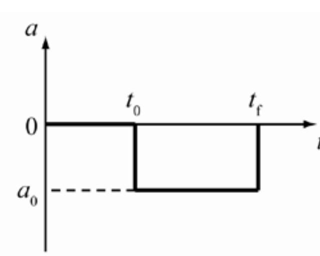
2.18. The acceleration due to gravity is always pointing downward to the center of the Earth.



It can be seen that the direction of velocity is opposite to the direction of acceleration when the ball is in flight upward. The direction of velocity is the same as the direction of acceleration when the ball is in flight downward.

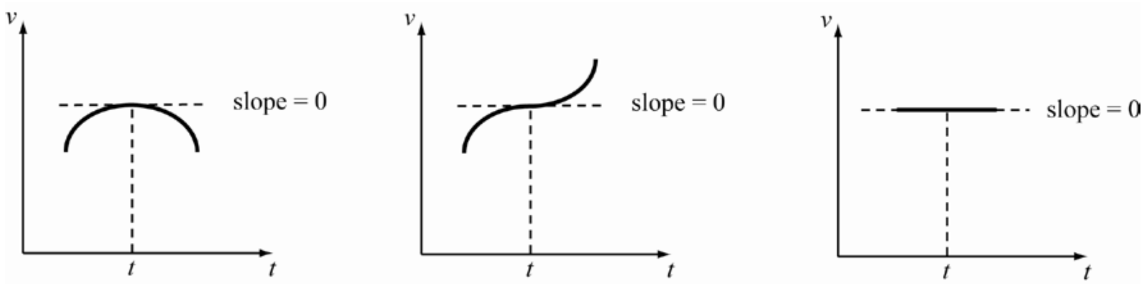
2.19. The car, before the brakes are applied, has a constant velocity, v_0, and zero acceleration. After the brakes are applied, the acceleration is constant and in the direction opposite to the velocity. In velocity versus time and acceleration versus time graphs, the motion is described in the figures below.





2.20. There are two cars, car 1 and car 2. The decelerations are $a_1 = 2a_2 = -a_0$ after applying the brakes. Before applying the brakes, the velocities of both cars are the same, $v_1 = v_2 = v_0$. When the cars have completely stopped, the final velocities are zero, $v_f = 0$. $v_f = v_0 + at = 0 \Rightarrow t = -\frac{v_0}{a}$. Therefore, the ratio of time taken to stop is Ratio = $\frac{\text{time of car 1}}{\text{time of car 2}} = \frac{-v_0 / -a_0}{-v_0 / \left(-\frac{1}{2}a_0\right)} = \frac{1}{2}$. So the ratio is one half.

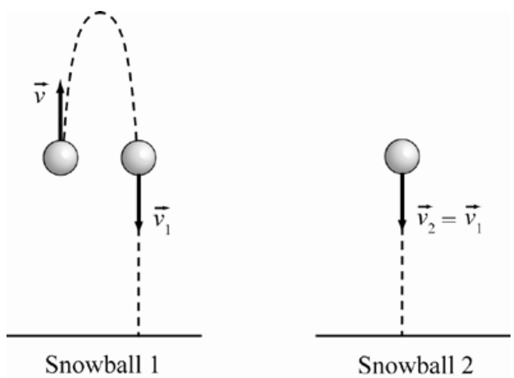
2.21. Here a and v are instantaneous acceleration and velocity. If $a = 0$ and $v \neq 0$ at time t, then at that moment the object is moving at a constant velocity. In other words, the slope of a curve in a velocity versus time plot is zero at time t. See the plots below.



2.22. The direction of motion is determined by the direction of velocity. Acceleration is defined as a change in velocity per change in time. The change in velocity, Δv, can be positive or negative depending on the values of initial and final velocities, $\Delta v = v_{\text{f}} - v_{\text{i}}$. If the acceleration is in the opposite direction to the motion, it means that the magnitude of the objects velocity is decreasing. This occurs when an object is slowing down.

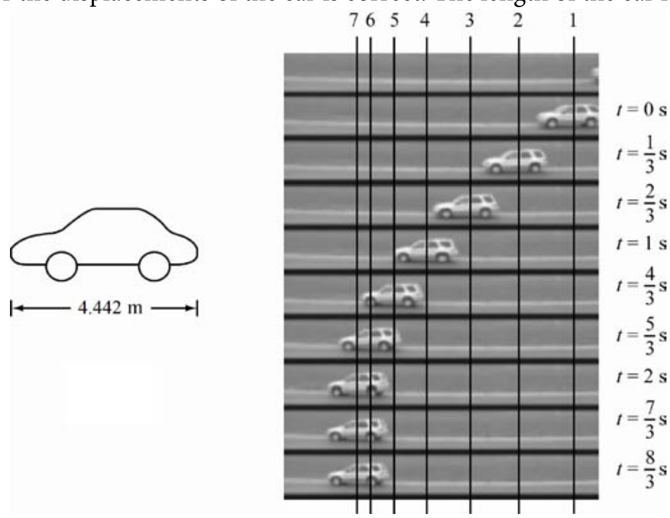
2.23. If there is no air resistance, then the acceleration does not depend on the mass of an object. Therefore, both snowballs have the same acceleration. Since initial velocities are zero, and the snowballs will cover the same distance, both snowballs will hit the ground at the same time. They will both have the same speed.

2.24. Acceleration is independent of the mass of an object if there is no air resistance.

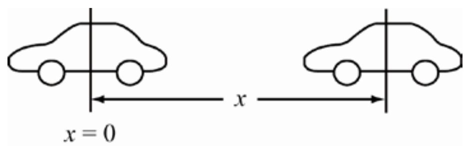


Snowball 1 will return to its original position after Δt, and then it falls in the same way as snowball 2. Therefore snowball 2 will hit the ground first since it has a shorter path. However, both snowballs have the same speed when they hit the ground.

2.25. Make sure the scale for the displacements of the car is correct. The length of the car is 174.9 in = 4.442 m.



Measuring the length of the car in the figure above with a ruler, the car in this scale is 0.80 ± 0.05 cm. Draw vertical lines at the center of the car as shown in the figure above. Assume line 7 is the origin ($x = 0$).



Assume a constant acceleration $a = a_0$. Use the equations $v = v_0 + at$ and $x = x_0 + v_0 t + (1/2)at^2$. When the car has completely stopped, $v = 0$ at $t = t_0$.

$$0 = v_0 + at_0 \;\Rightarrow\; v_0 = -at_0$$

Use the final stopping position as the origin, $x = 0$ at $t = t_0$.

$$0 = x_0 + v_0 t_0 + \frac{1}{2}at_0^2$$

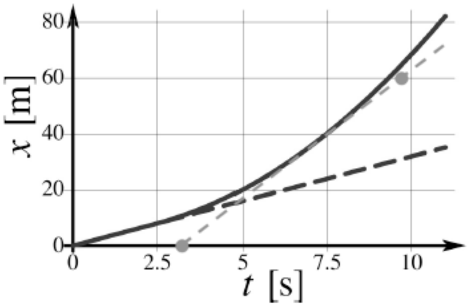
Substituting $v_0 = -at_0$ and simplifying gives

$$x_0 - at_0^2 + \frac{1}{2}at_0^2 = 0 \;\Rightarrow\; x_0 - \frac{1}{2}at_0^2 = 0 \;\Rightarrow\; a = \frac{2x_0}{t_0^2}$$

Note that time t_0 is the time required to stop from a distance x_0.First measure the length of the car. The length of the car is 0.80 cm. The actual length of the car is 4.442 m, therefore the scale is $\frac{4.442\text{ m}}{0.80\text{ cm}} = 5.5\text{ m/cm}$. The error in measurement is (0.05 cm) 5.5 m/cm ≈ 0.275 m (round at the end). So the scale is 5.5 ± 0.275 m/cm. The farthest distance of the car from the origin is 2.9 ± 0.05 cm. Multiplying by the scale, 15.95 m, $t_0 = (0.333)(6\text{ s}) = 1.998\text{ s}$. The acceleration can be found using $a = 2x_0 / t_0^2$: $a = \frac{2(15.95\text{ m})}{(1.998\text{ s})^2} = 7.991\text{ m/s}^2$. Because the scale has two significant digits, round the result to two significant digits: $a = 8.0\text{ m/s}^2$. Since the error in the measurement is $\Delta x_0 = 0.275\text{ m}$, the error of the acceleration is

$$\Delta a = \frac{2\Delta x_0}{t_0^2} = \frac{2(0.275\text{ m})}{(1.998\text{ s})^2} \approx 0.1\text{ m/s}^2.$$

2.26. Velocity can be estimated by computing the slope of a curve in a distance versus time plot.



Velocity is defined by $v = \Delta x / \Delta t$. If acceleration is constant, then $a = \frac{v_f - v_i}{t_f - t_i} = \frac{\Delta v}{\Delta t}$. (a) Estimate the slope of the dashed blue line. Pick two points: it is more accurate to pick a point that coincides with horizontal lines of the grid. Choosing points t = 0 s, x = 0 m and t = 6.25 s, x = 20 m:

$$v = \frac{20.\text{ m} - 0\text{ m}}{6.25\text{ s} - 0\text{ s}} = 3.2\text{ m/s}$$

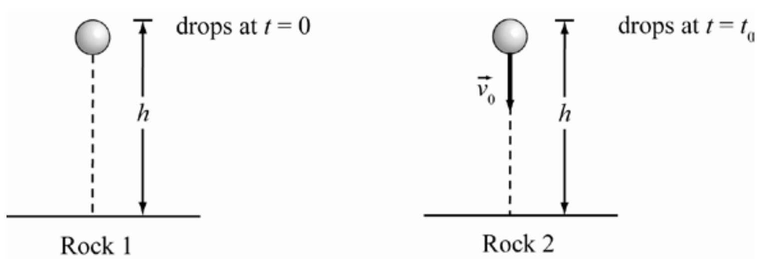
(b) Examine the sketch. There is a tangent to the curve at t = 7.5 s. Pick two points on the line. Choosing points: t = 3.4 s, x = 0 m and t = 9.8 s, x = 60 m:

$$v = \frac{60.\text{ m} - 0\text{ m}}{9.8\text{ s} - 3.4\text{ s}} = 9.4\text{ m/s}$$

(c) From (a), v = 3.2 m/s at t = 2.5 s and from (b), v = 9.4 m/s at t = 7.5 s. From the definition of constant acceleration,

$$a = \frac{9.4\text{ m/s} - 3.2\text{ m/s}}{7.5\text{ s} - 2.5\text{ s}} = \frac{6.2\text{ m/s}}{5.0\text{ s}} = 1.2\text{ m/s}^2.$$

2.27. There are two rocks, rock 1 and rock 2. Both rocks are dropped from height h. Rock 1 has initial velocity $v = 0$ and rock 2 has $v = v_0$ and is thrown at $t = t_0$.



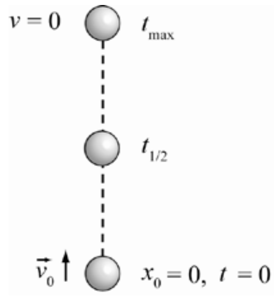
Rock 1: $h = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2h}{g}}$

Rock 2: $h = v_0(t - t_0) + \frac{1}{2}g(t - t_0)^2 \Rightarrow \frac{1}{2}g(t - t_0)^2 + v_0(t - t_0) - h = 0$

This equation has roots $t - t_0 = \frac{-v_0 \pm \sqrt{v_0^2 + 2gh}}{g}$. Choose the positive root since $(t - t_0) > 0$. Therefore $t_0 = t + \frac{v_0 - \sqrt{v_0^2 + 2gh}}{g}$. Substituting $t = \sqrt{\frac{2h}{g}}$ gives:

$$t_0 = \sqrt{\frac{2h}{g}} + \frac{v_0}{g} - \frac{\sqrt{v_0^2 + 2gh}}{g} \text{ or } \sqrt{\frac{2h}{g}} + \frac{v_0}{g} - \sqrt{\left(\frac{v_0}{g}\right)^2 + \frac{2h}{g}}.$$

2.28. I want to know when the object is at half its maximum height. The wrench is thrown upwards with an initial velocity $v(t=0)=v_0.$, $x=x_0+v_0t-\frac{1}{2}gt^2$, $v=v_0-gt$, and $g=9.81\text{ m/s}^2$.



At maximum height, $v = 0$. $v=v_0-gt \Rightarrow 0=v_0-gt_{max} \Rightarrow v_0=gt_{max}$. Substitute $t_{max}=v_0/g$ into $x=x_0+v_0t-(1/2)gt^2$.

$$x_{max}=v_0\left(\frac{v_0}{g}\right)-\frac{1}{2}g\left(\frac{v_0}{g}\right)^2=\frac{v_0^2}{g}-\frac{1}{2}\left(\frac{v_0^2}{g}\right)=\frac{v_0^2}{g}\left(1-\frac{1}{2}\right)=\frac{v_0^2}{2g}$$

Therefore, half of the maximum height is $x_{1/2}=\frac{v_0^2}{4g}$. Substitute this into the equation for x.

$$x_{1/2}=\frac{v_0^2}{4g}=v_0t_{1/2}-\frac{1}{2}gt_{1/2}^2 \Rightarrow \frac{1}{2}gt_{1/2}^2-v_0t_{1/2}+\frac{v_0^2}{4g}=0$$

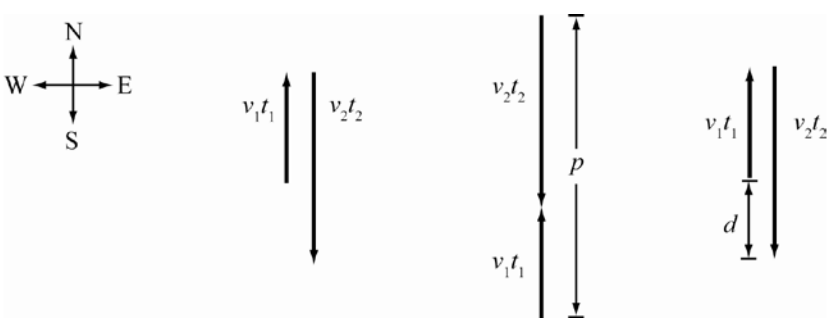
This is a quadratic equation with respect to $t_{1/2}$. The solutions to this equation are:

$$t_{1/2}=\frac{v_0\pm\sqrt{v_0^2-4\left(\frac{1}{2}g\right)\left(\frac{v_0^2}{4g}\right)}}{2\left(\frac{1}{2}g\right)}=\frac{v_0\pm\sqrt{v_0^2-\frac{1}{2}v_0^2}}{g}=\frac{v_0\pm v_0\left(\frac{1}{\sqrt{2}}\right)}{g}=\frac{v_0}{g}\left(1\pm\frac{1}{\sqrt{2}}\right)$$

Exercises

2.29. **THINK:** What is the distance traveled, p, and the displacement d if $v_1=30.0\text{ m/s}$ due north for $t_1=10.0\text{ min}$ and $v_2=40.0\text{ m/s}$ due south for $t_2=20.0\text{ min}$? Times should be in SI units: $t_1=10.0\text{ min}(60\text{ s/min})=6.00\cdot 10^2\text{ s}$, $t_2=20.0\text{ min}(60\text{ s/min})=1.20\cdot 10^3\text{ s}$.

SKETCH:



RESEARCH: The distance is equal to the product of velocity and time. The distance traveled is $p=v_1t_1+v_2t_2$ and the displacement is the distance between where you start and where you finish, $d=v_1t_1-v_2t_2$.

SIMPLIFY: There is no need to simplify.

CALCULATE: $p = v_1t_1 + v_2t_2 = (30.\text{ m/s})(6.00\cdot 10^2\text{ s}) + (40.\text{ m/s})(1.20\cdot 10^3\text{ s}) = 66{,}000.\text{ m}$

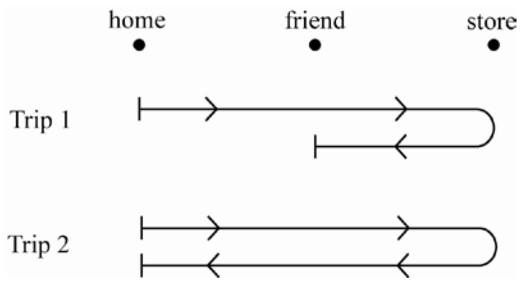
$d = v_1t_1 - v_2t_2 = (30.\text{ m/s})(6.00\cdot 10^2\text{ s}) - (40.\text{ m/s})(1.20\cdot 10^3\text{ s}) = -30{,}000.\text{ m}$

ROUND: The total distance traveled is 66.0 km, and the displacement is 30.0 km in southern direction.

DOUBLE-CHECK: The distance traveled is larger than the displacement as expected. Displacement is also expected to be towards the south since the second part of the trip going south is faster and has a longer duration.

2.30. **THINK:** I want to find the displacement and the distance traveled for a trip to the store, which is 1000. m away, and back. Let $l = 1000.$ m.

SKETCH:



RESEARCH: displacement (d) = final position – initial position
distance traveled = distance of path taken

SIMPLIFY:

(a) $d = \frac{1}{2}l - 0 = \frac{1}{2}l$

(b) $p = l + \frac{1}{2}l = \frac{3}{2}l$

(c) $d = 0 - 0 = 0$

(d) $p = l + l = 2l$

CALCULATE:

(a) $d = \frac{1}{2}l = \frac{1}{2}(1000.\text{ m}) = 500.0\text{ m}$

(b) $p = \frac{3}{2}l = \frac{3}{2}(1000.\text{ m}) = 1500.\text{ m}$

(c) $d = 0\text{ m}$

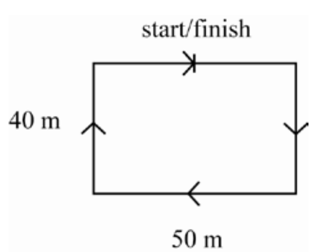
(d) $p = 2l = 2(1000.\text{ m}) = 2000.\text{ m}$

ROUND: No rounding is necessary.

DOUBLE-CHECK: These values are reasonable: they are of the order of the distance to the store.

2.31. **THINK:** I want to find the average velocity when I run around a rectangular 50 m by 40 m track in 100 s.

SKETCH:



RESEARCH: average velocity = $\dfrac{\text{final position} - \text{initial position}}{\text{time}}$

SIMPLIFY: $\bar{v} = \dfrac{x_f - x_i}{t}$

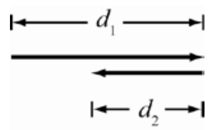
CALCULATE: $\bar{v} = \dfrac{0 \text{ m} - 0 \text{ m}}{100 \text{ s}} = 0 \text{ m/s}$

ROUND: Rounding is not necessary, because the result of 0 m/s is exact.

DOUBLE-CHECK: Since the final and initial positions are the same point, the average velocity will be zero. The answer may be displeasing at first since someone ran around a track and had no average velocity. Note that the speed would not be zero.

2.32. **THINK:** I want to find the average velocity and the average speed of the electron that travels $d_1 = 2.42$ m in $t_1 = 2.91 \cdot 10^{-8}$ s in the positive x-direction then $d_2 = 1.69$ m in $t_2 = 3.43 \cdot 10^{-8}$ s in the opposite direction.

SKETCH:



RESEARCH:

(a) average velocity = $\dfrac{\text{final position} - \text{initial position}}{\text{time}}$

(b) speed = $\dfrac{\text{total distance traveled}}{\text{time}}$

SIMPLIFY:

(a) $\bar{v} = \dfrac{d_1 - d_2}{t_1 + t_2}$

(b) $s = \dfrac{d_1 + d_2}{t_1 + t_2}$

CALCULATE:

(a) $\bar{v} = \dfrac{d_1 - d_2}{t_1 + t_2} = \dfrac{2.42 \text{ m} - 1.69 \text{ m}}{2.91 \cdot 10^{-8} \text{ s} + 3.43 \cdot 10^{-8} \text{ s}} = 11{,}514{,}195 \text{ m/s}$

(b) $s = \dfrac{d_1 + d_2}{t_1 + t_2} = \dfrac{2.42 \text{ m} + 1.69 \text{ m}}{2.91 \cdot 10^{-8} \text{ s} + 3.43 \cdot 10^{-8} \text{ s}} = 64{,}826{,}498 \text{ m/s}$

ROUND:

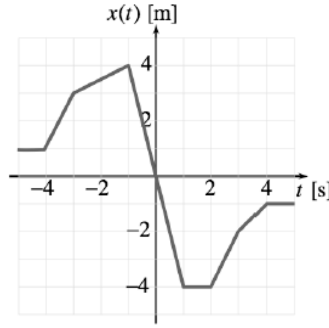
(a) $\bar{v} = 1.15 \cdot 10^7$ m/s

(b) $s = 6.48 \cdot 10^7$ m/s

DOUBLE-CHECK: The average velocity is less than the speed, which makes sense since the electron changes direction.

2.33. **THINK:** The provided graph must be used to answer several questions about the speed and velocity of a particle. Questions about velocity are equivalent to questions about the slope of the position function.

SKETCH:



RESEARCH: The velocity is given by the slope on a distance versus time graph. A steeper slope means a greater speed.

$$\text{average velocity} = \frac{\text{final position} - \text{initial position}}{\text{time}}, \quad \text{speed} = \frac{\text{total distance traveled}}{\text{time}}$$

(a) The largest speed is where the slope is the steepest.
(b) The average velocity is the total displacement over the time interval.
(c) The average speed is the total distance traveled over the time interval.
(d) The ratio of the velocities is $v_1 : v_2$.
(e) A velocity of zero is indicated by a slope that is horizontal.

SIMPLIFY:

(a) The largest speed is given by the steepest slope occurring between –1 s and +1 s.

$$s = \frac{|x(t_2) - x(t_1)|}{t_2 - t_1}, \text{ with } t_2 = 1 \text{ s and } t_1 = -1 \text{ s.}$$

(b) The average velocity is given by the total displacement over the time interval.

$$\bar{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}, \text{ with } t_2 = 5 \text{ s and } t_1 = -5 \text{ s.}$$

(c) In order to calculate the speed in the interval –5 s to 5 s, the path must first be determined. The path is given by starting at 1 m, going to 4 m, then turning around to move to –4 m and finishing at –1 m. So the total distance traveled is

$$\begin{aligned} p &= \left|(4 \text{ m} - 1 \text{ m})\right| + \left|((-4 \text{ m}) - 4 \text{ m})\right| + \left|(-1 \text{ m} - (-4 \text{ m}))\right| \\ &= 3 \text{ m} + 8 \text{ m} + 3 \text{ m} \\ &= 14 \text{ m} \end{aligned}$$

This path can be used to find the speed of the particle in this time interval.

$$s = \frac{p}{t_2 - t_1}, \text{ with } t_2 = 5 \text{ s and } t_1 = -5 \text{ s.}$$

(d) The first velocity is given by $v_1 = \dfrac{x(t_3) - x(t_2)}{t_3 - t_2}$ and the second by $v_2 = \dfrac{x(t_4) - x(t_3)}{t_4 - t_3}$,

(e) The velocity is zero in the regions 1 s to 2 s, -5 s to -4 s, and 4 s to 5 s.

CALCULATE:

(a) $s = \dfrac{|-4 \text{ m} - 4 \text{ m}|}{1 \text{ s} - (-1 \text{ s})} = 4.0 \text{ m/s}$

(b) $\bar{v} = \dfrac{-1 \text{ m} - 1 \text{ m}}{5 \text{ s} - (-5 \text{ s})} = -0.20 \text{ m/s}$

(c) $\bar{s} = \dfrac{14\text{ m}}{5\text{ s} - (-5\text{ s})} = 1.4\text{ m/s}$

(d) $v_1 = \dfrac{(-2\text{ m}) - (-4\text{ m})}{3\text{ s} - 2\text{ s}} = 2.0\text{ m/s}$, $v_2 = \dfrac{(-1\text{ m}) - (-2\text{ m})}{4\text{ s} - 3\text{ s}} = 1.0\text{ m/s}$, so $v_1 : v_2 = 2:1$.

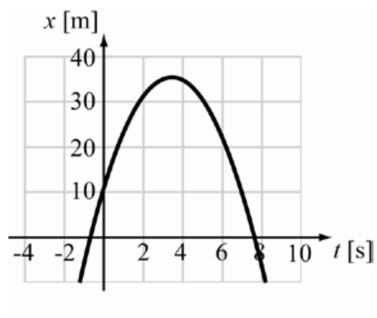
(e) There is nothing to calculate.

ROUND: Rounding is not necessary in this case, because we can read the values of the positions and times off the graph to at least 2 digit precision.

DOUBLE-CHECK: The values are reasonable for a range of positions between –4 m and 4 m with times on the order of seconds. Each calculation has the expected units.

2.34. **THINK:** I want to find the average velocity of a particle whose position is given by the equation $x(t) = 11 + 14t - 2.0t^2$ during the time interval t = 1.0 s to t = 4.0 s.

SKETCH:



RESEARCH: The average velocity is given by the total displacement over the time interval. $\bar{v} = \dfrac{x(t_2) - x(t_1)}{t_2 - t_1}$, with t_2 = 4.0 s and t_1 = 1.0 s.

SIMPLIFY: $\bar{v} = \dfrac{x(t_2) - x(t_1)}{t_2 - t_1} = \dfrac{\left(11 + 14t_2 - 2.0t_2^{\,2}\right) - \left(11 + 14t_1 - 2.0t_1^{\,2}\right)}{t_2 - t_1} = \dfrac{14(t_2 - t_1) - 2.0(t_2^{\,2} - t_1^{\,2})}{t_2 - t_1}$

CALCULATE: $\bar{v} = \dfrac{14(4.0\text{ s} - 1.0\text{ s}) - 2.0((4.0\text{ s})^2 - (1.0\text{ s})^2)}{4.0\text{ s} - 1.0\text{ s}} = 4.0\text{ m/s}$

ROUND: The values given are all accurate to two significant digits, so the answer is given by two significant digits: v = 4.0 m/s.

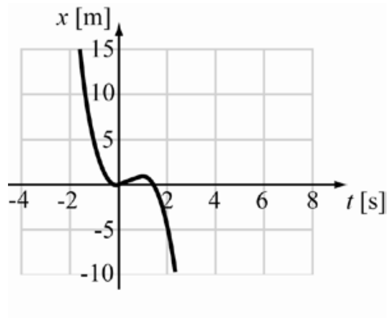
DOUBLE-CHECK: A reasonable approximation of the average velocity from $t = 1$ to $t = 4$ is to look at the instantaneous velocity at the midpoint. The instantaneous velocity is given by the derivative of the position, which is

$$v = \frac{d}{dt}(11 + 14t - 2.0t^2) = 0 + 1(14) - 2(2.0t) = 14 - 4.0t.$$

The value of the instantaneous velocity at $t = 2.5$ s is $14 - 4.0(2.5) = 4.0$ m/s. The fact that the calculated average value matches the instantaneous velocity at the midpoint lends support to the answer.

2.35. **THINK:** I want to find the position of a particle when it reaches its maximum speed. I know the equation for the position as a function of time: $x = 3.0t^2 - 2.0t^3$. I will need to find the expression for the velocity and the acceleration to determine when the speed will be at its maximum. The maximum speed in the x-direction will occur at a point where the acceleration is zero.

SKETCH:



RESEARCH: The velocity is the derivative of the position function with respect to time. In turn, the acceleration is given by derivative of the velocity function with respect to time. The expressions can be found using the formulas:

$$v(t) = \frac{d}{dt}x(t), \; a(t) = \frac{d}{dt}v(t).$$

Find the places where the acceleration is zero. The maximum speed will be the maximum of the speeds at the places where the acceleration is zero.

SIMPLIFY: $v(t) = \frac{d}{dt}x(t) = \frac{d}{dt}(3.0t^2 - 2.0t^3) = 2 \cdot 3.0t^{2-1} - 3 \cdot 2.0t^{3-1} = 6.0t - 6.0t^2$

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}(6.0t - 6.0t^2) = 6.0t^{1-1} - 2 \cdot 6.0t^{2-1} = 6.0 - 12t$$

CALCULATE: Solving for the value of t where a is zero:

$$0 = 6.0 - 12t \;\Rightarrow\; 6.0 = 12t \;\Rightarrow\; t = 0.50 \text{ s}$$

This time can now be used to solve for the position:

$$x(0.50) = 3.0(0.50)^2 - 2.0(0.50)^3 = 0.500 \text{ m}$$

Since there is only one place where the acceleration is zero, the maximum speed in the positive x-direction must occur here.

ROUND: Since all variables and parameters are accurate to 2 significant digits, the answer should be too: $x = 0.50$ m.

DOUBLE-CHECK: The validity of the answer can be confirmed by checking the velocity at $t = 0.50$ s and times around this point. At $t = 0.49$ s, the velocity is 1.4994 m/s, and at $t = 0.51$ s the velocity is also 1.4994 m/s. Since these are both smaller than the velocity at 0.50 s ($v = 1.5$ m/s), the answer is valid.

2.36. **THINK:** I want to find the time it took for the North American and European continents to reach a separation of 3000 mi if they are traveling at a speed of 10 mm/yr. First convert units:

$$d = (3000 \text{ mi})(1609 \text{ m/mi}) = 4827000 \text{ m}, \quad v = (10 \text{ mm/yr})(10^{-3} \text{ m/mm}) = 0.01 \text{ m/yr}.$$

SKETCH:

RESEARCH: The time can be found using the familiar equation: $d = vt$.
SIMPLIFY: The equation becomes $t = d / v$.

CALCULATE: $t = \dfrac{4827000 \text{ m}}{0.01 \text{ m/yr}} = 482{,}700{,}000 \text{ yr}$

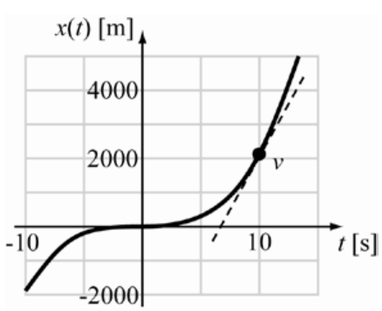
ROUND: The values given in the question are given to one significant digit, thus the answer also should only have one significant digit: $t = 5 \cdot 10^8$ yr .

DOUBLE-CHECK: The super continent Pangea existed about 250 million years ago or $2.5 \cdot 10^8$ years. Thus, this approximation is in the ballpark.

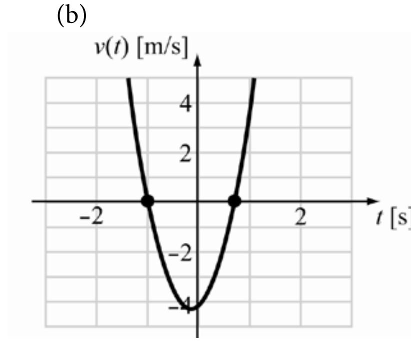
2.37. **THINK:**
(a) I want to find the velocity at t = 10.0 s of a particle whose position is given by the function $x(t) = At^3 + Bt^2 + Ct + D$, where A = 2.10 m/s^3 , B = 1.00 m/s^2 , C = –4.10 m/s, and D = 3.00 m. I can differentiate the position function to derive the velocity function.
(b) I want to find the time(s) when the object is at rest. The object is at rest when the velocity is zero. I'll solve the velocity function I obtain in (a) equal to zero.
(c) I want to find the acceleration of the object at t = 0.50 s. I can differentiate the velocity function found in part (a) to derive the acceleration function, and then calculate the acceleration at t = 0.50 s.
(d) I want to plot the function for the acceleration found in part (c) between the time range of –10.0 s to 10.0 s.

SKETCH:

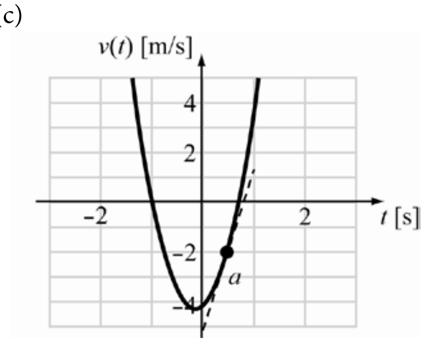
(a)



(b)



(c)



(d) The plot is part of CALCULATE.

RESEARCH:

(a) The velocity is given by the time derivative of the positive function $v(t) = \dfrac{d}{dt} x(t)$.

(b) To find the time when the object is at rest, set the velocity to zero, and solve for t. This is a quadratic equation of the form $ax^2 + bx + c = 0$, whose solution is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

(c) The acceleration is given by the time derivative of the velocity: $a(t)=\frac{d}{dt}v(t)$.

(d) The equation for acceleration found in part (c) can be used to plot the graph of the function.

SIMPLIFY:

(a) $v(t)=\frac{d}{dt}x(t)=\frac{d}{dt}(At^3+Bt^2+Ct+D)=3At^2+2Bt+C$

(b) Set the velocity equal to zero and solve for t using the quadratic formula:

$$t=\frac{-2B\pm\sqrt{4B^2-4(3A)(C)}}{2(3A)}=\frac{-2B\pm\sqrt{4B^2-12AC}}{6A}$$

(c) $a(t)=\frac{d}{dt}v(t)=\frac{d}{dt}(3At^2+2Bt+C)=6At+2B$

(d) There is no need to simplify this equation.

CALCULATE:

(a) $v(t=10.0\ s)=3(2.10\text{ m/s}^3)(10.0\text{ s})^2+2(1.00\text{ m/s}^2)(10.0\text{ s})-4.10\text{ m/s}=645.9\text{ m/s}$

(b) $t=\dfrac{-2(1.00\text{ m/s}^2)\pm\sqrt{4(1.00\text{ m/s}^2)^2-12(2.10\text{ m/s}^3)(-4.10\text{ m/s})}}{6(2.1\ 0\text{m/s}^3)}$

$=0.6634553\text{ s},-0.9809156\text{ s}$

(c) $a(t=0.50\text{ s})=6(2.10\text{ m/s}^3)(0.50\text{ s})+2(1.00\text{ m/s}^2)=8.30\text{ m/s}^2$

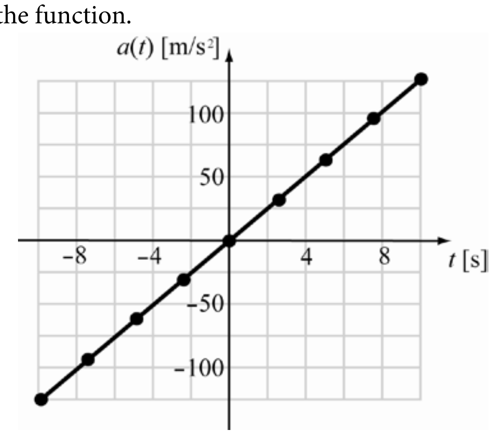
(d) The acceleration function, $a(t) = 6At + 2B$, can be used to compute the acceleration for time steps of 2.5 s. For example:

$$a(t=-2.5\text{ s})=6(2.10\text{ m/s}^3)(-2.5\text{ s})+2(1.00\text{ m/s}^2)=-29.5\text{ m/s}^2$$

The result is given in the following table.

t **[s]**	–10.0	–7.5	–5.0	–2.5	0.0	2.5	5.0	7.5	10.0
a **[m/s^2]**	–124.0	–92.5	–61.0	–29.5	2.0	33.5	65.0	96.5	128.0

These values are used to plot the function.



ROUND:

(a) The accuracy will be determined by the factor $3(2.10\text{ m/s}^3)(10.0\text{ s})^2$, which only has two significant digits. Thus the velocity at 10.0 s is 646 m/s.

(b) The parameters are accurate to two significant digits, thus the solutions will also have three significant digits: t = 0.663 s and –0.981 s

(c) The accuracy is limited by the values with the smallest number of significant figures. This requires three significant figures. The acceleration is then a = 8.30 m/s^2.

(d) No rounding is necessary.

DOUBLE-CHECK:

(a) This result is reasonable given the parameters. For example, $t^2 = (10.0 \text{ s})^2 = 100. \text{ s}$, so the velocity should be in the hundreds of meters per second.

(b) Since the function is quadratic, there should be two solutions. The negative solution means that the object was at rest 0.98 seconds before the time designated $t = 0$ s.

(c) These values are consistent with the parameters.

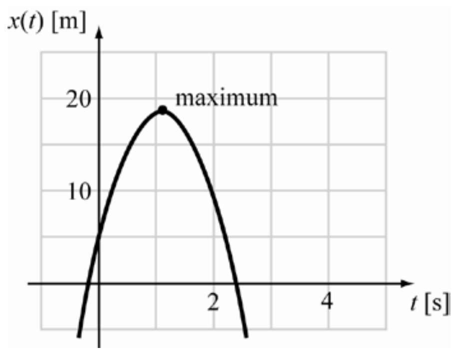
(d) The function for the acceleration is linear which the graph reflects.

2.38. **THINK:** I want to determine the time when a particle will reach its maximum displacement and what the displacement will be at that time. The equation of the object's displacement is given as:

$$x(t) = 4.35 \text{ m} + (25.9 \text{ m/s})t - \left(11.79 \text{ m/s}^2\right)t^2$$

Differentiating x with respect to t gives the equation for the velocity. This is important since the time at which the velocity is zero is the moment at which the object has reached its maximum displacement.

SKETCH:



RESEARCH: The velocity is the derivative: $v = \frac{d}{dt}x(t)$. Find the value of t that makes the velocity zero. Then, for part (b), substitute that value of t back into $x(t)$.

SIMPLIFY: $v = \frac{d}{dt}[4.35 \text{ m} + (25.9 \text{ m/s})t - (11.79 \text{ m/s}^2)t^2]$

$= 25.9 \text{ m/s} - 2(11.79 \text{ m/s}^2)t$

Time for the maximum displacement is found by solving for t in the equation: $25.9 \text{ m/s} - 2(11.79 \text{ m/s}^2)t = 0$.

CALCULATE:

(a) $t = \dfrac{25.9 \text{ m/s}}{2(11.79 \text{ m/s}^2)} = 1.0984 \text{ s}$

(b) $x(t) = 4.35 \text{ m} + (25.9 \text{ m/s})t - (11.79 \text{ m/s}^2)t^2$

$= 4.35 \text{ m} + (25.9 \text{ m/s})(1.10 \text{ s}) - (11.79 \text{ m/s}^2)(1.10 \text{ s})^2$

$= 18.5741 \text{ m}$

ROUND:

(a) The accuracy of this time is limited by the parameter 25.9 m/s, thus the time is $t = 1.10$ s.

(b) The least accurate term in the expression for $x(t)$ is accurate to the nearest tenth, so $x_{max} = 18.6 \text{ m}$.

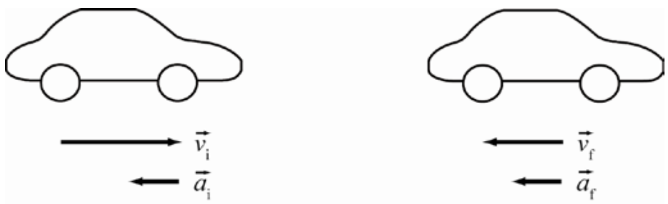
DOUBLE-CHECK: Consider the positions just before and after the time $t = 1.10$ s. $x = 18.5$ m for $t = 1.00$ s, and $x = 18.5$ m for $t = 1.20$ s. These values are less than the value calculated for x_{max}, which confirms the accuracy of the result.

2.39. **THINK:** I want to calculate the average acceleration of the bank robbers getaway car. He starts with an initial speed of 45 mph and reaches a speed of 22.5 mph in the opposite direction in 12.4 s.
First convert the velocities to SI units:

$$v_{\mathrm{i}} = (45 \text{ mph})\left(0.447 \ \frac{\text{m/s}}{\text{mph}}\right) = 20.115 \text{ m/s}$$

$$v_{\mathrm{f}} = (-22.5 \text{ mph})\left(0.447 \ \frac{\text{m/s}}{\text{mph}}\right) = -10.0575 \text{ m/s}$$

SKETCH:



RESEARCH: average acceleration = $\frac{\text{change in velocity}}{\text{change in time}}$

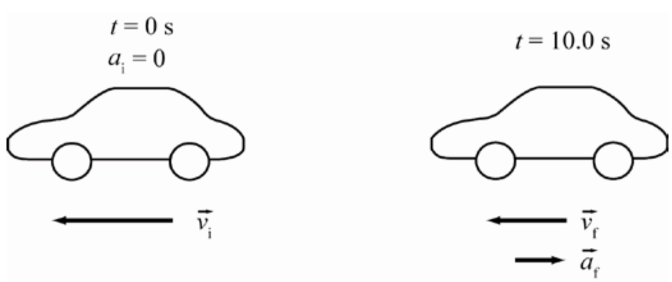
SIMPLIFY: $\bar{a} = \frac{v_{\mathrm{f}} - v_{\mathrm{i}}}{t}$

CALCULATE: $\bar{a} = \frac{(-10.0575 \text{ m/s}) - (20.115 \text{ m/s})}{12.4 \text{ s}} = -2.433 \text{ m/s}^2$

ROUND: The least precise of the velocities given in the question had two significant figures. Therefore, the final answer should also have two significant figures. The acceleration is $\bar{a} = -2.4 \text{ m/s}^2$, or 2.4 m/s^2 in the backward direction.

DOUBLE-CHECK: A top-of-the-line car can accelerate from 0 to 60 mph in 3 s. This corresponds to an acceleration of 8.94 m/s^2 . It is reasonable for a getaway car to be able to accelerate at a fraction of this value.

2.40. **THINK:** I want to find the magnitude and direction of average acceleration of a car which goes from 22.0 m/s in the west direction to 17.0 m/s in the west direction in 10.0 s: $v_{\mathrm{f}} = 17.0$ m/s, $v_{\mathrm{i}} = 22.0$ m/s, $t = 10.0$ s.
SKETCH:



RESEARCH: $\bar{a} = \frac{v_{\mathrm{f}} - v_{\mathrm{i}}}{t}$

SIMPLIFY: There is no need to simplify the above equation.

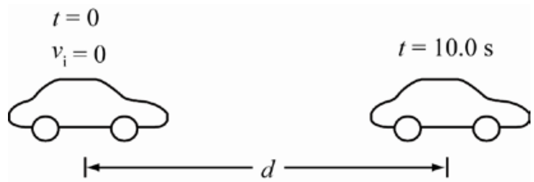
CALCULATE: $\bar{a} = \frac{17.0 \text{ m/s} - 22.0 \text{ m/s}}{10.0 \text{ s}} = -0.5000 \text{ m/s}^2$. The negative indicates the acceleration is east.

ROUND: The average acceleration is $\bar{a} = 0.500 \text{ m/s}^2$ east.

DOUBLE-CHECK: An acceleration of -0.500 m/s^2 is reasonable since a high performance car can accelerate at about 9 m/s^2.

2.41. **THINK:** I want to find the magnitude of the constant acceleration of a car that goes 0.500 km in 10.0 s: $d = 0.500$ km, $t = 10.0$ s.

SKETCH:



RESEARCH: The position of the car under constant acceleration is given by $d = \frac{1}{2}at^2$.

SIMPLIFY: Solving for acceleration gives $a = \frac{2d}{t^2}$.

CALCULATE: $a = \frac{2(0.500 \text{ km})}{(10.0 \text{ s})^2} = 0.0100 \text{ km/s}^2$

ROUND: The values all have three significant figures. Thus, the average acceleration is $a = 0.0100 \text{ km/s}^2$, which is 10.0 m/s^2.

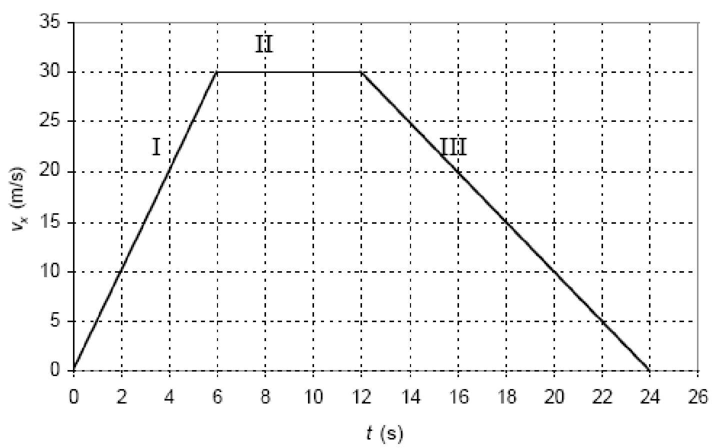
DOUBLE-CHECK: This acceleration is on the order of a high performance car which can accelerate from 0 to 60 mph in 3 seconds, or 9 m/s^2.

2.42. **THINK:**

(a) I want to find the average acceleration of a car and the distance it travels by analyzing a velocity versus time graph. Each segment has a linear graph. Therefore, the acceleration is constant in each segment.

(b) The displacement is the area under the curve of a velocity versus time graph.

SKETCH:



RESEARCH:

(a) The acceleration is given by the slope of a velocity versus time graph.

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

(b) The displacement is the sum of the areas of two triangles and one rectangle. Recall the area formulas:

$$\text{area of a triangle} = \frac{\text{base} \times \text{height}}{2}$$

$$\text{area of a rectangle} = \text{base} \times \text{height}$$

SIMPLIFY:

(a) $a_{\text{I}} = \dfrac{v_{\text{I}_2} - v_{\text{I}_1}}{t_{\text{I}_2} - t_{\text{I}_1}}$, $a_{\text{II}} = \dfrac{v_{\text{II}_2} - v_{\text{II}_1}}{t_{\text{II}_2} - t_{\text{II}_1}}$, $a_{\text{III}} = \dfrac{v_{\text{III}_2} - v_{\text{III}_1}}{t_{\text{III}_2} - t_{\text{III}_1}}$

(b) $x = \dfrac{1}{2} v_{\text{I}_2}(t_{\text{I}_2} - t_{\text{I}_1}) + v_{\text{II}_2}(t_{\text{II}_2} - t_{\text{II}_1}) + \dfrac{1}{2} v_{\text{III}_1}(t_{\text{III}_2} - t_{\text{III}_1})$

CALCULATE:

(a) $a_{\text{I}} = \dfrac{30.0 \text{ m/s} - 0 \text{ m/s}}{6.0 \text{ s} - 0 \text{ s}} = 5.0 \text{ m/s}^2$, $a_{\text{II}} = \dfrac{30.0 \text{ m/s} - 30.0 \text{ m/s}}{12.0 \text{ s} - 6.0 \text{ s}} = 0.0 \text{ m/s}^2$,

$a_{\text{III}} = \dfrac{0.0 \text{ m/s} - 30.0 \text{ m/s}}{24.0 \text{ s} - 12.0 \text{ s}} = -2.50 \text{ m/s}^2$

(b) $x = \dfrac{1}{2}(30.0 \text{ m/s})(6.0 \text{ s} - 0.0 \text{ s}) + (30.0 \text{ m/s})(12.0 \text{ s} - 6.0 \text{ s}) + \dfrac{1}{2}(30.0 \text{ m/s})(24.0 \text{ s} - 12.0 \text{ s}) = 450.0 \text{ m}$

ROUND:

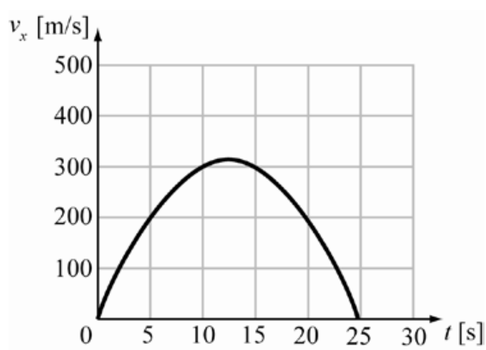
(a) Rounding is not necessary in this case, because the values of the velocities and times can be read off the graph to at least two digit precision.

(b) The answer is limited by the value 6.0 s, giving x = 450 m.

DOUBLE-CHECK: The accelerations calculated in part (a) are similar to those of cars. The distance of 450 m is reasonable. The acceleration in I should be -2 times the acceleration in III, since the change in velocities are opposites, and the time in III for the change in velocity is twice the change in time that occurs in I.

2.43. **THINK:** I want to find the acceleration of a particle when it reaches its maximum displacement. The velocity of the particle is given by the equation $v_x = 50.0t - 2.0t^3$. The maximum displacement must occur when the velocity is zero. The expression for the acceleration can be found by differentiating the velocity with respect to time.

SKETCH:



RESEARCH: The acceleration is the derivative of the velocity: $a = \dfrac{d}{dt} v_x$ **.** The maximum displacement will occur at a point where the velocity is zero. So, I can find the time at which the displacement is maximal by solving $v_x = 50.0t - 2.0t^3 = 0$ for t. The question says to consider after $t = 0$, so I will reject zero and negative roots. Then differentiate v with respect to t to obtain a formula for the acceleration. Evaluate the acceleration at the time where the displacement is maximized (which is when the velocity is zero).

SIMPLIFY: No simplification is required.

CALCULATE: Solving $v_x = 50.0t - 2.0t^3 = 0$ for t: $0 = 2.0t(25 - t^2)$, so t = 0, ±5.0. So, take $t = 5$. Now, differentiate v with respect t to find the expression for the acceleration.

$$a = \frac{d}{dt}(50.0t - 2.0t^3)$$
$$= 50.0 - 6.0t^2$$

Substitute $t = 5.0$ s into the expression for acceleration:

$$a = 50.0 - 6.0t^2 = 50.0 - 6.0(5.0 \text{ s})^2 = -100 \text{ m/s}^2$$

ROUND: The solution is limited by the accuracy of $6.0t^2$, where $t = 5.0$ s, so it must be significant to two digits. This gives $50.0 - 150 = -100$ m/s^2, which is also accurate to two significant figures. Therefore, the acceleration must be accurate to two significant figures: $a = -1.0 \cdot 10^2$ m/s^2.

DOUBLE-CHECK: The acceleration must be negative at this point, since the displacement would continue to increase if a was positive.

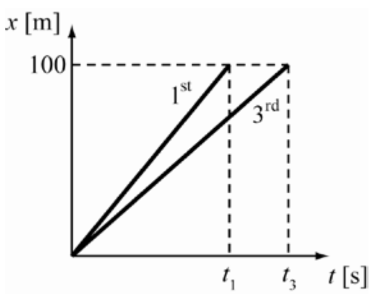
2.44. **THINK:**

(a) I want to know the distance between the first and third place runner when the first crosses the finish line, assuming they run at their average speeds throughout the race. The race is 100. m and the first place runner completes the race in 9.77 s while the third place runner takes 10.07 s to reach the finish line: $d = 100.$ m, $t_1 = 9.77$ s, and $t_3 = 10.07$ s.

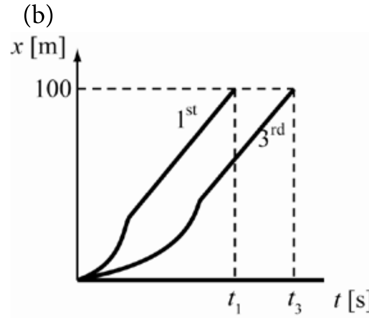
(b) I want to know the distance between the two runners when the first crosses the finish line, assuming they both accelerate to a top speed of 12 m/s: $d = 100.$ m, $t_1 = 9.77$ s, $t_3 = 10.07$ s, and $v = 12$ m/s.

SKETCH:

(a)



(b)



RESEARCH:

(a) First the average speed of each runner must be calculated: $\bar{s} = d/t$. From this the distance between the two runners can be found: $\Delta d = d_1 - d_2$, where d_1 is 100. m and d_2 is the position of the third place runner at 9.77 s.

(b) Since both runners are running at 12 m/s at the end of the race, the distance between the runners will be the distance the 3rd place runner runs after the first place runner crosses the line: $\Delta d = v\Delta t$.

SIMPLIFY:

(a) $\Delta d = d_1 - d_2 = d_1 - \bar{s}_3 t_1 = d_1 - \left(\frac{d_1}{t_3}\right)t_1 = d_1 - d_1\left(\frac{t_1}{t_3}\right) = d_1\left(1 - \frac{t_1}{t_3}\right)$

(b) $\Delta d = v(t_3 - t_1)$

CALCULATE:

(a) $\Delta d = d_1\left(1 - \frac{t_1}{t_3}\right) = (100.\text{ m})\left(1 - \frac{9.77 \text{ s}}{10.07 \text{ s}}\right) = 2.9791 \text{ m}$

(b) $\Delta d = (12 \text{ m/s})(10.07 \text{ s} - 9.77 \text{ s}) = 3.6 \text{ m}$

ROUND:

(a) The answer is limited to 3 significant figures from 9.77 s so $\Delta d = 2.98$ m.

(b) The distance then is 3.60 m between the first and third place runners.

DOUBLE-CHECK:

The two calculated distances are a small fraction (about 3%) of the race. It is reasonable for the third place runner to finish a small fraction of the track behind the first place finisher.

2.45. THINK:

(a) Since the motion is all in one direction, the average speed equals the distance covered divided by the time taken. I want to know the distance between the place where the ball was caught and midfield. I also want to know the time taken to cover this distance. The average speed will be the quotient of those two quantities.

(b) Same as in (a), but now I need to know the distance between midfield and the place where the run ended.

(c) I do not need to calculate the acceleration over each small time interval, since all that matters is the velocity at the start of the run and at the end. The average acceleration is the difference between those two quantities, divided by the time taken.

SKETCH:

In this case a sketch is not needed, since the only relevant quantities are those describing the runner at the start and end of the run, and at midfield.

RESEARCH:

The distance between two positions can be represented as $\Delta d = d_\text{f} - d_\text{i}$, where d_i is the initial position and d_f is the final one. The corresponding time difference is $\Delta t = t_\text{f} - t_\text{i}$. The average speed is $\Delta d / \Delta t$.

(a) Midfield is the 50-yard line, so $d_\text{i} = -1$ yd, $d_\text{f} = 50$ yd, $t_\text{i} = 0.00$ s, and $t_\text{f} = 5.73$ s.

(b) The end of the run is 1 yard past d = 100 yd, so $d_\text{i} = 50$ yd, $d_\text{f} = 101$ yd, $t_\text{i} = 5.73$ s, and $t_\text{f} = 12.01$ s.

(c) The average velocity is $\Delta v / \Delta t = \left(v_\text{f} - v_\text{i}\right)/\left(t_\text{f} - t_\text{i}\right)$. For this calculation, $t_\text{i} = 0$, $t_\text{i} = 12.01$ s, and $v_\text{i} = v_\text{f} = 0$ m/s, since the runner starts and finishes the run at a standstill.

SIMPLIFY:

(a), (b) $\dfrac{\Delta d}{\Delta t} = \dfrac{d_\text{f} - d_\text{i}}{t_\text{f} - t_\text{i}}$

(c) No simplification needed

CALCULATE:

(a) $\dfrac{\Delta d}{\Delta t} = \dfrac{(50\text{ yd}) - (-1\text{ yd})}{(5.73\text{ s}) - (0.00\text{ s})} = 8.900522356\text{ yd/s}\left(\dfrac{3\text{ ft}}{1\text{ yd}}\right)\left(\dfrac{0.3048\text{ m}}{1\text{ ft}}\right) = 8.138638743\text{ m/s}$

(b) $\dfrac{\Delta d}{\Delta t} = \dfrac{(101\text{ yd}) - (50\text{ yd})}{(12.01\text{ s}) - (5.73\text{ s})} = 8.121019108\text{ yd/s}\left(\dfrac{3\text{ ft}}{1\text{ yd}}\right)\left(\dfrac{0.3048\text{ m}}{1\text{ ft}}\right) = 7.425859873\text{ m/s}$

(c) $\dfrac{\Delta v}{\Delta t} = \dfrac{v_\text{f} - v_\text{i}}{t_\text{f} - t_\text{i}} = \dfrac{0\text{ m/s} - 0\text{ m/s}}{12.01\text{ s} - 0.00\text{ s}} = 0\text{ m/s}^2$

ROUND:

(a) We assume that the yard lines are exact, but the answer is limited to 3 significant figures by the time data. So the average speed is 8.14 m/s.

(b) The average speed is 7.43 m/s.

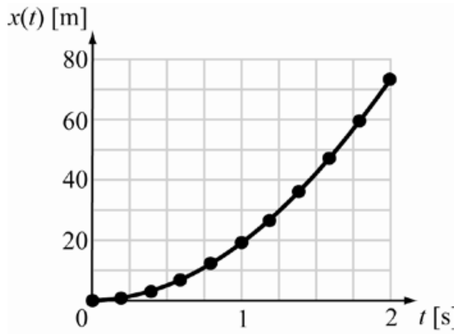
(c) The average velocity is 0 m/s^2.

DOUBLE-CHECK:

The average speeds in parts (a) and (b) are reasonable speeds (8.9 ft/s is about 18 mph), and it makes sense that the average speed during the second half of the run would be slightly less than during the first half, due to fatigue. In part (c) it is logical that average acceleration would be zero, since the net change in velocity is zero.

2.46. **THINK:** Use the difference formula to find the average velocity, and then the average acceleration of the jet given its position at several times, and determine whether the acceleration is constant.

SKETCH:



RESEARCH: The difference formula $m = \dfrac{\text{final point} - \text{initial point}}{\text{final time} - \text{initial time}}$.

SIMPLIFY: For velocity the difference formula is $v = (x_f - x_i)/(t_f - t_i)$ and the corresponding difference formula for the acceleration is $a = (v_f - v_i)/(t_f - t_i)$.

CALCULATE: As an example, $v = (6.6 \text{ m} - 3.0 \text{ m})/(0.60 \text{ s} - 0.40 \text{ s}) = 18.0 \text{ m/s}$, and the acceleration is $a = (26.0 \text{ m/s} - 18.0 \text{ m/s})/(0.80 \text{ s} - 0.60 \text{ s}) = 40.0 \text{ m/s}^2$.

t **[s]**	***x*** **[m]**	***v*** **[m/s]**	**a [m/s²]**
0.00	0	0.0	
0.20	0.70	3.5	17.5
0.40	3.0	11.5	40
0.60	6.6	18	32.5
0.80	11.8	26	40
1.00	18.5	33.5	37.5
1.20	26.6	40.5	35
1.40	36.2	48	37.5
1.60	47.3	55.5	37.5
1.80	59.9	63	37.5
2.00	73.9	70	35

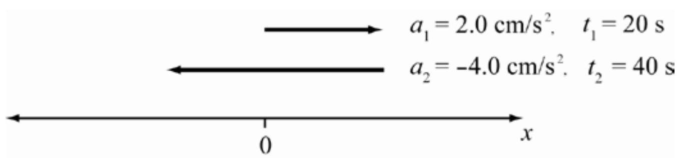
ROUND: The position measurements are given to the nearest tenth of a meter, and the time measurements are given to two significant figures. Therefore each of the stated results for velocity and acceleration should be rounded to two significant figures.

t [s]	*x* [m]	*v* [m/s]	a [m/s^2]
0.00	0.0	0.0	
0.20	0.70	3.5	18
0.40	3.0	12	40.
0.60	6.6	18	33
0.80	11.8	26	40.
1.00	18.5	34	38
1.20	26.6	41	35
1.40	36.2	48	38
1.60	47.3	56	38
1.80	59.9	63	38
2.00	73.9	70.	35

DOUBLE-CHECK: The final speed of the jet is 70. m/s, which is equivalent to 250 km/hr, the typical take-off speed of a commercial jet airliner.

2.47. **THINK:** I want to find the position of a particle after it accelerates from rest at $a_1 = 2.00\text{ cm/s}^2$ for $t_1 = 20.0\text{ s}$ then accelerates at $a_2 = -4.00\text{ cm/s}^2$ for $t_2 = 40.0\text{ s}$.

SKETCH:



RESEARCH: The position of a particle undergoing constant acceleration is given by the formula $x = x_0 + v_0 t + \frac{1}{2}at^2$. The same particle's velocity is given by $v = v_0 + at$. The final speed at the end of the first segment is the initial speed for the second segment.

SIMPLIFY: For the first 20 s the particle's position is $x_1 = \frac{1}{2}a_1 t_1^2$. This is the initial position for the second segment of the particle's trip. For the second segment, the particle is no longer at rest but has a speed of $v = a_1 t_1$.

$$x = x_1 + v_0 t_2 + \frac{1}{2}a_2 t_2^2 = \frac{1}{2}a_1 t_1^2 + a_1 t_1 t_2 + \frac{1}{2}a_2 t_2^2$$

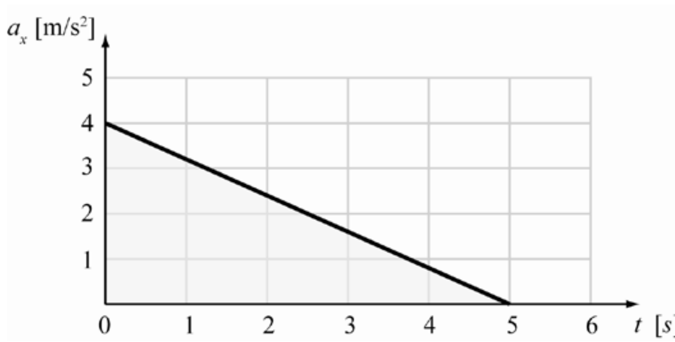
CALCULATE:

$$x = \frac{1}{2}(2.00\text{ cm/s}^2)(20.0\text{ s})^2 + (2.00\text{ cm/s}^2)(20.0\text{ s})(40.0\text{ s}) + \frac{1}{2}(-4.00\text{ cm/s}^2)(40.0\text{ s})^2 = -1200\text{ cm}$$

ROUND: The variables are given with three significant figures. Therefore, the particle is $-1.20 \cdot 10^3$ cm from its original position.

DOUBLE-CHECK: Note that the second phase of the trip has a greater magnitude of acceleration than the first part. The duration of the second phase is longer; thus the final position is expected to be negative.

2.48. **THINK:** The car has a velocity of +6 m/s and a position of +12 m at $t = 0$. What is its velocity at $t = 5.0$ s? The change in the velocity is given by the area under the curve in an acceleration versus time graph.
SKETCH:



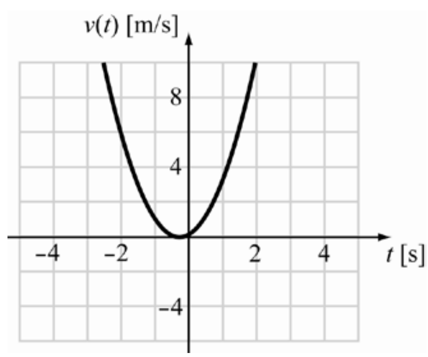
RESEARCH: $v = v_0 +$ area, area of triangle = $\dfrac{\text{base} \cdot \text{height}}{2}$

SIMPLIFY: $v = v_0 + \dfrac{\Delta a \Delta t}{2}$

CALCULATE: $v = 6 \text{ m/s} + \dfrac{(4.0 \text{ m/s}^2)(5.0 \text{ s})}{2} = 16 \text{ m/s}$

ROUND: The function can only be accurate to the first digit before the decimal point. Thus $v = 16$ m/s.
DOUBLE-CHECK: 16 m/s is approximately 58 km/h, which is a reasonable speed for a car.

2.49. **THINK:** I want to find the position of a car at $t_f = 3.0$ s if the velocity is given by the equation $v = At^2 + Bt$ with $A = 2.0 \text{ m/s}^3$ and $B = 1.0 \text{ m/s}^2$.
SKETCH:



RESEARCH: The position is given by the integral of the velocity function: $x = x_0 + \int_0^{t_f} v(t)\,dt$.

SIMPLIFY: Since the car starts at the origin, $x_0 = 0$ m.

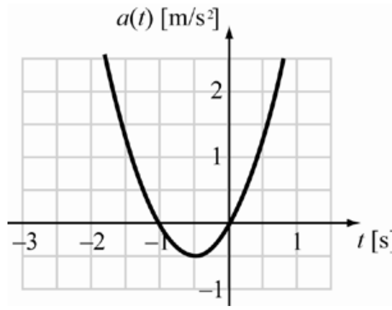
$$x = \int_0^{t_f} v(t)\,dt = \int_0^{t_f} \left(At^2 + Bt\right)dt = \frac{1}{3}At_f^3 + \frac{1}{2}Bt_f^2$$

CALCULATE: $x = \dfrac{1}{3}(2.0 \text{ m/s}^3)(3.0 \text{ s})^3 + \dfrac{1}{2}(1.0 \text{ m/s}^2)(3.0 \text{ s})^2 = 22.5 \text{ m}$

ROUND: The parameters are given to two significant digits, and so the answer must also contain two significant digits: $x = 23$ m.
DOUBLE-CHECK: This is a reasonable distance for a car to travel in 3.0 s.

2.50. **THINK:** An object starts at rest (so $v_0 = 0$ m/s) and has an acceleration defined by $a(t) = Bt^2 - (1/2)Ct$, where $B = 2.0 \text{ m/s}^4$ and $C = -4.0 \text{ m/s}^3$. I want to find its velocity and distance traveled after 5.0 s. Measure the position from the starting point $x_0 = 0$ m.

SKETCH:



RESEARCH:

(a) The velocity is given by integrating the acceleration with respect to time: $v = \int a(t)dt$.

(b) The position is given by integrating the velocity with respect to time: $x = \int v(t)dt$.

SIMPLIFY:

$$v = \int a(t)dt = \int\left(Bt^2 - \frac{1}{2}Ct\right)dt = \frac{1}{3}Bt^3 - \frac{1}{4}Ct^2 + v_0, \text{ and}$$

$$x = \int vdt = \int\left(\frac{1}{3}Bt^3 - \frac{1}{4}Ct^2 + v_0\right)dt = \frac{1}{12}Bt^4 - \frac{1}{12}Ct^3 + v_0t + x_0$$

CALCULATE:

$$v = \frac{1}{3}Bt^3 - \frac{1}{4}Ct^2 + v_0 = \frac{1}{3}(2.0 \text{ m/s}^4)(5.0 \text{ s})^3 - \frac{1}{4}(-4.0 \text{ m/s}^3)(5.0 \text{ s})^2 + 0 \text{ m/s} = 108.33 \text{ m/s}$$

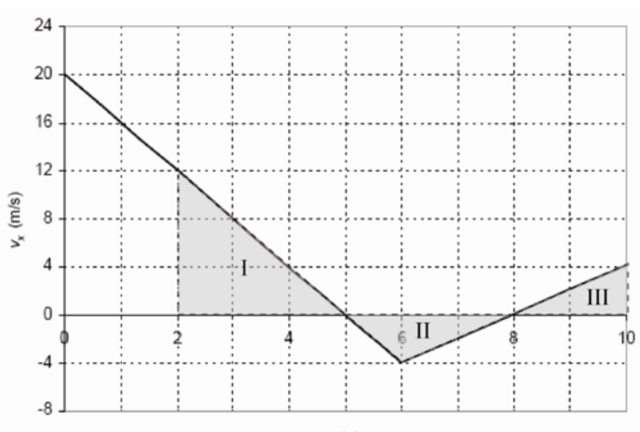
$$x = \frac{1}{12}(2.0 \text{ m/s}^4)(5.0 \text{ s})^4 - \frac{1}{12}(-4.0 \text{ m/s}^3)(5.0 \text{ s})^3 + (0 \text{ m/s})(5.0 \text{ s}) + (0 \text{ m}) = 145.83 \text{ m}$$

ROUND: All parameters have two significant digits. Thus the answers should also have two significant figures: at t = 5.0 s, $v = 110$ m/s and x = 150 m.

DOUBLE-CHECK: The distance traveled has units of meters, and the velocity has units of meters per second. These are appropriate units for a distance and velocity, respectively.

2.51. **THINK:** A car is accelerating as shown in the graph. At t_0= 2.0 s, its position is x_0 = 2.0 m. I want to determine its position at t = 10.0 s.

SKETCH:



RESEARCH: The change in position is given by the area under the curve of the velocity versus time graph plus the initial displacement: $x = x_0 +$ area . Note that region II is under the t-axis will give a negative area. Let A_1 be the area of region I, let A_2 be the area of region II, and let A_3 be the area of region III.

SIMPLIFY: $x = x_0 + A_{\text{I}} + A_{\text{II}} + A_{\text{III}}$

CALCULATE:

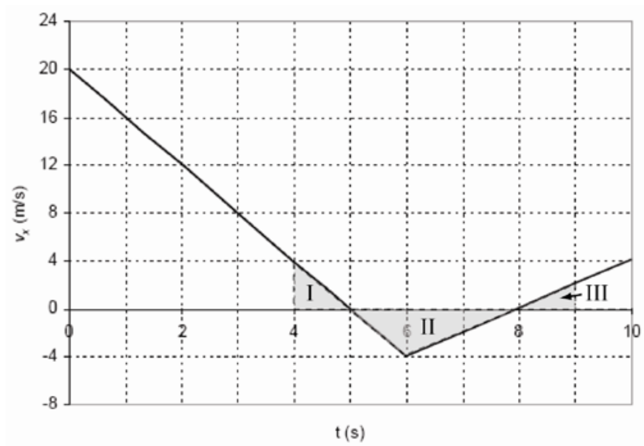
$$x = 2.0 \text{ m} + \frac{1}{2}(12.0 \text{ m/s})(5.0 \text{ s} - 2.0 \text{ s}) + \frac{1}{2}(-4.0 \text{ m/s})(8.0 \text{ s} - 5.0 \text{ s}) + \frac{1}{2}(4.0 \text{ m/s})(10.0 \text{ s} - 8.0 \text{ s}) = 18 \text{ m}$$

ROUND: The answer should be given to the least accurate calculated area. These are all accurate to the meter, thus the position is x = 18 m.

DOUBLE-CHECK: The maximum velocity is 12 m/s. If this were sustained over the 8 second interval, the distance traveled would be $2.0 \text{ m} + (12 \text{ m/s})(8.0 \text{ s}) = 98 \text{ m}.$ Since there was a deceleration and then an acceleration, we expect that the actual distance will be much less than the value 98 m.

2.52. **THINK:** A car is accelerating as shown in the graph. I want to determine its displacement between t = 4 s and t = 9 s.

SKETCH:



RESEARCH: The change in position is given by the area under the curve of a velocity versus time graph. Note that it is hard to read the value of the velocity at t = 9.0 s. This difficulty can be overcome by finding the slope of the line for this section. Using the slope, the velocity during this time can be determined: $\Delta x = \text{Area}$, $m = \frac{\text{rise}}{\text{run}}$. Let A_1 be the area of region I, let A_2 be the area of region II, and let A_3 be the area of region III.

SIMPLIFY: $\Delta x = A_{\text{I}} + A_{\text{II}} + A_{\text{III}}$

CALCULATE: $m = \dfrac{4.0 \text{ m/s} - (-4.0 \text{ m/s})}{10.0 \text{ s} - 6.0 \text{ s}} = 2.0 \text{ m/s}^2$

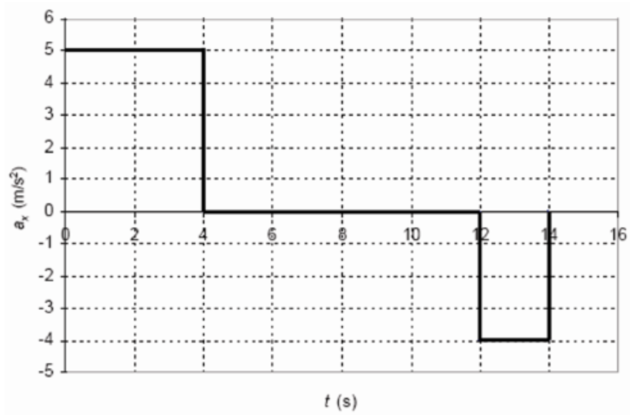
$$\Delta x = \frac{1}{2}(4.0 \text{ m/s})(5.0 \text{ s} - 4.0 \text{ s}) + \frac{1}{2}(-4.0 \text{ m/s})(8.0 \text{ s} - 5.0 \text{ s}) + \frac{1}{2}(2.0 \text{ m/s})(9.0 \text{ s} - 8.0 \text{ s}) = -3.0 \text{ m}$$

ROUND: $\Delta x = -3.0 \text{ m}$

DOUBLE-CHECK: The car will end up with a negative displacement since the area of region II is larger than the combined areas of regions I and III. The overall displacement is less than if the car had traveled constantly at its maximum velocity of 4 m/s (when the displacement would have been 20 m).

2.53. **THINK:** A motorcycle is accelerating at different rates as shown in the graph. I want to determine (a) its speed at t = 4.00 s and t = 14.0 s, and (b) its total displacement between t = 0 and t = 14.0 s.

SKETCH:



RESEARCH:

(a) The velocity of the motorcycle is defined by the area under the curve of the acceleration versus time graph. This area can be found by counting the blocks under the curve then multiply by the area of one block: 1 block = (2 s) 1 m/s² = 2 m/s.

(b) The displacement can be found by separating the acceleration into three parts: The first phase has an acceleration of $a_1 = 5 \text{ m/s}^2$ for times between 0 to 4 seconds. The second phase has no acceleration, thus the motorcycle has a constant speed. The third phase has a constant acceleration of $a_3 = -4 \text{ m/s}^2$. Recall the position and velocity of an object under constant acceleration is $x = x_0 + v_0 t + (1/2)at^2$ and $v = v_0 + at$, respectively.

SIMPLIFY: At t = 4.00 s and 14.0 s, there are 10 blocks and 6 blocks respectively. Recall that blocks under the time axis are negative. In the first phase the position is given by $x = (1/2)a_1(\Delta t_1)^2$ where Δt is the duration of the phase. The velocity at the end of this phase is $v = a_1 \Delta t_1$. The position and velocity of the first phase gives the initial position and velocity for the second phase.

$$x = x_0 + v_0 \Delta t_2 = \frac{1}{2}a_1(\Delta t_1)^2 + a_1 \Delta t_1 \Delta t_2$$

Since the velocity is constant in the second phase, this value is also the initial velocity of the third phase.

$$x = x_0 + v_0 \Delta t_3 + \frac{1}{2}a_3(\Delta t_3)^2 = \frac{1}{2}a_1(\Delta t_1)^2 + a_1 \Delta t_1 \Delta t_2 + a_1 \Delta t_1 \Delta t_3 + \frac{1}{2}a_3(\Delta t_3)^2$$

CALCULATE:

(a) $v(t = 4.00 \text{ s}) = 10(2.00 \text{ m/s}) = 20.0 \text{ m/s}$, $v(t = 14.0 \text{ s}) = 6(2.00 \text{ m/s}) = 12.0 \text{ m/s}$

(b)

$$x = \frac{1}{2}(5.0 \text{ m/s}^2)(4.00 \text{ s} - 0 \text{ s})^2 + (5.0 \text{ m/s}^2)(4.00 \text{ s} - 0 \text{ s})(12.0 \text{ s} - 4.0 \text{ s}) + (5.0 \text{ m/s}^2)(4.00 \text{ s} - 0 \text{ s})(14.0 \text{ s} - 12.0 \text{ s})$$
$$+ \frac{1}{2}(-4.0 \text{ m/s}^2)(14.0 \text{ s} - 12.0 \text{ s})^2$$
$$= 232 \text{ m}$$

ROUND:

(a) Rounding is not necessary in this case, because the values of the accelerations and times can be read off the graph to at least two digit precision.

(b) The motorcycle has traveled 232 m in 14.0 s.

DOUBLE-CHECK: The velocity of the motorcycle at t = 14 s is less than the speed at t = 4 s, which makes sense since the bike decelerated in the third phase. Since the bike was traveling at a maximum speed of 20 m/s, the most distance it could cover in 14 seconds would be 280 m. The calculated value is less than this, which makes sense since the bike decelerated in the third phase.

2.54. **THINK:** I want to find the time it takes the car to accelerate from rest to a speed of v = 22.2 m/s. I know that $v_0 = 0$ m/s, $v = 22.2$ m/s, distance = 243 m, and a is constant.

SKETCH:

RESEARCH: Recall that given constant acceleration, $d = (1/2)(v_0 + v)t$.

SIMPLIFY: $t = \dfrac{2d}{v_0 + v}$

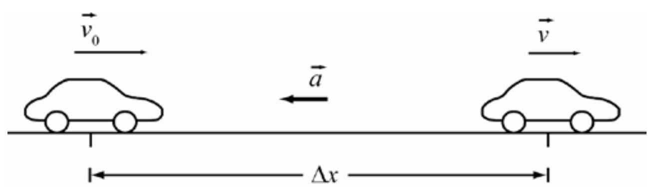
CALCULATE: $t = \dfrac{2(243 \text{ m})}{0.0 \text{ m/s} + 22.2 \text{ m/s}} = 21.8919 \text{ s}$

ROUND: Therefore, t = 21.9 s since each value used in the calculation has three significant digits.

DOUBLE-CHECK: The units of the solution are units of time, and the calculated time is a reasonable amount of time for a car to cover 243 m.

2.55. **THINK:** I want to determine (a) how long it takes for a car to decelerate from $v_0 = 31.0$ m/s to $v = 12.0$ m/s over a distance of 380. m and (b) the value of the acceleration.

SKETCH:



RESEARCH: Since the acceleration is constant, the time can be determined using the equation: $\Delta x = (1/2)(v_0 + v)t$, and the acceleration can be found using $v^2 = v_0^2 + 2a\Delta x$.

SIMPLIFY:

(a) $\Delta x = \dfrac{1}{2}(v_0 + v)t \;\Rightarrow\; (v_0 + v)t = 2\Delta x \;\Rightarrow\; t = \dfrac{2\Delta x}{v_0 + v}$

(b) $v^2 = v_0^2 + 2a\Delta x \;\Rightarrow\; 2a\Delta x = v^2 - v_0^2 \;\Rightarrow\; a = \dfrac{v^2 - v_0^2}{2\Delta x}$

CALCULATE:

(a) $t = \dfrac{2\Delta x}{v_0 + v} = \dfrac{2(380. \text{ m})}{(31.0 \text{ m/s} + 12.0 \text{ m/s})} = 17.674 \text{ s}$

(b) $a = \dfrac{v^2 - v_0^2}{2\Delta x} = \dfrac{(12.0 \text{ m/s})^2 - (31.0 \text{ m/s})^2}{2(380. \text{ m})} = -1.075 \text{ m/s}^2$

ROUND: Each result is limited to three significant figures as the values used in the calculations each have three significant figures.

(a) t = 17.7 s

(b) a = –1.08 m/s^2

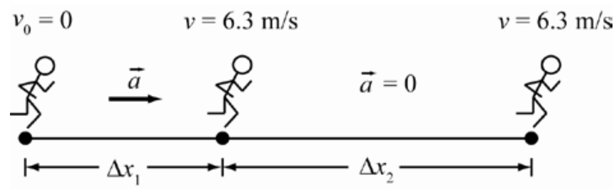
DOUBLE-CHECK:

(a) The resulting time has appropriate units and is reasonable for the car to slow down.

(b) The acceleration is negative, indicating that it opposes the initial velocity, causing the car to slow down.

2.56. **THINK:** I want to find (a) the total distance covered in time t = 59.7 s, and (b) the velocity of the runner at t = 59.7 s. It will be useful to know the time taken to accelerate, t_1, and the time taken to run at the achieved constant velocity, t_2. Note that the mass of the runner is irrelevant.

SKETCH:



RESEARCH: The runner accelerates from rest to some velocity, v, then continues to run at this constant velocity. The total distance covered, Δx, will be the sum of the distance covered while accelerating, Δx_1, and the distance covered while at constant velocity, Δx_2: $\Delta x = \Delta x_1 + \Delta x_2$. The distance Δx_1 is determined by $v^2 = v_0^2 + 2a\Delta x_1$. The distance Δx_2 is determined by $\Delta x_2 = vt_2$. It will be necessary to know the time taken to run this distance Δx_2. This time, t_2, and the time taken to cover the distance Δx_1, t_1, must sum to the given total time of 59.7 s: $t_{\text{total}} = t_1 + t_2$. The time t_1 can be determined using the equation: $v = v_0 + at_1$.

SIMPLIFY:

$$v^2 = v_0^2 + 2a\Delta x_1 \;\Rightarrow\; \Delta x_1 = \frac{v^2 - v_0^2}{2a}, \quad v = v_0 + at_1 \;\Rightarrow\; t_1 = \frac{v - v_0}{a}$$

$$\Delta x_2 = vt_2 \;\Rightarrow\; \Delta x_2 = v(t_{\text{total}} - t_1) \;\Rightarrow\; \Delta x_2 = v\left(t_{\text{total}} - \frac{v - v_0}{a}\right)$$

Finally, the total distance covered is $\Delta x = \Delta x_1 + \Delta x_2 = \dfrac{v^2 - v_0^2}{2a} + v\left(t_{\text{total}} - \dfrac{v - v_0}{a}\right) = \dfrac{v^2}{2a} + v\left(t_{\text{total}} - \dfrac{v}{a}\right)$.

CALCULATE:

(a) $$\Delta x = \frac{(6.3 \text{ m/s})^2}{2(1.25 \text{ m/s}^2)} + (6.3 \text{ m/s})\left(59.7 \text{ s} - \frac{6.3 \text{ m/s}}{1.25 \text{ m/s}^2}\right)$$
$$= 360.234 \text{ m}$$

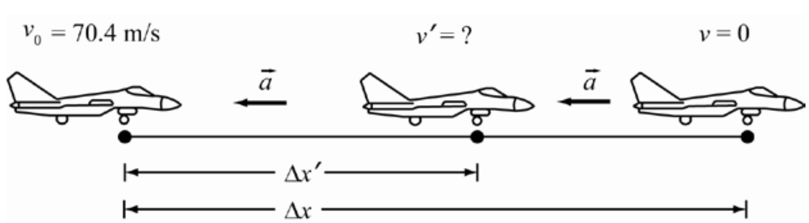
(b) Since $t_1 = \dfrac{6.3 \text{ m/s} - 0 \text{ m/s}}{1.25 \text{ m/s}^2} = 5.0$ s is the time taken to reach the final velocity, the velocity of the runner at $t_{\text{total}} = 59.7$ s is 6.3 m/s.

ROUND: Since v has only two significant digits, $\Delta x = 360$ m, or $3.6 \cdot 10^2$ m.

DOUBLE-CHECK: This seems like a reasonable distance to cover in the total time, given most of the distance is covered at the constant velocity 6.3 m/s. Since the runner stops accelerating after 5.0 s, the velocity of the runner is still 6.3 m/s at 59.7 s.

2.57. **THINK:** I am given $v_0 = 70.4$ m/s, $v = 0$, $\Delta x = 197.4$ m, and constant acceleration. I am asked to find the velocity v' when the jet is 44.2 m from its stopping position. This means the jet has traveled $\Delta x' = 197.4 \text{ m} - 44.2 \text{ m} = 153.2$ m on the aircraft carrier.

SKETCH:



RESEARCH: The initial and final velocities are known, as is the total distance traveled. Therefore the equation $v^2 = v_0^{\ 2} + 2a\Delta x$ can be used to find the acceleration of the jet. Once the acceleration is known, the intermediate velocity v' can be determined using $(v')^2 = v_0^{\ 2} + 2a\Delta x'$.

SIMPLIFY: First find the constant acceleration using the total distance traveled, Δx, the initial velocity, v_0, and the final velocity, v: $a = \dfrac{v^2 - v_0^{\ 2}}{2\Delta x} = -\dfrac{v_0^{\ 2}}{2\Delta x}$ (since $v = 0$ m/s). Next, find the requested intermediate velocity, v':

$$(v')^2 = v_0^{\ 2} + 2a\Delta x' \Rightarrow (v')^2 = v_0^{\ 2} + 2\left(-\frac{v_0^{\ 2}}{2\Delta x}\right)\Delta x' \Rightarrow v' = \sqrt{v_0^{\ 2} - \frac{v_0^{\ 2}}{\Delta x}\Delta x'}$$

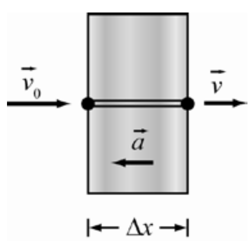
CALCULATE: $v' = \sqrt{(70.4 \text{ m/s})^2 - \dfrac{(70.4 \text{ m/s})^2}{(197.4 \text{ m})}(153.2 \text{ m})} = 33.313 \text{ m/s}$

ROUND: At $\Delta x' = 153.2$ m, the velocity is $v' = 33.3$ m/s.

DOUBLE-CHECK: This v' is less than v_0, but greater than v, and therefore makes sense.

2.58. **THINK:** I want to find the acceleration of a bullet passing through a board, given that $\Delta x = 10.0$ cm $= 0.100$ m, $v_0 = 400.$ m/s, and $v = 200.$ m/s. I expect the acceleration to be negative, since the bullet is slowing down.

SKETCH:



RESEARCH: $v^2 = v_0^{\ 2} + 2a\Delta x$

SIMPLIFY: $a = \dfrac{v^2 - v_0^{\ 2}}{2\Delta x}$

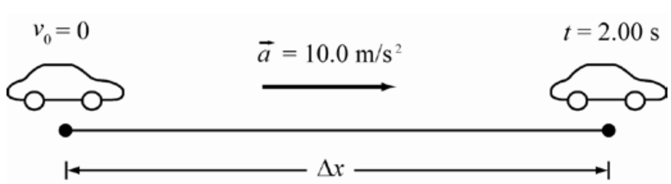
CALCULATE: $a = \dfrac{(200. \text{ m/s})^2 - (400. \text{ m/s})^2}{2(0.100 \text{ m})} = -600{,}000. \text{ m/s}^2$

ROUND: Since each velocity is given to three significant digits, $a = -6.00 \cdot 10^5 \text{ m/s}^2$.

DOUBLE-CHECK: That a is negative indicates it is in the opposite direction of the initial velocity, so the bullet slows down. The speed of the bullet decreases by 200 m/s in 0.1 m, so I am not surprised to get such a large value for the acceleration.

2.59. **THINK:** A car accelerates from rest with $a = 10.0 \text{ m/s}^2$. I want to know how far it travels in 2.00 s.

SKETCH:



RESEARCH: $\Delta x = v_0 t + \dfrac{1}{2}at^2$

SIMPLIFY: Since $v_0 = 0$ m/s, $\Delta x = \dfrac{1}{2}at^2$.

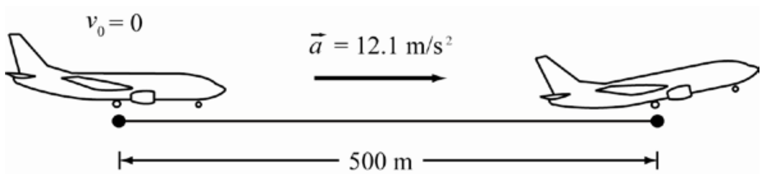
CALCULATE: $\Delta x = \frac{1}{2}(10.0 \text{ m/s}^2)(2.00 \text{ s})^2 = 20.0 \text{ m}$

ROUND: $\Delta x = 20.0 \text{ m}$

DOUBLE-CHECK: This seems like a reasonable distance to cover within 2.00 s given $a = 10.0 \text{ m/s}^2$.

2.60. **THINK:** A airplane accelerates from rest at $a = 12.1 \text{ m/s}^2$. I want to know its velocity at 500. m.

SKETCH:



RESEARCH: $v^2 = v_0^2 + 2a\Delta x$; $v_0 = 0$, $a = 12.1 \text{ m/s}^2$, $\Delta x = 500$ m

SIMPLIFY: $v = \sqrt{v_0^2 + 2a\Delta x}$
$= \sqrt{2a\Delta x}$

CALCULATE: $v = \sqrt{2(12.1 \text{ m/s}^2)(500. \text{ m})} = 110.00 \text{ m/s}$

ROUND: $v = 110. \text{ m/s}$

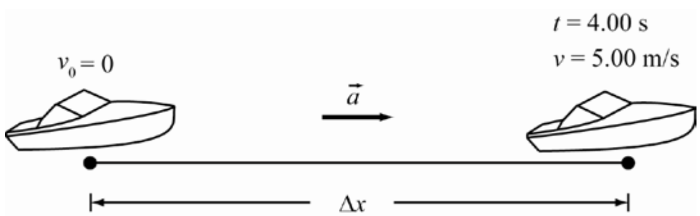
DOUBLE-CHECK: This take-off speed is about 400 kph, which is reasonable for a small plane.

2.61. **THINK:**

(a) I know that $v_0 = 0 \text{ m/s}$, $v = 5.00 \text{ m/s}$, and a is constant. I want to find v_{avg}.

(b) $t = 4.00$ s is given. I want to find Δx.

SKETCH:



RESEARCH:

(a) $v_{\text{avg}} = \frac{v_0 + v}{2}$

(b) a is unknown, so use $\Delta x = \frac{1}{2}(v_0 + v)t$

SIMPLIFY: It is not necessary to simplify the equations above.

CALCULATE:

(a) $v_{\text{avg}} = \frac{5.00 \text{ m/s} + 0 \text{ m/s}}{2} = 2.50 \text{ m/s}$

(b) $\Delta x = \frac{1}{2}(5.00 \text{ m/s} + 0 \text{ m/s})(4.00 \text{ s}) = 10.00 \text{ m}$

ROUND:

(a) v is precise to three significant digits, so $v_{\text{avg}} = 2.50$ m/s.

(b) Each v and t have three significant digits, so $\Delta x = 10.0$ m.

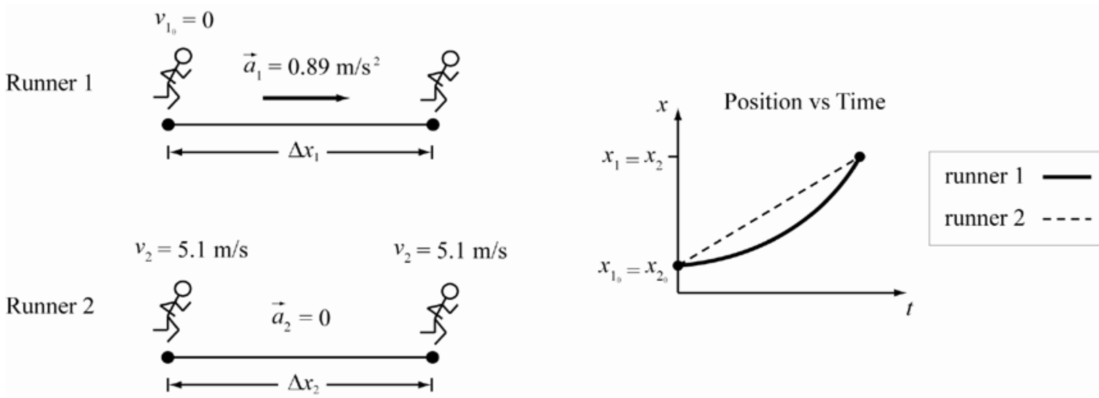
DOUBLE-CHECK:

(a) This v_{avg} is between the given v_0 and v, and therefore makes sense.

(b) This is a reasonable distance to cover in 4.00 s when $v_{\text{avg}} = 2.50$ m/s.

2.62. **THINK:** I have been given information on two runners. Runner 1 has an initial velocity $v_{1_0} = 0$ and an acceleration $a_1 = 0.89 \text{ m/s}^2$. Runner 2 has a constant velocity of $v_2 = 5.1 \text{ m/s}$. I want to know the distance traveled by runner 1 before he catches up to runner 2. Note that both runners cover the same distance, that is, $\Delta x_1 = \Delta x_2$, in the same time, *t*.

SKETCH:



RESEARCH: For runner 1, $\Delta x_1 = v_{1_0} t + (1/2) a_1 t^2$. For runner 2, $\Delta x_2 = v_2 t$.

SIMPLIFY: Since the time is not given, substitute the equation for runner 2 for the value of *t*: $t = \Delta x_2 / v_2$. Then for runner 1:

$$\Delta x_1 = v_{1_0} t + \frac{1}{2} a_1 t^2 \;\Rightarrow\; \Delta x_1 = \frac{1}{2} a_1 t^2 \;\Rightarrow\; \Delta x_1 = \frac{1}{2} a_1 \left(\frac{\Delta x_2}{v_2} \right)^2$$

Since $\Delta x_1 = \Delta x_2$, I write:

$$\Delta x_1 = \frac{1}{2} a_1 \left(\frac{\Delta x_1}{v_2} \right)^2 \;\Rightarrow\; \frac{1}{2} a_1 \frac{\Delta x_1^{\,2}}{v_2^{\,2}} - \Delta x_1 = 0 \;\Rightarrow\; \Delta x_1 \left(\frac{1}{2} a_1 \frac{\Delta x_1}{v_2^{\,2}} - 1 \right) = 0$$

Observe that one solution is $\Delta x_1 = 0$. This is true when runner 2 first passes runner 1. The other solution occurs when runner 1 catches up to runner 2: $\frac{1}{2} a_1 \frac{\Delta x_1}{v_2^{\,2}} - 1 = 0$. Then $\Delta x_1 = \frac{2 v_2^{\,2}}{a_1}$.

CALCULATE: $\Delta x_1 = \dfrac{2(5.1 \text{ m/s})^2}{(0.89 \text{ m/s}^2)} = 58.449 \text{ m}$

ROUND: $\Delta x_1 = 58 \text{ m}$

DOUBLE-CHECK: A runner might catch up to another runner on a race track in 58 m.

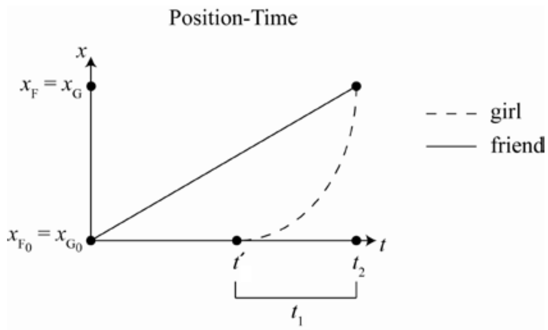
2.63. **THINK:**

(a) The girl is initially at rest, so $v_{1_0} = 0$, and then she waits $t' = 20 \text{ s}$ before accelerating at $a_1 = 2.2 \text{ m/s}^2$. Her friend has constant velocity $v_2 = 8.0 \text{ m/s}$. I want to know the time required for the girl to catch up with her friend, t_1. Note that both people travel the same distance: $\Delta x_1 = \Delta x_2$. The time the girls spends riding her bike is t_1. The friend, however, has a t' head-start; the friend travels for a total time of $t_2 = t' + t_1$.

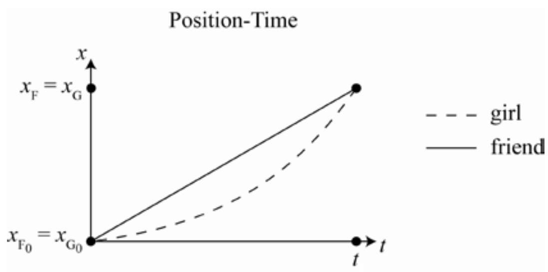
(b) The initial conditions of the girl have changed. Now $v_{1_0} = 1.2 \text{ m/s}$. The initial conditions of the friend are the same: $v_2 = 8.0 \text{ m/s}$. Now there is no time delay between when the friend passes the girl and when the girl begins to accelerate. The time taken to catch up is that found in part a), $t = 20 \text{ s}$. I will use $t = 16.2 \text{ s}$ for my calculations, keeping in mind that *t* has only two significant figures. I want to know the acceleration of the girl, a_1, required to catch her friend in time *t*.

SKETCH:

(a)



(b)



RESEARCH:

(a) The distance the girl travels is $\Delta x_1 = v_{1_0}t_1 + \frac{1}{2}a_1 t_1^{\,2}$. The distance her friend travels is $\Delta x_2 = v_2 t_2$.

(b) $\Delta x_1 = v_{1_0}t + \frac{1}{2}a_1 t^2$, $\Delta x_2 = v_2 t$

SIMPLIFY:

(a) Since $v_{1_0} = 0$, $\Delta x_1 = \frac{1}{2}a_1 t_1^{\,2}$. Also, since $t_2 = t' + t_1$, $\Delta x_2 = v_2\left(t' + t_1\right)$. Recall that $\Delta x_1 = \Delta x_2$. This leads to $\frac{1}{2}a_1 t_1^{\,2} = v_2\left(t' + t_1\right)$. Now solve for t_1: $\frac{1}{2}a_1 t_1^{\,2} = v_2 t' + v_2 t_1 \;\Rightarrow\; \frac{1}{2}a_1 t_1^{\,2} - v_2 t_1 - v_2 t' = 0.$

The quadratic formula gives:

$$t_1 = \frac{v_2 \pm \sqrt{v_2^{\,2} - 4\left(\frac{1}{2}a_1\right)\left(-v_2 t'\right)}}{2\left(\frac{1}{2}a_1\right)} = \frac{v_2 \pm \sqrt{v_2^{\,2} + 2a_1 v_2 t'}}{a_1}$$

(b) As in part (a), $\Delta x_1 = \Delta x_2$, and so $v_{1_0}t + \frac{1}{2}a_1 t^2 = v_2 t$. Solving for a_1 gives:

$$\frac{1}{2}a_1 t^2 = v_2 t - v_{1_0}t \;\Rightarrow\; a_1 = \frac{2\left(v_2 - v_{1_0}\right)}{t}$$

CALCULATE:

(a) $$t_1 = \frac{(8.0\text{ m/s}) \pm \sqrt{(-8.0\text{ m/s})^2 + 2(2.2\text{ m/s}^2)(8.0\text{ m/s})(20\text{ s})}}{2.2\text{ m/s}^2}$$

$$= \frac{(8.0\text{ m/s}) \pm \sqrt{64\text{ m}^2/\text{s}^2 + 704\text{ m}^2/\text{s}^2}}{2.2\text{ m/s}^2}$$

$$= \frac{(8.0\text{ m/s}) \pm 27.7\text{ m/s}}{2.2\text{ m/s}^2}$$

$$= 16.2272, -8.9545$$

(b) $$a_1 = \frac{2(8.0\text{ m/s} - 1.2\text{ m/s})}{16.2\text{ s}} = 0.840\text{ m/s}^2$$

ROUND:

(a) Time must be positive, so take the positive solution, $t_1 = 16$ s.

(b) $a_1 = 0.84\text{ m/s}^2$

DOUBLE-CHECK:

(a) The units of the result are those of time. This is a reasonable amount of time to catch up to the friend who is traveling at $v_2 = 8.0$ m/s.

(b) This acceleration is less than that in part (a). Without the 20 s head-start, the friend does not travel as far, and so the acceleration of the girl should be less in part (b) than in part a), given the same time.

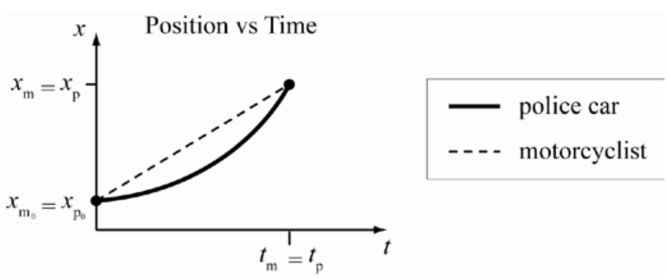
2.64. **THINK:** The motorcyclist is moving with a constant velocity $v_{\text{m}} = 36.0$ m/s. The police car has an initial velocity $v_{\text{p}_0} = 0$, and acceleration $a_{\text{p}} = 4.0\text{ m/s}^2$.

(a) I want to find the time required for the police car to catch up to the motorcycle. Note both the police car and the motorcycle travel for the same amount of time: $t_{\text{p}} = t_{\text{m}}$.

(b) I want to find the final speed of the police car, v_{p}.

(c) I want to find the distance traveled by the police car at the moment when it catches up to the motorcycle. Note the motorcyclist and the police car will have both traveled the same distance from the police car's initial position, once the police car catches up to the motorcycle. That is, $\Delta x_{\text{m}} = \Delta x_{\text{p}}$.

SKETCH:



RESEARCH:

(a) To find t_{p}, use $\Delta x_{\text{p}} = v_{\text{p}_0} t_{\text{p}} + (1/2)a_{\text{p}} t_{\text{p}}^2$ for the police car and $\Delta x_{\text{m}} = v_{\text{m}} t_{\text{m}}$ for the motorcycle.

(b) To find v_{p}, use $v_{\text{p}} = v_{\text{p}_0} + a_{\text{p}} t_{\text{p}}$ for the police car.

(c) Since $\Delta x_{\text{p}} = \Delta x_{\text{m}}$, $\Delta x_{\text{p}} = v_{\text{m}} t_{\text{m}}$ for the police car.

SIMPLIFY:

(a) Since $\Delta x_{\text{p}} = \Delta x_{\text{m}}$:

$$v_{\text{p}_0} t_{\text{p}} + \frac{1}{2} a_{\text{p}} t_{\text{p}}^{\ 2} = v_{\text{m}} t_{\text{m}}$$

$$v_{\text{p}_0} t_{\text{p}} + \frac{1}{2} a_{\text{p}} t_{\text{p}}^{\ 2} = v_{\text{m}} t_{\text{p}} \qquad \text{Since } t_{\text{m}} = t_{\text{p}},$$

$$\frac{1}{2} a_{\text{p}} t_{\text{p}}^{\ 2} = v_{\text{m}} t_{\text{p}} \qquad \text{Since } v_{\text{p}_0} = 0,$$

$$\frac{1}{2} a_{\text{p}} t_{\text{p}}^{\ 2} - v_{\text{m}} t_{\text{p}} = 0$$

$$t_{\text{p}} \left(\frac{1}{2} a_{\text{p}} t_{\text{p}} - v_{\text{m}} \right) = 0$$

There are two solutions for t_{p} here: $t_{\text{p}} = 0$ or $\left(\frac{1}{2} a_{\text{p}} t_{\text{p}} - v_{\text{m}} \right) = 0$. The first solution corresponds to the time when the motorcycle first passes the stationary police car. The second solution gives the time when the police car catches up to the motorcycle. Rearranging gives: $t_{\text{p}} = 2v_{\text{m}} / a_{\text{p}}$.

(b) $v_{\text{p}} = v_{\text{p}_0} + a_{\text{p}} t_{\text{p}} \;\Rightarrow\; v_{\text{p}} = a_{\text{p}} t_{\text{p}}$, since $v_{\text{p}_0} = 0$. Substituting $t_{\text{p}} = 2v_{\text{m}} / a_{\text{p}}$ into this equation gives:

$$v_{\text{p}} = a_{\text{p}} \left(2v_{\text{m}} / a_{\text{p}} \right) = 2v_{\text{m}}.$$

(c) No simplification is necessary.

CALCULATE:

(a) $t_{\text{p}} = \dfrac{2(36.0 \text{ m/s})}{4.0 \text{ m/s}^2} = 18.0 \text{ s}$

(b) $v_{\text{p}} = 2(36.0 \text{ m/s}) = 72.0 \text{ m/s}$

(c) $\Delta x_{\text{p}} = (36.0\text{m/s})(18.0\text{s}) = 648 \text{ m}$

ROUND:

(a) a_{p} has only two significant figures, so $t_{\text{p}} = 18$ s.

(b) v_{m} has three significant digits, so $v_{\text{p}} = 72.0$ m/s.

(c) a_{p} has only two significant digits, so $\Delta x_{\text{p}} = 650$ m.

DOUBLE-CHECK:

(a) The calculated time is reasonable for the police car to catch the motorcyclist.

(b) The calculated speed is fast, but it is a realistic speed for a police car to achieve while chasing a speeding vehicle.

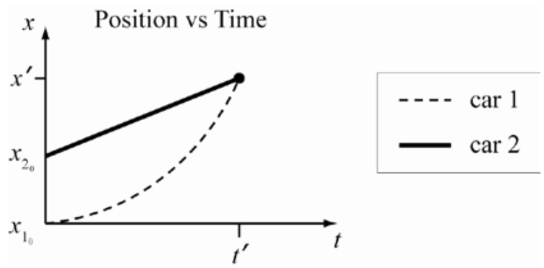
(c) The distance is a reasonable distance to cover in 18 s given that the average speed of the police car is $v_{\text{avg}} = (1/2)\left(v_{\text{p}_0} + v_{\text{p}}\right) = (1/2)v_{\text{p}} = 36.0$ m/s.

2.65. **THINK:** Since no information is given about the direction of the second car, it is assumed that both cars travel in the same direction. The first car accelerates from rest with $a_1 = 2.00 \text{ m/s}^2$. The second car moves with constant velocity $v_2 = 4.00 \text{ m/s}$. The cars are 30.0 m apart. Take the initial position of car 1 to be $x_{1_0} = 0$. Then the initial position of car 2 is $x_{2_0} = 30.0$ m. Both cars will have the same final position: $x_1 = x_2 = x'$. Both cars will travel for the same amount of time: $t_1 = t_2 = t'$.

(a) I want to know the position of the collision, x'.

(b) I want to know the time at which the collision occurs, t'.

SKETCH:



RESEARCH: car 1: $\Delta x_1 = v_{1_0} t_1 + \frac{1}{2} a_1 t_1^2$; car 2: $\Delta x_2 = v_2 t_2$

SIMPLIFY:

(a) To solve for x', use $\Delta x_1 = v_{1_0} t_1 + (1/2) a_1 t_1^2 \Rightarrow x' - x_{1_0} = v_{1_0} t_1 + (1/2) a_1 t_1^2$. Since $x_{1_0} = 0$ and $v_{1_0} = 0$, $x' = (1/2) a_1 t_1^2$. Time t_1 is not known, but $t_1 = t_2$ and $\Delta x_2 = v_2 t_2$, therefore, $t_1 = t_2 = \Delta x_2 / v_2 = \left(x' - x_{2_0}\right) / v_2$. Inserting this t_1 into the first equation yields

$$x' = \frac{1}{2} a_1 \left(\frac{x' - x_{2_0}}{v_2} \right)^2 = \frac{a_1}{2 v_2^2} \left((x')^2 - 2 x_{2_0} x' + x_{2_0}^2 \right)$$

Rearranging gives:

$$\frac{a_1}{2 v_2^2} (x')^2 - \left(\frac{a_1 x_{2_0}}{v_2^2} + 1 \right) x' + \frac{a_1 x_{2_0}^2}{2 v_2^2} = 0 \Rightarrow (x')^2 - \left(2 x_{2_0} + \frac{2 v_2^2}{a_1} \right) x' + x_{2_0}^2 = 0.$$

This is a quadratic equation. Solving for x':

$$x' = \frac{\left(2 x_{2_0} + \frac{2 v_2^2}{a_1} \right) \pm \sqrt{\left(2 x_{2_0} + \frac{2 v_2^2}{a_1} \right)^2 - 4 x_{2_0}^2}}{2}$$

(b) To solve for t', use $t' = t_2 = \frac{x' - x_{2_0}}{v_2}$, from above.

CALCULATE:

(a) $$x' = \frac{\left(2(30.0 \text{ m}) + \frac{2(4.00 \text{ m/s})^2}{2.00 \text{ m/s}^2} \right) \pm \sqrt{\left(2(30.0 \text{ m}) + \frac{2(4.00 \text{ m/s})^2}{2.00 \text{ m/s}^2} \right)^2 - 4(30.0 \text{ m})^2}}{2}$$

$= 14.68$ m, 61.32 m

The first solution may be disregarded; with both cars moving in the same direction, the position of the collision cannot be between their two initial positions. That is, x' cannot be between $x_{1_0} = 0$ and $x_{2_0} = 30$ m.

(b) $t' = \frac{x' - x_{2_0}}{v_2} = \frac{61.32 \text{ m} - 30.0 \text{ m}}{4.00 \text{ m/s}} = 7.830 \text{ s}$

ROUND:

(a) $x' = 61.3$ m

(b) $t' = 7.83$ s

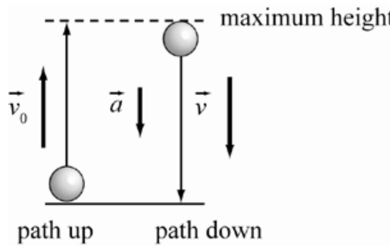
DOUBLE-CHECK:

(a) This collision position has units of distance, and is greater than the initial positions of both cars, as it should be.

(b) The time is reasonable since this is the time required for car 2 to travel $\Delta x_2 = x' - x_{2_0} = 30$ m at a speed of $v_2 = 4.0$ m/s.

2.66. **THINK:** I know that $v_0 = 26.4\text{ m/s}$ and $a = -g = -9.81\text{ m/s}^2$. I want to find t_{total}. Note that once the ball gets back to the starting point, $v = -26.4\text{ m/s}$, or $v = -v_0$.

SKETCH:



RESEARCH: $v = v_0 + at$

SIMPLIFY: $t = \dfrac{v - v_0}{a} = \dfrac{-v_0 - v_0}{-g} = \dfrac{2v_0}{g}$

CALCULATE: $t = \dfrac{2(26.4\text{ m/s})}{9.81\text{ m/s}^2} = 5.38226\text{ s}$

ROUND: Since all the values given have three significant digits, $t = 5.38$ s.

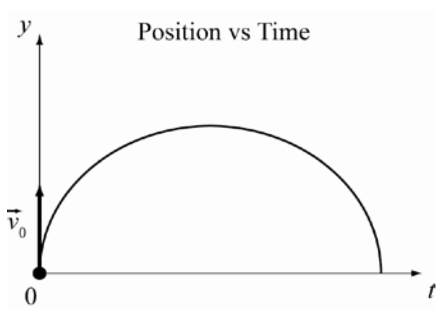
DOUBLE-CHECK: This seems like a reasonable amount of time for the ball to be up in the air.

2.67. **THINK:** I know that $v_0 = 10.0\text{ m/s}$, $a = -g = -9.81\text{ m/s}^2$, and $y_0 = 0$ m.

(a) I want to find the velocity v at $t = 0.50$ s.

(b) I want to find the height h of the stone at $t = 0.50$ s.

SKETCH:



RESEARCH:

(a) $v = v_0 + at$

(b) $\Delta y = v_0 t + \dfrac{1}{2}at^2$ and $\Delta y = h$

SIMPLIFY:

(a) $v = v_0 - gt$

(b) $h = v_0 t + \dfrac{1}{2}at^2 = v_0 t - \dfrac{1}{2}gt^2$

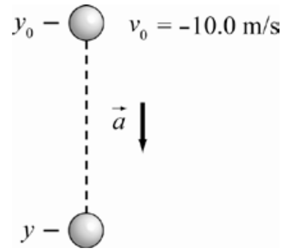
CALCULATE:

(a) $v = 10.0\text{ m/s} - (9.81\text{ m/s}^2)(0.50\text{ s})$
$= 10.0\text{ m/s} - 4.905\text{m/s}$
$= 5.095\text{ m/s}$

(b) $h = (10.0\text{ m/s})(0.50\text{ s}) - \dfrac{1}{2}(9.81\text{ m/s}^2)(0.50\text{ s})^2$
$= 5.0\text{ m} - 1.226\text{ m}$
$= 3.774\text{ m}$

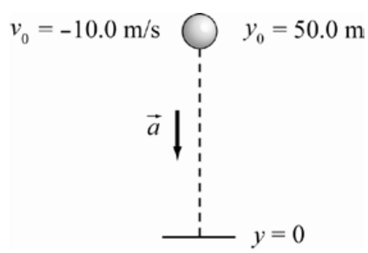
ROUND:
(a) Subtracting two numbers is precise to the least precise decimal place of the numbers. Therefore, $v = 5.1$ m/s .
(b) $h = 3.8$ m
DOUBLE-CHECK:
(a) $v < v_0$, and this makes sense since speed decreases as the object rises.
(b) This is a reasonable height for a ball to achieve in 0.50 s after it is thrown upward.

2.68. **THINK:** I know that $v_0 = -10.0$ m/s , and $a = -g = -9.81$ m/s^2. I want to find v at t = 0.500 s.
SKETCH:



RESEARCH: $v = v_0 + at$
SIMPLIFY: $v = v_0 - gt$
CALCULATE: $v = -10.0 \text{ m/s} - (9.81 \text{ m/s}^2)(0.500 \text{ s}) = -10.0 \text{ m/s} - 4.905 \text{ m/s} = -14.905 \text{ m/s}$
ROUND: Subtracting two numbers is precise to the least precise decimal place of the numbers. Therefore, $v = -14.9$ m/s.
DOUBLE-CHECK: A negative v indicates that the stone is (still) falling downward. This makes sense, since the stone was thrown downward.

2.69. **THINK:** Take "downward" to be along the negative y-axis. I know that $v_0 = -10.0$ m/s , $\Delta y = -50.0$ m , and $a = -g = -9.81$ m/s^2. I want to find t, the time when the ball reaches the ground.
SKETCH:



RESEARCH: $\Delta y = v_0 t + \frac{1}{2}at^2$

SIMPLIFY: $\frac{1}{2}at^2 + v_0 t - \Delta y = 0$. This is a quadratic equation. Solving for t:

$$t = \frac{-v_0 \pm \sqrt{v_0^2 - 4\left(\frac{1}{2}a\right)(-\Delta y)}}{2\left(\frac{1}{2}a\right)} = \frac{-v_0 \pm \sqrt{v_0^2 - 2g\Delta y}}{-g}$$

CALCULATE: $t = \dfrac{-(-10.0 \text{ m/s}) \pm \sqrt{(-10.0 \text{ m/s})^2 - 2(9.81 \text{ m/s}^2)(-50.0 \text{ m})}}{-9.81 \text{ m/s}^2}$

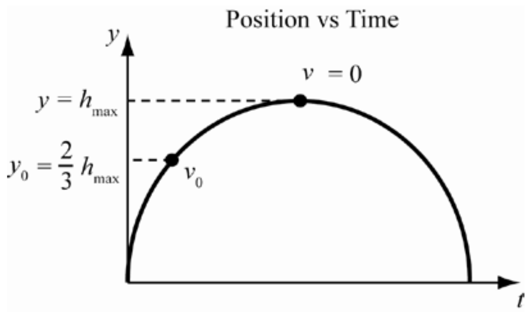
$= -4.3709 \text{ s}, \ 2.3322 \text{ s}$

The time interval has to be positive, so t = 2.3322 s.
ROUND: All original quantities are precise to three significant digits, therefore t = 2.33 s.

DOUBLE-CHECK: A negative v indicates that the stone is (still) falling downward. This makes sense, since the stone was thrown downward. The velocity is even more negative after 0.500 s than it was initially, which is consistent with the downward acceleration.

2.70. **THINK:** I know that $v_0 = 20.0$ m/s, $y_0 = (2/3)h_{max}$, and $a = -g = -9.81$ m/s^2. I want to find h_{max}. Note that when $y = h_{max}$, the velocity is $v = 0$.

SKETCH:



RESEARCH: $v^2 = v_0^2 + 2a(y - y_0)$

SIMPLIFY: $v^2 = v_0^2 - 2g\left(h_{max} - \frac{2}{3}h_{max}\right) \Rightarrow v^2 - v_0^2 = -2g\left(\frac{1}{3}h_{max}\right) \Rightarrow h_{max} = -\frac{3\left(v^2 - v_0^2\right)}{2g} \Rightarrow h_{max} = \frac{3v_0^2}{2g}$

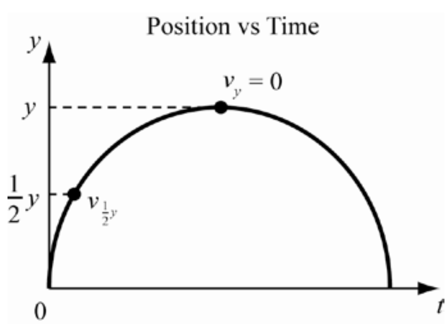
CALCULATE: $h_{max} = \frac{3(20.0 \text{ m/s})^2}{2\left(9.81 \text{ m/s}^2\right)} = 61.16$ m

ROUND: $h_{max} = 61.2$ m

DOUBLE-CHECK: h_{max} is positive which is consistent with the sketch. This seems like a reasonable height to achieve by throwing the ball upward.

2.71. **THINK:** I know the final height is y and the initial velocity is v_0. The velocity at this height is zero: $v_y = 0$. Also, $a_y = -g$. I want to know the velocity at half of the final height, $v_{\frac{1}{2}y}$. Assume $y_0 = 0$.

SKETCH:



RESEARCH: $v_y^{\ 2} = v_0^{\ 2} + 2a\left(y - y_0\right)$

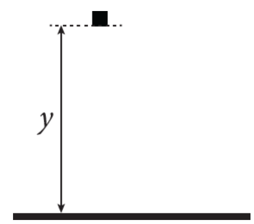
SIMPLIFY: The initial velocity, v_0, is $v_0 = \sqrt{v_y^{\ 2} - 2a\left(y - y_0\right)} = \sqrt{2gy}$. Then $v_{\frac{1}{2}y}$, in terms of the maximum height y, is

$$\left(v_{\frac{1}{2}y}\right)^2 = v_0^{\ 2} + 2a\left(\left(\frac{1}{2}y\right) - y_0\right) \Rightarrow v_{\frac{1}{2}y}^{\ \ 2} = \left(\sqrt{2gy}\right)^2 - 2g\left(\frac{1}{2}y\right) \Rightarrow v_{\frac{1}{2}y}^{\ \ 2} = 2gy - gy \Rightarrow v_{\frac{1}{2}y} = \sqrt{gy}$$

CALCULATE: This step is not necessary.
ROUND: This step is not necessary.

DOUBLE-CHECK: The units of $v_{\frac{1}{2}y}$ are: $\left[v_{\frac{1}{2}y}\right]=\sqrt{\left(\frac{\text{m}}{\text{s}^2}\right)(\text{m})}=\sqrt{\frac{\text{m}^2}{\text{s}^2}}=\text{m/s}$, which is a unit of velocity.

2.72. **THINK:** The acceleration of an object due to gravity on the surface of the Moon is independent of the mass of the object.
SKETCH:



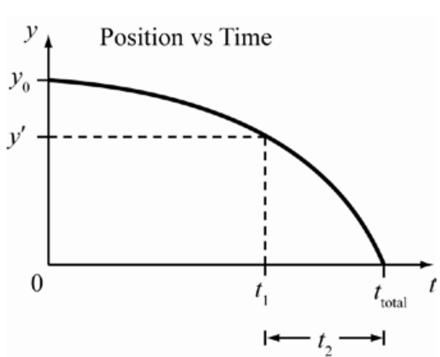
RESEARCH: We can use $y=\frac{1}{2}gt^2$, where y is the distance the objects fall, t is the time it takes for the objects to fall, and g is the acceleration of gravity on the Moon.

SIMPLIFY: We can solve our equation for g: $y=\frac{1}{2}gt^2 \Rightarrow g=\frac{2y}{t^2}$.

CALCULATE: $g=\frac{2y}{t^2}=\frac{2(1.6\text{ m})}{(1.4\text{ s})^2}=1.6327\text{ m/s}^2$.

ROUND: The values given are all accurate to two significant digits, so the answer is given two by two significant digits: $g = 1.6\text{ m/s}^2$.
DOUBLE-CHECK: The Moon is smaller and less dense than the Earth, so it makes sense that the acceleration of gravity on the surface of the Moon is about 6 times less that the acceleration of gravity on the surface of the Earth.

2.73. **THINK:** The bowling ball is released from rest. In such a case we have already studied the relationship between vertical distance fallen and time in Example 2.5, "Reaction Time", in the book. With this result in our arsenal, all we have to do here is to compute the time t_{total} it takes the ball to fall from Bill's apartment down to the ground and subtract from it the time t_1 it takes the ball to fall from Bill's apartment down to John's apartment.
SKETCH:



RESEARCH: We will use the formula $t=\sqrt{2h/g}$ from Example 2.5. If you look at the sketch, you see that $t_{\text{total}}=\sqrt{2h_{\text{total}}/g}=\sqrt{2y_0/g}$ and that $t_1=\sqrt{2h_1/g}=\sqrt{2(y_0-y')/g}$.
SIMPLIFY: Solving for the time difference gives:

$$t_2=t_{\text{total}}-t_1=\sqrt{2y_0/g}-\sqrt{2(y_0-y')/g}$$

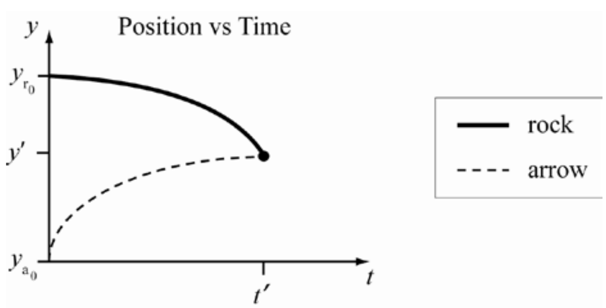
CALCULATE: $t_2 = \sqrt{2(63.17 \text{ m})/(9.81 \text{ m/s}^2)} - \sqrt{2(63.17 \text{ m} - 40.95 \text{ m})/(9.81 \text{ m/s}^2)}$
$= 1.4603 \text{ s}$

ROUND: We round to $t_2 = 1.46$ s. , because *g* has three significant figures.

DOUBLE-CHECK: The units of the solution are those of time, which is already a good minimum requirement for a valid solution. But we can do better! If we compute the time it takes an object to fall 40.95 m from rest, we find from again using $t = \sqrt{2h/g}$ that this time is 2.89 s. In the problem here the bowling ball clearly already has a significant downward velocity as it passes the height of 40.95 m, and so we expect a time t_2 shorter than 2.89 s, which is clearly fulfilled for our solution.

2.74. **THINK:** The information known for the rock is the initial velocity, $v_{r_0} = 0$ and the initial height, $y_{r_0} = 18.35 \text{ m}$. The information known for the arrow is the initial velocity, $v_{a_0} = 47.4 \text{ m/s}$ and the initial height, $y_{a_0} = 0$. For both, $a = -g = -9.81 \text{ m/s}^2$. Note that both the rock and the arrow will have the same final position, y', and both travel for the same time, t'. I want to find t'.

SKETCH:



RESEARCH: $\Delta y = v_0 t + \frac{1}{2}at^2$

SIMPLIFY: For the rock, $y_r - y_{r_0} = v_{r_0}t' + (1/2)a(t')^2 \Rightarrow y_r = -\frac{1}{2}g(t')^2 + y_{r_0}$. For the arrow, $y_a - y_{a_0} = v_{a_0}t' + (1/2)a(t')^2 \Rightarrow y_a = v_{a_0}t' - (1/2)g(t')^2$. As the final positions for each are the same, we know $y_r = y_a \Rightarrow -\frac{1}{2}g(t')^2 + y_{r_0} = v_{a_0}t' - \frac{1}{2}g(t')^2 \Rightarrow y_{r_0} = v_{a_0}t' \Rightarrow t' = \frac{y_{r_0}}{v_{a_0}}$.

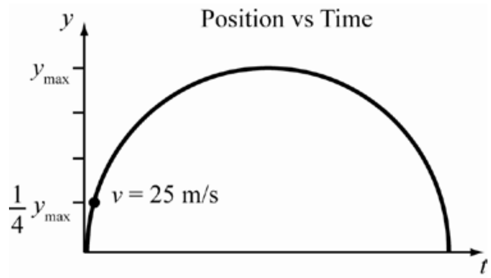
CALCULATE: $t' = \frac{18.35 \text{ m}}{47.4 \text{ m/s}} = 0.38713 \text{ s}$

ROUND: v_{a_0} is given to three significant figures, so $t' = 0.387$ s.

DOUBLE-CHECK: This is a reasonable time for an arrow of initial velocity 47.4 m/s to rise to a height less than 18.35 m (the height from which the rock was dropped).

2.75. **THINK:** At $y=(1/4)y_{max}$, $v=25$ m/s. Also, $a=-g=-9.81$ m/s^2 and $y_0=0$. I want to find v_0. It will be useful to know y_{max}. At y_{max}, $v'=0$.

SKETCH:



RESEARCH: $v^2=v_0^2+2a(y-y_0)$

SIMPLIFY: $v^2=v_0^2+2a(y-y_0)=v_0^2-2g\left(\frac{1}{4}y_{max}\right)$

$$\Rightarrow v_0^2=v^2+2g\left(\frac{1}{4}y_{max}\right)=v^2+\frac{1}{2}gy_{max}$$

Now I must find y_{max}. When y_{max} is achieved, the velocity v' is zero. Then

$$(v')^2=v_0^2+2a(y_{max}-y_0)\ \Rightarrow\ 0=v_0^2-2gy_{max}\ \Rightarrow\ y_{max}=\frac{v_0^2}{2g}.$$

Inserting this into the equation above gives

$$v_0^2=v^2+\frac{1}{2}g\left(\frac{v_0^2}{2g}\right)\ \Rightarrow\ v_0^2=v^2+\frac{1}{4}v_0^2\ \Rightarrow\ \frac{3}{4}v_0^2=v^2\ \Rightarrow\ v_0^2=\frac{4}{3}v^2\ \Rightarrow\ v_0=\frac{2}{\sqrt{3}}v.$$

CALCULATE: $v_0=\frac{2}{\sqrt{3}}(25\text{ m/s})=28.87$ m/s

ROUND: The value for v limits the calculation to two significant figures. So $v_0=29$ m/s.

DOUBLE-CHECK: v_0 is greater than $v=25$ m/s, as it should be.

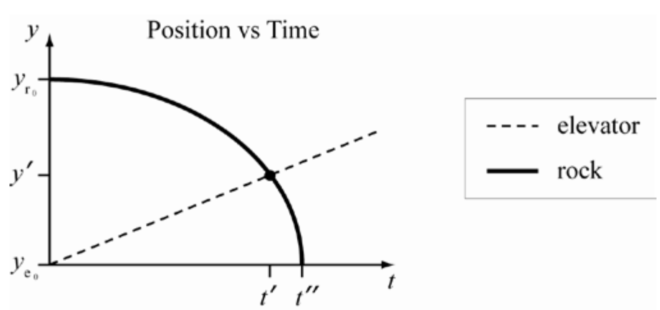
2.76. **THINK:** For the elevator, the velocity is $v_e=1.75$ m/s, the acceleration is $a_e=0$, and the initial height is $y_{e_0}=0$. For the rock, the initial velocity is $v_{r_0}=0$, the acceleration is $a_r=-g=-9.81$ m/s^2, and the initial height is $y_{r_0}=80.0$ m.

(a) I need to find the time it takes the rock to intercept the elevator, t'.

(b) I need to find the time it takes the rock to hit the ground at $y_r=0$, t''.

When the rock intercepts the elevator, both are at the same position y', and have taken the same time, t', to arrive there.

SKETCH:



RESEARCH: The elevator position is determined from $\Delta y_e=v_et$. For the rock, $\Delta y_r=v_{r_0}t+(1/2)a_rt^2$.

SIMPLIFY: For the elevator, $y_\text{e} - y_{\text{e}_0} = v_\text{e}t \Rightarrow y_\text{e} = v_\text{e}t$. For the rock, $y_\text{r} - y_{\text{r}_0} = v_{\text{r}_0}t + (1/2)a_\text{r}t^2 \Rightarrow$ $y_\text{r} - y_{\text{r}_0} = -(1/2)gt^2$.

(a) Both objects take the same time to intercept each other, and both have the same position at interception:

$$y_\text{e} = y_\text{r} \Rightarrow v_\text{e}t' = y_{\text{r}_0} - \frac{1}{2}g(t')^2 \Rightarrow \frac{1}{2}g(t')^2 + v_\text{e}t' - y_{\text{r}_0} = 0.$$

Solving for t'. in the quadratic equation gives:

$$t' = \frac{-v_\text{e} \pm \sqrt{v_\text{e}^2 - 4\left(\frac{1}{2}\text{g}\right)\left(-y_{\text{r}_0}\right)}}{2\left(\frac{1}{2}\text{g}\right)} = \frac{-v_\text{e} \pm \sqrt{v_\text{e}^2 + 2\text{g}y_{\text{r}_0}}}{\text{g}}.$$

(b) The total fall time, t'', for the rock is $\Delta y_\text{r} = y_r - y_{\text{r}_0} = v_{\text{r}_0}t'' + (1/2)a_\text{r}(t'')^2$. The final position is $y_r = 0$. With $v_{\text{r}_0} = 0$, $-y_{\text{r}_0} = -(1/2)\text{g}(t'')^2 \Rightarrow t'' = \sqrt{2y_{\text{r}_0}/\text{g}}$.

CALCULATE:

(a) $$t' = \frac{-(1.75 \text{ m/s}) \pm \sqrt{(1.75 \text{ m/s})^2 + 2(9.81 \text{ m/s}^2)(80.0 \text{ m})}}{9.81 \text{ m/s}^2} = 3.8641 \text{ s}, -4.2209 \text{ s}$$

(b) $$t'' = \sqrt{\frac{2(80.0 \text{ m})}{9.81 \text{ m/s}^2}} = 4.0386 \text{ s}$$

ROUND: The values given have three significant figures, so the final answers will also have three significant figures.

(a) Taking the positive solution for time, $t' = 3.86$ s.

(b) $t'' = 4.04$ s

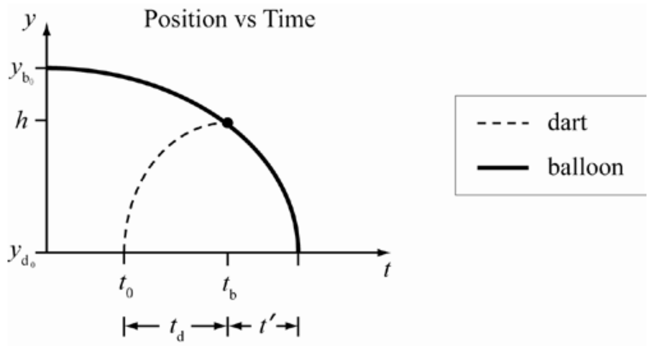
DOUBLE-CHECK: The total time to fall is greater than the intercept time, as it should be.

2.77. **THINK:** It is probably a good idea to read through the solution of the "Melon Drop" problem, Solved Problem 2.5 in the textbook before getting started with the present problem. The present problem has the additional complication that the water balloon gets dropped some time before the dart get fired, whereas in the "Melon Drop" problem both projectiles get launched simultaneously. For the first 2 seconds, only our water balloon is in free fall, and we can calculate its position y_{b_0} and velocity v_{b_0} at the end of this time interval.

a) After the initial two seconds the dart also gets launched, and then both objects (water balloon and dart) are in free-fall. Their initial distance is y_{b_0}, and their relative velocity is the difference between the initial velocity of the dart and v_{b_0}. The time until the two objects meet is then simply the ratio of the initial distance and the relative velocity.

b) For this part we simply calculate the time it takes for the balloon to free-fall the entire height h and subtract our answer form part a).

SKETCH:



RESEARCH:

(a) The position and velocity of the balloon after the time $t_0 = 2$ s are

$$y_{b_0} = h - \tfrac{1}{2}gt_0^{\ 2}$$
$$v_{b_0} = -gt_0$$

The time is takes then for the balloon and the dart to meet is the ratio of their initial distance to their initial relative velocity:

$$t_d = y_{b_0} / (v_{d_0} - v_{b_0})$$

Our answer for part a) is the sum of the time t_0, during which the balloon was in free-fall alone, and the time t_1, $t_b = t_d + t_0$.

(b) The total time it takes for the balloon to fall all the way to the ground is

$$t_{total} = \sqrt{2h/g}$$

We get our answer for part b) by subtracting the result of part a) from this total time:

$$t' = t_{total} - t_d$$

SIMPLIFY:

(a) If we insert the expressions for the initial distance and relative speed into $t_d = y_{b_0} / (v_{d_0} - v_{b_0})$, we find $t_d = y_{b_0} / (v_{d_0} - v_{b_0}) = (h - \tfrac{1}{2}gt_0^{\ 2}) / (v_{d_0} + gt_0)$. Adding t_0 then gives us our final answer:

$$t_b = t_0 + (h - \tfrac{1}{2}gt_0^{\ 2}) / (v_{d_0} + gt_0)$$

(b) For the time between the balloon being hit by the dart and the water reaching the ground we find by inserting $t_{total} = \sqrt{2h/g}$ into $t' = t_{total} - t_b$:

$$t' = \sqrt{2h/g} - t_b$$

CALCULATE:

(a) $t_b = 2.00 \text{ s} + \dfrac{80.0 \text{ m-}\tfrac{1}{2}\left(9.81 \text{ m/s}^2\right)\left(2.00 \text{ s}\right)^2}{20.0 \text{ m/s} + \left(9.81 \text{ m/s}^2\right)\left(2.00 \text{ s}\right)} = 3.524 \text{ s}$

(b) $t' = \sqrt{2\left(80.0 \text{ m}\right)/(9.81 \text{ m/s}^2)} - 3.524 \text{ s} = 0.515 \text{ s}.$

ROUND:

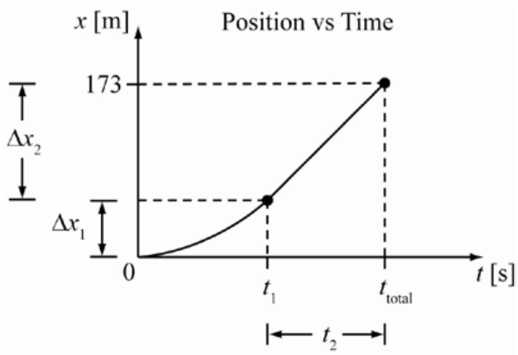
(a) $t_b = 3.52s$

(b) $t' = 0.515$ s

DOUBLE-CHECK: The solution we showed in this problem is basically the double-check step in Solved Problem 2.5. Conversely, we can use the solution method of Solved Problem 2.5 as a double-check for what we have done here. This is left as an exercise for the reader.

2.78. **THINK:** I know the runner's initial velocity, $v_0 = 0,$ her acceleration, $a = 1.23 \text{ m/s}^2,$ her final velocity, $v = 5.10 \text{ m/s},$ and the distance she traveled, $\Delta x = 173 \text{ m}.$ I want to know the total time t_{total}. Note that Δx is composed of a displacement Δx_1 which occurs while accelerating and a displacement Δx_2 which occurs at a constant velocity. That is, $\Delta x = \Delta x_1 + \Delta x_2$. Mass is irrelevant.

SKETCH:



RESEARCH: The total time is the sum of the times for each displacement. Let $t_{\text{total}} = t_1 + t_2$ with t_1 the time for displacement Δx_1 and t_2 the time for displacement Δx_2. For t_1, use $v = v_0 + at_1$. For Δx_1, use $\Delta x_1 = (1/2)(v + v_0)t_1$. For t_2, use $\Delta x_2 = vt_2$.

SIMPLIFY: Note that

$$\Delta x_1 = \frac{1}{2}(v + v_0)t_1 = \frac{1}{2}vt_1 = \frac{1}{2}v\frac{v}{a} = \frac{v^2}{2a}.$$

To compute the value of t_{total}, first simplify expressions for t_1 and t_2:

$$t_1 = \frac{v - v_0}{a} = \frac{v}{a} \text{ and } t_2 = \frac{\Delta x_2}{v} = \frac{\Delta x - \Delta x_1}{v} = \frac{\Delta x - \frac{v^2}{2a}}{v} = \frac{\Delta x}{v} - \frac{v}{2a}.$$

Using the last two equations t_{total} can be calculated as follows:

$$t_{\text{total}} = t_1 + t_2 = \frac{v}{a} + \frac{\Delta x}{v} - \frac{v}{2a} = \frac{v}{2a} + \frac{\Delta x}{v}.$$

CALCULATE: $t_{\text{total}} = \dfrac{5.10 \text{ m/s}}{2\left(1.23 \text{ m/s}^2\right)} + \dfrac{173 \text{ m}}{5.10 \text{ m/s}} = 35.995 \text{ s}$

ROUND: Each initial value has three significant figures, so $t_{\text{total}} = 36.0 \text{ s}$

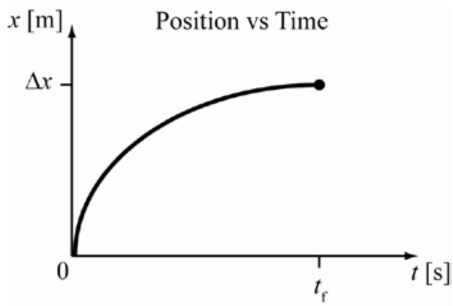
DOUBLE-CHECK: This is a reasonable amount of time required to run 173 m.

2.79. **THINK:** Let the moment the jet touches down correspond to the time $t = 0.$ The initial velocity is

$$v_0 = \left(\frac{142.4 \text{ mi}}{1 \text{ hr}}\right)\left(\frac{1 \text{ hr}}{3600 \text{ s}}\right)\left(\frac{1609.3 \text{ m}}{1 \text{ mi}}\right) = 63.66 \text{ m/s}.$$

The jet comes to rest in a time of $t_{\text{f}} = 12.4 \text{ s}.$ The jet comes to a complete stop, which makes the final velocity zero, so $v_{\text{f}} = 0.$ I want to compute the distance the jet travels after it touches down, $\Delta x.$

SKETCH:



RESEARCH: To determine the distance traveled, the following equation can be used: $\Delta x = \frac{1}{2}(v_0 + v)t$.

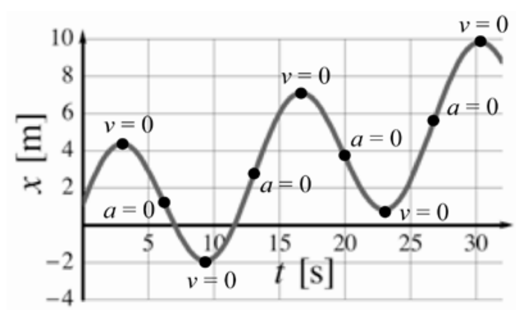
SIMPLIFY: With $v = 0$, the equation becomes $\Delta x = (v_0 t)/2$.

CALCULATE: $\Delta x = \frac{1}{2}(63.66 \text{ m/s})(12.4 \text{ s}) = 394.7 \text{ m}$.

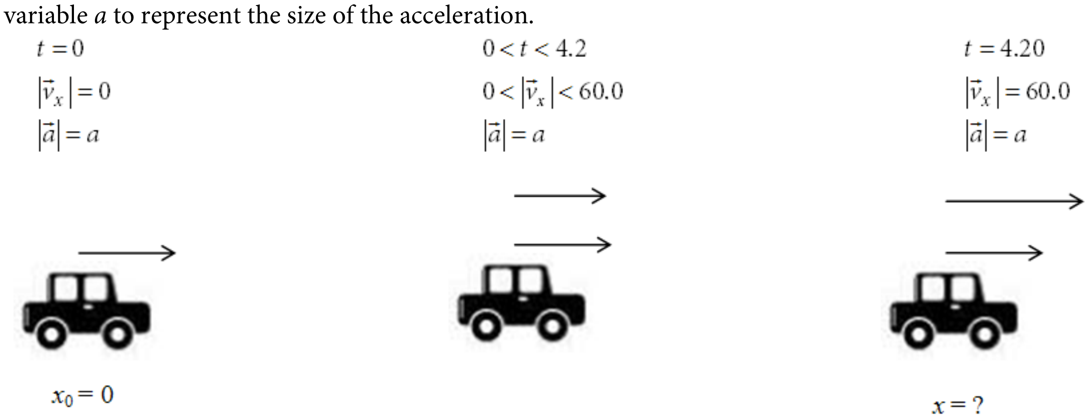
ROUND: Since t has three significant digits, the result should be rounded to $\Delta x = 395 \text{ m}$.

DOUBLE-CHECK: This is a reasonable distance to decelerate from 63.66 m/s in 12.4 s.

2.80. Velocity is the slope of the position versus time graph. Therefore, $v = 0$ at the local maxima and minima. Acceleration is the slope of the velocity versus time graph. On a position versus time graph, acceleration, a is zero at inflection points on the curve that are not maxima or minima, i.e. $a = 0$ as the slope of x vs. t approaches a constant value over some non-zero time interval, Δt :



2.81. **THINK:** The acceleration is the derivative of the velocity with respect to time, that is, the instantaneous change in velocity. Since the car is stopped and then accelerates to 60.0 miles per hour, we can infer that the acceleration is positive in the direction of the car's motion.
SKETCH: Sketch the motion of the car at 0 s and time 4.20 s. Since the acceleration is unknown, use the variable a to represent the size of the acceleration.



RESEARCH: Since this problem involves motion with constant acceleration, use equation (2.23). (a) The velocity in the positive x direction at time t is equal to the velocity in the positive x direction at time 0 plus the acceleration in the x direction multiplied by the time, $v_x = v_{x0} + a_x t$. (b) Having found the acceleration in the positive x direction, use the equation $x = x_0 + v_{x0} + \frac{1}{2}a_x t^2$ to find the position x at time t = 4.20 s. To make the calculations simple and straightforward, take the position of the car at time t = 0 s to be the zero of our coordinate system, so x_0 = 0 miles.

SIMPLIFY: (a) Use the velocity at times t = 0 and t = 4.2, solve the equation $v_x = v_{x0} + a_x t$ for a_x to get:

$$v_x = v_{x0} + a_x t$$
$$-v_{x0} + v_x = -v_{x0} + v_{x0} + a_x t = a_x t$$
$$\left(v_x - v_{x0}\right)/t = \left(a_x t\right)/t$$
$$\frac{v_x - v_{x0}}{t} = a_x$$

(b) Using the expression for a_x and algebra to find an expression for the total distance traveled:

$$\begin{aligned} x &= x_0 + v_{x0}t + \tfrac{1}{2}a_x t^2 \\ &= x_0 + v_{x0}t + \tfrac{1}{2}\left(\frac{v_x - v_{x0}}{t}\right)t^2 \\ &= x_0 + v_{x0}t + \tfrac{1}{2}\left(v_x - v_{x0}\right)t \end{aligned}$$

CALCULATE: (a) Since the car starts at rest, $v_{x0} = 0$ mph. Also, the velocity at time t = 4.20 s is $v_x = 60.0$ mph in the positive x direction. Using these values gives $a_x = \dfrac{v_x - v_{x0}}{t} = \dfrac{60.0 - 0}{4.2} = \dfrac{100}{7}\text{mph}\cdot\text{s}^{-1}$.

Since time t is given in miles per hour, it is necessary to convert this to miles per second to make the units consistent. Convert this to a more convenient set of units, such as mi·s^{-2} to make future calculations easier: $\dfrac{100}{7}\cdot\dfrac{\text{mi}}{\text{hour}\cdot\text{sec}}\cdot\dfrac{1\text{ hour}}{3600\text{ sec}} = \dfrac{1}{252}\cdot\dfrac{\text{mi}}{\text{sec}^2}$ (b) Plug in values to find x, the location of the car at time t = 4.20 s. This gives:

$$\begin{aligned} x &= x_0 + v_{x0}t + \tfrac{1}{2}a_x t^2 \\ &= 0 + 0 + \tfrac{1}{2}\tfrac{1}{252}\cdot(4.20)^2 \\ &= \frac{7}{200} = 0.035\text{ mi} \end{aligned}$$

ROUND: Since the measured values have 3 significant figures, the answers in both parts will have 3 significant figures. (a) For the acceleration, $\dfrac{1}{252}\dfrac{\text{mi}}{\text{sec}^2} \approx 0.00397\dfrac{\text{mi}}{\text{sec}^2}$ or $3.97\times10^{-3}\dfrac{\text{mi}}{\text{sec}^2}$. (b) Using scientific notation, $0.035 = 3.50\times10^{-2}$ mi. Note also that if we convert to SI units, we obtain (a) 6.39 m/s^2 for the acceleration and (b) 56.3 m for the distance.

DOUBLE-CHECK:

(a) Accelerating at a constant rate of $3.97\times10^{-3}\dfrac{\text{mi}}{\text{sec}^2}$ for 4.20 seconds from a standing start means that the car will be going $3.97\times10^{-3}\dfrac{\text{mi}}{\text{sec}^2}\cdot 4.2\text{ sec}\cdot\dfrac{3600\text{ sec}}{\text{hour}}$ or 60.0 mph after 4.20 seconds. This agrees with the question statement.

(b) Since the car is at position $x = \frac{1}{2}a_x t^2 = \frac{1}{504}t^2$ miles at time t seconds, the derivative with respect to time gives the velocity as a function of time: $\frac{dx}{dt} = \frac{d\left(\frac{1}{504}t^2\right)}{dt} = \frac{1}{504}\cdot 2t = \frac{1}{252}t$ miles per second, or $\frac{1}{252}t\frac{\text{mi}}{\text{sec}}\cdot\frac{3600\text{ sec}}{\text{hour}} = \frac{100}{7}t\frac{\text{mi}}{\text{hour}}$ at time t. At time t = 4.20 s, this gives a velocity of $\frac{100}{7}(4.2) = 60.0$ mph, which agrees with the setup for this problem.

2.82. **THINK:** It is known for a car that when the initial velocity is

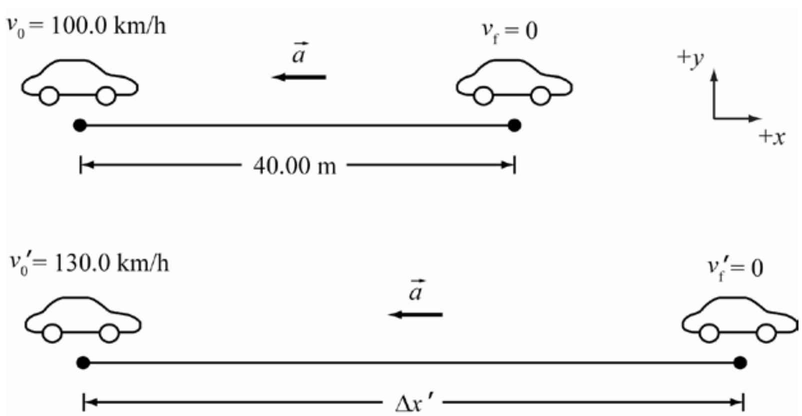
$$v_0 = 100.0\ \frac{\text{km}}{\text{h}}\cdot\frac{1\text{ h}}{3600\text{ s}}\cdot\frac{1000\text{ m}}{1\text{ km}} = 27.7778\text{ m/s},$$

and the final velocity is $v_f = 0$, the stopping distance is $\Delta x = 40$ m. Determine the stopping distance, $\Delta x'$ when the initial velocity is

$$v_0' = 130.0\ \frac{\text{km}}{\text{h}}\cdot\frac{1\text{ h}}{3600\text{ s}}\cdot\frac{1000\text{ m}}{1\text{ km}} = 36.1111\text{ m/s},$$

and the final velocity is $v_f' = 0$. The road conditions are the same in each case, so it can be assumed that the acceleration does not change.

SKETCH:



RESEARCH: The acceleration, a can be determined from the original conditions with $v^2 = v_0^2 + 2a\Delta x$. Substitute the value of the acceleration computed from the first set of conditions as the acceleration in the second conditions to determine $\Delta x'$.

SIMPLIFY: With $v = 0$, $0 = v_0^2 + 2a\Delta x \Rightarrow a = -v_0^2/(2\Delta x)$. Then, $v_f'^2 = v_0'^2 + 2a\Delta x'$. With $v_f' = 0$,

$$\Delta x' = -\frac{v_0'^2}{2a} = -\frac{v_0'^2}{2\left(-\frac{v_0^2}{2\Delta x}\right)} = \frac{v_0'^2}{v_0^2}\Delta x.$$

CALCULATE: $\Delta x' = \frac{(36.1111\text{ m/s})^2}{(27.7778\text{ m/s})^2}(40.00\text{ m}) = 67.5999\text{ m}$

Note that the unit conversion from km/h to m/s was not necessary as the units of velocity cancel each other in the ratio.

ROUND: $\Delta x' = 67.60$ m

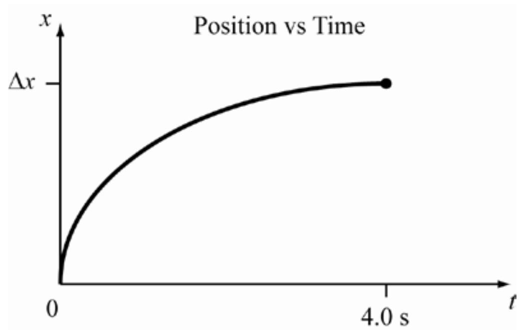
DOUBLE-CHECK: The stopping distance for the larger initial velocity is greater than the stopping distance for the small initial velocity, as it should be.

2.83. **THINK:** The initial velocity is

$$v_0 = 60.0\ \frac{\text{km}}{\text{h}} \cdot \frac{1\ \text{h}}{3600\ \text{s}} \cdot \frac{1000\ \text{m}}{1\ \text{km}} = 16.67\ \text{m/s}.$$

The final velocity is $v = 0$. The stop time is $t = 4.00$ s. The deceleration is uniform. Determine (a) the distance traveled while stopping, Δx and (b) the deceleration, a. I expect $a < 0$.

SKETCH:



RESEARCH:

(a) To determine the stopping distance, use $\Delta x = t\left(v_0 + v\right)/2$.

(b) To determine a, use $v = v_0 + at$.

SIMPLIFY:

(a) With $v = 0$, $\Delta x = v_0 t/2$.

(b) With $v = 0$, $0 = v_0 + at \Rightarrow a = -v_0/t$.

CALCULATE:

(a) $\Delta x = \dfrac{\left(16.67\ \text{m/s}\right)\left(4.00\ \text{s}\right)}{2} = 33.34\ \text{m}$

(b) $a = -\dfrac{16.67\ \text{m/s}}{4.00\ \text{s}} = -4.167\ \text{m/s}^2$

ROUND:

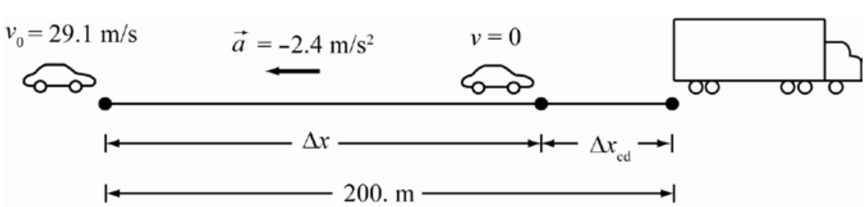
(a) $\Delta x = 33.3\ \text{m}$

(b) $a = -4.17\ \text{m/s}^2$

DOUBLE-CHECK: The distance traveled while stopping is of an appropriate order of magnitude. A car can reasonably stop from 60 km/h in a distance of about 30 m. The acceleration is negative, indicating that the car is slowing down from its initial velocity.

2.84. **THINK:** The car's initial velocity is $v_0 = 29.1\ \text{m/s}$. The deceleration is $a = -2.4\ \text{m/s}^2$. Assume that the final velocity is $v = 0$, that is the car does not hit the truck. The truck is a distance $d = 200.0$ m when the car begins to decelerate. Determine (a) the final distance between the car and the truck, Δx_{cd} and (b) the time it takes to stop, t.

SKETCH:



RESEARCH:

(a) The distance to the truck is the difference between the initial distance d and the stopping distance Δx: $\Delta x_{cd} = d - \Delta x$. Δx can be determined from $v^2 = v_0^2 + 2a\Delta x$.

(b) The stop time is determined from $v = v_0 + at$.

SIMPLIFY:

(a) With $v = 0$, $v^2 = v_0^2 + 2a\Delta x \Rightarrow 0 = v_0^2 + 2a\Delta x \Rightarrow \Delta x = -v_0^2 / 2a$. Then, $\Delta x_{cd} = d + \left(v_0^2 / 2a\right)$.

(b) With $v = 0$, $t = -v_0 / a$.

CALCULATE:

(a) $\Delta x_{cd} = 200.0 \text{ m} + \dfrac{(29.1 \text{ m/s})^2}{2\left(-2.4 \text{ m/s}^2\right)} = 200.0 \text{ m} - 176.4 \text{ m} = 23.6 \text{ m}$

(b) $t = -\dfrac{(29.1 \text{ m/s})}{\left(-2.4 \text{ m/s}^2\right)} = 12.13 \text{ s}$

ROUND:

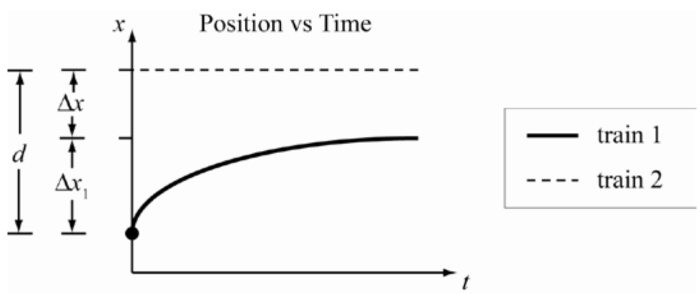
(a) Since the acceleration has two significant figures, $\Delta x_{cd} = 24 \text{ m}$

(b) Rounding to two significant figures, $t = 12$ s.

DOUBLE-CHECK: The stopping time and distance are realistic for a car decelerating from 29.1 m/s.

2.85. **THINK:** For train 1, it is known that $v_{1,0} = 40.0 \text{ m/s}$, $a_1 = -6.0 \text{ m/s}^2$ and $v_1 = 0$. For train 2, it is known that $v_2 = 0$ and $a_2 = 0$. The distance between the trains is d = 100.0 m. Determine the distance between the trains after train 1 stops, Δx.

SKETCH:



RESEARCH: The final distance between the trains, Δx is the difference between the initial distance, d and the stopping distance of train 1, Δx_1: $\Delta x = d - \Delta x_1$.

SIMPLIFY: With $v_1 = 0$, $v_1^2 = v_{1,0}^2 + 2a_1\Delta x_1 \Rightarrow 0 = v_{1,0}^2 + 2a_1\Delta x_1 \Rightarrow \Delta x_1 = -\dfrac{v_{1,0}^2}{2a_1}$. Then, $\Delta x = d + \dfrac{v_{1,0}^2}{2a_1}$.

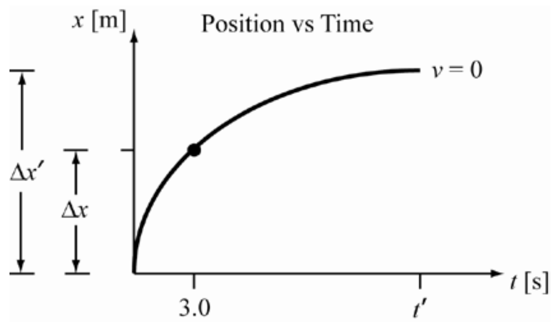
CALCULATE: $\Delta x = 100.0 \text{ m} + \dfrac{(40.0 \text{ m/s})^2}{2\left(-6.0 \text{ m/s}^2\right)} = 100.0 \text{ m} - 133.3 \text{ m} = -33.3 \text{ m}$

ROUND: Note that Δx is determined to be a negative value. This is due to the stopping distance being greater than the initial distance between the trains. This implies that train 1 actually collides with train 2. Then the distance between the two trains is zero.

DOUBLE-CHECK: It is reasonable for train 1 to collide with train 2 given the initial separation of only 100.0 m and an initial velocity for train 1 of 40.0 m/s.

2.86. **THINK:** The initial velocity is $v_0 = 25.0$ m/s. The acceleration is $a = -1.2 \text{ m/s}^2$. Determine (a) the distance Δx traveled in t = 3.0 s, (b) the velocity, v after traveling this distance, (c) the stopping time, t' and (d) the stopping distance, $\Delta x'$. Note when the car is stopped, $v' = 0$.

SKETCH:



RESEARCH:

(a) To determine Δx, use $\Delta x = v_0 t + \left(at^2\right)/2.$

(b) To determine *v*, use $v = v_0 + at.$

(c) To determine t', use $v = v_0 + at.$

(c) To determine $\Delta x'$, use $v^2 = v_0^{\ 2} + 2a\Delta x.$

SIMPLIFY:

(a) It is not necessary to simplify.
(b) It is not necessary to simplify.
(c) With $v' = 0$, $v' = v_0 + at' \Rightarrow t' = -v_0 / a.$
(d) With $v' = 0$, $v'^2 = v_0^{\ 2} + 2a\Delta x' \Rightarrow \Delta x' = -v_0^{\ 2} / 2a.$

CALCULATE:

(a) $\Delta x = (25.0 \text{ m/s})(3.0 \text{ s}) + \frac{1}{2}\left(-1.2 \text{ m/s}^2\right)(3.0 \text{ s})^2 = 69.6 \text{ m}$

(b) $v = 25.0 \text{ m/s} + \left(-1.2 \text{ m/s}^2\right)(3.0 \text{ s}) = 21.4 \text{ m/s}$

(c) $t' = -\frac{(25.0 \text{ m/s})}{\left(-1.2 \text{ m/s}^2\right)} = 20.83 \text{ s}$

(d) $\Delta x' = -\frac{(25.0 \text{ m/s})^2}{2\left(-1.2 \text{ m/s}^2\right)} = 260.4 \text{ m}$

ROUND: Both the acceleration and the time have two significant figures, so the results should be rounded to $\Delta x = 70.$ m , *v* = 21 m/s, $t' = 21$ s and $\Delta x' = 260$ m.

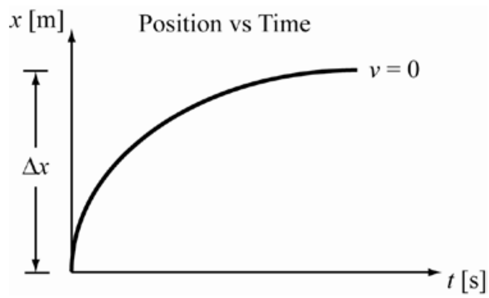
DOUBLE-CHECK: The car travels 70 m while decelerating, which is less than the 75 m it would have traveled in the same time if it had not been decelerating. The velocity after decelerating is less than the initial velocity. The stopping distance is greater than the distance traveled in 3.0 s, and the stopping time is greater than the intermediate time of 3.0 s. All of these facts support the calculated values.

2.87. **THINK:** The initial velocity is

$$v_0 = 212.809 \text{ mph}\left(\frac{1 \text{ h}}{3600 \text{ s}}\right)\left(\frac{1609.3 \text{ m}}{\text{mile}}\right) = 95.1315 \text{ m/s}.$$

The acceleration is $a = -8.0 \text{ m/s}^2$. The final speed is *v* = 0. Determine the stopping distance, Δx.

SKETCH:



RESEARCH: Use $v^2 = v_0^2 + 2a\Delta x.$

SIMPLIFY: With $v = 0$, $0 = v_0^2 + 2a\Delta x \Rightarrow \Delta x = -v_0^2 / 2a.$

CALCULATE: $\Delta x = -\dfrac{(95.1315 \text{ m/s})^2}{2(-8.0 \text{ m/s}^2)} = 565.6 \text{ m}$

ROUND: The acceleration has two significant figures, so the result should be rounded to $\Delta x = 570$ m.

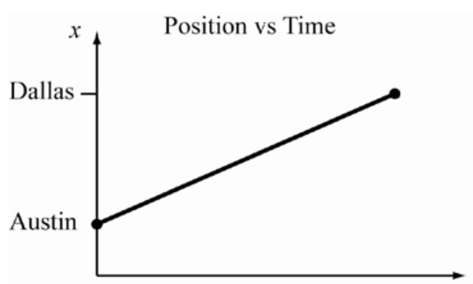
DOUBLE-CHECK: The initial velocity is large and the deceleration has a magnitude close to that of gravity. A stopping distance greater than half of a kilometer is reasonable.

2.88. **THINK:** The velocity can be converted to SI units as follows:

$$v_0 = 245 \text{ mph}\left(\frac{1 \text{ h}}{3600 \text{ s}}\right)\left(\frac{1609.3 \text{ m}}{\text{mile}}\right) = 109.5 \text{ m/s}.$$

The distance is $\Delta x = 362 \text{ km} = 3.62 \cdot 10^5$ m. Determine the time, t to travel the distance, Δx. Note the acceleration is $a = 0$.

SKETCH:



RESEARCH: For $a = 0$, use $\Delta x = vt$.

SIMPLIFY: $t = \dfrac{\Delta x}{v}$

CALCULATE: $t = \dfrac{3.62 \cdot 10^5 \text{ m}}{109.5 \text{ m/s}} = 3306 \text{ s}$

ROUND: The distance Δx has three significant figures, so the result should be rounded to $t = 3310$ s.

DOUBLE-CHECK: The time in hours is

$$3310 \text{ s}\left(\frac{1 \text{ h}}{3600 \text{ s}}\right) = 0.919 \text{ h}.$$

An hour is a reasonable amount of time to fly a distance of 362 km.

2.89. The position is given by $x = at^3 + bt^2 + c$, where $a = 2.0 \text{ m/s}^3$, $b = 2.0 \text{ m/s}^2$ and $c = 3.0$ m.

(a) Determine the sled's position between $t_1 = 4.0$ s and $t_2 = 9.0$ s.

$x(4.0 \text{ s}) = (2.0 \text{ m/s}^3)(4.0 \text{ s})^3 + (2.0 \text{ m/s}^2)(4.0 \text{ s})^2 + 3.0 \text{ m} = 163 \text{ m} \approx 160 \text{ m}$

$x(9.0\text{ s}) = (2.0\text{ m/s}^3)(9.0\text{ s})^3 + (2.0\text{ m/s}^2)(9.0\text{ s})^2 + 3.0\text{ m} = 1623\text{ m} \approx 1600\text{ m}$

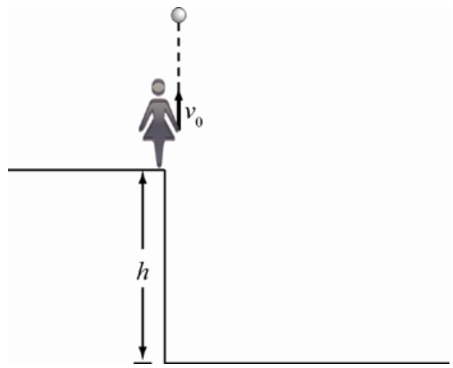
The sled is between x = 160 m and x = 1600 m.

(b) Determine the sled's average speed over this interval.

$$V_{\text{avg}} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{1623\text{ m} - 163\text{ m}}{9.0\text{ s} - 4.0\text{ s}} = \frac{1460\text{ m}}{5.0\text{ s}} \approx 292\text{ m/s} = 290\text{ m/s}$$

2.90. **THINK:** The cliff has a height of h = 100. m above the ground. The girl throws a rock straight up with a speed of $v_0 = 8.00$ m/s. Determine how long it takes for the rock to hit the ground and find the speed, v of the rock just before it hits the ground. The acceleration due to gravity is $a = -g = -9.81\text{ m/s}^2$.

SKETCH:



RESEARCH: The total displacement in the vertical direction is given by $\Delta y = y_f - y_i$. If the top of the cliff is taken to be the origin of the system, then $y_i = 0$ and $y_f = -h = -100.$ m. Therefore, $\Delta y = -h$.

(a) $\Delta y = v_0 t + \frac{1}{2}at^2$

(b) $v^2 = v_0^2 + 2a\Delta y$

SIMPLIFY:

(a) The quadratic equation can be used to solve for t from the equation $gt^2/2 - v_0 t + \Delta y = 0$:

$$t = \frac{v_0 \pm \sqrt{v_0^2 - 4\left(\frac{g}{2}\right)(-h)}}{2\left(\frac{g}{2}\right)} = \frac{v_0 \pm \sqrt{v_0^2 + 2gh}}{g}.$$

(b) $v = \sqrt{v_0^2 + 2gh}$

CALCULATE:

(a) $t = \dfrac{8.00\text{ m/s} \pm \sqrt{(8.00\text{ m/s})^2 + 2(9.81\text{ m/s}^2)(100.\text{ m})}}{(9.81\text{ m/s}^2)}$

$= 5.40378\text{ s or } -3.77\text{ s}$

The negative time is impossible.

(b) $v = \sqrt{(8.00\text{ m/s})^2 + 2(9.81\text{ m/s}^2)(100.\text{ m})} = 45.011\text{ m/s}$

ROUND:

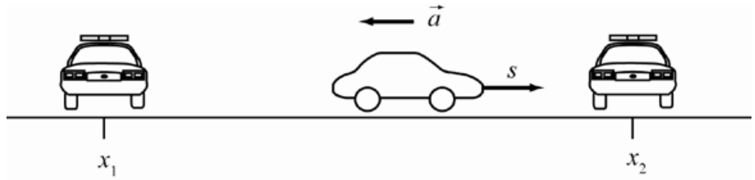
(a) t = 5.40 s

(b) v = 45.0 m/s

DOUBLE-CHECK: The calculated time and speed for the rock are reasonable considering the height of the cliff. Also, the units are correct units for time and speed.

2.91. **THINK:** The police have a double speed trap set up. A sedan passes the first speed trap at a speed of $s_1 = 105.9$ mph. The sedan decelerates and after a time, $t = 7.05$ s it passes the second speed trap at a speed of $s_2 = 67.1$ mph. Determine the sedan's deceleration and the distance between the police cruisers.

SKETCH:



RESEARCH:

(a) Convert the speeds to SI units as follows:

$$s_1 = 105.9 \text{ mph}\left(\frac{1 \text{ h}}{3600 \text{ s}}\right)\left(\frac{1609.3 \text{ m}}{\text{mile}}\right) = 47.34 \text{ m/s}$$

$$s_2 = 67.1 \text{ mph}\left(\frac{1 \text{ h}}{3600 \text{ s}}\right)\left(\frac{1609.3 \text{ m}}{\text{mile}}\right) = 29.996 \text{ m/s}.$$

The sedan's velocity, v can be written in terms of its initial velocity, v_0 the time t, and its acceleration a: $v = v_0 + at$. Substitute s_1 for v_0 and s_2 for v.

(b) The distance between the cruisers is given by: $\Delta x = x_2 - x_1 = v_0 t + (1/2)at^2$.

SIMPLIFY:

(a) $a = \dfrac{v - v_0}{t} = \dfrac{s_2 - s_1}{t}$

(b) Substitute s_1 for v_0 and the expression from part (a) for a: $\Delta x = s_1 t + (1/2)at^2$

CALCULATE:

(a) $a = \dfrac{29.996 \text{ m/s} - 47.34 \text{ m/s}}{7.05 \text{ s}} = -2.4602 \text{ m/s}^2$

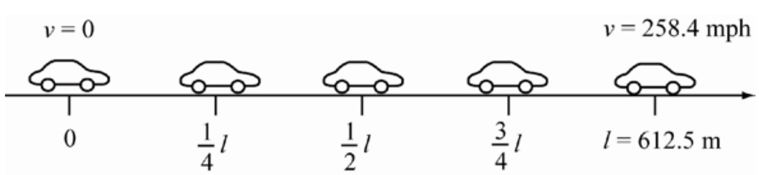
(b) $\Delta x = (47.34 \text{ m/s})(7.05 \text{ s}) + \dfrac{1}{2}(-2.4602 \text{ m/s}^2)(7.05 \text{ s})^2 = 272.6079 \text{ m}$

ROUND: The least number of significant figures provided in the problem are three, so the results should be rounded to $a = -2.46 \text{ m/s}^2$ and $\Delta x = 273$ m.

DOUBLE-CHECK: The sedan did not have its brakes applied, so the values calculated are reasonable for the situation. The acceleration would have been larger, and the distance would have been much smaller, if the brakes had been used. The results also have the proper units.

2.92. **THINK:** The initial speed of a new racecar is $v_0 = 0$ (standing start). The car accelerates with a constant acceleration and reaches a speed of $v = 258.4$ mph at a distance of $l = 612.5$ m. Determine a relationship between the speed and distance.

SKETCH:



RESEARCH: The acceleration is constant, so there are two expressions for velocity and distance: $v = v_0 + at$, $x = x_0 + v_0 t + (1/2)at^2$.

SIMPLIFY: It is given that $v_0 = 0$ and $x_0 = 0$, so the above expressions simplify to $v = at$, $x = \dfrac{1}{2}at^2$. Thus, $t = \sqrt{2x/a}$. Substituting this expression into $v = at$,

$$v = a\sqrt{\frac{2x}{a}} = \sqrt{2ax}.$$

CALCULATE: (1) The speed at a distance of $x = l/4$ is given by:

$$v_{l/4} = \sqrt{2a\frac{l}{4}} = \sqrt{\frac{1}{4}}\left(\sqrt{2al}\right) = \frac{1}{2}\sqrt{2al}.$$

Note that $v = \sqrt{2al}$, therefore, $v_{l/4} = v/2.$

$$v_{l/4} = \frac{1}{2}\left(258.4 \text{ mph}\right) = 129.2 \text{ mph}$$

(2) Similarly, substituting $x = l/2$ into $v = \sqrt{2ax}$,

$$v_{l/2} = \sqrt{\frac{1}{2}}v = \sqrt{\frac{1}{2}}\left(258.4 \text{ mph}\right) = 182.716 \text{ mph}.$$

(3) Substituting $x = 3l/4$ into $v = \sqrt{2ax}$,

$$v_{3l/4} = \sqrt{\frac{3}{4}}v = \sqrt{\frac{3}{4}}\left(258.4 \text{ mph}\right) = 223.781 \text{ mph}.$$

ROUND: Initially there are four significant figures, so the results should be rounded to $v_{l/4} = 129.2$ mph , $v_{l/2} = 182.7$ mph and $v_{3l/4} = 223.8$ mph.

DOUBLE-CHECK: Note that $v_{l/4} < v_{l/2} < v_{3l/4} < v$ as expected.

2.93. **THINK:** An expression of y as a function of t is given. Determine the speed and acceleration from this function, $y(t)$. The first derivative of $y(t)$ yields speed as a function of time, $v = dy/dt$, and the second derivative yields acceleration as a function of time, $a = dv/dt$.

SKETCH: A sketch is not needed to solve the problem.

RESEARCH: From a table of common derivatives: $\dfrac{d\sin\left(\alpha t + \beta\right)}{dt} = \alpha\cos\left(\alpha t + \beta\right)$, and $\dfrac{d\cos\left(\alpha t + \beta\right)}{dt} = -\alpha\sin\left(\alpha t + \beta\right).$

SIMPLIFY: It is not necessary to simplify.

CALCULATE:

(a)
$$\begin{aligned} v &= \frac{d}{dt}\left(3.8\sin\left(0.46t/\text{s} - 0.31\right) \text{ m} - 0.2t \text{ m/s} + 5.0 \text{ m}\right) \\ &= 3.8\left(0.46\right)\cos\left(0.46t/\text{s} - 0.31\right) \text{ m/s} - 0.2 \text{ m/s} \\ &= 1.748\cos\left(0.46t/\text{s} - 0.31\right) \text{ m/s} - 0.2 \text{ m/s} \end{aligned}$$

$$\begin{aligned} a &= \frac{dv}{dt} = \frac{d}{dt}\left(1.748\cos\left(0.46t/\text{s} - 0.31\right) \text{ m/s} - 0.2 \text{ m/s}\right) \\ &= -1.748\left(0.46\right)\sin\left(0.46t/\text{s} - 0.31\right) \text{ m/s}^2 \\ &= -0.80408\sin\left(0.46t/\text{s} - 0.31\right) \text{ m/s}^2 \end{aligned}$$

(b) Set $a = 0$: $0 = -0.80408\sin\left(0.46t/\text{s} - 0.31\right) \text{ m/s}^2 \Rightarrow \sin\left(0.46t/\text{s} - 0.31\right) = 0.$ It is known that $\sin\alpha = 0$ when $\alpha = n\pi$ and n is an integer. Therefore,

$$0.46t/\text{s} - 0.31 = n\pi \Rightarrow t = \frac{n\pi + 0.31}{0.46} \text{ s}.$$

The times between 0 and 30 s that satisfy a = 0 are:

$$\begin{aligned} t &= 6.8295n + 0.6739 \text{ s} \\ &= 0.6739 \text{ s for } n = 0 \\ &= 7.5034 \text{ s for } n = 1 \\ &= 14.3329 \text{ s for } n = 2 \\ &= 21.1624 \text{ s for } n = 3 \\ &= 27.9919 \text{ s for } n = 4. \end{aligned}$$

ROUND: Rounding to two significant figures,

(a) $v = 1.7\cos\left(0.46t / \text{s} - 0.31\right) \text{ m/s} - 0.2 \text{ m/s}, \quad a = -0.80\sin\left(0.46t / \text{s} - 0.31\right) \text{ m/s}^2$

(b) t = 0.67 s, 7.5 s, 14 s, 21 s and 28 s.

DOUBLE-CHECK: For oscillatory motion, where the position is expressed in terms of a sinuous function, the velocity is always out of phase with respect to the position. Out of phase means if $x = \sin t$, then $v = \cos t = \sin\left(t + \pi / 2\right)$. The acceleration is proportional to the position function. For example, if $x = A\sin t$, $a = -A\sin t$.

2.94. **THINK:** An expression for position as a function of time is given as $x\left(t\right) = 4t^2$.

SKETCH: A sketch is not needed to solve the problem.

RESEARCH: $\left(a+b\right)^2 = a^2 + 2ab + b^2$

SIMPLIFY: Simplification is not necessary.

CALCULATE:

(a) $x\left(2.00\right) = 4\left(2.00\right)^2 \text{ m} = 16.00 \text{ m}$

(b) $x\left(2.00 + \Delta t\right) = 4\left(2.00 + \Delta t\right)^2 \text{ m} = 4\left(4.00 + 4.00\Delta t + \Delta t^2\right) \text{ m}$

$$= \left(16.00 + 16.00\Delta t + 4\Delta t^2\right) \text{ m}$$

(c) $$\begin{aligned} \frac{\Delta x}{\Delta t} &= \frac{x\left(2.00 + \Delta t\right) - x\left(2.00\right)}{\Delta t} \\ &= \frac{16.00 + 16.00\Delta t + 4\Delta t^2 - 16.00}{\Delta t} \frac{\text{m}}{\text{s}} \\ &= (16.00 + 4\Delta t) \text{ m/s} \end{aligned}$$

Taking the limit as $\Delta t \to 0$: $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} 16.00 + \lim_{\Delta t \to 0} 4\Delta t = 16.00 \text{ m/s}.$

ROUND: Rounding to three significant figures,

(a) $x\left(2.00\right) = 16.0 \text{ m}$

(b) $x\left(2.00 + \Delta t\right) = \left(16.0 + 16.0\Delta t + 4\Delta t^2\right) \text{ m}$

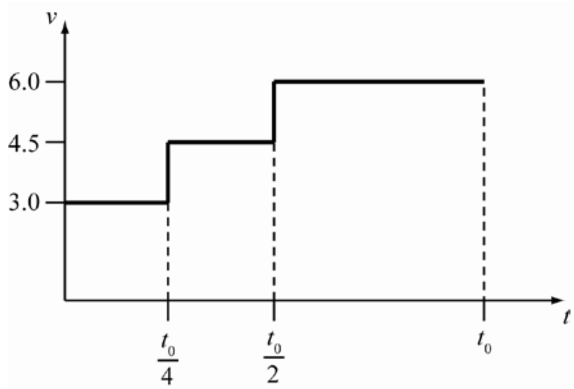
(c) $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = 16.0 \text{ m/s}$

DOUBLE-CHECK: The value of the position function near t = 2.00 s coincides with its value at t = 2.00 s. This should be the case, since the position function is continuous. The value of the velocity can also be found from the derivative: $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \frac{d}{dt}\left(4t^2\right) = 8t$. Substitute t = 2.00 s,

$\left.\frac{dx}{dt}\right|_{t=2.00} = 8\left(2.00\right) = 16.00 \text{ m/s}$. This value agrees with what was calculated in part (c).

2.95. **THINK:** The distance to the destination is 199 miles or 320 km. To solve the problem it is easiest to draw a velocity versus time graph. The distance is then given by the area under the curve.

SKETCH:



RESEARCH: For a constant speed, the distance is given by $x = vt$.

SIMPLIFY: To simplify, divide the distance into three parts.

Part 1: from $t = 0$ to $t = t_0 / 4$.

Part 2: from $t = t_0 / 4$ to $t = t_0 / 2$.

Part 3: from $t = t_0 / 2$ to $t = t_0$.

CALCULATE:

(a) The distances are $x_1 = 3.0t_0 / 4$, $x_2 = 4.5t_0 / 4$ and $x_3 = 6.0t_0 / 2$. The total distance is given by

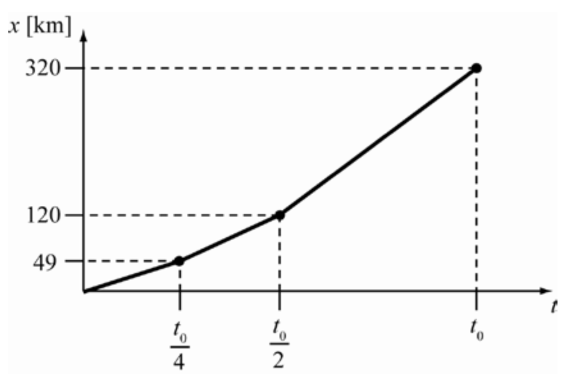
$$x = x_1 + x_2 + x_3 = \frac{(3.0 + 4.5 + 12)t_0}{4} \text{ m} = \frac{19.5t_0}{4} \text{ m} \Rightarrow t_0 = \frac{4x}{19.5} \text{ s.}$$

$$t_0 = \frac{4(320 \cdot 10^3)}{19.5} \text{ s} = 65.6410 \cdot 10^3 \text{ s} = 65641 \text{ s} \Rightarrow t_0 = 18.2336 \text{ h}$$

(b) The distances are:

$$x_1 = 3.0\left(\frac{65641}{4}\right) \text{ m} = 49.23 \text{ km}, \quad x_2 = 4.5\left(\frac{65641}{4}\right) \text{ m} = 73.85 \text{ km}, \quad x_3 = 6.0\left(\frac{65641}{2}\right) \text{ m} = 196.92 \text{ km}.$$

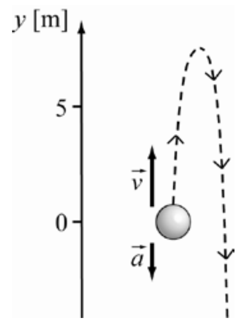
ROUND: Since the speeds are given to two significant figures, the results should be rounded to $x_1 = 49$ km, $x_2 = 74$ km and $x_3 = 2.0 \cdot 10^2$ km. $x_1 + x_2 = 123 \text{ km} \approx 120 \text{ km}$, and then $x = x_1 + x_2 + x_3 = 323 \text{ km} \approx 320 \text{ km}$.



DOUBLE-CHECK: The sum of the distances x_1, x_2 and x_3 must be equal to the total distance of 320 km: $x_1 + x_2 + x_3 = 49.23 + 73.85 + 196.92 = 320$ km as expected. Also, note that $x_1 < x_2 < x_3$ since $v_1 < v_2 < v_3$.

2.96. **THINK:** The initial speed is $v_0 = 15.0$ m/s. Assume there is no air resistance. The acceleration due to gravity is given by $g = 9.81\text{ m/s}^2$. t_1 is the time taken from the original position to the 5.00 m position on the way up. The time it takes from the initial position to 5.00 m on its way down is t_2.

SKETCH:



RESEARCH: For motion with a constant acceleration, the expressions for speed and distances are $v = v_0 + at$, $y = y_0 + v_0 t + \frac{1}{2}at^2$. The acceleration due to gravity is $a = -g$.

SIMPLIFY:

(a) At the maximum height, the velocity is $v = 0$. Using $y_0 = 0$:

$$0 = v_0 - gt \;\Rightarrow\; t = \frac{v_0}{g},$$

$$y = y_{max} = v_0 t - \frac{1}{2}gt^2.$$

Substituting $t = v_0 / g$,

$$y_{max} = v_0\left(\frac{v_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{2g}.$$

(b) If the motion of the ball starts from the maximum height, there is free fall motion with $v_0 = 0$.

$$v = -gt \;\Rightarrow\; t = -\frac{v}{g}$$

$$y = y_{max} + v_0 t - \frac{1}{2}gt^2 = y_{max} - \frac{1}{2}gt^2$$

Substituting $t = v/g$:

$$y = y_{max} - \frac{v^2}{2g} \;\Rightarrow\; v = \sqrt{(y_{max} - y)2g}.$$

CALCULATE:

(a) $y_{max} = \dfrac{(15.0\text{ m/s})^2}{2(9.81\text{ m/s}^2)} = 11.468$ m

(b) $v = \sqrt{(11.468 - 5.00)(2)(9.81\text{ m/s}^2)} = 11.265$ m/s. Thus the speed at this point is 11.265 m/s.

(c,d) Using $y = y_0 + v_0 t + (at^2)/2$, $y = v_0 t - (1/2)gt^2$. Using $v_0 = 15.0$ m/s, $g = 9.81\text{ m/s}^2$ and $y = 5.00$ m, the quadratic equation is $(1/2)(9.81\text{ m/s}^2)t^2 - 15.0t + 5.00\text{ m} = 0$. Solving the quadratic equation:

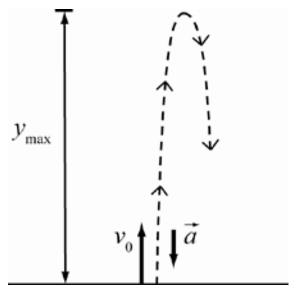
$$t = \frac{15.0 \pm \sqrt{(15.0)^2 - 2(9.81)5}}{9.81}\text{ s} = \frac{15.0 \pm 11.265}{9.81}\text{ s} = 1.529 \pm 1.1483 = 2.6773\text{ s and }0.3807\text{ s}$$

ROUND:

(a) Rounding to three significant figures, $y_{max} = 11.5$ m.

(b) All the numerical values have three significant figures, so the result is rounded to $v = 11.3$ m/s. Note the speed on the way up is the same as the speed on the way down.
(c) Rounding the values to three significant figures, $t_1 = 0.381$ s.
(d) $t_2 = 2.68$ s.
DOUBLE-CHECK: The speed at $t_1 = 0.381$ s and $t_2 = 2.68$ s must be the same and it is equal to the speed determined in part (b).
$v = v_0 - gt$
$v_1 = 15.0 - (9.81)0.381 = 11.2624 \text{ m/s} \approx 11.3 \text{ m/s}$
$v_2 = 15.0 - (9.81)2.68 = -11.2908 \text{ m/s} \approx -11.3 \text{ m/s}$
As can be seen, $|v_1| = |v_2|$ is equal to the result in part (b).

2.97. **THINK:** The maximum height is $y_{\text{max}} = 240 \text{ ft} = 73.152 \text{ m}$. The acceleration due to gravity is given by $g = 9.81 \text{ m/s}^2$.
SKETCH:



RESEARCH: To solve this constant acceleration problem, use $v = v_0 - gt$ and $y = y_0 + v_0 t - (gt^2/2)$. $y_0 = 0$.
SIMPLIFY:
(a) At a maximum height, the velocity v is zero.

$$v_0 - gt = 0 \Rightarrow t = \frac{v_0}{g}$$

$$y_{\text{max}} = v_0\left(\frac{v_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{2g} \Rightarrow v_0 = \sqrt{2gy_{\text{max}}}$$

(b) If the motion is considering as starting from the maximum height y_{max}, there is free fall motion with $v_0 = 0$.

$$v = -gt \Rightarrow t = \frac{v}{g}$$

$$y = y_{\text{max}} - \frac{1}{2}gt^2 = y_{\text{max}} - \frac{1}{2}g\left(\frac{v}{g}\right)^2 = y_{\text{max}} - \frac{v^2}{2g} \Rightarrow v = \sqrt{(y_{\text{max}} - y)2g}$$

(c) Note that v_0 is equal to the speed in part (b), $v_0 = -26.788$ m/s and v is equal to the original speed but in the opposite direction, $v = -37.884$ m/s.

$$t = \frac{v_0 - v}{g}$$

CALCULATE:
(a) $v_0 = \sqrt{2(9.81)73.152} = 37.885$ m/s

(b) $y=\frac{y_{\text{max}}}{2}$, so $v=\sqrt{\left(y_{\text{max}}-\frac{y_{\text{max}}}{2}\right)2g}=\sqrt{gy_{\text{max}}}=\sqrt{(9.81)73.152}=26.788$ m/s. Choose the positive root because the problem asks for the speed, which is never negative.

(c) $t=\frac{37.884\text{ m/s}-26.788\text{ m/s}}{\left(9.81\text{ m/s}^2\right)}=1.131\text{ s}$

ROUND:
(a) Rounding to three significant figures, $v_0=37.9$ m/s.
(b) Rounding to three significant figures, $v=26.8$ m/s.
(c) Rounding to three significant figures, t = 1.13 s.

DOUBLE-CHECK: It is known that $v=\sqrt{2gy}$. This means that the ratio of two speeds is:

$$\frac{v_1}{v_2}=\frac{\sqrt{2gy_1}}{\sqrt{2gy_2}}=\sqrt{\frac{y_1}{y_2}}.$$

The result in part (b) is for $y=y_{\text{max}}/2$, so the ratio is

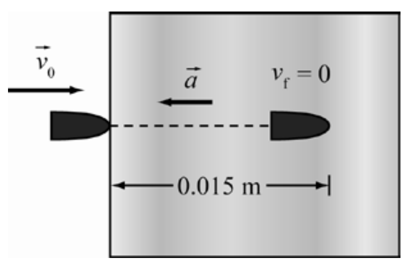
$$\frac{v_{1/2}}{v_0}=\sqrt{\frac{\frac{1}{2}y_{\text{max}}}{y_{\text{max}}}}=\sqrt{\frac{1}{2}}=0.7071.$$

Using the results in parts (a) and (b):

$$\frac{v_{1/2}}{v_0}=\frac{26.8\text{ m/s}}{37.9\text{ m/s}}=0.7071\text{ as expected.}$$

2.98. **THINK:** The initial velocity is $v_0=200.$ m/s. There is constant acceleration and the maximum distance is $x_{\text{max}}=1.5\text{ cm}=0.015\text{ m}$.

SKETCH:



RESEARCH: To solve a constant acceleration motion, use $v=v_0+at$. There is a deceleration of *a*. $x=v_0t+\frac{1}{2}at^2$

SIMPLIFY: At the final position, $v = 0$.

$$v_0-at=0\ \Rightarrow\ a=\frac{v_0}{t}$$

Substituting $a=v_0/t$ into $x_{\text{max}}=v_0t-(1/2)at^2$ gives:

$$x_{\text{max}}=v_0t-\frac{1}{2}\frac{v_0}{t}t^2=\frac{1}{2}v_0t\ \Rightarrow\ t=\frac{2x_{\text{max}}}{v_0}$$

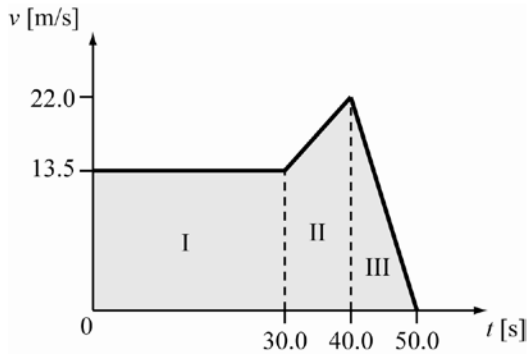
CALCULATE: $t=\frac{2(0.015\text{ m})}{200.\text{ m/s}}=1.5\cdot 10^{-4}\text{ s}$

ROUND: Rounding to two significant figures yields the same result, $t=1.5\cdot 10^{-4}$ s

DOUBLE-CHECK: It is expected the resulting time is small for the bullet to stop at a short distance.

2.99. **THINK:** $v_1 = 13.5$ m/s for $\Delta t = 30.0$ s. $v_2 = 22.0$ m/s after $\Delta t = 10.0$ s (at t = 40.0 s). $v_3 = 0$ after $\Delta t = 10.0$ s (at t = 50.0 s). It will be easier to determine the distance from the area under the curve of the velocity versus time graph.

SKETCH:



RESEARCH: Divide and label the graph into three parts as shown above.

SIMPLIFY: The total distance, d is the sum of the areas under the graph, $d = A_1 + A_2 + A_3$.

CALCULATE: $d = (13.5 \text{ m/s})(30.0 \text{ s}) + \frac{1}{2}(13.5 \text{ m/s} + 22.0 \text{ m/s})(10.0 \text{ s}) + \frac{1}{2}(22.0 \text{ m/s})(10.0 \text{ s})$

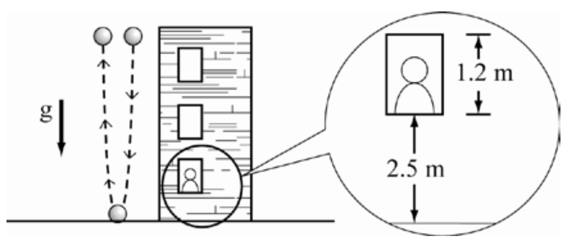
$= 405 \text{ m} + 177.5 \text{ m} + 110 \text{ m} = 692.5 \text{ m}$

ROUND: The speeds are given in three significant figures, so the result should be rounded to $d = 693$ m.

DOUBLE-CHECK: From the velocity versus time plot, the distance can be estimated by assuming the speed is constant for all time, t: $d = (13.5 \text{ m/s})(50.0 \text{ s}) = 675$ m. This estimate is in agreement with the previous result.

2.100. **THINK:** It is given that the initial velocity is $v_0 = 0$. The time for the round trip is t = 5.0 s.

SKETCH:



RESEARCH: $a = -g$. Using two expressions for velocity and distance:

(a) $v = v_0 + at$

(b) $y = y_0 + v_0 t + \frac{1}{2}at^2$

SIMPLIFY:

(a) $y_0 = y_{\max}$, $v = -gt$

(b) $y = y_{\max} - \frac{1}{2}gt^2$

(c) The distance from the top of the window to the ground is 1.2 + 2.5 = 3.7 m. From part (b),

$$y = y_{\max} - \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2(y_{\max} - y)}{g}}.$$

CALCULATE: The time taken from the roof to the ground is half the time of the round trip, t = 5.0/2 = 2.5 s.

(a) The velocity before the ball hits the ground is $v = -(9.81)(2.5) = -24.525$ m/s. So the speed is 24.525 m/s.

(b) $y = 0$ (ground), and t is the time from the roof to the ground.

$$0 = y_{\text{max}} - \frac{1}{2}gt^2 \Rightarrow y_{\text{max}} = \frac{1}{2}gt^2 \Rightarrow y_{\text{max}} = \frac{1}{2}\left(9.81\text{ m/s}^2\right)\left(2.5\text{ s}\right)^2 = 30.656\text{ m}$$

(c) $t = \sqrt{\dfrac{2\left(30.656 - 3.7\right)}{\left(9.81\right)}} = 2.3443\text{ s}$

ROUND: Rounding to two significant figures, $|v| = 25\text{ m/s}$, $y_{\text{max}} = 31\text{ m}$ and $t = 2.3$ s.

DOUBLE-CHECK: The speed in part (a) is consistent with an object accelerating uniformly due to gravity. The distance in (b) is a reasonable height for a building. For the result of part (c), the time must be less than 2.5 s, which it is.

2.101. From a mathematical table: $\dfrac{d}{dt}e^{\alpha t} = \alpha e^{\alpha t}$.

(a) $x(t) = 2x_0 = \dfrac{1}{4}x_0 e^{3\alpha t} \Rightarrow e^{3\alpha t} = 8 \Rightarrow 3\alpha t = \ln 8 \Rightarrow t = \dfrac{1}{3\alpha}\ln 8$

(b) $v(t) = \dfrac{dx}{dt} = \dfrac{3\alpha}{4}x_0 e^{3\alpha t}$

(c) $a(t) = \dfrac{dv}{dt} = \dfrac{(3\alpha)^2}{4}x_0 e^{3\alpha t} = \dfrac{9\alpha^2}{4}x_0 e^{3\alpha t}$

(d) αt must be dimensionless. Since the units of t are s, the units of α are s^{-1}.

2.102. Note that $\dfrac{d}{dt}\left(t^n\right) = nt^{n-1}$.

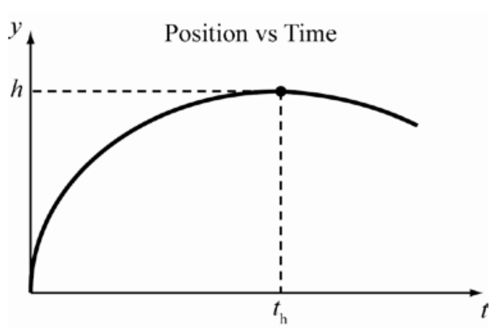
(a) $v = \dfrac{dx}{dt} = 4At^3 - 3Bt^2$

(b) $a = \dfrac{dv}{dt} = 12At^2 - 6Bt$

Multi-Version Exercises

2.103. **THINK:** The initial velocity is $v_0 = 28.0$ m/s. The acceleration is $a = -g = -9.81\text{ m/s}^2$. The velocity, v is zero at the maximum height. Determine the time, t to achieve the maximum height.

SKETCH:



RESEARCH: To determine the velocity use $v = v_0 + at_h$.

SIMPLIFY: $at_h = v - v_0 \Rightarrow t_h = \dfrac{v - v_0}{a} = \dfrac{-v_0}{-g} = \dfrac{v_0}{g}$

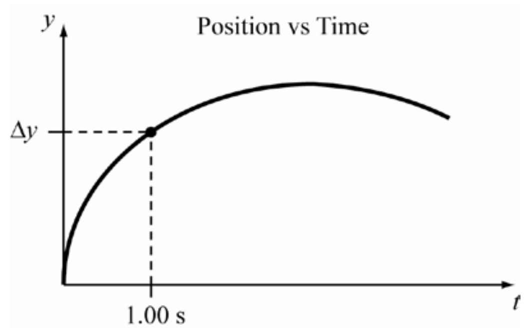
CALCULATE: $t_h = \dfrac{28.0\text{ m/s}}{9.81\text{ m/s}^2} = 2.8542\text{ s}$

ROUND: The initial values have three significant figures, so the result should be rounded to $t_h = 2.85$ s.

DOUBLE-CHECK: The initial velocity of the object is about 30 m/s, and gravity will cause the velocity to decrease about 10 m/s per second. It should take roughly three seconds for the object to reach its maximum height.

2.104. **THINK:** The initial velocity is $v_0 = 28.0$ m/s. The time is t = 1.00 s. The acceleration is $a = -g = -9.81 \text{ m/s}^2$. Determine the height above the initial position, Δy.

SKETCH:



RESEARCH: To determine the height use $\Delta y = v_0 t + \left(at^2\right)/2$.

SIMPLIFY: $\Delta y = v_0 t - \frac{1}{2}\left(gt^2\right)$

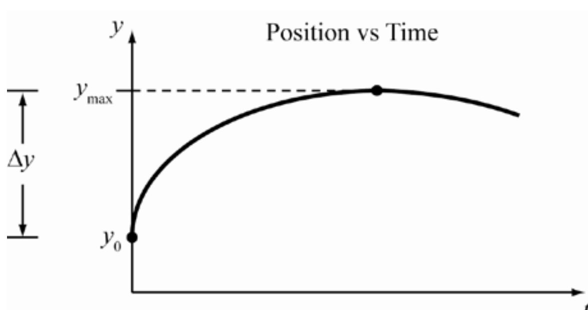
CALCULATE: $\Delta y = (28.0 \text{ m/s})(1.00 \text{ s}) - \frac{1}{2}\left(\left(9.81 \text{ m/s}^2\right)(1.00 \text{ s})^2\right) = 23.095 \text{ m}$

ROUND: As all initial values have three significant figures, the result should be rounded to $\Delta y = 23.1$ m.

DOUBLE-CHECK: The displacement Δy is positive, indicating that the final position is higher than the initial position. This is consistent with the positive initial velocity.

2.105. **THINK:** The initial velocity is $v_0 = 28.0$ m/s. The acceleration is $a = -g = -9.81 \text{ m/s}^2$. The velocity, v is zero at the maximum height. Determine the maximum height, Δy above the projection point.

SKETCH:



RESEARCH: The maximum height can be determined from the following equation: $v^2 = {v_0}^2 + 2a\Delta y$.

SIMPLIFY: With $v = 0$, $0 = {v_0}^2 - 2g\Delta y \Rightarrow \Delta y = \frac{{v_0}^2}{2g}$.

CALCULATE: $\Delta y = \frac{(28.0 \text{ m/s})^2}{2\left(9.81 \text{ m/s}^2\right)} = 39.96 \text{ m}$

ROUND: $v_0 = 28.0$ m/s has three significant figures, so the result should be rounded to $\Delta y = 40.0$ m.

DOUBLE-CHECK: The height has units of meters, which are an appropriate unit of distance. The calculated value is a reasonable maximum height for an object launched with a velocity of 28 m/s to achieve.

2.106. **THINK:** Since the rock is dropped from a fixed height and allowed to fall to the surface of Mars, this question involves free fall. It is necessary to impose a coordinate system. Choose $y = 0$ to represent the surface of Mars and $t_0 = 0$ to be the time at which the rock is released.

SKETCH: Sketch the situation at time $t_0 = 0$ and time t, when the rock hits the surface.

RESEARCH: For objects in free fall, equations 2.25 can be used to compute velocity and position. In particular, the equation $y = y_0 + v_{y0}t - \frac{1}{2}gt^2$ can be used. In this case, $y_0 = 1.013$ m. Since the object is not thrown but dropped with no initial velocity, $v_{y0} = 0$ m/s, and $g = 3.699$ m/s^2 on the surface of Mars.

SIMPLIFY: The starting position and velocity ($y_0 = 1.013$ m and $v_{y0} = 0$ m/s), final position ($y = 0$ m) and gravitational acceleration are known. Using the fact that $v_{y0} = 0$ m/s and solving the equation for t gives:

$$0 = y_0 + 0t - \tfrac{1}{2}(g)t^2 =$$
$$\frac{g}{2}t^2 = y_0 \Rightarrow$$
$$t^2 = \frac{2 \cdot y_0}{g}$$
$$t = \sqrt{\frac{2 \cdot y_0}{g}}$$

CALCULATE: On Mars, the gravitational acceleration $g = 3.699$ m/s^2. Since the rock is dropped from a height of 1.013 m, $y_0 = 1.013$ m. Plugging these numbers into our formula gives a time $t = \sqrt{\frac{2.026}{3.699}}$ s.

ROUND: In this case, all measured values are given to four significant figures, so our final answer has four significant digits. Using the calculator to find the square root gives a time $t = 0.7401$ s.

DOUBLE-CHECK: First note that the answer seems reasonable. The rock is not dropped from an extreme height, so it makes sense that it would take less than one second to fall to the Martian surface. To check the answer by working backwards, first note that the velocity of the rock at time t is given by the equation $v_y = v_{y0} - gt = 0 - gt = -gt$ in this problem. Plug this and the value $v_{y0} = 0$ into the equation to find the average velocity $\bar{v}_y = \frac{1}{2}(v_y + 0) = \frac{1}{2}(-gt)$. Combining this with the expression for position gives:

$$y = y_0 + \bar{v}_y t$$
$$= y_0 + \left(\tfrac{1}{2}(-gt)\right)t$$

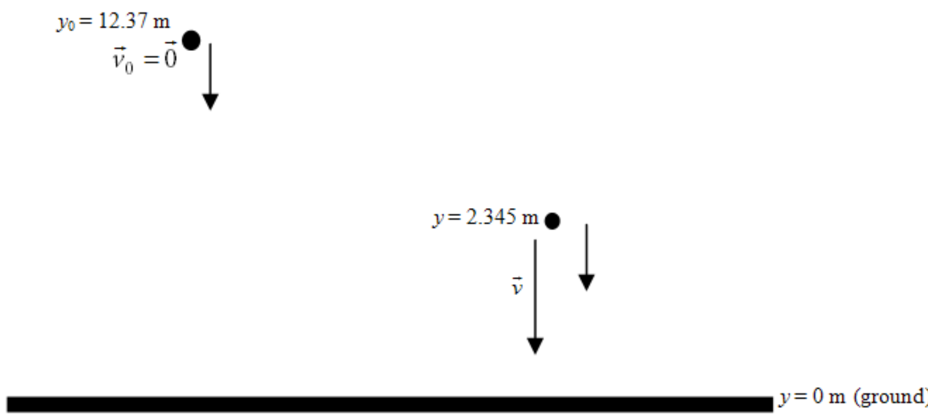
Using the fact that the rock was dropped from a height of $y_0 = 1.013$ m and that the gravitational acceleration on Mars is $g = 3.699$ m/s^2, it is possible to confirm that the height of the rock at time $t = 0.7401$ s is $y = 1.013 + \frac{1}{2}(-3.699)(0.7401^2) = 0$, which confirms the answer.

2.107. The time that the rock takes to fall is related to the distance it falls by $y = \frac{1}{2}gt^2$.

$$y = \frac{1}{2}gt^2 = \frac{1}{2}\left(3.699 \text{ m/s}^2\right)\left(0.8198 \text{ s}\right)^2 = 1.243 \text{ m}$$

2.108. **THINK:** Since the ball is dropped from a fixed height with no initial velocity and allowed to fall freely, this question involves free fall. It is necessary to impose a coordinate system. Choose $y = 0$ to represent the ground. Let $t_0 = 0$ be the time when the ball is released from height $y_0 = 12.37$ m and t_1 be the time the ball reaches height $y_1 = 2.345$ m.

SKETCH: Sketch the ball when it is dropped and when it is at height 2.345 m.



RESEARCH: Equations (2.25) are used for objects in free fall. Since the ball is released with no initial velocity, we know that $v_{0t} = 0$. We also know that on Earth, the gravitational acceleration is 9.81 m/s^2. In this problem, it is necessary to find the time that the ball reaches 2.345 m and find the velocity at that time. This can be done using equations (2.25) part (i) and (iii):

$$\text{(i)} \quad y = y_0 - \tfrac{1}{2}gt^2$$
$$\text{(iii)} \quad v_y = -gt$$

SIMPLIFY: We use algebra to find the time t_1 at which the ball will reach height $v_1 = 2.345$ m in terms of the initial height y_0 and gravitational acceleration g:

$$y_1 = y_0 - \tfrac{1}{2}g\left(t_1\right)^2 \Rightarrow$$
$$\tfrac{1}{2}g\left(t_1\right)^2 = y_0 - y_1 \Rightarrow$$
$$\left(t_1\right)^2 = \tfrac{2}{g}\left(y_0 - y_1\right) \Rightarrow$$
$$t_1 = \sqrt{\tfrac{2}{g}\left(y_0 - y_1\right)}$$

Combining this with the equation for velocity gives $v_{y1} = -gt_1 = -g\sqrt{\tfrac{2}{g}\left(y_0 - y_1\right)}$.

CALCULATE: The ball is dropped from an initial height of 12.37 m above the ground, and we want to know the speed when it reaches 2.345 m above the ground, so the ball is dropped from an initial height of 12.37 m above the ground, and we want to know the speed when it reaches 2.345 m above the ground, so $y_0 = 12.37$ and $y_1 = 2.345$ m. Use this to calculate $v_{y1} = -9.81\sqrt{\tfrac{2}{9.81}\left(12.37 - 2.345\right)}$ m/s.

ROUND: The heights above ground (12.37 and 2.345) have four significant figures, so the final answer should be rounded to four significant figures. The speed of the ball at time t_1 is then $-9.81\sqrt{\tfrac{2}{9.81}\left(12.37 - 2.345\right)} = -14.02$ m/s. The velocity of the ball when it reaches a height of 2.345 m above the ground is 14.02 m/s towards the ground.

DOUBLE-CHECK: To double check that the ball is going 14.02 m/s towards the ground, we use equation (2.25) (v) to work backwards and find the ball's height when the velocity is 14.02 m/s. We know that:

$$v_y^2 = v_{y0}^2 - 2g(y - y_0) \Rightarrow$$
$$v_y^2 = 0^2 - 2g(y - y_0) = -2g(y - y_0) \Rightarrow$$
$$\frac{v_y^2}{-2g} = y - y_0 \Rightarrow$$
$$\frac{v_y^2}{-2g} + y_0 = y$$

We take the gravitational acceleration g = 9.81 m/s^2 and the initial height y_0 = 12.37 m, and solve for y when v_y = −14.02 m/s. Then $y = \frac{v_y^2}{-2g} + y_0 = \frac{(-14.02)^2}{-2(9.81)} + 12.37 = 2.352$ m above the ground. Though this doesn't match the question exactly, it is off by less than 4 mm, so we are very close to the given value. In fact, if we keep the full accuracy of our calculation without rounding, we get that the ball reaches a velocity of 14.0246… m/s towards the ground at a height of 2.345 m above the ground.

2.109. Using the results noted in the double-check step of the preceding problem,

$$y_0 - y = \frac{v^2}{2g}$$
$$y = y_0 - \frac{v^2}{2g} = 13.51 \text{ m} - \frac{(14.787 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} = 2.37 \text{ m}.$$

By the rule for subtraction, the result is significant to the hundredths place.

2.110. Again using the results from the double-check step of the earlier problem,

$$y_0 - y = \frac{v^2}{2g}$$
$$y_0 = y + \frac{v^2}{2g} = 2.387 \text{ m} + \frac{(15.524 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} = 14.670 \text{ m}.$$

Note that if the value of g is treated as exact, by the addition rule the result has five significant figures.

Chapter 3: Motion in Two and Three Dimensions

Concept Checks

3.1. b **3.2.** a **3.3.** b, f **3.4.** d **3.5.** b **3.6.** a **3.7.** b **3.8.** b

Multiple-Choice Questions

3.1. c **3.2.** d **3.3.** d **3.4.** d **3.5.** c **3.6.** d **3.7.** a **3.8.** c **3.9.** a **3.10.** c **3.11.** a **3.12.** d **3.13.** a **3.14.** b **3.15.** a

Conceptual Questions

3.16. For ideal projectile motion, both a_x and a_y are constant and a_x is in fact zero since no horizontal force acts on the projectile. This results in v_x remaining constant since there is no horizontal acceleration. From the same logic, since a_y is a non-zero constant ($a_y = -9.81 \text{ m/s}^2$), v_y does not remain constant. Both x and y do not remain constant when the angle is between 0° and 90°.

3.17. If air resistance is neglected, the ball will land back in the passenger's hands since both the ball and the passenger have the same speed in the *x*-direction as the ball leaves the passenger's hand. If the train is accelerating, the answer does change. After the ball leaves the passenger's hand, it no longer has the same acceleration as the train and passenger in the *x*-direction and as result lands behind the passenger.

3.18. Even though the rock was thrown, this would only add initial velocity to the rock. After leaving the thrower's grasp the only force acting on the rock is due to gravity. Therefore, according to Newton's second law, the rock's acceleration is identical to the acceleration due to gravity.

3.19. Since the balls all start with an initial vertical velocity of zero ($v_y = 0$) and acceleration due to gravity is constant, they all take the same amount of time to reach the ground.

3.20. The maximum height is given by $y_{\max} = \dfrac{\left(-v_0^2 \sin^2\theta\right)}{g}$. In order to maximize the height you want $\sin\theta = 1$, which occurs when $\theta = 90^\circ$.

3.21. (a) Neglecting air resistance, the package travels at horizontal speed v_x. Since the package and the plane travel at the same horizontal velocity, they will both have traveled the same horizontal distance when the package hits the lake. The distance then between the package and the plane is the altitude *h*.

(b) When the package hits the lake, the horizontal component of the velocity vector remains v_x.

(c) The vertical component of the package's velocity is

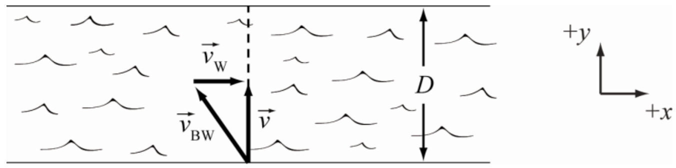
$$v_y = gt = \sqrt{2hg}, \quad \text{since } h = \frac{1}{2}gt^2, \text{ and } t = \sqrt{\frac{2h}{g}}.$$

The speed *s* is given by $s = \sqrt{v_x^2 + v_y^2} = \sqrt{v_x^2 + 2hg}$.

3.22. First the answer for vacuum: Here, it is possible to simply apply ideal trajectory considerations, which neglect air resistance. The cannonballs are launched with the same velocity at the same angle, and the equation for the range does not depend on mass; therefore the material of the cannon balls could not be distinguished based on trajectories or range. However, in the presence of air resistance one can distinguish between the lead and the wood ball. While the force of air friction is the same on both balls, the lead ball has a much greater mass and thus a much smaller acceleration due to air resistance than the wood ball. Thus the lead cannonball is the one which deviates less from the ideal projectile motion trajectory.

3.23. In order to minimize impact, speed should be minimized. The person should jump in the opposite direction of the vehicle's motion. This will reduce the magnitude of v_x and thus the impact is reduced.

3.24. (a)



The velocity of the boat will be $v=\sqrt{v_{BW}^2-v_W^2}$ by the Pythagorean Theorem. The time is then equal to: $t=\frac{d}{v}=\frac{2D}{\sqrt{v_{BW}^2-v_W^2}}$.

(b)

For the first part of the trip the velocity is $v_1=v_{BW}+v_W$, while on the way back the velocity is $v_2=v_{BW}-v_W$, so the time of the trip is $t=\frac{D}{v_1}+\frac{D}{v_2}=\frac{D}{v_{BW}+v_W}+\frac{D}{v_{BW}-v_W}$

$$=\frac{D(v_{BW}-v_W)}{(v_{BW}+v_W)(v_{BW}-v_W)}+\frac{D(v_{BW}+v_W)}{(v_{BW}+v_W)(v_{BW}-v_W)}=\frac{D(v_{BW}-v_W)+D(v_{BW}+v_W)}{v_{BW}^2-v_W^2}=\frac{2Dv_{BW}}{v_{BW}^2-v_W^2}.$$

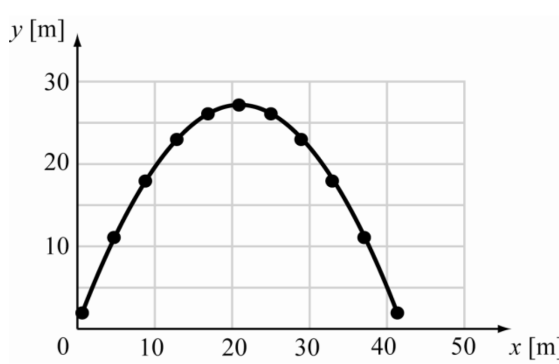
3.25. $x=x_0+v_xt$; $y=y_0+v_{0_y}t+\frac{1}{2}a_yt^2$. a_y can be found by calculating the slope of the v_y versus time graph.

$$a_y=\frac{v_f-v_i}{t_f-t_i}=\frac{-10 \text{ m/s}-10 \text{ m/s}}{10 \text{ s}-0 \text{ s}}=-2 \text{ m/s}^2$$

Now, use the equations with values to get different points.

$$x=1 \text{ m}+4t \text{ m/s} \text{ and } y=2 \text{ m}+10t \text{ m/s}+\frac{1}{2}(-2 \text{ m/s}^2)t^2$$

t(s)	x(m)	y(m)
0	1	2
1	5	11
2	9	18
3	13	23
4	17	26
5	21	27
6	25	26
7	29	23
8	33	18
9	37	11
10	41	2



3.26. The projection of the object's trajectory onto the xy-plane is uniform linear motion along the diagonal ($x = y$). The projection onto the z-axis has a motion is parabolic as a function of time. The components of the velocity are $v_x(t) = \sqrt{2}/2$, $v_y(t) = \sqrt{2}/2$ and $v_z(t) = \sqrt{3} - 9.8t$, since the velocity is the time derivative of position. The components of the acceleration are the time derivatives of velocity: $a_x = 0, a_y = 0, a_z = -9.8.$ This represents projectile motion in the vertical plane that bisects the xy-plane, with gravity acting in the $-z$ direction.

3.27. Differentiate the position components with respect to time to determine the velocity components.

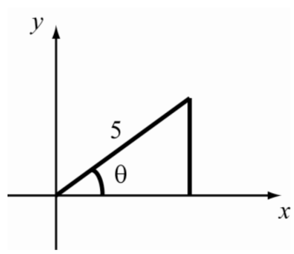
$$v_x(t) = \frac{dx(t)}{dt} = \frac{d}{dt}(4.9t^2 + 2t + 1) = 9.8t + 2, \quad v_y(t) = \frac{dy(t)}{dt} = \frac{d}{dt}(3t + 2) = 3$$

From these equations, differentiate again with respect to time to find the acceleration components.

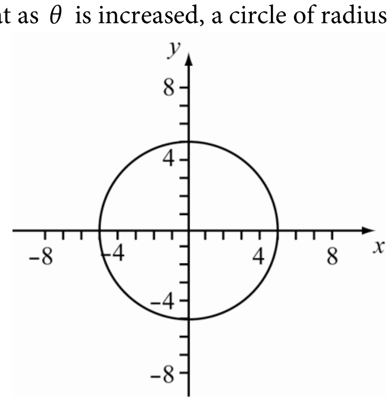
$$a_x(t) = \frac{dv_x(t)}{dt} = \frac{d}{dt}(9.8t + 2) = 9.8, \quad a_y(t) = \frac{dv_y(t)}{dt} = \frac{d}{dt}(3) = 0$$

The acceleration vector at $t = 2$ s is 9.8 m/s^2 in the x-direction.

3.28. (a) Observe the following sketch.



From this sketch, it can be seen that as θ is increased, a circle of radius 5 is drawn.

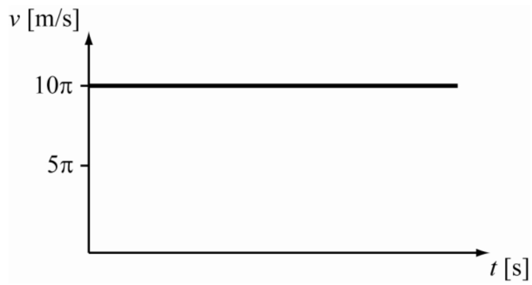


(b) Differentiate $x(t)$ and $y(t)$ with respect to time to find the velocity components.

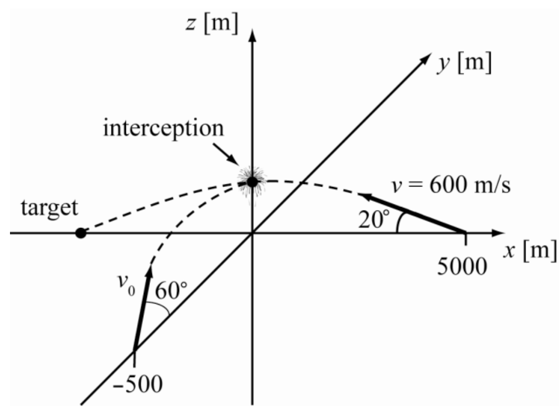
$$v_x(t) = \frac{dx(t)}{dt} = \frac{d}{dt}(5\cos 2\pi t) = -10\pi \sin 2\pi t, \quad v_y(t) = \frac{dy(t)}{dt} = \frac{d}{dt}(5\sin 2\pi t) = 10\pi \cos 2\pi t$$

(c) The particle's speed is given by:

$$v=\sqrt{v_x^{\,2}+v_y^{\,2}}=\sqrt{(10\pi)^2\sin^2(2\pi t)+(10\pi)^2\cos^2(2\pi t)}=10\pi\sqrt{\sin^2 2\pi t+\cos^2 2\pi t}=10\pi\sqrt{1}=10\pi$$



3.29. A diagram of the situation is shown.



When the missile reaches the z-axis, it has traveled for $t=\dfrac{x_0}{v_0\cos\theta_0}$. This means it has attained a height of:

$$z_{\text{m}}=v_0\sin\theta_0 t+\frac{1}{2}at^2=\frac{v_0\sin\theta_0 x_0}{v_0\cos\theta_0}-\frac{gx_0^{\,2}}{2v_0^{\,2}\cos^2\theta_0}$$

Inserting values, it is determined that t = 8.868 s and z_{m} = 1434 m. Also:

$$z_r=y_0\tan\alpha_0-\frac{gy_0^{\,2}}{2v_r^{\,2}\cos^2\alpha_0}\ \Rightarrow\ v_r=\sqrt{\frac{-gy_0^{\,2}}{2\cos^2\alpha_0\left(z_r-|y_0|\tan\alpha_0\right)}}$$

For the rocket to collide with the missile, $z_r=z_{\text{m}}$. This results is an imaginary number for v_r because $z_r>|y_0|\tan\theta_r$ and $g=9.81\text{ m/s}^2$. This means that there is no speed that will intercept the missile at this distance and angle.

3.30. The range is given by $x=-\dfrac{\left(2v_0^2\cos\theta\sin\theta\right)}{g}$. The height is given by $y=-\dfrac{\left(v_0^2\sin^2\theta\right)}{2g}$. Note that $\cos 45^\circ=\sin 45^\circ=1/\sqrt{2}$.

$$x=-\frac{\left[2v_0^{\,2}\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\right]}{g}=-\frac{v_0^{\,2}}{g};\ y=-\frac{\frac{1}{2}v_0^{\,2}}{2g}=-\frac{v_0^{\,2}}{4g}\ \Rightarrow\ \frac{x}{y}=\frac{\left(\frac{-v_0^{\,2}}{g}\right)}{\left(\frac{-v_0^{\,2}}{4g}\right)}=4.$$

The ratio is not dependent on the initial velocity so the answer does not change when the speed is doubled.

3.31. (a) The range is given by $x = -\frac{\left(2v_0^2\cos\theta\sin\theta\right)}{g}$. The height is given by $y = -\frac{\left(v_0^2\sin^2\theta\right)}{2g}$.

$$-\frac{2v_0^2}{g}\cos\theta\sin\theta = -\frac{v_0^2}{2g}\sin^2\theta \Rightarrow 4 = \frac{\sin\theta}{\cos\theta} \Rightarrow 4 = \tan\theta \Rightarrow \theta = \tan^{-1}(4) = 75.96^\circ$$

The launch angle should be 75.96°.

(b) The range is given by $x = -\frac{\left(2v_0^2\cos\theta\sin\theta\right)}{g}$. Half this range is given by:

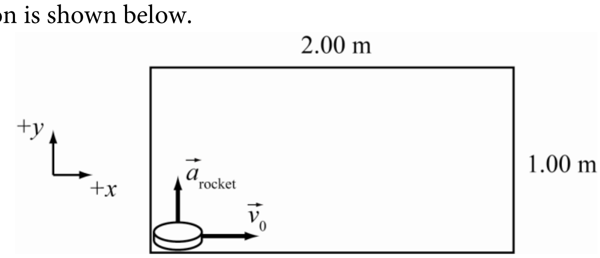
$$\frac{x_0}{2} = -\frac{v_0^2}{g}\cos\theta_0\sin\theta_0 = -\frac{2v_0^2}{g}\cos\theta\sin\theta,$$

since $(1/2)\cos\theta_0\sin\theta_0 = \cos\theta\sin\theta$. Using the trigonometric identity $\sin 2x = 2\sin x\cos x$:

$$\frac{1}{4}\sin 2\theta_0 = \frac{1}{2}\sin 2\theta$$
$$\sin 2\theta_0 = 2\sin 2\theta$$
$$\theta_0 = \frac{1}{2}\sin^{-1}\left(2\sin 2\theta\right)$$
$$\theta_0 = \frac{1}{2}\sin^{-1}\left[2\sin\left(2\tan^{-1}4\right)\right]$$
$$\theta_0 = 35.13^\circ$$

For the range to be half of what it was in part (a), the angle should be 35.13°.

3.32. A diagram of the situation is shown below.



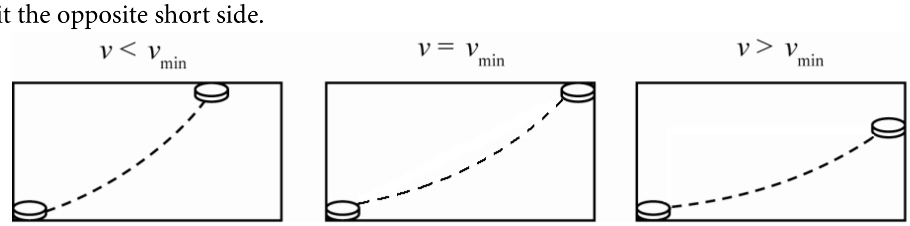
For the original velocity to be the minimum so that the puck does not hit the long side, it must reach the end of the table in the time it takes the puck to cross the short side. The x and y components can be considered separately, since the acceleration acts only in the y direction. The time it takes the rocket to push the puck across the width of the table is:

$$y_f = \frac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{2y_f}{a}}$$

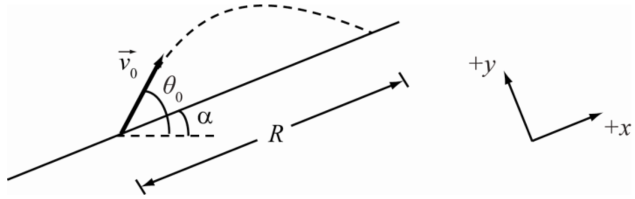
Since there is no acceleration in the x-direction,

$$x_f = v_x t \Rightarrow v_x = \frac{x_f}{t} = \frac{x_f}{\sqrt{2y_f/a}} = \frac{2.00}{\sqrt{2\cdot 1.00/2.00}} = 2.00 \text{ m/s}$$

The trajectory is a parabola. If $v < v_{\min}$, the puck will hit the opposite long side of the table. If $v > v_{\min}$, the puck will hit the opposite short side.



3.33. A diagram of the situation is shown.



The projectile's position is given by $x = v_0 \cos\theta_0 t$ and $y = v_0 \sin\theta_0 t - \frac{1}{2}gt^2$. The possible values of x and y along the slope can be found by $x = R\cos\alpha$ and $y = R\sin\alpha$, where R is the range along the hill. From this, $R\cos\alpha = v_0\cos\theta_0 t$ and $R\sin\alpha = v_0 \sin\theta_0 t - \frac{1}{2}gt^2$. The two unknowns in these equations are t and R. First, solve for t:

$$R\cos\alpha = v_0\cos\theta_0 t \;\Rightarrow\; t = \frac{R\cos\alpha}{v_0\cos\theta_0}.$$

Putting this result into the second equation:

$$R\sin\alpha = \left(v_0\sin\theta_0\right)\frac{R\cos\alpha}{v_0\cos\theta_0} - \frac{1}{2}g\frac{R^2\cos^2\alpha}{v_0^2\cos^2\theta_0}$$

$$R\sin\alpha = \tan\theta_0 R\cos\alpha - \frac{1}{2}g\frac{R^2\cos^2\alpha}{v_0^2\cos^2\theta_0}$$

$$0 = R\left(\tan\theta_0\cos\alpha - \sin\alpha - \frac{g\cos^2\alpha}{2v_0^2\cos^2\theta_0}R\right)$$

The horizontal range should be positive, so $R > 0$. Then:

$$R = \frac{2\left(\sin\alpha - \tan\theta_0\cos\alpha\right)v_0^2\cos^2\theta_0}{g\cos^2\alpha}.$$

Using the horizontal range equation from the text (equation 3.25), $R' = \frac{v_0^2\sin\left(2\theta_0\right)}{g} = \frac{2v_0^2\cos\theta_0\sin\theta_0}{g}$, the objective is to find a comparative factor β so that $R = R'\beta$.

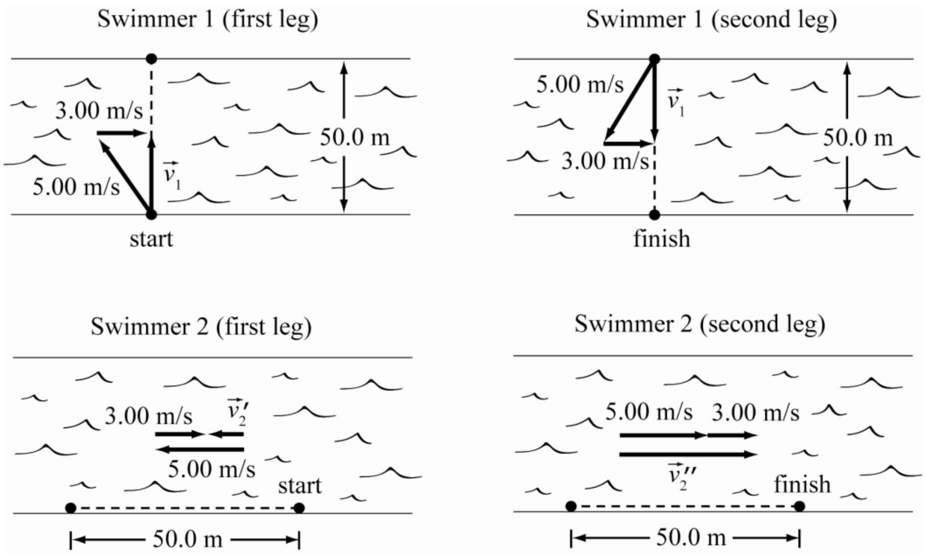
$$R = \frac{2v_0^2\cos\theta_0\sin\theta_0}{g}\cdot\frac{\cos\theta_0}{\sin\theta_0}\cdot\frac{\sin\alpha - \tan\theta_0\cos\alpha}{\cos^2\alpha}$$

$$= R'\frac{\cot\theta_0}{1}\frac{\tan\alpha - \tan\theta_0}{\cos\alpha}$$

$$= R'\frac{\cot\theta_0\tan\alpha - 1}{\cos\alpha}$$

Set $\frac{\cot\theta_0\tan\alpha - 1}{\cos\alpha}\;\beta = \frac{\cot\theta_0}{1}\frac{\tan\alpha - \tan\theta_0}{\cos\alpha}$ to obtain the comparative factor.

3.34. A diagram of the situation is shown.



The first swimmer crosses at a speed $v_1 = \sqrt{5.00^2 - 3.00^2} = 4.00$ m/s both ways. The total trip time for this swimmer is $t = 2d / v_1 = 2(50.0 \text{ m})/(4.00 \text{ m/s}) = 25.0$ s. The second swimmer travels at $v_2' = 5 \text{ m/s} - 3 \text{ m/s} = 2 \text{ m/s}$ and $v_2'' = 5 \text{ m/s} + 3 \text{ m/s} = 8 \text{ m/s}$ for the first and second part of the trip, respectively. The total time for the trip is

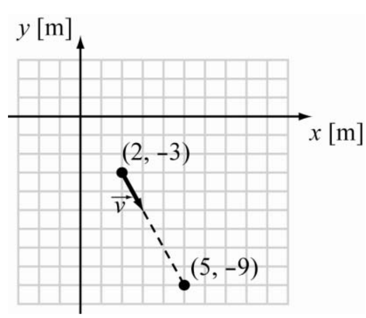
$$t = \frac{d}{v_2'} + \frac{d}{v_2''} = \frac{50.0 \text{ m}}{2.00 \text{ m/s}} + \frac{50.0 \text{ m}}{8.00 \text{ m/s}} = 31.3 \text{ s.}$$

The first swimmer gets back faster.

Exercises

3.35. **THINK:** To calculate the magnitude of the average velocity between $x = 2.0$ m, $y = -3.0$ m and $x = 5.0$ m, $y = -9.0$ m, the distance between these coordinates must be calculated, then the time interval can be used to determine the average velocity.

SKETCH:



RESEARCH: The equation to find the distance is $d = \sqrt{(x_\text{f} - x_\text{i})^2 + (y_\text{f} - y_\text{i})^2}$. The average velocity is given by $v = d / t$.

SIMPLIFY: $|\vec{v}| = \frac{\sqrt{(x_\text{f} - x_\text{i})^2 + (y_\text{f} - y_\text{i})^2}}{t}$

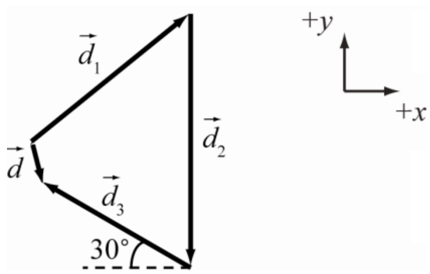
CALCULATE: $\left|\vec{v}\right| = \frac{\sqrt{\left(5.0\text{ m}-2.0\text{ m}\right)^2+\left(\left(-9.0\text{ m}\right)-\left(-3.0\text{ m}\right)\right)^2}}{2.4\text{ s}} = 2.7951\text{ m/s}$

ROUND: Rounding to two significant figures, $\left|\vec{v}\right| = 2.8$ m/s.

DOUBLE-CHECK: This result is on the same order of magnitude as the distances and the time interval.

3.36. **THINK:** The given distances must be converted to SI units; but let's save this until after the calculations. If you convert and round too early, this calculation turns into total nonsense. The magnitude and direction of the total displacement are to be determined.

SKETCH:



RESEARCH: Break the distances into north (+y) and east (+x) components. The displacement is given by $\vec{d} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3$. The magnitude is given by $\left|\vec{d}\right| = \sqrt{d_x^{\,2} + d_y^{\,2}}$ and the direction is given by $\theta = \tan^{-1}\left(d_y / d_x\right)$.

SIMPLIFY: $d_x = d_{1x} + d_{2x} + d_{3x} = d_1\cos 45° - d_3\cos 30° = \left(1/\sqrt{2}\right)d_1 - \left(\sqrt{3}/2\right)d_3$

$d_y = d_{1y} + d_{2y} + d_{3y} = d_1\sin 45° - d_2 + d_3\sin 30° = \left(1/\sqrt{2}\right)d_1 - d_2 + \left(1/2\right)d_3$

CALCULATE: $d_x = \left(1/\sqrt{2}\right)\left(10\text{ mi}\right) - \left(\sqrt{3}/2\right)\left(8\text{ mi}\right) = 0.14286\text{ mi}$

$d_y = \left(1/\sqrt{2}\right)\left(10\text{ mi}\right) - 12\text{ mi} + \left(1/2\right)\left(8\text{ mi}\right) = -0.92893\text{ mi}$

$\left|\vec{d}\right| = \sqrt{\left(0.14286\text{ mi}\right)^2 + \left(0.92893\text{ mi}\right)^2} = 0.93985\text{ mi} = 1.5122\text{ km}$

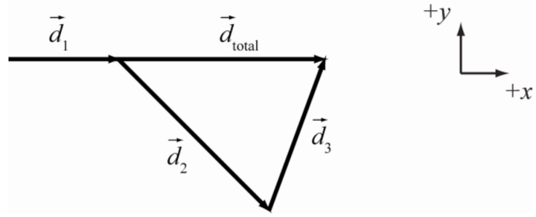
$\theta = \tan^{-1}\left(\frac{-0.92893\text{ mi}}{0.14286\text{ mi}}\right) = -81.257°$

ROUND: If you are driving, your odometer usually does not give you the distances to a greater precision than 1/10$^{\text{th}}$ of a mile. Rounding to two significant figures, the resultant displacement is 1.5 km. The direction is $-81°$, or $81°$ below the positive side of the x-axis.

DOUBLE-CHECK: Our sketch already tells us that the man will end up close to his house and almost exactly south of it, a conclusion which the sketch supports.

3.37. **THINK:** Determine the third vector $\vec{d_3}$ for a sail boat ride that results in a displacement of $\vec{d}_{\text{total}} = 6.00\text{ km}$ east when the first two legs of the journey are given as $\vec{d_1} = 2.00\text{ km}$ east and $\vec{d_2} = 4.00$ km southeast.

SKETCH:



RESEARCH: The component representation of vectors can be used to determine the x and y components of each vector: $d_x = d\cos\theta$ and $d_y = d\sin\theta$. The other equations that can be used are: $\vec{d}_{\text{total}} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3$; $\left|\vec{d}\right| = \sqrt{d_x^{\ 2} + d_y^{\ 2}}$; and $\theta = \tan^{-1}\left(d_y / d_x\right)$.

SIMPLIFY: $\vec{d}_3 = \vec{d}_{\text{total}} - \vec{d}_1 - \vec{d}_2$

$$d_{3x} = d_{\text{total}\,x} - d_{1x} - d_{2x} = d_{\text{total}}\cos 0° - d_1\cos 0° - d_2\cos 45° = d_{\text{total}} - d_1 - d_2/\sqrt{2}$$

$$d_{3y} = d_{total\,y} - d_{1y} - d_{2y} = d_{tot}\sin 0° - d_1\sin 0° - d_2\sin 45° = 0 - 0 + d_2/\sqrt{2} = d_2/\sqrt{2}$$

$$\left|\vec{d}_3\right| = \sqrt{\left(d_{\text{total}} - d_1 - d_2/\sqrt{2}\right)^2 + \left(d_2/\sqrt{2}\right)^2}, \quad \theta = \tan^{-1}\left(\frac{d_2/\sqrt{2}}{d_{\text{total}} - d_1 - d_2/\sqrt{2}}\right)$$

CALCULATE: $\left|\vec{d}_3\right| = \sqrt{\left(6.00\text{ km} - 2.00\text{ km} - 4.00\text{ km}/\sqrt{2}\right)^2 + \left(4.00\text{ km}/\sqrt{2}\right)^2} = 3.0615\text{ km}$

$$\theta = \tan^{-1}\left(\frac{4.00\text{ km}/\sqrt{2}}{6.00\text{ km} - 2.00\text{ km} - 4.00\text{ km}/\sqrt{2}}\right) = 67.500°$$

ROUND: Rounding to three significant figures, $\left|\vec{d}_3\right| = 3.06$ km and $\theta = 67.5°$. The missing part of the trip was 3.06 km 67.5° North of East.

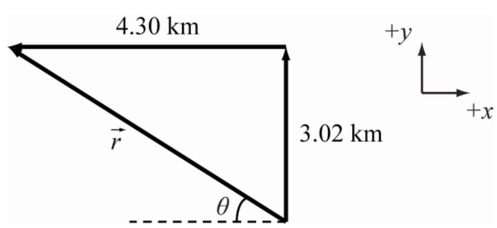
DOUBLE-CHECK: The result is on the same order of magnitude as the other parts of the trip and by looking at the sketch, the angle is reasonable.

3.38. **THINK:**

(a) The net displacement vector must be found for a truck that drives 3.02 km north then drives another 4.30 km west.

(b) Determine the average velocity of the trip if it takes 5.00 minutes to complete. Convert the time to seconds: $t = 5.00\text{ min}\left(60\text{ s/min}\right) = 300.\text{ s}$.

SKETCH:



RESEARCH:

(a) The displacement is given by $\vec{r} = \left(x, y\right)$.

(b) The magnitude of the average velocity is $\left|\vec{v}\right| = d/t$, where $d = \left|\vec{r}\right| = \sqrt{x^2 + y^2}$.

SIMPLIFY:

(a) It is not necessary to simplify.

(b) $\left|\vec{v}\right| = \dfrac{d}{t} = \dfrac{\sqrt{x^2 + y^2}}{t}$

CALCULATE:

(a) $\vec{r} = \left(-4.30\text{ km},\ 3.02\text{ km}\right)$

(b) $\left|\vec{v}\right| = \dfrac{\sqrt{\left(-4.30\right)^2 + \left(3.02\right)^2}}{300.\text{ s}} = 0.017515\text{ km/s}$

ROUND:

(a) It is not necessary to round since the data is simply restated.

(b) Rounding to three significant figure, $\left|\vec{v}\right| = 0.0175$ km/s.

DOUBLE-CHECK:

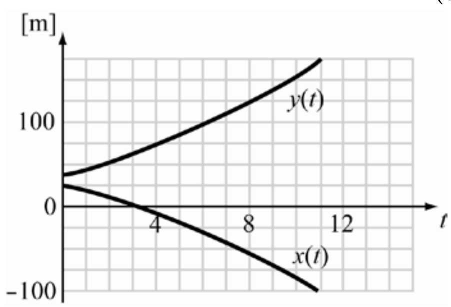
(a) The distance is reasonable for a truck to travel.

(b) This speed corresponds to 63 km/h or 39 mph, which is reasonable for a truck.

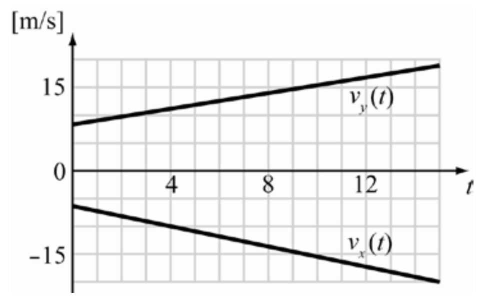
3.39. **THINK:** The position components $x(t) = -0.45t^2 - 6.5t + 25$ and $y(t) = 0.35t^2 + 8.3t + 34$ can be used to find the magnitude and direction of the position at t = 10.0 s. The velocity and acceleration at t = 10.0 s must then be determined.

SKETCH:

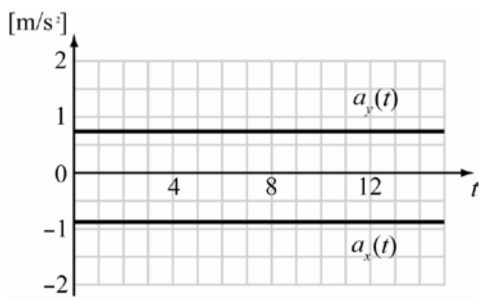
(a)



(b)



(c)



RESEARCH:

(a) Insert t = 10 s into the given equations, then use $|\vec{r}| = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$.

(b) Differentiate the given components with respect to time to get functions of velocity.

(c) Differentiate the velocity functions with respect to time to get functions of acceleration.

SIMPLIFY:

(a) It is not necessary to simplify.

(b) $v_x(t) = \dfrac{dx(t)}{dt} = \dfrac{d(-0.45t^2 - 6.5t + 25)}{dt} = (-0.90t - 6.5)$ m/s

$v_y(t) = \dfrac{dy(t)}{dt} = \dfrac{d(0.35t^2 + 8.3t + 34)}{dt} = (0.70t + 8.3)$ m/s

(c) $a_x(t) = \dfrac{dv_x(t)}{dt} = \dfrac{d(-0.90t - 6.5)}{dt} = -0.90 \text{ m/s}^2$, $a_y(t) = \dfrac{dv_y(t)}{dt} = \dfrac{d(0.70t + 8.3)}{dt} = 0.70 \text{ m/s}^2$

CALCULATE:

(a) $x(10.0) = -0.45(10.0)^2 - 6.5(10.0) + 25 = -85$ m, $y(10.0) = 0.35(10.0)^2 + 8.3(10.0) + 34 = 152$ m

Now, insert these values into the magnitude and distance equations:

$$|\vec{r}| = \sqrt{(-85 \text{ m})^2 + (152 \text{ m})^2} = 174 \text{ m}, \quad \theta = \tan^{-1}\left(\frac{152}{-85}\right) = -60.786°$$

(b) $v_x(10.0) = -0.90(10.0) - 6.5 = -15.5$ m/s, $v_y(10.0) = 0.70(10.0) + 8.3 = 15.3$ m/s

$$|\vec{v}| = \sqrt{(-15.5)^2 + (15.3)^2} = 21.8 \text{ m/s}, \quad \theta = \tan^{-1}\left(\frac{15.3}{-15.5}\right) = -44.6°$$

(c) Since there is no time dependence, the acceleration is always $\vec{a} = (a_x, a_y) = (-0.90 \text{ m/s}^2, 0.70 \text{ m/s}^2)$.

The $|\vec{a}| = \sqrt{(-0.90 \text{ m/s}^2)^2 + (0.70 \text{ m/s}^2)^2} = 1.140 \text{ m/s}^2$, $= \tan^{-1}\left(\frac{0.70}{-0.90}\right) = -37.87°$.

ROUND:

(a) Both distances and the magnitude are accurate to the meter, $|\vec{r}| = 174$ m. Round the angle to three significant figures, $60.8°$ north of west (note: west is used because x was negative).

(b) The equation's parameters are accurate to a tenth of a meter. The rabbit's velocity is then 21.8 m/s, $44.6°$ north of west.

(c) It is not necessary to consider the significant figures since the original parameters of the function are used. The rabbit's velocity is 1.14 m/s^2, $37.9°$ north of west.

DOUBLE-CHECK:

(a) 174 m in 10 s seems reasonable for a rabbit, considering the world record for the 100 m dash is about 10 s.

(b) The velocity of a rabbit ranges from 12 m/s to 20 m/s. This rabbit would be at the top of that range.

(c) A rabbit may accelerate at this rate but it can not sustain this acceleration for too long.

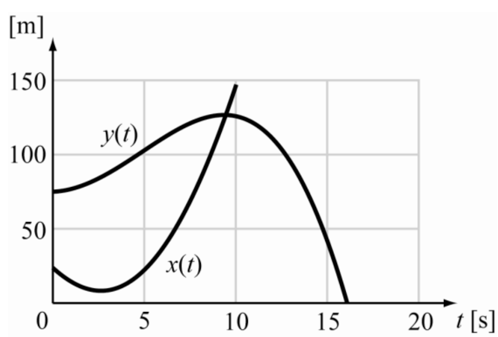
3.40. **THINK:** (a) The position vector is given. The distance the car is away from the origin at t = 5.00 s is to be determined.

(b) Now, the velocity vector for the car is to be determined.

(c) The speed (magnitude of the velocity) is to be determined at 5 s.

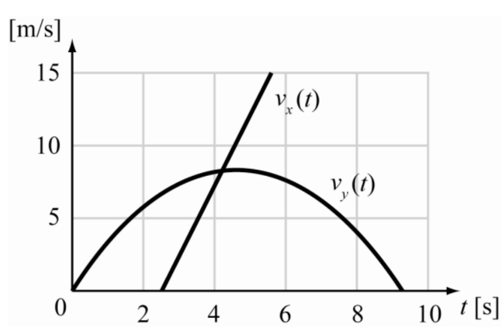
SKETCH:

(a)



(b) See sketch above.

(c)



RESEARCH:
(a) The time can be inserted into the position vector and the magnitude can then be found using $d = |r| = \sqrt{x^2 + y^2}$.

(b) The velocity vector is given by derivative of the position vector with respect to time, $v(t) = \frac{d\vec{r}(t)}{dt}$.

(c) To determine the speed, use the magnitude equation $|\vec{v}| = \sqrt{{v_x}^2 + {v_y}^2}$.

SIMPLIFY:
(a) It is not necessary to simplify.

(b) $\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{d}{dt}\left(\left[24.4 - 12.3t + 2.43t^2\right], \frac{d}{dt}\left[74.4 + 1.80t^2 - 0.130t^3\right]\right)$
$= \left(-12.3 \text{ m/s} + 4.86t \text{ m/s}^2,\ 3.60t \text{ m/s}^2 - 0.390t^2 \text{ m/s}^3\right)$

(c) It is not necessary to simplify.

CALCULATE:
(a) $\vec{r}(5) = (24.4 \text{ m}) - 5(12.3 \text{ m/s}) + (5)^2(2.43 \text{ m/s}^2), (74.4 \text{ m}) + (5)^2(1.80 \text{ m/s}^2) - (5)^3(0.130 \text{ m/s}^3)$
$= (23.65 \text{ m},\ 103.150 \text{ m})$

Now, insert the components into the equation for the magnitude:

$$d = |r| = \sqrt{(23.65 \text{ m})^2 + (103.150 \text{ m})^2} = 105.8265 \text{ m}.$$

(b) There are no other calculations to be done.

(c) $v(5) = \left(-12.3 \text{ m/s} + 4.86(5) \text{ m/s}^2,\ 3.60(5) \text{ m/s}^2 - 0.390(5)^2 \text{ m/s}^3\right) = (12.0 \text{ m/s, } 8.25 \text{ m/s})$

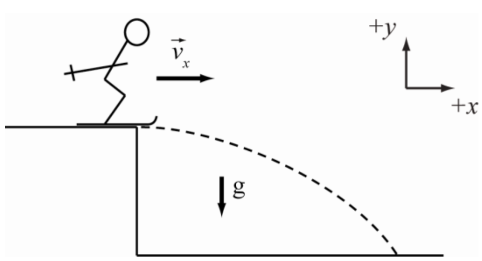
$|\vec{v}| = \sqrt{(12.0 \text{ m/s})^2 + (8.25 \text{ m/s})^2} = 14.5624 \text{ m/s}$

ROUND:
(a) The equation is accurate to four significant figures so the distance is now $d = 105.8$ m.
(b) The significant figures remain the same for the parameters.
(c) Rounding to the first place after the decimal, $|\vec{v}| = 14.6$ m/s.

DOUBLE-CHECK:
(a) The distance is reasonable for a car to travel.
(b) The derivatives were done correctly.
(c) The speed is reasonable for a car.

3.41. **THINK:** Ignoring air resistance, the skier's horizontal velocity will remain unchanged, while her vertical velocity is influenced solely by gravity. $v_x = 30.0$ m/s , $g = 9.81 \text{ m/s}^2$ and $t = 2.00$ s.

SKETCH:



RESEARCH: $v_{ix} = v_{fx}$ and $v_{fy} = v_{iy} + at$.

SIMPLIFY: $v_{fy} = 0 - gt = -gt$

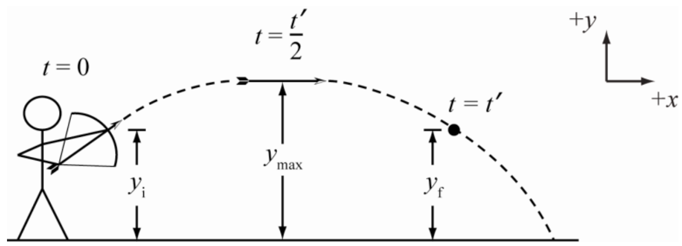
CALCULATE: $v_{fx} = 30.0$ m/s and $v_{fy} = -(9.81 \text{ m/s}^2)(2.00 \text{ s}) = -19.62$ m/s.

ROUND: $|v_{fx}| = 30.0$ m/s and $|v_{fy}| = 19.6$ m/s.

DOUBLE-CHECK: The order of magnitude is reasonable.

3.42. **THINK:** When the arrow is horizontal it is at its maximum height. This occurs when the vertical velocity is zero. The time to reach maximum height is half the time it takes to fall back to the same height. $y_0 = 1.14$ m, $v_0 = 47.5$ m/s, $\theta = 35.2°$, and $g = 9.81 \text{ m/s}^2$.

SKETCH:



RESEARCH: $y - y_0 = v_{y0}t + \frac{1}{2}at^2$; $v_{y0} = v_0 \sin\theta$

SIMPLIFY: For $y = y_0$, use $t = t'$:

$$0 = v_0 \sin\theta (t') - \frac{1}{2}g(t')^2.$$

Since $t' \neq 0$, $t' = \frac{2v_0 \sin\theta}{g}$. Therefore, at the maximum height, $t = \frac{t'}{2} = \frac{v_0 \sin\theta}{g}$.

CALCULATE: $t = \frac{(47.5 \text{ m/s})\sin(35.2°)}{9.81 \text{ m/s}^2} = 2.7911$ s

ROUND: To three significant figures, $t = 2.79$ s.

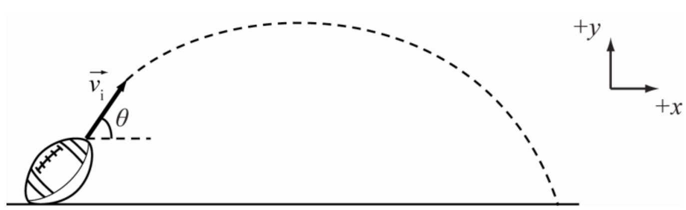
DOUBLE-CHECK: The arrow is horizontal when its vertical velocity is zero:

$$v_y = 0 = v_{y0} + at = v_0 \sin\theta - gt \Rightarrow t = \frac{v_0 \sin\theta}{g}$$

This is the same as the result obtained above.

3.43. **THINK:** Assume the ball starts on the ground so that the initial and final heights are the same. The initial velocity of the ball is $v_i = 27.5$ m/s, with $\theta = 56.7°$ and $g = 9.81 \text{ m/s}^2$.

SKETCH:



RESEARCH: $y_f - y_i = v_{iy}t + \frac{1}{2}at^2$ and $v_{iy} = v_i \sin\theta$.

SIMPLIFY: $0 = v_i(\sin\theta)t - \frac{1}{2}gt^2 \Rightarrow v_i \sin\theta = \frac{1}{2}gt \Rightarrow t = \frac{2v_i \sin\theta}{g}$

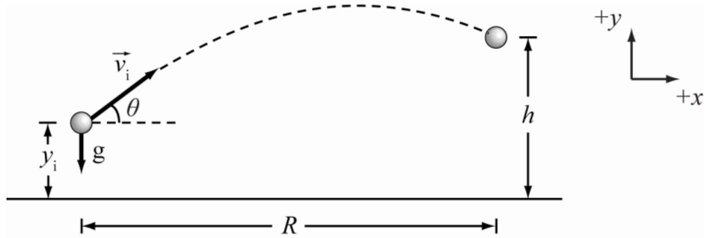
CALCULATE: $t = \frac{2(27.5 \text{ m/s})\sin(56.7°)}{9.81 \text{ m/s}^2} = 4.6860$ s

ROUND: Rounding to three significant figures, $t = 4.69$ s.

DOUBLE-CHECK: Given the large angle the ball was kicked, about 5 seconds is a reasonable amount of time for it to remain in the air.

3.44. **THINK:** Use the horizontal distance and velocity to determine the time it takes for the ball to reach the net. Then, use this time to determine the vertical height of the ball. Use this vertical height to determine if the ball clears the net and by how much it does or does not.

SKETCH:



RESEARCH: $v_{ix} = v_i\cos\theta$; $v_{iy} = v_i\sin\theta$; $R = v_{ix}t$; and $y_f - y_i = v_{iy}t + \frac{1}{2}at^2$.

SIMPLIFY: $t = \frac{R}{v_i\cos\theta}$ and $y_f = y_i + (v_i\sin\theta)t - \frac{1}{2}gt^2 = y_i + R\tan\theta - \frac{gR^2}{2v_i^2\cos^2\theta}$. In order to compare the height of the net, h, subtract h from the final height, y_f: $\Delta h = y_f - h = y_i - h + R\tan\theta - \frac{gR^2}{2v_i^2\cos^2\theta}$.

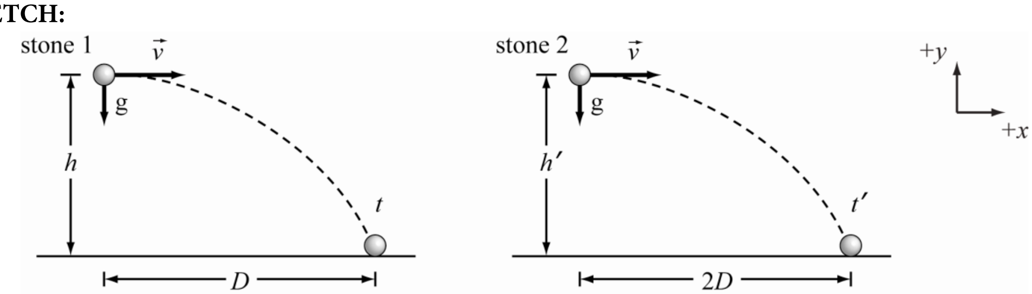
CALCULATE: $\Delta h = (1.8\text{ m} - 1.07\text{ m}) + (11.83\text{ m})\tan(7.00°) - \frac{(9.81\text{ m/s}^2)(11.83\text{ m})^2}{2(18.0\text{ m/s})^2\cos^2(7.00°)} = 0.031929\text{ m}$

ROUND: Rounding the result to two significant figures, the ball clears the net by 3.2 cm.

DOUBLE-CHECK: Given the long distance to the net and the small angle that the ball is hit at, it is reasonable that the ball would clear the net by such a small distance.

3.45. **THINK:** Simply find a relation between the height of the building and the distance traveled. With the use of this equation, determine how the height is affected when the distance is doubled. No values are needed.

SKETCH:



RESEARCH: $x_f - x_i = vt$ and $y_f - y_i = v_{iy}t + \frac{1}{2}at^2$.

SIMPLIFY: $D = vt \Rightarrow t = \frac{D}{v}$, $2D = vt' \Rightarrow t' = \frac{2D}{v}$, $-h = 0 - \frac{1}{2}gt^2 = -\frac{gD^2}{2v^2}$.

$$-h' = 0 - \frac{1}{2}g(t')^2 = -\frac{4gD^2}{2v^2} = -\frac{2gD^2}{v^2}$$

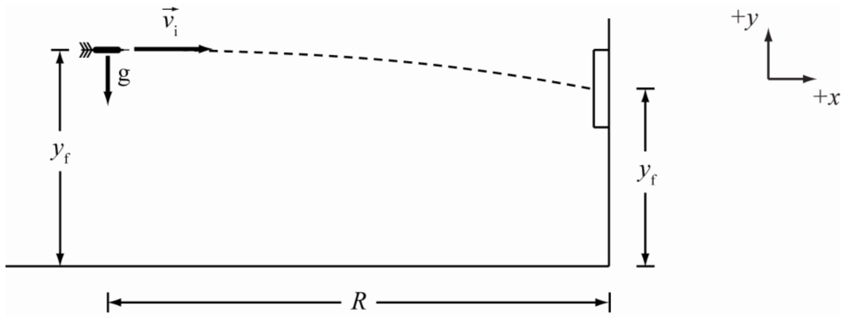
CALCULATE: $\frac{h'}{h} = \frac{(2gD^2/v^2)}{(gD^2/2v^2)} = 4$

ROUND: The ratio of the heights of the buildings is 4 to 1.

DOUBLE-CHECK: Seeing as the paths of the stones are parabolic, the ratio of the heights is proportional to the square of the ratio of the distances, so the result makes sense.

3.46. **THINK:** Assume there is no air resistance and the horizontal velocity remains constant. Also, assume there is no initial vertical velocity. $R = 3.0 \text{ m}$, $y_\text{i} = 2.0 \text{ m}$, $y_\text{f} = 1.65 \text{ m}$ and $g = 9.81 \text{ m/s}^2$.

SKETCH:



RESEARCH:

(a) $y_\text{f} - y_\text{i} = v_{\text{i}y}t + \frac{1}{2}at^2$ and $v_{\text{i}y} = 0$ and $a = -g$.

(b) $v_{\text{i}x} = v_{\text{f}x}$ and $R = v_{\text{i}x}t$.

(c) $\vec{v}_\text{f} = v_{\text{f}x}\hat{x} + v_{\text{f}y}\hat{y}$; $v_{\text{f}y} = v_{\text{i}y} + at$; and $\left|\vec{v}_\text{f}\right| = \sqrt{v_{\text{f}x}^2 + v_{\text{f}y}^2}$.

SIMPLIFY:

(a) $y_\text{f} - y_\text{i} = 0 - \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{-2\left(y_\text{f} - y_\text{i}\right)}{g}}$

(b) $v_\text{i} = v_{\text{i}x} = \frac{R}{t}$

(c) $v_{\text{f}y} = -gt \Rightarrow \vec{v}_\text{f} = v_\text{i}\hat{x} - gt\hat{y} \Rightarrow \left|\vec{v}_\text{f}\right| = \sqrt{v_\text{i}^2 + \left(-gt\right)^2}$

CALCULATE:

(a) $t = \sqrt{\frac{-2\left(1.65 \text{ m} - 2.00 \text{ m}\right)}{9.81 \text{ m/s}^2}} = 0.26712 \text{ s}$

(b) $v_\text{i} = \frac{3.0 \text{ m}}{0.26712 \text{ s}} = 11.231 \text{ m/s}$

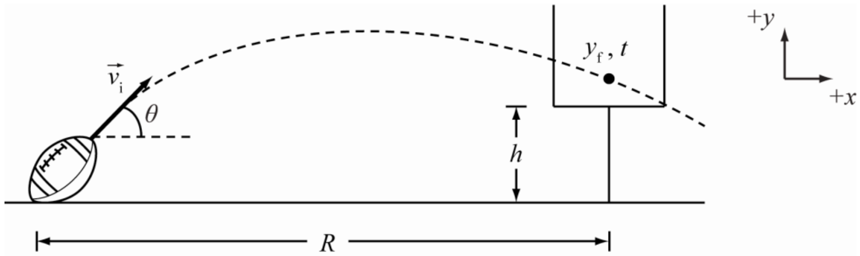
(c) $\left|\vec{v}_\text{f}\right| = \sqrt{\left(11.231 \text{ m/s}\right)^2 + \left(\left(-9.81 \text{ m/s}^2\right)\left(0.26712 \text{ s}\right)\right)^2} = 11.532 \text{ m/s}$

ROUND: Rounding to two significant figures, $t = 0.27 \text{ s}$, $v_\text{i} = 11 \text{ m/s}$ and $\left|\vec{v}_\text{f}\right| = 12 \text{ m/s}$.

DOUBLE-CHECK: The initial and final velocities of the dart are reasonable for a player to achieve. At these velocities, the flight time is also reasonable.

3.47. **THINK:** Use the horizontal distance and velocity to determine the time it takes to reach the post. Use the time to determine the height of the ball at that point. Then compare the height of the ball to the height of the goal post. Vertical velocity at this point can be determined as well: $v_\text{i} = 22.4 \text{ m/s}$, $\theta = 49.0°$, $R = 39.0 \text{ m}$, $h = 3.05 \text{ m}$, $g = 9.81 \text{ m/s}^2$. Assume the ball is kicked off the ground, $y_\text{i} = 0 \text{ m}$.

SKETCH:



RESEARCH: $R = v_{ix}t$; $y_f - y_i = v_{iy}t + \frac{1}{2}at^2$; $v_{fy} = v_{iy} + at$; $v_{ix} = v_i \cos\theta$; and $v_{iy} = v_i \sin\theta$.

SIMPLIFY:

(a) $t = \frac{R}{v_i \cos\theta}$ and $y_f = (v_i \sin\theta)t - \frac{1}{2}gt^2 = R\tan\theta - \frac{gR^2}{2v_i^2\cos^2\theta}$.

In order to compare the height of the ball to the height of the goal post, subtract h from both sides of the equation,

$$\Delta h = y_f - h = -h + R\tan\theta - \frac{gR^2}{2v_i^2\cos^2\theta}.$$

(b) $v_{fy} = v_i \sin\theta - \frac{gR}{v_i \cos\theta}$

CALCULATE:

(a) $\Delta h = -3.05 \text{ m} + (39.0 \text{ m})\tan(49.0°) - \frac{(9.81 \text{ m/s}^2)(39.0 \text{ m})^2}{2(22.4 \text{ m/s})^2\cos^2(49.0°)} = 7.2693 \text{ m}$

(b) $v_{fy} = (22.4 \text{ m/s})\sin(49.0°) - \frac{(9.81 \text{ m/s}^2)(39.0 \text{ m})}{(22.4 \text{ m/s})\cos(49.0°)} = -9.1286 \text{ m/s}$

ROUND: Round to the appropriate three significant figures:

(a) The ball clears the post by 7.27 m.

(b) The ball is heading downward at 9.13 m/s.

DOUBLE-CHECK: The initial velocity certainly seems high enough to clear the goal post from about 1/3 of the field away. It also makes sense that the vertical velocity at this point is lower than the initial velocity and the ball is heading down.

3.48. **THINK:** Since the time of the last portion of the flight and the horizontal displacement during that time are given, the x component of the initial velocity can be determined, because the horizontal velocity component remains constant throughout the flight. The initial velocity can then be determined, because we also know the initial angle of $\theta = 35.0°$. The vertical displacement of the projectile during the last flight phase is also given. However, since we do not know the relative altitude of the beginning and end of the trajectory, the vertical displacement provides no useful information and is thus a distractor, which we can and should ignore.

SKETCH: A sketch is not needed in this case.

RESEARCH: $v_{ix} = v_{fx} = v_i \cos\theta$; $v_{ix} = \frac{d}{\Delta t}$

SIMPLIFY: $v_i \cos\theta = \frac{d}{\Delta t} \Rightarrow v_i = \frac{d}{\cos\theta \Delta t}$.

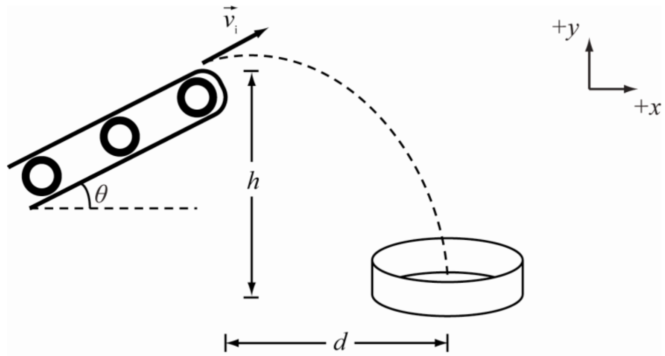
CALCULATE: $v_i = (10.0 \text{ m})/\left[\cos(35.0°)(1.50 \text{ s})\right] = 8.138497 \text{ m/s}$

ROUND: Rounding to three significant figures, $v_i = 8.14$ m/s.

DOUBLE-CHECK: This speed is reasonable.

3.49. **THINK:** Assume the sand moves as a single object and that the sand is moving fast enough to be projected from the conveyor belt at the top. To solve for the distance, the quadratic equation must be used. $\theta = 14.0°$, $h = 3.00$ m, $v_i = 7.00$ m/s and $g = 9.81$ m/s^2.

SKETCH:



RESEARCH: $v_{ix}t = d$; $v_{ix} = v_i\cos\theta$; $y_f - y_i = v_{iy}t + \frac{1}{2}at^2$; and $v_{iy} = v_i\sin\theta$.

SIMPLIFY: $v_{ix}t = d \Rightarrow v_i\cos\theta\, t = d \Rightarrow t = \dfrac{d}{v_i\cos\theta}$

$-h = v_i\sin\theta\, t - \frac{1}{2}gt^2 = d\tan\theta - \dfrac{gd^2}{2v_i^2\cos^2\theta} \Rightarrow \left(\dfrac{g}{2v_i^2\cos^2\theta}\right)d^2 - (\tan\theta)d - h = 0$, so

$$d = \frac{\tan\theta \pm \sqrt{\tan^2\theta + 2\left(gh/\left(v_i^2\cos^2\theta\right)\right)}}{g/\left(v_i^2\cos^2\theta\right)}.$$

CALCULATE:

$$d = \frac{\tan(14.0°) \pm \sqrt{\tan^2(14.0°) + 2\left(9.81 \text{ m/s}^2\right)(3.00 \text{ m})/\left[(7.00 \text{ m/s})^2\cos^2(14.0°)\right]}}{9.81 \text{ m/s}^2\left[(7.00 \text{ m/s})^2\cos^2(14.0°)\right]}$$

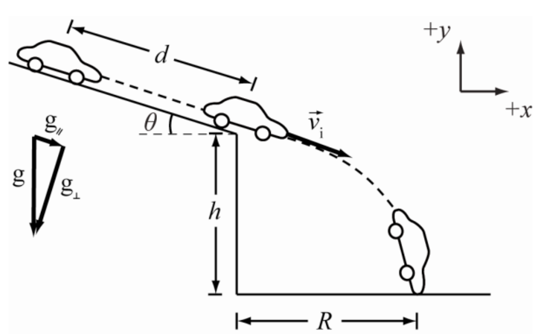
$= 6.6122$ m or -4.2672 m

ROUND: d = 6.61 m (the answer must be a positive value)

DOUBLE-CHECK: If launched at $\theta = 0°$, the sand would travel about 6 m, so 7 m for a small angle is reasonable.

3.50. **THINK:** Since gravity is a vector, it can be broken up into components. This means that a portion of gravity accelerates the car down the incline. Once the car reaches the edge, it is under the whole force of gravity. $g = 9.81$ m/s^2, d = 29.0 m, $\theta = 17.0°$, h = 55.0 m. The quadratic equation can be used to solve for distance.

SKETCH:



RESEARCH: $g_{\parallel} = g\sin\theta$; $v_{\text{f}}^2 = v_{\text{i}}^2 + 2ad$; $R = v_{\text{i}x}t$; $y_{\text{f}} - y_{\text{i}} = v_{\text{i}y}t + \frac{1}{2}gt^2$; $v_{\text{i}x} = v_{\text{i}}\cos\theta$; and $v_{\text{i}y} = v_{\text{i}}\sin\theta$.

SIMPLIFY: $v_{\text{i}}^2 = 0 + 2g_{\parallel}d = 2(g\sin\theta)d \Rightarrow v_{\text{i}} = \sqrt{2gd\sin\theta}$

(a) $-h = -(v_{\text{i}}\sin\theta)t - \frac{1}{2}gt^2$

$$-h = -v_{\text{i}}(\sin\theta)\left(\frac{R}{v_{\text{i}}\cos\theta}\right) - \frac{1}{2}g\left(\frac{R}{v_{\text{i}}\cos\theta}\right)^2 \qquad \left(\text{since } t = \frac{R}{v_{\text{i}}\cos\theta}\right)$$

$$-h = -R\tan\theta - \frac{gR^2}{2v_{\text{i}}^2\cos^2\theta}$$

$$0 = \frac{gR^2}{2\cos^2\theta(2gd\sin\theta)} + R\tan\theta - h \qquad \left(\text{since } v_{\text{i}} = \sqrt{2gd\sin\theta}\right)$$

$$0 = \left(\frac{1}{4d\cos^2\theta\sin\theta}\right)R^2 + (\tan\theta)R - h$$

Therefore, using the quadratic formula, $R = \dfrac{-\tan\theta \pm \sqrt{\tan^2\theta + \dfrac{h}{d\cos^2\theta\sin\theta}}}{1/\left(2d\cos^2\theta\sin\theta\right)}$.

(b) $t = \dfrac{R}{v_{\text{i}}\cos\theta} = \dfrac{R}{\sqrt{2gd\sin\theta}\cos\theta}$

CALCULATE:

(a) $$R = \frac{-\tan(17°) \pm \sqrt{\tan^2(17°) + 55.0 \text{ m}/\left((29.0 \text{ m})\cos^2(17°)\sin(17°)\right)}}{1/\left(2(29.0 \text{ m})\cos^2(17°)\sin(17°)\right)}$$

$$= 36.8323 \text{ m or } -46.3148 \text{ m}$$

(b) $$t = \frac{36.8323 \text{ m}}{\left(\sqrt{2(9.81 \text{ m/s}^2)(29.0 \text{ m})\sin(17°)}\right)\cos(17°)} = 2.9862 \text{ s}$$

ROUND:

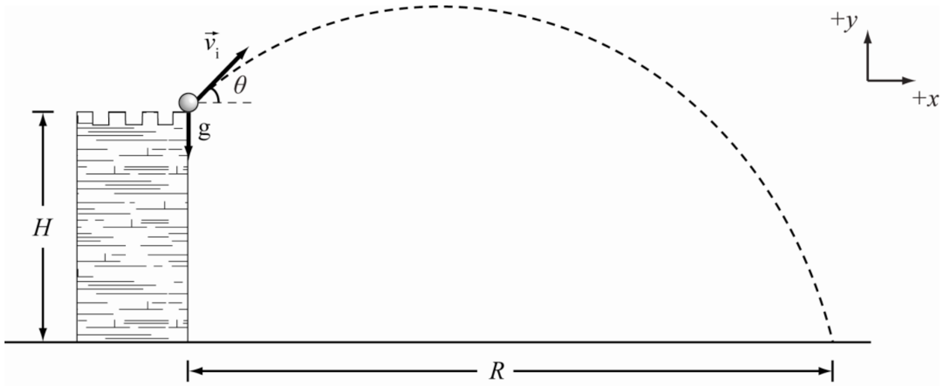
(a) The car falls 36.8 m from the base of the cliff.

(b) The car is in the air for 2.99 seconds.

DOUBLE-CHECK: If the car went over the cliff horizontally, it would travel about 43 m and take about 3.3 seconds to fall. The fact that the car travels a shorter distance in a shorter period of time with a small initial vertical velocity makes sense.

3.51. **THINK:** Though not explicitly stated, assume the launch angle is not zero. The projectile's height as a function of time is given. This function can be related to a more general function and the specifics of the motion can be determined. $g = 9.81 \text{ m/s}^2$ and $v_{\text{i}} = 20.0 \text{ m/s}$.

SKETCH:



RESEARCH: $y(t) = -4.90t^2 + 19.32t + 60.0$; $R = v_{ix}t$; $y_f - y_i = v_{iy}t + \frac{1}{2}at^2$; $v_{iy} = v_i \sin\theta$; and $v_{ix} = v_i \cos\theta$.

SIMPLIFY:

(a) $y_f = y_i + v_i \sin\theta t - \frac{1}{2}gt^2$, where $(1/2)g = 4.90 \text{ m/s}^2$, $v_i \sin\theta = 19.32 \text{ m/s}$, and $y_i = 60.0$ m.

(b) $\sin\theta = \dfrac{19.32 \text{ m/s}}{v_i} \Rightarrow \theta = \sin^{-1}\left(\dfrac{19.32}{v_i}\right)$

(c) $0 = y_i + v_i \sin\theta t - \frac{1}{2}gt^2 = y_i + R\tan\theta - \dfrac{gR^2}{2v_i^2\cos^2\theta}$ $\quad \left(\text{since } t = \dfrac{R}{v_i \cos\theta}\right)$

Therefore, $\left(\dfrac{g}{2v_i^2\cos^2\theta}\right)R^2 - (\tan\theta)R - y_i = 0$. Using the quadratic formula,

$$R = \frac{\tan\theta \pm \sqrt{\tan^2\theta + \dfrac{2y_i g}{v_i^2\cos^2\theta}}}{g/\left(v_i^2\cos^2\theta\right)}.$$

CALCULATE:

(a) $y_i = H = 60.0$ m

(b) $\theta = \sin^{-1}\left(\dfrac{19.32 \text{ m/s}}{20.0 \text{ m/s}}\right) = 75.02°$

(c) $$R = \frac{\tan(75.02°) \pm \sqrt{\tan^2(75.02°) + \dfrac{2(60.0 \text{ m})(9.81 \text{m/s}^2)}{(20.0 \text{ m/s})^2\cos^2(75.02°)}}}{\dfrac{9.81 \text{m/s}^2}{(20.0 \text{ m/s}^2)\cos^2(75.02°)}} = 30.9386 \text{ m or } -10.5715 \text{ m}$$

ROUND:

(a) The building height is 60.0. m.

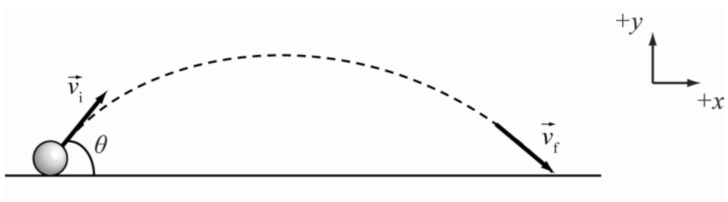
(b) The launch angle is 75.0°.

(c) The object travels 31.0 m (the positive value must be chosen).

DOUBLE-CHECK: Given the large launch angle, it makes sense that the object doesn't travel too far.

3.52. **THINK:** The x-component of the velocity will remain unchanged, while the y-component of the velocity will remain the same in magnitude but opposite in direction. $\Delta\vec{v} = -20\hat{y}$ m/s and $\theta = 60°$.

SKETCH:



RESEARCH: $\Delta\vec{v} = \vec{v}_f - \vec{v}_i$; $\vec{v} = v_x\hat{x} + v_y\hat{y}$; $v_{ix} = v_{fx}$; $v_{iy} = -v_{fy}$; $v_x = v\cos\theta$; and $v_y = v\sin\theta$.

SIMPLIFY: $\Delta\vec{v} = \left(v_{fx}\hat{x} + v_{fy}\hat{y}\right) - \left(v_{ix}\hat{x} + v_{iy}\hat{y}\right) = \left(v_{fx} - v_{ix}\right)\hat{x} + \left(v_{fy} - v_{iy}\right)\hat{y} = -2v_{iy}\hat{y} = -2v_i\sin\theta\,\hat{y}$

Therefore, $v_i = \dfrac{\Delta v}{-2\sin\theta}$ and $v_{ix} = v_i\cos\theta = \dfrac{\Delta v}{-2\tan\theta}$.

CALCULATE: $-2v_{iy} = -20$ m/s $\Rightarrow$ $v_{iy} = 10$ m/s. Therefore, $v_{fy} = -10$ m/s and

$$v_{ix} = \frac{-20 \text{ m/s}}{-2\tan\left(60°\right)} = 5.7735 \text{ m/s}.$$

ROUND: $\vec{v}_i = \left(5.8\hat{x} + 10\hat{y}\right)$ m/s and $\vec{v}_f = \left(5.8\hat{x} - 10\hat{y}\right)$ m/s.

DOUBLE-CHECK: Because the projectile is shot on level ground, it makes sense that the initial and final y velocities are equal in magnitude and opposite in direction. Since the angle is greater than 45°, it also makes sense that the vertical velocity is greater than the horizontal velocity.

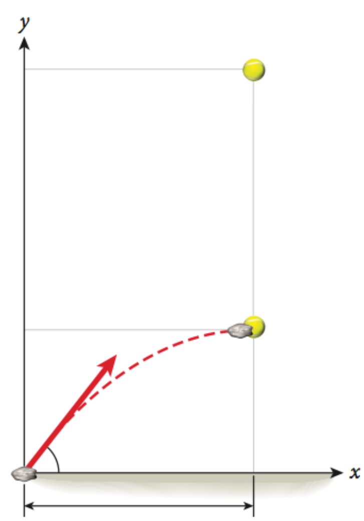
3.53. **THINK:** At its most basic this problem is just the "Shoot the Monkey" problem (Example 3.1) again. The tennis ball plays the role of the monkey, and the rock that of the tranquilizer dart.

The time and the position in space where the collision occurs are given, so the information about the tennis ball is irrelevant. But note that the initial height of the golf ball (h = 54.1 m) and the height (y = 10.0 m) and time (t = 3.00 s) at which the collision occurs are consistent, because the tennis ball drops a distance $\frac{1}{2}gt^2 = 0.5(9.81 \text{ m/s}^2)(3 \text{ s})^2 = 44.1$ m during 3.00 s, which is exactly $h - y$.

In order to calculate the horizontal velocity component of the rock, we have to keep in mind that it has to travel a given distance x in a given time. Note that the horizontal velocity component remains unchanged during the flight of the rock.

For the vertical velocity component, we know that the rock also has to travel a given distance y in the same time, but that it is in free-fall during that time period.

SKETCH:



RESEARCH: For the initial velocity components we use $x - x_0 = v_{x0}t$ and $y - y_0 = v_{y0}t - \frac{1}{2}gt^2$ with $x_0 = y_0 = 0$. For the final velocity components at the collision moment we use $v_x = v_{x0}$ and $v_y = v_{y0} - gt$.

SIMPLIFY: We solve the first two equations for the initial velocity components and find

$v_{x0} = x/t$ and $v_{y0} = (y + \frac{1}{2}gt^2)/t = y/t + \frac{1}{2}gt$

The two equations for the final velocity components are already in the form we can use for inserting numbers.

CALCULATE: $v_x = v_{x0} = \dfrac{50.0 \text{ m}}{3.00 \text{ s}} = 16.667 \text{ m/s},$

$$v_{y0} = \frac{10.0 \text{ m} + \frac{1}{2}(9.81 \text{ m/s}^2)(3.00 \text{ s})^2}{3.00 \text{ s}} = 18.0483 \text{ m/s},$$

$$|\vec{v}_0| = \sqrt{(16.667 \text{ m/s})^2 + (18.0483 \text{ m/s})^2} = 24.57 \text{ m/s}, \quad \theta_0 = \tan^{-1}\left(\frac{18.0483}{16.6667}\right) = 47.28°$$

$$v_y = 18.0483 \text{ m/s} - (9.81 \text{ m/s}^2)(3.00 \text{ s}) = -11.3817 \text{ m/s}$$

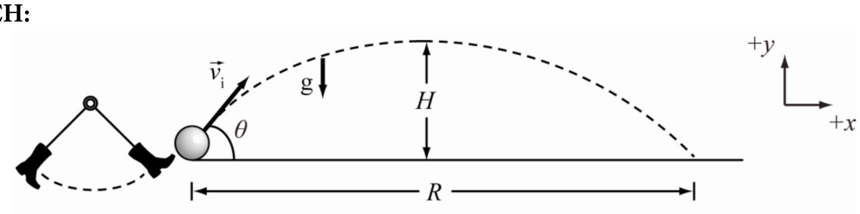
$$|\vec{v}| = \sqrt{(16.667 \text{ m/s})^2 + (-11.3817 \text{ m/s})^2} = 20.18 \text{ m/s}, \quad \theta = \tan^{-1}\left(\frac{-11.3817}{16.6667}\right) = -34.33°$$

ROUND: The initial velocity is v_0 = 24.6 m/s at 47.3° above horizontal. The final velocity is v = 20.2 m/s at 34.3° below horizontal.

DOUBLE-CHECK: In Example 3.1, "Shoot the Monkey", we learned that in order to hit the monkey during its free-fall, one has to aim directly in a straight line at it at the beginning. This means that the initial velocity vector has to point from the origin directly at the release point of the ball (x,h). This implies that the tangent of the initial velocity vector is given by $\tan v_0 \equiv v_{y0}/v_{x0} = h/x$. From the values given in the problem you can see that $h/x = (54.1 \text{ m})/(50.0 \text{ m}) = 1.08$, and our numerical answer yields $v_{y0}/v_{x0} = (18.0 \text{ m/s})/(16.7 \text{ m/s}) = 1.08$.

3.54. **THINK:** Since the initial and final heights are the same, the range and maximum height equations can be used. Assume the only acceleration is gravity so the horizontal velocity remains unchanged. When the golf ball is at the maximum height, its vertical velocity is zero. $g = 9.81 \text{ m/s}^2$, $\theta = 31.5°$ and $v_\text{i} = 11.2$ m/s.

SKETCH:



RESEARCH: $R = \dfrac{v_\text{i}^2 \sin(2\theta)}{g}$; $H = y_0 + \dfrac{v_\text{iy}^2}{2g}$; $v_\text{ix} = v_\text{fx} = v_\text{i}\cos\theta$; and $v_\text{iy} = v_\text{i}\sin\theta$.

SIMPLIFY:

(a) $R = \dfrac{v_\text{i}^2 \sin(2\theta)}{g}$

(b) $H = 0 + \dfrac{v_\text{i}^2 \sin^2\theta}{2g} = \dfrac{v_\text{i}^2 \sin^2\theta}{2g}$

(c) $\vec{v} = v_\text{ix}\hat{x} + 0\hat{y} = v_\text{i}\cos\theta\,\hat{x}$

(d) $\vec{a} = 0\hat{x} - g\hat{y} = -g\hat{y}$

CALCULATE:

(a) $R = \dfrac{(11.2\text{ m/s})^2 \sin(63°)}{9.81\text{ m/s}^2} = 11.393\text{ m}$

(b) $H = \dfrac{(11.2\text{ m/s})^2 \sin^2(31.5°)}{2(9.81\text{ m/s}^2)} = 1.7454\text{ m}$

(c) $v = (11.2\text{ m/s})\cos(31.5°)\hat{x} = 9.5496\hat{x}\text{ m/s}$

(d) $\vec{a} = -9.81\hat{y}\text{ m/s}^2$

ROUND:

(a) $R = 11.4\text{ m}$

(b) $H = 1.75\text{ m}$

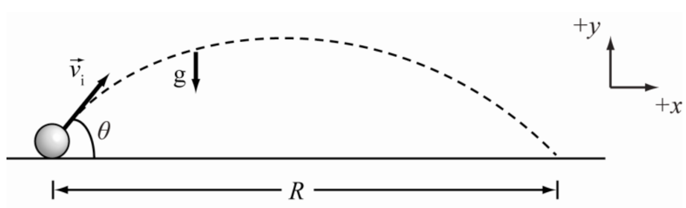
(c) $v = 9.55\hat{x}\text{ m/s}$

(d) $\vec{a} = -9.81\hat{y}\text{ m/s}^2$

DOUBLE-CHECK: Given the initial velocity and angle, the height and range are reasonable.

3.55. **THINK:** The question does not specify a launch angle. However, for maximum distance, the launch angle is $45°$. Assume the initial and final heights are the same so the range equation can be used. $R = 0.67$ km, $g = 9.81\text{ m/s}^2$ and $\theta = 45°$.

SKETCH:



RESEARCH: $R = \dfrac{v_i^2 \sin(2\theta)}{g}$

SIMPLIFY: $v_i = \sqrt{\dfrac{gR}{\sin(2\theta)}}$

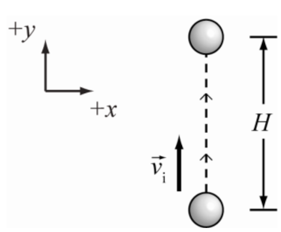
CALCULATE: $v_i = \sqrt{\dfrac{(9.81\text{ m/s}^2)(670\text{ m})}{\sin(2\cdot 45°)}} = 81.072\text{ m/s}$

ROUND: Rounding to two significant figures: $v_i = 81$ m/s.

DOUBLE-CHECK: This speed is equivalent to about 300 km/h, which seems reasonable for a catapult.

3.56. **THINK:** In order to attain maximum height, the launch angle must be exactly $90°$. The mass of the object is irrelevant to the kinematic equations. $v_i = 80.3\text{ m/s}$, $\theta = 90.0°$ and $g = 9.81\text{ m/s}^2$.

SKETCH:



RESEARCH: $H = y_0 + \dfrac{v_i^2 \sin^2\theta}{2g}$

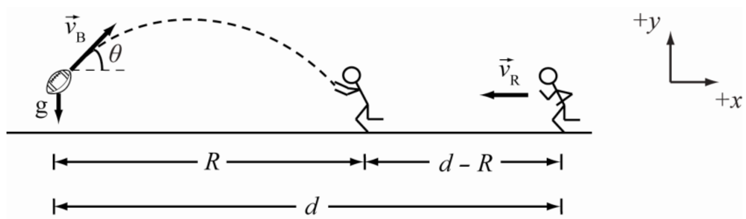
SIMPLIFY: $H = \dfrac{v_i^2 \sin^2\theta}{2g}$

CALCULATE: $H = \dfrac{(80.3\text{ m/s})^2 \sin^2(90.0°)}{2(9.81\text{ m/s}^2)} = 328.65\text{ m}$

ROUND: Rounding to three significant figures, $H = 329$ m.
DOUBLE-CHECK: Given the initial velocity, this height is reasonable.

3.57. **THINK:** Since the initial and final heights are the same, the range equation can be used to determine where the runner catches the ball. The time of flight of the ball must also be determined, thus the runner must run the remaining distance in the same amount of time. $v_B = 25.0$ m/s, $\theta = 35.0°$, d = 70.0 m and $g = 9.81\text{ m/s}^2$.

SKETCH:



RESEARCH: $R = \dfrac{v_i^2 \sin 2\theta}{g}$; $y_f - y_i = v_{iy}t + \dfrac{1}{2}at^2$; $v_{ix} = v_{fx}$; $v_{ix} = v_i \cos\theta$; $v_{iy} = v_i \sin\theta$.

SIMPLIFY: $R = \dfrac{v_B^2 \sin 2\theta}{g}$ and $0 = v_B \sin\theta t - \dfrac{1}{2}gt^2 \Rightarrow t = \dfrac{2v_B \sin\theta}{g}$.

$$v_R = \frac{d - R}{t} = \frac{d - \left(\dfrac{v_B^2 \sin 2\theta}{g}\right)}{\left(\dfrac{2v_B \sin\theta}{g}\right)} = \frac{dg}{2v_B \sin\theta} - v_B \cos\theta$$

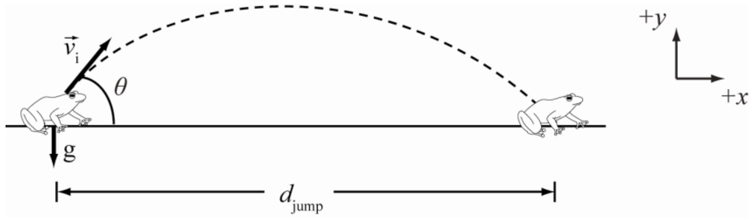
CALCULATE: $v_R = \dfrac{(70.0\text{ m})(9.81\text{ m/s}^2)}{2(25.0\text{ m/s})\sin(35.0°)} - (25.0\text{ m/s})\cos(35.0°) = 3.466\text{ m/s}$

ROUND: Rounding to three significant figures, $v_R = 3.47$ m/s.

DOUBLE-CHECK: The maximum speed a person can run at is around 12 m/s (see Chapter 1); so the result obtained here is possible.

3.58. **THINK:** If the frog is to maximize its distance, it must jump at an angle of 45°. Since the initial and final heights are the same, the range equation can be used to determine the velocity it jumps with. This velocity can then be used to determine the amount of time of one jump, and then the number of jumps in an hour. This means the total distance traveled is the number of jumps multiplied by the distance per jump. $d_{\text{jump}} = 1.3$ m, $g = 9.81\text{ m/s}^2$ and $\theta = 45°$. The total jump time is $0.8(1\text{ h}) = 2880$ s.

SKETCH:



RESEARCH: $d_{jump} = R_{max} = \dfrac{v_i^2}{g}$ and $v_{ix} = v_i \cos 45° = v_i / \sqrt{2} = \dfrac{d_{jump}}{t_{jump}}$.

SIMPLIFY: $v_i = \sqrt{g d_{jump}}$ and $v_{ix} = \sqrt{g d_{jump} / 2} = \dfrac{d_{jump}}{t_{jump}}$. Therefore,

$$t_{jump} = \frac{d_{jump}}{\sqrt{g d_{jump}/2}}.$$

The number of jumps, n, is equal to 2880 s / t_{jump}. This implies that the total distance, D, is

$$D = n d_{jump} = \frac{2880 \text{ s}}{t_{jump}} d_{jump} = d_{jump} \frac{2880 \text{ s}}{d_{jump}} \sqrt{g d_{jump}/2} = (2880 \text{ s})\sqrt{g d_{jump}/2}.$$

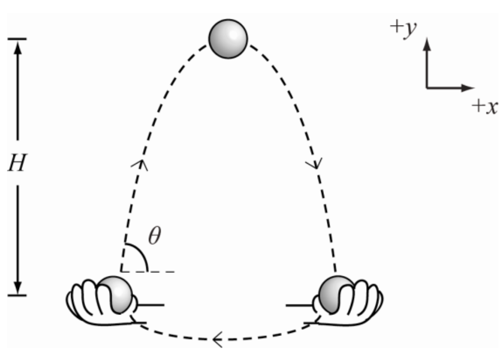
CALCULATE: $D = (2880 \text{ s})\sqrt{\left(9.81 \text{ m/s}^2\right)\left(1.3 \text{ m}\right)/2} = 7272.5 \text{ m}$

ROUND: The frog jumps a total of 7.3 km.

DOUBLE-CHECK: This result is the same distance a person walking briskly at 2 m/s would cover in an hour, so it is reasonable that a frog could cover the same distance in the same amount of time.

3.59. **THINK:** If the juggler has a ball in her left hand, he can also have one in his right hand, assuming her right hand is throwing the ball up. This means if the juggler has x number of balls, the minimum time between balls is 0.200 s. If the time between any two balls is less than 0.200 s, then the hands can't act fast enough and she'll drop the balls. The given information must be used to determine the time of flight, and thus the total time of one ball going around the loop. Then determine the largest integer, n, which can be multiplied by $t_{pass} = 0.200$ s which is still less than the total time. H = 90.0 cm, $\theta = 75.0°$ and $g = 9.81 \text{ m/s}^2$.

SKETCH:



RESEARCH: $H = y_i + \dfrac{v_i^2 \sin^2 \theta}{2g}$ and $y_f - y_i = v_{yi} t + \dfrac{1}{2} a t^2$.

SIMPLIFY: $H = \dfrac{\left(v_i \sin\theta\right)^2}{2g} \Rightarrow v_i \sin\theta = \sqrt{2gH}$. Use this when finding t_{throw}:

$$0 = v_i \sin\theta t_{throw} - \frac{1}{2} g t_{throw}^2 \Rightarrow t_{throw} = \frac{2 v_i \sin\theta}{g} = \frac{2\sqrt{2gH}}{g} = \sqrt{\frac{8H}{g}}$$

Therefore, $nt_{\text{pass}} \le \left(t_{\text{pass}} + t_{\text{throw}}\right) \Rightarrow n \le \dfrac{t_{\text{pass}} + t_{\text{throw}}}{t_{\text{pass}}} \Rightarrow n \le \dfrac{t_{\text{pass}} + \sqrt{\dfrac{8H}{g}}}{t_{\text{pass}}}$.

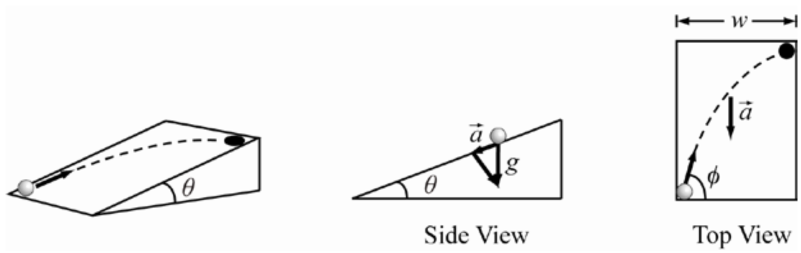
CALCULATE: $n \le \dfrac{0.2\text{ s} + \sqrt{\dfrac{8(0.9\text{ m})}{9.81\text{ m/s}^2}}}{0.2\text{ s}} = 5.2835$

ROUND: The maximum number of balls is $n = 5$.

DOUBLE-CHECK: It is reasonable for a person to juggle five balls at a time.

3.60. **THINK:** Since the plane is at an angle with the horizontal, it will tend to accelerate down the incline at only a fraction of gravity. Looking directly down the board, it can be considered as regular projectile motion with a vertical acceleration less than gravity. If the ball is to land in a hole on the opposite corner, the ball can't overshoot the corner, so the ball must be at its maximum height in the trajectory to make it in the hole. $\theta = 30.0°$, $\phi = 45.0°$, $w = 50.0$ cm, $g = 9.81\text{ m/s}^2$.

SKETCH:



RESEARCH: $a = g\sin\theta$; $R = \dfrac{v_{\text{i}}^2 \sin(2\phi)}{g}$; and $w = \dfrac{R}{2}$.

SIMPLIFY: $2w = \dfrac{v_{\text{i}}^2 \sin(2\phi)}{g\sin\theta} \Rightarrow v_{\text{i}} = \sqrt{\dfrac{2wg\sin\theta}{\sin(2\phi)}}$.

CALCULATE: $v_{\text{i}} = \sqrt{\dfrac{2(0.500\text{ m})(9.81\text{ m/s}^2)\sin(30.0°)}{\sin(90.0°)}} = 2.2147\text{ m/s}$

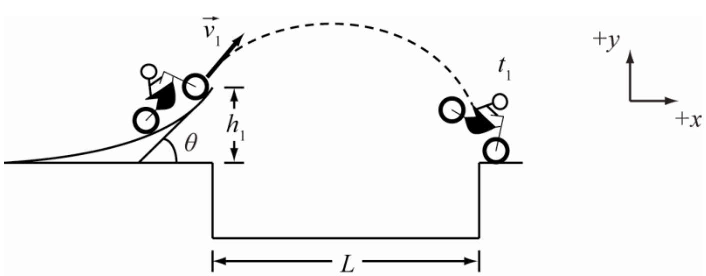
ROUND: $v_{\text{i}} = 2.21\text{ m/s}$

DOUBLE-CHECK: The speed is typical for a pinball game, and so we have confidence that our solution as the right order of magnitude.

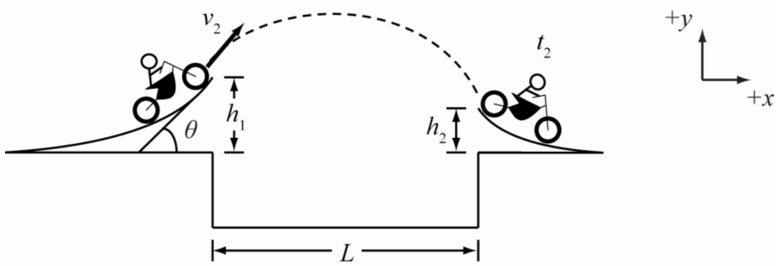
3.61. **THINK:** Ignoring air resistance, the biker's horizontal velocity will remain unchanged. Initially the biker lands on flat land and afterwards he tries again with a landing ramp on the edge. $L = 400.$ m, $\theta = 45.0°$, $h_1 = 8.00$ m, $h_2 = 3.00$ m and $g = 9.81\text{ m/s}^2$.

SKETCH:

(a)



(b)



RESEARCH: $v_{xi}t = x_f - x_i$; $y_f - y_i = v_{yi}t + \frac{1}{2}at^2$; $v_{xi} = v_i\cos\theta$; and $v_{yi} = v_i\sin\theta$.

SIMPLIFY:

(a) $\frac{L}{v_1\cos\theta} = t_1$ and $0 - h_1 = (v_1\sin\theta)t_1 - \frac{1}{2}gt_1^2$. Therefore, $-h_1 = L\tan\theta - \frac{gL^2}{2v_1^2\cos^2\theta}$ or

$\frac{gL^2}{2v_1^2\cos^2\theta} = L\tan\theta + h_1$. Solving for v_1 this gives $v_1 = \sqrt{\frac{gL^2}{2\cos^2\theta\left(L\tan\theta + h_1\right)}}$.

(b) $\frac{L}{v_2\cos\theta} = t_2$ and $h_2 - h_1 = (v_2\sin\theta)t_2 - \frac{1}{2}gt_2^2$. So, $h_2 - h_1 = L\tan\theta - \frac{gL^2}{2v_2^2\cos^2\theta}$. Solving for v_2 gives

$v_2 = \sqrt{\frac{gL^2}{2\cos^2\theta\left(L\tan\theta + h_1 - h_2\right)}}$.

CALCULATE:

(a) $v_1 = \sqrt{\frac{\left(9.81\text{ m/s}^2\right)\left(400.\text{ m}\right)^2}{2\cos^2\left(45.0°\right)\left(\left(400.\text{ m}\right)\tan\left(45.0°\right) + 8.00\text{ m}\right)}} = 62.025\text{ m/s}$

(b) $v_2 = \sqrt{\frac{\left(9.81\text{ m/s}^2\right)\left(400.\text{ m}\right)^2}{2\cos^2\left(45.0°\right)\left(\left(400.\text{ m}\right)\tan\left(45.0°\right) + 8.00\text{ m} - 3.00\text{ m}\right)}} = 62.254\text{ m/s}$

ROUND:

(a) $v_1 = 62.0\text{ m/s}$

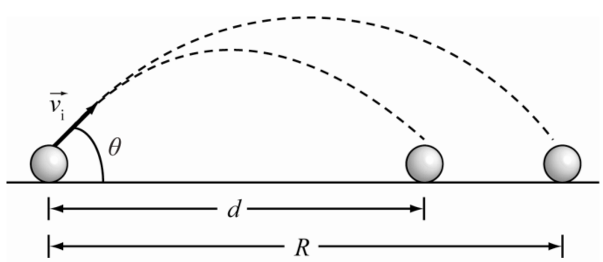
(b) $v_2 = 62.3\text{ m/s}$

DOUBLE-CHECK: The initial speed is approximately 220 km/h (around 140 mph), which is achievable with a motorcycle. Also note that we find that the required initial speeds for parts (a) and (b) are essentially the same. The change of 3.00 m does not matter much compared to 400. m.

3.62. **THINK:** Determine the distance the golf ball should travel, then compare to the distance it actually travels. Assume the initial and final heights of the ball are the same, so the range equation can be used. $\theta = 35.5°$, $d = 86.8$ m and $g = 9.81\text{ m/s}^2$.

$$v_i = 83.3\text{ mph}\cdot\frac{1.602\text{ km}}{1\text{ mile}}\cdot\frac{1\text{ h}}{3600\text{ s}}\cdot\frac{1000\text{ m}}{1\text{ km}} = 37.0685\text{ m/s}$$

SKETCH:



RESEARCH: $R = \dfrac{v_i^2 \sin 2\theta}{g}$ and $\Delta d = R - d$.

SIMPLIFY: It is not necessary to simplify.

CALCULATE: $R = \dfrac{(37.0685 \text{ m/s})^2 \sin(71.0°)}{9.81 \text{ m/s}^2} = 132.44 \text{ m}$

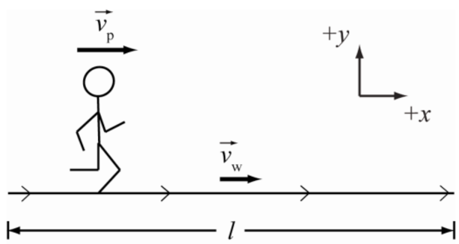
$\Delta d = 132.44 \text{ m} - 86.8 \text{ m} = 45.64 \text{ m}$

ROUND: Rounding to three significant figures, 45.6 m.

DOUBLE-CHECK: The ball lost about 1/3 of its distance to the wind. If you are a golfer or if you watch golf on TV, then you know that golf shots do not quite follow parabolic trajectories, and that the result found here is quite possible.

3.63. **THINK:** If an object is moving in a reference frame that is in motion, then to a stationary observer the object moves at a sum of the velocities. $l = 59.1$ m, $v_w = 1.77$ m/s and $v_p = 2.35$ m/s.

SKETCH:



RESEARCH: $x_f - x_i = v_x t$

SIMPLIFY: $t = \dfrac{l}{v_w + v_p}$

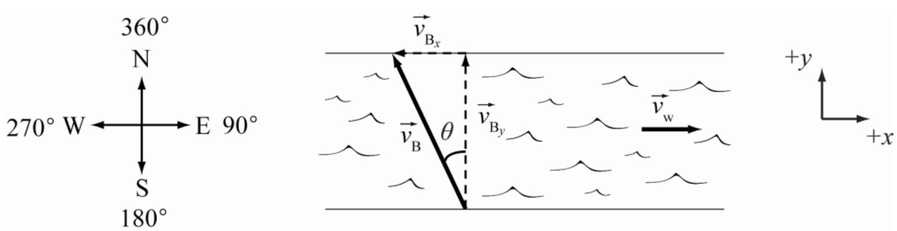
CALCULATE: $t = \dfrac{59.1 \text{ m}}{2.35 \text{ m/s} + 1.77 \text{ m/s}} = 14.345 \text{ s}$

ROUND: Rounding to three significant figures, $t = 14.3$ s.

DOUBLE-CHECK: Given the long length of the walkway and the slow speed, a large time is reasonable.

3.64. **THINK:** If the captain wants to get directly across the river, he must angle the boat so that the component of the boat's velocity that counters the river is the same in magnitude and opposite in direction. $v_w = 1.00$ m/s and $v_B = 6.10$ m/s.

SKETCH:



RESEARCH: $\vec{v} = v_x \hat{x} + v_y \hat{y}$; $\left|-\vec{v}_{Bx}\right| = \left|\vec{v}_w\right|$; $v_x = v \sin\theta$; and $v_y = v \cos\theta$.

SIMPLIFY: $\left|-\vec{v}_B\right| \sin\theta = \left|\vec{v}_w\right| \Rightarrow \theta = \sin^{-1}\left(\dfrac{v_w}{v_B}\right)$

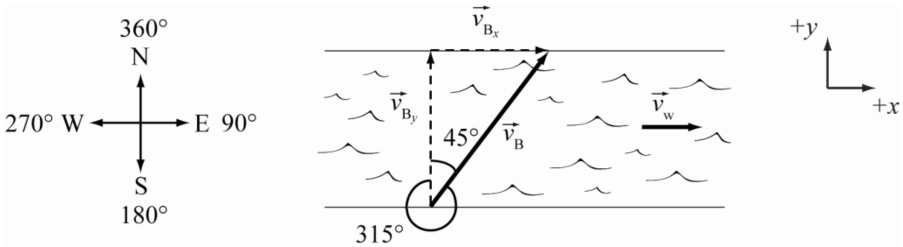
CALCULATE: $\theta = \sin^{-1}\left(\dfrac{1.00 \text{ m/s}}{6.10 \text{ m/s}}\right) = 9.4353°$

ROUND: Round to three significant figures, as the boat must travel at $9.44°$ west of north.

DOUBLE-CHECK: Since the velocity of the water is small compared to the boat's velocity, a small angle is expected.

3.65. **THINK:** If the captain wants to get directly across the river, he must angle the boat so that the component of the boat's velocity that counters the river is the same in magnitude and opposite in direction. $\theta = 315°$ and $v_B = 5.57$ m/s.

SKETCH:



RESEARCH: $\vec{v} = v_x\hat{x} + v_y\hat{y}$; $\left|-\vec{v}_{Bx}\right| = \left|\vec{v}_w\right|$; $v_x = v\sin\theta$; and $v_y = v\cos\theta$.

SIMPLIFY: $\left|\vec{v}_w\right| = \left|-\vec{v}_B\sin\theta\right|$

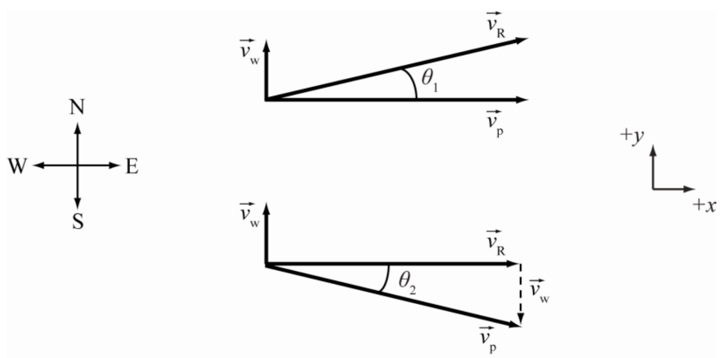
CALCULATE: $v_w = \left|(5.57 \text{ m/s})\sin(315°)\right| = 3.9386$ m/s

ROUND: Three significant figures: The river is flowing at 3.94 m/s.

DOUBLE-CHECK: Given the large angle that the boat makes with the river, it is reasonable that the water has this large velocity.

3.66. **THINK:** The components of the velocity vectors of the wind and the plane can be summed to determine a resultant vector of the plane's velocity with respect to the ground. If the pilot wants to travel directly east, the plane must travel in a direction such that the component of plane's velocity in the wind's direction is equal in magnitude to the wind speed. $v_p = 350.$ km/h and $v_w = 40.0$ km/h.

SKETCH:



RESEARCH: $\vec{v} = v_x\hat{x} + v_y\hat{y}$; $\left|\vec{v}\right| = \sqrt{v_x^2 + v_y^2}$; $\tan\theta = \left(\dfrac{v_y}{v_x}\right)$; and $\sin\theta = \left(\dfrac{v_y}{v_h}\right)$.

SIMPLIFY: $\vec{v}_R = v_p\hat{x} + v_w\hat{y} \Rightarrow \left|\vec{v}\right| = \sqrt{v_p^2 + v_w^2}$; $\theta_1 = \tan^{-1}\left(\dfrac{v_w}{v_p}\right)$; and $\theta_2 = \sin^{-1}\left(\dfrac{v_w}{v_p}\right)$.

CALCULATE:

$$\left|\vec{v}\right| = \sqrt{(350. \text{ km/h})^2 + (40.0 \text{ km/h})^2} = 352.28 \text{ km/h}$$

$$\theta_1 = \tan^{-1}\left(\frac{40.0 \text{ km/h}}{350. \text{km/h}}\right) = 6.5198°$$

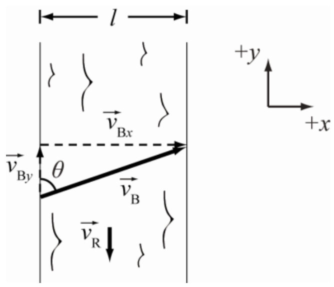
$$\theta_2 = \sin^{-1}\left(\frac{40.0\text{ km/h}}{350.\text{ km/h}}\right) = 6.5624°$$

ROUND: The plane has a velocity of 352 km/h at 6.52° north of east. To travel directly east, the plane must travel 6.56° south of east.

DOUBLE-CHECK: Given the small wind speed relative to the plane's speed, small angles are expected.

3.67. **THINK:** If the boaters want to travel directly over to the other side, they must angle the boat so that the component of the boat's velocity that counters the river is equal in magnitude. The time it takes to get over will then be based solely on the velocity perpendicular to the river flow. The minimum time to cross the river is when the boat is pointed exactly at the other side. Also, as long as the boat's velocity component is countering the river, any velocity in the perpendicular direction will get the boat across the river: l = 127 m, $v_B = 17.5$ m/s, and $v_R = 5.33$ m/s.

SKETCH:



RESEARCH: $\vec{v} = v_x\hat{x} + v_y\hat{y}$; $|v_{By}| = |-v_R|$; $v_x = v\sin\theta$; $v_y = v\cos\theta$; $x_f - x_i = v_x\Delta t$; $\tan\theta = \left(\frac{v_x}{v_y}\right)$; and $\cos\theta = \left(\frac{v_y}{v}\right)$. In part (e), the minimum speed, technically an infimum, will be when the angle θ is arbitrarily close to 0, and the component of the velocity directly across the stream is arbitrarily close to 0.

SIMPLIFY:

(a) $v_{By} = v_R \Rightarrow v_B\cos\theta = v_{By} \Rightarrow \theta = \cos^{-1}\left(\frac{v_R}{v_B}\right)$

(b) $l = (v_B\sin\theta)t \Rightarrow t = \frac{l}{v_B\sin\theta}$

(c) $\theta_{min} = 90°$

(d) $t_{min} = \frac{l}{v_B\sin\theta_{min}}$

(e) $\vec{v} \approx -\vec{v}_R$

CALCULATE:

(a) $\theta = \cos^{-1}\left(\frac{5.33\text{ m/s}}{17.5\text{ m/s}}\right) = 72.27°$

(b) $t = \frac{127\text{ m}}{(17.5\text{ m/s})\sin(72.27°)} = 7.619$ s

(c) $\theta = 90°$

(d) $t_{min} = \frac{127\text{ m}}{17.5\text{ m/s}} = 7.257$ s

(e) $v = 5.33$ m/s

ROUND:

(a) $\theta = 72.3°$

(b) $t = 7.62$ s

(c) $\theta = 90°$

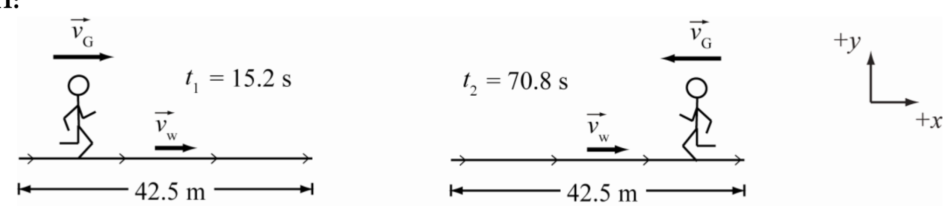
(d) $t = 7.26$ s

(e) $v = 5.33$ m/s

DOUBLE-CHECK: Given the width of the river and the velocities, these answers are reasonable.

3.68. **THINK:** When the girl walks in the same direction as the walkway, her velocity relative to the terminal is the sum of her velocity relative to the walkway and the velocity of the walkway relative to the terminal. When she walks in the opposite direction, her velocity relative to the terminal is the difference between her velocity relative to the walkway and the velocity of the walkway relative to the terminal. l = 42.5 m, $t_1 = 15.2$ s and $t_2 = 70.8$ s.

SKETCH:



RESEARCH: $x_f - x_i = v_x \Delta t$

SIMPLIFY: $l = v_x \Delta t \Rightarrow v_x = \frac{l}{t}$; $v_G + v_w = \frac{l}{t_1}$; and $v_G - v_w = \frac{l}{t_2}$.

$$v_w = \frac{l}{t_1} - v_G = v_G - \frac{l}{t_2} \Rightarrow l\left(\frac{1}{t_1} + \frac{1}{t_2}\right) = 2v_G \Rightarrow v_G = \frac{l}{2}\left(\frac{1}{t_1} + \frac{1}{t_2}\right)$$

Therefore,

$$v_w = \frac{l}{t_1} - \frac{l}{2}\left(\frac{1}{t_1} + \frac{1}{t_2}\right) = \frac{l}{2}\left(\frac{1}{t_1} - \frac{1}{t_2}\right).$$

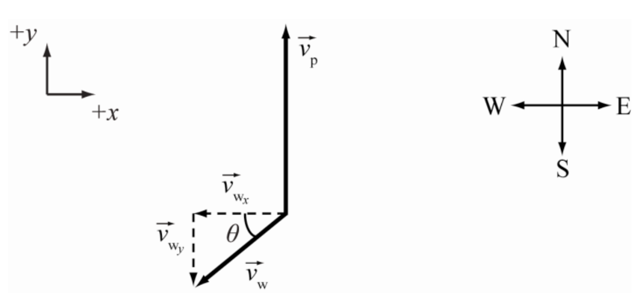
CALCULATE: $v_G = \frac{42.5 \text{ m}}{2}\left(\frac{1}{15.2 \text{ s}} + \frac{1}{70.8 \text{ s}}\right) = 1.698$ m/s, $\quad v_w = \frac{42.5 \text{ m}}{2}\left(\frac{1}{15.2 \text{ s}} - \frac{1}{70.8 \text{ s}}\right) = 1.0979$ m/s

ROUND: Rounding to three significant figures, $v_w = 1.70$ m/s and $v_G = 1.10$ m/s.

DOUBLE-CHECK: The velocities are small, which makes sense for a walkway.

3.69. **THINK:** Since the wind and plane velocities are vectors, simply add the components of the two vectors to determine the resultant vector. Southwest is 45° South of West. $v_p = 126.2$ m/s, $v_w = 55.5$ m/s and $\theta = 45°$.

SKETCH:



RESEARCH: $\vec{v} = v_x\hat{x} + v_y\hat{y}$; $v_x = v\cos\theta$; $v_y = v\sin\theta$; and $|\vec{v}| = \sqrt{{v_x}^2 + {v_y}^2}$.

SIMPLIFY: $v_x = v_w\cos\theta$ and $v_y = v_w\sin\theta$. Therefore, $\vec{v} = (-v_w\cos\theta)\hat{x} + (v_p - v_w\sin\theta)\hat{y}$ and $|\vec{v}| = \sqrt{(-v_w\cos\theta)^2 + (v_p - v_w\sin\theta)^2}$.

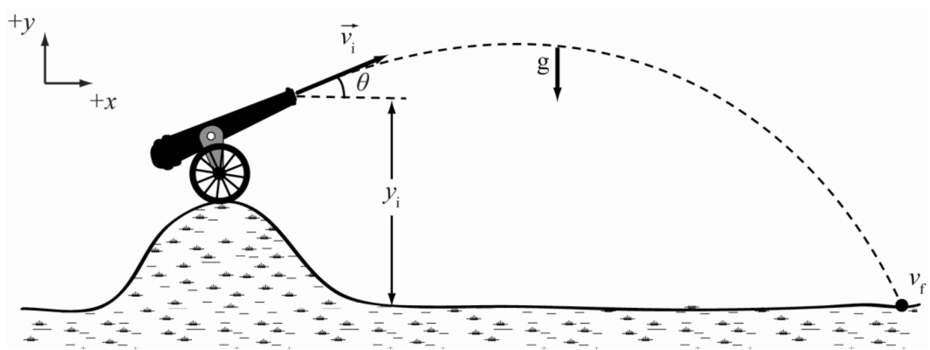
CALCULATE: $|\vec{v}| = \sqrt{\left(-55.5 \text{ m/s}\cos(45.0°)\right)^2 + \left(126.2 \text{ m/s} - 55.5 \text{ m/s}\sin(45.0°)\right)^2} = 95.401 \text{ m/s}$

ROUND: Rounding to three significant figures, $|\vec{v}| = 95.4$ m/s.

DOUBLE-CHECK: Given that the wind is blowing against the plane, the magnitude of the resultant velocity should be less than the plane's speed.

3.70. **THINK:** The horizontal velocity is constant. Determine the vertical velocity as it hits the ground and then determine the overall velocity. Mass is irrelevant. $y_i = 116.7$ m, $\theta = 22.7°$, $v_i = 36.1$ m/s and $g = 9.81 \text{ m/s}^2$.

SKETCH:



RESEARCH: $v_f^2 = v_i^2 + 2ad$; $v_x = v\cos\theta$; $v_y = v\sin\theta$; $v_{ix} = v_{fx}$; and $|\vec{v}| = \sqrt{v_x^2 + v_y^2}$.

SIMPLIFY: $v_{fx} = v_i\cos\theta$ and $v_{fy} = \sqrt{(v_i\sin\theta)^2 + 2gy_i}$. Also,
$|\vec{v}_f| = \sqrt{v_i^2\sin^2\theta + 2gy_i + v_i^2\cos^2\theta} = \sqrt{v_i^2 + 2gy_i}$.

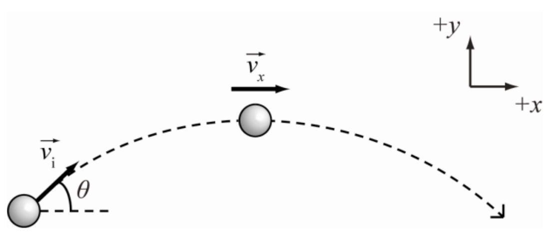
CALCULATE: $|\vec{v}_f| = \sqrt{(36.1 \text{ m/s})^2 + 2(9.81 \text{ m/s}^2)(116.7 \text{ m})} = 59.941 \text{ m/s}$

ROUND: Rounding to three significant figures, $|v_f| = 59.9$ m/s.

DOUBLE-CHECK: The final velocity is greater than the initial velocity, which makes sense.

3.71. **THINK:** The horizontal velocity remains unchanged at any point. $v_i = 31.1$ m/s and $\theta = 33.4°$.

SKETCH:



RESEARCH: $v_{ix} = v_{fx} = v_i\cos\theta$

SIMPLIFY: $v_x = v_i\cos\theta$

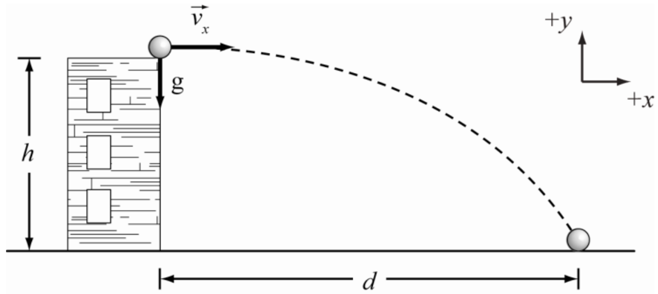
CALCULATE: $v_x = (31.1 \text{ m/s})\cos(33.4°) = 25.964 \text{ m/s}$

ROUND: Rounding to three significant figures, $v_x = 26.0$ m/s.

DOUBLE-CHECK: The result is smaller than the initial velocity, which makes sense.

3.72. **THINK:** Determine an equation that relates the height of the object being launched to how far horizontally it travels. $v_x = 10.1$ m/s , d = 57.1 m and $g = 9.81 \text{ m/s}^2$.

SKETCH:



RESEARCH: $v_x t = x_{\text{f}} - x_{\text{i}}$; $y_{\text{f}} - y_{\text{i}} = v_{\text{i}y} t + \frac{1}{2} a t^2$; and $v_{\text{i}y}\theta = v_{\text{i}} \sin$.

SIMPLIFY: $d = v_x t \;\Rightarrow\; t = \dfrac{d}{v_x}$ and $-h = 0 - \dfrac{1}{2} g t^2 \;\Rightarrow\; h = \dfrac{g d^2}{2 {v_x}^2}$.

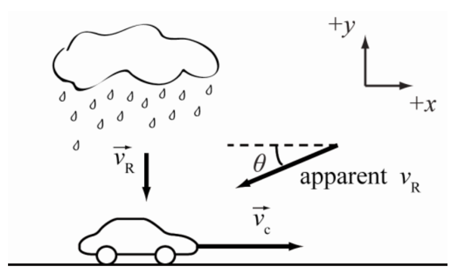
CALCULATE: $h = \dfrac{\left(9.81 \text{ m/s}^2\right)\left(57.1 \text{ m}\right)^2}{2\left(10.1 \text{ m/s}\right)^2} = 156.77 \text{ m}$

ROUND: Rounding to three significant figures, h = 157 m.

DOUBLE-CHECK: Given the velocity and distance, this height is reasonable.

3.73. **THINK:** If rain is falling straight down, to someone moving it would appear as if the rain was heading towards said person. $v_{\text{c}} = 19.3$ m/s and $v_{\text{R}} = 8.9$ m/s.

SKETCH:



RESEARCH: $\tan\theta = \left(\dfrac{v_y}{v_x}\right)$

SIMPLIFY: $\theta = \tan^{-1}\left(\dfrac{v_{\text{R}}}{v_{\text{c}}}\right)$

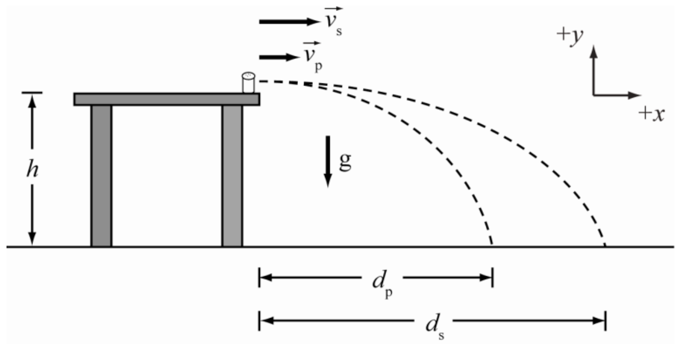
CALCULATE: $\theta = \tan^{-1}\left(\dfrac{8.9 \text{ m/s}}{19.3 \text{ m/s}}\right) = 24.756°$

ROUND: Rounding to two significant figures for angles gives the rain an angle of 25° with the horizontal.

DOUBLE-CHECK: The car is moving faster than the rain so this angle is reasonable.

3.74. **THINK:** Since both shakers leave the table with no initial vertical velocity, it should take both the same amount of time to hit the ground. $h = 0.85$ m, $v_\text{p} = 2.5$ m/s, $v_\text{s} = 5$ m/s and $g = 9.81$ m/s^2.

SKETCH:



RESEARCH: (a) The ratio of the times is 1:1 since the times are the same. (b) The ratio of the distances will be the ratio of the speeds.

SIMPLIFY:

(a) Not necessary.

(b) $d_\text{p} : d_\text{s} = v_\text{p} : v_\text{s}$

CALCULATE:

(a) 1:1

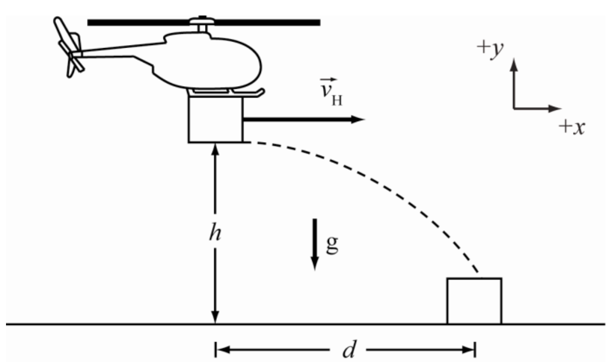
(b) $d_\text{p} : d_\text{s} = v_\text{p} : v_\text{s} = 2.5\,\text{m/s} : 5\,\text{m/s} = 1{:}2$

ROUND: Rounding is not necessary.

DOUBLE-CHECK: Since the horizontal component of the velocity is constant, and the initial horizontal velocity of the salt shaker is double that of the pepper shaker, it makes sense that the salt shaker travels twice as far as the pepper shaker.

3.75. **THINK:** Assuming the box has no parachute, at the time of drop it will have the same horizontal velocity as the velocity of the helicopter, $\vec{v}_\text{H}$, and this will remain constant throughout the fall. The initial vertical velocity is zero. The vertical component of the velocity will increase due to the acceleration of gravity. The final speed of the box at impact can be found from the horizontal and final vertical velocity of the box just before impact. $h = 500.$ m, $d = 150.$ m, and $g = 9.81$ m/s^2.

SKETCH:



RESEARCH: To find the speed of the helicopter we use: $y_\text{f} - y_\text{i} = v_{y\text{i}}t + \frac{1}{2}at^2$ and $x_\text{f} - x_\text{i} = v_x t$. To find the final speed of the box use $v_{\text{f}y} = v_{\text{i}y} + at$ as the vertical component, the helicopter speed as the horizontal component, and $|\vec{v}| = \sqrt{v_y^{\ 2} + v_x^{\ 2}}$ to find the final speed of the box when it hits the ground.

SIMPLIFY: $-h = 0 - \frac{1}{2}gt^2 \ \Rightarrow \ t = \sqrt{\frac{2h}{g}}$ and so $v_x = v_{\text{H}} = \frac{d}{t} = d\sqrt{\frac{g}{2h}}$.

$v_y = -gt = -g\sqrt{\frac{2h}{g}} = -\sqrt{2gh}$ and so $|\vec{v}| = \sqrt{\left(-\sqrt{2gh}\right)^2 + \left(d\sqrt{\frac{g}{2h}}\right)^2} = \sqrt{g\left(2h + d^2/[2h]\right)}$

CALCULATE: $v_{\text{H}} = (150.\text{ m})\sqrt{\frac{9.81\text{ m/s}^2}{2(500.\text{ m})}} = 14.857\text{ m/s}$

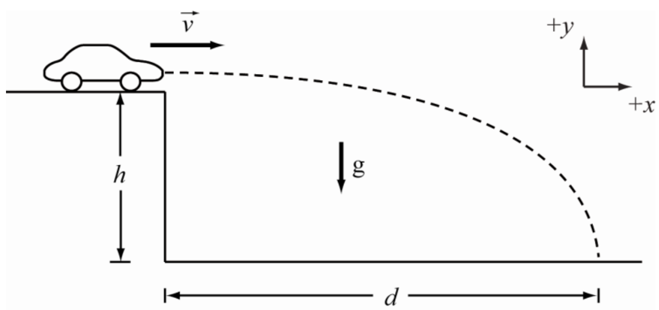
$|v| = \sqrt{9.81\text{ m/s}^2\left(2(500.\text{ m}) + (150.\text{ m})^2/\left[2(500.\text{ m})\right]\right)} = 100.15\text{ m/s}$

ROUND: $v_{\text{H}} = 14.9\text{ m/s}$ and $|\vec{v}| = 100.\text{ m/s}$.

DOUBLE-CHECK: The helicopter velocity is equivalent to about 50 km/h and the speed that the box has when it hits the ground is equivalent to about 360 km/h. Both are reasonable values.

3.76. **THINK:** Ignoring air resistance, the horizontal velocity remains constant. Assume the car had no initial vertical velocity when it went over the cliff. d = 150. m, h = 60.0 m, $g = 9.81\text{ m/s}^2$.

SKETCH:



RESEARCH: $x_{\text{f}} - x_{\text{i}} = v_x t$ and $y_{\text{f}} - y_{\text{i}} = v_{\text{i}y}t + \frac{1}{2}at^2$.

SIMPLIFY: $d = v_x t \ \Rightarrow \ v_x = \frac{d}{t}$ and $-h = 0 - \frac{1}{2}gt^2 \ \Rightarrow \ t = \sqrt{\frac{2h}{g}}$. Therefore, $v_x = d\sqrt{\frac{g}{2h}}$.

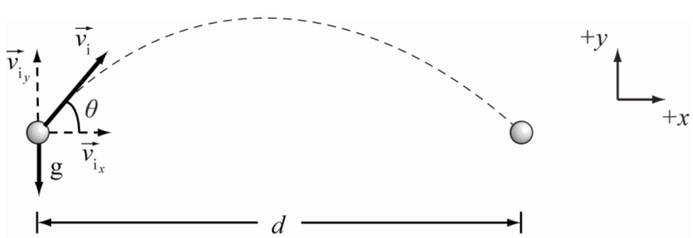
CALCULATE: $v_x = (150.\text{ m})\sqrt{\frac{9.81\text{ m/s}^2}{2(60.0\text{ m})}} = 42.888\text{ m/s}$

ROUND: $v_x = 42.9\text{ m/s}$

DOUBLE-CHECK: The vertical distance is smaller than the horizontal distance, so the car must have been going fast.

3.77. **THINK:** The initial horizontal velocity must be used to determine the time of flight, then this time can be used to determine the initial vertical velocity. $v_{\text{i}x} = 3.90\text{ m/s}$, d = 30.0 m, $g = 9.81\text{ m/s}^2$.

SKETCH:



RESEARCH: $x_f - x_i = v_x t$; $y_f - y_i = v_{iy} t + \frac{1}{2}at^2$; $v_{iy} = v_i \sin\theta$; and $v_{ix} = v_i \cos\theta$.

SIMPLIFY: $d = v_x t \Rightarrow t = \frac{d}{v_x}$ and so $0 = v_{iy} t - \frac{1}{2}gt^2 \Rightarrow v_{iy} = \frac{1}{2}gt = \frac{gd}{2v_x}$.

Also, $\frac{v_{iy}}{v_{ix}} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{v_{iy}}{v_{ix}}\right)$.

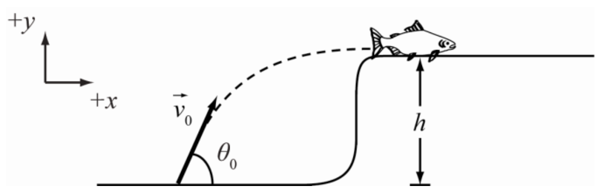
CALCULATE: $v_{iy} = \frac{(9.81 \text{ m/s}^2)(30.0 \text{ m})}{2(3.90 \text{ m/s})} = 37.73 \text{ m/s}, \quad \theta = \tan^{-1}\left(\frac{37.73 \text{ m/s}}{3.90 \text{ m/s}}\right) = 84.10°$

ROUND: The vertical velocity is 37.7 m/s and the launch angle is $84.1°$.

DOUBLE-CHECK: The exam bundle has a small horizontal velocity but travels 30 m, so the vertical velocity and launch angle must be large to allow the bundle to remain in the air longer.

3.78. **THINK:** Determine the initial speed of the jump v_0. The height of the waterfall is h = 1.05 m. The time taken for the jump is t = 2.1 s. The initial launch angle is $\theta_0 = 35°$.

SKETCH:



RESEARCH: Assuming an ideal parabolic trajectory, the kinematic equation for the vertical direction, $\Delta y = v_{y_0} t - \frac{1}{2}gt^2$ can be used to solve for v_0.

SIMPLIFY: $\Delta y = v_{y_0} t - \frac{1}{2}gt^2$ and $h = v_0 \sin\theta_0 t - \frac{1}{2}gt^2 \Rightarrow v_0 = \frac{h + \frac{1}{2}gt^2}{\sin\theta_0 t}$.

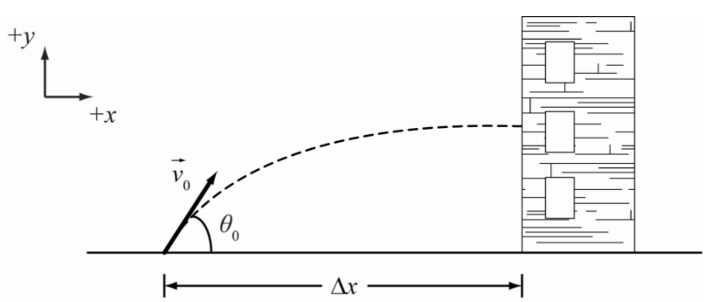
CALCULATE: $v_0 = \frac{1.05 \text{ m} + \frac{1}{2}(9.81 \text{ m/s}^2)(2.1 \text{ s})^2}{\sin(35°)(2.1 \text{ s})} = \frac{1.05 \text{ m} + 21.63 \text{ m}}{1.2045 \text{ s}} = 18.83 \text{ m/s}$

ROUND: There are two significant figures in both the sum in the numerator and the product in the denominator of the equation for v_0, so v_0 = 19 m/s.

DOUBLE-CHECK: Salmon migrate great distances (hundreds of kilometers) up rivers to spawn, and must overcome large obstacles to do so; this is a fast but reasonable speed for a salmon to exert on its journey.

3.79. **THINK:** Determine which floor of the building the water strikes (each floor is $h_f = 4.00$ m high). The horizontal distance between the firefighter and the building is $\Delta x = 60.0$ m. The initial angle of the water stream is $\theta_0 = 37.0°$. The initial speed is $v_0 = 40.3$ m/s.

SKETCH:



RESEARCH: To determine which floor the water strikes, the vertical displacement of the water with respect to the ground must be determined. The trajectory equation can be used, assuming ideal parabolic motion:

$$\Delta y = (\tan\theta_0)\Delta x - \frac{g(\Delta x)^2}{2v_0^2\cos^2\theta_0}.$$

The floor, at which the water strikes, n, is the total vertical displacement of the water divided by the height of each floor:

$$n = \frac{\Delta y}{h_\text{f}}.$$

SIMPLIFY: $n = \frac{\Delta y}{h_\text{f}} = \frac{1}{h_\text{f}}\left[(\tan\theta_0)\Delta x - \frac{g(\Delta x)^2}{2v_0^2\cos^2\theta_0}\right]$

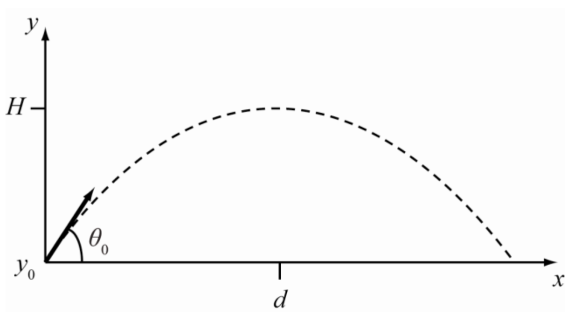
CALCULATE: $n = \frac{1}{4\text{ m}}\left[\tan(37°)(60\text{ m}) - \frac{(9.81\text{ m/s}^2)(60\text{ m})^2}{2(40.3\text{ m/s})^2\cos^2(37°)}\right] = 7.042$

ROUND: n = 7.042 floors above ground level (where n = 0) corresponds to ankle height on the 8th floor.

DOUBLE-CHECK: It is reasonable to expect the water from a high-pressure fire hose to reach the 8th floor of a building. Also, the answer is unitless, as it should be.

3.80. **THINK:** For a projectile, the launch angle is $\theta_0 = 68°$. In time t, it achieves a maximum height $y = H$ and travels a horizontal distance $\Delta x = d$. Assume $y_0 = 0$. Determine H/d.

SKETCH:



RESEARCH: Assuming ideal parabolic motion, the maximum height equation can be used,

$$H = y_0 + \frac{v_{y_0}^{\;2}}{2g}.$$

Note that at the maximum height, the horizontal distance traveled is half of the total range. That is, $d = R/2$ and thus, $R = \frac{v_0^2\sin 2\theta_0}{g}$.

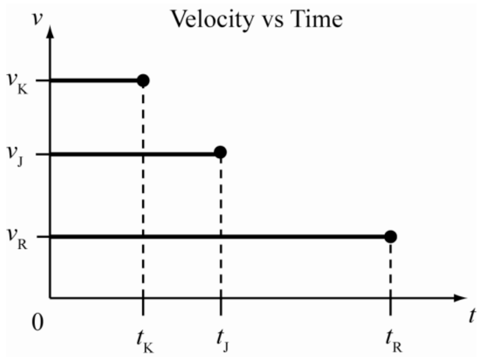
SIMPLIFY: $\frac{H}{d} = \frac{y_0 + v_{y_0}^2/(2g)}{R/2} = \frac{v_0^2\sin^2\theta_0}{2g(R/2)} = \frac{v_0^2\sin^2\theta_0}{g\left(\frac{v_0^2}{g}\sin 2\theta_0\right)} = \frac{\sin^2\theta_0}{\sin 2\theta_0} = \frac{\sin\theta_0\sin\theta_0}{2\sin\theta_0\cos\theta_0} = \frac{1}{2}\tan\theta_0$

CALCULATE: $\frac{H}{d} = \frac{1}{2}\tan 68° = 1.2375$

ROUND: The launch has two significant figures, so $H/d = 1.2$.

DOUBLE-CHECK: The ratio shows that the maximum height is 1.2 times the horizontal distance achieved in the same period of time. For such a steep launch angle, it is reasonable to expect $H > d$.

3.81. **THINK:** The time it takes Robert to cover the distance of the walkway is $t_R = 30.0$ s. John's time is $t_J = 13.0$ s. Kathy walks on the walkway at the same speed as Robert, $v_K = v_R$. Determine the time, t_K, that it takes Kathy to cover the distance of the walkway. Take the distance of the walkway to be Δx.
SKETCH:



RESEARCH: Kathy's velocity relative to the airport reference frame, v_{Ka}, is the sum of the walkway velocity, v_{wa}, and her velocity relative to the walkway, v_{Kw}. v_{wa} is actually John's velocity relative to the airport reference frame, v_{Ja}, and v_{Kw} is Robert's velocity relative to the airport, v_{Ra}. So, $v_{Ka} = v_{wa} + v_{Kw}$; $v_{wa} = v_{Ja}$; and $v_{Kw} = v_{Ra}$. The constant velocity equation $v = \Delta x / t$ can be used to find t_K.

SIMPLIFY: $v_{Ka} = \dfrac{\Delta x}{t_K}$, or $v_{Ka} = v_{wa} + v_{Kw} = v_{Ja} + v_{Ra} = \dfrac{\Delta x}{t_J} + \dfrac{\Delta x}{t_R}$. $\dfrac{\Delta x}{t_K} = \dfrac{\Delta x}{t_J} + \dfrac{\Delta x}{t_R}$ and $t_K = \left(\dfrac{1}{t_J} + \dfrac{1}{t_R}\right)^{-1}$.

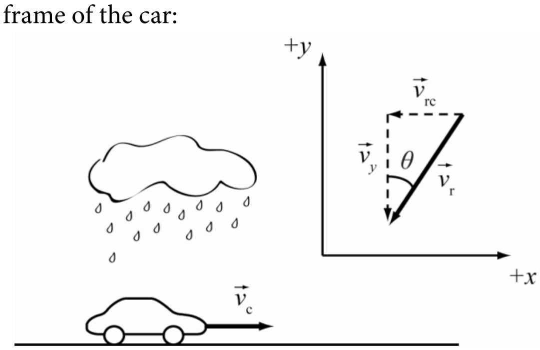
CALCULATE: $t_K = \left(\dfrac{1}{13.0 \text{ s}} + \dfrac{1}{30.0 \text{ s}}\right)^{-1} = 9.0698$ s

ROUND: Each time given has three significant figures, so $t_K = 9.07$ s.
DOUBLE-CHECK: Kathy's time should be the fastest.

3.82. **THINK:** The rain speed is $v_y = 7.00$ m/s downward. The car speed is

$$v_c = \left(\frac{60.0 \text{ km}}{1 \text{ h}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right)\left(\frac{1000\text{m}}{1 \text{ km}}\right) = 16.67 \text{ m/s}.$$

Relative to the car, the rain has a horizontal speed of $v_{rc} = 16.67$ m/s directed towards the car. Determine the angle, θ, from the vertical at which the rain appears to be falling relative to the traveling car.
SKETCH: In the reference frame of the car:



RESEARCH: The x and y components of the velocity of the rain, relative to the traveling car are known. These components make a right-triangle (shown above), such that $\tan\theta = v_{rc} / v_y$.

SIMPLIFY: $\theta = \tan^{-1}\left(\frac{v_{\text{rc}}}{v_y}\right)$

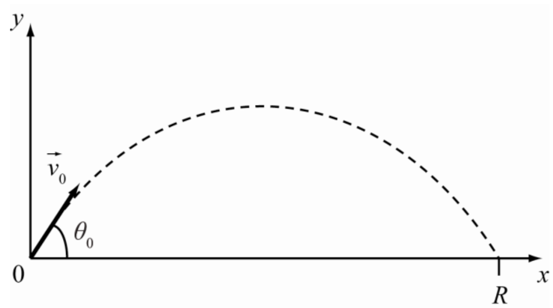
CALCULATE: $\theta = \tan^{-1}\left(\frac{16.67 \text{ m/s}}{7.00 \text{ m/s}}\right) = 67.22°$

ROUND: $\theta = 67.2°$ from the vertical.

DOUBLE-CHECK: This angle is less than $90°$ so it is reasonable that the rain is falling at such an angle.

3.83. **THINK:** Determine g when the range is R = 2165 m, $v_0 = 50.0$ m/s and $\theta_0 = 30.0°$.

SKETCH:



RESEARCH: Since the initial and final heights are equal, the range equation can be used:

$$R = \frac{v_0^2 \sin(2\theta_0)}{g}.$$

SIMPLIFY: $g = \frac{v_0^2 \sin(2\theta_0)}{R}$

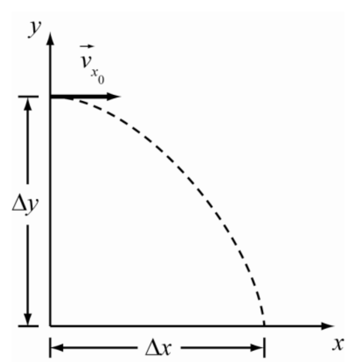
CALCULATE: $g = \frac{(50.0 \text{ m/s})^2 \sin(2(30.0°))}{2165 \text{ m}} = 1.00003 \text{ m/s}^2$

ROUND: $g = 1.00 \text{ m/s}^2$

DOUBLE-CHECK: The units of the calculated g are correct.

3.84. **THINK:** The height is $y_0 = 40.0$ m. The horizontal distance is $\Delta x = 7.00$ m. Determine the minimum initial horizontal speed v_{x_0}. Assume the diver does not jump up, but rather out ($v_{y_0} = 0$).

SKETCH:



RESEARCH: If $v_{y_0} = 0$ for the diver, the initial angle is also $\theta_0 = 0$. The trajectory equation can be used,

$$y = y_0 + (\tan\theta_0)\Delta x - \frac{g(\Delta x)^2}{2v_{x_0}^2}.$$

SIMPLIFY: Taking $y = 0$ (the base of the cliff),

$$0 = y_0 - \frac{g(\Delta x)^2}{2(v_{x_0})^2} \Rightarrow \frac{g(\Delta x)^2}{2v_{x_0}^2} = y_0 \Rightarrow v_{x_0} = \sqrt{\frac{g(\Delta x)^2}{2y_0}}.$$

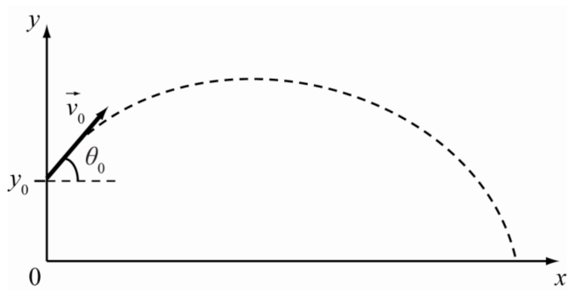
CALCULATE: $v_{x_0} = \sqrt{\frac{(9.81 \text{ m/s}^2)(7.00 \text{ m})^2}{2(40.0 \text{ m})}} = 2.451 \text{ m/s}$

ROUND: $v_{x_0} = 2.45 \text{ m/s}$

DOUBLE-CHECK: This is a reasonable velocity for a person to achieve at the start of their dive (i.e. a running start).

3.85. **THINK:** The initial velocity is $v_0 = 32.0 \text{ m/s}$, the launch angle is $\theta_0 = 23.0°$ and the initial height is $y_0 = 1.83 \text{ m}$. Determine the travel time t for the ball before it hits the ground at $y = 0$.

SKETCH:



RESEARCH: Assuming an ideal parabolic trajectory, the kinematic equation can be used for the vertical displacement, $y - y_0 = (v_0 \sin\theta_0)t - (1/2)gt^2$.

SIMPLIFY: $y - y_0 = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 \Rightarrow \frac{1}{2}gt^2 - (v_0 \sin\theta_0)t + y - y_0 = 0$

Solve for t using the quadratic formula:

$$t = \frac{v_0\sin\theta_0 \pm \sqrt{(v_0\sin\theta_0)^2 - 4\left(\frac{1}{2}g\right)(y - y_0)}}{2\left(\frac{1}{2}g\right)} = \frac{v_0\sin\theta_0 \pm \sqrt{(v_0\sin\theta_0)^2 + 2g(y_0 - y)}}{g}$$

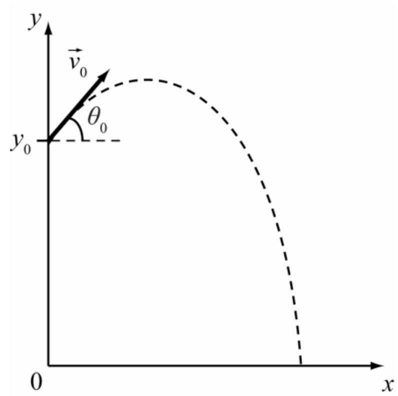
CALCULATE:
$$t = \frac{(32 \text{ m/s})\sin(23°) \pm \sqrt{\left((32 \text{ m/s})\sin(23°)\right)^2 + \left(2(9.81 \text{ m/s}^2)\right)(1.83 \text{ m} - 0)}}{9.81 \text{ m/s}^2}$$
$$= \frac{12.5034 \text{ m/s} \pm \sqrt{156.33 \text{ m}^2/\text{s}^2 + 35.90 \text{ m}^2/\text{s}^2}}{9.81 \text{ m/s}^2}$$
$$= 2.6879 \text{ s or } -0.1388 \text{ s}$$

ROUND: Round to three significant figures. Then, $t = 2.69$ s (choosing the positive solution).

DOUBLE-CHECK: The units of the result are units of time. This seems to be a reasonable flight time for a thrown baseball.

3.86. **THINK:** For the rock, the initial height is $y_0 = 34.9 \text{ m}$, the initial speed is $v_0 = 29.3 \text{ m/s}$ and the launch angle is $\theta_0 = 29.9°$. Determine the speed of the rock, v, when it hits the ground at the bottom of the cliff, $y = 0$.

SKETCH:



RESEARCH: Assuming an ideal parabolic trajectory, when the rock descends to the height of the cliff after reaching its maximum height, it has the same vertical speed as it was launched with, only directed downward. That is, $v_y' = -v_{y_0}$, where v_y' occurs on the trajectory at the vertical position y_0. In addition, $v_{y_0} = v_0 \sin\theta_0$. Since, due to gravity, the speed changes only in the vertical direction, the constant acceleration equation, $v_y^2 = v_{y_0}^2 + 2a_y\left(y - y_0\right)$, can be used to determine the vertical speed at the bottom of the cliff. As the horizontal speed v_x remains constant $\left(v_x = v_{0x} = v_0\cos\theta_0\right)$, the speed at the bottom is $v = \sqrt{v_x^2 + v_y^2}$.

SIMPLIFY: Take the initial vertical speed to be v_y'. Then

$$v_y^2 = \left(v_y'\right)^2 - 2g\left(-y_0\right) \Rightarrow v_y^2 = \left(-v_0\sin\theta_0\right)^2 + 2gy_0$$

Also, $v_x = v_0\cos\theta_0$. Then,

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(v_0\cos\theta_0\right)^2 + \left(-v_0\sin\theta_0\right)^2 + 2gy_0} = \sqrt{v_0^2\left(\cos^2\theta_0 + \sin^2\theta_0\right) + 2gy_0} = \sqrt{v_0^2 + 2gy_0}.$$

CALCULATE: $v = \sqrt{\left(29.3 \text{ m/s}\right)^2 + 2\left(9.81 \text{ m/s}^2\right)34.9 \text{ m}} = \sqrt{858.49 \text{ m}^2/\text{s}^2 + 684.74 \text{ m}^2/\text{s}^2}$

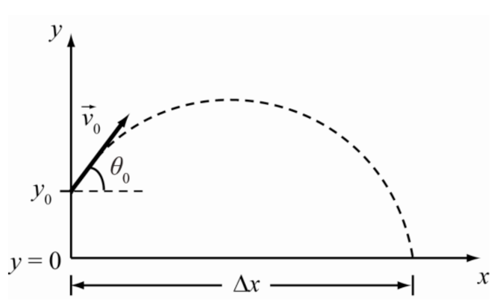
$= \sqrt{1543.2 \text{ m}^2/\text{s}^2} = 39.284 \text{ m/s}$

ROUND: The sum in the square root is precise to the ones-place and therefore has four significant figures. Thus, $v = 39.28$ m/s.

DOUBLE-CHECK: This velocity is greater than v_0. It should be, given the acceleration in the vertical direction due to gravity.

3.87. **THINK:** For the shot-put, the initial speed is $v_0 = 13.0$ m/s, the launch angle is $\theta_0 = 43.0°$ and the initial height is $y_0 = 2.00$ m. Determine (a) the horizontal displacement Δx and (b) the flight time *t*, after the shot hits the ground at $y = 0$.

SKETCH:



RESEARCH: Assuming ideal parabolic motion, find (a) Δx from the trajectory equation:

$$y = y_0 + \Delta x \tan\theta_0 - \frac{g(\Delta x)^2}{2v_0^2\cos^2\theta_0}$$

and (b) t from the equation $\Delta x = v_0 t\cos\theta_0$.

SIMPLIFY:

(a) With $y = 0$, $\frac{g}{2v_0^2\cos^2\theta_0}(\Delta x)^2 - (\tan\theta_0)\Delta x - y_0 = 0.$ Solving this quadratic equation yields:

$$\Delta x = \frac{\tan\theta_0 \pm \sqrt{\tan^2\theta_0 - 4\left(\frac{g}{2v_0^2\cos^2\theta_0}\right)(-y_0)}}{2\left(\frac{g}{2v_0^2\cos^2\theta_0}\right)} = \left(\tan\theta_0 \pm \sqrt{\tan^2\theta_0 + \left(\frac{2gy_0}{v_0^2\cos^2\theta_0}\right)}\right)\left(\frac{v_0^2\cos^2\theta_0}{g}\right).$$

(b) $t = \frac{\Delta x}{v_0\cos\theta_0}$

CALCULATE:

(a) $$\Delta x = \left(\tan(43°) \pm \sqrt{\tan^2(43°) + \left(\frac{2(9.81 \text{ m/s}^2)(2 \text{ m})}{(13.0 \text{ m/s})^2\cos^2(43°)}\right)}\right)\left(\frac{(13.0 \text{ m/s})^2\cos^2 43°}{9.81 \text{ m/s}^2}\right)$$

$$= \left(0.9325 \pm \sqrt{0.8696 + 0.4341}\right)(9.2145 \text{ m})$$

$$= \left(0.9325 \pm \sqrt{1.3037}\right)(9.2145 \text{ m})$$

$$= 19.114 \text{ m or } -1.928 \text{ m}$$

(b) $t = \frac{19.114 \text{ m}}{(13.0 \text{ m/s})\cos(43°)} = 2.0104 \text{ s}$

ROUND:

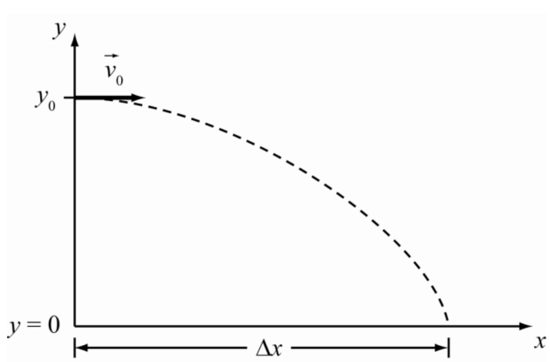
(a) The sum under the square root is precise to the tenth-place, and so has three significant figures. Then, $\Delta x = 19.1$ m (take the positive root).

(b) Since θ_0 and Δx have three significant figures, $t = 2.01$ s.

DOUBLE-CHECK: For near optimal launch angle (optimal being $\theta_0 = 45°$), a horizontal displacement of 19.1 m is reasonable. The flight time of 2.0 s is reasonable for this horizontal displacement.

3.88. **THINK:** For the phone, the initial height is $y_0 = 71.8$ m , the launch angle is $\theta_0 = 0°$ and the initial speed is $v_0 = 23.7$ m/s. Determine (a) the horizontal distance Δx and (b) the final speed v upon hitting the water at $y = 0$. Note: $v_{y_0} = 0$.

SKETCH:



RESEARCH: Assuming ideal parabolic motion,

(a) Δx can be determined from the trajectory equation, $y = y_0 + \tan\theta_0 \Delta x - \dfrac{g\Delta x^2}{2v_0^2 \cos^2\theta_0}$.

(b) $|\vec{v}| = \sqrt{v_x^2 + v_y^2}$. v_x does not change, as there is no horizontal acceleration; $v_x = v_{x_0} = v_0 \cos\theta_0$. Since the vertical kinematics are governed by free-fall under gravity, to find v_y use $v_y^2 = v_{y_0}^2 - 2g(y - y_0)$.

SIMPLIFY:

(a) With $y = 0$, the trajectory equation becomes

$$\frac{g\Delta x^2}{2v_0^2 \cos^2\theta_0} - \tan\theta_0 \Delta x - y_0 = 0.$$

With $\theta_0 = 0°$, this equation reduces to $\dfrac{g\Delta x^2}{2v_0^2} - y_0 = 0$. Thus, $\Delta x = \sqrt{\dfrac{2v_0^2 y_0}{g}}$.

(b) $v_x = v_0$. Since $v_{y_0} = 0$ and $y = 0$, $v_y^2 = 2gy_0$. Then, $|\vec{v}| = \sqrt{v_0^2 + 2gy_0}$.

CALCULATE:

(a) $\Delta x = \sqrt{\dfrac{2(23.7 \text{ m/s})^2(71.8 \text{ m})}{9.81 \text{ m/s}^2}} = 90.676 \text{ m}$

(b) $|\vec{v}| = \sqrt{(23.7 \text{ m/s})^2 + 2(9.81 \text{ m/s}^2)(71.8 \text{ m})} = \sqrt{561.69 \text{ m}^2/\text{s}^2 + 1408.716 \text{ m}^2/\text{s}^2} = 44.389 \text{ m/s}$

ROUND:

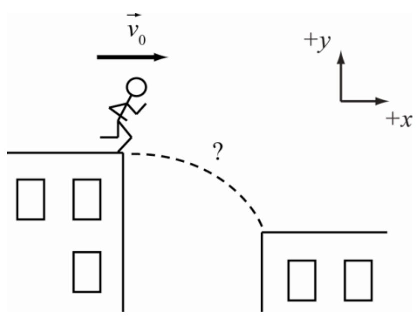
(a) All the values used in the product have three significant figures, so $\Delta x = 90.7$ m.

(b) The sum under the square root is precise to the tens-place, and has three significant figures, so $v = 44.4$ m/s.

DOUBLE-CHECK: The horizontal displacement is close to the height; this is reasonable for an object thrown from the given height. The final speed is greater than the initial speed, as it should be.

3.89. **THINK:** For the burglar, the initial speed is $v_0 = 4.20$ m/s and the angle is $\theta_0 = 0°$. Determine if the burglar can make it to the next roof, which is a horizontal distance of 5.50 m away and a vertical distance of 4.00 m lower.

SKETCH:



RESEARCH: There are two ways to approach this problem. Firstly, the vertical displacement can be determined when the horizontal displacement is $\Delta x = 5.50$ m. If the magnitude of Δy is less than 4.0 m, the burglar will reach the next rooftop. Secondly, the horizontal displacement Δx can be determined when the vertical displacement is $\Delta y = -4.00$ m. For this solution, Δy will be determined. The trajectory equation for ideal parabolic motion can be used:

$$y = y_0 + \tan\theta_0 \Delta x - \frac{g\Delta x^2}{2v_0^2 \cos^2\theta_0}.$$

SIMPLIFY: With $\theta_0 = 0°$, $y = -\dfrac{g\Delta x^2}{2v_0^{\,2}}$.

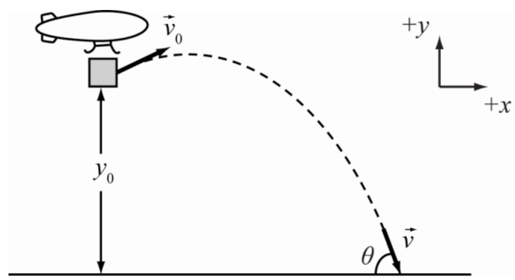
CALCULATE: $y = -\dfrac{(9.81\text{ m/s}^2)(5.5\text{ m})^2}{2(4.2\text{ m/s})^2} = -8.4114\text{ m}$

ROUND: The values used in the product have three significant figures, so $y = -8.41$ m. This means that after the burglar reaches a horizontal displacement of 5.50 m, he has dropped 8.41 m from the first rooftop and cannot reach the second rooftop.

DOUBLE-CHECK: The alternate method listed above also reveals that the burglar will not reach the second rooftop.

3.90. **THINK:** The initial vertical speed of the package is $v_{y_0} = 7.50\text{ m/s}$. Its horizontal speed is $v_{x_0} = v_x = 4.70\text{ m/s}$ and the initial height is $y_0 = 80.0$ m. Determine (a) the fall time, *t*, to the ground, *y* = 0 and (b) the magnitude and direction of the velocity upon impact.

SKETCH:



RESEARCH:

(a) To determine *t*, use the equation $y - y_0 = v_{y_0}t - (gt^2)/2$.

(b) The speed *v* is given by $v = \sqrt{v_x^{\,2} + v_y^{\,2}}$. v_x remains constant. To determine v_y, use the constant acceleration equation, $v_y^{\,2} = v_{y_0}^{\,2} + 2a\Delta y$. To determine the direction, note that $\tan\theta = v_y / v_x$, where θ is above the horizontal.

SIMPLIFY:

(a) With $y = 0$, $-y_0 = v_{y_0}t - \dfrac{gt^2}{2} \Rightarrow \dfrac{gt^2}{2} - v_{y_0}t - y_0 = 0$. Solving the quadratic gives:

$$t = \frac{v_{y_0} \pm \sqrt{v_{y_0}^{\,2} - 4\left(\frac{g}{2}\right)(-y_0)}}{2\left(\frac{g}{2}\right)} = \frac{v_{y_0} \pm \sqrt{v_{y_0}^{\,2} + 2gy_0}}{g}.$$

(b) $v_y^{\,2} = v_{y_0}^{\,2} - 2g(y - y_0) = v_{y_0}^{\,2} + 2gy_0$. Then, $v = \sqrt{v_x^{\,2} + v_{y0}^{\,2} + 2gy_0}$, and

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{-\sqrt{v_{y_0}^{\,2} + 2gy_0}}{v_x}\right),$$

where theta is measured with respect to the +*x*-axis and therefore comes out negative.

CALCULATE:

(a) $t = \dfrac{7.50\text{ m/s} \pm \sqrt{(7.50\text{ m/s})^2 + 2(9.81\text{ m/s}^2)(80.0\text{ m})}}{9.81\text{ m/s}^2} = 4.875\text{ s} \quad\text{or}\quad -3.346\text{ s}$

(b) $v = \sqrt{(4.70 \text{ m/s})^2 + (7.50 \text{ m/s})^2 + 2(9.81 \text{ m/s}^2)(80.0 \text{ m})} = 40.59 \text{ m/s}$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{(7.50 \text{ m/s})^2 + 2(9.81 \text{ m/s}^2)(80.0 \text{ m})}}{(4.70 \text{ m/s})}\right) = -83.35°$$

ROUND: $t = 4.88$ s and $\theta = -83.4°$.

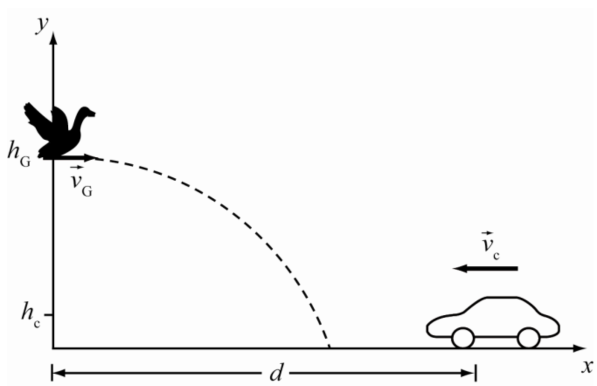
DOUBLE-CHECK: A five second fall is reasonable for such a height. The final speed is greater than the initial speed $v_0 = \sqrt{{v_{x_0}}^2 + {v_{y_0}}^2} = 8.9$ m/s. The impact angle is almost $-90°$, as it should be since $|v_y|$ is much greater than $|v_x|$ after the fall.

3.91. **THINK:** The height of the goose is $h_g = 30.0$ m. The height of the windshield is $h_c = 1.00$ m. The speed of the goose is $v_g = 15.0$ m/s. The speed of the car is

$$v_c = \frac{100.0 \text{ km}}{1 \text{ hr}} \cdot \frac{1 \text{ h}}{3600 \text{ s}} \cdot \frac{1000 \text{ m}}{1 \text{ km}} = 27.7778 \text{ m/s}.$$

The initial horizontal distance between the goose and the car is $d = 104.0$ m. The goose and the car move toward each other. Determine if (a) the egg hits the windshield and (b) the relative velocity of the egg with respect to the windshield, v_g'. Let θ be the angle the egg makes with the horizontal when it impacts.

SKETCH:



RESEARCH:

(a) The egg must have a vertical displacement $\Delta y_g = h_c - h_g$ to hit the windshield. Use $\Delta y_g = (v_0 \sin\theta_0)t - \frac{1}{2}(gt^2)$ to determine the fall time *t*. Use $\Delta x_g = (v_0 \cos\theta_0)t$ to determine the horizontal displacement of the egg. In order for the egg to collide with the windshield, the car must travel $\Delta x_c = d - \Delta x_g$ in time *t*. Use $\Delta x = vt$ to determine the car's travel distance Δx_c. Note that the launch angle of the egg is $\theta_0 = 0°$.

(b) The horizontal component of the egg's speed in the reference frame of the windshield will be $v_{gx}' = v_{gx} + v_c$ because the car is moving toward the egg in the horizontal direction. Because the car has no vertical speed, the vertical speed of the egg in the reference frame of the car is unchanged, $v_{gy}' = v_{gy}$. To determine v_{gy}, use ${v_y}^2 = {v_{y_0}}^2 + 2a\Delta y$. Then $v_g' = \sqrt{(v_{gx}')^2 + (v_{gy}')^2}$. The angle of impact is implicitly given by $\tan\theta = \frac{v'_{gy}}{v'_{gx}}$.

SIMPLIFY:

(a) Since $\theta_0 = 0°$ and $v_0 = v_g$, $v_{gy_0} = v_g \sin\theta_0 = 0$ and $v_{gx_0} = v_g \cos\theta_0 = v_g$. Then

$$\Delta y_g = -\frac{1}{2}gt^2 \Rightarrow h_c - h_g = -\frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2(h_g - h_c)}{g}},$$

and also $\Delta x_g = v_g t = v_g\sqrt{\frac{2(h_g - h_c)}{g}}$. In this time, the car travels $\Delta x_c = v_c t = v_c\sqrt{\frac{2(h_g - h_c)}{g}}$.

(b) $v_{gx}' = v_{gx} + v_c = v_g + v_c$ and $v_{gy}' = v_{gy} = \sqrt{2g\Delta y_g} = \sqrt{2g(h_g - h_c)}$. Then, substituting gives: $v_g' = \sqrt{(v_g + v_c)^2 + 2g(h_g - h_c)}$. The angle of impact relative to the car is given by the equation: $\theta = \arctan\left(\frac{v_{gy}}{v_{gx}'}\right) = \arctan\left(\frac{\sqrt{2g(h_g - h_c)}}{v_g + v_c}\right)$.

CALCULATE:

(a) $\Delta x_c = (27.778 \text{ m/s})\sqrt{\frac{2(30.0 \text{ m} - 1.00 \text{ m})}{9.81 \text{ m/s}^2}} = 67.542 \text{ m}$

$$d - \Delta x_g = d - v_g\sqrt{\frac{2(h_g - h_c)}{g}} = (104.0 \text{ m}) - (15.0 \text{ m/s})\sqrt{\frac{2(30.0 \text{ m} - 1.00 \text{ m})}{9.81 \text{ m/s}^2}} = 67.527 \text{ m}$$

(b) $v_g' = \sqrt{(15.0 \text{ m/s} + 27.7778 \text{ m/s})^2 + 2(9.81 \text{ m/s}^2)(30.0 \text{ m} - 1.00 \text{ m})} = \sqrt{2398.94 \text{ m}^2/\text{s}^2} = 48.98 \text{ m/s}$

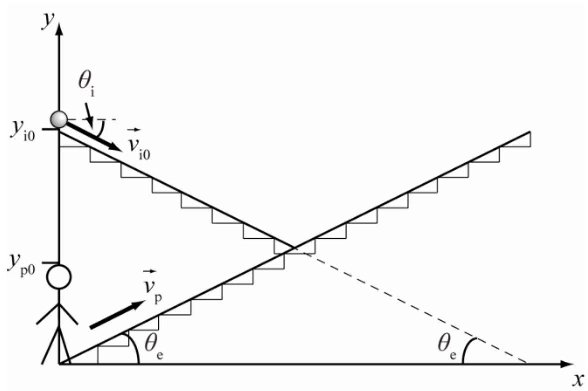
$$\theta = \arctan\left(\frac{\sqrt{2(9.81 \text{ m/s}^2)(30.0 \text{ m} - 1.00 \text{ m})}}{15.0 \text{ m/s} + 27.7778 \text{ m/s}}\right) = 29.144°$$

ROUND: Rounding to three significant figures, Δx_c = 67.5 m and $d - \Delta x_g = 67.5$ m. The egg hits the windshield at a speed of 49.0 m/s relative to the windshield at an angle of $29.1°$ above the horizontal.

DOUBLE-CHECK: The speed of the egg relative to the windshield is greater than the speed of the car and the goose.

3.92. THINK: The initial speed of the ice cream is $v_{i0} = 0.400$ m/s and the escalator angle is $\theta_e = 40.0°$. The ice cream launch angle is $\theta_i = -\theta_e$ (below the horizontal) and the initial height of the ice cream is $y_{i0} = 10.0$ m. The initial height of the professor's head is $y_{p0} = 1.80$ m, the professor's speed is $v_p = 0.400$ m/s, and the professor's direction is $\theta_p = \theta_e$. If the ice cream lands on the professor's head then the kinematic equations can be used to determine the ice cream flight time *t*, the final vertical position of the ice cream $y_i = y_p$, and the speed of the ice cream relative to the professor's head, v_{ip}.

SKETCH:



RESEARCH: Since the professor and the ice cream have the same horizontal starting point and horizontal speed, the ice cream does land on your professor's head because they always have the same horizontal position. At the time t of the collision, the final vertical positions of the ice cream and the professor are equal, $y_i = y_p \equiv y$. To find t use $y_i - y_{i0} = v_{i0,y}t - \frac{1}{2}gt^2$ and $y_p - y_{p0} = v_{p0,y}t$ (the professor does not accelerate). With t known, determine the vertical height y can be found. The speed of the ice cream v_{iy} can be found from $v_{iy}{}^2 = v_{i0,y}{}^2 - 2g(y - y_{i0})$.

SIMPLIFY: At time *t*, $y_i = y_p$:

$$y_i = y_{i0} + v_{i0,y}t - \frac{1}{2}gt^2; \quad y_p = y_{p0} + v_{p,y}t$$

$$y_{i0} + v_{i0,y}t - \frac{1}{2}gt^2 = y_{p0} + v_{p,y}t$$

$$y_{i0} + (-v_{i0}\sin\theta_e)t - \frac{1}{2}gt^2 = y_{p0} + (v_p\sin\theta_e)t$$

$$\frac{1}{2}gt^2 + 2v_{i0}\sin\theta_e t + (y_{p0} - y_{i0}) = 0,$$

since $v_{i0} = v_p$. Use the quadratic formula to solve for t:

$$t = \frac{-2v_{i0}\sin\theta_e \pm \sqrt{4v_{i0}^2\sin^2\theta_e + 2g(y_{i0} - y_{p0})}}{g}.$$

Recall that at time *t*, the professor's head and the ice cream are at the same vertical height y:

$$y = y_{p0} + v_p\sin\theta_e t.$$

The speed of the ice cream relative to the professor is $v_{ip} = \sqrt{(v_{ip,x})^2 + (v_{ip,y})^2}$, where $v_{ip,x} = 0$ (they have the same horizontal speed), $v_{ip,y} = v_{iy} + v_{py}$ (they move towards each other vertically), v_{iy} can be found from $v_{iy}{}^2 = v_{i0,y}{}^2 - 2g(y - y_{i0})$, and $v_{py} = v_{p,y}$ (since the professor does not accelerate). Putting this altogether gives:

$$v_{ip} = \sqrt{(-v_{i0}\sin\theta_e)^2 - 2g(y - y_{i0})} + v_p\sin\theta_e.$$

CALCULATE: The time that the ice cream lands on the professor's head is:

$$t = \frac{-2(0.400 \text{ m/s})\sin(40.0°) \pm \sqrt{4(0.400 \text{ m/s})^2\sin^2(40.0°) + 2(9.81 \text{ m/s}^2)(10.0 \text{ m} - 1.8 \text{ m})}}{9.81 \text{ m/s}^2}$$

$$= 1.2416 \text{ s} \text{ or } -1.3464 \text{ s}$$

The positive answer is correct. With t known, the vertical height at which the ice cream lands on the professor's head is:

$$y=(1.80\text{ m})+(0.400\text{ m/s})(\sin 40.0^\circ)(1.2416\text{ s})=2.1192\text{ m}.$$

The relative speed of the ice cream with respect to the professor's head at the time of impact is:

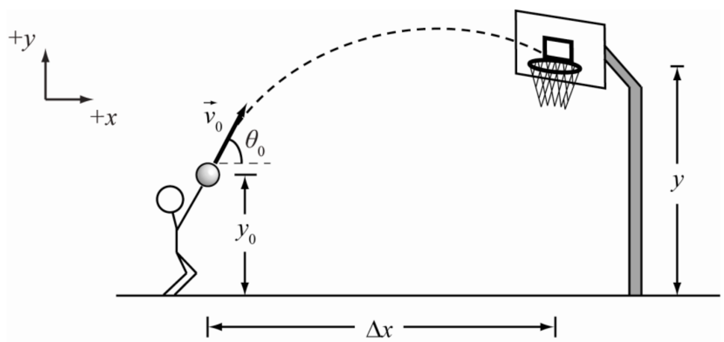
$$v_{\text{ip}}=\sqrt{(0.400\text{ m/s})^2\sin^2(40.0^\circ)-2(9.81\text{ m/s}^2)(2.1192\text{ m}-10.0\text{ m})}+(0.400\text{ m/s})\sin 40.0^\circ=12.694\text{ m/s}.$$

ROUND: $t=1.24$ s, $y=2.12$ m, and $v_{\text{ip}}=12.7$ m/s.

DOUBLE-CHECK: Considering the slow speed of the escalator and the original vertical positions, these values are reasonable.

3.93. **THINK:** The ball's horizontal distance from the hoop is $\Delta x=7.50$ m. The initial height is $y_0=2.00$ m. The final height is $y=3.05$ m. The launch angle is $\theta_0=48.0^\circ$. Determine the initial speed v_0.

SKETCH:



RESEARCH: To find v_0, use $\Delta y=\tan\theta_0\Delta x-\dfrac{g(\Delta x)^2}{2v_0^2\cos^2\theta_0}$.

SIMPLIFY:

$$\tan\theta_0\Delta x-\Delta y=\frac{g(\Delta x)^2}{2v_0^2\cos^2\theta_0}$$

$$\tan\theta_0\Delta x-(y-y_0)=\frac{g(\Delta x)^2}{2v_0^2\cos^2\theta_0}$$

$$2v_0^2\cos^2\theta_0=\frac{g(\Delta x)^2}{\tan\theta_0\Delta x-y+y_0}$$

$$v_0=\sqrt{\frac{g(\Delta x)^2}{2\cos^2\theta_0(\tan\theta_0\Delta x-y+y_0)}}$$

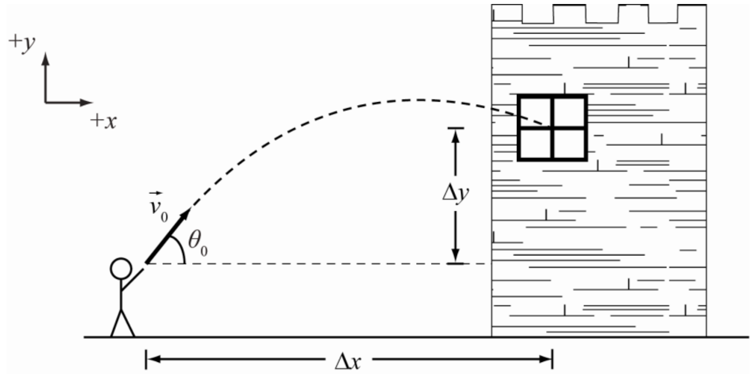
CALCULATE:
$$v_0=\sqrt{\frac{9.81\text{ m/s}^2(7.50\text{ m})^2}{2\cos^2(48.0^\circ)((7.50\text{ m})\tan(48.0^\circ)-3.05\text{ m}+2.00\text{ m})}}=9.2006\text{ m/s}$$

ROUND: As θ_0 has three significant figures, $v_0=9.20$ m/s.

DOUBLE-CHECK: The result has units of velocity. This is a reasonable speed to throw a basketball.

3.94. **THINK:** The pebble's launch angle is $\theta_0 = 37°$. The vertical displacement is $\Delta y = 7.0$ m. The horizontal displacement is $\Delta x = 10.0$ m. Determine the pebble's initial speed v_0.

SKETCH:



RESEARCH: To find v_0, use $\Delta y = \Delta x \tan\theta_0 - \dfrac{g(\Delta x)^2}{2v_0^2\cos^2\theta_0}$.

SIMPLIFY: $\tan\theta_0 \Delta x - \Delta y = \dfrac{g(\Delta x)^2}{2v_0^2\cos^2\theta_0}$, $\quad v_0 = \sqrt{\dfrac{g(\Delta x)^2}{2\cos^2\theta_0(\tan\theta_0\Delta x - \Delta y)}}$

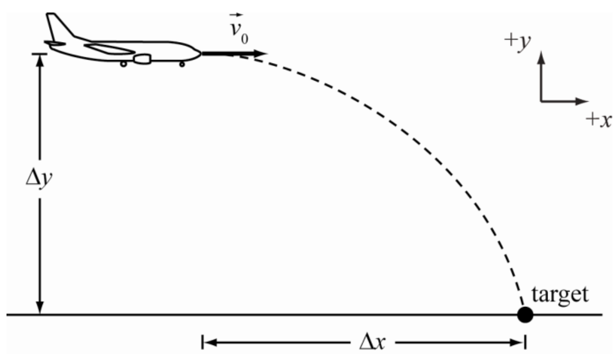
CALCULATE: $v_0 = \sqrt{\dfrac{9.81 \text{ m/s}^2(10.0 \text{ m})^2}{2\cos^2(37°)((10.0 \text{ m})\tan(37°) - 7.0 \text{ m})}} = 37.89$ m/s

ROUND: With two significant figures in θ_0, $v_0 = 38$ m/s.

DOUBLE-CHECK: This speed is fast, but reasonable if, for example, Romeo is a pitcher in the Major leagues.

3.95. **THINK:** The bomb's vertical displacement is $\Delta y = -5.00 \cdot 10^3$ m (falling down). The initial speed is $v_0 = 1000.$ km/h$(\text{h}/3600 \text{ s})(1000 \text{ m/km}) = 277.8$ m/s. The launch angle is $\theta_0 = 0°$. Determine the distance from a target, Δx, and the margin of error of the time Δt if the target is $d = 50.0$ m wide.

SKETCH:



RESEARCH: To find Δx, use the trajectory equation: $\Delta y = \Delta x \tan\theta_0 - \dfrac{g(\Delta x)^2}{2v_0^2\cos^2\theta_0}$.

To find Δt, consider the time it would take the bomb to travel the horizontal distance *d*; this is the margin of error in the time. The margin of error for the release time Δt can be determined from $d = v_0 \cos\theta_0 \Delta t$.

SIMPLIFY: Note, $\tan\theta_0 = 0$ and $\cos\theta_0 = 1$ for $\theta_0 = 0$.

$$\Delta y = -\frac{g(\Delta x)^2}{2v_0^{\,2}} \text{ and } \Delta x = \sqrt{\frac{-2v_0^{\,2}\Delta y}{g}}$$

For Δt, use $d = v_0 \cos\theta_0 \Delta t \Rightarrow \Delta t = d / v_0$.

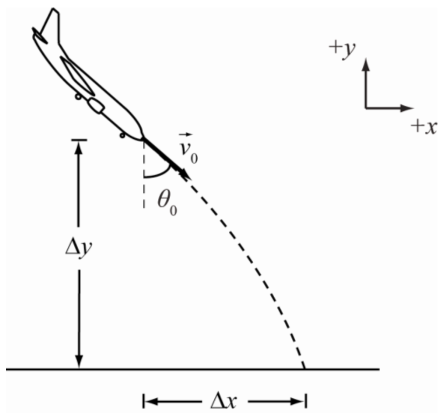
CALCULATE: $\Delta x = \sqrt{\dfrac{-2(277.8 \text{ m/s})^2(-5.00\cdot 10^3 \text{ m})}{9.81 \text{ m/s}^2}} = 8869 \text{ m}$, $\quad \Delta t = \dfrac{50.0 \text{ m}}{277.8 \text{ m/s}} = 0.1800 \text{ s}$

ROUND: $\Delta x = 8.87$ km and $\Delta t = 0.180$ s.

DOUBLE-CHECK: Considering the altitude and the terrific speed of the airplane, these values are reasonable.

3.96. **THINK:** The package's launch angle is $\theta_0 = 49.0°$ downward from the vertical. The vertical displacement is $\Delta y = -600.$ m. The flight time is $t = 3.50$ s. Determine the horizontal displacement Δx.

SKETCH:



RESEARCH: Note, $v_{x_0} = v_0 \sin\theta_0$ for the given θ_0 and $v_{y_0} = -v_0\cos\theta_0$. To determine Δx, use $\Delta x = v_{x_0} t$.

First, determine v_0 from $\Delta y = v_{y_0} t - \frac{1}{2}(gt^2)$.

SIMPLIFY: $\Delta y = v_{y_0} t - \dfrac{1}{2} g t^2 = -v_0 \cos\theta_0 t - \dfrac{1}{2} g t^2 \Rightarrow v_0 = -\dfrac{\left(\Delta y + \frac{1}{2} g t^2\right)}{\cos\theta_0 t}$

$$\Delta x = v_{x_0} t = v_0 \sin\theta_0 t = -\frac{\left(\Delta y + \frac{1}{2} g t^2\right)}{\cos\theta_0 t}\sin\theta_0 t = -\left(\Delta y + \frac{1}{2} g t^2\right)\tan\theta_0$$

CALCULATE: $\Delta x = -\left(-600. \text{ m} + \dfrac{1}{2}(9.81 \text{ m/s}^2)(3.50 \text{ s})^2\right)\tan(49.0°) = 621.1 \text{ m}$

ROUND: $\Delta x = 621$ m

DOUBLE-CHECK: The units for the result are units of distance.

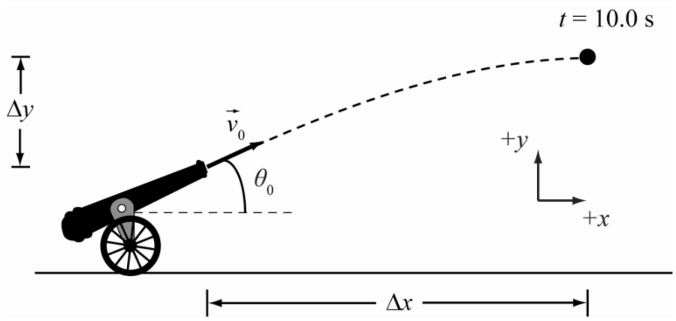
3.97. **THINK:** During the flight of the cannonball it moves with constant horizontal velocity component while at the same time undergoing free-fall in vertical direction.

(a) Since the time to hit a given point in space is given, we can easily extract the initial velocity from using our kinematic equations.

(b) If we know the initial velocity, then we can simply use our equation for the maximum height of the trajectory.

(c) If we know the initial velocity, then we can calculate the velocity at any given point in time, simply by applying our kinematic equations one more time.

SKETCH:



RESEARCH:

(a) The horizontal velocity component is $v_x = v_{x0} = x/t$. We can get the vertical component of the initial velocity from $y = y_0 + v_{y0}t - \frac{1}{2}gt^2$

(b) To determine *H*, use $H = y_0 + \dfrac{v_{y0}^{\;2}}{2g}$.

(c) Since we have determined the initial velocity components in part (a), we get the velocity components at any time from $v_x = v_{x0}$; $\quad v_y = v_{y0} - gt$.

SIMPLIFY:

(a) Since $y_0 = 0$ we get for the initial vertical velocity component $y + \frac{1}{2}gt^2 = v_{0y}t \Rightarrow v_{0y} = y/t + \frac{1}{2}gt$

Parts (b) and (c) are already in the shape we can use to put in numbers.

CALCULATE:

(a) $v_{x0} = (500.\text{ m})/(10.0\text{ s}) = 50.0\text{ m/s}$; $v_{y0} = (100.\text{ m})/(10.0\text{ s}) + \frac{1}{2}(9.81\text{ m/s}^2)(10.0\text{ s}) = 59.05\text{ m/s}$

$v_0 = \sqrt{(50.0\text{ m/s})^2 + (59.05\text{ m/s})} = 77.375\text{ m/s},\ \text{Angle} = \tan^{-1}\left((59.05\text{ m/s})/(50.0\text{ m/s})\right) = 49.7441°$

(b) $H = \dfrac{(59.05\text{ m/s})^2}{2(9.81\text{ m/s}^2)} = 177.72\text{ m}$

(c) $v_x = 50.0\text{ m/s}$; $v_y = 59.05\text{ m/s} - (9.81\text{ m/s}^2)(10.0\text{ s}) = -39.05\text{ m/s}$

$v = \sqrt{(50.0\text{ m/s})^2 + (-39.05\text{ m/s})} = 63.442\text{ m/s},\quad \text{Angle} = \tan^{-1}\left((-39.05\text{ m/s})/(50.0\text{ m/s})\right) = -37.9898°$

ROUND:

(a) $\vec{v}_0$ = 77.4 m/s at 49.7° above the horizontal

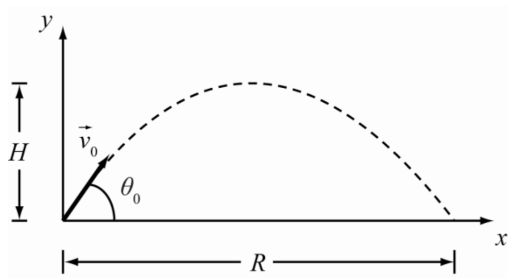
(b) H = 178 m

(c) v = 63.4 m/s at 38.0° below the horizontal

DOUBLE-CHECK: The velocity is reasonable for a cannon. $H \geq \Delta y$ as it must be, and $v < v_0$, which it should be with $y > y_0$.

3.98. **THINK:** When the ball is at height $y = 12.5\text{ m}$, its velocity is $\vec{v} = (v_x\hat{x} + v_y\hat{y}) = (5.6\hat{x} + 4.1\hat{y})$ m/s. The kinematic equations can be used to determine (a) the maximum height H, (b) the range R and (c) the magnitude and direction of the velocity when the ball hits the ground.

SKETCH:



RESEARCH:

(a) The maximum height that the soccer ball rises to is given by $H = y_0 + \frac{v_{y0}^2}{2g}$. The initial vertical velocity v_{y0} can be determined from $v_y^2 = v_{y0}^2 - 2g(y - y_0)$.

(b) The horizontal distance that the soccer ball travels is given by $R = \frac{v_0^2}{g}\sin 2\theta_0$.

(c) For ideal parabolic motion when the initial and final height of the projectile is the same, the initial and final speeds are the same. The angle of impact is the same, but it is below the horizontal.

SIMPLIFY:

(a) Since $y_0 = 0$,

$$v_{y0} = \sqrt{v_y^2 + 2gy} \Rightarrow H = \frac{v_y^2 + 2gy}{2g} = \frac{v_y^2}{2g} + y$$

(b) The initial velocity v_0 is given by $v_0 = \sqrt{v_x^2 + v_{y0}^2} = \sqrt{v_x^2 + v_y^2 + 2gy}$ and the launch angle θ_0 is given by $v_x = v_0 \cos\theta_0 \Rightarrow \theta_0 = \cos^{-1}\left(\frac{v_x}{v_0}\right)$.

(c) $v = v_0$ and $\theta = -\theta_0$.

CALCULATE:

(a) $H = \frac{(4.1 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} + 12.5 \text{ m} = 13.357 \text{ m}$

(b) $v_0 = \sqrt{(5.6 \text{ m/s})^2 + (4.1 \text{ m/s})^2 + 2(9.81 \text{ m/s}^2)(12.5 \text{ m})} = 17.13 \text{ m/s}, \quad \theta_0 = \cos^{-1}\left(\frac{5.6 \text{ m/s}}{17.13 \text{ m/s}}\right) = 70.92°,$

$$R = \frac{(17.13 \text{ m/s})^2}{(9.81 \text{m/s}^2)}\sin(2(70.92°)) = 18.48 \text{ m}$$

(c) $v = 17.13 \text{ m/s}, \quad \theta = -70.92°$

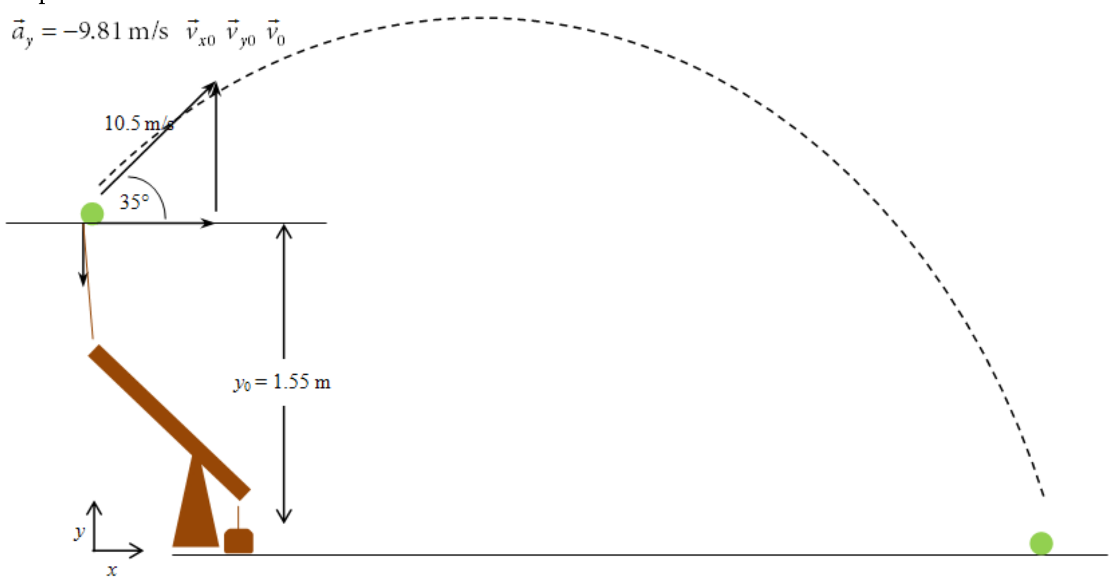
ROUND: (a) To the first decimal place, $H = 13.4$ m. To two significant figures, (b) $R = 18$ m and (c) $v = 17$ m/s, 71° below the horizontal.

DOUBLE-CHECK: With such a steep launch angle, the range is not expected to be large. Also, H is greater than $y = 12.5$ m, which is expected.

Multi-Version Exercises

3.99. **THINK:** The initial height, velocity, and angle of the tennis ball are known. To find the total horizontal distance covered before the tennis ball hits the ground it makes sense to decompose the motion into horizontal and vertical components. Then, find the time at which the tennis ball hits the ground (y = 0) and determine the horizontal position at that time. To make the problem as simple as possible, choose y = 0 m to be the ground and x = 0 m to be the location of the trebuchet where the ball is released.

SKETCH: A sketch helps to see exactly how to decompose the initial velocity into horizontal and vertical components.



RESEARCH: This problem involves ideal projectile motion. Since there is no horizontal acceleration and the tennis ball starts at x_0 = 0, the equation (3.11) for the horizontal position at time t is $x = v_{x0}t$. Equation (3.13) gives the vertical position as $y = y_0 + v_{y0}t - \frac{1}{2}gt^2$. To find a final answer, is necessary to determine the x- and y-components of the initial velocity, given by $v_{y0} = v_0 \sin\theta$ and $v_{x0} = v_0 \cos\theta$.

SIMPLIFY: To find the time when the tennis ball hits the ground, it is necessary to find a non-negative solution to the equation $0 = y_0 + v_{y0}t - \frac{1}{2}gt^2$. The quadratic formula gives a solution of $t = \dfrac{-v_{y0} \pm \sqrt{(v_{y0})^2 - 4(-\frac{1}{2}g)(y_0)}}{2(-\frac{1}{2}g)} = \dfrac{v_{y0} \pm \sqrt{(v_{y0})^2 + 2gy_0}}{g}$. It will be necessary to take the positive square root here: the tennis ball cannot land *before* it is released. This time can then be used with the equation for horizontal position to get the position when the tennis ball hits the ground at $x = v_{x0}t = v_{x0}\left(\dfrac{v_{y0} + \sqrt{(v_{y0})^2 + 2gy_0}}{g}\right)$. Combining this with the equations for the horizontal and vertical components of the initial velocity ($v_{y0} = v_0 \sin\theta$ and $v_{x0} = v_0 \cos\theta$) gives that the tennis ball lands at $x = (v_0 \cos\theta)\left(\dfrac{v_0 \sin\theta + \sqrt{(v_0 \sin\theta)^2 + 2gy_0}}{g}\right)$.

CALCULATE: The problem states that the initial height y_0 = 1.55 m. The initial velocity $v_0 = 10.5$ m/s at an angle of θ = 35° above the horizontal. The gravitational acceleration on Earth is –9.81 m/s². Thus the tennis ball lands at

$$x = (v_0 \cos\theta)\left(\frac{v_0 \sin\theta + \sqrt{(v_0 \sin\theta)^2 + 2gy_0}}{g}\right)$$

$$= \left[10.5\cos(35°)\right]\left(\frac{10.5\sin(35°) + \sqrt{(10.5\sin(35°))^2 + 2\cdot 9.81\cdot 1.55}}{9.81}\right)$$

$$= 12.43999628$$

ROUND: Since the measured values have 3 significant figures, the answer should also have three significant figures. Thus the tennis ball travels a horizontal distance of 12.4 m before it hits the ground.

DOUBLE-CHECK: Using equation (3.22) for the path of a projectile, it is possible to work backwards from the initial position (x_0, y_0) = (0, 1.55) and the position when the tennis ball lands (x, y) = (12.4, 0), and angle θ_0 = 35° to find the starting velocity, which should confirm what was given originally.

$$y = y_0 + (\tan\theta_0)x - \frac{g}{2v_0^2\cos^2\theta_0}x^2 \Rightarrow$$

$$0 = 1.55 + (\tan 35°)\cdot 12.4 - \frac{9.81}{2(v_0)^2\cos^2 35°}(12.4)^2 \Rightarrow$$

$$1.55 + (\tan 35°)\cdot 12.4 = \frac{9.81}{2(v_0)^2\cos^2 35°}(12.4)^2 \Rightarrow$$

$$\left(1.55 + (\tan 35°)\cdot 12.4\right)\left[\frac{(v_0)^2}{1.55 + (\tan 35°)\cdot 12.4}\right] = \frac{9.81\cdot(12.4)^2}{2(v_0)^2\cos^2 35°}\left[\frac{(v_0)^2}{1.55 + (\tan 35°)\cdot 12.4}\right] \Rightarrow$$

$$(v_0)^2 = \frac{9.81\cdot(12.4)^2}{2\cos^2 35°\left(1.55 + (\tan 35°)\cdot 12.4\right)} \Rightarrow$$

$$v_0 = \sqrt{\frac{9.81\cdot(12.4)^2}{2\cos^2 35°\left(1.55 + (\tan 35°)\cdot 12.4\right)}} \Rightarrow$$

$$v_0 = 10.4805539$$

Rounded to three significant figures, this becomes v_0 = 10.5, which confirms the answer.

3.100. The x-component of velocity does not change during flight. It is always equal to v_{x0}.

$$v_x = v_{x0} = v_0 \cos\theta = (10.5 \text{ m/s})\cos(35.0°) = 8.601096465 \text{ m/s} = 8.60 \text{ m/s}.$$

3.101. The time of travel was previously found to be $t = \frac{v_{y0} + \sqrt{(v_{y0})^2 + 2gy_0}}{g}$, and $v_{y0} = v_0 \sin\theta$. Therefore, the y-component of velocity just before impact is

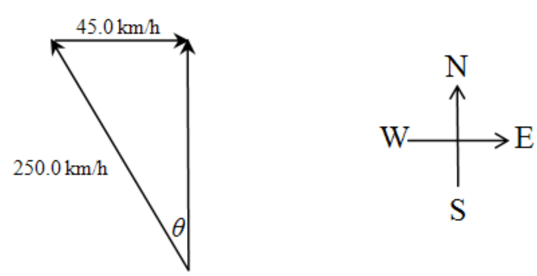
$$\begin{aligned} v_y &= v_{y0} - gt \\ &= v_{y0} - g\left(\frac{v_{y0} + \sqrt{(v_{y0})^2 + 2gy_0}}{g}\right) \\ &= -\sqrt{(v_{y0})^2 + 2gy_0} = -\sqrt{(v_0 \sin\theta)^2 + 2gy_0} \\ &= -\sqrt{\left[(10.5 \text{ m/s})\sin(35.0°)\right]^2 + 2(9.81 \text{ m/s}^2)(1.55 \text{ m})} \\ &= -8.165913274 \text{ m/s} = 8.17 \text{ m/s downward.} \end{aligned}$$

3.102. From the two preceding problems,

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(8.601096465 \text{ m/s})^2 + (-8.165913274 \text{ m/s})^2} = 11.86 \text{ m/s.}$$

3.103. **THINK:** This question involves flying a plane through the air. The speed of the airplane with respect to the wind and the velocity of the wind with respect to the ground are both given, so this problem involves relative motion. The vector sum of the plane's velocity with respect to the air and the velocity of the air with respect to the ground must point in the direction of the pilot's destination. Since the wind is blowing from the west to the east and the pilot wants to go north, the plane should head to the northwest.

SKETCH: The sketch should shows the velocity of the plane with respect to the wind ($\vec{v}_{\text{pa}}$), the velocity of the wind (which can be thought of as the velocity of the air with respect to the ground or $\vec{v}_{\text{ag}}$), and the velocity of the plane with respect to the ground ($\vec{v}_{\text{pg}}$). The distance the airplane is to travel will affect how long it takes the pilot to get to her destination, but will not effect in the $\vec{v}_{\text{pa}}$ $\vec{v}_{\text{pg}}$ $\vec{v}_{\text{ag}}$ direction she flies.



RESEARCH: To solve this problem, it is first necessary to note that the wind, blowing from West to East, is moving in a direction perpendicular to the direction the pilot wants to fly. Since these two vectors form a right angle, it is easy to use trigonometry to find the angle θ with the equation $\sin\theta = \frac{|\vec{v}_{\text{ag}}|}{|\vec{v}_{\text{pa}}|}$. Since North is 360° and West is 270°, the final answer will be a heading of $(360 - \theta)°$.

SIMPLIFY: To find the final answer, it is necessary to use the inverse sine function. The equation for the angle θ can be found using algebra and trigonometry:

$$\sin^{-1}(\sin\theta) = \sin^{-1}\left(\frac{|\vec{v}_{\text{ag}}|}{|\vec{v}_{\text{pa}}|}\right)$$

$$\theta = \sin^{-1}\left(\frac{|\vec{v}_{\text{ag}}|}{|\vec{v}_{\text{pa}}|}\right) \Rightarrow$$

$$360 - \theta = 360 - \sin^{-1}\left(\frac{|\vec{v}_{\text{ag}}|}{|\vec{v}_{\text{pa}}|}\right)$$

CALCULATE: To find the final answer in degrees, it is necessary first to make sure that the calculator or computer program is in degree mode. Note that the speed of the wind $|\vec{v}_{\text{ag}}| = 45.0$ km/h and the speed of the plane with respect to the air $|\vec{v}_{\text{pa}}| = 250.0$ km/h are given in the problem. Plugging these in and solving gives a heading of:

$$360 - \sin^{-1}\left(\frac{|\vec{v}_{\text{ag}}|}{|\vec{v}_{\text{pa}}|}\right) = 360° - \sin^{-1}\left(\frac{45.0 \text{ km/h}}{250.0 \text{ km/h}}\right)$$
$$= 349.6302402°$$

ROUND: Rounding to four significantfigures, the final heading is 349.6°.

DOUBLE-CHECK: Intuitively, this answer seems correct. The pilot wants to fly North and the wind is blowing from West to East, so she should head somewhere towards the Northeast. Since the speed of the airplane with respect to the air is much greater than the speed of the air with respect to the ground (wind speed), the East-West component of the airplane's velocity with respect to the air should be less than the North-South component. Resolving the motion of the plane into horizontal (East-West) and vertical components gives that the horizontal speed of the plane with respect to the ground $v_{\text{pa}x}$ is $250.0\frac{\text{km}}{\text{h}} \cdot \cos(349.6° - 270°)$ or 45.1 km/h from East to West, which is within rounding the same as the known wind speed.

3.104. As in the preceding problem, the velocity vectors form a right triangle, with the wind velocity as the east-west leg, the plane's ground velocity as the north-south leg, and the plane's velocity relative to the air as the hypotenuse. The plane's ground speed can therefore be found by applying the Pythagorean Theorem: $\sqrt{(250.0 \text{ km/h})^2 - (45.0 \text{ km/h})^2} = 245.9166525 \text{ km/h} = 245.9 \text{ km/h}$.

3.105. Using the result from the preceding problem, the time required will be

$$t = \frac{d}{v} = \frac{200.0 \text{ km}}{245.9166525 \text{ km/h}} = 0.8132836794 \text{ h} = 48.78 \text{ min}.$$

Chapter 4: Force

Concept Checks

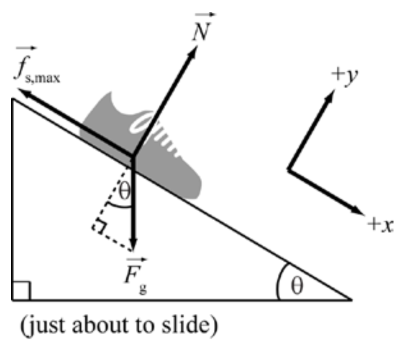
4.1. c **4.2.** c **4.3.** b **4.4.** c **4.5.** b **4.6.** d

Multiple-Choice Questions

4.1. d **4.2.** a **4.3.** d **4.4.** a **4.5.** a **4.6.** b and c **4.7.** b **4.8.** a **4.9.** a **4.10.** b **4.11.** b **4.12.** b **4.13.** c and d
4.14. a, b and d **4.15.** b **4.16.** a

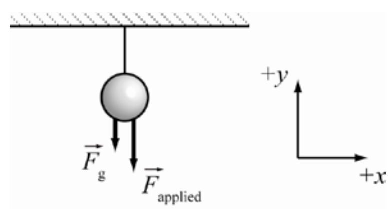
Conceptual Questions

4.17. Determine an expression for μ_s in terms of θ. Since the shoe just starts to slide, the maximum static friction has been achieved. Sketching shows the forces acting on the shoe:



In the y-direction: $F_{\text{net}y} = N - F_{gy} = 0 \Rightarrow N = F_{gy} = F_{\text{g}}\cos\theta$. In the x-direction: $F_{\text{net}x} = F_{gx} - f_{\text{s,max}} = 0$ $\Rightarrow f_{\text{s,max}} = F_{gx} = F_{\text{g}}\sin\theta$. With $f_{\text{s,max}} = \mu_{\text{s}}N$, $\mu_{\text{s}} = (F_{\text{g}}\sin\theta)/N$. With $N = F_{\text{g}}\cos\theta$, $\mu_{\text{s}} = (F_g\sin\theta)/(F_g\cos\theta) = \tan\theta$.

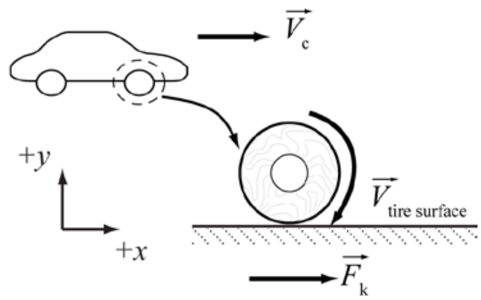
4.18.



If the lower string is pulled slowly, the upper string breaks due to the greater tension in the upper string. If the lower string is pulled fast enough, the lower string breaks due to the inertia of the ball. The quick jerk increases the tension in the lower string faster than in the upper string.

4.19. The car and the trailer are no longer in an inertial reference frame due to their acceleration up the hill. However, the intertial forces of the car-trailer system do not change. That is, $F_t = F_c$ is still true. The sum of the internal forces of a system should always be zero.

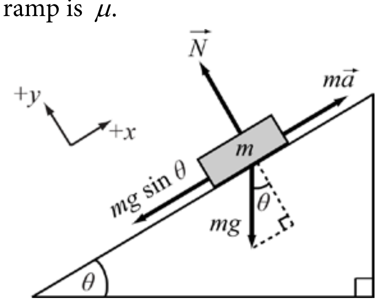
4.20. It is the frictional forces against the tires that drive the car. The direction of the force of the tire against the road is actually opposite to the direction the car travels in.



4.21. A force external to the two interacting bodies can cause them to accelerate. For example, when a horse pulls a cart, both objects exert a force on each other equal in magnitude and opposite in direction, but the friction force acting on the horse-cart system causes it to accelerate.

4.22. If the textbook is initially at rest and there is no net force acting on it, then it will remain at rest. However, if the textbook is moving across a frictionless table at a constant velocity $\left(d\vec{v}/dt=0\right)$, then it will remain in motion unless acted upon by an external force. The statement is false.

4.23. A mass, m is sliding on a ramp that is elevated at an angle, θ with respect to the floor. The coefficient of friction between the mass and ramp is μ.



(a) The forces acting on the box are as follows:

- The force due to gravity acting down the ramp, $mg\sin\theta$.
- The force of friction, $f_k=\mu_k N$, where the subscript *k* denotes kinetic friction, acting opposite to the direction of motion, in this case down the ramp.

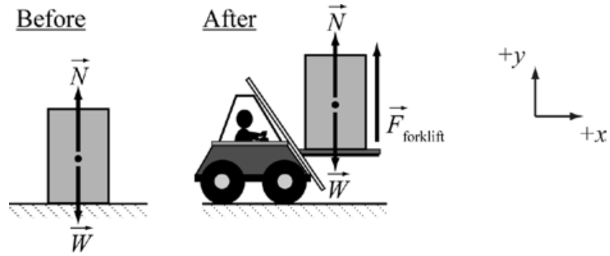
The sum of the force components along the ramp is $\sum F=ma=-mg\sin\theta-\mu_k N$. This expression can be simplified by noting that $N=mg\cos\theta$. Substituting this into the expression yields:

$$ma=-mg\sin\theta-mg\cos\theta\mu_k \Rightarrow a=-g\left(\sin\theta+\mu_k\cos\theta\right).$$

(b) If the mass is sliding down the ramp, the force of gravity remains the same, but the friction force still opposes the direction of motion and therefore now points up the ramp. Thus the expression for Newton's second law can now be written $ma=-mg\sin\theta+\mu_k N$. The equation for the acceleration is then:

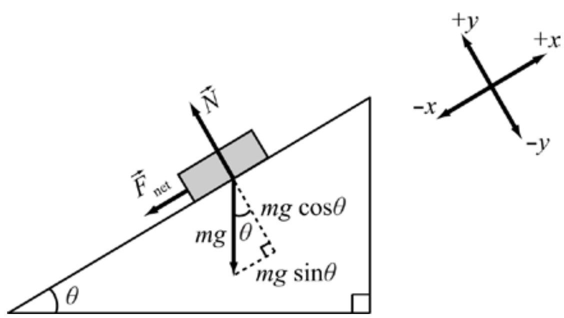
$$a=-g\left(\sin\theta-\mu_k\cos\theta\right).$$

4.24. The shipping crate has a weight of $w = 340$ N. The force of the forklift is $F_{\text{forklift}} = 500$ N.



The force due to gravity equals the weight of the crate. The mass of the crate, m is constant and so is the acceleration due to gravity, g. Therefore, the force due to gravity is $F_g = mg = w = 340$ N.

4.25. The near frictionless slope is at an incline of $\theta = 30.0°$. It will be useful to draw a diagram of the sliding block of mass, m, and label the forces on it. The diagram shows a block of mass, m, sliding down a (near) frictionless incline.

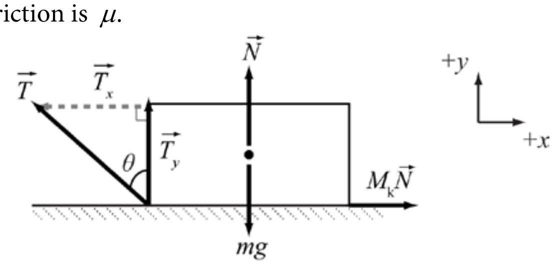


The magnitude of the net force acting on the block as it slides down the ramp is $F_{\text{net}} = mg\sin\theta$. The magnitude of the normal force is $N = mg\cos\theta$. This means that

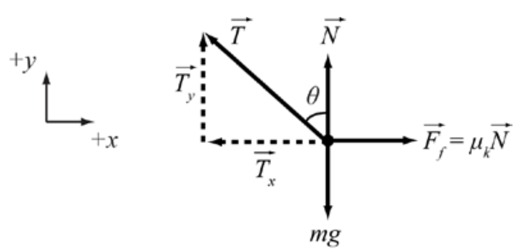
$$F_{\text{net}} / N = (mg\sin\theta)/(mg\cos\theta) = \tan\theta.$$

For an angle of $30°$ this means that $F_{\text{net}} / N = \tan 30° \approx 0.58$. The net force is approximately 58% of the normal force.

4.26. The tow truck's mass is M. The mass of shipping crate is m. The angle with respect to the vertical is θ. The coefficient of kinetic friction is μ.



(a)



(b) The truck pulls the crate at a constant speed, so the net force on the crate is $\vec{F}_{\text{net}} = m\vec{a} = 0$. The components of the net force can be written as $\sum F_x = 0$; $\sum F_y = 0$. The sum of the forces in the x direction are $\sum F_x = T_x - \mu_k N = 0$.

Trigonometry can be used to obtain the expression for T_x :

$$\sin\theta = \frac{T_x}{T} \Rightarrow T_x = T\sin\theta.$$

Therefore, $\sum F_x = \mu_k N - T\sin\theta = 0.$ The sum of the forces in the y direction is $\sum F_y = 0 = T_y + N - mg = 0.$ To simplify this expression, $T_y = T\cos\theta$ can be used. In summary, our two equation for the x and y directions yield:

$$(1)\ T\cos\theta + N - mg = 0$$

$$(2)\ \mu_k N - T\sin\theta = 0$$

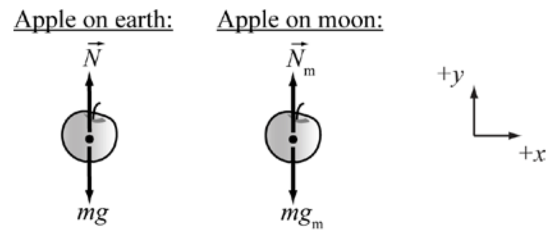
Equation (2) can be solved in terms of N to get $N = T\sin\theta / \mu_k$. Substituting this into equation (1) yields:

$$T\left(\cos\theta + \sin\theta / \mu_k\right) - mg = 0 \Rightarrow T = \frac{\mu_k mg}{\sin\theta + \mu_k \cos\theta}.$$

Exercises

4.27. **THINK:** The gravitational constant on the Moon is $g_{\text{m}} = g/6$, where g is the Earth's gravitational constant. The weight of an apple on the Earth is w = 1.00 N.

SKETCH:



RESEARCH: The gravitational constant on the Earth is $g = 9.81 \text{ m/s}^2$. The weight of the apple, w is given by its mass times the gravitational constant, $w = mg$.

SIMPLIFY:

(a) The weight of the apple on the Moon is $w_{\text{m}} = mg_{\text{m}}$. Simplify this expression by substituting $g_{\text{m}} = g/6$: $w_{\text{m}} = m\left(g/6\right)$. Mass is constant so I can write expressions for the mass of the apple on Earth and on the Moon and then equate the expressions to solve for m. On Earth, $m = w/g$. On the Moon, $m = 6w_m / g$. Therefore, $w/g = 6w_{\text{m}}/g \Rightarrow w_{\text{m}} = w/6$.

(b) The expression for the weight of an apple on Earth can be rearranged to solve for m: $m = w/g$.

CALCULATE:

(a) $w_{\text{m}} = \frac{1}{6}\left(1.00 \text{ N}\right) = 0.166667 \text{ N}$

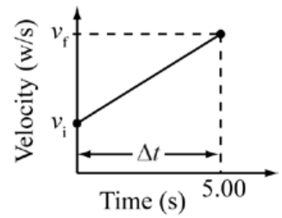
(b) $m = \frac{1.00 \text{ N}}{9.81 \text{ m/s}^2} = 0.101931 \text{ kg}$

ROUND: Rounding to three significant figures, (a) $w_{\text{m}} = 0.167 \text{ N}$ and (b) m = 0.102 kg.

DOUBLE-CHECK: It is expected that the weight of the apple on the Moon is much less than the weight of the apple on the Earth. Also, a mass of about 100 g is reasonable for an apple.

4.28. **THINK:** The go-cart is accelerated by having a force, $F = 423.5 \text{ N}$ applied to it. The initial velocity is given as $v_{\text{i}} = 10.4 \text{ m/s}$ and the final velocity is $v_{\text{f}} = 17.9 \text{ m/s}.$ The time interval over which the change in velocity occurs is $\Delta t = 5.00 \text{ s}.$ Determine the mass, m of the go-cart and the driver.

SKETCH:



RESEARCH: Because the force is constant, the acceleration is constant and the increase in velocity, v is linear. The equation for force is $F = ma,$ and the expression for the acceleration is $a = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}.$

SIMPLIFY: The expression for a can be substituted into the expression for force:

$$F = m\left(\frac{v_f - v_i}{t_f - t_i}\right) \Rightarrow m = \frac{F}{\left(\frac{v_f - v_i}{t_f - t_i}\right)}.$$

CALCULATE: $m = \frac{423.5 \text{ N}}{\left(\frac{17.9 \text{ m/s} - 10.4 \text{ m/s}}{5.00 \text{ s} - 0 \text{ s}}\right)} = 282.33 \text{ kg}$

ROUND: Rounding to three significant figures, m = 282 kg.

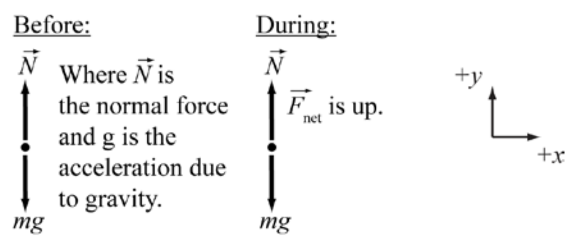
DOUBLE-CHECK: This mass is reasonable for a go-cart with a driver. The units are also correct.

4.29. **THINK:** The scale is in an elevator and the gym member has a mass of m = 183.7 lb when the elevator is stationary. The elevator accelerates upward with an acceleration of $a = 2.43 \text{ m/s}^2$. The mass must be converted to SI units:

$$m = 183.7 \text{ lb} \cdot \frac{1 \text{ kg}}{2.205 \text{ lb}} = 83.31 \text{ kg}.$$

Determine the weight, w measured by the scale as the elevator accelerates upward.

SKETCH: It will be useful to sketch free body diagrams for the forces acting on the scale before and during the elevator's acceleration.



RESEARCH: Before the elevator accelerates, the net force is $F_{net} = 0.$ The sum of the forces at this point is $F_{net,i} = 0 = N - mg$, therefore $N = mg.$ Once the elevator starts to accelerate upward, there is a net force, $F_{net,a} = ma = N - mg.$ Since the mass of the person, m, the gravitational acceleration, g, and the acceleration, a are known, the equation can be used to determine N, which is the normal force that the scale displays.

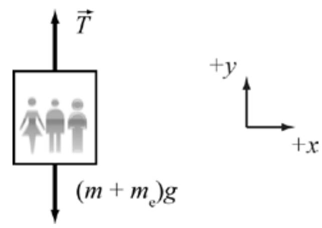
SIMPLIFY: While the elevator is accelerating, the net force acting on the scale can be written as $ma = N - mg.$ Rearranging to solve for N yields $N = m(a + g).$

CALCULATE: $N = 83.31\text{ kg}\left(2.43\text{ m/s}^2 + 9.81\text{ m/s}^2\right) = 1019.7144\text{ N}$. However, scales are calibrated to read weight based on the assumption of $a = g = 9.81\text{ m/s}^2$. The acceleration of the elevator "tricks" the scale. The gym member's weight as displayed on the scale will be

$$\frac{1019.7144\text{ N}}{9.81\text{ m/s}^2}\cdot\frac{2.205\text{ lb}}{1\text{ kg}} = 229.20\text{ lb}.$$

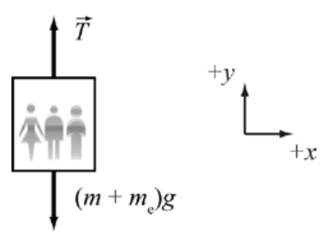
ROUND: The acceleration is given with three significant figures, so the gym member's weight is 229 lb.
DOUBLE-CHECK: Since the acceleration of the elevator is up, a person's apparent weight should increase. The weight of the person when the elevator is at rest is 183.7 pounds and the apparent weight of the person while the elevator is accelerating is 229 pounds, so the calculation makes sense.

4.30. **THINK:** The given quantities are the cabin mass of the elevator, $m_e = 358.1\text{ kg}$ and the combined mass of the people in the cabin, m = 169.2 kg. The elevator is undergoing a constant acceleration, $a = 4.11\text{ m/s}^2$ due to be being pulled by a cable. Determine the tension, T in the cable.
SKETCH:



RESEARCH: The sum of all the forces acting on the elevator is $F_{\text{net}} = \sum F_y = ma + m_e a = T - mg - m_e g$.
SIMPLIFY: Combine like terms: $a\left(m + m_e\right) = T - g\left(m + m_e\right)$. Rearrange the equation to solve for T: $T = \left(m + m_e\right)\left(a + g\right)$.
CALCULATE: $T = \left(169.2\text{ kg} + 358.1\text{ kg}\right)\left(4.11\text{ m/s}^2 + 9.81\text{ m/s}^2\right) = 7340.016\text{ N}$
ROUND: Rounding to three significant figures, $T = 7340$ N. The force of the tension is directed upward.
DOUBLE-CHECK: The units calculated for the tension are Newtons, which are units of force. Because the elevator is accelerating upward, it is expected that the tension in the cable should be higher than if the elevator was hanging at rest. The calculated tension for the cable when the elevator was accelerating is T = 7340 N. If the elevator is not accelerating, the expected tension is approximately 5173 N, so the answer is reasonable.

4.31. **THINK:** The mass of the elevator cabin is $m_e = 363.7\text{ kg}$ and the total mass of the people in the elevator is m = 177.0 kg. The elevator is being pulled upward by a cable which has a tension of $T = 7638$ N. The acceleration of the elevator is to be determined.
SKETCH:



RESEARCH: Force is equal to mass times acceleration, $\vec{F} = m\vec{a}$. The sum of all the forces acting on the elevator will give the net force that acts upon the elevator, $F_{\text{net},y} = \sum F_y$. In this case, $F_{\text{net},y} = ma + m_e a = T - mg - m_e g$. The gravitational acceleration is $g = 9.81\text{ m/s}^2$.

SIMPLIFY: Group like terms, $a(m+m_e)=T-g(m+m_e)$. Rearrange to solve for *a*:

$$a=\frac{T-(m+m_e)g}{(m+m_e)}.$$

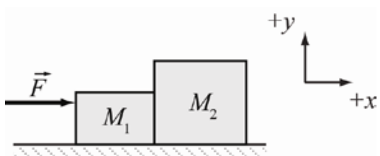
CALCULATE: $a=\dfrac{7638\text{ N}-(177.0\text{ kg}+363.7\text{ kg})9.81\text{ m/s}^2}{(177.0\text{ kg}+363.7\text{ kg})}=4.3161\text{ m/s}^2$

ROUND: Rounding to three significant figures because that is the precision of *g*, $a=4.32\text{ m/s}^2$.
DOUBLE-CHECK: The units of the result are correct. Also, the value determined for a is approximately 45% of the acceleration due to gravity, so the answer is reasonable.

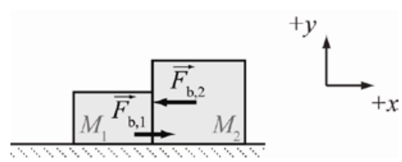
4.32. **THINK:** Two blocks are in contact, moving across a frictionless table with a constant acceleration of $a=2.45\text{ m/s}^2$. The masses of the two blocks are $M_1=3.20\text{ kg}$ and $M_2=5.70\text{ kg}$. Determine (a) the magnitude of the applied force, *F*, (b) the contact force between the blocks, F_b and (c) the net force acting on block 1, $F_{\text{net},1}$.

SKETCH:

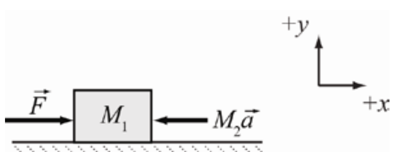
(a)



(b)



(c)



RESEARCH:
(a) Since the blocks are on a frictionless surface, there is no force due to friction. The net force is $F=M_1a_1+M_2a_2$. But the blocks are in contact, so $a_1=a_2=a$. The equation can now be reduced to $F=(M_1+M_2)a$.
(b) The force that block 1 feels due to block 2 is equal and opposite to the force that block 2 feels due to block 1. The contact force is $F_b=M_2a$.
(c) The net force acting on block 1 is the sum of all the forces acting on it, $F_{\text{net},1}=F-M_2a$.
SIMPLIFY: This step is not necessary.
CALCULATE:

(a) $F=(3.20\text{ kg}+5.70\text{ kg})(2.45\text{ m/s}^2)=21.805\text{ N}$

(b) $F_b=(5.70\text{ kg})(2.45\text{ m/s}^2)=13.965\text{ N}$

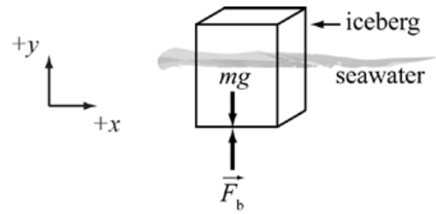
(c) $F_{\text{net},1}=21.805\text{ N}-(5.70\text{ kg})(2.45\text{ m/s}^2)=7.84\text{ N}$

ROUND: Three significant figures were provided in the question, so (a) $F=21.8\text{ N}$, (b) $F_b=14.0\text{ N}$ and (c) $F_{\text{net},1}=7.84\text{ N}$.
DOUBLE-CHECK: All three results have the correct units of Newtons, which are units of force.

4.33. **THINK:** The force that the ocean exerts on the iceberg keeps the iceberg floating. Since the iceberg floats, the net force on it must be zero. Therefore the magnitude of the force of the ocean on the iceberg must be equal to its weight.

The given quantities are the density of ice, $\rho_i = 917 \text{ kg/m}^3$ and the density of seawater, $\rho_s = 1024 \text{ kg/m}^3$. The volume of the iceberg above the sea is $v_{\text{above}} = 4205.3 \text{ m}^3$. This volume is 10.45% of the total volume of the iceberg. Determine the force, F_b that the seawater exerts on the iceberg.

SKETCH:



RESEARCH: The weight of the iceberg is $F = m_{\text{ice}}g$. The mass is the product of density and Volume, $m_{\text{ice}} = \rho_{\text{ice}} V_{\text{ice}}$. We were given the volume above the surface and were told that it represents 10.45% of the overall volume. So the total volume of the ice is: $V_{\text{ice}} = V_{\text{above}} / 0.1045$

SIMPLIFY: The weight of the iceberg is $F = m_{\text{ice}} g = \rho_{\text{ice}} V_{\text{ice}} g = \rho_{\text{ice}} g V_{\text{above}} / 0.1045$

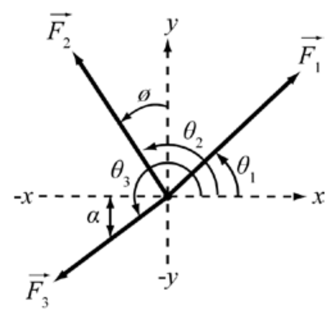
CALCULATE: $F = (917 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(4205.3 \text{ m}^3)/0.1045 = 3.620087 \cdot 10^8 \text{ N}$

ROUND: There are three significant figures provided in the question, so the answer should be written as $F = 3.62 \cdot 10^8 \text{ N}$

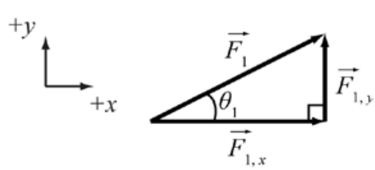
DOUBLE-CHECK: The units of Newtons that were calculated are correct units of force. Also, an iceberg is a very massive object, so it reasonable that the seawater must exert a large force to keep it floating.

4.34. **THINK:** There are three massless ropes attached at one point with the following forces applied to them: $F_1 = 150. \text{ N}$ at $\theta_1 = 60.0°$, $F_2 = 200. \text{ N}$ at $\theta_2 = 100.°$ and $F_3 = 100. \text{ N}$ at $\theta_3 = 190.°$. Determine the magnitude and direction of a fourth force that is necessary to keep the system in equilibrium.

SKETCH:



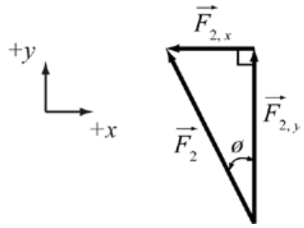
RESEARCH: For the system to remain stationary, the sum of the forces in the x and y directions must be zero, $\sum F_x = 0$ and $\sum F_y = 0$. The known forces must be broken into components to complete the calculations.



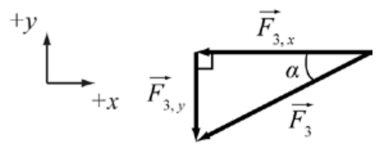
$\theta_1 = 60.0°$ with respect to the x-axis.

$F_{1,x} = F_1 \cos\theta_1$

$F_{1,y} = F_1 \sin\theta_1$



$\phi = -10.0°$ with respect to the y-axis.
$F_{2,x} = -F_2 \sin\phi$
$F_{2,y} = F_2 \cos\phi$



$\alpha = -10.0°$ with respect to the negative x-axis.
$F_{3,x} = -F_3 \cos\alpha$
$F_{3,y} = -F_3 \sin\alpha$

The equations for the sum of the forces in the x and y directions are:

$$\sum F_x = 0 = F_1 \cos\theta_1 - F_2 \sin\phi - F_3 \cos\alpha + F_{4,x}, \quad \sum F_y = 0 = F_1 \sin\theta_1 + F_2 \cos\phi - F_3 \sin\alpha + F_{4,y}.$$

The angle is $\theta_4 = \tan^{-1}\left(F_{4,y} / F_{4,x}\right)$. The magnitude of F_4 is given by $\left|F_4\right| = \sqrt{F_{4,x}{}^2 + F_{4,y}{}^2}$.

SIMPLIFY: The equations can be rearranged to solve for the components of the fourth force: $F_{4,x} = F_2 \sin\phi + F_3 \cos\alpha - F_1 \cos\theta_1$ and $F_{4,y} = F_3 \sin\alpha - F_1 \sin\theta_1 - F_2 \cos\phi$.

CALCULATE: $F_{4,x} = (200.)\sin(10.0°)\text{ N} + (100.)\cos(10.0°)\text{ N} - (150.)\cos(60.0°)\text{ N} = 58.2104\text{ N}$

$F_{4,y} = (100.)\sin(10.0°)\text{ N} - (150.)\sin(60.0°)\text{ N} - (200.)\cos(10.0°)\text{ N} = -309.500\text{ N}$

$\theta_4 = \tan^{-1}\left(\dfrac{-309.500\text{N}}{58.2104\text{ N}}\right) = -79.3483°$ with respect to the positive x-axis.

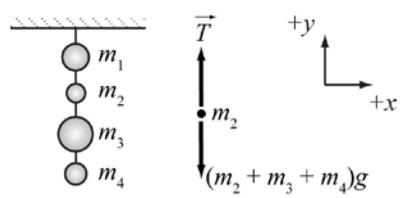
$\left|F_4\right| = \sqrt{(58.2104\text{ N})^2 + (-309.500\text{N})^2} = 314.9265\text{ N}$

ROUND: The given value for θ_1 has three significant figures, so the answers must be written as $\left|F_4\right| = 315\text{ N}$ and $\theta_4 = 79.3°$ below the positive x-axis.

DOUBLE-CHECK: The direction that force F_4 is applied is consistent with the diagram and the magnitude of the force is reasonable.

4.35. **THINK:** The given quantities are the masses of the four weights, $m_1 = 6.50$ kg, $m_2 = 3.80$ kg, $m_3 = 10.70$ kg and $m_4 = 4.20$ kg. Determine the tension in the rope connected m_1 and m_2.

SKETCH: Focus on an arbitrary point between m_1 and m_2.



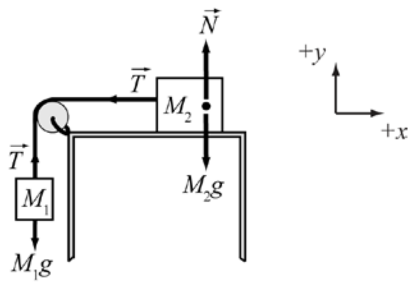
RESEARCH: The masses are in equilibrium, so the sum of the forces in the vertical direction is equal to zero. Therefore the tension, T in the rope between m_1 and m_2 is equal to the force exerted by gravity due to masses m_2, m_3 and m_4: $T - m_2 g - m_3 g - m_4 g = 0$.

SIMPLIFY: $T = \left(m_2 + m_3 + m_4\right)g$

CALCULATE: $T = (3.80\text{ kg} + 10.70\text{ kg} + 4.20\text{ kg})9.81\text{ m/s}^2 = 183.447\text{ N}$

ROUND: There are three significant figures provided in the question so the answer should be written $T = 183$ N.
DOUBLE-CHECK: Tension is a force and the result has units of force (Newtons). The value of the tension is also reasonable considering the masses of the objects.

4.36. **THINK:** The value of the hanging mass is $M_1 = 0.50$ kg and the mass that is on the frictionless table is $M_2 = 1.50$ kg. The masses are attached by a light string with a mass that can be neglected. Determine the magnitude of the acceleration, a of M_2.
SKETCH:



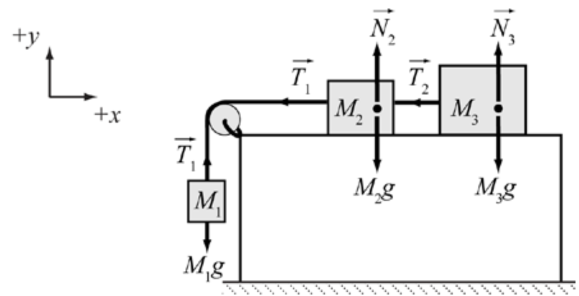
RESEARCH: Force is a vector so the components must be considered separately. Equations can be written for the components of the force, $\vec{F}_2$ acting on mass, m_2: $\sum F_{2,y} = 0 = N - M_2 g$, $\sum F_{2,x} = M_2 a = T$. The two masses are connected, so they accelerate at the same rate. Consider the components of the force, $\vec{F}_1$ acting on the mass, M_1: $\sum F_{1,x} = 0$, $\sum F_{1,y} = -M_1 a = T - M_1 g$. The two expressions of interest are $T - M_2 a = 0$ and $T - M_1 g = -M_1 a$.
SIMPLIFY: To determine a, T must be eliminated. Since the masses are rigidly connected by the string, the tension, T in the rope is constant so both equations can be solved for T, then equated to solve for a: $T = M_2 a$ (1), $T = M_1 g - M_1 a$ (2). Therefore,

$$M_2 a = M_1 g - M_1 a \Rightarrow a(M_2 + M_1) = M_1 g \Rightarrow a = \frac{M_1 g}{(M_2 + M_1)}.$$

CALCULATE: $a = \dfrac{0.50 \text{ kg}(9.81 \text{ m/s}^2)}{(1.50 \text{ kg} + 0.50 \text{ kg})} = 2.4525 \text{ m/s}^2$
ROUND: The least number of significant figures provided in the question is two, so the answer should be written $\vec{a} = 2.5 \text{ m/s}^2$.
DOUBLE-CHECK: The answer has the correct units for acceleration. Also, the value is approximately ¼ of the acceleration due to gravity, so it is reasonable.

4.37. **THINK:** The given quantities are the masses $M_1 = 0.500$ kg, $M_2 = 1.50$ kg and $M_3 = 2.50$ kg. The masses are connected by a light string with a mass that can be neglected. The string attaching M_1 is routed over a frictionless pulley. M_2 and M_3 rest on a frictionless table. Determine the magnitude of the acceleration of block 3, a, and the tension in the string between M_1 and M_2.

SKETCH:



RESEARCH: The forces acting on each of the three masses need to be considered separately. The objects are rigidly connected so they accelerate at the same rate, $a_1 = a_2 = a_3 = a.$ The forces on M_3 are $F_{3,y} = 0 = N_3 - M_3g$ (1), and $F_{3,x} = M_3a = T_2$ (2). The forces on M_2 are $F_{2,y} = 0 = N_2 - M_2g$ (3) and $F_{2,x} = M_2a = T_1 - T_2 \Rightarrow T_2 = T_1 - M_2a$ (4). The forces on M_1 are $F_{1,y} = -M_1a = T_1 - M_1g$ (5) and $F_{1,x} = 0$ (6). Substituting equation (4) into equation (2) yields $M_3a = T_1 - M_2a$ (7). This implies that $T_1 = M_3a + M_2a = (M_3 + M_2)a.$

SIMPLIFY:

(a) $a = \dfrac{M_1g}{M_3 + M_2 + M_1}$

(b) The tension between M_1 and M_2 is $T_1 = (M_3 + M_2)a.$

CALCULATE:

(a) $a = \dfrac{(0.500\text{ kg})(9.81\text{ m/s}^2)}{(2.50\text{ kg} + 1.50\text{ kg} + 0.500\text{ kg})} = 1.09\text{ m/s}^2$

(b) $T_1 = (2.50\text{ kg} + 1.50\text{ kg})(1.09\text{ m/s}^2) = 4.36\text{ N}$

ROUND: The number of significant figures given in the question was three.

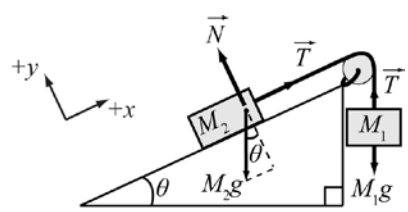
(a) $a = 1.09\text{ m/s}^2$

(b) $T_1 = 4.36\text{ N}$

DOUBLE-CHECK: The units calculated for the acceleration and the tension are appropriate SI units. Also, the values determined are reasonable considering the masses involved.

4.38. **THINK:** The given quantities are the masses, $M_1 = 0.400\text{ kg}$ and $M_2 = 1.20\text{ kg}.$ The hanging mass, M_1 is attached by a light rope to mass, M_2 across a frictionless pulley. M_2 is initially at rest on a frictionless ramp that is elevated at an angle of $\theta = 30.0°$ above the horizontal. Determine the magnitude and direction of the acceleration of M_2.

SKETCH:



RESEARCH: The forces acting on the masses must be considered separately. The forces acting on M_2 are $F_{2,p} = 0 = N - M_2g\cos\theta$ (1) and $F_{2,R} = -M_2a_2 = -T + M_2g\sin\theta$ (2). We have to choose a direction for acceleration so we chose for M_2 to move up the ramp – if we are wrong then the value for the acceleration will be negative. If M_2 moves up the ramp then M_1 must be descending, so the forces acting on M_1 are

$F_{1,y} = -M_1 a_1 = T - M_1 g$ (3). Since the two masses are rigidly attached, they both accelerate at the same rate, $a_1 = a_2 = a$. Also, the tension, *T*, in the rope between the masses is the same in equations (2) and (3).

SIMPLIFY: From equation (2), $T = M_2 g \sin\theta + M_2 a$ (4). From equation (3), $T = M_1 g - M_1 a$ (5). Therefore $M_2 g \sin\theta + M_2 a = M_1 g - M_1 a$. Solving for *a* gives: $a_2 = a = \dfrac{g(M_1 - M_2 \sin\theta)}{M_1 + M_2}$.

CALCULATE: $a_2 = \dfrac{9.81 \text{ m/s}^2 \left(0.400 \text{ kg} - \left(1.20 \text{ kg}\right)\sin\left(30.0°\right)\right)}{\left(1.20 \text{ kg} + 0.400 \text{ kg}\right)} = -1.226 \text{ m/s}^2$

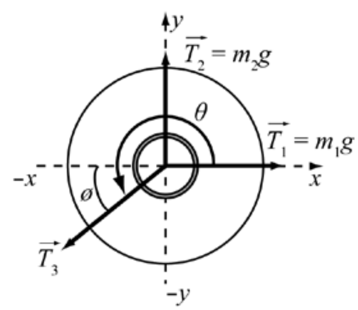
Therefore, $a_2 = 1.226 \text{ m/s}^2$ down the ramp.

ROUND: Rounding to three significant figures, the answer should be written $a_2 = 1.23 \text{ m/s}^2$ down the ramp.

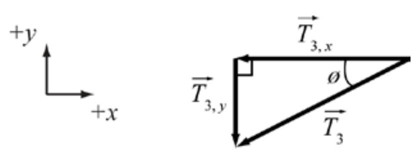
DOUBLE-CHECK: The calculated acceleration has the correct units. Also, it makes sense that a_2 is down the ramp because the force due to gravity pulling M_2 down the ramp is greater than the force exerted on M_2 up the ramp due to the force of gravity on M_1. In addition we find the limit of the acceleration of the Atwood machine in the limit of $\theta = 90°$ (See Example 4.5) and the limit of Example 4.4, Two Blocks Connected by a Rope, for $\theta = 0°$ as limiting cases of our answer. This gives us additional confidence in our solution.

4.39. **THINK:** The given quantities are the hanging masses m_1 and m_2 and the direction of the horizontal forces cause by the hanging masses on the ring. The strings that attach the hanging masses to the ring can be considered massless and the pulleys that the strings are routed through are frictionless. Determine the mass, m_3, and the angle, θ, that will result in the ring being balanced in the middle of the table.

SKETCH: Top-down view:



RESEARCH: A sketch of the *x* and *y* components of T_3 is shown below.



$T_{3,y} = T_3 \sin\phi = m_3 g \sin\phi$

$T_{3,x} = T_3 \cos\phi = m_3 g \cos\phi$

The angle counterclockwise from the positive *x*-axis, θ is given by $\theta = 180° + \phi$. For the ring to be balanced, the sum of the forces in the *x* and *y* directions must be balanced:

$$\sum F_y = 0 = T_2 - T_3 \sin\phi = m_2 g - m_3 g \sin\phi \quad (1)$$

$$\sum F_x = 0 = T_1 - T_3 \cos\phi = m_1 g - m_3 g \cos\phi \quad (2)$$

SIMPLIFY: Solve equation (1) in terms of m_3 and substitute into equation (2) to solve for ϕ. $m_3 = m_2 g / g \sin\phi = m_2 / \sin\phi$ substituted into (2) yields:

$$m_1 g = m_3 g \cos\phi \;\Rightarrow\; m_1 = \frac{m_2 \cos\phi}{\sin\phi} \;\Rightarrow\; \frac{\sin\phi}{\cos\phi} = \frac{m_2}{m_1} \;\Rightarrow\; \tan\phi = \frac{m_2}{m_1} \;\Rightarrow\; \phi = \tan^{-1}\left(\frac{m_2}{m_1}\right).$$

CALCULATE: $\phi = \tan^{-1}\left(\frac{0.0300 \text{ kg}}{0.0400 \text{ kg}}\right) = 36.8698°$, $\theta = 180° + 36.8698° = 216.8698°$,

$$m_3 = \frac{0.030 \text{ kg}}{\sin(36.8698°)} = 0.05000 \text{ kg}$$

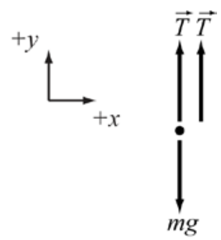
ROUND: Four significant figures are provided in the question, so the answers should be written $\theta = 216.8°$ and $m_3 = 0.0500$ kg.

DOUBLE-CHECK: By observing the sketch, it can be seen that the value of θ is reasonable to balance the forces. The mass is also a reasonable value.

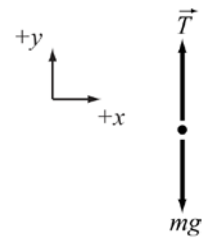
4.40. **THINK:** Consider the combined mass, m of the monkey and the wood plate when m = 100. kg. Assume the rope's mass is negligible and the system is frictionless. In part (b), the acceleration is $a = 2.45 \text{ m/s}^2$.

SKETCH:

For monkey/plate system:



For second monkey of mass, m_2 pulling on rope, free body diagram for m becomes:



RESEARCH:

(a) The minimum force the monkey needs to apply to lift off the ground is the force T required to balance the force of gravity due to the combined mass, m: $2T - mg = 0 \Rightarrow T = mg/2$.

(b) $\sum F_y = ma = 2T - mg$. There is now a net upward force due to the monkey.

(c) If there is a second monkey, then in part (a), $T = mg$ and the equation in (b) becomes $T = m(a+g)$.

SIMPLIFY:

(a) It is not necessary to simplify.

(b) $T = \frac{ma + mg}{2} = \frac{m(a+g)}{2}$

(c) It is not necessary to simplify.

CALCULATE:

(a) $T = \frac{(100. \text{ kg})(9.81 \text{ m/s}^2)}{2} = 490.5 \text{ N}$

(b) $T = \frac{(100. \text{ kg})(2.45 \text{ m/s}^2 + 9.81 \text{ m/s}^2)}{2} = 613 \text{ N}$

(c) $T(\text{no } a) = (100. \text{ kg})(9.81 \text{ m/s}^2) = 981.0 \text{ N}$

$T(a = 2.45 \text{ m/s}^2) = (100. \text{ kg})(2.45 \text{ m/s}^2 + 9.81 \text{ m/s}^2) = 1226 \text{ N}$

ROUND: The acceleration has three significant figures, so the answers should be written:

(a) T = 491 N

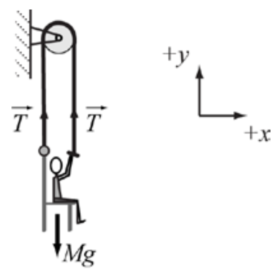
(b) T = 613 N

(c) T = 981 N, 1230 N

DOUBLE-CHECK: The calculated values seem reasonable for the mass of the system of interest. It is reasonable that a larger force must be applied to give the system an upward acceleration. If a second monkey pulled the rope from the ground, the tension, T would be due to the full gravitational force of mass, m, so it is reasonable that the answers for part (c) doubled.

4.41. **THINK:** The rope has negligible mass and the pulley is frictionless. The chair and boatswain have a combined mass, M, of 90.0 kg. Consider two cases: (a) The magnitude of the force the boatswain must pull on the rope with to achieve constant velocity, $\vec{v}_c$ and (b) the maximum force the boatswain must pull on the rope with to achieve an acceleration of $a_{\text{max}} = 2.00 \text{ m/s}^2$.

SKETCH:



RESEARCH:

(a) If the boatswain is moving at a constant velocity, then there is no net force on the system because $dv/dt = 0$ for a constant v. For this case, the sum of the forces is $\sum F = 0 = 2T - Mg$. The force the boatswain must pull with is $F = T$.

(b) In the case where the boatswain is accelerating, the maximum force, F_{max} can be substituted in to the sum of the forces equation, $2F_{\text{max}} - Mg = Ma_{\text{max}}$.

SIMPLIFY:

(a) $2T = Mg$. Substitute the force, F the boatswain must pull on the rope into the equation: $F = Mg/2$.

(b) Rearrange to solve for F_{max}: $F_{\text{max}} = \dfrac{M\left(g + a_{\text{max}}\right)}{2}$.

CALCULATE:

(a) $F = \dfrac{\left(90.0 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right)}{2} = 441.45 \text{ N}$

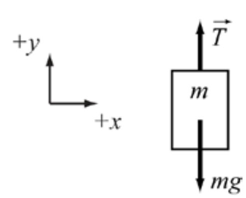
(b) $F_{\text{max}} = \dfrac{90.0 \text{ kg}\left(9.81 \text{ m/s}^2 + 2.00 \text{ m/s}^2\right)}{2} = 531.45 \text{ N}$

ROUND: To three significant figures, the answers are (a) $F = 441 \text{ N}$ and (b) $F_{\text{max}} = 531 \text{ N}$.

DOUBLE-CHECK: The unit of Newtons is a correct unit of force. It is reasonable that the boatswain had to pull on the rope with more force to cause a net acceleration than when the system moved at constant speed.

4.42. **THINK:** A granite block of mass, m = 3311 kg is suspended from a pulley system. The rope is wound around the pulley system six times. Assume the rope is massless and the pulley is frictionless. Determine the force, F, the rope would have to be pulled with to hold m in equilibrium.

SKETCH:



RESEARCH: The diagram suggests that only the rope attaches to the block, which means that for the block to be in equilibrium, $T - mg = 0$. The tension in the end of the rope that is pulled on is the same as the tension in the rope where it attaches to the block. The tension in the rope of a pulley system is given by $T = mg/2n$, where n is the number of loops the rope makes around the pulley system.

SIMPLIFY: The force required to hold the system in equilibrium must be equal to the tension:

$$F = T = \frac{mg}{2n}.$$

CALCULATE: $F = \dfrac{(3311\text{ kg})(9.81\text{ m/s}^2)}{(2)(6)} = 2707\text{ N}$

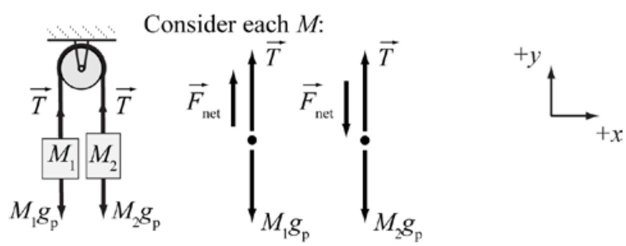
ROUND: The value of g used in the calculation has three significant figures, so the answer is rounded to $F = 2710$ N.

DOUBLE-CHECK: The units of the result are Newtons, which are a unit of force. The calculated value is reasonable, considering the mass of the block, although it would be very difficult for one person to hold this mass in equilibrium.

4.43. **THINK:** Two masses, $M_1 = 100.0$ g and $M_2 = 200.0$ g are placed on an Atwood device. Each mass moves a distance, $\Delta y = 1.00$ m in a time interval of $\Delta t = 1.52$ s. Determine the gravitational acceleration, g_p for the planet and the tension, T in the string. The string is massless and the pulley is frictionless. M_1 and M_2 should be converted to the SI unit of kg.

$$M_1 = (100.0\text{ g})\left(\frac{1\text{ kg}}{1000\text{ g}}\right) = 0.1000\text{ kg},\quad M_2 = (200.0\text{ g})\left(\frac{1\text{ kg}}{1000\text{ g}}\right) = 0.2000\text{ kg}$$

SKETCH:



RESEARCH:

(a) The masses are initially at rest, therefore their initial speed $v_0 = 0$. Because the masses are rigidly connected, they accelerate at the same rate, a. The net force for one mass is upward and downward for the other. The value of a can be determined using the kinematic equation $\Delta y = v_0 t + (at^2)/2$. Because the masses are initially at rest, the equation reduces to $\Delta y = (at^2)/2$ or $a = 2\Delta y/t^2$. If the forces on mass M_1 are considered, the net force equation is $F_\text{net} = M_1 a = T - M_1 g_\text{p}$. For mass M_2, the net force equation is $F_\text{net} = -M_2 a = T - M_2 g_\text{p}$.

(b) Solve for g_p and substitute into the force equation to solve for T.

SIMPLIFY:

(a) $M_1 a = T - M_1 g_\text{p}$ (1), $-M_2 a = T - M_2 g_\text{p}$ (2)

Because the tensions in the ends of the rope are the same, solve equations (1) and (2) in terms of T and equate the expressions.

$$M_2 g_\text{p} - M_2 a = M_1 g_\text{p} + M_1 a \Rightarrow a(M_1 + M_2) = g_\text{p}(M_2 - M_1) \Rightarrow g_\text{p} = a\left(\frac{M_1 + M_2}{M_2 - M_1}\right)$$

Substitute for a using $\Delta y = (at^2)/2$ to get $g_\text{p} = \dfrac{2\Delta y}{t^2}\left(\dfrac{M_1 + M_2}{M_2 - M_1}\right)$.

(b) $T = M_2\left(g_{\text{p}} - a\right) = M_2\left(g_{\text{p}} - \dfrac{2\Delta y}{t^2}\right)$

CALCULATE:

(a) $g_{\text{p}} = \dfrac{2(1.00\text{ m})}{(1.52\text{ s})^2}\left(\dfrac{0.1000\text{ kg} + 0.2000\text{ kg}}{0.2000\text{ kg} - 0.1000\text{ kg}}\right) = 2.59695\text{ m/s}^2$

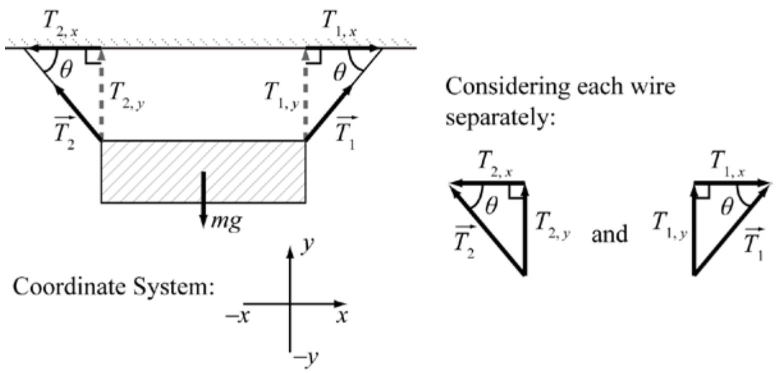
(b) $T = 0.2000\text{ kg}\left(2.59695\text{ m/s}^2 - \dfrac{2(1.00\text{ m})}{(1.52\text{ s})^2}\right) = 0.346260\text{ N}$

ROUND: To three significant figures, the answers should be (a) $g_{\text{p}} = 2.60\text{ m/s}^2$ and (b) $T = 0.346$ N.

DOUBLE-CHECK: The units for the calculated answers were the correct units of acceleration and force. The small tension calculated is reasonable, considering the small masses.

4.44. **THINK:** The mass of the sign is given as $m = 4.25$ kg. The sign is hung by 2 wires that each makes an angle of $\theta = 42.4°$ with the ceiling. Determine the tension in each wire.

SKETCH:



RESEARCH: Because the sign is in equilibrium, the sum of the forces in the x- and y-directions must equal zero. The sum of the forces in the y-direction is $\sum F_y = 0 = T_{1,y} + T_{2,y} - mg$, where $T_{1,y} = T_1 \sin\theta$ and $T_{2,y} = T_2 \sin\theta$. Inserting, the expression can be written $T_1 \sin\theta + T_2 \sin\theta = mg$. The sum of the forces in the x-direction can be written $\sum F_x = 0 = T_{1,x} - T_{2,x}$, where $T_{1,x} = T_1 \cos\theta$ and $T_{2,x} = T_2 \cos\theta$. Therefore, $T_1 \cos\theta = T_2 \cos\theta$ or $T_1 = T_2$.

SIMPLIFY: Because $T_1 = T_2$, the tension can simply be called T. The forces in the y-direction can then be simplified, $2T\sin\theta = mg \;\Rightarrow\; T = mg/\left(2\sin\theta\right)$.

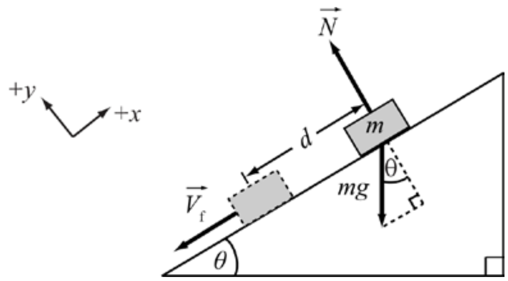
CALCULATE: $T = \dfrac{(4.25\text{ kg})(9.81\text{ m/s}^2)}{2\sin(42.4°)} = 30.9153\text{ N}$

ROUND: Rounding to three significant figures, $T = 30.9$ N.

DOUBLE-CHECK: The calculated tension has Newtons as the unit, which is the correct SI unit for force. The calculated value is reasonable considering the given mass and angle. Note that in the limit of $\theta \to 0$ we recover $T = mg/2$, i.e. the tension in each wire is equal to half of the weight of the sign, as expected. For all values $\theta > 0$ the tensions in the wires are larger than this value at $\theta \to 0$, which is also comforting. Finally, as $\theta \to 90°$ the tensions in the wires become infinitely large, which is also expected.

4.45. **THINK:** A crate of oranges with mass m slides down a frictionless incline. The crate has an initial velocity $v_{\text{i}} = 0$ and a final velocity $v_{\text{f}} = 5.832$ m/s after sliding a distance, $d = 2.29$ m. Determine the angle of inclination, θ with respect to the horizontal.

SKETCH:



RESEARCH: The net acceleration, *a,* of *m* down the ramp can be determined from the equation $v_f^2 = v_i^2 + 2ad.$ Sum all of the forces acting on the crate down the ramp: $\sum F_{\text{ramp}} = ma = mg\sin\theta.$

SIMPLIFY: $v_i = 0$, therefore $a = v_f^2 / 2d.$ Substitute this relation into the equation for the forces acting down the ramp:

$$\frac{mv_f^2}{2d} = mg\sin\theta \Rightarrow \sin\theta = \frac{v_f^2}{2dg} \Rightarrow \theta = \sin^{-1}\left(\frac{v_f^2}{2dg}\right).$$

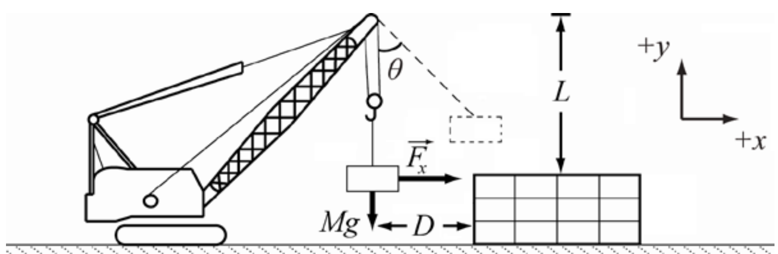
CALCULATE: $\theta = \sin^{-1}\left(\frac{(5.832 \text{ m/s})^2}{2(2.29 \text{ m})(9.81 \text{ m/s}^2)}\right) = 49.20114669°$

ROUND: Rounding to three significant figures, $\theta = 49.2°.$

DOUBLE-CHECK: The inclination of the plane is reasonable.

4.46. **THINK:** The bricks have a mass $M = 200.0$ kg and are attached to a crane by a cable of negligible mass and length, L = 3.00 m. In the initial vertical position, the bricks are a distance D = 1.50 m from the wall. Determine the magnitude of the horizontal force, F_x, that must be applied to the bricks to position them directly above the wall.

SKETCH:



RESEARCH: The given lengths can be used to solve for the angle, θ, that the bricks move through: $\tan\theta = D / L$ (1). By similar reasoning, the angle in the force vector diagram is given by:
$\tan\theta = F_x / (Mg)$ (2).

SIMPLIFY: Because θ is the same angle in both cases, equations (1) and (2) can be equated:

$$\frac{D}{L} = \frac{F_x}{Mg} \Rightarrow F_x = \frac{MgD}{L}.$$

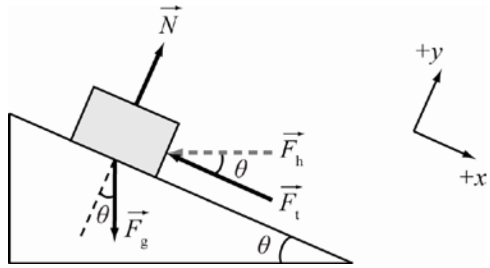
CALCULATE: $F_x = \frac{(200.0 \text{ kg})(9.81 \text{ m/s}^2)(1.50 \text{ m})}{3.00 \text{ m}} = 981 \text{ N}$

ROUND: To three significant figures $F_x = 981$ N.

DOUBLE-CHECK: The unit of the calculated value is Newtons, which is a SI unit for force. The result is reasonable considering the mass of the bricks.

4.47. **THINK:** The problem asks for the force needed to hold the block in place. This means that the net force on the block has to be zero in each case, $F_{\text{net}\,x} = F_{\text{net}\,y} = 0$. The only forces to consider are the force of gravity, which act straight downward, the normal force from the plane, which is perpendicular to the plane, and the third external force we are asked to apply in parts (a) and (b). We do not need to consider friction forces, because the problem stipulates a "frictionless ramp".

SKETCH:



RESEARCH:

(a) To find F_t , the forces acting in the *x* direction on the block must be balanced.

(b) Note that now F_t and F_h are related by $F_t = F_h \cos\theta$.

SIMPLIFY:

(a) $F_{\text{net}x} = -F_t + F_{gx} = 0, \;\; F_t = F_{gx} = mg\sin\theta$

(b) $F_h = F_t / \cos\theta = mg\sin\theta / \cos\theta = mg\tan\theta$

CALCULATE:

(a) $F_t = (80.0 \text{ kg})(9.81 \text{ m/s}^2)\sin(36.9°) = 471.2 \text{ N}$

(b) $F_h = (80.0 \text{ kg})(9.81 \text{ m/s}^2)\tan(36.9°) = 589.2 \text{ N}$

ROUND: To three significant figures,

(a) $F_t = 471$ N and

(b) $F_h = 589$ N.

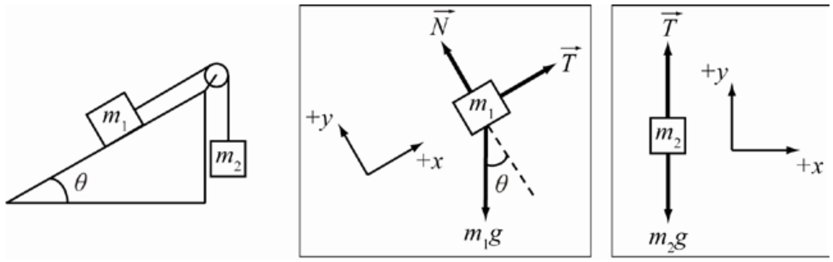
DOUBLE-CHECK: With almost all problems involving inclined planes, such as this one, one can obtain great insight and perform easy checks of the algebra by considering the limiting cases of the angle θ of the plane approaching 0 and 90 degrees.

In the case of $\theta \to 0°$ the block will simply sit on a horizontal surface, and no external force should be required to hold it in that position. Our calculations are compatible with this, because $\sin 0° = \tan 0° = 0$.

In the case of $\theta \to 90°$ our results for parts (a) and (b) should be very different. In part (a) the force acts long the plane and so will be straight up in this limit, thus balancing the weight of the block all by itself. Therefore, as $\theta \to 90°$, we expect our force to approach $F_t(\theta \to 90°) = mg$. This is satisfied in our solution because $\sin 90° = 1$. In part (b) the external force will act perpendicular to the plane in the limit of $\theta \to 90°$. Thus almost no part of it will be available to balance the weight of the block, and consequently an infinitely big force magnitude should be required. This is also born out by our analytic result for part (b), because $\tan 90° \to \infty$.

4.48. **THINK:** The mass, $m_1 = 20.0$ kg. The ramp angle is $\theta = 30.0°$. The acceleration of the masses is $a_1 = a_2 = 0$.

SKETCH:



RESEARCH: For m_2, $F_{\text{net},y} = T - m_2 g = 0$. Determine T from the sum of forces on m_1.

SIMPLIFY: For m_1, $F_{\text{net},x} = T - F_{\text{g}1,x} = 0$, $T - m_1 g\sin\theta = 0$, and $T = m_1 g \sin\theta$. Then for m_2, $T = m_2 g \Rightarrow m_2 = T / g = m_1 \sin\theta$.

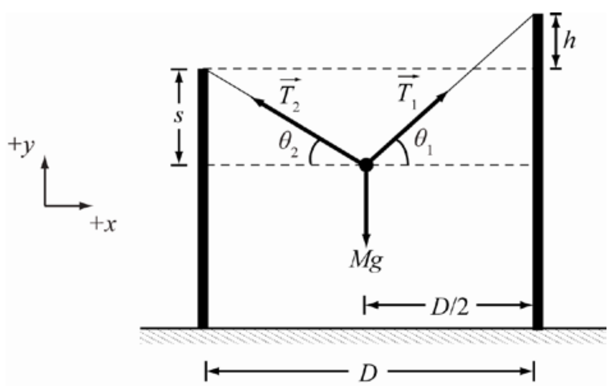
CALCULATE: $m_2 = (20.0 \text{ kg})\sin(30.0°) = 10.0 \text{ kg}$

ROUND: $m_2 = 10.0$ kg.

DOUBLE-CHECK: m_2 is the same order of magnitude as m_1.

4.49. **THINK:** The piñata's mass is M = 8.00 kg. The distance between the poles is D = 2.00 m. The difference in pole heights is h = 0.500 m. The vertical distance between the shorter pole and the piñata is s = 1.00 m. Horizontally, the piñata is $D/2$ from each pole. Determine the tension in each part of the rope, T_1 and T_2. Note, $F_{\text{net}x} = F_{\text{net}y} = 0$.

SKETCH:



RESEARCH: To find T_1 and T_2, balance the forces on the piñata in each direction. θ_1 and θ_2 can be determined from trigonometry.

SIMPLIFY: Find θ_1: $\tan\theta_1 = \dfrac{h+s}{D/2} \Rightarrow \theta_1 = \tan^{-1}\left(\dfrac{2(h+s)}{D}\right)$. Similarly, $\theta_2 = \tan^{-1}\left(\dfrac{2s}{D}\right)$.

$$F_{\text{net},x} = T_{1x} - T_{2x} = 0 \Rightarrow T_1\cos\theta_1 - T_2\cos\theta_2 = 0 \quad (1)$$

$$F_{\text{net},y} = T_{1y} + T_{2y} - mg = 0 \Rightarrow T_1\sin\theta_1 + T_2\sin\theta_2 - mg = 0 \quad (2)$$

Solve for T_1 in terms of T_2 in (1) and substitute into (2):

$$\left(T_2\frac{\cos\theta_2}{\cos\theta_1}\right)\sin\theta_1 + T_2\sin\theta_2 - mg = 0 \Rightarrow T_2\left(\cos\theta_2\tan\theta_1 + \sin\theta_2\right) = mg \Rightarrow T_2 = \frac{mg}{\left(\cos\theta_2\tan\theta_1 + \sin\theta_2\right)}.$$

CALCULATE: $\theta_1 = \tan^{-1}\left(\dfrac{2(0.500\text{ m}+1.00\text{ m})}{2.00\text{ m}}\right) = 56.31°,\ \theta_2 = \tan^{-1}\left(\dfrac{2.00\text{ m}}{2.00\text{ m}}\right) = 45.0°$

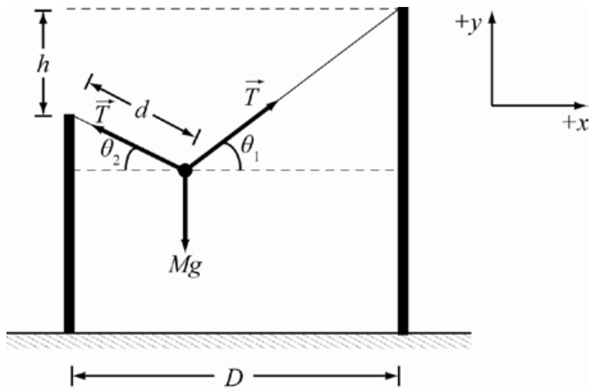
$$T_2 = \frac{(8.00\text{ kg})(9.81\text{ m/s}^2)}{\left(\cos(45.0°)\tan(56.31°)+\sin(45.0°)\right)} = 44.39\text{ N},\ T_1 = (44.39\text{ N})\frac{\cos(45.0°)}{\cos(56.31°)} = 56.59\text{ N}$$

ROUND: With all the given values containing three significant figures, $T_1 = 56.6\text{ N}$ and $T_2 = 44.4\text{ N}$.

DOUBLE-CHECK: Both T_1 and T_2 are less than the weight of the piñata and are reasonable values.

4.50. **THINK:** The piñata's mass is M = 12 kg. The distance between the poles is D = 2.0 m. The difference in pole height is h = 0.50 m. The rope length is L = 3.0 m. The piñata is in equilibrium, so $F_{\text{net},x} = F_{\text{net},y} = 0$. Determine (a) the distance from the top of the lower pole to the ring, d and (b) the tension in the rope, T.

SKETCH:



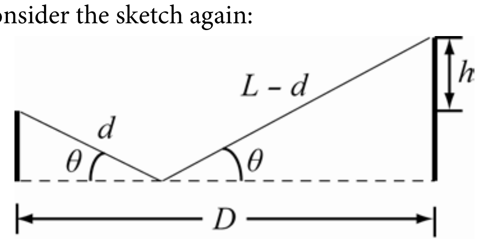
RESEARCH:

(a) Trigonometry can be used to find d. Also, a relation between the angles θ_1 and θ_2 can be established by balancing the horizontal forces on the ring: $F_{\text{net},x} = \sum F_x = 0$.

(b) To find T, the horizontal and vertical forces on the ring must be balanced. The magnitude of T is the same on each side of the ring because it is the same rope.

SIMPLIFY:

(a) First, determine how θ_1 and θ_2 relate: $F_{\text{net},x} = T\cos\theta_1 - T\cos\theta_2 = 0 \Rightarrow T\cos\theta_1 = T\cos\theta_2$. Then, $\theta_1 = \theta_2 = \theta$. To find d, consider the sketch again:



$$d\cos\theta + (L-d)\cos\theta = D \Rightarrow L\cos\theta = D \Rightarrow \theta = \cos^{-1}(D/L)$$

$$(L-d)\sin\theta - d\sin\theta = h \Rightarrow L\sin\theta - 2d\sin\theta = h \Rightarrow d = \frac{L\sin\theta - h}{2\sin\theta}$$

(b) From (a), $\theta = 48.19°$. To determine T, consider: $F_{\text{net},y} = 2T\sin\theta - Mg = 0$. Then, $T = \dfrac{Mg}{2\sin\theta}$.

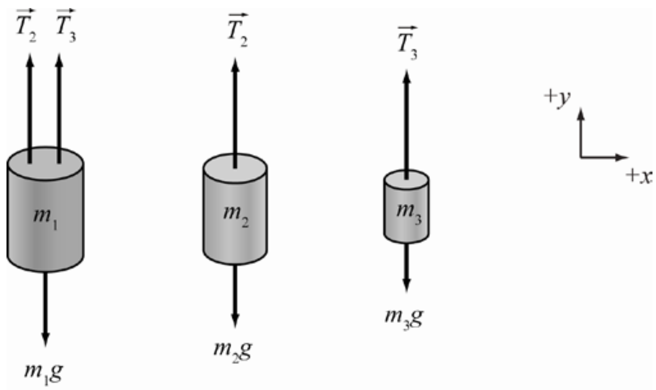
CALCULATE:

(a) $\theta = \cos^{-1}\left(\dfrac{2.0\text{ m}}{3.0\text{ m}}\right) = 48.19°,\ d = \dfrac{(3.0\text{ m})\sin(48.19°) - 0.50\text{ m}}{2\sin(48.19°)} = \dfrac{1.736\text{ m}}{1.4907} = 1.1646\text{ m}$

(b) $T = \dfrac{(12\text{ kg})(9.81\text{ m/s}^2)}{2\sin(48.19^\circ)} = 78.97\text{ N}$

ROUND:
(a) Rounding to two significant figures, d = 1.2 m.
(b) Rounding to two significant figures, T = 79 N.
DOUBLE-CHECK: As they should be, d is less than D and T is less than Mg.

4.51. **THINK:** The masses are given as $m_1 = 36.5\text{ kg}$, $m_2 = 19.2\text{ kg}$ and $m_3 = 12.5\text{ kg}$. Determine the acceleration of m_1, a_1. As there are no forces in the x-direction, only the y-direction needs to be considered.
SKETCH:



RESEARCH: To determine a_1, use $F_{1\text{net},y} = \sum F_{1y}$. Determine T_2 and T_3 by summing the forces on m_2 and m_3. A key idea is that $a_1 = -a_2 = -a_3$, as all the masses are connected (and ignoring any tipping of m_1). With $m_1 > m_2 + m_3$, it can be seen that m_1 moves downward while m_2 and m_3 move upward.
SIMPLIFY: $m_2: F_{2\text{net}} = T_2 - m_2 g \Rightarrow m_2 a_2 = T_2 - m_2 g \Rightarrow T_2 = m_2(a_2 + g)$.
$m_3: F_3 = T_3 - m_3 g \Rightarrow m_3 a_3 = T_3 - m_3 g \Rightarrow T_3 = m_3(a_3 + g)$
$m_1: F_{1\text{net}} = T_2 + T_3 - m_1 g \Rightarrow m_1 a_1 = m_2(a_2 + g) + m_3(a_3 + g) - m_1 g$
With $a_1 = -a_2 = -a_3$,

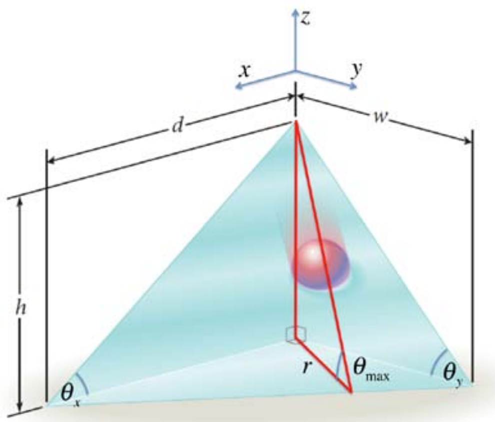
$$\begin{aligned} m_1 a_1 &= m_2(-a_1 + g) + m_3(-a_1 + g) - m_1 g \\ m_1 a_1 + m_2 a_1 + m_3 a_1 &= m_2 g + m_3 g - m_1 g \\ a_1(m_1 + m_2 + m_3) &= g(m_2 + m_3 - m_1) \\ a_1 &= \frac{g(m_2 + m_3 - m_1)}{(m_1 + m_2 + m_3)} \end{aligned}$$

CALCULATE: $a_1 = \dfrac{9.81\text{ m/s}^2(19.2\text{ kg} + 12.5\text{ kg} - 36.5\text{ kg})}{(36.5\text{ kg} + 19.2\text{ kg} + 12.5\text{ kg})} = -0.69044\text{ m/s}^2$

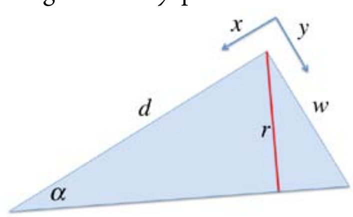
ROUND: There are two significant figures in the sum in the numerator, so the answer should be written, $a_1 = 0.69\text{ m/s}^2$ downward.
DOUBLE-CHECK: a_1 is less than g in magnitude, which it should be for this system.

4.52. **THINK:** The block's dimensions are w = 1.165 m, d = 1.648 m and h = 1.051 m. The paperweight's mass is m = 16.93 kg. Determine the paperweight's acceleration, a, down the incline. We first need to realize that the paperweight will slide down the plane in the direction of the steepest slope, i.e. the direction that has the smallest angle with the vertical. (This is the same as for a golf ball rolling down the side of a hill on a putting green.) The most difficult part of this problem is to find which way the steepest slope points. At the edges of our slope the angles are $\theta_x = \tan^{-1}(h/d)$ and $\theta_y = \tan^{-1}(h/w)$. But the question is if there is a steeper angle somewhere in between the x- and y-directions. One may be tempted to think that this leads to a minimization problem involving some sort of derivative. However, there is a shortcut, if one realizes that the angle of steepest descent, $\theta_{\max}$, is the one for which the bottom of the slope has the shortest distance, r, to the corner of the block.

SKETCH: We use the figure in the problem and indicate our coordinate system at the top. The red triangle containing the angle $\theta_{\max}$ is a right triangle with side lengths h along the z-direction and r in the xy-plane. The hypotenuse of this triangle is the path along which the paperweight slides down.



It is also instructive to draw a top view of the bottom triangle in the xy-plane, because it helps us to determine the length of the distance r. From this drawing we see that the direction of r has to be such that it forms the height of the right triangle in the xy-plane.



RESEARCH: From trigonometry we know that the angle $\alpha = \tan^{-1}(w/d)$ and that the length of r is then given by $r = d\sin\alpha$. Once we have r, we can compute the angle $\theta_{\max} = \tan^{-1}(h/r)$. The magnitude of the acceleration is then calculated as $a = g\sin\theta_{\max}$, which is universally the case for inclined plane problems.

SIMPLIFY: Inserting all intermediate results, we find

$$\begin{aligned} a = g\sin\theta_{\max} &= g\sin\left(\tan^{-1}\left[\frac{h}{r}\right]\right) \\ &= g\sin\left(\tan^{-1}\left[\frac{h}{d\sin\alpha}\right]\right) \\ &= g\sin\left(\tan^{-1}\left[\frac{h}{d\sin\left(\tan^{-1}(w/d)\right)}\right]\right) \end{aligned}$$

CALCULATE: Inserting our given numbers results in

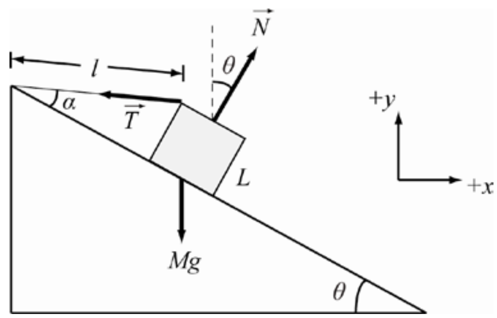
$$a = (9.81\text{ m/s}^2)\sin\left(\tan^{-1}\left[\frac{105.1}{164.8\sin\left(\tan^{-1}(116.5/164.8)\right)}\right]\right) = 7.27309\text{ m/s}^2$$

ROUND: Since the gravitational acceleration is only specified to three significant digits, we also round our result to $a = 7.27 \text{ m/s}^2$.

DOUBLE-CHECK: Our result is smaller than the free-fall value of the gravitational acceleration, which is comforting. What about limits? As $h \to 0$ we find $a \to 0$, as expected. Also, we find that in either limit, $d \to 0$ or $w \to 0$, that our acceleration value approaches $a \to g$, which is also expected.

4.53. **THINK:** The only forces that act on the block are the force of gravity, the normal force from the ramp, and the tension force from the rope. Gravity, as always, points straight down. The normal force, also as always, is perpendicular to the plane. And the tension force points in the direction of the rope. No friction is present. The block is held in position, which implies force equilibrium. The only direction that motion could occur in is along the plane, and so we should try to compute the force components along the direction of the plane and make sure they add up to 0, as required by Newton's first law.

SKETCH:



RESEARCH: Note that we have drawn the angle α, where $\alpha = \sin^{-1}(L/l)$. It is the angle between the rope and the plane. The component of the tension force along the plane is $T\cos\alpha$, where we have defined the direction up the plane as positive. As always in inclined plane problems, the force component of gravity along the plane is $Mg\sin\theta$. Since these are the only two fore components along the plane, we find:

$$T\cos\alpha - Mg\sin\theta = 0$$

SIMPLIFY: It is now fairly straightforward to solve for the magnitude of the tension in the string:

$$T = \frac{Mg\sin\theta}{\cos\alpha} = \frac{Mg\sin\theta}{\cos\left(\sin^{-1}(L/l)\right)}$$

We can use the trigonometric identity $\cos\phi = \sqrt{1-\sin^2\phi}$ (which is valid for any angle) and then find finally

$$T = \frac{Mg\sin\theta}{\cos\alpha} = \frac{Mg\sin\theta}{\sqrt{1-(L/l)^2}}$$

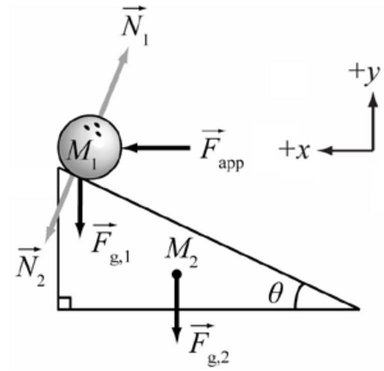
CALCULATE: $T = \dfrac{64.0 \text{ kg}\left(9.81 \text{ m/s}^2\right)\sin(26.0°)}{\sqrt{1-(0.400/1.60)^2}} = 284.2532 \text{ N}$

ROUND: There are three significant figures in all given values, so the answer should be written as $T = 284$ N.

DOUBLE-CHECK: There are two limiting cases that we can study to see if our solution makes sense. First we can see what happens in the limit that L approaches 0. We can see that our solution then reduces to $T = Mg\sin\theta$, which is as expected, because then the rope is along the plane and the tension is simply equal to the component of gravity along the plane. For any value of $L > 0$, the tension has to increase. In the limiting case that the side length of the cube approaches the length of the rope, the tension in the rope has to become infinitely large, which is also born out by our analytic result. This gives us added confidence that we have solved the problem properly.

4.54. **THINK:** The bowling ball's mass is $M_1 = 6.00$ kg. The initial velocity is $v_{b0} = 0$. The wedge's mass is $M_2 = 9.00$ kg and it sits on a frictionless surface. The wedge's angle is $\theta = 36.9°$. Determine (a) the force on the ball, F_{app}, to maintain the ball's vertical position, and (b) the magnitude of the wedge's acceleration, a_2, if $F_{app} = 0$.

SKETCH:



RESEARCH:

(a) $F_{1\text{ net},y} = 0$ in order to keep the ball in the same vertical position. Also, the ball and wedge must have the same velocity in the x direction at all times in order to prevent the ball from rolling down (or up) the wedge $\Rightarrow a_{1,x} = a_{2,x} = a_x$. To determine $\vec{F}_{app}$, sum the forces in the x direction acting on M_2: $F_{2\text{net},x} = \sum F_{2,x}$. The normal force of the ball on the wedge N_2 must be determined. By Newton's third law, $\vec{N}_2 = -\vec{N}_1$. $\vec{N}_1$ can be determined by summing the forces on M_1.

(b) The wedge's acceleration, a_2, in the absence of any external force can be found by writing one equation relating accelerations in the x-direction and three equations relating force to acceleration in the x- and y-directions. The resulting system of four equations can be solved for a_2.

SIMPLIFY:

(a) First, consider M_1: In the y direction, $F_{1\text{ net},y} = 0$ to keep the ball in the same vertical position. Then,

$$F_{1\text{ net},y} = N_{1,y} - F_{g,1} = 0 \ \Rightarrow \ N_1 \cos\theta = M_1 g \ \Rightarrow \ N_1 = \frac{M_1 g}{\cos\theta}.$$

In the x-direction, $F_{1\text{ net},x} = F_{app} - N_{1,x} = F_{app} - N_1 \sin\theta = F_{app} - \dfrac{M_1 g}{\cos\theta}\sin\theta = F_{app} - M_1 g \tan\theta$. Now consider M_2: The net force in the x- direction is $F_{2\text{net},x} = N_{2,x} = N_2 \sin\theta = M_1 g \tan\theta$, where we have used $|\vec{N}_2| = |\vec{N}_1|$ from Newton's third law. To keep the ball stationary with respect to the wedge,

$$a_{1,x} = a_{2,x} \Rightarrow \frac{F_{1\text{net},x}}{M_1} = \frac{F_{2\text{net},x}}{M_2} \Rightarrow \frac{F_{app} - M_1 g \tan\theta}{M_1} = \frac{M_1 g \tan\theta}{M_2}$$

Solving for F_{app} yields $F_{app} = M_1\left(\dfrac{M_1 g \tan\theta}{M_1} + \dfrac{M_1 g \tan\theta}{M_2}\right) = M_1\left(1 + \dfrac{M_1}{M_2}\right) g \tan\theta.$

(b) To determine a_2 when $F_{app} = 0$, note that $v_2 - v_{1,x}$ is the relative velocity at which the wedge is moves horizontally out from under the ball, and therefore $(v_2 - v_{1,x})\tan\theta$ is the rate at which the surface of the wedge drops downward beneath the ball. Since the ball drops, too, maintaining contact with the wedge, this is also the vertical speed of the ball, $|v_{1,y}|$. Using the sign convention in the figure to write the appropriate equation, and then *taking the time derivative of all each side*, we obtain

$$\left(v_2 - v_{1,x}\right)\tan\theta = -v_{1,y}$$
$$\left(a_2 - a_{1,x}\right)\tan\theta = -a_{1,y}$$
$$a_{1,y} = \left(a_{1,x} - a_2\right)\tan\theta \qquad (1)$$

Now, knowing that $|N_1| = |N_2| = N,$ we relate force to acceleration in the x-direction for the wedge, and in both the x- and y-directions for the ball:

$$N\sin\theta = M_2 a_2 \qquad (2)$$
$$N\sin\theta = -M_1 a_{1,x} \qquad (3)$$
$$N\cos\theta - M_1 g = M_1 a_{1,y} \qquad (4)$$

We eliminate N using equations (2) and (3):

$$M_2 a_2 = -M_1 a_{1,x}$$
$$a_{1,x} = -\frac{M_2}{M_1}a_2 \qquad (5)$$

We also eliminate N using equations (2) and (4):

$$\frac{M_2 a_2}{\sin\theta}\cos\theta - M_1 g = M_1 a_{1,y}$$
$$M_2 a_2 \cot\theta - M_1 g = M_1 a_{1,y}$$
$$a_{1,y} = \frac{M_2}{M_1}a_2\cot\theta - g \qquad (6)$$

Finally, we eliminate $a_{1,y}$ using equations (1) and (6), substitute for $a_{1,x}$ using (5), and solve for a_2.

$$\left(a_{1,x} - a_2\right)\tan\theta = \frac{M_2}{M_1}a_2\cot\theta - g$$
$$\left(-\frac{M_2}{M_1}a_2 - a_2\right)\tan\theta = \frac{M_2}{M_1}a_2\cot\theta - g$$
$$-a_2\left(\frac{M_2}{M_1}+1\right)\tan\theta - \frac{M_2}{M_1}a_2\cot\theta = -g$$
$$a_2 = \frac{g}{\left(\frac{M_2}{M_1}+1\right)\tan\theta + \frac{M_2}{M_1}\cot\theta}$$
$$a_2 = \frac{M_1 g}{\left(M_1 + M_2\right)\tan\theta + M_2\cot\theta}$$

Note that $a_2 = a_{2,x},$ as the wedge does not accelerate in the y direction.

CALCULATE:

(a) $F_{\text{app}} = \left(6.00\text{ kg}\right)\left(1 + \frac{6.00\text{ kg}}{9.00\text{ kg}}\right)\left(9.81\text{ m/s}^2\right)\tan\left(36.9°\right) = 73.66\text{ N}$

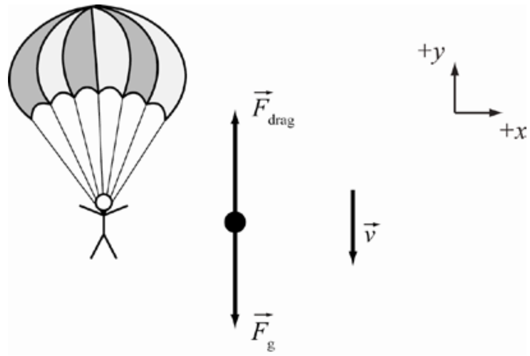
(b) $a_{\text{w}} = \frac{\left(6.00\text{ kg}\right)\left(9.81\text{ m/s}^2\right)}{\left(6.00\text{ kg} + 9.00\text{ kg}\right)\tan\left(36.9°\right) + \left(9.00\text{ kg}\right)\cot\left(36.9°\right)} = 2.532\text{ m/s}^2$

ROUND: There are three significant figures in both masses, so the results should be written as $F_{\text{app}} = 73.7\text{ N}$ and $a_{\text{w}} = 2.53\text{ m/s}^2$.

DOUBLE-CHECK: The applied force seems reasonable for the bowling ball on an inclined wedge. A positive a_2 implies that the wedge accelerates to the left in the sketch, as it should.

4.55. **THINK:** The skydiver's total mass is m = 82.3 kg. The drag coefficient is $c_d = 0.533$. The parachute area is $A = 20.11 \text{ m}^2$. The density of air is $\rho = 1.14 \text{ kg/m}^3$. The skydiver has reached terminal velocity $\left(a_{\text{net}} = 0\right)$. Determine the drag force of the air, F_{drag}.

SKETCH:



RESEARCH: The skydiver has achieved terminal velocity, that is $F_{\text{net,y}} = 0$. By balancing the forces in y, F_{drag} can be determined.

SIMPLIFY: $F_{\text{net},y} = F_{\text{drag}} - F_{\text{g}} = 0 \Rightarrow F_{\text{drag}} = F_{\text{g}} = mg$

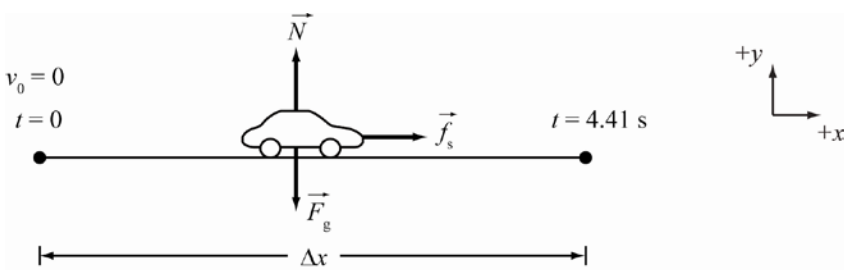
CALCULATE: $F_{\text{drag}} = \left(82.3 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right) = 807.36 \text{ N}$

ROUND: Since the mass has three significant figures, $F_{\text{drag}} = 807 \text{ N}$.

DOUBLE-CHECK: Since the skydiver has reached terminal velocity, the air's drag force should be equal to her weight, *mg*.

4.56. **THINK:** The dragster's initial speed is $v_0 = 0$. The distance traveled is $\Delta x = 402 \text{ m}$ in time, $t = 4.441 \text{ s}$. Determine the coefficient of static friction, μ_{s} necessary to achieve this result.

SKETCH:



RESEARCH: $f_{\text{s}} = \mu_{\text{s}} N$. To determine f_{s} and N, the forces acting along the x and y directions must be balanced: $F_{\text{net},x} = \sum F_x$ and $F_{\text{net,y}} = \sum F_y$. Note, the vertical acceleration is zero, so $F_{\text{net},y} = 0$. $F_{\text{net},x} = ma_{\text{net},x}$. Determine $a_{\text{net},x}$ by assuming a constant acceleration and using the equation $\Delta x = v_0 t + \left(at^2\right)/2$.

SIMPLIFY: To determine $a_{\text{net},x}$: $\Delta x = v_0 t + \left(a_{\text{net},x} t^2\right)/2 = \left(a_{\text{net},x} t^2\right)/2$ with $v_0 = 0$. Then, $a_{\text{net},x} = 2\Delta x / t^2$. Sum the forces in the vertical direction on the dragster: $F_{\text{net,y}} = N - F_{\text{g}} = 0 \Rightarrow N = F_{\text{g}} = mg$. Sum the forces in the horizontal direction: $F_{\text{net},x} = f_{\text{s}} \Rightarrow ma_{\text{net},x} = \mu_{\text{s}} N \Rightarrow ma_{\text{net},x} = \mu_{\text{s}} mg$. So,

$$\mu_s = \frac{a_{\text{net},x}}{g} = \frac{2\Delta x}{gt^2}.$$

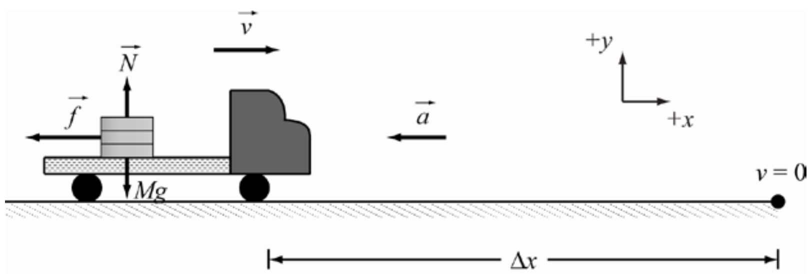
CALCULATE: $\mu_s = \dfrac{2(402 \text{ m})}{(9.81 \text{ m/s}^2)(4.441 \text{ s})^2} = 4.1555$

ROUND: There are three significant figures in the distance as given, so the result should be written $\mu_s = 4.16$.

DOUBLE-CHECK: The coefficient of friction that we have calculated would be extremely high for regular tires. If you ever walked around a car racetrack, you might have noticed that the road surface feels sticky. Why? The answer is that racing tires are very sticky, with the resulting higher coefficients of friction. However, the value we calculated here is too high, even for real racing tires, because we assumed that the only contribution to the normal force is the weight of the dragster. In real race situations, however, spoilers, wings, and other aerodynamics adjustments can convert some of the wind resistance into a downward force, which adds to the normal force. In addition, top fuel dragsters point their exhaust pipes almost straight up. The superchargers in the dragsters' engine expel the exhaust with very large velocities, and pointing the exhaust pipes up generates more downward force on the car. The net effect of all of these corrections is that real-life coefficients of friction do not have to be as high as our calculated result in order to achieve the accelerations reached.

4.57. **THINK:** The initial speed of the truck is $v = 30.0$ m/s. The final speed of the truck is $v = 0$. The mass of the block is M. The coefficient of static friction is $\mu_s = 0.540$. Determine the minimum distance, Δx the truck can travel while stopping without causing the block to slide.

SKETCH:



RESEARCH: The minimum stopping distance occurs at the maximum acceleration the truck can undergo without causing the block to slide. Use the equation $v^2 = v_0^{\ 2} + 2a\Delta x$ to determine Δx. The acceleration is found from balancing the forces in the horizontal direction acting on the block.

SIMPLIFY: For the block, when it is just about to slide, $F_{\text{net},x} = -f_{\text{s,max}}$. Then, $Ma_{\text{net},x} = -\mu_s N = -\mu_s Mg$ $\Rightarrow a_{\text{net},x} = -\mu_s g$. Since the block and the truck remain in contact, they form a single system with the same acceleration. With $v = 0$,

$$0 = v_0^{\ 2} + 2a_{\text{net},x}\Delta x \Rightarrow \Delta x = \frac{-v_0^{\ 2}}{2a_{\text{net},x}} = \frac{-v_0^{\ 2}}{2(-\mu_s g)} = \frac{v_0^{\ 2}}{2\mu_s g}.$$

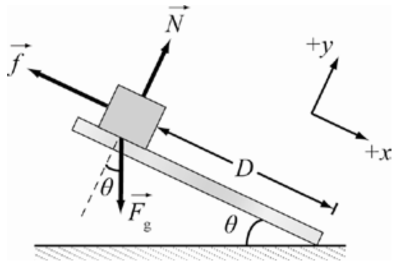
CALCULATE: $\Delta x = \dfrac{(30.0 \text{ m/s})^2}{2(0.540)(9.81 \text{ m/s}^2)} = 84.95$ m

ROUND: To three significant figures, the result should be written $\Delta x = 84.9$ m.

DOUBLE-CHECK: The displacement is positive, which is consistent with how the sketch is set up. This is a reasonable stopping distance.

4.58. **THINK:** The box's distance from the end of the board is $D = 0.540$ m. The coefficients of friction are $\mu_s = 0.320$ and $\mu_k = 0.250$. Determine the speed of the box after it reaches the end of the board, v. It is useful to know the angle of the board with respect to the horizontal, θ.

SKETCH:



RESEARCH: The final speed, v can be found from the equation $v^2 = v_0^2 + 2a\Delta x$. Note, $v_0 = 0$ as the box starts from rest. The acceleration a is the net acceleration in the x direction. This is found by balancing the forces in the horizontal direction when the book is sliding. N is determined by balancing the forces in the vertical direction. The angle, θ can be determined from the equation $f_{\text{s,max}} = \mu_{\text{s}} N$, just before the box begins to slide. Note that when the box is stationary, $F_{\text{net},x} = F_{\text{net},y} = 0$.

SIMPLIFY: First, determine θ. When the box is at rest, just about to slide, $F_{\text{net},x} = F_{\text{g},x} - f_{s,\max} = 0$ $\Rightarrow F_{\text{g},x} = f_{s,\max} \Rightarrow F_{\text{g}} \sin\theta = \mu_{\text{s}} N$. Since F_{g} is unknown, use the equation:

$$F_{\text{net},y} = N - F_{\text{g},y} = 0 \Rightarrow N = F_{\text{g}} \cos\theta \Rightarrow F_{\text{g}} = \frac{N}{\cos\theta}.$$

Then, $(N/\cos\theta)\sin\theta = \mu_s N$. Rearranging, $\mu_{\text{s}} = \tan\theta \Rightarrow \theta = \tan^{-1}(\mu_{\text{s}})$. Once the box is sliding, there is kinetic friction and a net acceleration in the horizontal direction. Determine $a_{\text{net},x}$:

$$F_{\text{net},x} = ma_{\text{net},x} = F_{\text{g},x} - f_{\text{k}} = mg\sin\theta - \mu_{\text{k}} N.$$

From above, $N = F_{\text{g}} \cos\theta = mg\cos\theta$. Then, $ma_{\text{net},x} = mg\sin\theta - \mu_{\text{k}} mg\cos\theta$. This can be reduced:

$$a_{\text{net},x} = g\left(\sin\theta - \mu_{\text{k}}\cos\theta\right) \Rightarrow a_{\text{net},x} = g\left(\sin\left(\tan^{-1}(\mu_{\text{s}})\right) - \mu_{\text{k}}\cos\left(\tan^{-1}(\mu_{\text{s}})\right)\right).$$

With $a_{\text{net}x}$ known, $v^2 = v_0^2 + 2a\Delta x \Rightarrow v = \sqrt{2a_{\text{net},x} D}$.

CALCULATE: $a_{\text{net},x} = \left(9.81 \text{ m/s}^2\right)\left(\sin\left(\tan^{-1}(0.320)\right) - (0.250)\cos\left(\tan^{-1}(0.320)\right)\right) = 0.6540 \text{ m/s}^2$,

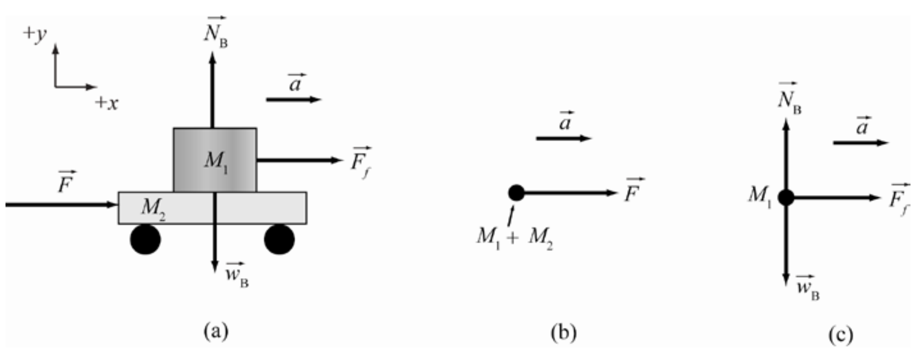
$v = \sqrt{2\left(0.654 \text{ m/s}^2\right)\left(0.540 \text{ m}\right)} = 0.8404 \text{ m/s}$

ROUND: Due to the difference in values that appear in the equation for $a_{\text{net},x}$, there are two significant figures. The result should be written as $v = 0.84 \text{ m/s}$.

DOUBLE-CHECK: This is a reasonable speed to achieve after the box slides down the incline.

4.59. **THINK:** It is given that there is a block of mass, $M_1 = 0.640$ kg at rest on a cart of mass, $M_2 = 0.320$ kg. The coefficient of static friction between the block and the cart is $\mu_{\text{s}} = 0.620$. Determine the maximum force on the cart and block such that the block does not slip.

SKETCH:



RESEARCH: Use Newton's second law: $\sum F_x = ma_x$, $\sum F_y = ma_y$. The force of friction is given by $F_f = \mu_s N$. First, consider a composite body (block and cart system, free-body diagram (b)). Applying Newton's second law, $\sum F_x = ma \Rightarrow F = (M_1 + M_2)a$. Note that both the block and the cart accelerate at the same rate. Second, consider only the block, applying Newton's second law in the horizontal and vertical directions: $\sum F_{B,y} = ma_y \Rightarrow N - w_B = 0 \Rightarrow N = w_B = M_1 g \;\; (a_y = 0)$, $\sum F_{B,x} = ma_x \Rightarrow F_f = M_1 a$.

SIMPLIFY: $F = (M_1 + M_2)a$, $F_f = M_1 a$, $N = M_1 g$. The maximum magnitude of F is when the acceleration is at a maximum. This means also that the force of friction is maximum which is equal to $F_f = \mu_s N = \mu_s M_1 g$. Note that when $F_f > \mu_s N$, the block starts to slip. $F_f = \mu_s M_1 g = M_1 a_{max} \Rightarrow a_{max} = \mu_s g$. Therefore, $F_{max} = (M_1 + M_2)a_{max} \Rightarrow F_{max} = (M_1 + M_2)\mu_s g$.

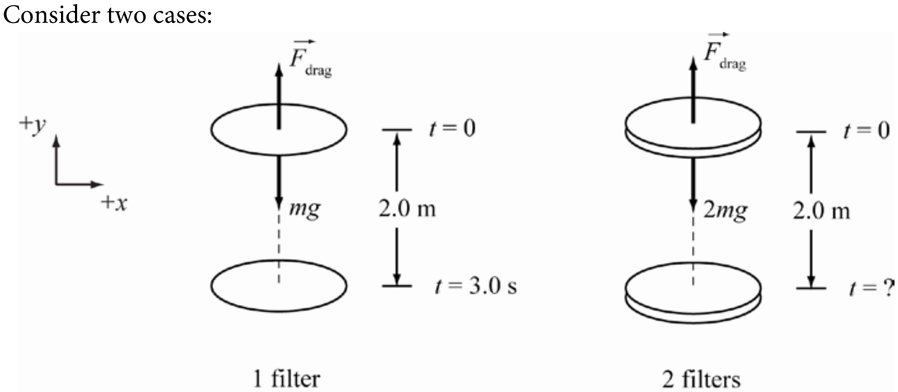
CALCULATE: $F_{max} = (0.640 \text{ kg} + 0.320 \text{ kg})(0.620)(9.81 \text{ m/s}^2) = 5.83891 \text{ N}$

ROUND: There are three significant figures initially, so the result should be $F_{max} = 5.84$ N.

DOUBLE-CHECK: By checking the masses given and the coefficient of static friction, it can be determined that the result should be the same order of magnitude as gravity. This is indeed the case.

4.60. **THINK:** A coffee filter is dropped from a height of 2.0 m. The coffee filter reaches the ground after 3.0 s. What happens when there are two coffee filters? A drag force is $F_{drag} = Kv^2$. The drag constant, K does not change if there are one or two filters. Since the cross-sectional areas are the same for the filters, the drag force for two filters remains $F_{drag} = Kv^2$.

SKETCH: Consider two cases:



RESEARCH: Use Newton's second law to determine the acceleration of the system:

$$\sum F_y = ma \Rightarrow F_{drag} - mg = ma.$$

However, the brief period when the filters are accelerating has been neglected. This means consider only when the filters reach terminal velocity. This occurs when $a = 0$.

$$F_{drag} = Kv^2 = mg \Rightarrow v^2 = \frac{mg}{K} \Rightarrow v = \sqrt{\frac{mg}{K}}$$

Because of constant speed, use $y = y_0 - vt$ ($y = 0$ is the ground).

SIMPLIFY: $y_0 = vt \Rightarrow t = y_0 / v = y_0 / \sqrt{mg / K}$. For one filter, $m = m_0$: $t_1 = \dfrac{y_0}{\sqrt{\dfrac{m_0 g}{K}}}$.

For two filters, $m = 2m_0$:

$$t_2 = \frac{y_0}{\sqrt{\dfrac{2m_0 g}{K}}} = \frac{y_0}{\sqrt{2}\sqrt{\dfrac{m_0 g}{K}}} = \frac{t_1}{\sqrt{2}}.$$

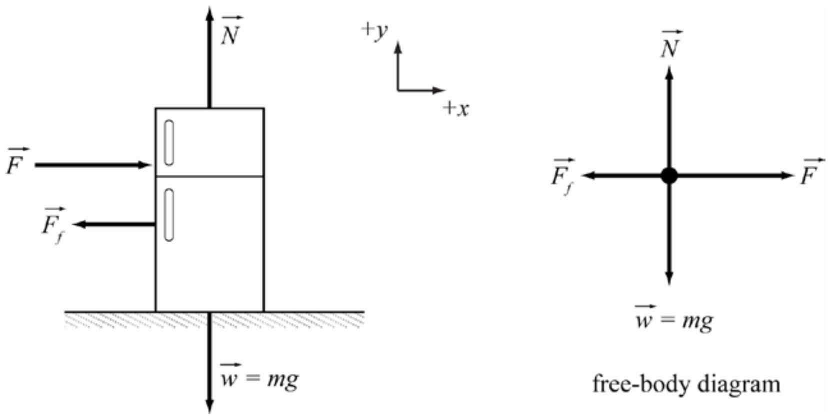
CALCULATE: $t_2 = \frac{3.0 \text{ s}}{\sqrt{2}} = 2.1213 \text{ s}$

ROUND: The first time has two significant figures, so the result should be rounded to $t_2 = 2.1$ s.

DOUBLE-CHECK: It can be seen that t is inversely proportional to $\sqrt{m}$. This implies that if the mass of the object is increased, the time will be decreased. t_2 is less than t_1 as expected.

4.61. **THINK:** The refrigerator has a mass of m = 112.2 kg. The coefficients of static and kinetic friction are $\mu_s = 0.460$ and $\mu_k = 0.370$. Determine the force of friction if the refrigerator is pushed with a force F.

SKETCH:



RESEARCH: Use Newton's second law: $\sum F_x = ma_x \Rightarrow F - F_f = ma_x$, $\sum F_y = ma_y \; (a_y = 0)$, $N - w = 0 \Rightarrow N = mg$. To move the refrigerator, the maximum static friction needs to be overcome. The maximum static friction is given by $F_f = f_{s,\max} = \mu_s N = \mu_s mg$. After the refrigerator has moved, the force applied needs to be larger than the force of kinetic friction in order to keep the refrigerator moving. The force of kinetic friction is given by $f_k = \mu_k N = \mu_k mg$.

SIMPLIFY: $f_{s,\max} = \mu_s mg$ and $f_k = \mu_k mg$.

CALCULATE: $f_{s,\max} = 0.460(112.2 \text{ kg})(9.81 \text{ m/s}^2) = 506.31 \text{ N}$,

$f_k = 0.370(112.2 \text{ kg})(9.81 \text{ m/s}^2) = 407.25 \text{ N}$

ROUND: Rounding to three significant figures, since gravity has three significant figures, the results are $f_{s,\max} = 506$ N and $f_k = 407$ N.

(a) Here, F is less than $f_s = 506$ N. This means that the force of friction balances the force F. Therefore, the force of friction is 300. N.

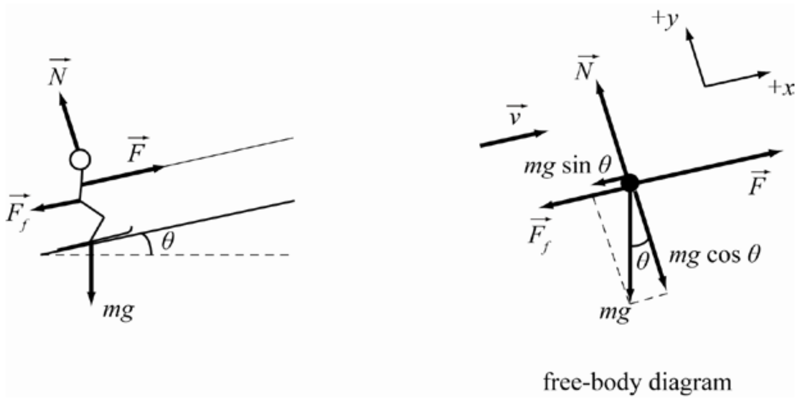
(b) F is still less than $f_s = 506$ N. The force of friction is 500. N.

(c) F is larger than $f_s = 506$ N. This means that initially the force of friction is 506 N, but after the refrigerator is in motion, the force of friction is the force of kinetic friction, $F_f = f_k = 407$ N.

DOUBLE-CHECK: The force of friction must be equal or less than the force acting on an object, $F_f \le F$. Also, the maximum static friction is always larger than kinetic friction, $f_{s,\max} > f_k$.

4.62. **THINK:** A towrope pulls the skiers with a constant speed of 1.74 m/s. The slope of the hill is 12.4°. A child with a mass of 62.4 kg is pulled up the hill. The coefficients of static and kinetic friction are 0.152 and 0.104, respectively. What is the force of the towrope acting on the child? Constant speed means zero acceleration, $a = 0$.

SKETCH:



RESEARCH: Use Newton's second law: $\sum F_x = ma_x$, $\sum F_y = ma_y$. The maximum force of static friction is given by $f_{\text{s,max}} = \mu_\text{s} N$ and the force of kinetic friction is given by $f_\text{k} = \mu_\text{k} N$. Initially, the force of the towrope must overcome the maximum static friction in order to move the child. $\sum F_y = ma_y = 0$ since $a_y = 0$. So $N - mg\cos\theta = 0 \Rightarrow N = mg\cos\theta$. $\sum F_x = ma_x = 0$ since $a_x = 0$. So $F - F_\text{f} - mg\sin\theta = 0$ $\Rightarrow F = F_\text{f} + mg\sin\theta$. Also, $F_\text{f} = f_{\text{s,max}} = \mu_\text{s} N = \mu_\text{s} mg\cos\theta$.

SIMPLIFY: $N = mg\cos\theta$, $F = f_{\text{s,max}} + mg\sin\theta = \mu_\text{s} mg\cos\theta + mg\sin\theta = mg\left(\mu_\text{s}\cos\theta + \sin\theta\right)$. After the child is in motion with a speed of 1.74 m/s, μ_s above is replaced by μ_k. Therefore, $F = mg\left(\mu_\text{k}\cos\theta + \sin\theta\right)$.

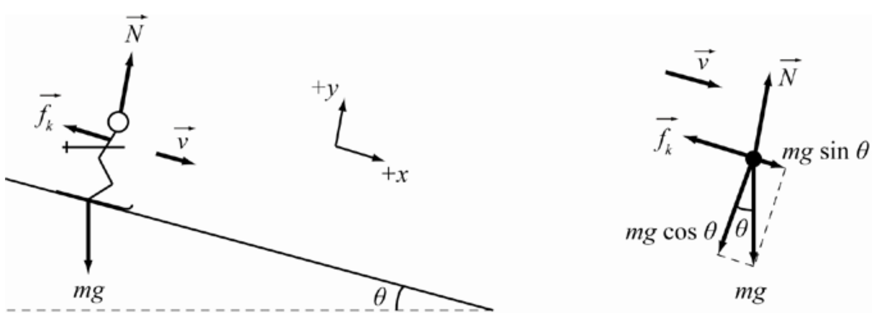
CALCULATE: $F = 62.4 \text{ kg}\left(9.81 \text{ m/s}^2\right)\left(0.152\cos\left(12.4°\right) + \sin\left(12.4°\right)\right) = 222.324 \text{ N}$. After the child is in motion, $F = 62.4 \text{ kg}\left(9.81 \text{ m/s}^2\right)\left(0.104\cos\left(12.4°\right) + \sin\left(12.4°\right)\right) = 193.63 \text{ N}$.

ROUND: Rounding to three significant figures, before movement, F = 222 N and after movement begins, F = 194 N.

DOUBLE-CHECK: The initial force F = 222 N must be larger than the force after the child is in motion $F = 194$ N.

4.63. **THINK:** A skier moves down a slop with an angle of 15.0°. The initial speed is 2.00 m/s. The coefficient of kinetic friction is 0.100. Determine the speed after 10.0 s. First, the acceleration of the skier must be determined.

SKETCH:



RESEARCH: Assume the direction of motion is the positive direction of the x axis. The force of kinetic friction is given by $f_\text{k} = \mu_\text{k} N$. Use Newton's second law to determine the acceleration of the skier:

$$\sum F_x = ma_x \Rightarrow mg\sin\theta - f_\text{k} = ma_x \Rightarrow ma_x = mg\sin\theta - \mu_\text{k} N$$

$$\sum F_y = ma_y \quad \left(a_y = 0\right) \Rightarrow N - mg\cos\theta = 0 \Rightarrow N = mg\cos\theta$$

SIMPLIFY: $ma_x = mg\sin\theta - \mu_k mg\cos\theta \Rightarrow a_x = g(\sin\theta - \mu_k\cos\theta)$. The speed after the time interval Δt is: $v = v_0 + a_x\Delta t = v_0 + g(\sin\theta - \mu_k\cos\theta)\Delta t$.

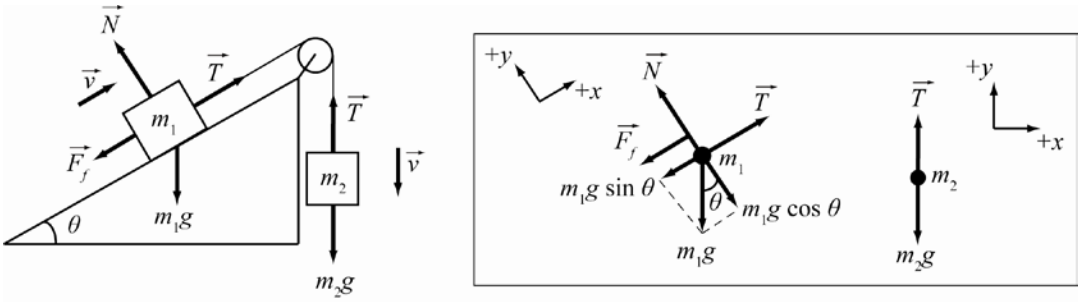
CALCULATE: $v = 2.00 \text{ m/s} + (9.81 \text{ m/s}^2)(\sin 15.0° - 0.100\cos 15.0°)(10.0 \text{ s}) = 17.91 \text{ m/s}$

ROUND: Since v_0 has three significant figures, round the result to v = 17.9 m/s.

DOUBLE-CHECK: 17.9 m/s is equivalent to about 64.4 km/h, which is a reasonable speed.

4.64. **THINK:** A block of mass, $m_1 = 21.9$ kg is connected to another block of mass, $m_2 = 25.1$ kg on an inclined plane with $\theta = 30.0°$. The coefficients of friction are $\mu_s = 0.109$ and $\mu_k = 0.086$. Determine the displacement of block 2 after 1.51 s. Because block 2 is heavier than block 1, the displacement of block 2 is either downward or zero.

SKETCH:



The force of friction always opposes a motion. This means its direction is opposite to the direction of motion of the system.

RESEARCH: Use Newton's second law: $\sum F_x = ma_x$, $\sum F_y = ma_y$.

Block 2: $\sum F_y = m_2 a \Rightarrow T - m_2 g = m_2 a$.

Block 1: $\sum F_y = 0 \ (a_y = 0) \Rightarrow N - m_1 g\cos\theta = 0 \Rightarrow N = m_1 g\cos\theta$

$\sum F_x = m_1 a \Rightarrow m_1 g\sin\theta - T + F_f = m_1 a.$

SIMPLIFY: $T - m_2 g = m_2 a \Rightarrow T = m_2 g + m_2 a$, $F_f = \mu N = \mu m_1 g\cos\theta$ ($\mu = \mu_s$ if the blocks are at rest and $\mu = \mu_k$ if the blocks are in motion). So,

$$m_1 g\sin\theta - T + F_f = m_1 a$$
$$m_1 g\sin\theta - m_2 g - m_2 a + F_f = m_1 a$$
$$m_1 g\sin\theta - m_2 g + F_f = (m_1 + m_2)a$$
$$m_1 g\sin\theta - m_2 g + \mu m_1 g\cos\theta = (m_1 + m_2)a$$
$$a = \frac{m_1 g\sin\theta - m_2 g + \mu m_1 g\cos\theta}{(m_1 + m_2)}$$

Before a is calculated, it must be determined if the net force (excluding friction) is larger than the maximum force of static friction.

$$m_2 g - m_1 g\sin\theta > \mu_s m_1 g\cos\theta \Rightarrow m_2 > m_1\sin\theta + \mu_s m_1\cos\theta \Rightarrow m_2 > m_1(\sin\theta + \mu_s\cos\theta)$$

Because $(\sin\theta + \mu_s\cos\theta) < 1$ and $m_2 > m_1$, the above condition is satisfied. So the above equation for the acceleration can be used. Displacement after t is $y = y_0 + v_0 t + (1/2)at^2$. $y_0 = 0$ and $v_0 = 0$.

$$y = \frac{1}{2}at^2 = \frac{1}{2}t^2\left(\frac{m_1 g\sin\theta - m_2 g + \mu_k m_1 g\cos\theta}{(m_1 + m_2)}\right)$$

CALCULATE: Displacement is

$$y=\frac{1}{2}(1.51\text{ s})^2\left(\frac{21.9\text{ kg}(9.81\text{ m/s}^2)\sin(30°)-25.1\text{ kg}(9.81\text{ m/s}^2)+(0.086)21.9\text{ kg}(9.81\text{ m/s}^2)\cos(30°)}{(21.9\text{ kg}+25.1\text{ kg})}\right)$$
$$=-2.9789\text{ m}$$

ROUND: Since the coefficient of kinetic friction has two significant figures, round the result to $y=-3.0$ m.

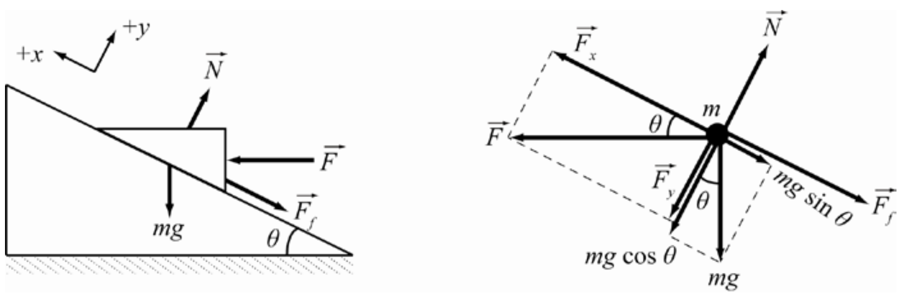
DOUBLE-CHECK: It is understandable that the displacement is negative, since m_2 is larger than m_1. If it assumed the acceleration is equal to gravity, then

$$y=-\frac{1}{2}gt^2=-\frac{1}{2}(9.81\text{ m/s}^2)(1.51)^2=-11.2\text{ m}.$$

The result is much smaller than −11.2 m, since the acceleration must be smaller than the acceleration due to gravity.

4.65. **THINK:** A wedge has a mass, m = 36.1 kg. The angle is $\theta=21.3°$ and the force is F = 302.3 N. The coefficient of kinetic friction is $\mu_k=0.159.$ Determine the acceleration.

SKETCH:



RESEARCH: Use Newton's second law: $\sum F_x=ma_x$ and $\sum F_y=ma_y$. $F_x=F\cos\theta,$ $F_y=F\sin\theta,$ and $F_f=\mu_k N.$

$$\sum F_y=ma_y=0,\text{ since }a_y=0\ \Rightarrow\ N-F_y-mg\cos\theta=0\ \Rightarrow\ N=F_y+mg\cos\theta=F\sin\theta+mg\cos\theta$$

$$\sum F_x=ma_x\ \Rightarrow\ F_x-F_f-mg\sin\theta=ma\ \Rightarrow\ ma=F\cos\theta-\mu_k N-mg\sin\theta$$

SIMPLIFY: $a=\dfrac{F\cos\theta-\mu_k(F\sin\theta+mg\cos\theta)-mg\sin\theta}{m}=\dfrac{F\cos\theta-\mu_k F\sin\theta-\mu_k mg\cos\theta-mg\sin\theta}{m}$

$$=\frac{F}{m}(\cos\theta-\mu_k\sin\theta)-g(\mu_k\cos\theta+\sin\theta)$$

CALCULATE: $a=\dfrac{302.3\text{ N}}{36.1\text{ kg}}(\cos(21.3°)-0.159\sin(21.3°))-(9.81\text{ m/s}^2)(0.159\cos(21.3°)+\sin(21.3°))$

$$=2.3015\text{ m/s}^2$$

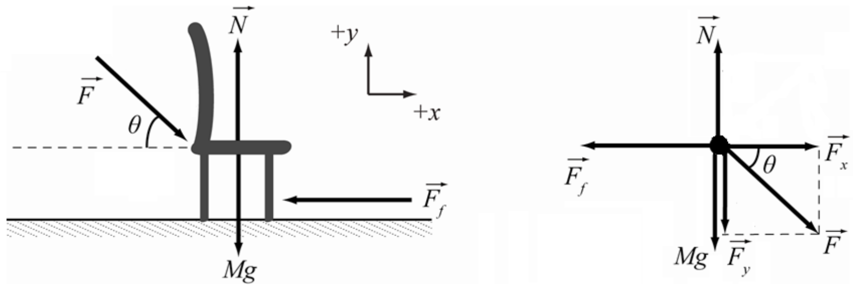
ROUND: Rounding to three significant figures, $a=2.30\text{ m/s}^2$.

DOUBLE-CHECK: If it is assumed that $\theta=0$, then the acceleration is $a = F/m$. $a=\dfrac{302.2}{36.1}=8.37\text{ m/s}^2$

Therefore the result is reasonably less than this value.

4.66. **THINK:** The chair has a mass of M. The coefficient of static friction is $\mu_s=0.560.$ A force F is acting at an angle θ to the horizontal. Determine the range of θ so that the chair does not move. The condition requires that the horizontal component of the force F be equal to or less than the maximum force of static friction, $f_{s,max}=\mu_s N$.

SKETCH:



RESEARCH: Use Newton's second law:

$$\sum F_y = ma_y = 0 \;\Rightarrow\; N - F_y - Mg = 0 \;\text{ and }\; \sum F_x = ma_x \;\Rightarrow\; F_x - F_f = ma$$

SIMPLIFY: $N = Mg + F\sin\theta$ and $F\cos\theta - F_f = ma$. The chair not moving means that $a = 0$, so $\sum F_x = 0$. At the minimum angle,

$$\begin{aligned}
F_x &= f_{s,\max} \\
F\cos\theta &= F_{f,\max} \\
F\cos\theta &= \mu_s N \\
F\cos\theta &= \mu_s\left(Mg + F\sin\theta\right) \\
F\cos\theta &= \mu_s Mg + \mu_s F\sin\theta \\
F\left(\cos\theta - \mu_s\sin\theta\right) &= \mu_s Mg \\
\cos\theta - \mu_s\sin\theta &= \frac{\mu_s Mg}{F}
\end{aligned}$$

Since $\mu_s Mg / F$ is greater than or equal to zero, the critical value occurs when $\cos\theta - \mu_s\sin\theta = 0$. Solving for θ, $\cos\theta = \mu_s\sin\theta \;\Rightarrow\; \dfrac{1}{\mu_s} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta \;\Rightarrow\; \tan^{-1}\left(\dfrac{1}{\mu_s}\right) = \theta.$

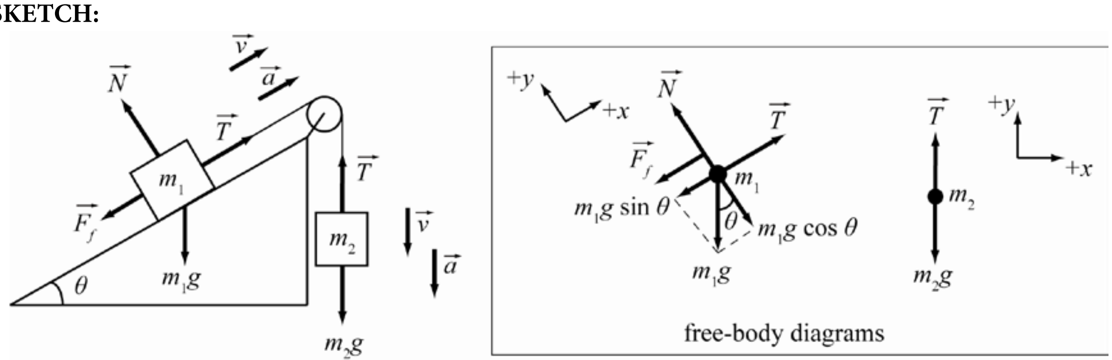
CALCULATE: $\theta = \tan^{-1}\left(\dfrac{1}{0.560}\right) = 60.751°$

ROUND: Rounding to three significant figures, $\theta_c = 60.8°$. Thus, the minimum angle at which any force can be applied and the chair will not move across the floor is $60.8°$.

DOUBLE-CHECK: If $\theta = 90°$, the chair is pushed straight down and there are no horizontal forces, therefore the chair does not move.

4.67. **THINK:** The two blocks have masses $m_1 = 0.2500$ kg and $m_2 = 0.5000$ kg. The coefficients of static and kinetic friction are 0.250 and 0.123. The angle of the incline is $\theta = 30.0°$. The blocks are initially at rest.

SKETCH:



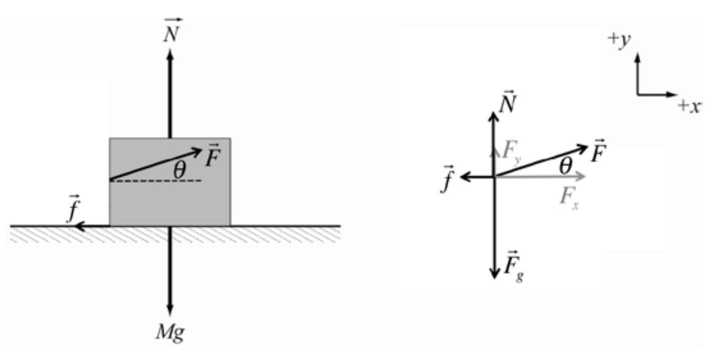
RESEARCH:
(a) If there is no friction, $F_f = 0$, it is given that $m_2 > m_1$. This would cause block 1 to move up and block 2 to move down. The same motion occurs when there is friction, only the acceleration is less when there is friction.
(b) Use Newton's second law to determine the acceleration:
Body 1: $\sum F_y = 0$ and $a_y = 0$, so $N - m_1 g\cos\theta = 0 \Rightarrow N = m_1 g\cos\theta$.
Also, $\sum F_x = m_1 a$ so $T - m_1 g\sin\theta - F_f = m_1 a$.
Body 2: $\sum F_y = m_2 a$ so $m_2 g - T = m_2 a \Rightarrow T = m_2 g - m_2 a$.
SIMPLIFY: (b) $T - m_1 g\sin\theta - F_f = m_1 a$ and $F_f = \mu_k N$, so

$$m_2 g - m_2 a - m_1 g\sin\theta - \mu_k N = m_1 a$$
$$m_2 g - m_1 g\sin\theta - \mu_k m_1 g\cos\theta = (m_1 + m_2)a$$
$$a = \frac{m_2 g - m_1 g\sin\theta - \mu_k m_1 g\cos\theta}{(m_1 + m_2)}$$
$$a = g\frac{\left(m_2 - m_1\left(\sin\theta + \mu_k\cos\theta\right)\right)}{(m_1 + m_2)}$$

CALCULATE: (b) $a = \left(9.81\text{ m/s}^2\right)\dfrac{\left(0.5000\text{ kg} - 0.2500\text{ kg}\left(\sin(30.0°) + 0.123\cos(30.0°)\right)\right)}{\left(0.5000\text{ kg} + 0.2500\text{ kg}\right)} = 4.5567\text{ m/s}^2$

ROUND:
(b) Rounding to three significant figures,
$a = 4.56\text{ m/s}^2$.
DOUBLE-CHECK: The result is reasonable since it is less than the acceleration due to gravity. In addition we find the limit of the acceleration of the Atwood machine in the limit of $\theta = 90°$ (See Example 4.4) and the limit of Example 4.8, Two Blocks Connected by a Rope – with Friction, for $\theta = 0°$ as limiting cases of our answer. This gives us additional confidence in our solution.

4.68. **THINK:** Let's first consider the case where there is no friction; then the force F obviously has to act horizontally direction, because that is the direction of the intended acceleration. What, then, changes with the presence of friction? The friction force is always in the opposite direction of the motion of the block, and it is proportional to the magnitude of the normal force acting on the block from the supporting surface. Thus, if we direct the force downward, with an angle below the horizontal ($\theta < 0$), then our vertical force component adds to the normal force, which causes a large friction force … which is bad. However, if we direct the force upward, then we *reduce* the normal force and thus reduce the friction force, and we have a chance to have a higher acceleration of the block than what we would have if the external force acted horizontally.
SKETCH:



RESEARCH: Use Newton's second law:
y-direction: $N - Mg + F_y = 0 \Rightarrow N = Mg - F_y = Mg - F\sin\theta$
x-direction: $F_x - f = Ma$ and $f = \mu_k N$, so $F_x - \mu_k N = Ma \Rightarrow F\cos\theta - \mu_k N = Ma.$
SIMPLIFY:

$$F\cos\theta - \mu_k\left(Mg - F\sin\theta\right) = Ma \Rightarrow a = \frac{F\cos\theta - \mu_k Mg + \mu_k F\sin\theta}{M} = \frac{F}{M}\left(\cos\theta + \mu_k\sin\theta\right) - \mu_k g$$

The acceleration is maximized when $da/d\theta = 0$.

$$\frac{da}{d\theta} = \frac{F}{M}\left(-\sin\theta + \mu_k\cos\theta\right) = 0 \Rightarrow$$
$$-\sin\theta + \mu_k\cos\theta = 0$$
$$\tan\theta = \mu_k$$
$$\theta = \tan^{-1}\left(\mu_k\right)$$

CALCULATE:
(a) $\theta = \tan^{-1}\left(0.41\right) = 22.2936°$

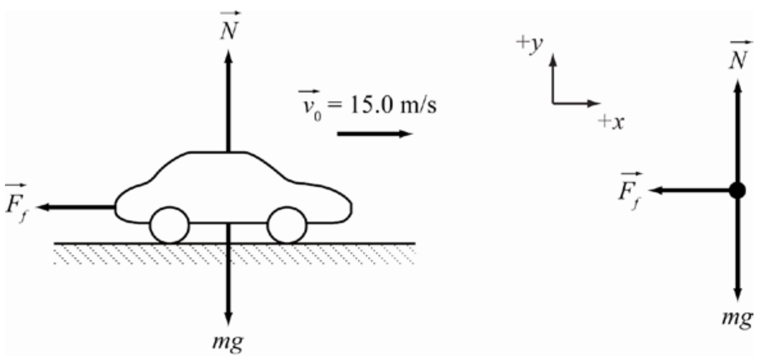
(b) $a = \dfrac{10.0\text{ N}}{0.5000\text{ kg}}\left(\cos\left(22.29°\right) + \left(0.41\right)\sin\left(+22.29°\right)\right) - \left(0.41\right)\left(9.81\text{ m/s}^2\right) = 17.5936\text{ m/s}^2$

ROUND: Rounding to three significant figures, $\theta = 22.3°$ and $a = 17.6\text{ m/s}^2$.
DOUBLE-CHECK: The fact that the force is directed upward makes sense because the y-component of this applied force will reduce the normal force, acting to reduce the force of friction. Assuming no friction, the maximum acceleration is when $\theta = 0.$ This means the acceleration is $a = F/M = 10.0\text{ N}/0.500\text{ kg} = 20.0\text{ m/s}^2$. If there is friction and $\theta = 0,$ the acceleration is

$$a = \frac{\left(F - \mu_k Mg\right)}{M} = \frac{F}{M} - \mu_k g = 20.0\text{ m/s}^2 - 0.41\left(9.81\text{ m/s}^2\right) = 16.4\text{ m/s}^2.$$

The part (b) result is between 16.4 m/s^2 and 20.0 m/s^2, as it should be.

4.69. **THINK:** A car is initially moving at a speed of 15.0 m/s then hits the brakes to make a sudden stop. The coefficients of static and kinetic friction are 0.550 and 0.430. Determine the acceleration and the distance traveled before the car stops.
SKETCH:



RESEARCH: Using Newton's second law: $\sum F_x = ma_x \Rightarrow F_f = ma \Rightarrow \mu_k N = ma$ and $\sum F_y = ma_y = 0 \Rightarrow N - mg \Rightarrow N = mg$. Also, $v^2 = {v_0}^2 - 2ax$.
SIMPLIFY:
(a) $\mu_k mg = ma \Rightarrow a = \mu_k g$

(b) $v^2 = {v_0}^2 - 2ax \Rightarrow x = \dfrac{v_0^2 - v^2}{2a}$

CALCULATE:

(a) $a = 0.430\left(9.81 \text{ m/s}^2\right) = 4.2183 \text{ m/s}^2$

(b) $x = \dfrac{\left(15.0 \text{ m/s}\right)^2 - 0^2}{2\left(4.2183 \text{ m/s}^2\right)} = 26.6695 \text{ m}$

ROUND: Rounding to three significant figures,

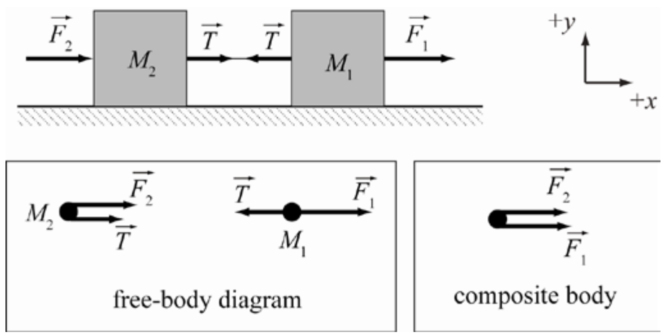
(a) $a = 4.22 \text{ m/s}^2$ and

(b) $x = 26.7$ m.

DOUBLE-CHECK: The acceleration is less than the acceleration due to gravity, as expected.

4.70. **THINK:** There are two blocks with masses, $M_1 = 2.00$ kg and $M_2 = 6.00$ kg. The are two forces, $F_1 = 10.0$ N and $F_2 = 5.00$ N.

SKETCH:



RESEARCH: Using Newton's second law: $\sum F_x = ma_x$. Consider the composite body $\left(M_1 + M_2\right)$:

$F_1 + F_2 = \left(M_1 + M_2\right)a$

SIMPLIFY:

(a) $a = \dfrac{\left(F_1 + F_2\right)}{\left(M_1 + M_2\right)}$; Consider M_1: $\sum F_x = ma,\ F_1 - T = m_1 a.$ Consider M_2: $\sum F_x = ma,\ F_2 + T = m_2 a.$

(b) $T = F_1 - m_1 a,\ T = m_2 a - F_2$

(c) The net force acting on M_1 is $\sum F_x = F_1 - T = m_1 a.$

CALCULATE:

(a) $a = \dfrac{\left(10.0 \text{ N} + 5.00 \text{ N}\right)}{\left(2.00 \text{ kg} + 6.00 \text{ kg}\right)} = 1.875 \text{ m/s}^2$

(b) $T = 10.0 \text{ N} - \left(2.00 \text{ kg}\right)\left(1.875 \text{ m/s}^2\right) = 6.25 \text{ N}$

(c) $\sum F = \sum F_x = \left(2.00 \text{ kg}\right)\left(1.875 \text{ m/s}^2\right) = 3.75 \text{ N}$

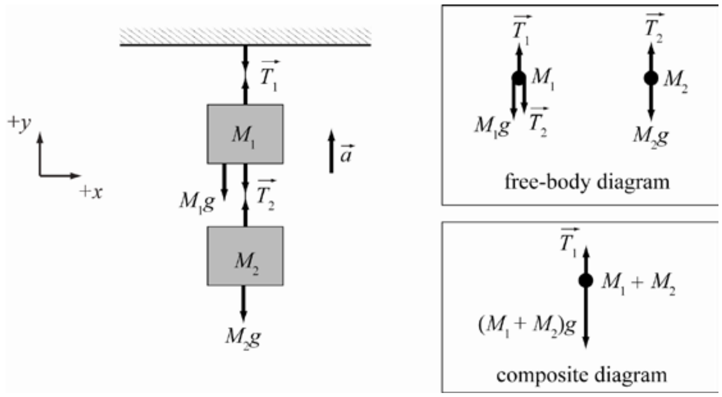
ROUND: To three significant figures, $a = 1.88 \text{ m/s}^2$, and $T = 6.25$ N. The sum of the forces acting on M_1 is $\sum F = 3.75 \text{ N}$.

DOUBLE-CHECK: The tension T can also be calculated by $T = m_2 a - F_2$.

$T = \left(6.0 \text{ kg}\right)\left(1.875 \text{ m/s}^2\right) - 5.0 \text{ N} = 6.25 \text{ N}$, which agrees with the previous result.

4.71. **THINK:** There are two masses, $M_1 = 2.00$ kg and $M_2 = 4.00$ kg. For part (a): a constant velocity means zero acceleration.

SKETCH:



RESEARCH: Using Newton's second law:

Mass 1: $\sum F_y = m_1 a \Rightarrow T_1 - T_2 - m_1 g = m_1 a$

Mass 2: $\sum F_y = m_2 a \Rightarrow T_2 - m_2 g = m_2 a$

SIMPLIFY: $T_2 = m_2(a+g)$. Substitute into the following equation:

$$T_1 = T_2 + m_1 a + m_1 g \Rightarrow T_1 = m_2(a+g) + m_1(a+g) = (m_1+m_2)(a+g)$$

The composite mass (m_1+m_2): $\sum F_y = ma \Rightarrow T_1 - (m_1+m_2)g = (m_1+m_2)a \Rightarrow T_1 = (m_1+m_2)(a+g)$.

CALCULATE:

(a) $a = 0$, so $T_1 = (2.00 \text{ kg} + 4.00 \text{ kg})(9.81 \text{ m/s}^2) = 58.86 \text{ N}$

(b) $a = 3.00 \text{ m/s}^2$, so $T_1 = (2.00 \text{ kg} + 4.00 \text{ kg})(3.00 \text{ m/s}^2 + 9.81 \text{ m/s}^2) = 76.86 \text{ N}$

ROUND: Since the masses have three significant figures, the results should be rounded to:

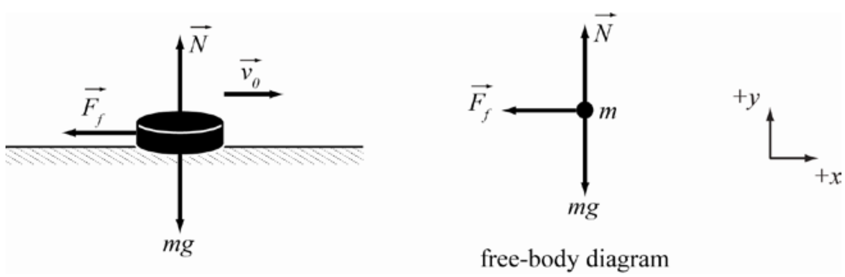
(a) $T_1 = 58.9$ N

(b) $T_1 = 76.9$ N

DOUBLE-CHECK: The tension increases as acceleration increases (assuming the acceleration is upward). As a check, the tension in part (a) is less than the tension in part (b).

4.72. **THINK:** The initial speed of a hockey puck is $v_0 = 12.5$ m/s. The puck stops after sliding a distance of 60.5 m. Determine the acceleration and then the coefficient of kinetic friction.

SKETCH:



RESEARCH: Using Newton's second law and the relation $v^2 = v_0^2 - 2ax$:

$\sum F_x = ma_x \Rightarrow F_f = ma \Rightarrow F_f = \mu_k N$

$\sum F_y = ma_y = 0$ (since $a_y = 0$) $\Rightarrow N - mg = 0 \Rightarrow N = mg$

SIMPLIFY: $ma = \mu_k N = \mu_k mg \Rightarrow a = \mu_k g \Rightarrow \mu_k = a/g$. The final speed is zero since the puck has stopped; $v = 0$.

$$2ax = v_0^2 \Rightarrow a = \frac{v_0^2}{2x} \Rightarrow \mu_k = \frac{v_0^{\ 2}}{2gx}$$

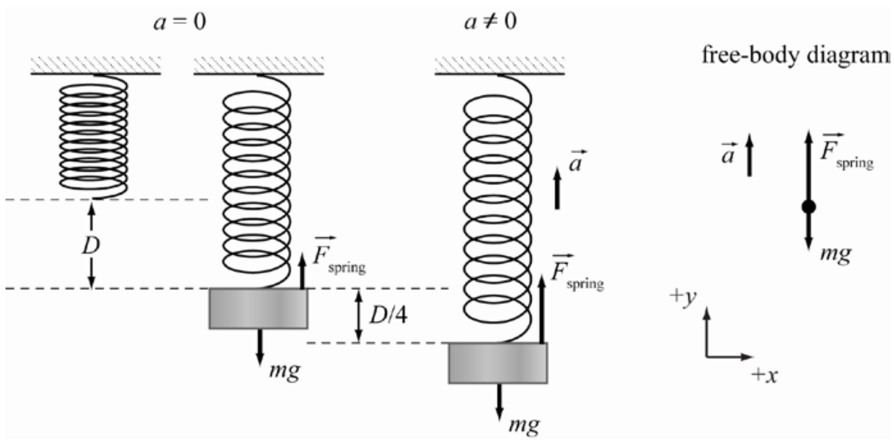
CALCULATE: $\mu_k = \dfrac{(12.5 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)60.5 \text{ m}} = 0.13163$

ROUND: Rounding to three significant figures, $\mu_k = 0.132$.

DOUBLE-CHECK: If the puck is on ice, it is expected that the coefficient of kinetic friction is small since the ice surface is smooth and slippery.

4.73. **THINK:** A mass M is attached to a massless spring. The spring stretches a distance D after the mass is attached. It stretches an additional $D/4$ after the elevator accelerates. Assume $F_{\text{spring}} = k\Delta x$.

SKETCH:



RESEARCH: Using Newton's second law and the equation for the force of the spring:

$$\sum F_y = ma \Rightarrow F_{\text{spring}} - mg = ma \Rightarrow F_{\text{spring}} = m(a+g) = k\Delta x.$$

SIMPLIFY: When $a = 0$, $\Delta x = D$. When $a \neq 0$, $\Delta x = D + D/4 = 5D/4$.

$a = 0 \Rightarrow kD = mg \Rightarrow k = mg/D$

$$a \neq 0 \Rightarrow k\frac{5D}{4} = m(a+g) \Rightarrow \left(\frac{mg}{D}\right)\frac{5D}{4} = m(a+g) \Rightarrow \frac{5mg}{4} = m(a+g) \Rightarrow \frac{5g}{4} = a+g \Rightarrow a = \frac{g}{4}$$

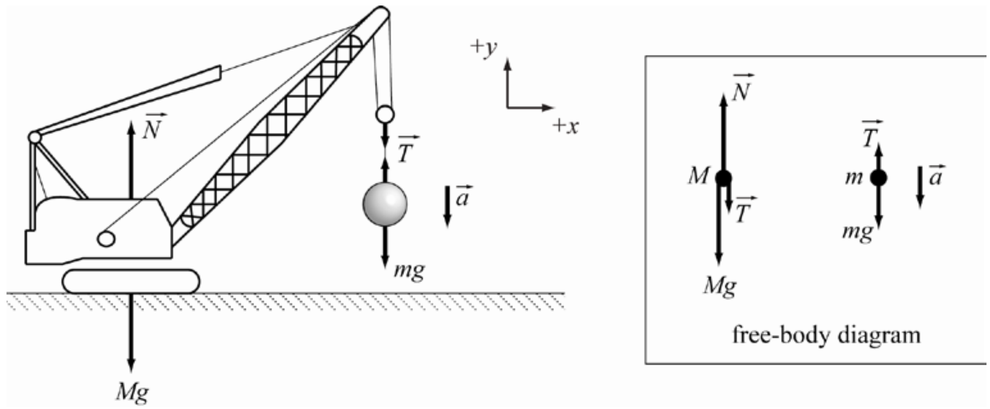
CALCULATE: $a = \dfrac{(9.81 \text{ m/s}^2)}{4} = 2.4525 \text{ m/s}^2$

ROUND: $a = 2.45 \text{ m/s}^2$

DOUBLE-CHECK: It makes sense that a = g/4, since it produces D/4 displacement and D is proportional to g.

4.74. **THINK:** The mass of a crane is $M = 1.00 \cdot 10^4$ and the ball has a mass, m = 1200. kg. Determine the normal force exerted on the crane by the ground when (a) the acceleration is zero and (b) the acceleration is not zero.

SKETCH:



RESEARCH: Use Newton's second law and the equation $v^2 = v_0^2 - 2ax$.

For the crane: $\sum F_y = ma_y$, $a_y = 0$ because the crane does not move. $N - T - Mg = 0 \Rightarrow N = T + Mg$.

For the ball: $\sum F_y = ma_y$, $a_y = -a$ (deceleration). $T - mg = -ma \Rightarrow T = m(g-a)$

SIMPLIFY: $N = m(g-a) + Mg \Rightarrow N = (m+M)g - ma$. $v = 0$ (the ball has stopped), so $a = v_0^2/(2x)$.

$$N = (m+M)g - m\frac{v_0^2}{2x}$$

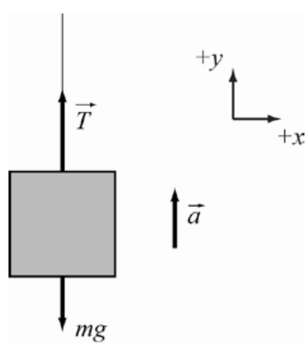
CALCULATE:

(a) Constant speed means $a = 0$. $N = (1.00\cdot 10^4 \text{ kg} + 1200. \text{ kg})(9.81 \text{ m/s}^2) = 109872 \text{ N}$.

(b) $N = 109872 \text{ N} - (1200. \text{ kg})\dfrac{(1.00 \text{ m/s})^2}{2(0.250 \text{ m})} = 107472 \text{ N}$

ROUND: Rounding to three significant figures, (a) $N = 1.10\cdot 10^5$ N and (b) $N = 1.07\cdot 10^5$ N.
DOUBLE-CHECK: Because the ball is decelerating, it is understandable that the normal force in part (a) is larger than the normal force in part (b). This is a similar situation to measuring weight in an elevator.

4.75. **THINK:** A block of mass $m = 20.0$ kg is initially at rest and then pulled upward with a constant acceleration, $a = 2.32 \text{ m/s}^2$.
SKETCH:



RESEARCH: Using Newton's second law and the equation $v^2 = v_0^2 - 2ax$: $v_0 = 0 \Rightarrow v^2 = 2ax$.

$\sum F_y = ma_y \Rightarrow T - mg = ma \Rightarrow T = m(a+g)$.

SIMPLIFY: $v^2 = 2ax \Rightarrow v = \sqrt{2ax}$
CALCULATE:

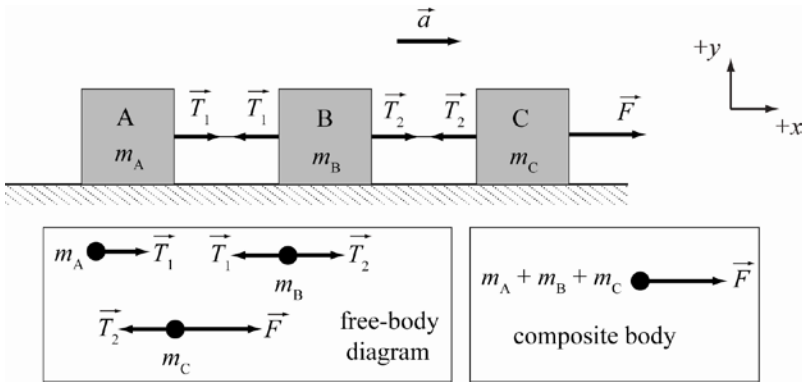
(a) $T = (20.0 \text{ kg})(2.32 \text{ m/s}^2 + 9.81 \text{ m/s}^2) = 242.6 \text{ N}$

(b) $\sum F_y = (20.0 \text{ kg})(2.32 \text{ m/s}^2) = 46.4 \text{ N}$

(c) $v = \sqrt{2(2.32 \text{ m/s}^2)(2.00 \text{ m})} = 3.04631 \text{ m/s}$

ROUND: Rounding to three significant figures, the results are
(a) $T = 243 \text{ N}$,
(b) $\sum F_y = 46.4 \text{ N}$ and
(c) $v = 3.05 \text{ m/s}$.
DOUBLE-CHECK: Since a is about g/4, the net force must be about T/5.

4.76. **THINK:** There are three blocks A, B and C. A force F = 12 N is pulling block C. Determine the tension in the string between blocks B and C.
SKETCH:



RESEARCH: First, consider the three blocks as a composite block. Using Newton's second law:

$$\sum F_x = ma_x \Rightarrow F - T_2 = m_c a \Rightarrow T_2 = F - ma.$$

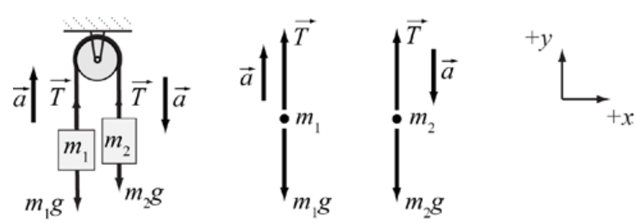
SIMPLIFY: $T_2 = F - m\left(\frac{F}{3m}\right) = F - \frac{F}{3} = \frac{2F}{3}$

CALCULATE: $T_2 = \frac{2(12 \text{ N})}{3} = 8.0 \text{ N}$

ROUND: It is not necessary to round. $T_2 = 8.0 \text{ N}$

DOUBLE-CHECK: It is reasonable that $T_2 = 2F/3$ since T_2 pulls two blocks.

4.77. **THINK:** There are two masses, $m_1 = 3.00 \text{ kg}$ and $m_2 = 4.00 \text{ kg}$ arranged as an Atwood machine. Determine the acceleration of the blocks.
SKETCH:



RESEARCH: Using Newton's second law, block 1: $\sum F_y = ma_y$, $a_y = a$, and $T - m_1 g = m_1 a$. Block 2: $\sum F_y = ma_y$, $a_y = -a$, and $T - m_2 g = -m_2 a$.

SIMPLIFY:
$$T = m_1(a+g)$$
$$T - m_2 g = -m_2 a$$
$$m_1 a + m_1 g - m_2 g + m_2 a = 0$$
$$(m_1 + m_2)a = (m_2 - m_1)g$$
$$a = \frac{(m_2 - m_1)}{(m_1 + m_2)}g$$

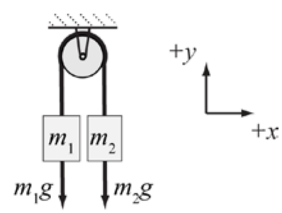
CALCULATE: $a = \frac{(4.00\text{ kg} - 3.00\text{ kg})}{(4.00\text{ kg} + 3.00\text{ kg})}(9.81\text{ m/s}^2) = 1.4014\text{ m/s}^2$

ROUND: Rounding to three significant figures, $a = 1.40\text{ m/s}^2$.

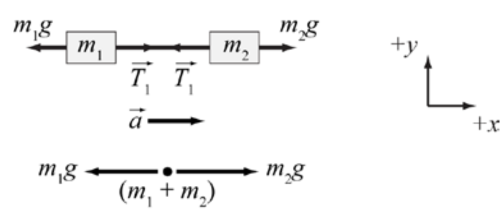
DOUBLE-CHECK: If the sum of the forces along the string are considered,
$$\sum F = (m_1 + m_2)a = m_2 g - T + T - m_1 g \;\Rightarrow\; a = \frac{(m_2 - m_1)}{(m_1 + m_2)}g.$$
This is the same as above.

4.78. **THINK:** There are two blocks of masses, $m_1 = 3.50$ kg and m_2 arranged as an Atwood's machine. The acceleration of the blocks is $a_0 = 0.400g$. Determine the mass, m_2.

SKETCH:



If the axis along the string is considered, the problem can be redrawn.



RESEARCH: The magnitude of the acceleration is given but not the direction of motion. Consider the two values of acceleration, $a = \pm a_0 = \pm 0.4g$. Using Newton's second law:
$$\sum F = ma \;\Rightarrow\; m_2 g - m_1 g = (m_1 + m_2)a \;\Rightarrow\; m_2(g-a) = m_1(g+a) \;\Rightarrow\; m_2 = m_1\frac{(g+a)}{(g-a)}.$$

SIMPLIFY: If $a = +a_0$, $m_2 = m_1\frac{(g+a_0)}{(g-a_0)}$. If $a = -a_0$, $m_2 = m_1\frac{(g-a_0)}{(g+a_0)}$.

CALCULATE: $m_2 = (3.50\text{ kg})\frac{(9.81\text{ m/s}^2 + 0.400g)}{(9.81\text{ m/s}^2 - 0.400g)} = 8.1667$ kg,

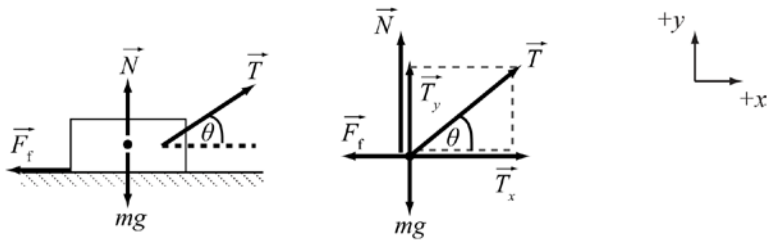
or, $m_2 = (3.50\text{ kg})\frac{(9.81\text{ m/s}^2 - 0.400g)}{(9.81\text{ m/s}^2 + 0.400g)} = 1.5$ kg

ROUND: Keeping three significant figures, the mass is either $m_2 = 8.17$ kg or $m_2 = 1.50$ kg.

DOUBLE-CHECK: There are two masses calculated for m_2 and one of them must be larger than m_1 and the other must be smaller than m_1. This is what we have found, so the answers are reasonable.

4.79. **THINK:** The sled has a mass, M = 1000. kg. The coefficient of kinetic friction is $\mu_k = 0.600$. The sled is pulled at an angle $\theta = 30.0°$ above the horizontal. Determine magnitude of the tension in the rope when the acceleration is $a = 2.00\text{ m/s}^2$.

SKETCH:



RESEARCH: $T_y = T\sin\theta$, $T_x = T\cos\theta$ and $F_f = \mu_k N$. Using Newton's second law: $\sum F_y = ma_y$ and $a_y = 0$, so $N + T_y - mg = 0 \Rightarrow N = mg - T_y = mg - T\sin\theta.$ Also, $\sum F_x = ma \Rightarrow T_x - F_f = ma$ $\Rightarrow T_x = F_f + ma.$

SIMPLIFY:

$$T\cos\theta = \mu_k N + ma$$
$$T\cos\theta = \mu_k\left(mg - T\sin\theta\right) + ma$$
$$T\cos\theta + \mu_k T\sin\theta = \mu_k mg + ma$$
$$T\left(\cos\theta + \mu_k\sin\theta\right) = \mu_k mg + ma$$
$$T = \frac{m\left(\mu_k g + a\right)}{\left(\cos\theta + \mu_k\sin\theta\right)}$$

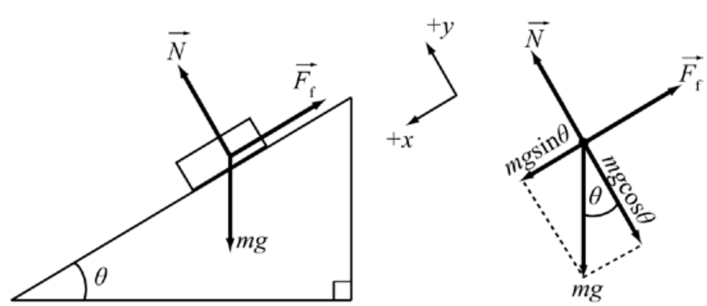
CALCULATE: $T = \dfrac{\left(1000.\text{ kg}\right)\left(0.600\left(9.81\text{ m/s}^2\right) + 2.00\text{ m/s}^2\right)}{\left(\cos\left(30.0°\right) + 0.600\sin\left(30.0°\right)\right)} = 6763.15\text{ N}$

ROUND: Rounding to three significant figures, T = 6760 N.

DOUBLE-CHECK: If there is no friction, $T\cos\theta = ma \Rightarrow T = \dfrac{ma}{\cos\theta} = \dfrac{\left(1000.\text{ kg}\right)\left(2.00\text{ m/s}^2\right)}{\cos 30.0°} = 2309\text{ N}.$ Since friction was considered previously, the result was larger.

4.80. **THINK:** A block with a mass of m = 2.00 kg is on an inclined plane with an angle $\theta = 20.0°$. The coefficient of static friction is $\mu_s = 0.60$.

SKETCH:



RESEARCH:

(a) The three forces are the normal, frictional and the gravitational forces.

(b) The maximum force of friction is given by $F_{f,max} = f_s = \mu_s N$. To determine the normal force, use Newton's second law: $\sum F_y = ma_y = 0$ and $a_y = 0$, so $N - mg\cos\theta = 0 \Rightarrow N = mg\cos\theta.$ Also, $\sum F_x = ma_x \Rightarrow mg\sin\theta - F_f = ma.$

SIMPLIFY:
(c) The block is moving if $mg\sin\theta > F_{\text{f}} = \mu_{\text{s}}N$ or $mg\sin\theta - \mu_{\text{s}}N > 0.$

$$\sum F_x = mg\sin\theta - \mu_{\text{s}}mg\cos\theta = mg\left(\sin\theta - \mu_{\text{s}}\cos\theta\right)$$

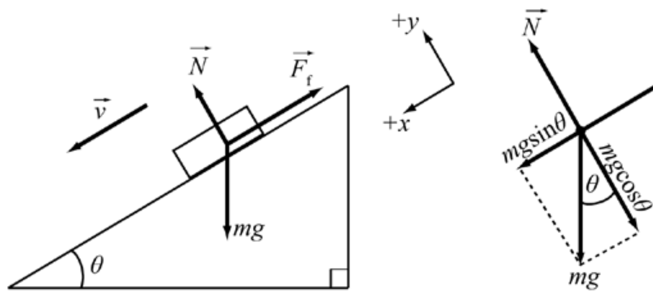
CALCULATE:
(b) $N = 2.00 \text{ kg}\left(9.81 \text{ m/s}^2\right)\cos 20.0° = 18.437 \text{ N}$

(c) $\sum F_x > 0$ if $\sin\theta - \mu_{\text{s}}\cos\theta > 0.$

$$\sin\theta - \mu_{\text{s}}\cos\theta = \sin\left(20.0°\right) - \left(0.60\right)\cos\left(20.0°\right) = -0.2218 < 0$$

Therefore, $mg\sin\theta < \mu_s N.$ This means the block does not move.
ROUND: (b) $N = 18.437$ N. Rounding to two significant figures, N = 18 N.
DOUBLE-CHECK: The critical angle of the inclined plane such that the blocks starts to move is $\tan\theta_c = \mu_s \Rightarrow \theta_c = \tan^{-1}\left(\mu_s\right).$ Here, $\theta_c = \tan^{-1}\left(0.60\right) = 30.96°.$ Since $\theta < \theta_c$, the block is not moving.

4.81. **THINK:** A block of mass, m = 5.00 kg is sliding down an inclined plane of angle $\theta = 37.0°$ at a constant velocity (a = 0). Determine the frictional force and the coefficient of kinetic friction.
SKETCH:



RESEARCH: There is no acceleration in any direction, $a_x = a_y = 0.$ The force of friction is given by $F_{\text{f}} = \mu_{\text{k}}N.$ Using Newton's second law: $\sum F_y = ma_y = 0$ and $a_y = 0,$ so $N - mg\cos\theta = 0$ $\Rightarrow N = mg\cos\theta.$ Also, $\sum F_x = ma_x = 0$ and $a_x = 0,$ so $mg\sin\theta - F_{\text{f}} = 0 \Rightarrow F_{\text{f}} = mg\sin\theta.$

SIMPLIFY: $\mu_{\text{k}}N = mg\sin\theta \Rightarrow \mu_{\text{k}}mg\cos\theta = mg\sin\theta \Rightarrow \mu_{\text{k}} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$

CALCULATE:
(a) $F_{\text{f}} = \left(5.00 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right)\sin 37.0° = 29.519 \text{ N}$
(b) $\mu_{\text{k}} = \tan\left(37.0°\right) = 0.75355$
ROUND: Rounding to three significant figures,
(a) $F_{\text{f}} = 29.5.$ N and
(b) $\mu_{\text{k}} = 0.754.$
DOUBLE-CHECK: μ_{k} is less than 1 and does not depend on the mass of the block.

4.82. **THINK:** The mass of the skydiver, $m = 83.7$ kg, is given as well as her drag coefficient, $c_{\text{d}} = 0.587,$ and her surface area, $A = 1.035 \text{ m}^2.$ We need to determine the terminal velocity then the time to reach a distance 296.7 m. Air density is $\rho = 1.14 \text{ kg/m}^3$ and $g = 9.81 \text{ m/s}^2.$
SKETCH: Not needed.
RESEARCH: Terminal speed is $v_{\text{t}} = \sqrt{2mg / c_{\text{d}}\rho A},$ and for constant velocity $v = x/t,$ where x is the distance traveled.

SIMPLIFY: $t = \dfrac{x}{v_\mathrm{t}} = x\sqrt{\dfrac{c_\mathrm{d}\rho A}{2mg}}$

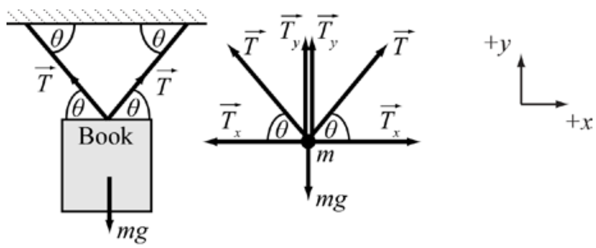
CALCULATE: $t = (296.7\text{ m})\sqrt{\dfrac{0.587(1.14\text{ km/m}^3)(1.035\text{ m}^2)}{2(83.7\text{ kg})(9.81\text{ m/s}^2)}} = 6.09322\text{ s}$

ROUND: Keeping only three significant digits, $t \approx 6.09$ s.

DOUBLE-CHECK: The typical speed of a sky diver is 60 m/s. $t = (296.7\text{ m})/(60\text{ m/s}) = 4.945$ s. The result is comparable to this value.

4.83. **THINK:** A book has a mass of m = 0.500 kg. The tension on each wire is T = 15.4 N. Determine the angle of the wires with the horizontal.

SKETCH:



RESEARCH: There is no acceleration in any direction, $a_x = a_y = 0$. Using Newton's second law: $\sum F_y = ma_y = 0 \Rightarrow 2T_y - mg = 0$ and $\sum F_x = ma_x = 0 \Rightarrow T_x - T_x = 0$.

SIMPLIFY: $T_y = T\sin\theta,$ so $2T\sin\theta = mg \Rightarrow \sin\theta = \dfrac{mg}{2T} \Rightarrow \theta = \sin^{-1}\left(\dfrac{mg}{2T}\right).$

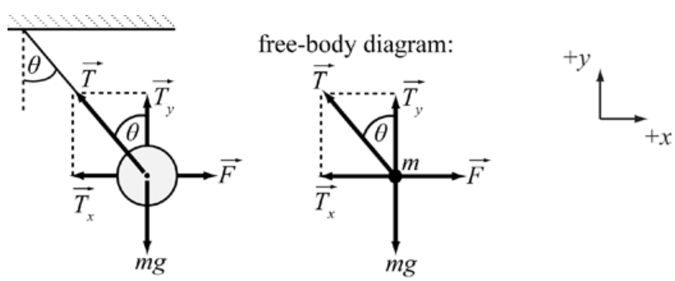
CALCULATE: $\theta = \sin^{-1}\left(\dfrac{0.500\text{ kg}(9.81\text{ m/s}^2)}{2(15.4\text{ N})}\right) = 9.1635°$

ROUND: Rounding to three significant figures, $\theta = 9.16°$.

DOUBLE-CHECK: If the angle is $\theta = 90°$, the tension required is $T = mg/2 = 0.500(9.81)/2 = 2.45$ N. It is reasonable that a smaller angle requires more tension.

4.84. **THINK:** A bob has a mass of m = 0.500 kg. The angle is $\theta = 30.0°$.

SKETCH:



RESEARCH: There is no acceleration in any direction, so $a_x = a_y = 0$. Using Newton's second law: $\sum F_x = 0 \Rightarrow F - T_x = 0$ and $\sum F_x = 0 \Rightarrow T_y - mg = 0$.

SIMPLIFY: $T_x = T\sin\theta$ and $T_y = T\cos\theta$. $F = T_x = T\sin\theta$ and $T\cos\theta = mg \Rightarrow T = mg/\cos\theta.$ So,

$$F = \left(\frac{mg}{\cos\theta}\right)\sin\theta = mg\tan\theta$$

CALCULATE:

(a) $F=(0.500\text{ kg})(9.81\text{ m/s}^2)\tan(30.0°)=2.8319\text{ N}$

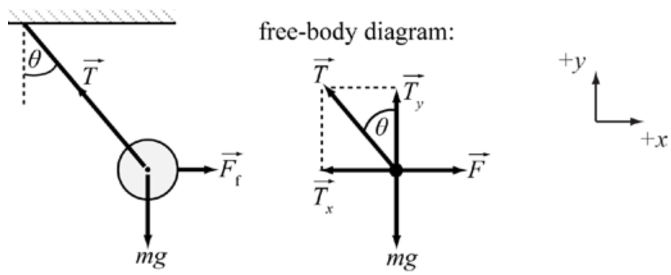
(b) $T=\dfrac{0.500\text{ kg}(9.81\text{ m/s}^2)}{\cos(30.0°)}=5.66381\text{ N}$

ROUND: Rounding to three significant figures, F = 2.83 N and T = 5.66 N.

DOUBLE-CHECK: The ratio F/T is equal to $\sin 30.0°=0.5$, which is correct.

4.85. **THINK:** A ping-pong ball with a mass, $m=2.70\cdot 10^{-3}$ kg is suspended by a string at an angle of $\theta=15.0°$ with the vertical. The force of friction is proportional to the square of the speed of the air stream, $v=$ 20.5 m/s.

SKETCH:



RESEARCH: Use the equation $F_\text{f}=cv^2$, where c is a constant. There is no acceleration in any direction, so $a_x=a_y=0$. $T_y=T\cos\theta$, $T_x=T\sin\theta$. Using Newton's second law: $\sum F_x=ma_x=0 \Rightarrow F_\text{f}-T_x=0$ and $\sum F_y=ma_y=0 \Rightarrow T_y-mg=0$.

SIMPLIFY: $F_\text{f}=T\sin\theta$, and $T\cos\theta=mg \Rightarrow T=\dfrac{mg}{\cos\theta}$. And so,

$$F_\text{f}=cv^2=\left(\frac{mg}{\cos\theta}\right)\sin\theta \Rightarrow c=\frac{mg}{v^2}\left(\frac{\sin\theta}{\cos\theta}\right)=\frac{mg}{v^2}\tan\theta$$

CALCULATE:

(a) $c=\dfrac{2.70\cdot 10^{-3}\text{ kg}(9.81\text{ m/s}^2)\tan(15.0°)}{(20.5\text{ m/s})^2}=1.688\cdot 10^{-5}\text{ kg/m}$

(b) $T=\dfrac{2.70\cdot 10^{-3}\text{ kg}(9.81\text{ m/s}^2)}{\cos(15.0°)}=0.027421\text{ N}$

ROUND: Rounding to three significant figures, the results should be

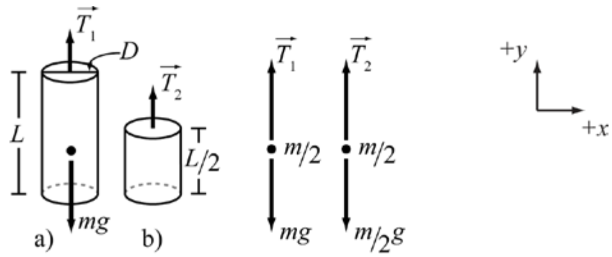
(a) $c=1.69\cdot 10^{-5}\text{ kg/m}$ and

(b) $T=0.0274$ N.

DOUBLE-CHECK: Because the mass of the ping pong ball is small, a small value for the tension is expected. Also, the cross-sectional area of the ball is small so the coefficient of friction is expected to be small.

4.86. **THINK:** A silicon nanowire has a length of 100.0 nm and a diameter of 5.0 nm. The density of silicon is $\rho = 2.33 \text{ g/cm}^3$.

SKETCH:



RESEARCH: Using Newton's second law:

(a) $\sum F_y = 0 \;\Rightarrow\; T_1 - mg = 0 \;\Rightarrow\; T_1 = mg$

(b) $\sum F_y = 0 \;\Rightarrow\; T_2 - \frac{m}{2}g = 0 \;\Rightarrow\; T_2 = \frac{mg}{2} = \frac{T_1}{2}$. The mass of the nanowire is $m = \rho V = \rho\left(\pi R^2 L\right)$.

SIMPLIFY: $m = \rho\pi\left(\frac{D}{2}\right)^2 L = \rho\frac{\pi}{4}D^2 L$

CALCULATE: $\rho = 2.33 \text{ g/cm}^3 = \frac{2.33\cdot 10^{-3} \text{ kg}}{\left(10^{-2} \text{ m}\right)^3} = 2.33\cdot 10^3 \text{ kg/m}^3, \quad D = 5.0 \text{ nm} = 5\cdot 10^{-9} \text{ m},$

$L = 100.0 \text{ nm} = 1.00\cdot 10^{-7} \text{ m}, \qquad m = \left(2.33\cdot 10^3 \text{ kg/m}^3\right)\frac{\pi}{4}\left(5.0\cdot 10^{-9} \text{ m}\right)^2\left(1.00\cdot 10^{-7} \text{ m}\right) = 4.575\cdot 10^{-21} \text{ kg}$

(a) $T_1 = 4.575\cdot 10^{-21} \text{ kg}\left(9.81 \text{ m/s}^2\right) = 4.488\cdot 10^{-20} \text{ N}$

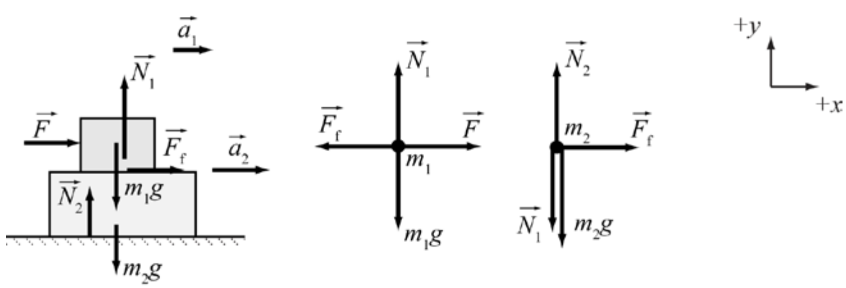
(b) $T_2 = \frac{4.488\cdot 10^{-20} \text{ N}}{2} = 2.244\cdot 10^{-20} \text{ N}$

ROUND: Rounding to two significant figures, $T_1 = 4.5\cdot 10^{-20}$ N and $T_2 = 2.2\cdot 10^{-20}$ N.

DOUBLE-CHECK: Since the volume of a nanowire is very small, it is expected to get a very small tension.

4.87. **THINK:** Two blocks have masses of $m_1 = 2.50$ kg and $m_2 = 3.75$ kg. The coefficients of static and kinetic friction between the two blocks are 0.456 and 0.380. A force, F is applied horizontally on m_1. Determine the maximum force, F, such that m_1 does not slide, and also the acceleration of m_1 and m_2 when $F =$ 24.5 N.

SKETCH:



RESEARCH: The force of friction is given by $F_f = \mu_s N_1$. First, consider m_1. Using Newton's second law:
$\sum F_x = ma_x \;\Rightarrow\; F - F_f = m_1 a_1 \;\Rightarrow\; F = F_f + m_1 a_1$ and $\sum F_y = ma_y = 0 \;\Rightarrow\; N_1 - m_1 g = 0 \;\Rightarrow\; N_1 = m_1 g$.
Then, consider m_2: $\sum F_x = ma_x$, $F_f = m_2 a_2$, and $N_2 - N_1 - m_2 g = 0$.

(a) The force is maximum when $F_f = \mu_s N_1$ and $a_1 = a_2 = a$.

(b) If F = 24.5 N is larger than F_{max}, then m_1 slides on m_2.

SIMPLIFY:

(a) $F_f = \mu_s N_1 = m_2 a_2 \Rightarrow \mu_s m_1 g = m_2 a_2 \Rightarrow a = \mu_s (m_1 / m_2) g$

$F_{max} = F_f + m_1 a = \mu_s m_1 g + m_1 \mu_s (m_1 / m_2) g = \mu_s m_1 g (1 + m_1 / m_2)$

(b) The force of friction is given by $F_f = \mu_k N_1$. Using the equations, $F = F_f + m_1 a_1$ and $F_f = m_2 a_2$:

$a_2 = \dfrac{F_f}{m_2} = \dfrac{\mu_k m_1 g}{m_2}$ and $a_1 = \dfrac{F - F_f}{m_1} = \dfrac{F - \mu_k m_1 g}{m_1} = \dfrac{F}{m_1} - \mu_k g$.

CALCULATE:

(a) $F_{max} = 0.456(2.50 \text{ kg})(9.81 \text{ m/s}^2)\left(1 + \dfrac{2.50 \text{ kg}}{3.75 \text{ kg}}\right) = 18.639 \text{ N}$

(b) $a_1 = \dfrac{24.5 \text{ N}}{2.50 \text{ kg}} - 0.380(9.81 \text{ m/s}^2) = 6.0722 \text{ m/s}^2$ and $a_2 = \dfrac{(0.380)(2.50 \text{ kg})(9.81 \text{ m/s}^2)}{3.75 \text{ kg}} = 2.4852 \text{ m/s}^2$

ROUND: Rounding to three significant figures,

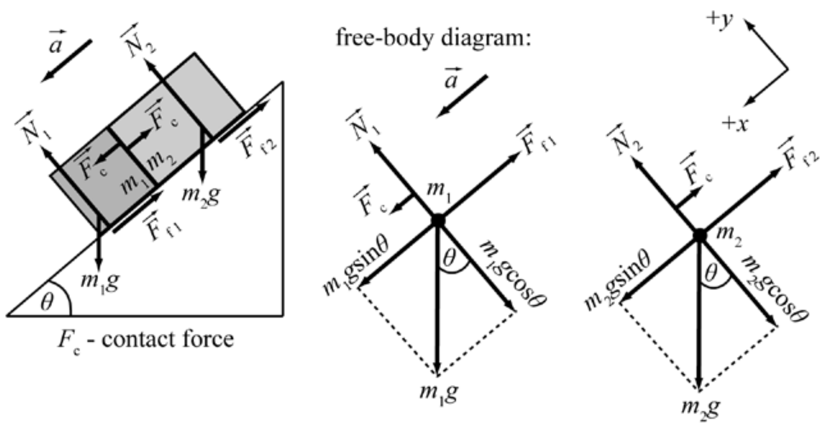
(a) $F_{max} = 18.6 \text{ N}$,

(b) $a_1 = 6.07 \text{ m/s}^2$ and $a_2 = 2.49 \text{ m/s}^2$.

DOUBLE-CHECK: For part (b), it is known that m_1 slides on m_2. This means that a_1 is larger than a_2.

4.88. **THINK:** Two blocks with masses of $m_1 = 1.23 \text{ kg}$ and $m_2 = 2.46 \text{ kg}$ are glued together, moving on an inclined plane of angle $\theta = 40.0°$. The coefficients of kinetic friction are $\mu_{1k} = 0.23$ and $\mu_{2k} = 0.35$. Determine the acceleration of the blocks.

SKETCH:



RESEARCH: The forces of friction are given by $F_{f1} = \mu_{1k} N_1$ and $F_{f2} = \mu_{2k} N_2$. First, consider m_1. Using Newton's second law:

$\sum F_x = ma_x \Rightarrow m_1 g \sin\theta + F_c - F_{f1} = m_1 a \Rightarrow m_1 g \sin\theta + F_c - \mu_{1k} N_1 = m_1 a$

$\sum F_y = ma_y = 0 \Rightarrow N_1 - m_1 g \cos\theta = 0 \Rightarrow N_1 = m_1 g \cos\theta$

Consider m_2:

$\sum F_x = ma_x \Rightarrow m_2 g \sin\theta - F_c - F_{f2} = m_2 a \Rightarrow m_2 g \sin\theta - F_c - \mu_{2k} N_2 = m_2 a$

$\sum F_y = ma_y = 0 \Rightarrow N_2 - m_2 g \cos\theta = 0 \Rightarrow N_2 = m_2 g \cos\theta$

SIMPLIFY:

$$\begin{array}{l} m_1 g \sin\theta + F_c - \mu_{1k} N_1 = m_1 a \\ +\ m_2 g \sin\theta - F_c - \mu_{2k} N_2 = m_2 a \\ \hline (m_1 + m_2) g \sin\theta - \mu_{1k} N_1 - \mu_{2k} N_2 = (m_1 + m_2) a \end{array}$$

This implies $a = g\sin\theta - \dfrac{\mu_{1k}m_1 g\cos\theta + \mu_{2k}m_2 g\cos\theta}{(m_1+m_2)} = g\sin\theta - g\cos\theta\left[\dfrac{\mu_{1k}m_1 + \mu_{2k}m_2}{(m_1+m_2)}\right].$

CALCULATE: $a = (9.81\text{ m/s}^2)\sin(40.0°) - (9.81\text{ m/s}^2)\cos(40°.0)\left[\dfrac{0.23(1.23\text{ kg}) + 0.35(2.46\text{ kg})}{(1.23\text{ kg} + 2.46\text{ kg})}\right]$

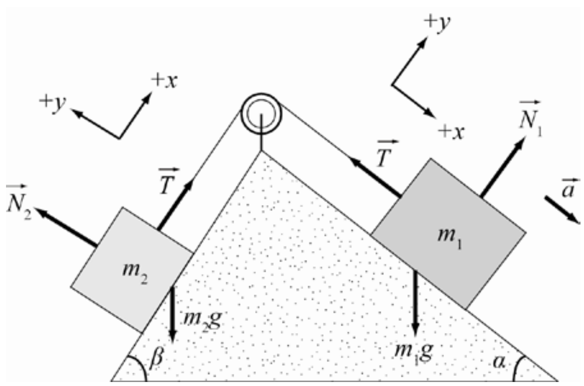
$= 3.976\text{ m/s}^2$

ROUND: Rounding to two significant figures, $a = 4.0\text{ m/s}^2$.

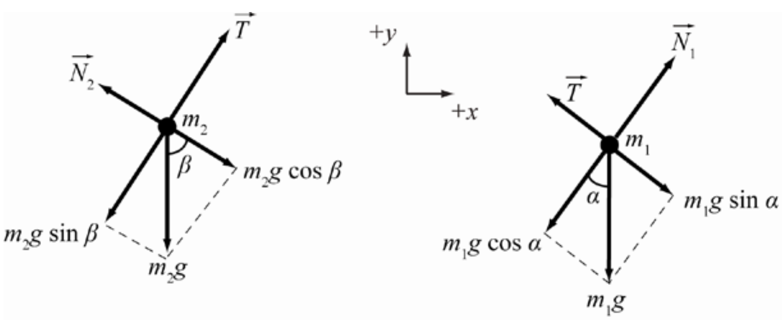
DOUBLE-CHECK: For a one block system, the acceleration is given by $a = g\sin\theta - \mu_k g\cos\theta$. $\dfrac{\mu_{1k}m_1 + \mu_{2k}m_2}{(m_1+m_2)}$ is an effective coefficient for two blocks.

4.89. **THINK:** Two blocks with masses, $m_1 = 567.1\text{ kg}$ and $m_2 = 266.4\text{ kg}$ are on inclined planes with angles $\alpha = 39.3°$ and $\beta = 53.2°$. Assume there is no friction. Determine the acceleration of the marble block, m_1. The marble block, m_1, is heavier than the granite block, m_2, but angle β is larger than angle α. It is impossible to guess which block will move up the incline, and which block will move down. The assumption will be made that the marble block accelerates down the incline, and choose that to be the positive direction. If the assumption is correct, the acceleration will be positive, and if it is incorrect, the acceleration will be negative.

SKETCH:



Assume the motion is in the positive *x* direction.



RESEARCH: First, consider m_1. Using Newton's second law: $\sum F_y = ma_y = 0,\quad N_1 - m_1 g\cos\alpha = 0,$ $\sum F_x = ma_x$, and $m_1 g\sin\alpha - T = m_1 a.$ Then, consider m_2: $\sum F_y = ma_y = 0,\quad N_2 - m_2 g\cos\beta = 0,$ $\sum F_x = ma_x$, and $T - m_2 g\sin\beta = m_2 a.$

SIMPLIFY: $T = m_2 g\sin\beta + m_2 a,$ and so:

$$m_1 g\sin\alpha - T = m_1 a$$
$$m_1 g\sin\alpha - m_2 g\sin\beta - m_2 a = m_1 a$$
$$m_1 g\sin\alpha - m_2 g\sin\beta = \left(m_1 + m_2\right)a$$
$$a = g\frac{m_1\sin\alpha - m_2\sin\beta}{\left(m_1 + m_2\right)}$$

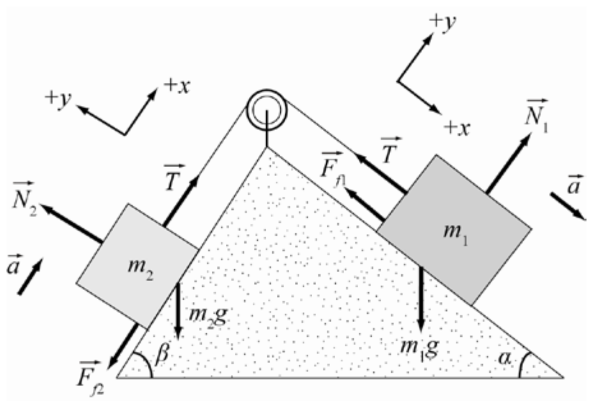
CALCULATE: $a = \left(9.81\text{ m/s}^2\right)\dfrac{\left(567.1\text{ kg}\right)\sin\left(39.3°\right) - \left(266.4\text{ kg}\right)\sin\left(53.2°\right)}{\left(567.1\text{ kg} + 266.4\text{ kg}\right)} = 1.7169\text{ m/s}^2$

ROUND: Rounding to three significant figures, $a = 1.72\text{ m/s}^2$.

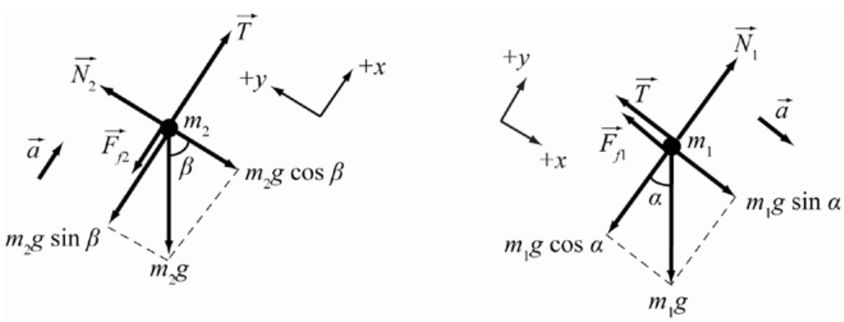
DOUBLE-CHECK: As we expect, the acceleration must be less than $\text{g} = 9.81\text{ m/s}^2$.

4.90. **THINK:** There are two blocks with masses, $m_1 = 559.1\text{ kg}$ and $m_2 = 128.4\text{ kg}$. The angles are $\alpha = 38.3°$ and $\beta = 57.2°$. The coefficients of friction are $\mu_1 = 0.13$ and $\mu_2 = 0.31$. Determine the accelertation of the blocks.

SKETCH:



Assume motion is in the positive x-direction.



RESEARCH: The forces of friction are given by $F_{\text{f1}} = \mu_1 N_1$ and $F_{\text{f2}} = \mu_2 N_2$. First, consider m_1. Using Newton's second law:

$\sum F_x = ma_x$ and $m_1 g\sin\alpha - T - F_{\text{f1}} = m_1 a \Rightarrow m_1 g\sin\alpha - T - \mu_1 m_1 = m_1 a$

$\sum F_y = ma_y = 0,$ $a_y = 0,$ and $N_1 - m_1 g\cos\alpha = 0 \Rightarrow N_1 = m_1 g\cos\alpha$

Then, consider m_2:

$\sum F_x = ma_x$ and $T - m_2 g\sin\beta - F_{\text{f2}} = m_2 a \Rightarrow T - m_2 g\sin\beta - \mu_2 N_2 = m_2 a$

$\sum F_y = ma_y = 0$ and $N_2 - m_2 g\cos\beta = 0 \Rightarrow N_2 = m_2 g\cos\beta$

SIMPLIFY: $m_1 g\sin\alpha - T - \mu_1 m_1 g\cos\alpha = m_1 a$ and $T - m_2 g\sin\beta - \mu_2 m_2 g\cos\beta = m_2 a$, which implies $T = m_2 a + m_2 g\sin\beta + \mu_2 m_2 g\cos\beta$. Eliminate T:

$$m_1 g\sin\alpha - m_2 a - m_2 g\sin\beta - \mu_2 m_2 g\cos\beta = \mu_1 m_1 g\cos\alpha + m_1 a$$

$$m_1 g\sin\alpha - m_2 g\sin\beta - \mu_2 m_2 g\cos\beta - \mu_1 m_1 g\cos\alpha = \left(m_1 + m_2\right)a$$

$$a = g\frac{m_1\sin\alpha - m_2\sin\beta - \mu_2 m_2\cos\beta - \mu_1 m_1\cos\alpha}{\left(m_1 + m_2\right)}$$

$$a = g\left(\frac{m_1\left(\sin\alpha - \mu_1\cos\alpha\right)}{m_1 + m_2} - \frac{m_2\left(\sin\beta + \mu_2\cos\beta\right)}{m_1 + m_2}\right)$$

CALCULATE:

$$a = 9.81\text{ m/s}^2\left(\frac{(559.1\text{ kg})(\sin 38.3° - 0.13\cos 38.3°)}{(559.1\text{ kg} + 128.4\text{ kg})} - \frac{(128.4\text{ kg})(\sin 57.2° + 0.31\cos 57.2°)}{(559.1\text{ kg} + 128.4\text{ kg})}\right)$$

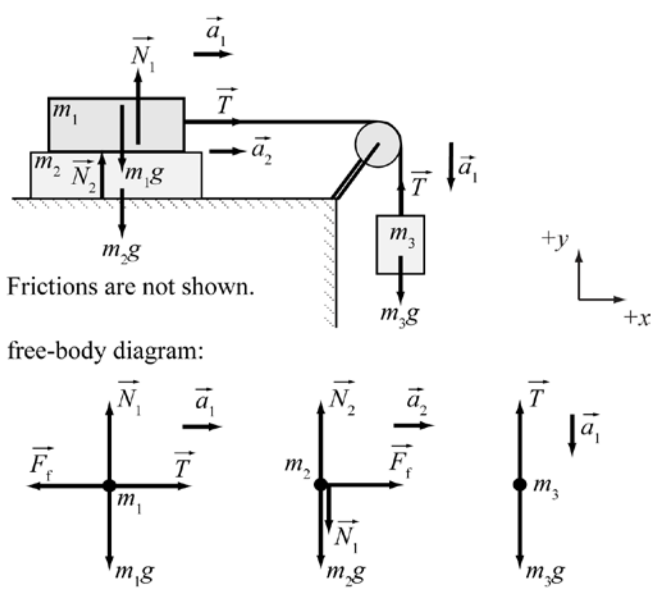
$$= 2.283\text{ m/s}^2$$

ROUND: Rounding to two significant figures, $a = 2.3\text{ m/s}^2$.

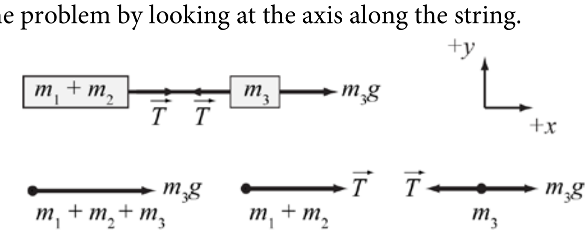
DOUBLE-CHECK: The acceleration must be less than $g = 9.81\text{ m/s}^2$.

4.91. **THINK:** Three blocks have masses, $m_1 = 3.50$ kg, $m_2 = 5.00$ kg and $m_3 = 7.60$ kg. The coefficients of static and kinetic friction between m_1 and m_2 are 0.600 and 0.500. Determine the accelerations of m_1 and m_2, and tension of the string. If m_1 does not slip on m_2, then the accelerations of both blocks will be the same. First, make the assumption that the blocks do not slide. Then, it must be determined whether the acting force of friction, F_f, is less than or greater than the maximum force of friction.

SKETCH:



RESEARCH: Simplify the problem by looking at the axis along the string.



Using Newton's second law: $\sum F = ma \;\Rightarrow\; m_3 g = (m_1 + m_2 + m_3)a$ and $\sum F = ma \;\Rightarrow\; T = (m_1 + m_2)a$.

SIMPLIFY:

(a) $a = \dfrac{m_3 g}{(m_1 + m_2 + m_3)}$

(b) $T = \dfrac{(m_1 + m_2)m_3 g}{(m_1 + m_2 + m_3)}$

CALCULATE:

(a) $a = \dfrac{7.60\text{ kg}(9.81\text{ m/s}^2)}{(3.50\text{ kg} + 5.00\text{ kg} + 7.60\text{ kg})} = 4.631\text{ m/s}^2$

(b) $T = (3.50\text{ kg} + 5.00\text{ kg})4.631\text{ m/s}^2 = 39.362\text{ N}$

Now, it must be determined if the force of friction, F_f is less than the maximum force of static friction. From the free-body diagram of m_2: $\sum F_x = m_2 a \;\Rightarrow\; F_\text{f} = m_2 a = 5.00(4.631\text{ m/s}^2) = 23.15\text{ N}.$ The maximum force of static friction is $f_{\text{s,max}} = \mu_\text{s} N_1$ where $N_1 = m_1 g$. $f_{\text{s,max}} = (0.600)(3.50)(9.81) = 20.60\text{ N}.$ $F_\text{f} > f_{\text{s,max}}$, so block 1 slips on block 2. Some parts of the question must be reconsidered.

RESEARCH: The force of friction is given by $F_\text{f} = \mu_\text{k} N_1$. First, consider m_1. Using Newton's second law:

$\sum F_x = ma_x \;\Rightarrow\; T - F_\text{f} = m_1 a_1 \;\Rightarrow\; T - \mu_\text{k} N_1 = m_1 a_1 \;\Rightarrow\; T - \mu_\text{k} m_1 g = m_1 a_1$

$\sum F_y = ma_y = 0 \;\Rightarrow\; N_1 - m_1 g = 0 \;\Rightarrow\; N_1 = m_1 g$

Now, consider m_2: $\sum F_x = ma_x \;\Rightarrow\; F_\text{f} = m_2 a_2 \;\Rightarrow\; a_2 = \dfrac{\mu_\text{k} N_1}{m_2} = \mu_\text{k}\dfrac{m_1 g}{m_2}$. Finally, consider m_3:

$\sum F_y = ma_y \;\Rightarrow\; m_3 g - T = m_3 a_1.$

SIMPLIFY: $T - \mu_\text{k} m_1 g = m_1 a_1$ and $m_3 g - T = m_3 a_1$ can be used to eliminate *T*:

$$m_3 g - m_1 a_1 - \mu_\text{k} m_1 g = m_3 a_1 \;\Rightarrow\; m_3 g - \mu_\text{k} m_1 g = (m_1 + m_3)a_1 \;\Rightarrow\; a_1 = g\frac{(m_3 - \mu_k m_1)}{(m_1 + m_3)}$$

Also, $a_2 = \dfrac{\mu_\text{k} m_1}{m_2} g$ and $T = m_1(a_1 + \mu_\text{k} g) = m_3(g - a_1)$.

CALCULATE:

(a) $a_1 = (9.81\text{ m/s}^2)\dfrac{(7.60\text{ kg} - 0.500(3.50\text{ kg}))}{(7.60\text{ kg} + 3.50\text{ kg})} = 5.1701\text{ m/s}^2$,

$a_2 = \dfrac{0.500(3.50\text{ kg})}{5.00\text{ kg}}(9.81\text{ m/s}^2) = 3.4335\text{ m/s}^2$

(b) $T = 7.60\text{ kg}(9.81\text{ m/s}^2 - 5.1701\text{ m/s}^2) = 35.26\text{ N}$

ROUND: Rounding to three significant figures,

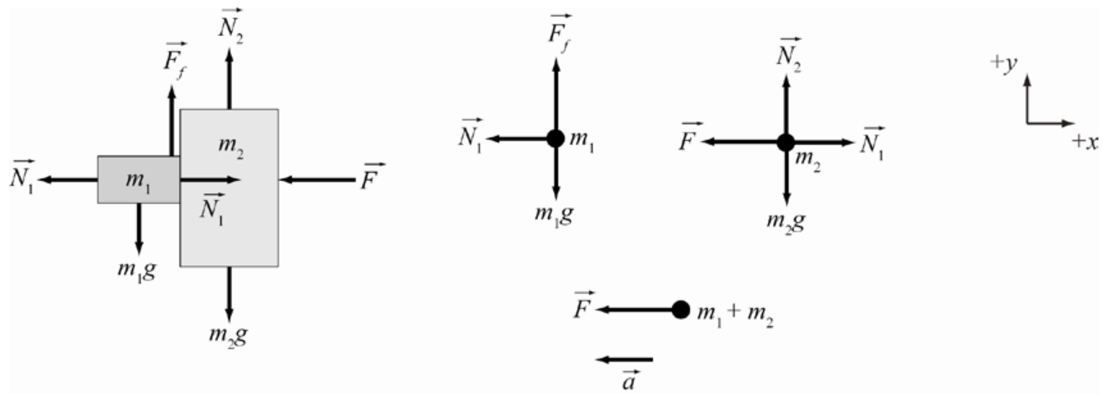
(a) $a_1 = 5.17\text{ m/s}^2$, $a_2 = 3.43\text{ m/s}^2$ and

(b) $T = 35.3\text{ N}$.

DOUBLE-CHECK: Because block 1 slips on block 2, a_1 is larger than a_2.

4.92. **THINK:** There are two blocks with masses, $m_1 = 2.30 \text{ kg}$ and $m_2 = 5.20 \text{ kg}$. The coefficient of static friction between the blocks is 0.65.

SKETCH:



RESEARCH:

(a) From the above diagram, there are three forces acting on block 1; the normal, frictional and gravitational forces.

(b) Consider block 1. Using Newton's second law:

$$\sum F_x = ma_x \Rightarrow N_1 = m_1 a \text{ and } \sum F_y = ma_y = 0 \Rightarrow F_f - m_1 g = 0 \Rightarrow F_f = m_1 g$$

There is a minimum force *F*, when there is a minimum acceleration and $F_f = \mu_s N_1$.

SIMPLIFY: $F_f = \mu_s N_1 = m_1 g \Rightarrow N_1 = \dfrac{m_1 g}{\mu_s} \Rightarrow \mu_s m_1 a = m_1 g \Rightarrow a = \dfrac{g}{\mu_s}$. Consider a block composed of block 1 and block 2: $\sum F_x = ma_x \Rightarrow F = (m_1 + m_2)a \Rightarrow F = \dfrac{(m_1 + m_2)g}{\mu_s}$.

(c) $N_1 = \dfrac{m_1 g}{\mu_s}$

(d) $\sum F_x = m_2 a = \dfrac{m_2 g}{\mu_s}$

CALCULATE:

(b) $F = \dfrac{(2.30 \text{ kg} + 5.20 \text{ kg})(9.81 \text{ m/s}^2)}{0.65} = 113.19 \text{ N}$

(c) $N_1 = \dfrac{2.30 \text{ kg}(9.81 \text{ m/s}^2)}{0.65} = 34.71 \text{ N}$

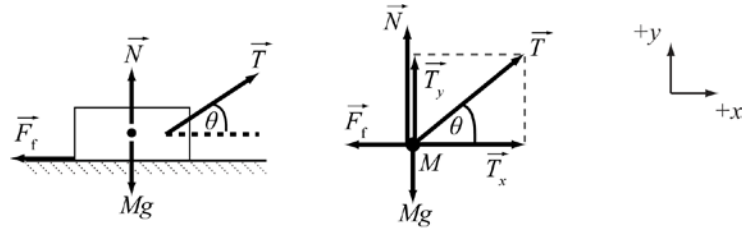
(d) $\sum F_x = \dfrac{5.20 \text{ kg}(9.81 \text{ m/s}^2)}{0.65} = 78.48 \text{ N}$

ROUND: Rounding to two significant figures, *F* = 110 N, $N_1 = 35 \text{ N}$ and $\sum F_x = 78 \text{ N}$.

DOUBLE-CHECK: The net force on block 2 added to the contact force N_1 should equal *F*. *F* = 78 N + 35 N = 113 N, as expected.

4.93. **THINK:** The suitcase has a weight of $M_g = 450.\text{ N}$. The coefficient of kinetic friction is $\mu_k = 0.640$. Determine the angle that minimizes the force to pull the suitcase.

SKETCH:



RESEARCH: The force of friction is given by $F_\text{f} = \mu_\text{k} N$. Also, $T_x = T\cos\theta$ and $T_y = T\sin\theta$. Using Newton's second law:

$\sum F_x = ma_x$, and $a_x = 0$, so $T_x - F_\text{f} = 0 \Rightarrow T\cos\theta = F_\text{f} = \mu_\text{k} N$

$\sum F_y = ma_y = 0 \Rightarrow T_y + N - Mg = 0 \Rightarrow T\sin\theta + N - Mg = 0 \Rightarrow N = Mg - T\sin\theta$

SIMPLIFY:

$$T\cos\theta = \mu_\text{k}\left(Mg - T\sin\theta\right) = \mu_\text{k} Mg - \mu_\text{k} T\sin\theta$$

$$T\left(\cos\theta + \mu_k \sin\theta\right) = \mu_\text{k} Mg$$

$$T = \mu_\text{k} Mg\left(\cos\theta + \mu_\text{k}\sin\theta\right)^{-1}$$

Differentiate T with respect to θ; $\dfrac{dT}{d\theta} = -\mu_k Mg\left(-\sin\theta + \mu_\text{k}\cos\theta\right)\left(\cos\theta + \mu_\text{k}\sin\theta\right)^{-2}$. The minimum tension is when $dT/d\theta = 0$: $-\mu_k Mg\left(-\sin\theta + \mu_\text{k}\cos\theta\right)\left(\cos\theta + \mu_\text{k}\sin\theta\right)^{-2} = 0$, which simplifies to $-\sin\theta + \mu_\text{k}\cos\theta = 0$, or $\tan\theta = \mu_\text{k}$. Thus, $\theta = \tan^{-1}\left(\mu_\text{k}\right)$.

CALCULATE:

(a) $\theta = \tan^{-1}\left(0.640\right) = 32.6192°$

(b) $T = \dfrac{0.640\left(450.\text{ N}\right)}{\left(\cos\left(32.6°\right) + 0.640\sin\left(32.6°\right)\right)} = 242.5\text{ N}$

ROUND: Rounding to three significant figures,

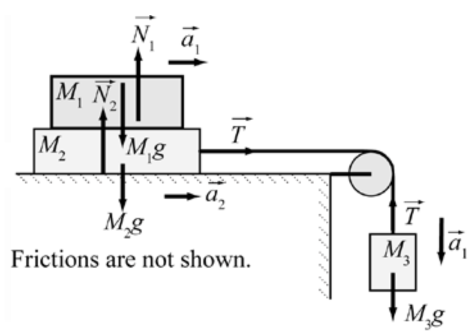
(a) $\theta = 32.6°$ and

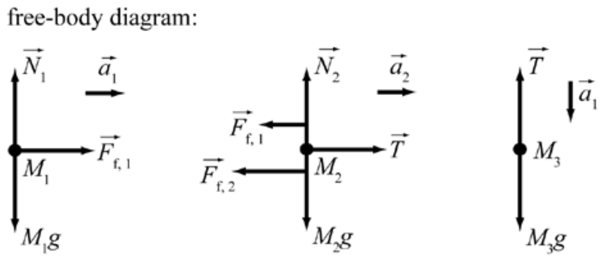
(b) $T = 243$ N.

DOUBLE-CHECK: As a comparison, if $\theta = 0°$, $T = \mu_k M_g = 0.640\left(450.\text{ N}\right) = 288\text{ N}$. The minimum tension calculated is less than this value.

4.94. **THINK:** There are three blocks with masses, $M_1 = 0.450\text{ kg}$, $M_2 = 0.820\text{ kg}$ and M_3. The coefficients of static and kinetic friction are $\mu_\text{s} = 0.560$ and $\mu_\text{k} = 0.340$.

SKETCH:





RESEARCH: The force of friction is given by $F_f = \mu N$. It is given that block 1 does not slide on M_2, which means $a_1 = a_2 = a$. First, consider M_1. Using Newton's second law:

$$\sum F_x = ma_x \Rightarrow F_{f,1} = M_1 a \text{ and } \sum F_y = ma_y = 0 \Rightarrow N_1 - M_1 g = 0 \Rightarrow N_1 = M_1 g$$

Then, consider M_2: $\sum F_x = ma_x \Rightarrow T - F_{f,1} - F_{f,2} = M_2 a$

$$\text{and} \sum F_y = ma_y = 0 \Rightarrow N_2 - N_1 - M_2 g = 0 \Rightarrow N_2 = (M_1 + M_2)g$$

Finally, consider M_3: $\sum F_y = ma_y \Rightarrow M_3 g - T = M_3 a.$

SIMPLIFY: The maximum force of friction before M_1 will slip is given by $F_{f,1} = f_{s,max} = \mu_s N_1 = \mu_s M_1 g.$

$$F_{f,1} = \mu_s M_1 g = M_1 a \Rightarrow a = \mu_s g \quad (1)$$

Mass 2 is slipping along the table so $F_{f,2} = f_k = \mu_k N_2 = \mu_k (M_1 + M_2)g$; therefore, the equation for the x-direction yields $T - \mu_s M_1 g - \mu_k (M_1 + M_2)g = M_2 a \Rightarrow T = M_2 a + \mu_s M_1 g + \mu_k (M_1 + M_2)g$ (2).

Substituting (1) in (2) yields $T = M_2 \mu_s g + \mu_s M_1 g + \mu_k (M_1 + M_2)g = \mu_s (M_1 + M_2)g + \mu_k (M_1 + M_2)g$ (3).

Solving the equation x-direction equation for M_3 yields, $M_3 = \dfrac{T}{g - a}$. Substituting (1) and (3) for a and T, respectively, yields $M_3 = \dfrac{\mu_s (M_1 + M_2)g + \mu_k (M_1 + M_2)g}{g - \mu_s g} = \dfrac{(\mu_s + \mu_k)(M_1 + M_2)}{1 - \mu_s}.$

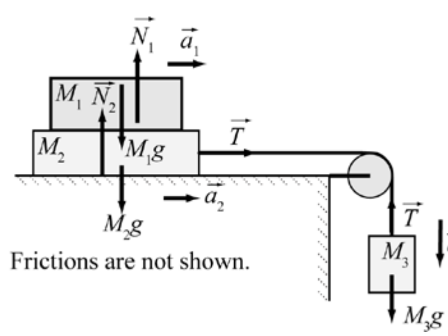
CALCULATE: $M_3 = \dfrac{(0.560 + 0.340)(0.450 \text{ kg} + 0.820 \text{ kg})}{1 - 0.560} = 2.5977 \text{ kg}$

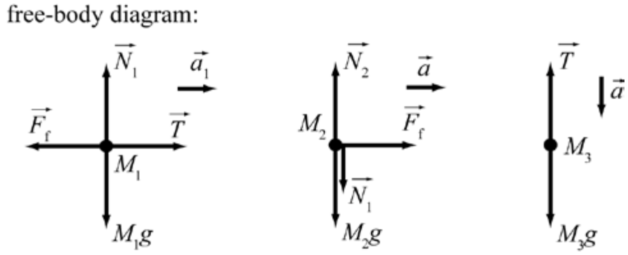
ROUND: Rounding to three significant figures, $M_3 = 2.60$ kg.

DOUBLE-CHECK: The order of magnitude of M_3 is reasonable in comparison to M_1 and M_2.

4.95. **THINK:** The three blocks have masses, $M_1 = 0.250$ kg, $M_2 = 0.420$ kg and $M_3 = 1.80$ kg. The coefficient of kinetic friction is $\mu_k = 0.340$.

SKETCH:





RESEARCH: The force of friction is given by $F_f = \mu_k N$. First, consider M_1. Using Newton's second law:

$$\sum F_x = ma_x \Rightarrow F_{f,1} = M_1 a_1 \text{ and } \sum F_y = ma_y = 0 \Rightarrow N_1 - M_1 g = 0 \Rightarrow N_1 = M_1 g$$

Then, consider M_2: $\sum F_x = ma_x \Rightarrow T - F_{f,1} - F_{f,2} = M_2 a$ and

$$\sum F_y = ma_y = 0 \Rightarrow N_2 - N_1 - M_2 g = 0 \Rightarrow N_2 = (M_1 + M_2)g$$

Finally, consider M_3: $\sum F_y = ma_y \Rightarrow M_3 g - T = M_3 a.$

SIMPLIFY:

(a) $F_{f,1} = \mu_k N_1 = \mu_k M_1 g = M_1 a_1 \Rightarrow a_1 = \mu_k g$, which is the acceleration of the block.

(b) For the slab (M_2) the friction with the table is $F_{f,2} = f_k = \mu_k N_2 = \mu_k (M_1 + M_2)g$; therefore, the equation for the x direction yields $T - \mu_k M_1 g - \mu_k (M_1 + M_2)g = M_2 a$. From the equation for M_3, $T = M_3(g - a)$. Substituting this for T yields the following.

$$M_3(g-a) - \mu_k M_1 g - \mu_k (M_1 + M_2)g = M_2 a \Rightarrow (M_2 + M_3)a = M_3 g - \mu_k (2M_1 + M_2)g$$

$$\Rightarrow a = \frac{\left[M_3 - \mu_k (2M_1 + M_2)\right]g}{M_2 + M_3}$$

CALCULATE:

(a) $a_1 = (0.340)(9.81 \text{ m/s}^2) = 3.335 \text{ m/s}^2$

(b) $a = \dfrac{\left[1.80 \text{ kg} - (0.340)(2 \cdot 0.250 \text{ kg} + 0.420 \text{ kg})\right](9.81 \text{ m/s}^2)}{0.420 \text{ kg} + 1.80 \text{ kg}} = 6.572 \text{ m/s}^2$

ROUND: Rounding to three significant figures,

(a) $a_1 = 3.34 \text{ m/s}^2$ and

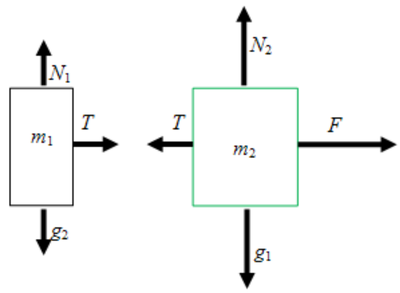
(b) $a = 6.57 \text{ m/s}^2$.

DOUBLE-CHECK: Because M_1 slides on M_2, it is expected that a_1 is less than a. Both a_1 and a must be less than g.

Multi-Version Exercises

4.96. **THINK:** This problem involves two blocks sliding along a frictionless surface. For these types of problems, use Newton's laws. Also note that the tension force from block 1 must be exactly equal and opposite from the force or block 2.

SKETCH: Start with the image from the text to create free body diagrams for each block.



RESEARCH: Since the blocks are lying on a frictionless surface, there is no friction force acting on the blocks. It is necessary to find the tension T on the rope in terms of the force F acting on the second block and the masses m_1 and m_2. Since force F is acting on the blocks and rope as a single system, Newton's Second Law gives that $F=(m_1+m_2)a$, where a is the acceleration of the blocks. Looking at the horizontal forces on block 1 gives $T=m_1a$.

SIMPLIFY: It is necessary to find an expression for the tension in terms of the outside force F acting on block 2 and the masses of the two blocks. Rewriting $T=m_1a$ as $a=\frac{T}{m_1}$ and combining with $F=(m_1+m_2)a$ gives $F=(m_1+m_2)\frac{T}{m_1}$. This expression can be re-written to give the tension in terms of known quantities: $T=F\frac{m_1}{m_1+m_2}$.

CALCULATE: Using the masses and force given in the question statement gives a tension force of:

$$T=12.61\text{ N}\left(\frac{1.267\text{ kg}}{1.267\text{ kg}+3.557\text{ kg}}\right)$$
$$=3.311954809\text{ N}$$

ROUND: The masses of the blocks are given to four significant figures, and their sum also has four significant figures. The only other measured quantity is the external force acting on the second block, which also has four significant figures. This means that the final answer should be rounded to four significant figures, giving a total tension of T = 3.312 N.

DOUBLE-CHECK: Think of the tension on the rope transmitting the force from block 2 to block 1. Since block 2 is much more massive than block 1, block 1 represents about one fourth of the total mass of the system. So, it makes sense that only about one fourth of the force will be transmitted along the string to the second block. About 12.61/4 or 3.153 N will be transmitted, which is pretty close to our calculated value of 3.312 N.

4.97.

$$F=(m_1+m_2)\frac{T}{m_1}$$
$$m_1F=m_1T+m_2T$$
$$m_1(F-T)=m_2T$$
$$m_1=m_2\frac{T}{F-T}=(3.577\text{ kg})\frac{4.094\text{ N}}{13.89\text{ N}-4.094\text{ N}}=1.495\text{ kg}$$

4.98. $F = (m_1 + m_2)\dfrac{T}{m_1}$

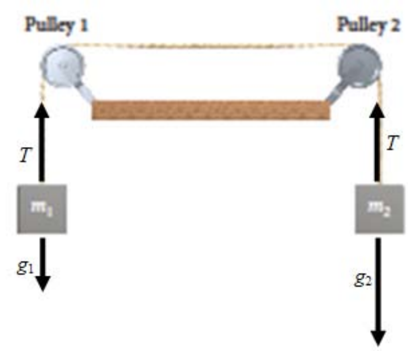
$m_1 F = m_1 T + m_2 T$

$m_2 = \dfrac{m_1 F - m_1 T}{T} = m_1 \dfrac{F - T}{T} = (1.725 \text{ kg})\dfrac{15.17 \text{ N} - 4.915 \text{ N}}{4.915 \text{ N}} = 3.599 \text{ kg}$

4.99. $F = (m_1 + m_2)\dfrac{T}{m_1} = T\dfrac{m_1 + m_2}{m_1} = (5.777 \text{ N})\dfrac{1.955 \text{ kg} + 3.619 \text{ kg}}{1.955 \text{ kg}} = 16.47 \text{ N}$

4.100. **THINK:** This problem involves two hanging masses and two frictionless pulleys. The rope is massless, so it is not necessary to include the gravitational force on the spring. The forces on this system will be the gravitational force and the tension in the string. To solve this problem, it will be necessary to use Newton's laws.

SKETCH: Begin with the sketch from the text. Then, draw free body diagrams for both masses, keeping in mind that the forces exerted by the rope on the masses must be equal.



RESEARCH: First note that the two blocks will accelerate at the same rate, but in opposite directions. First note that the gravitational force on each block can be given in terms of the mass and gravitational acceleration: $g_1 = gm_1$ and $g_2 = gm_2$. It is possible to use Newton's Second law on each block individually to get two equations relating the tension *T*, the masses of the blocks m_1 and m_2, and the acceleration *a*. For the first block we have that $(T - m_1 g) = m_1 a$. Likewise, $(T - m_2 g) = m_2 \cdot (-a)$ because block 2 is accelerating in the opposite direction from block 1. With these two equations, we should be able to solve for either of the unknown quantities *a* or *T*.

SIMPLIFY: Since the problems asks for acceleration *a*, first find an equation for the tension *T* in terms of the other quantities. $T - m_1 g = m_1 a$ means that $T = m_1 a + m_1 g$. Substitute this expression for *T* into the equation $(T - m_2 g) = m_2 \cdot (-a)$ to get:

$$-m_2 a = (m_1 a + m_1 g) - m_2 g \Rightarrow$$
$$-m_1 a - m_2 a = -m_1 a + m_1 a + m_1 g - m_2 g \Rightarrow$$
$$(-m_1 - m_2)a = m_1 g - m_2 g \Rightarrow$$
$$a = \frac{m_1 g - m_2 g}{-m_1 - m_2}$$

CALCULATE: The masses m_1 and m_2 are given in the problem. The gravitational acceleration *g* is about 9.81 m·s^{-2}. Using these values gives the acceleration *a*:

$$a = \frac{m_1 g - m_2 g}{-m_1 - m_2}$$
$$= \frac{1.183 \text{ kg} \cdot 9.81 \text{ m s}^{-2} - 3.639 \text{ kg} \cdot 9.81 \text{ m s}^{-2}}{-1.183 \text{ kg} - 3.639 \text{ kg}} = 4.99654915 \text{ m s}^{-2}$$

ROUND: The masses of the blocks are given to four significant figures, and their sum also has four significant figures. Since they are the only measured quantities in this problem, the final answer should have four significant figures, giving a final answer of 4.997 m/s^2.

DOUBLE-CHECK: In this situation, it is intuitively obvious that the heavier mass will fall towards the ground less quickly than if it were in free fall. So it is reasonable that our calculated value of 4.997 m/s^2 is less than the 9.81 m/s^2, which is the rate at which objects on the Earth's surface generally accelerate towards the ground.

4.101. $m_1 g + m_1 a = m_2 g - m_2 a$

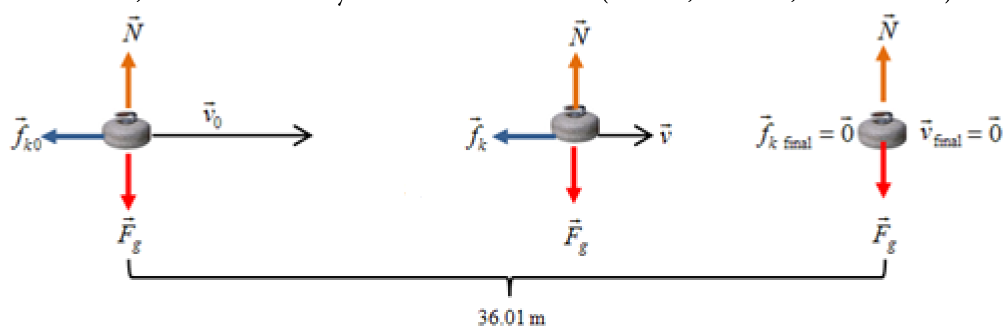
$$m_2 = m_1 \frac{g+a}{g-a} = (1.411\text{ kg})\frac{9.81\text{ m/s}^2 + 4.352\text{ m/s}^2}{9.81\text{ m/s}^2 - 4.352\text{ m/s}^2} = 3.661\text{ kg}$$

4.102. $m_1 g + m_1 a = m_2 g - m_2 a$

$$m_1 = m_2 \frac{g-a}{g+a} = (3.681\text{ kg})\frac{9.81\text{ m/s}^2 - 3.760\text{ m/s}^2}{9.81\text{ m/s}^2 + 3.760\text{ m/s}^2} = 1.641\text{ kg}$$

4.103. **THINK:** This problem involves friction, so the only forces acting on the curling stone are gravity, the normal force, and the frictional force. Since gravity and the normal force act in the vertical direction, the only force slowing the horizontal movement of the curling stone is the frictional force. It is necessary to come up with a way to relate the initial velocity to the mass, coefficient of friction, and total distance traveled by the stone. Since the curling stone is slowing to a stop, there is a net external force, so it will be necessary to use Newton's Second law.

SKETCH: It is helpful to draw the free body diagrams for the curling stone at three different times: the moment the curling stone is released, part of the way along its path, and after it has stopped. When the stone is at rest, there is no velocity and no kinetic friction (there is, however, static friction).



RESEARCH: The only unknown force acting on the curling stone is the kinetic friction force. The magnitude is given by $f_k = \mu_k N$. But the normal force is equal and opposite to the gravitational force ($\vec{N} = -\vec{F}_g = -(mg)$), so the magnitude of the normal force is $N = mg$. The initial kinetic energy of the stone is $K = \frac{1}{2}mv_0^2$ and the kinetic energy of the stone at rest is zero. So, all of the kinetic energy has been dissipated by friction. The energy dissipated by friction is equal to the magnitude of the force times the distance traveled, $f_k d = K$. This will allow us to find the initial velocity in terms of known quantities.

SIMPLIFY: First, it is necessary to combine $f_k d = K$ with the equations for the kinetic energy, normal force, and frictional force. So:

$$\begin{aligned} f_k d &= K \Rightarrow \\ (\mu_k N)d &= \tfrac{1}{2}mv_0^2 \Rightarrow \\ \mu_k (mg)d &= \tfrac{1}{2}mv_0^2 \end{aligned}$$

Use algebra to find an expression for the initial velocity in terms of known quantities:

$$\mu_k(mg)d = \tfrac{1}{2}mv_0^2 \Rightarrow$$
$$\frac{2}{m}\mu_k mgd = \frac{2}{m}\tfrac{1}{2}mv_0^2 \Rightarrow$$
$$2\mu_k gd = v_0^2 \Rightarrow$$
$$\sqrt{2\mu_k gd} = v_0$$

CALCULATE: The mass of the curling stone m = 19.00 kg, the coefficient of kinetic friction between the stone and the ice μ_k = 0.01869 and the total distance traveled d = 36.01 m. The acceleration due to gravity on the surface of the Earth is not given in the problem, but it is g = 9.81 m/s^2. Using these values,

$$v_0 = \sqrt{2\mu_k gd} = \sqrt{2\cdot 0.01869\cdot 9.81\text{ m/s}^2\cdot 36.01\text{ m}} = 3.633839261\text{ m/s}$$

ROUND: The mass of the curling stone, distance traveled, and coefficient of friction between the ice and the stone are all given to four significant figures. These are the only measured values in the problem, so the answer should also be given to four significant figures. The initial velocity was 3.634 m/s.

DOUBLE-CHECK: Since the gravitational and normal forces are perpendicular to the direction of the motion and cancel one another exactly, they will not affect the velocity of the curling stone. Between the time the stone is released and the moment it stops, the frictional force acts in the opposite direction of the velocity and is proportional to the normal force, so this is one-dimensional motion with constant acceleration. It is possible to check this problem by working backward from the initial velocity and force to find an expression for velocity as a function of time, and then use that to find the total distance traveled. Newton's Second law and the equation for the frictional force can be combined to find the acceleration of the curling stone: $f_k = -\mu_k mg = ma_x \Rightarrow a_x = -\mu_k g.$ If the spot where the curling stone was released is x_0 = 0, then the equations for motion in one dimension with constant acceleration become:

$$v = v_{x0} + at = 3.634 - 0.01869\cdot 9.81t$$

and

$$d = x_0 + v_{x0}t + \frac{1}{2}at^2 = 0 + 3.634t - \frac{1}{2}0.01869\cdot 9.81t^2$$

Solving the first equation for v = 0 gives $0 = 3.634 - 0.01869\cdot 9.81t \Rightarrow t = \dfrac{3.634}{0.01869\cdot 9.81} = 19.82$ sec. (The stone was in motion 19.82 seconds, which is reasonable for those who are familiar with the sport of curling.) This value can be used to compute the distance traveled by the stone at the moment it stops as

$$d = 0 + 3.634t - \frac{1}{2}0.01869\cdot 9.81t^2 = 0 + 3.634(19.82) - \frac{1}{2}0.01869\cdot 9.81(19.82)^2 = 36.01\text{ m}$$

This confirms the calculated result.

4.104. $(\mu_k mg)d = \dfrac{1}{2}mv_0^2$

$$d = \frac{mv_0^2}{2(\mu_k mg)} = \frac{v_0^2}{2\mu_k g} = \frac{(2.788\text{ m/s})^2}{2(0.01097)(9.81\text{ m/s}^2)} = 36.11\text{ m}$$

4.105. $(\mu_k mg)d = \dfrac{1}{2}mv_0^2$

$$\mu_k = \frac{v_0^2}{2gd} = \frac{(3.070\text{ m/s})^2}{2(9.81\text{ m/s}^2)(36.21\text{ m})} = 0.01327$$

Chapter 5: Kinetic Energy, Work, and Power

Concept Checks

5.1. c **5.2.** b **5.3.** d **5.4.** a) True b) False c) True

Multiple-Choice Questions

5.1. c **5.2.** d **5.3.** a **5.4.** d **5.5.** e **5.6.** e **5.7.** c **5.8.** b **5.9.** b **5.10.** c **5.11.** e **5.12.** d **5.13.** b **5.14.** a

Conceptual Questions

5.15. If the net work done on a particle is zero, then the net force on the particle must also be zero. If the net force is zero, then the acceleration is also zero. Hence, the particle's speed is constant.

5.16. If Paul and Kathleen both start from rest at a height h, then, by conservation of energy, they will have the same speed when they reach the bottom. That is, their initial energy is pure potential energy mgh and their final energy is pure kinetic energy $(1/2)mv^2$. Since energy is conserved (if we neglect friction!) then $mgh=(1/2)mv^2 \Rightarrow v=\sqrt{2gh}$. Their final velocity is independent of both their mass and the shape of their respective slides! They will in general not reach the bottom at the same time. From the figure, Kathleen will likely reach the bottom first since she will accelerate faster initially and will attain a larger speed sooner. Paul will start off much slower, and will acquire the bulk of his acceleration towards the end of his slide.

5.17. No. The gravitational force that the Earth exerts on the Moon is perpendicular to the Moon's displacement and so no work is done.

5.18. When the car is travelling at speed v_1, its kinetic energy is $(1/2)mv_1^2$. The brakes do work on the car causing it to stop over a distance d_1. The final velocity is zero, so the work Fd_1 is given by the initial kinetic energy: $(1/2)mv_1^2=Fd_1$. Similarly, when the car is travelling at speed v_2, the brakes cause the car to stop over a distance d_2, so we have $(1/2)mv_2^2=Fd_2$. Taking the ratio of the two equations, we have

$$\frac{(1/2)mv_2^2=Fd_2}{(1/2)mv_1^2=Fd_1}\rightarrow d_2=d_1\frac{v_2^2}{v_1^2}=d_1\frac{(2v_1)^2}{v_1^2}=4d_1.$$

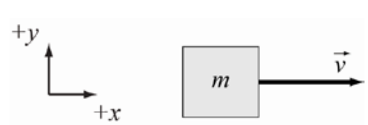
Thus the braking distance increases by a factor of 4 when the initial speed is increased by a factor of 2.

Exercises

5.19. **THINK:** Kinetic energy is proportional to the mass and to the square of the speed. m and v are known for all the objects:

(a) $m=10.0$ kg, $v=30.0$ m/s
(b) $m=100.0$ g, $v=60.0$ m/s
(c) $m=20.0$ g, $v=300.$ m/s

SKETCH:



RESEARCH: $K=\frac{1}{2}mv^2$

SIMPLIFY: $K=\frac{1}{2}mv^2$ is already in the right form.

CALCULATE:

(a) $K=\frac{1}{2}(10.0\text{ kg})(30.0\text{ m/s})^2=4500\text{ J}$

(b) $K=\frac{1}{2}(100.0\cdot 10^{-3}\text{ kg})(60.0\text{ m/s})^2=180\text{ J}$

(c) $K=\frac{1}{2}(20.0\cdot 10^{-3}\text{ kg})(300.\text{ m/s})^2=900\text{ J}$

ROUND:

(a) 3 significant figures: $K=4.50\cdot 10^3\text{ J}$

(b) 3 significant figures: $K=1.80\cdot 10^2\text{ J}$

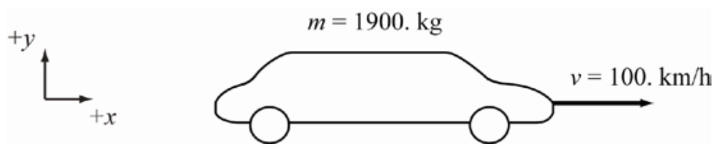
(c) 3 significant figures: $K=9.00\cdot 10^2\text{ J}$

DOUBLE-CHECK: The stone is much heavier so it has the greatest kinetic energy even though it is the slowest. The bullet has larger kinetic energy than the baseball since it moves at a much greater speed.

5.20. **THINK:** I want to compute kinetic energy, given the mass ($m=1900.\text{ kg}$) and the speed ($v=100.\text{ km/h}$). I must first convert the speed to m/s .

$$100.\frac{\text{km}}{\text{h}}\cdot 10^3\frac{\text{m}}{\text{km}}\cdot\frac{1\text{ h}}{60\text{ min}}\cdot\frac{1\text{ min}}{60\text{ s}}=\frac{100.\cdot 10^3\text{ m}}{3600\text{ s}}=27.778\text{ m/s}$$

SKETCH:



RESEARCH: $K=mv^2/2$

SIMPLIFY: No simplification needed.

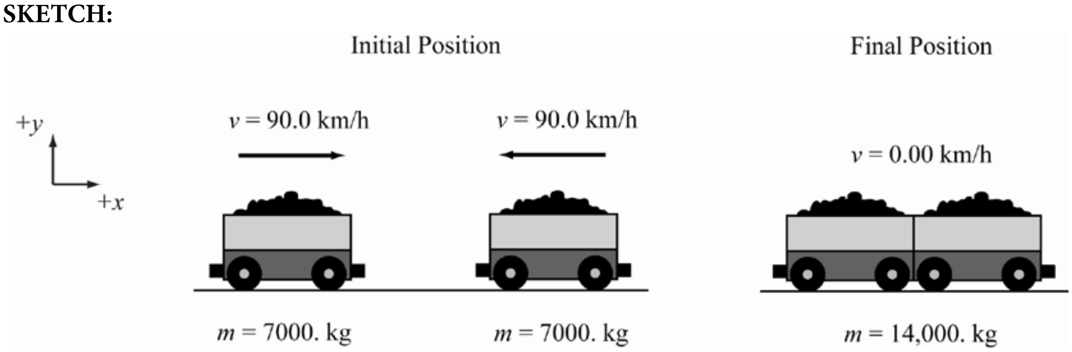
CALCULATE: $K=(1/2)mv^2=(1/2)(1900.\text{ kg})(27.778\text{ m/s})^2=7.3302\cdot 10^5\text{ J}$

ROUND: 100. km/h has three significant figures. Round the result to three significant figures: $K=7.33\cdot 10^5\text{ J}$.

DOUBLE-CHECK: This is a very large energy. The limo is heavy and is moving quickly.

5.21. **THINK:** Since both cars come to rest, the final kinetic energy of the system is zero. All the initial kinetic energy of the two cars is lost in the collision. The mass ($m=7000.\text{ kg}$) and the speed $v=90.0\frac{\text{km}}{\text{h}}\cdot\frac{10^3\text{ m}}{\text{km}}\cdot\frac{1\text{ h}}{3600\text{ s}}=25.0\text{ m/s}$ of each car is known. The total energy lost is the total initial kinetic energy.

SKETCH:



RESEARCH: $K_{\text{lost}} = \frac{1}{2}mv^2 + \frac{1}{2}mv^2$

SIMPLIFY: $K_{\text{lost}} = 2\left(\frac{1}{2}mv^2\right) = mv^2$

CALCULATE: $K_{\text{lost}} = 7000.\text{ kg}(25.0\text{ m/s})^2 = 4.375 \cdot 10^6\text{ J}$

ROUND: To three significant figures: $K_{\text{lost}} = 4.38 \cdot 10^6\text{ J}$.

DOUBLE-CHECK: Such a large amount of energy is appropriate for two colliding railroad cars.

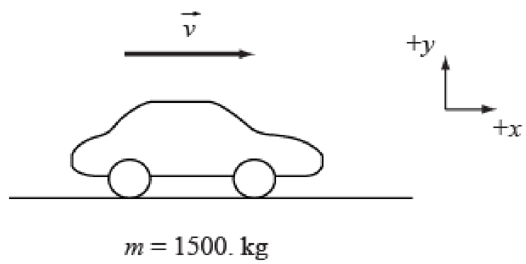
5.22. **THINK:** The mass and the speed are given:

(a) $m = 1500.\text{ kg},\ v = 15.0\text{ m/s}$

(b) $m = 1500.\text{ kg},\ v = 30.0\text{ m/s}$

With this information, I can compute the kinetic energy and compare the results.

SKETCH:



RESEARCH: $K = \frac{1}{2}mv^2$

SIMPLIFY: b) The change in kinetic energy is the difference of the kinetic energies.

CALCULATE:

(a) $K = \frac{1}{2}(1500.\text{ kg})(15.0\text{ m/s})^2 = 1.688 \cdot 10^5\text{ J}$

(b) $K = \frac{1}{2}(1500.\text{ kg})(30.0\text{ m/s})^2 = 6.750 \cdot 10^5\text{ J}$, so the change is $6.750 \cdot 10^5\text{ J} - 1.688 \cdot 10^5\text{ J} = 5.062 \cdot 10^5\text{ J}$.

ROUND: Three significant figures:

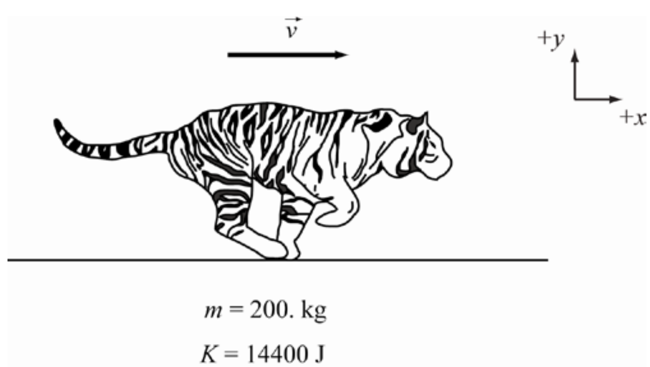
(a) $K = 1.69 \cdot 10^5\text{ J}$

(b) $\Delta K = 5.06 \cdot 10^5\text{ J}$

DOUBLE-CHECK: Such large energies are reasonable for a car. Also, when the speed doubles, the kinetic energy increases by a factor of 4, as it should since $K \propto v^2$.

5.23. **THINK:** Given the tiger's mass, $m = 200.\text{ kg}$, and energy, $K = 14400\text{ J}$, I want to determine its speed. I can rearrange the equation for kinetic energy to obtain the tiger's speed.

SKETCH:



RESEARCH: $K = \frac{1}{2}mv^2$

SIMPLIFY: $v = \sqrt{\frac{2K}{m}}$

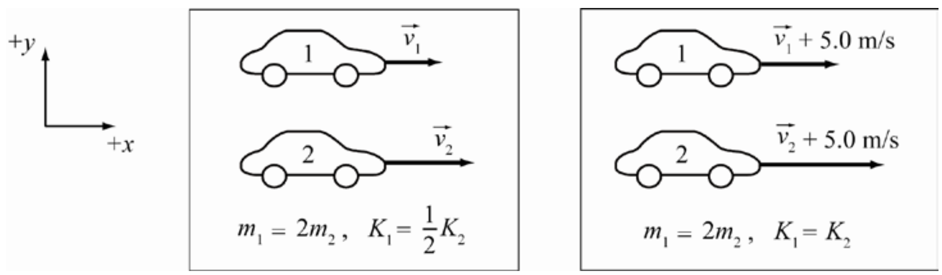
CALCULATE: $v = \sqrt{\frac{2(14400 \text{ J})}{200. \text{ kg}}} = 12.0 \text{ m/s}$

ROUND: Three significant figures: $v = 12.0$ m/s.

DOUBLE-CHECK: $(10 \text{ m/s}) \cdot \frac{10^{-3} \text{ km}}{1 \text{ m}} \cdot \frac{3600 \text{ s}}{1 \text{ h}} = 36$ km/h. This is a reasonable speed for a tiger.

5.24. **THINK:** I want to calculate the original speeds of both cars. I know only the ratios of their masses and kinetic energies. For speeds v_1 and v_2: $m_1 = 2m_2$, $K_1 = (1/2)K_2$. For speeds ($v_1 + 5.0$ m/s) and ($v_2 + 5.0$ m/s): $m_1 = 2m_2$, $K_1 = K_2$.

SKETCH:



RESEARCH: $K = \frac{1}{2}mv^2$; $m_1 = 2m_2$; $K_1 = \frac{1}{2}K_2$

SIMPLIFY: $K_1 = \frac{1}{2}m_1v_1^2 = \frac{1}{2}\left(\frac{1}{2}m_2v_2^2\right) \Rightarrow \frac{1}{2}\left(2m_2v_1^2\right) = \frac{1}{4}m_2v_2^2 \Rightarrow v_1 = \frac{1}{2}v_2$

When the speeds are increased by Δv, the kinetic energies are equal:

$$K_1 = \frac{1}{2}m_1\left(v_1 + \Delta v\right)^2 = K_2 = \frac{1}{2}m_2\left(v_2 + \Delta v\right)^2$$

$$\frac{1}{2}(2m_2)\left(\frac{1}{2}v_2 + \Delta v\right)^2 = \frac{1}{2}m_2\left(v_2 + \Delta v\right)^2 \Rightarrow (2)\left(\frac{1}{2}v_2 + \Delta v\right)^2 = \left(v_2 + \Delta v\right)^2$$

$$\Rightarrow v_2\left(1 - \frac{\sqrt{2}}{2}\right) = \left(\sqrt{2} - 1\right)\Delta v \Rightarrow v_2 = \Delta v\frac{\left(\sqrt{2}-1\right)}{\left(1 - \frac{\sqrt{2}}{2}\right)}$$

$$\Rightarrow \sqrt{2}\left(\frac{1}{2}v_2 + \Delta v\right) = \left(v_2 + \Delta v\right) \Rightarrow \frac{\sqrt{2}}{2}v_2 + \sqrt{2}\Delta v = v_2 + \Delta v$$

CALCULATE: $v_2 = (5.0 \text{ m/s})\frac{\left(\sqrt{2}-1\right)}{\left(1 - \frac{\sqrt{2}}{2}\right)} = 7.0711 \text{ m/s}, \quad v_1 = \frac{7.0711 \text{ m/s}}{2} \Rightarrow v_1 = 3.5355 \text{ m/s}$

ROUND: Two significant figures: $v_1 = 3.5$ m/s, and $v_2 = 7.1$ m/s.

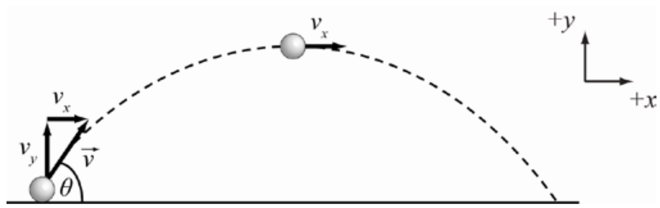
DOUBLE-CHECK: If $m_1 = 2m_2$ and $v_1 = (1/2)v_2$, then $K_1 = (1/2)K_2$:

$$K_1 = \frac{1}{2}m_1v_1^2 = \frac{1}{2}(2m_2)(\frac{1}{2}v_2)^2 = \frac{1}{4}m_2v_2^2 = \frac{1}{2}\left(\frac{1}{2}m_2v_2^2\right) = \frac{1}{2}K_2$$

The results are consistent.

5.25. **THINK:** At the apex of the projectile's trajectory, the only velocity is in the horizontal direction. The only force is gravity, which acts in the vertical direction. Hence the horizontal velocity is constant. This velocity is simply the horizontal component of the initial velocity or 27.3 m/s at an angle of 46.9°. The mass of the projectile is 20.1 kg.

SKETCH:



RESEARCH: $E = \frac{1}{2}mv_x^{\ 2}$ (at the apex, $v_y = 0$); $v_x = v\cos\theta$

SIMPLIFY: $K = \frac{1}{2}mv^2\cos^2\theta$

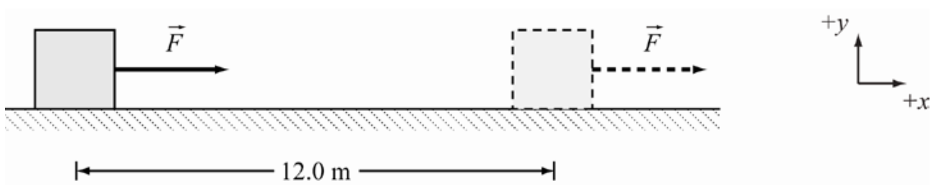
CALCULATE: $E = \frac{1}{2}(20.1 \text{ kg})(27.3 \text{ m/s})^2\cos^2(46.9°) = 3.497 \cdot 10^3$ J

ROUND: 3 significant figures: $K = 3.50 \cdot 10^3$ J.

DOUBLE-CHECK: For such a heavy mass moving at a large speed, the result is reasonable.

5.26. **THINK:** A force $F = 5.00$ N moves an object a distance $d = 12.0$ m. The object moves parallel to the force.

SKETCH:



RESEARCH: $W = Fd\cos\theta$

SIMPLIFY: $\theta = 0 \Rightarrow W = Fd$

CALCULATE: $W = (5.00 \text{ N})(12.0 \text{ m}) = 60.0$ J

ROUND: Three significant figures: $W = 60.0$ J.

DOUBLE-CHECK: This is a relatively small force over a moderate distance, so the work done is likewise moderate.

5.27. **THINK:** The initial speeds are the same for the two balls, so they have the same initial kinetic energy. Since the initial height is also the same for both balls, the gravitational force does the same work on them on their way down to the ground, adding the same amount of kinetic energy in the process. This automatically means that they hit the ground with the same value for their final kinetic energy. Since the balls have the same mass, they consequently have to have the same speed upon ground impact. This means that the difference in speeds that the problem asks for is 0. No further steps are needed in this solution.

SKETCH: Not necessary.

RESEARCH: Not necessary.

SIMPLIFY: Not necessary.

CALCULATE: Not necessary.

ROUND: Not necessary.

DOUBLE-CHECK: Even though our arguments based on kinetic energy show that the impact speed is identical for both balls, you may not find this entirely convincing. After all, most people expect the ball throw directly downward to have a higher impact speed. If you still want to perform a double-check, then you can return to the kinematic equations of chapter 3 and calculate the answer for both cases. Remember that the motion in horizontal direction is one with constant horizontal velocity component, and the motion in vertical direction is free-fall. In both cases we thus have:

$$v_x = v_{0x}$$
$$v_y = \sqrt{v_{0y}^2 + 2gh}$$

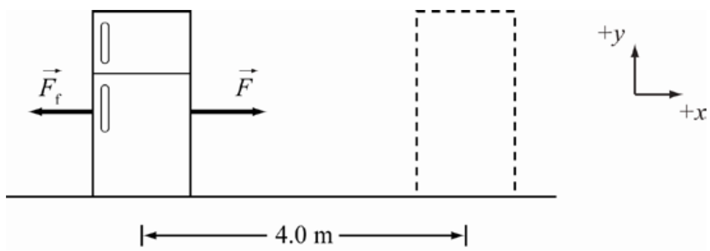
If you now square each equation and add them, you get:

$$v^2 = v_x^2 + v_y^2 = \left(v_{0x}^2\right) + \left(v_{0y}^2 + 2gh\right) = \left(v_{0x}^2 + v_{0y}^2\right) + 2gh = v_0^2 + 2gh$$

Then you see that indeed we have each time for the final speed $v = \sqrt{v_0^2 + 2gh}$, independent of the direction of the initial velocity vector. What we can learn from this double-check step is two-fold. First, our energy and work considerations yield the exact same results as our kinematic equations from Chapter 3 did. Second, and perhaps more important, the energy and work considerations required much less computational effort to arrive at the same result.

5.28. **THINK:** The object moves at constant speed so the net force is zero. The force applied is then equal to the force of friction. $F_{\text{f}} = 180$ N , $d = 4.0$ m , $m = 95$ kg.

SKETCH:



RESEARCH: $W = Fd\cos\theta$

SIMPLIFY: $\theta = 0 \Rightarrow W = -Fd = F_{\text{f}}d$

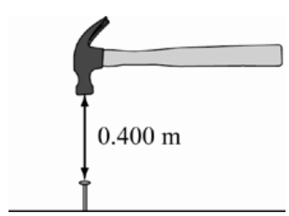
CALCULATE: $W = F_{\text{f}}d = (180 \text{ N})(4.0 \text{ m}) = 720 \text{ J}$

ROUND: 2 significant figures: $W = 720$ J

DOUBLE-CHECK: If we applied a force greater than 180 N, the object would accelerate. 720 J is reasonable for pushing a refrigerator 4.0 m.

5.29. **THINK:** The maximum amount of work that the hammerhead can do on the nail is equal to the work that gravity does on the hammerhead during the fall. $h = 0.400$ m and $m = 2.00$ kg.

SKETCH:



RESEARCH: The work done by gravity is $W_{\text{g}} = mgh$ and this is equal to the maximum work W that the hammerhead can do on the nail.

SIMPLIFY: $W = mgh$

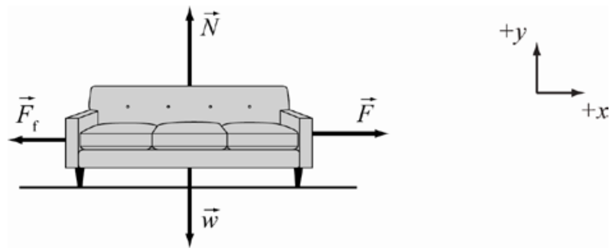
CALCULATE: $W = (2.00 \text{ kg})\left(9.81 \text{ m/s}^2\right)(0.400 \text{ m}) = 7.848 \text{ J}$

ROUND: Three significant figures: $W = 7.85$ J.

DOUBLE-CHECK: This result is reasonable. If the hammerhead had an initial velocity, more work could be done.

5.30. **THINK:** Only those forces that have a component along the couch's displacement contribute to the force. You push your couch with a force of $F = 200.0$ N a distance of $d = 4.00$ m. The frictional force opposes the motion, so the direction of F_f is opposite to F.

SKETCH:



RESEARCH: $W = Fd\cos\theta$. Friction: $\theta = 180°$, $W_f = -F_f d$; You: $\theta = 0°$, $W_{you} = Fd$; Gravity: $\theta = 90°$, $W_g = Fd_y$; Net: $W_{net} = Fd - F_f d$.

SIMPLIFY: $W_{you} = Fd$

$W_f = -F_f d$

$W_g = Fd\cos 90.°$

$W_{net} = d(F - F_f)$

CALCULATE: $W_{you} = (4.00 \text{ m})(200.0 \text{ N}) = 800.00 \text{ J}$

$W_f = -(4.00 \text{m})(150.0 \text{ N}) = -600.0 \text{ J}$

$W_g = 0$

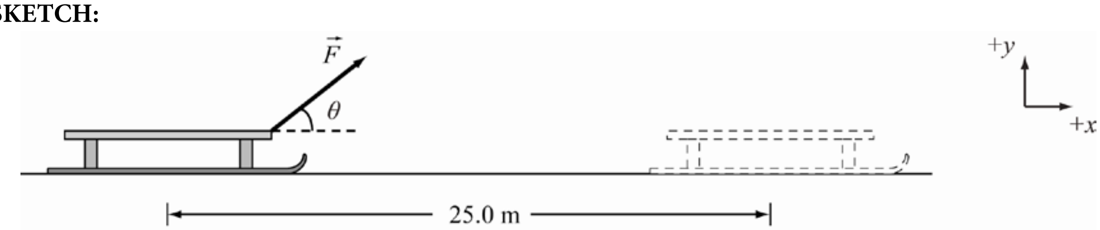
$W_{net} = (4.00 \text{ m})(200.0 \text{ N} - 150.0 \text{ N}) = 200.0 \text{ J}$

ROUND: Since the distance is given to three significant figures, $W_{you} = 8.00 \cdot 10^2$ J, $W_f = -6.00 \cdot 10^2$ J, $W_g = 0$, and $W_{net} = 2.00 \cdot 10^2$ J.

DOUBLE-CHECK: The work done by the person is greater than the work done by friction. If it was not, the couch would not move. The units of the work calculations are Joules, which are appropriate for work.

5.31. **THINK:** Only the component of the force parallel to the displacement does work.

SKETCH:



RESEARCH: $W = Fd\cos\theta$

SIMPLIFY: Not applicable.

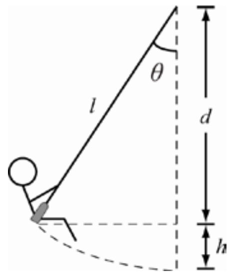
CALCULATE: $W = Fd\cos\theta = (25.0 \text{ N})(25.0 \text{ m})\cos 30.0° = 5.4127 \cdot 10^2$ J

ROUND: Three significant figures: $W = 5.41 \cdot 10^2$ J.

DOUBLE-CHECK: The magnitude of the work done by the person is greater than the magnitude of the work done by friction. The units of the work calculations are joules, which are appropriate for work.

5.32. **THINK:** Neglect friction and use conservation of energy. Take the zero of gravitational potential energy to be the bottom of the swinging arc. Then, the speed at the bottom of the swinging motion can be determined from the fact that the initial potential energy is all converted to kinetic energy.

SKETCH:



RESEARCH: $\frac{1}{2}mv^2 + mgh = C$

SIMPLIFY: Initial: $v = 0,\ h = l - d = l(1-\cos\theta),\ \frac{1}{2}mv^2 + mgh = 0 + mgl(1-\cos\theta) = E$

Final: $h = 0,\ \frac{1}{2}mv^2 + 0 = E$

$E_\text{i} = E_\text{f} \ \Rightarrow\ \frac{1}{2}mv^2 = mgl(1-\cos\theta) \ \Rightarrow\ v = \sqrt{2gl(1-\cos\theta)}$

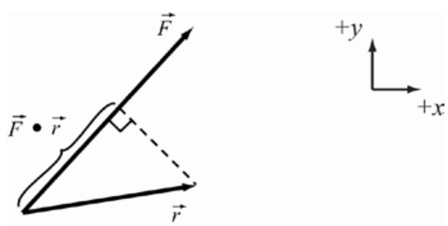
CALCULATE: $v = \sqrt{2\left(9.81 \text{ m/s}^2\right)(3.00 \text{ m})(1-\cos 33.6°)} = 3.136 \text{ m/s}$

ROUND: Three significant figures: $v = 3.14$ m/s.

DOUBLE-CHECK: The result is independent of mass here because both potential and kinetic energy depend linearly on mass.

5.33. **THINK:** The scalar product can be used to determine the work done, since the vector components of the force $\vec{F} = (4.79, -3.79, 2.09)$ N and the displacement $\vec{r} = (4.25,\ 3.69,\ -2.45)$ m, are given.

SKETCH:



RESEARCH: $W = \vec{F} \bullet \vec{r}$

SIMPLIFY: $W = F_x r_x + F_y r_y + F_z r_z$

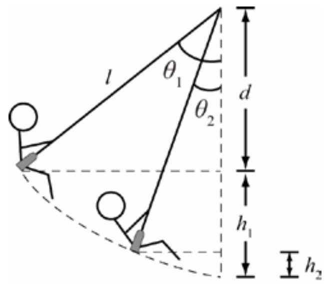
CALCULATE: $W = \left[(4.79)(4.25) + (-3.79)(3.69) + (2.09)(-2.45)\right] = 1.2519$ J

ROUND: 3 significant figures: $W = 1.25$ J .

DOUBLE-CHECK: The work done is much less than $\left|\vec{F}\right| \cdot \left|\vec{r}\right|$ since the force and the displacement are not parallel.

5.34. **THINK:** Take the zero of gravitational potential energy to be at the bottom of the swinging arc. The final speed can then be determined using conservation of energy.

SKETCH:



RESEARCH: $\frac{1}{2}mv^2 + mgh = \text{constant}$

SIMPLIFY: At θ_1: $v = 0,\ h = h_1 \ \Rightarrow\ \frac{1}{2}m(0)^2 + mgh_1 = \text{const.} = E \ \Rightarrow\ mgh_1 = E$

At θ_2: $h = h_2 \ \Rightarrow\ \frac{1}{2}mv^2 + mgh_2 = E = mgh_1 \ \Rightarrow\ \frac{1}{2}v^2 = g(h_1 - h_2) \ \Rightarrow\ v = \sqrt{2g(h_1 - h_2)}$

$\Rightarrow\ v = \sqrt{2g[l(1-\cos\theta_1) - l(1-\cos\theta_2)]} = \sqrt{2gl(-\cos\theta_1 + \cos\theta_2)}$

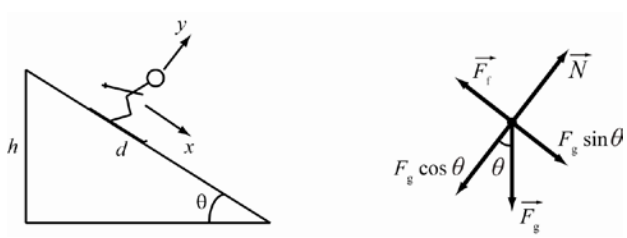
CALCULATE: $v = \sqrt{2(9.81 \text{ m/s}^2)(3.50 \text{ m})(-\cos 35.0° + \cos 15.0°)} = 3.1747 \text{ m/s}$

ROUND: 3 significant figures: $v = 3.17 \text{ m/s}$

DOUBLE-CHECK: The result is independent of mass and the final velocity seems reasonable.

5.35. **THINK:** The work done by gravity is mgh. In the absence of friction, the potential energy mgh will be converted to kinetic energy. The actual kinetic energy, when friction is included is less than this. The "missing" energy is the work done by friction. If the work done by friction is known, the frictional force and the coefficient of friction can be determined.

SKETCH:



RESEARCH: $W = W_g + W_f = \frac{1}{2}mv^2$, $W_g = mgh$, $W_f = F_f d$

SIMPLIFY: $W_f = W - W_g \ \Rightarrow\ F_f d = \frac{1}{2}mv^2 - mgh \ \Rightarrow\ -\mu N d = m\left(\frac{1}{2}v^2 - gh\right)$

But $N = F_g \cos\theta = mg\cos\theta \ \Rightarrow\ -\mu mg\cos\theta d = m\left(\frac{1}{2}v^2 - gh\right) \ \Rightarrow\ \mu = \frac{1}{gd\cos\theta}\left(gh - \frac{1}{2}v^2\right)$

CALCULATE: $g = (9.81 \text{ m/s}^2)\left(\frac{1 \text{ ft}}{0.3048 \text{ m}}\right) = 32.185 \text{ ft/s}^2$,

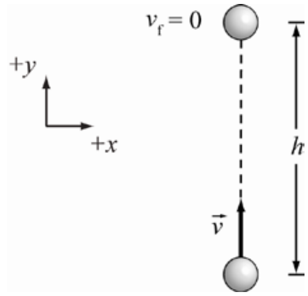
$$\mu = \frac{(32.185 \text{ ft/s}^2)(80.0\sin 30.0° \text{ ft}) - (0.5)(45.0 \text{ ft/s})^2}{(32.185 \text{ ft/s}^2)(80.0 \text{ ft})\cos 30.0°} = 0.123282$$

ROUND: Three significant figures: $\mu = 0.123$.

DOUBLE-CHECK: This is a reasonable result for the friction coefficient. If I had used SI units, the result would be the same because μ is dimensionless.

5.36. **THINK:** The molecule's initial speed can be determined from its mass and kinetic energy. At the highest point all the initial kinetic energy has been converted to potential energy.

SKETCH:



RESEARCH: $E = \frac{1}{2}mv^2 + mgh = \text{constant}$. When $h = 0$, $E_i = \frac{1}{2}mv^2$. When $v_f = 0$, $E_f = mgh$.

SIMPLIFY: E is known: $E = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2E}{m}}$. At the maximum height $E = mgh \Rightarrow h = \frac{E}{mg}$.

% of Earth's radius: $p = \frac{h}{R_E} \times 100\%$

CALCULATE: $v = \sqrt{\frac{2(6.2 \cdot 10^{-21}\text{ J})}{4.7 \cdot 10^{-26}\text{ kg}}} = 513.64\text{ m/s}$

$$h = \frac{6.2 \cdot 10^{-21}\text{ J}}{(4.7 \cdot 10^{-26}\text{ kg})\left(9.81\text{ m/s}^2\right)} = 1.3447 \cdot 10^4\text{ m}$$

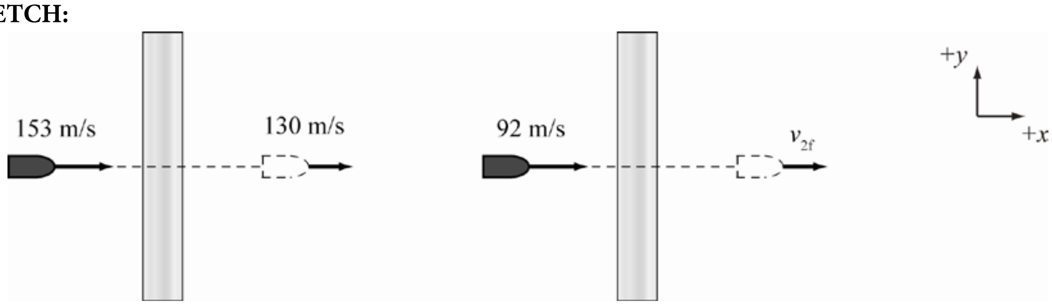
$$p = \frac{(1.3447 \cdot 10^4\text{ m})}{(6.37 \cdot 10^6\text{ m})} \times 100\% = 0.21110\%$$

ROUND: 2 significant figures: $v = 510\text{ m/s}$, $h = 1.3 \cdot 10^4$ m, and $p = 0.21\%$.

DOUBLE-CHECK: The particle is not expected to escape the Earth's gravity, or to reach relativistic speeds. This lends support to the reasonableness of the answers.

5.37. **THINK:** If the resistance of the plank is independent of the bullet's speed, then both bullets should lose the same amount of energy while passing through the plank. From the first bullet, the energy loss can be determined. This can then be used to determine the second bullet's final speed.

SKETCH:



RESEARCH: $K = \frac{1}{2}mv^2$, $\Delta K = \frac{1}{2}m({v_{1f}}^2 - {v_{1i}}^2) = \frac{1}{2}m({v_{2f}}^2 - {v_{2i}}^2)$

SIMPLIFY: $({v_{1f}}^2 - {v_{1i}}^2) = ({v_{2f}}^2 - {v_{2i}}^2)$, $v_{2f} = \sqrt{{v_{1f}}^2 - {v_{1i}}^2 + {v_{2i}}^2}$

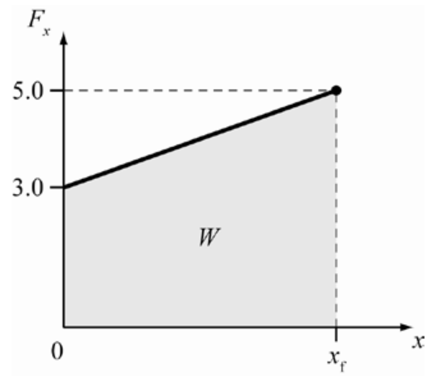
CALCULATE: $v_{2f} = \sqrt{(130.\text{ m/s})^2 - (153\text{ m/s})^2 + (92.0\text{ m/s})^2} = 44.215\text{ m/s}$

ROUND: By the rule for subtraction, the expression inside the square root has two significant figures. Rounding to two significant figures: $v_{2f} = 44$ m/s.

DOUBLE-CHECK: The final speed should be positive because the bullet is still moving to the right. The final speed should also be less than the initial speed. The answer is reasonable.

5.38. **THINK:** An expression F_x as a function of *x* is given, $F_x = (3.0 + 0.50x)$ N. The work done by the force must be determined.

SKETCH:



RESEARCH: Recall that the work done by a variable force is $W = \int_{x_i}^{x_f} F(x)dx$, or the area under the curve of *F* versus *x* plot.

SIMPLIFY: $W = \int_{x_i}^{x_f} F(x)dx = \int_0^4 (3.0 + 0.50x)dx$

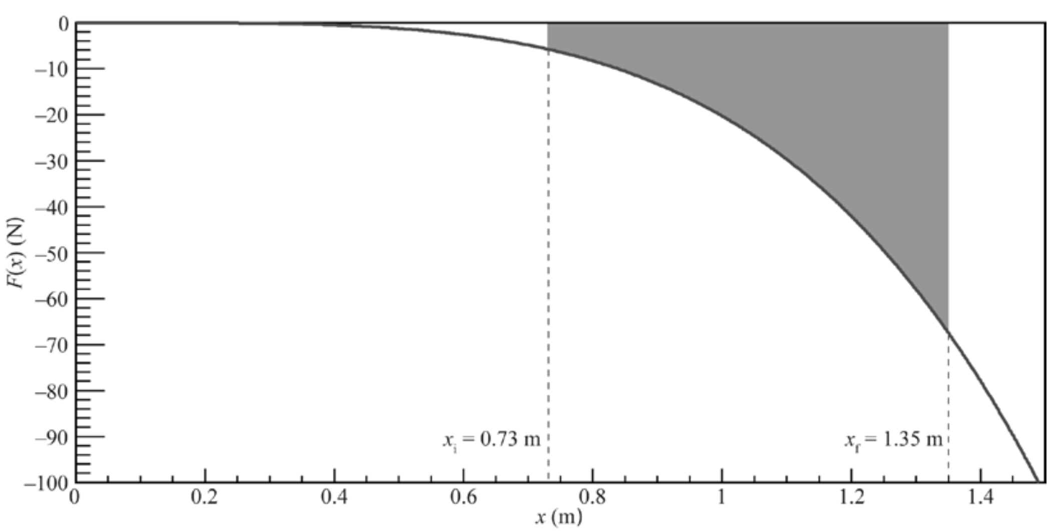
CALCULATE: $W = \int_0^{4.0} (3.0 + 0.50x)dx = \left[3x + \frac{1}{4}x^2\right]_{x=0}^{x=4.0} = 3(4) + \frac{1}{4}(4)^2 - 0 = 12 + 4 = 16 \text{ J}$

ROUND: $W = 16$ J

DOUBLE-CHECK: Since F_x is a linear function of *x*, $W = \vec{F}(\Delta x)$. $\vec{F}$ is the average force. $\vec{F}$ = 4.0 N and $\Delta x = 4.0$ m. So $W = 4.0(4.0) = 16$ J, as expected.

5.39. **THINK:** Determine the work necessary to change displacement from 0.730 m to 1.35 m for a force of $F(x) = -kx^4$ with a constant $k = 20.3 \text{ N/m}^4$.

SKETCH:



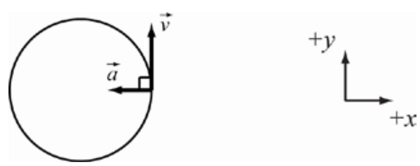
RESEARCH: The work done by the available force is $W = \int_{x_i}^{x_f} F(x)dx$.

SIMPLIFY: $W = \int_{x_i}^{x_f} -kx^4 dx = \left[-\frac{k}{5}x^5\right]_{x_i}^{x_f} = \frac{k}{5}x_i^{\ 5} - \frac{k}{5}x_f^{\ 5} = \frac{k}{5}\left(x_i^{\ 5} - x_f^{\ 5}\right)$

CALCULATE: $W = \frac{20.3 \text{ N/m}^4}{5}\left[(0.730 \text{ m})^5 - (1.35 \text{ m})^5\right] = -17.364 \text{ J}$

ROUND: Due to the difference, the answer has three significant figures. The work done *against* the spring force is the negative of the work done *by* the spring force: W = 17.4 J.
DOUBLE-CHECK: The negative work in this case is similar to the work done by a spring force.

5.40. **THINK:** Find a relationship between $\vec{a}(t)$ and $\vec{v}(t)$ for a body of mass m moving along a trajectory $\vec{r}(t)$ at constant kinetic energy.
SKETCH: Not necessary.
RESEARCH: Kinetic energy $K(t) = \text{constant}$. Therefore, the work done by a force $\vec{F} = m\vec{a}$ is zero since $W = \Delta K = 0$ at all times. This means $P = dW / dt = 0$.
SIMPLIFY: $P = \vec{F} \bullet \vec{v} = m\vec{a} \bullet \vec{v} = 0 \Rightarrow \vec{a} \bullet \vec{v} = 0$ **.** The acceleration vector is perpendicular to the velocity vector.
CALCULATE: Not necessary.
ROUND: Not necessary.
DOUBLE-CHECK: If a particle is moving in a circular motion at constant speed the kinetic energy is constant. The acceleration vector is always perpendicular to the velocity vector.



5.41. **THINK:** $\vec{F}(x) = 5x^3\hat{x}$ N/m^3 $F(x) = 5x^3\hat{x}$ N/m^3 is acting on a 1.00 kg mass. The work done from x = 2.00 m to x = 6.00 m must be determined.
SKETCH: Not applicable.
RESEARCH:

(a) Work done by a variable force is $W = \int_{x_i}^{x_f} F(x)dx$.

(b) Work-kinetic energy relation is $W = \Delta K$.
SIMPLIFY:

(a) $W = \int_{x_i}^{x_f} (5x^3)dx = \left[\frac{5}{4}x^4\right]_{x_i}^{x_f} = \frac{5}{4}\left(x_f^{\ 4} - x_i^{\ 4}\right)$

(b) $W = \Delta K = K_f - K_i = \frac{1}{2}mv_f^{\ 2} - \frac{1}{2}mv_i^{\ 2} = \frac{1}{2}m\left(v_f^{\ 2} - v_i^{\ 2}\right) \Rightarrow v_f^{\ 2} = \frac{2W}{m} + v_i^{\ 2} \Rightarrow v_f = \sqrt{\frac{2W}{m} + v_i^{\ 2}}$

CALCULATE:

(a) $W = \left(\frac{5}{4}\ \frac{\text{N}}{\text{m}^3}\right)\left[(6.00 \text{ m})^4 - (2.00 \text{ m})^4\right] = 1600 \text{ J}$

(b) $v_f = \sqrt{\frac{2(1600 \text{ J})}{1.00 \text{ kg}} + (2.00 \text{ m/s})^2} = 56.6039 \text{ m/s}$

ROUND: Quantities in the problem are given to three significant figures.
(a) $W = 1.60 \cdot 10^3$ J

(b) $v_\text{f} = 56.6$ m/s

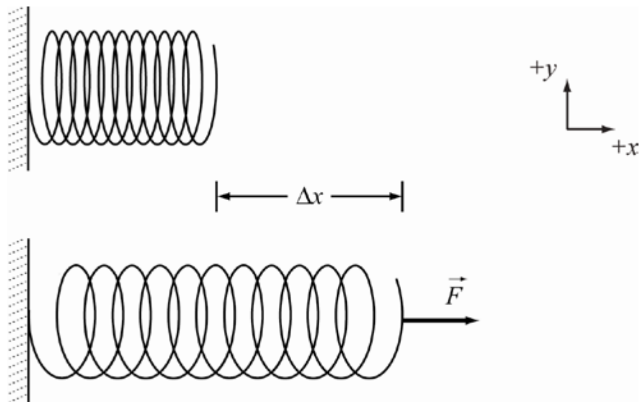
DOUBLE-CHECK: Since $v = dx/dt$, $a = dv/dt = (dv/dx)(dx/dt) = v(dv/dx)$. So, $vdv = adx$.

$$\int_{v_\text{i}}^{v_\text{f}} vdv = \int_{x_\text{i}}^{x_\text{f}} adx \Rightarrow \frac{1}{2}(v_\text{f}^2 - v_\text{i}^2) = \int_{x_\text{i}}^{x_\text{f}} \frac{F}{m}dx \Rightarrow \frac{1}{2}m(v_\text{f}^2 - v_\text{i}^2) = \int_{x_\text{i}}^{x_\text{f}} Fdx \Rightarrow \frac{1}{2}m(v_\text{f}^2 - v_\text{i}^2) = W$$

This is the same as above.

5.42. **THINK:** The spring has a constant k = 440 N/m. The displacement from its equilibrium must be determined for W = 25 J.

SKETCH:



RESEARCH: $W = \frac{1}{2}kx^2$

SIMPLIFY: $x = \sqrt{\frac{2W}{k}}$

CALCULATE: $x = \sqrt{\frac{2(25 \text{ J})}{440 \text{ N/m}}} = 0.3371 \text{ m}$

ROUND: x = 0.34 m

DOUBLE-CHECK: Because the value of k is large, a small displacement is expected for a small amount of work.

5.43. **THINK:** The spring constant must be determined given that it requires 30.0 J to stretch the spring 5.00 cm $= 5.00 \cdot 10^{-2}$ m. Recall that the work done by the spring force is always negative for displacements from equilibrium.

SKETCH: Not necessary.

RESEARCH: $W_\text{s} = -\frac{1}{2}kx^2$

SIMPLIFY: $k = -\frac{2W_\text{s}}{x^2}$

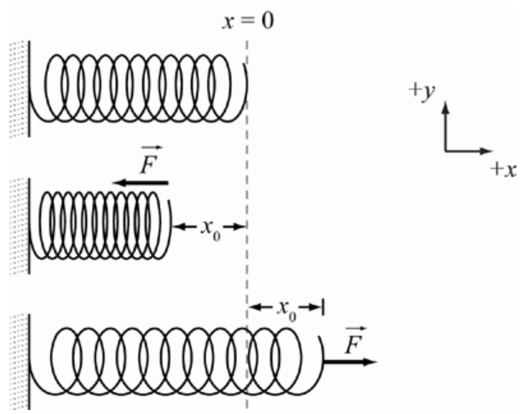
CALCULATE: $k = -\frac{2(-30.0 \text{ J})}{(5.00 \cdot 10^{-2} \text{ m})^2} = 2.40 \cdot 10^4 \text{ N/m}$

ROUND: Variables in the question are given to three significant figures, so the answer remains $k = 2.40 \cdot 10^4$ N/m.

DOUBLE-CHECK: Because the displacement is in the order 10^{-2} m, the spring constant is expected to be in the order of $1/\left(10^{-2}\right)^2 \approx 10^4$.

5.44. **THINK:** The ratio of work done on a spring when it is stretched and compressed by a distance x_0 is to be determined.

SKETCH:



RESEARCH: $W = (1/2)kx^2$. When the spring is stretched, the work done is $W_s = (1/2)k{x_0}^2$. When the spring is compressed, the work done is $W_c = (1/2)k{x_0}^2$.

SIMPLIFY: $\text{Ratio} = \dfrac{W_s}{W_c}$

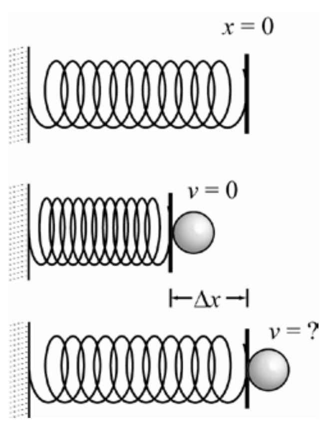
CALCULATE: $\text{Ratio} = \dfrac{W_s}{W_c} = \left(\dfrac{1}{2}k{x_0}^2\right) / \left(\dfrac{1}{2}k{x_0}^2\right) = 1$

ROUND: Not necessary.

DOUBLE-CHECK: The work done on a spring is the same, regardless if it compressed or stretched; provided the displacement is the same.

5.45. **THINK:** The spring has a constant of 238.5 N/m and $\Delta x = 0.231$ m . The steel ball has a mass of 0.0413 kg. The speed of the ball as it releases from the spring must be determined.

SKETCH:



RESEARCH: $W = \dfrac{1}{2}k{x_\text{i}}^2 - \dfrac{1}{2}k{x_\text{f}}^2$, $W = \Delta K = K_\text{f} - K_\text{i}$

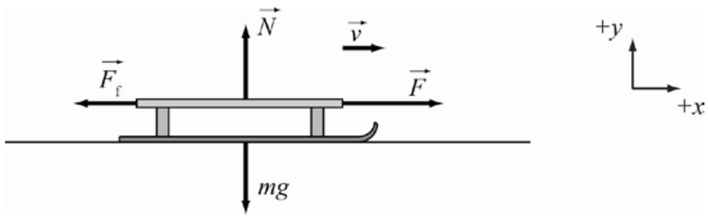
SIMPLIFY: $x_\text{f} = 0$, $K_\text{i} = 0$ and $K_\text{f} = \dfrac{1}{2}m{v_\text{f}}^2$. It follows that:

$$W = \frac{1}{2}k{x_\text{i}}^2 - 0 = K_\text{f} - 0 \Rightarrow \frac{1}{2}k{x_\text{i}}^2 = \frac{1}{2}m{v_\text{f}}^2 \Rightarrow v_\text{f} = \sqrt{\frac{k{x_\text{i}}^2}{m}} \Rightarrow v_\text{f} = |x_\text{i}|\sqrt{\frac{k}{m}}$$

CALCULATE: $x_\text{i} = -\Delta x = -0.231$ m , $v_\text{f} = 0.231 \text{ m}\sqrt{\dfrac{238.5 \text{ N/m}}{0.0413 \text{ kg}}} = 17.554$ m/s

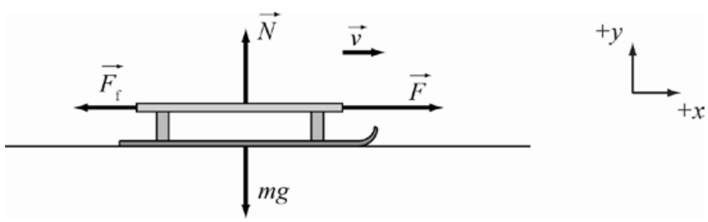
ROUND: $v_{\text{f}} = 17.6$ m/s (three significant figures)
DOUBLE-CHECK: Energy stored in the spring is transferred to kinetic energy of the ball, $(1/2)kx^2 = (1/2)mv^2$.

5.46. **THINK:** Determine the power needed to move a sled and load with a combined mass of 202.3 kg at a speed of 1.785 m/s if the coefficient of friction between the sled and snow is 0.195.
SKETCH:



RESEARCH: Use Newton's second law, $F_{\text{f}} = \mu N$ and $P = \vec{F} \bullet \vec{v}$. Since $a_x = 0$, $\sum F_x = 0 \Rightarrow F - F_{\text{f}} = 0 \Rightarrow F = F_{\text{f}} = \mu N$. Also, $\sum F_y = ma_y = 0 \Rightarrow N - mg = 0 \Rightarrow N = mg$.
SIMPLIFY: $F = \mu mg$ and $P = \mu mgv$.
CALCULATE: $P = (0.195)(202.3 \text{ kg})(9.81 \text{ m/s}^2)(1.785 \text{ m/s}) = 690.78 \text{ W}$
ROUND: $P = 691$ W
DOUBLE-CHECK: 1 horse power (hp) = 746 W. Our result is about 1 hp, which is reasonable since the sled is drawn by a horse.

5.47. **THINK:** Determine the constant speed of a sled drawn by a horse with a power of 1.060 hp. The coefficient of friction is 0.115 and the mass of the sled with a load is 204.7 kg. $P = 1.060 \text{ hp}(746 \text{ W/hp}) = 791 \text{ W}$.
SKETCH:



RESEARCH: Use Newton's second law, $F_{\text{f}} = \mu N$ and $P = \vec{F} \bullet \vec{v}$. $\sum F_x = 0$, since $a_x = 0$. So $F - F_{\text{f}} = 0 \Rightarrow F = F_{\text{f}} = \mu N$ and $\sum F_y = ma_y = 0 \Rightarrow N - mg = 0 \Rightarrow N = mg$.
SIMPLIFY: $P = Fv \Rightarrow v = \dfrac{P}{F} = \dfrac{P}{\mu N} = \dfrac{P}{\mu mg}$
CALCULATE: $v = \dfrac{791 \text{ W}}{(0.115)(204.7 \text{ kg})(9.81 \text{ m/s}^2)} = 3.42524 \text{ m/s}$
ROUND: $v = 3.43$ m/s (three significant figures)
DOUBLE-CHECK: $v = 3.43$ m/s = 12.3 km/h, which is a reasonable speed.

5.48. **THINK:** Determine the power supplied by a towline with a tension of 6.00 kN which tows a boat at a constant speed of 12 m/s.
SKETCH: Not necessary.
RESEARCH: $P = Fv$
SIMPLIFY: Not required.
CALCULATE: $P = (6.00 \cdot 10^3 \text{ N})(12 \text{ m/s}) = 72.0 \cdot 10^3 \text{ W} = 72.0 \text{ kW}$
ROUND: Not necessary.

DOUBLE-CHECK: $P = 72{,}000\text{ W}\left(\dfrac{1\text{ hp}}{746\text{ W}}\right) = 96\text{ hp}$, which is a reasonable value.

5.49. **THINK:** A car with a mass of 1214.5 kg is moving at 62.5 mph$\left(\dfrac{0.447\text{ m/s}}{1\text{ mph}}\right)$ = 27.94 m/s and comes to rest in 0.236 s. Determine the average power in watts.

SKETCH: Not required.

RESEARCH: Work, $W = \Delta K$ and the average power $\bar{P} = \dfrac{W}{\Delta t}$.

SIMPLIFY: $|P| = \left|\dfrac{W}{\Delta t}\right| = \left|\dfrac{\Delta K}{\Delta t}\right| = \left|\dfrac{\frac{1}{2}m(v_\text{f}^{\,2} - v_\text{i}^{\,2})}{\Delta t}\right|$. $v_\text{f} = 0$, so $|P| = \left|\dfrac{-\frac{1}{2}mv_\text{i}^{\,2}}{\Delta t}\right| = \dfrac{\frac{1}{2}mv_\text{i}^{\,2}}{\Delta t}$

CALCULATE: $|P| = \dfrac{\frac{1}{2}(1214.5\text{ kg})(27.94\text{ m/s})^2}{0.236\text{ s}} = 2.0087\cdot 10^6\text{ W}$

ROUND: Three significant figures: $|P| = 2.01\cdot 10^6\text{ W}$

DOUBLE-CHECK: Without the absolute values, the value would have been negative. The omitted negative sign on the power would indicate that the power is released by the car. It is expected that to make a car stop in a short time a large amount of power must be expended.

5.50. **THINK:** Determine the retarding force acting on a car travelling at 15.0 m/s with an engine expending $40.0\text{ hp}\left(\dfrac{746\text{W}}{1\text{ hp}}\right) = 29{,}840\text{W}$.

SKETCH: Not necessary.

RESEARCH: $P = Fv$

SIMPLIFY: $F = \dfrac{P}{v}$

CALCULATE: $F = \dfrac{29840\text{ W}}{15.0\text{ m/s}} = 1989.33\text{ N}$

ROUND: $F = 1990\text{ kN}$

DOUBLE-CHECK: Assume the mass of the car is 1000 kg and the coefficient of friction is about 0.1. The force of friction is about: $F = \mu N = \mu mg = 0.1(1000)(9.81) = 981\text{ N} \approx 1\text{ kN}$. So, the result is reasonable.

5.51. **THINK:** Determine the speed of a 942.4 kg car after 4.55 s, starting from rest with a power output of 140.5 hp. $140.5\text{ hp}(746\text{ W/hp}) = 104{,}813\text{ W}$.

SKETCH: Not necessary.

RESEARCH: Use the definition of average power, $\bar{P} = \dfrac{W}{\Delta t}$, and the work-kinetic energy relation, $W = \Delta K$.

SIMPLIFY: $\bar{P} = \dfrac{W}{\Delta t} = \dfrac{\Delta K}{\Delta t} = \dfrac{K_\text{f} - K_\text{i}}{\Delta t} = \dfrac{\frac{1}{2}mv_\text{f}^2 - \frac{1}{2}mv_\text{i}^2}{\Delta t}$. $v_\text{i} = 0$, so $\bar{P} = \dfrac{mv_\text{f}^2}{2\Delta t} \Rightarrow v_\text{f} = \sqrt{\dfrac{2\bar{P}\Delta t}{m}}$.

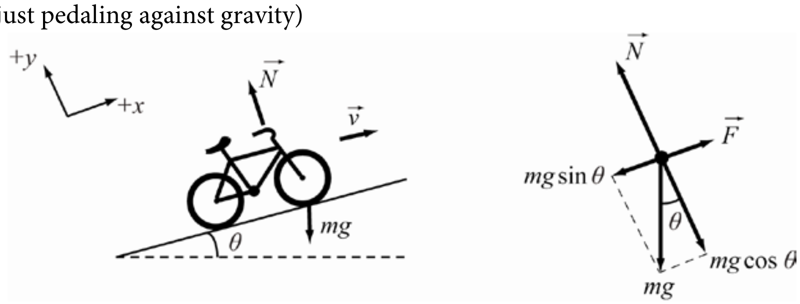
CALCULATE: $v_\text{f} = \sqrt{\dfrac{2(104{,}813\text{ W})(4.55\text{ s})}{942.4\text{ kg}}} = 31.81\text{ m/s}$

ROUND: $v_\text{f} = 31.8\text{ m/s}$

DOUBLE-CHECK: $v_\text{f} = 31.8\text{ m/s} = 114\text{ km/h}$. This represents a large acceleration, but the car is very light. This is consistent with a high performance sports car.

5.52. **THINK:** If you ride your bicycle on a horizontal surface and stop pedaling, you slow down to a stop. The force that causes this is the combination of friction in the mechanical components of the bicycle, air resistance, and rolling friction between the tires and the ground. In the first part of the problem statement we learn that the bicycle rolls down the hill at a constant speed. This automatically implies that the net force acting on it is zero. (Newton's First Law!) The force along the slope and downward is $mg\sin\theta$ (see sketch). For the net force to be zero this force has to be balanced by the force of friction and air resistance, which acts opposite to the direction of motion, in this case up the slope. So we learn from this first statement that the forces of friction and air resistance have exactly the same magnitude in this case as the component of the gravitational force along the slope. But if you go up the same slope, then gravity and the forces of air resistance and friction point in the same direction. Thus we can calculate the total work done against all forces in this case (and only in this case!) by just calculating the work done against gravity, and then simply multiplying by a factor of 2.

SKETCH: (for just pedaling against gravity)



RESEARCH: Again, let's just calculate the work done against gravity, and then in the end multiply by 2. The component of the gravitational force along the slope is $mg\sin\theta$. F is the force exerted by the bicyclist. Power $= Fv$. Using Newton's second law:

$$\sum F_x = ma_x = 0 \;\Rightarrow\; F - mg\sin\theta = 0 \;\Rightarrow\; F = mg\sin\theta$$

SIMPLIFY: Power $= 2Fv = 2(mg\sin\theta)v$

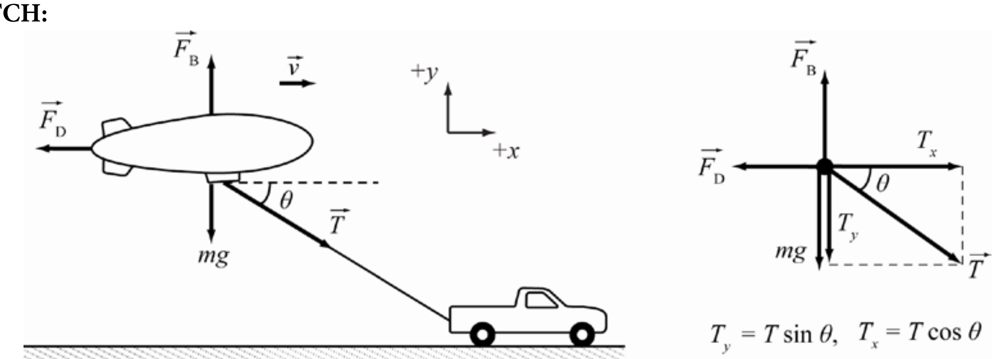
CALCULATE: $P = 2{\cdot}75\text{ kg}\left(9.81\text{ m/s}^2\right)\sin(7.0°)\left(5.0\text{ m/s}\right) = 896.654\text{ W}$

ROUND: $P = 0.90\text{ kW}$

DOUBLE-CHECK: $P = 900\text{ W}\left(\dfrac{1\text{ hp}}{746\text{ W}}\right) = 1.2\text{ hp}$. As this shows, going up a 7 degree slope at 5 m/s requires approximately 1.2 horsepower, which is what a good cyclist can expend for quite some time. (But it's hard!)

5.53. **THINK:** A blimp with a mass of 93.5 kg is pulled by a truck with a towrope at an angle 53.3° from the horizontal. The height of the blimp is $h = 19.5$ m and the truck moves for 840.5 m at a constant velocity $v = 8.90$ m/s. The drag coefficient of air is $k = 0.500$ kg/m. Determine the work done by the truck.

SKETCH:



RESEARCH: The tension in the towrope can be determined using Newton's second law.

$$\sum F_x = T_x - F_D = ma_x = 0$$
$$T_x = T\cos\theta = F_D = Kv^2$$

SIMPLIFY: The work done by the truck is: $W = \vec{F}_D \bullet \vec{d} = (T\cos\theta)(d) = Kv^2 d.$

CALCULATE: $W = (0.500 \text{ kg/m})(8.90 \text{ m/s})^2 (840.5 \text{ m}) = 33{,}288 \text{ J}$

ROUND: Rounding to three significant digits, $W = 3.33 \cdot 10^4$ J.
DOUBLE-CHECK: It is expected that the work is large for a long distance *d*.

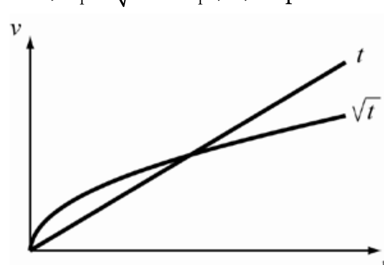
5.54. **THINK:** A car of mass *m* accelerates from rest with a constant engine power *P*, along a track of length *x*.
(a) Find an expression for the vehicle's velocity as a function of time, $v(t)$.
(b) A second car has a constant acceleration *a*. I want to know which car initially takes the lead, and whether the other car overtakes it.
(c) Find the minimum power output, P_0, required to win a race against a car that accelerates at a constant rate of $a = 12 \text{ m/s}^2$. This minimum value occurs when the distance at which my car overtakes the other car is equal to the length of the track.
SKETCH: Not necessary.
RESEARCH:
(a) Use the relation between power and work, $P = W / \Delta t$ and $W = \Delta K$.
(b) $v_2 = at + v_0$,
(c) Use the result from part *b*, $x = \dfrac{512 P_o^2}{81 m_1^2 a^3} \Rightarrow P_0 = \sqrt{\dfrac{81 x m_1^2 a^3}{512}}.$

The typical track is a quarter mile long. $x = 0.250 \text{ mi}\left(\dfrac{1609 \text{ m}}{\text{mi}}\right) = 402.25 \text{ m} = 402 \text{ m}.$

SIMPLIFY:

(a) $P = \dfrac{W}{\Delta t} = \dfrac{\Delta E}{\Delta t} = \dfrac{\frac{1}{2} m \Delta v^2}{\Delta t}$, $v = \sqrt{\dfrac{2P\Delta t}{m}}$

(b) $v_0 = 0 \Rightarrow v_2 = at$. As a comparison, $v_1 = \sqrt{2P/m_1}\sqrt{t}$, so plot $v = t$ and $v = \sqrt{t}$.



By looking at the area under the curve for the distance traveled that the first car initially takes the lead but after a time *t*, the second car overtakes the first car. Assume this occurs at distances $x_1 = x_2$.

$$x = \int_0^t v\,dt + x_0, \; x_0 = 0 \Rightarrow x = \int_0^t v\,dt$$

$$\text{So, } x_1 = \sqrt{\frac{2P_0}{m_1}}\int_0^t t^{1/2}dt = \sqrt{\frac{2P_0}{m_1}}\left(\frac{2}{3}\right)t^{3/2}, \; x_2 = a\int_0^t t\,dt = \frac{1}{2}at^2, \; x_1 = x_2 = x_0.$$

(c) $P_0 = m_1\sqrt{\dfrac{81}{512}x_0 a_0^3}$ (see below)

CALCULATE:
(a) Not necessary.

(b) $x_1 = x_2 \Rightarrow \sqrt{\frac{2P}{m_1}}\left(\frac{2}{3}\right)t^{3/2} = \frac{1}{2}at^2 \Rightarrow \sqrt{\frac{2P}{m_1}}\frac{4}{3a} = t^{1/2} \Rightarrow t = \left(\frac{2P}{m_1}\right)\left(\frac{4}{3a}\right)^2 = \frac{32P}{9m_1a^2}$

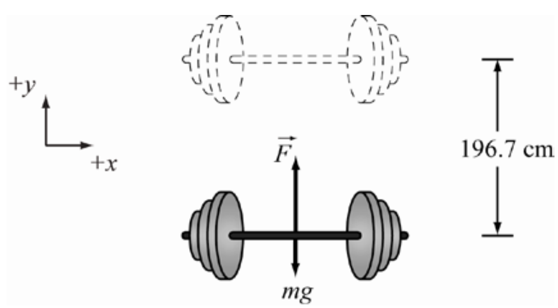
$$x_0 = \frac{1}{2}at^2 = \frac{1}{2}a\left(\frac{32P}{9m_1a^2}\right)^2 = \frac{512P^2}{81m_1^2a^3}$$

(c) $P_0 = (1000.\text{ kg})\sqrt{\frac{81}{512}(402\text{ m})(12.0\text{ m/s}^2)^3} = 331507\text{ W}\left(\frac{1\text{ hp}}{746\text{ W}}\right) = 444.38\text{ hp}$

ROUND:
(a) Not necessary.
(b) Not necessary.
(c) $P_o = 444$ hp

DOUBLE-CHECK:
(a) Not necessary.
(b) Not necessary.
(c) Even a typical sports car does not have an overall power band of 444 hp. This car must be some kind of professional drag racer.

5.55. **THINK:** Determine the work done by an athlete that lifted 472.5 kg to a height of 196.7 cm.
SKETCH:



RESEARCH: Use $W = Fd$. F is the combined force needed to the lift the weight, which is $F = mg$.

SIMPLIFY: $W = mgd$

CALCULATE: $W = (472.5\text{ kg})(9.81\text{ m/s}^2)(1.967\text{ m}) = 9117.49\text{ J}$

ROUND: Rounding to three significant figures, $W = 9.12$ kJ.
DOUBLE-CHECK: A large amount of work is expected for such a large weight.

5.56. **THINK:** Determine the amount of work done in lifting a 6 kg weight a distance of 20 cm.
SKETCH: Not necessary.
RESEARCH: Use $W = Fd$ and $F = mg$.

SIMPLIFY: $W = mgd$

CALCULATE: $W = (6.00\text{ kg})(9.81\text{ m/s}^2)(0.200\text{ m}) = 11.772\text{ J}$

ROUND: $W = 11.8$ J
DOUBLE-CHECK: For such a small distance, a small amount of work is expected.

5.57. **THINK:** Determine the power in kilowatts and horsepower developed by a tractor pulling with a force of 14.0 kN and a speed of 3.00 m/s.
SKETCH: Not necessary.
RESEARCH: $P = Fv$
SIMPLIFY: Not necessary.

CALCULATE: $P=(14.0\text{ kN})(3.00\text{ m/s})=42.0\text{ kW}=42000\text{ W}\left(\dfrac{1\text{ hp}}{746\text{ W}}\right)=56.3\text{ hp}$

ROUND: Variables in the question are given to three significant figures, so the answers remain $P=42.0\text{ kW}=56.3\text{ hp}$.

DOUBLE-CHECK: This is a reasonable value for a tractor.

5.58. **THINK:** It is given that a mass of m = 7.3 kg with initial speed $v_{\text{i}}=0$ is accelerated to a final speed of $v_{\text{f}}=14\text{ m/s}$ in 2.0 s. Determine the average power of the motion.

SKETCH: Not necessary.

RESEARCH: $W=\Delta K=\frac{1}{2}m(v_{\text{f}}^2-v_{\text{i}}^2),\ \ P=\dfrac{W}{\Delta t}$

SIMPLIFY: $P=\dfrac{W}{\Delta t}=\dfrac{\Delta K}{\Delta t}=\dfrac{\frac{1}{2}m(v_{\text{f}}^2-v_{\text{i}}^2)}{\Delta t}.\ \ v_{\text{i}}=0,\text{ so }\ P=\dfrac{\frac{1}{2}mv_{\text{f}}^2}{\Delta t}.$

CALCULATE: $P=\dfrac{\frac{1}{2}(7.3\text{ kg})(14\text{ m/s})^2}{2.0\text{ s}}=357.7\text{ W}$

ROUND: $P=360\text{ W}$

DOUBLE-CHECK: 360 W is equivalent to about half a horsepower, so this is a reasonable result.

5.59. **THINK:** A car with mass $m=1200.$ kg can accelerate from rest to a speed of 25.0 m/s in 8.00 s. Determine the average power produced by the motor for this acceleration.

SKETCH: Not necessary.

RESEARCH: $W=\Delta K,\ \ P=\dfrac{W}{\Delta t}$

SIMPLIFY: $P=\dfrac{W}{\Delta t}=\dfrac{\Delta K}{\Delta t}=\dfrac{\frac{1}{2}m(v_{\text{f}}^2-v_{\text{i}}^2)}{\Delta t}.\ \ v_{\text{i}}=0,\text{ so }\ P=\dfrac{\frac{1}{2}mv_{\text{f}}^2}{\Delta t}.$

CALCULATE: $P=\dfrac{\frac{1}{2}(1200.\text{ kg})(25.0\text{ m/s})^2}{8.00\text{ s}}=46875\text{ W}\left(\dfrac{1\text{ hp}}{746\text{ W}}\right)=62.835\text{ hp}$

ROUND: Three significant figures: $P=62.8\text{ hp}$

DOUBLE-CHECK: An average car motor has a power between 100 and 500 hp. This result is reasonable for a small car.

5.60. **THINK:** Determine the work that must be done to stop a car of mass m = 1250 kg traveling at a speed v_0 = 105 km/h (29.2 m/s).

SKETCH: Not necessary.

RESEARCH: $W=\Delta K,\ \ v_{\text{i}}=29.2\text{ m/s},\ v_{\text{f}}=0$

SIMPLIFY: $W=\frac{1}{2}m(v_{\text{f}}^2-v_{\text{i}}^2)=\frac{1}{2}m(0-v_{\text{i}}^2)=-\frac{1}{2}mv_{\text{i}}^2$

CALCULATE: $W=-\frac{1}{2}(1250\text{ kg})(29.2\text{ m/s})^2=-532900\text{ J}$

ROUND: $W=-533\text{ kJ}$

DOUBLE-CHECK: A negative amount of work means that the force to stop the car must be in the opposite direction to the velocity. This value is reasonable to stop a car moving at this speed.

5.61. **THINK:** A bowstring exerts an average force F = 110. N on an arrow with a mass m = 0.0880 kg over a distance d = 0.780 m. Determine the speed of the arrow as it leaves the bow.

SKETCH: Not necessary.

RESEARCH: $W = Fd = \Delta K = \frac{1}{2}m(v_f^{\,2} - v_i^{\,2})$

SIMPLIFY: $Fd = \frac{1}{2}m(v_f^{\,2} - 0) \;\Rightarrow\; v_f = \sqrt{\frac{2Fd}{m}}$

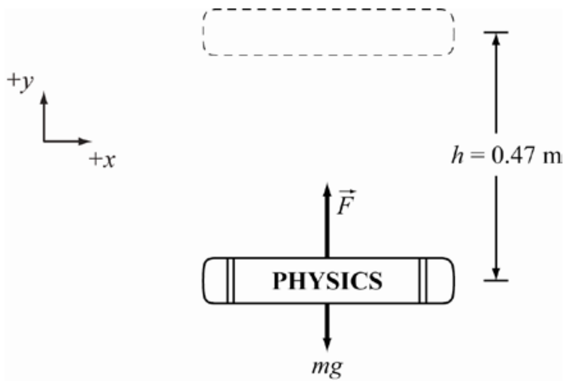
CALCULATE: $v_f = \sqrt{\frac{2(110.\text{ N})(0.780\text{ m})}{0.0880\text{ kg}}} = 44.159\text{ m/s}$

ROUND: $v_f = 44.2\text{ m/s}$

DOUBLE-CHECK: As a comparison, the speed of a rifle bullet is about 1000 m/s and the speed of sound is 343 m/s. This result is reasonable.

5.62. **THINK:** A textbook with a mass m = 3.4 kg is lifted to a height h = 0.47 m at a constant speed of v = 0.27 m/s.

SKETCH:



RESEARCH:

(a) Work is given by $W = Fh\cos\theta$, where θ is the angle between F and h. The force of gravity is given by $F_g = mg$.

(b) Power is given by $P = Fv$.

SIMPLIFY:

(a) $W_g = F_g h\cos\theta,\; \theta = 180° \;\Rightarrow\; W_g = -mgh$

(b) $P = F_g v$. From (a), $F_g = mg \;\Rightarrow\; P = mgv$.

CALCULATE:

(a) $W_g = -(3.4\text{ kg})(9.81\text{ m/s}^2)(0.47\text{ m}) = -15.676\text{ J}$

(b) $P = (3.4\text{ kg})(9.81\text{ m/s}^2)(0.27\text{ m/s}) = 9.006\text{ W}$

ROUND:

(a) $W_g = -16\text{ J}$

(b) $P = 9.0\text{ W}$

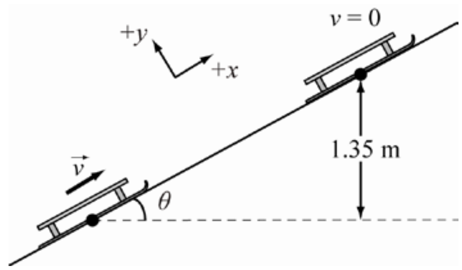
DOUBLE-CHECK:

(a) This is a reasonable result for a relatively light textbook moved a short distance.

(b) This result is much less than the output power of human muscle, which is of the order of 10^2 W.

5.63. **THINK:** Determine the initial speed of a sled which is shoved up an incline that makes an angle of 28.0° with the horizontal and comes to a stop at a vertical height of h = 1.35 m.

SKETCH:



RESEARCH: The work done by gravity must be equal to the change in kinetic energy: $W = \Delta K$.

SIMPLIFY: $W_{\text{g}} = -mgh = K_{\text{f}} - K_{\text{i}} = \frac{1}{2}m(v_{\text{f}}^2 - v_{\text{i}}^2)$. $v_{\text{f}} = 0$, so $-mgh = \frac{1}{2}m(0 - v_{\text{i}}^2) \Rightarrow v_{\text{i}} = \sqrt{2gh}$

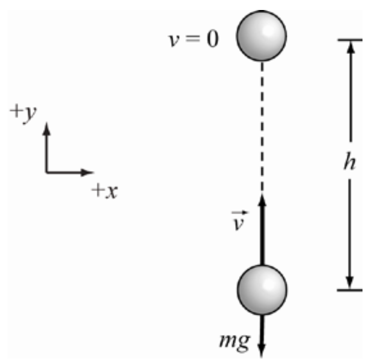
CALCULATE: $v_{\text{i}} = \sqrt{2\left(9.81 \text{ m/s}^2\right)\left(1.35 \text{ m}\right)} = 5.1466$ m/s

ROUND: The angle was given to three significant figures; so you may think that our result needs to be rounded to three digits. This is not correct, because the angle did not even enter into our calculations. The height was given to three digits, and so we round v_{i} = 5.15 m/s.

DOUBLE-CHECK: $v_{\text{i}} = 5.15 \text{ m/s} = 18.5 \text{ km/h}$ is a reasonable value.

5.64. **THINK:** Determine the maximum height h that a rock of mass m = 0.325 kg reaches when thrown straight up and a net amount of work, W_{net} = 115 J is done on the rock.

SKETCH:



RESEARCH: The amount of work done by the person's arm must equal the work done by gravity: $W_{\text{net}} = -W_{\text{g}}$.

SIMPLIFY: $W_{\text{g}} = -mgh,\;\; W_{\text{net}} = mgh \;\Rightarrow\; h = \frac{W_{\text{net}}}{mg}$

CALCULATE: $h = \frac{115 \text{ J}}{0.325 \text{ kg}\left(9.81 \text{ m/s}^2\right)} = 36.0699$ m

ROUND: $h = 36.1$ m

DOUBLE-CHECK: This is just under 120 ft—fairly high, but it is not unreasonable that an object with a small mass can be thrown this high.

5.65. **THINK:** Since we know the displacement, and we know that the car travels at constant velocity, the force must act in the same direction as the displacement. Then the work is simply the product of force times displacement.

SKETCH: Not necessary

RESEARCH: $W = Fx$

SIMPLIFY: $F = \dfrac{W}{x}$

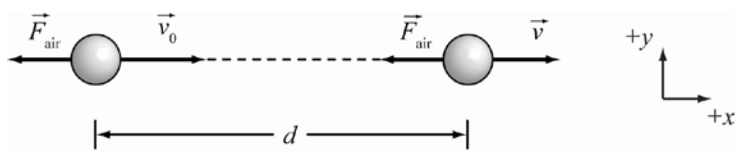
CALCULATE: $F = \dfrac{7.00 \cdot 10^4 \text{ J}}{2.8 \cdot 10^3 \text{ m}} = 25.0 \text{ N}$

ROUND: The variables in the question were given to three significant figures, so the answer remains $F = 25.0$ N.

DOUBLE-CHECK: It should take about 200 seconds to travel this distance. The average power is the net work done divided by the time interval, which under this assumption would compute to 350 W, which is realistic for a small car at relatively slow cruising speeds. The mass of the car should be around 1000 kg. A force of 25.0 N could accelerate it at 0.025 m/s^2, if it was not for friction and air resistance. These numbers are all of the right magnitude for a small passenger car, which gives us confidence in our solution.

5.66. **THINK:** A softball of mass m = 0.250 kg is pitched at an initial speed of v_0 = 26.4 m/s. Air resistance causes the ball to slow down by 10.0% over a distance $d = 15.0$ m. I want to determine the average force of air resistance, F_{air}, which causes the ball to slow down.

SKETCH:



RESEARCH: $W = Fd$ and $W = \Delta K$.

SIMPLIFY: Work done by air resistance: $W = -F_{\text{air}}d = \Delta K = \dfrac{1}{2}m(v_{\text{f}}^2 - v_{\text{i}}^2) \Rightarrow F_{\text{air}} = \dfrac{m}{2d}(v_{\text{i}}^2 - v_{\text{f}}^2)$.

CALCULATE: $v_{\text{f}} = 0.900(26.4 \text{ m/s}) = 23.76 \text{ m/s}$, $F_{\text{air}} = \dfrac{0.250 \text{ kg}}{2(15.0 \text{ m})}\left[(26.4 \text{ m/s})^2 - (23.76 \text{ m/s})^2\right] = 1.104 \text{ N}$

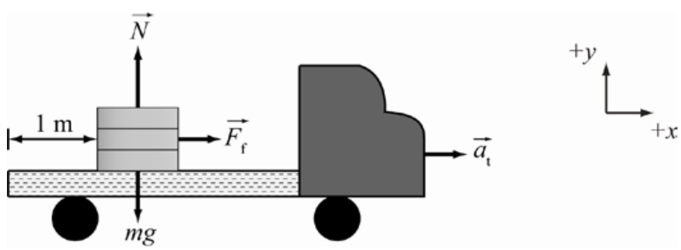
ROUND: Rounding to three significant figures, $F_{\text{air}} = 1.10$ N .

DOUBLE-CHECK: As a comparison, the force of gravity on the softball is

$$F_{\text{g}} = 0.25 \text{ kg}\left(9.81 \text{m/s}^2\right) = 2.45 \text{ N}$$

5.67. **THINK:** The stack of cement sacks has a combined mass m = 1143.5 kg. The coefficients of static and kinetic friction between the sacks and the bed of the truck are 0.372 and 0.257, respectively. The truck accelerates from rest to $56.6 \text{ mph}\left(\dfrac{0.447 \text{ m/s}}{\text{mph}}\right) = 25.3 \text{ m/s}$ in $\Delta t = 22.9$ s. Determine if the sacks slide and the work done on the stack by the force of friction.

SKETCH:



RESEARCH: The acceleration of the truck a_t and the acceleration of the stack a_{c} must be determined: $a_{\text{t}} = v / \Delta t$. The maximum acceleration that will allow the cement sacks to stay on the truck is calculated by: $F_{\text{f,max}} = ma_{\text{c,max}} = \mu_{\text{s}}N$.

SIMPLIFY: $F_{\text{f,max}} = ma_{\text{c,max}} = \mu_{\text{s}}mg \Rightarrow a_{\text{c,max}} = \mu_{\text{s}}g$

CALCULATE: $a_t = \dfrac{25.3 \text{ m/s}}{22.9 \text{ s}} = 1.1048 \text{ m/s}^2$, $a_{c,max} = (0.372)(9.81 \text{ m/s}^2) = 3.649 \text{ m/s}^2$

$a_{c,max}$ is larger than a_t. This means that the stack does not slide on the truck bed and $F_f < \mu_s N$. The acceleration of the stack must be the same as the acceleration of the truck $a_c = a_t = 1.10 \text{ m/s}^2$. The work done on the stack by the force of friction is calculated using $W = \Delta K$:

$$W = K_f - K_i = \frac{1}{2}m(v_f^2 - v_i^2). \text{ Since } v_i = 0,\ W = \frac{1}{2}mv_f^2 = \frac{1}{2}(1143.5 \text{ kg})(25.3 \text{ m/s})^2 = 365971 \text{ J}.$$

ROUND: $W = 366$ kJ

DOUBLE-CHECK: The work done by the force of friction can also be calculated by $W = F_f d$; where $F_f = ma_c$ and $d = \frac{1}{2}a_c t^2$:

$$W = ma_c\left(\frac{1}{2}a_c t^2\right) = \frac{1}{2}ma_c^2 t^2 = \frac{1}{2}m(a_c t)^2. \text{ Using } v_f = a_c t,\ W = \frac{1}{2}m(v_f)^2 \text{ as before.}$$

5.68. **THINK:** Determine the power needed to keep a car of mass m = 1000. kg moving at a constant velocity v = 22.2 m/s. When the car is in neutral, it loses power such that it decelerates from 25.0 m/s to 19.4 m/s in t = 6.00 s. The average velocity over the period of deceleration is 22.2 m/s. Therefore, the power required to maintain this velocity is equal in magnitude to the power lost during the deceleration.

SKETCH: Not necessary.

RESEARCH: The power is given by the change in energy over time, $P = (K_f - K_i)/t$. The energy is kinetic energy, $K = (1/2)mv^2$.

SIMPLIFY: $P = \dfrac{K_f - K_i}{t} = \dfrac{\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2}{t} = \dfrac{m}{2t}\left(v_f^2 - v_i^2\right)$

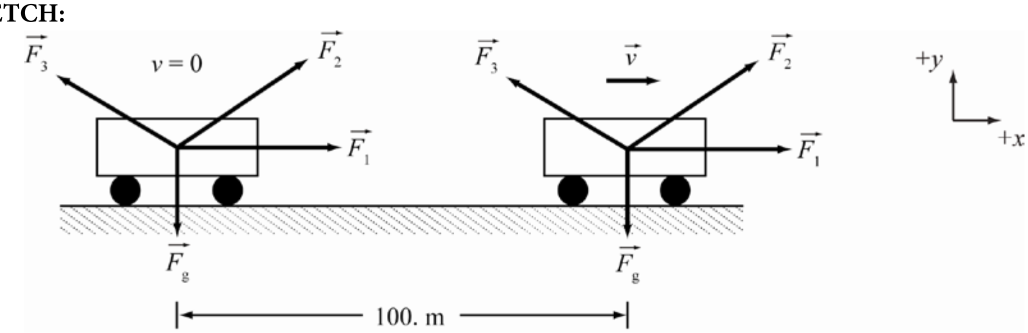
CALCULATE: $P = \dfrac{1000. \text{ kg}}{2(6.00 \text{ s})}\left[(25.0 \text{ m/s})^2 - (19.4 \text{ m/s})^2\right] = 20720 \text{ W}$

ROUND: Rounding to three significant figures, P = 20.7 kW.

DOUBLE-CHECK: A 300 hp engine is equivalent to $300 \text{ hp}(746 \text{ W/hp}) = 223800 \text{ W} = 223 \text{ kW}$. Since the solution is smaller than 300 hp, the calculation is reasonable.

5.69. **THINK:** There are four forces acting on a 125 kg cart at rest. These forces are $\vec{F}_1 = 300.$ N at 0°, $\vec{F}_2 = 300.$ N at 40.0°, $\vec{F}_3 = 200.$ N at 150.° and $\vec{F}_g = mg$ downward. The cart does not move up or down, so the force of gravity, and the vertical components of the other forces, need not be considered. The horizontal components of the forces can be used to determine the net work done on the cart, and the Work-Kinetic Energy Theorem can be used to determine the velocity of the cart after 100. m.

SKETCH:



RESEARCH: $F_{1,x} = F_1 \cos\theta_1$, $F_{2,x} = F_2 \cos\theta_2$, $F_{3,x} = F_3 \cos\theta_3$, $W = \sum_{i=1}^{n} F_{i,x} \bullet \Delta x$, $W = \Delta K = K_f - K_i$,
$\sum_{i=1}^{3} F_{i,x} = F_1 \cos\theta_1 + F_2 \cos\theta_2 + F_3 \cos\theta_3 = F_x$, $\Delta K = K_f - K_i = \frac{1}{2}m\left(v_f^2 - v_i^2\right)$

SIMPLIFY: $W = \sum_{i=1}^{3} F_{i,x} \bullet \Delta x = K_f - K_i$, $v_f = \sqrt{\frac{2F_x\Delta x}{m}} = \sqrt{\frac{2(F_1\cos\theta_1 + F_2\cos\theta_2 + F_3\cos\theta_3)\Delta x}{m}}$.

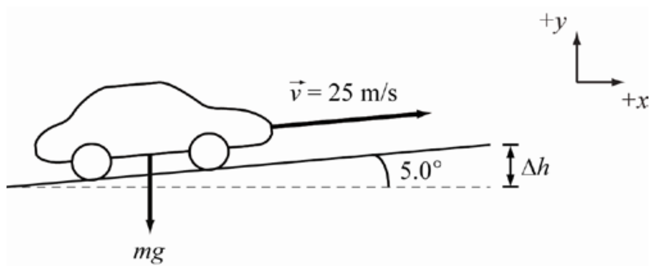
CALCULATE: $v_f = \sqrt{\frac{2\left[(300.\text{ N})\cos 0° + (300.\text{ N})\cos 40.0° + (200.\text{ N})\cos 150.°\right](100.\text{ m})}{125\text{ kg}}}$
$= 23.89$ m/s in the direction of F_1.

ROUND: Rounding to three significant figures, $v_f = 23.9$ m/s in the direction of F_1.

DOUBLE-CHECK: If only F_1 was acting on the cart, the velocity would be 21.9 m/s. This is close to the answer above, so the answer is reasonable.

5.70. **THINK:** Determine the power required to propel a 1000.0 kg car up a slope of $5.0°$.

SKETCH:



RESEARCH: Since the speed is constant, the power is given by the change in potential energy over time, $P = \frac{W}{\Delta t} = \frac{\Delta E}{\Delta t} = \frac{mg\Delta h}{\Delta t}$.

SIMPLIFY: $P = \frac{\Delta E}{\Delta t} = \frac{mg\Delta h}{\Delta t} = mgv\sin\theta$

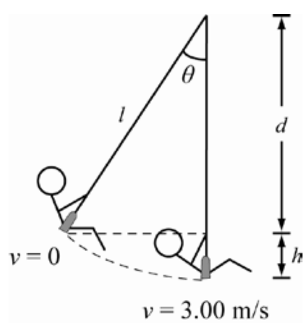
CALCULATE: $P = 1000.0\text{ kg}\left(9.81\text{ m/s}^2\right)\left(25.0\text{ m/s}\right)\sin\left(5.0°\right) = 21{,}374.95$ W

ROUND: Rounding to two significant figures, $P = 21$ kW.

DOUBLE-CHECK: This is a reasonable amount of power for a car.

5.71. **THINK:** Determine the angle θ that the granddaughter is released from to reach a speed of 3.00 m/s at the bottom of the swinging motion. The granddaughter has a mass of m = 21.0 kg and the length of the swing is l = 2.50 m.

SKETCH:



RESEARCH: The energy is given by the change in the height from the top of the swing, mgh. It can be seen from the geometry that $h = l - d = l - l\cos\theta = l(1-\cos\theta)$. At the bottom of the swinging motion, there is only kinetic energy, $K = (1/2)mv^2$.

SIMPLIFY: Equate the energy at the release point to the energy at the bottom of the swinging motion and solve for θ:

$$mgh = \frac{1}{2}mv^2 \ \Rightarrow\ gl(1-\cos\theta) = \frac{1}{2}v^2 \ \Rightarrow\ \theta = \cos^{-1}\left(1-\frac{v^2}{2gl}\right)$$

CALCULATE: $\theta = \cos^{-1}\left(1-\dfrac{(3.00\text{ m/s})^2}{2(9.81\text{ m/s}^2)(2.50\text{ m})}\right) = 35.263°$

ROUND: Rounding to three significant figures, $\theta = 35.3°$.

DOUBLE-CHECK: This is a reasonable angle to attain such a speed on a swing.

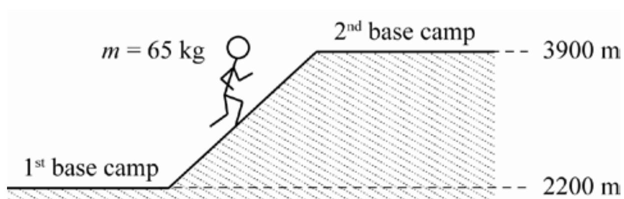
5.72 **THINK:**

(a) Determine the work done against gravity by a 65 kg hiker in climbing from height $h_1 = 2200$ m to a height $h_2 = 3900$ m.

(b) The trip takes $t = 5.0\text{ h}(3600\text{ s/h}) = 18{,}000$ s. Determine the average power output.

(c) Determine the energy input rate assuming the body is 15% efficient.

SKETCH:



RESEARCH:

(a) The work done against gravity is $W = mg(h_2 - h_1)$.

(b) $P = \dfrac{E_f - E_i}{t} = \dfrac{\Delta E}{t}$

(c) The energy output is given by $E_{in} \times \%\text{ conversion} = E_{out}$.

SIMPLIFY:

(a) Not necessary.

(b) Not necessary.

(c) $E_{in} = \dfrac{E_{out}}{\%\text{ conversion}}$

CALCULATE:

(a) $W = 65\text{ kg}(9.81\text{ m/s}^2)(3900\text{ m} - 2200\text{ m}) = 1{,}084{,}005$ J

(b) $P = \dfrac{1{,}084{,}005\text{ J}}{18{,}000\text{ s}} = 60.22$ W

(c) $E_{in} = \dfrac{1{,}084{,}005\text{ J}}{0.15} = 7{,}226{,}700$ J

ROUND:

(a) Rounding to two significant figures, $W = 1.1 \cdot 10^6$ J.

(b) Rounding to two significant figures, $P = 60.$ W.

(c) Rounding to two significant figures, $E_{in} = 7.2 \cdot 10^6$ J.

DOUBLE-CHECK:

(a) This is a reasonable value for such a long distance traveled.

(b) This value is reasonable for such a long period of time.

(c) The daily caloric requirements for a 65 kg man is 2432 calories, which is about $1.0\cdot 10^7$ J. This is on the same order of magnitude as the result.

5.73. For work done by a force that varies with location, $W=\int_{x_1}^{x_2} F_x dx$. In order to oppose the force, equal work must be done opposite the direction of F_x.

$$W=\int_{x_1}^{x_2} F_x dx=\int_{x_1}^{x_2}(-cx^3)dx=-\frac{c}{4}\left[x^4\right]_{x_1}^{x_2}=\frac{c}{4}\left[x_1^4-x_2^4\right]$$

This evaluates to:

$$W=\frac{19.1\text{ N/m}^3}{4}\left[(0.810\text{ m})^4-(1.39\text{ m})^4\right]=-15.77\text{ J}$$

Therefore the work required to oppose F_x is the opposite: $W=15.77$ J or 15.8 J when rounded to three significant figures.

5.74. Apply Hooke's law to find the spring constant k:

$$F=-kx_0\rightarrow|k|=\frac{F}{x_0}$$

The work done to compress the spring further is equal to the change in spring energy.

$$W=\Delta E=\frac{1}{2}k\left[x_f^2-x_0^2\right]=\frac{1}{2}\frac{F}{x_0}\left[x_f^2-x_0^2\right]$$

$$W=\frac{1}{2}\left(\frac{63.5\text{ N}}{0.0435\text{ m}}\right)\left[(0.0815\text{ m})^2-(0.0435\text{ m})^2\right]=3.47\text{ J}$$

5.75. The amount of power required to overcome the force of air resistance is given by $P=F\cdot v$. And the force of air resistance is given by the Ch. 4 formula

$$F_d=\left(\frac{1}{2}c_dA\rho\right)v^2$$

$$\Rightarrow P=\left(\frac{1}{2}c_dA\rho v^2\right)\cdot v=\frac{1}{2}c_dA\rho v^3$$

This evaluates as:

$$P=\frac{1}{2}(0.333)(3.25\text{ m}^2)(1.15\text{ kg/m}^3)(26.8\text{ m/s})^3=11{,}978.4\text{ W}=(11{,}978.4\text{ W})\left(\frac{1\text{ hp}}{745.7\text{ W}}\right)=16.06\text{ hp}$$

To three significant figures, the power is 16.1 hp.

Multi-Version Exercises

5.76. **THINK:** This problem involves a variable force. Since we want to find the change in kinetic energy, we can find the work done as the object moves and then use the work-energy theorem to find the total work done.

SKETCH:

RESEARCH: Since the object started at rest, it had zero kinetic energy to start. Use the work-energy theorem $W=\Delta K$ to find the change in kinetic energy. Since the object started with zero kinetic energy, the total kinetic energy will equal the change in kinetic energy: $\Delta K = K$. The work done by a variable force in the x-direction is given by $W=\int_{x_0}^{x} F_x\left(x'\right)dx'$ and the equation for our force is $F_x\left(x'\right)=A\left(x'\right)^6$. Since the object starts at rest at 1.093 m and moves to 4.429 m, we start at x_0 = 1.093 m and end at x = 4.429 m.
SIMPLIFY: First, find the expression for work by substituting the correct expression for the force: $W=\int_{x_0}^{x} A\left(x'\right)^6 dx'$. Taking the definite integral gives $W=\frac{A}{7}\left(x'\right)^7\Big|_{x_0}^{x}=\frac{A}{7}\left(x^7-x_0^7\right)$. Combining this with the work-energy theorem gives $\frac{A}{7}\left(x^7-x_0^7\right)=W=K$.
CALCULATE: The problem states that A = 11.45 N/m^6, that the object starts at x_0 = 1.093 m and that it ends at x = 4.429 m. Plugging these into the equation and calculating gives:

$$\begin{aligned}K&=\frac{A}{7}\left(x^7-x_0^7\right)\\&=\frac{11.45\text{ N/m}^6}{7}\left(\left(4.429\text{ m}\right)^7-\left(1.093\text{ m}\right)^7\right)\\&=5.467930659\cdot 10^4\text{ J}\end{aligned}$$

ROUND: The measured values in this problem are the constant A in the equation for the force and the two distances on the x-axis. All three of these are given to four significant figures, so the final answer should have four significant figures: 5.468·10^4 J or 54.68 kJ.
DOUBLE-CHECK: Working backwards, if a variable force in the +x-direction changes the kinetic energy from zero to 5.468·10^4 J, then the object will have moved

$$\begin{aligned}x&=\sqrt[7]{\frac{7\left(5.468\cdot 10^4\text{ J}\right)}{11.45\text{ N/m}^6}+1.093^7}\\&=4.429008023\text{ m.}\end{aligned}$$

This is, within rounding error, the 4.429 m given in the problem, so it seems that the calculations were correct.

5.77. $K=\frac{A}{7}\left(x^7-x_0^7\right)$

$$\frac{7K}{A}=x^7-x_0^7$$

$$x=\sqrt[7]{\frac{7K}{A}+x_0^7}=\sqrt[7]{\frac{7\left(5.662\cdot 10^3\text{ J}\right)}{13.75\text{ N/m}^6}+\left(1.105\text{ m}\right)^7}=3.121\text{ m}$$

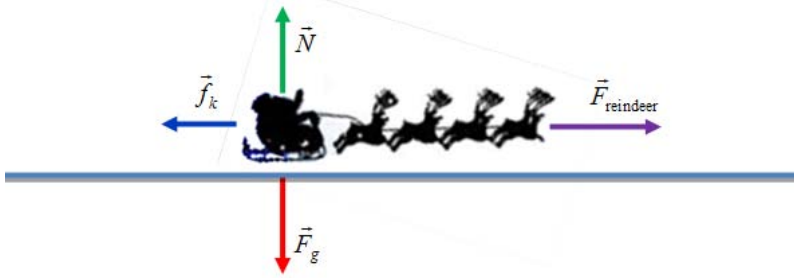
5.78. $K=\frac{A}{7}\left(x^7-x_0^7\right)$

$$\frac{7K}{A}=x^7-x_0^7$$

$$x_0=\sqrt[7]{x^7-\frac{7K}{A}}=\sqrt[7]{\left(3.313\right)^7-\frac{7\left(1.00396\cdot 10^4\text{ J}\right)}{16.05\text{ N/m}^6}}=1.114\text{ m}$$

5.79. **THINK:** In this problem, the reindeer must pull the sleigh to overcome the friction between the runners of the sleigh and the snow. Express the friction force in terms of the speed and weight of the sleigh, and the coefficient of friction between the sleigh and the ground. It is then possible to find the power from the force and velocity.

SKETCH: Draw a free-body diagram for the sleigh:



RESEARCH: Since the sleigh is moving with a constant velocity, the net forces on the sleigh are zero. This means that the normal force and the gravitational force are equal and opposite ($\vec{N} = -\vec{F}_g$), as are the friction force and the force from the reindeer ($\vec{F}_{\text{reindeer}} = -\vec{f}_k$). From the data given in the problem, it is possible to calculate the friction force $f_k = \mu_k mg$. The power required to keep the sleigh moving at a constant speed is given by $P = F_{\text{reindeer}}v$. Eventually, it will be necessary to convert from SI units (Watts) to non-standard units (horsepower or hp). This can be cone using the conversion factor 1 hp = 746 W.

SIMPLIFY: To find the power required for the sleigh to move, it is necessary to express the force from the reindeer in terms of known quantities. Since the force of the reindeer is equal in magnitude with the friction force, use the equation for frictional force to find:

$$\begin{aligned} \left|\vec{F}_{\text{reindeer}}\right| &= \left|-\vec{f}_k\right| \\ &= f_k \\ &= \mu_k mg \end{aligned}$$

Use this and the speed of the sleigh to find that $P = F_{\text{reindeer}}v = \mu_k mgv$.

CALCULATE: With the exception of the gravitational acceleration, all of the needed values are given in the question. The coefficient of kinetic friction between the sleigh and the snow is 0.1337, the mass of the system (sleigh, Santa, and presents) is 537.3 kg, and the speed of the sleigh is 3.333 m/s. Using a gravitational acceleration of 9.81 m/s gives:

$$\begin{aligned} P &= \mu_k mgv \\ &= 0.1337 \cdot 537.3 \text{ kg} \cdot 9.81 \text{ m/s}^2 \cdot 3.333 \text{ m/s} \\ &= 2348.83532 \text{ W} \end{aligned}$$

This can be converted to horsepower: $2348.83532 \text{ W} \cdot \dfrac{1 \text{ hp}}{746 \text{ W}} = 3.148572815 \text{ hp}$.

ROUND: The measured quantities in this problem are all given to four significant figures. Though the conversion from watts to horsepower and the gravitational acceleration have three significant figures, they do not count for the final answer. The power required to keep the sleigh moving is 3.149 hp.

DOUBLE-CHECK: Generally, it is thought that Santa has 8 or 9 reindeer (depending on how foggy it is on a given Christmas Eve). This gives an average of between 0.3499 and 0.3936 horsepower per reindeer, which seems reasonable. Work backwards to find that, if the reindeer are pulling the sled with 3.149 hp, then the speed they are moving must be (rounding to four significant figures):

$$v=\frac{3.149\text{ hp}}{\mu_k mg}$$
$$=\frac{3.149\text{ hp}\cdot 746\text{ W/hp}}{0.1337\cdot 537.3\text{ kg}\cdot 9.81\text{ m/s}^2}$$
$$=3.333452207\frac{\text{W}}{\text{kg}\cdot\text{m/s}^2}$$
$$=3.333\frac{\text{kg}\cdot\text{m}^2/\text{s}^3}{\text{kg}\cdot\text{m/s}^2}=3.333\text{ m/s}$$

This matches the constant velocity from the problem, so the calculations were correct.

5.80. $P=\mu_\text{k}mgv$

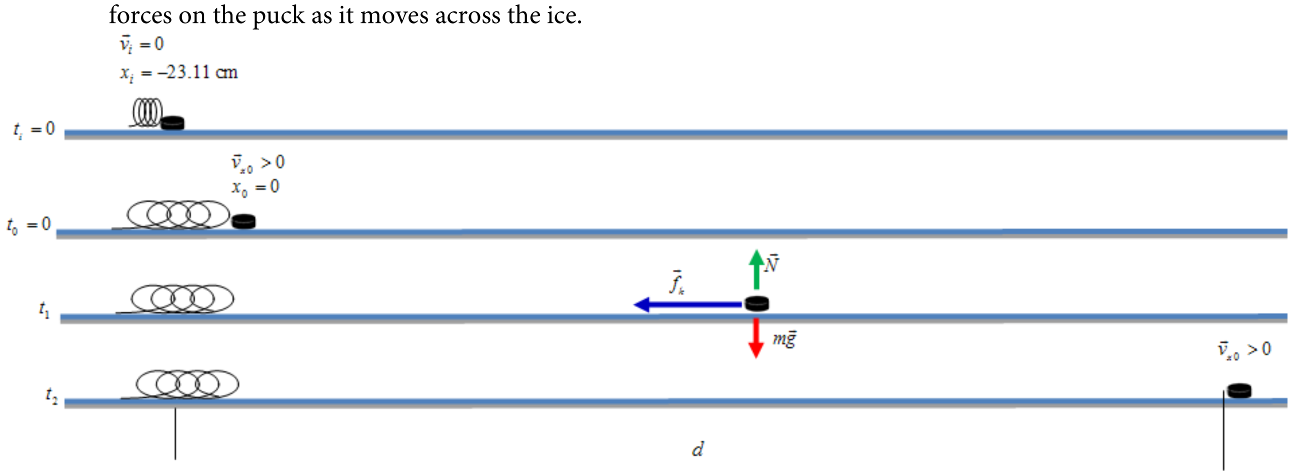
$$\mu_\text{k}=\frac{P}{mgv}=\frac{(2.666\text{ hp})\frac{746\text{ W}}{\text{hp}}}{(540.3\text{ kg})(9.81\text{ m/s}^2)(2.561\text{ m/s})}=0.1465$$

5.81. $P=\mu_\text{k}mgv$

$$m=\frac{P}{\mu_\text{k}gv}=\frac{(3.182\text{ hp})\frac{746\text{ W}}{\text{hp}}}{(0.1595)(9.81\text{ m/s}^2)(2.791\text{ m/s})}=543.6\text{ kg}$$

5.82. **THINK:** In this problem, the energy stored in the spring is converted to kinetic energy as the puck slides across the ice. The spring constant and compression of the spring can be used to calculate the energy stored in the spring. This is all converted to kinetic energy of the puck. The energy is dissipated as the puck slides across the ice. It is necessary to compute how far the puck must slide to dissipate all of the energy that was, originally, stored in the spring.

SKETCH: Sketch the puck when the spring is fully compressed, when it leaves contact with the spring, as it moves across the ice, and at the moment it comes to a stop. Include a free body diagram showing the forces on the puck as it moves across the ice.



RESEARCH: The potential energy stored in the spring is $U=\frac{1}{2}kx^2$, where x is the compression of the spring. The energy dissipated by the force of friction is $\Delta U=Fd$. The force of friction on the puck is given by $F=\mu_k mg$. It is necessary to find the total distance traveled d.

SIMPLIFY: First, find the energy dissipated by the force of friction in terms of known quantities $\Delta U=\mu_k mgd$. This must equal the energy that was stored in the spring, $U=\frac{1}{2}kx^2$. Setting $\Delta U=U$, solve for the total distance traveled in terms of known quantities:

$$\Delta U = U$$
$$\mu_k mgd = \frac{1}{2}kx^2$$
$$d = \frac{kx^2}{2\mu_k mg}$$

It is important to note that x represents the compression of the spring before the puck was released, and d is the total distance traveled from the time that the puck was released (not from the time the puck left contact with the spring).

CALCULATE: Before plugging the values from the question into the equation above, it is important to make sure that all of the units are the same. In particular, note that it is easier to solve the equation directly if the compression is changed from 23.11 cm to 0.2311 m and the mass used is 0.1700 kg instead of 170.0 g. Then the distance is:

$$\begin{aligned} d &= \frac{kx^2}{2\mu_k mg} \\ &= \frac{15.19 \text{ N/m} \cdot (-0.2311 \text{ m})^2}{2 \cdot 0.02221 \cdot 0.1700 \text{ kg} \cdot 9.81 \text{ m/s}^2} \\ &= 10.95118667 \text{ m} \end{aligned}$$

Of this distance, 0.2311 m is the distance the spring was compressed. So the distance traveled by the puck after leaving the spring is 10.95118667 m – 0.2311 m = 10.72008667 m.

ROUND: The measured values are all given to four significant figures, so the final answer is that the hockey puck traveled 10.72 m.

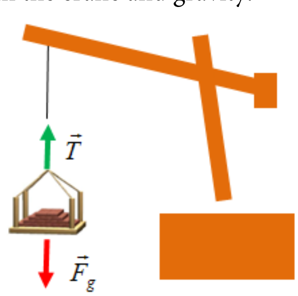
DOUBLE-CHECK: Working backwards, if the hockey puck weighs 0.1700 kg and traveled 10.95 m across the ice (including spring compression) with a coefficient of kinetic friction of 0.02221, then the energy dissipated was $\Delta U = \mu_k mgd = 0.0221 \cdot 9.81 \text{ m/s}^2 \cdot 0.1700 \text{ kg} \cdot 10.95 \text{ m} = 0.4056 \text{ J}$. Since the energy stored in this spring is $U = \frac{1}{2}kx^2 = \frac{15.19 \text{ N/m}}{2}x^2$, it is necessary for the spring to have been compressed by $x = \sqrt{\frac{0.4056 \text{ J} \cdot 2}{15.19 \text{ N/m}}} = 0.231092 \text{ m}$, within rounding of the value of 23.11 cm given in the problem.

5.83. $$d = \frac{kx^2}{2\mu_k mg}$$

$$\mu_k = \frac{kx^2}{2mgd} = \frac{(17.49 \text{ N/m})(0.2331 \text{ m})^2}{2(0.1700 \text{ kg})(9.81 \text{ m/s}^2)(12.13 \text{ m} + 0.2331 \text{ m})} = 0.02305$$

5.84. **THINK:** Since the bricks travel at a low, constant speed, use the information given in the problem to find the tension force that the crane exerts to raise the bricks. The power can be computed by finding the scalar product of the force vector and the velocity vector.

SKETCH: A free body diagram of the bricks as they are raised to the top of the platform is helpful. The only forces are tension from the crane and gravity.



RESEARCH: The average power is the scalar product of the force exerted by the crane on the bricks and the velocity of the bricks: $P = \vec{F} \bullet \vec{v}$, where the force is the tension from the crane. (The speed of the bricks is low, so air resistance is negligible in this case.) The bricks are moving at a constant velocity, so the sum of the forces is zero and $\vec{T} = -\vec{F}_g = -m\vec{g}$. The velocity is constant and can be computed as the distance divided by the time $\vec{v} = \dfrac{\Delta\vec{d}}{\Delta t}$.

SIMPLIFY: Instead of using vector equations, note that the tension force and the velocity are in the same direction. The equation for the power then becomes $P = \vec{F} \bullet \vec{v} = Fv\cos\alpha$, where α is the angle between the velocity and force. Since $T = mg$ and $v = \dfrac{d}{t}$, the power is given by the equation $P = \dfrac{mgd}{t}\cos\alpha$.

CALCULATE: The mass, distance, and time are given in the problem. The velocity of the bricks is in the same direction as the tension force, so $\alpha = 0$.

$$\begin{aligned} P &= \frac{mgd}{t}\cos\alpha \\ &= \frac{75.0 \text{ kg} \cdot 9.81 \text{ m/s}^2 \cdot 45.0 \text{ m}}{52.0 \text{ s}}\cos 0° \\ &= 636.7067308 \text{ W} \end{aligned}$$

ROUND: The mass of the bricks, height to which they are raised, and time are all given to three significant figures, and the answer should have four significant figures. The average power of the crane is 637 W.

DOUBLE-CHECK: To check, note that the average power is the work done divided by the elapsed time: $\bar{P} = \dfrac{W}{\Delta t}$. Combine this with the equation for the work done by the constant tension force $W = |\vec{F}||\Delta\vec{r}|\cos\alpha$ to find an equation for the average power: $\vec{P} = \dfrac{|\vec{F}||\Delta\vec{r}|\cos\alpha}{\Delta t}$. Plug in the values for the tension force $\vec{T} = -\vec{F}_g = -mg$ and distance $\Delta\vec{r} = 45.0$ m upward to find:

$$\begin{aligned} P &= \frac{mgd}{t}\cos\alpha \\ &= \frac{75.0 \text{ kg} \cdot 9.81 \text{ m/s}^2 \cdot 45.0 \text{ m}}{52.0 \text{ s}}\cos 0° \\ &= 636.7067308 \text{ W} \end{aligned}$$

When this is rounded to three decimal places, it confirms the calculations.

5.85. $t = \dfrac{mgd}{P} = \dfrac{(75.0 \text{ kg})(9.81 \text{ m/s}^2)(45.0 \text{ m})}{725 \text{ W}} = 45.7 \text{ s}.$

5.86. $d = \dfrac{Pt}{mg} = \dfrac{(815 \text{ W})(52.0 \text{ s})}{(75.0 \text{ kg})(9.81 \text{ m/s}^2)} = 57.6 \text{ m}.$

Chapter 6: Potential Energy and Energy Conservation

Concept Checks

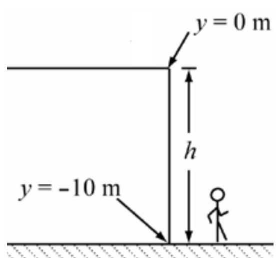
6.1. b **6.2.** a **6.3.** c **6.4.** e **6.5.** e **6.6.** d **6.7.** b

Multiple-Choice Questions

6.1. a **6.2.** c **6.3.** e **6.4.** e **6.5.** d **6.6.** e **6.7.** d **6.8.** e **6.9.** a **6.10.** d **6.11.** c **6.12.** c **6.13.** b

Conceptual Questions

6.14. The kinetic energy, *K*, of an object is proportional to the mass, *m*, of the object and the square of its speed, *v*. The formula is $K = mv^2/2$. The mass is always positive, and the square of the velocity is non-negative. Since the product of non-negative numbers is non-negative, the kinetic energy of an object cannot be negative. However, the potential energy of an object can be negative because it is a relative value. An example of negative potential energy is gravitational potential energy, given by the formula $U = mgh$, where *m* is the mass of the object, *g* is the acceleration due to gravity, and *h* is the vertical distance above the ground. Consider a person standing at the base of a bridge as in the figure below.



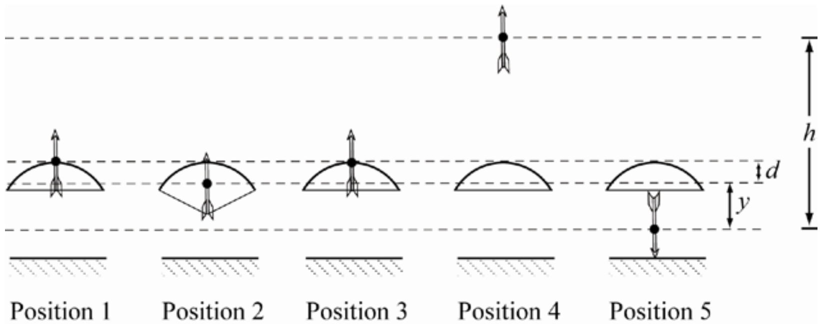
In this coordinate system, the person has potential energy due to gravity of $U = m\left(9.81 \text{ m/s}^2\right)\left(-10 \text{ m}\right)$ relative to the reference point of the bridge. Since a mass is always positive, the potential energy of the person standing on the ground relative to the bridge above has a negative value.

6.15. (a) If a person jumps off a table onto the floor, mechanical energy is not conserved. Mechanical energy is conserved while you are falling towards the floor (assuming energy lost to air resistance is ignored) because gravitational potential energy is being converted to kinetic energy. However, once you land on the floor all of the kinetic energy is absorbed by your body on impact. The energy is lost to non-conservative forces such as friction within your body and heat expelled by your muscles.

(b) The car's mechanical energy is not conserved. Assume a car is on a level plane so it has no gravitational potential energy. The car is in motion so its energy is in the form of kinetic energy. The energy is lost to non-conservative forces such as friction on the tires, thermal energy on the car's brakes and energy dissipated as the car's body is bent by the tree.

6.16. Work is defined as the dot product of force and displacement. This is indicated in the formula $W = \vec{F}\cdot\Delta\vec{r} = \left|\vec{F}\right|\left|\Delta\vec{r}\right|\cos\theta$, where $\vec{F}$ is the applied force, $\vec{r}$ is the displacement of the object, and θ is the angle between the vectors $\vec{F}$ and $\vec{r}$. When you are standing still, the bag of groceries does not travel any distance, i.e. $\left|\vec{r}\right| = 0$, so there is no work done. Assuming that you do not lift or lower the bag of groceries when you carry the bag a displacement $\vec{r}$ across the parking lot, then you do not do any work. This is because the applied force $\vec{F}$ is perpendicular to the displacement $\vec{r}$. Using $\theta = 90°$ in the formula gives $W = \left|\vec{F}\right|\left|\Delta\vec{r}\right|\cos 90° = \left|\vec{F}\right|\left|\Delta\vec{r}\right|\cdot 0 = 0 \text{ J}$.

6.17. The energy in the system, *E*, is the sum of the energy stored in the bow by flexing it, E_b, the kinetic energy of the arrow, *K*, and the gravitational potential energy of the arrow, *U*. Let the arrow have mass *m* and the bow have spring constant *k*. Five separate positions of the arrow and bow system will be considered. Position 1 is where the arrow is put in the bow. Position 2 is where the arrow is pulled back in the bow. Position 3 is where the bow has returned to its relaxed position and the arrow is leaving the bowstring. Position 4 is where the arrow has reached its maximum height *h*. Position 5 is where the arrow has stuck in the ground.



At position 1 the arrow has gravitational potential energy $U = mg(y+d)$ (refer to diagram) relative to the ground. The total energy in the system at this position is $E_1 = mg(y+d)$. At position 2, the arrow now has gravitational potential energy $U = mgy$ and the elastic energy stored in the bow is $E_b = kd^2/2$ due to the downward displacement *d*. The total energy in the system at this position is $E_2 = mgy + \left(kd^2/2\right)$. The work done by the bowstring during this displacement is $E_{tot} = 2.0$ J At position 3, the bow's tension is released and the arrow is launched with a velocity, *v*. The total energy is given by $E_3 = \left(mv^2/2\right) + mg\left(y+d\right)$. The work done on the arrow by the bow is $W_3 = kd^2/2$. At position 4, the arrow has reached its maximum height, *h*. At this position, the velocity of the arrow is zero, so the kinetic energy is zero. The total energy is given by $E_4 = mgh$. The work done on the arrow by gravity is equal to the change in kinetic energy, $W_4 = \Delta K = 0 - mv^2/2$. At position 5, the arrow has hit the ground and stuck in. The total energy is $E_5 = 0$. When the arrow hits the ground the energy of the system is dissipated by friction between the arrow and the ground. The work done on the arrow by gravity during its fall is given by $W_5 = \Delta K = \left(mv^2/2\right) - 0$. This is equal to the kinetic energy of the arrow just before it strikes the ground.

6.18. (a) Assuming both billiard balls have the same mass, *m*, the initial energies, E_{Ai} and E_{Bi} are given by $E_{Ai} = mgh$ and $E_{Bi} = mgh$. The final energy is all due to kinetic energy, so the final energies are $E_{Af} = \left(mv_A^{\ 2}\right)/2$ and $E_{Bf} = \left(mv_B^{\ 2}\right)/2$. By conservation of energy (assuming no loss due to friction), $E_i = E_f$. For each ball the initial and final energies are equal. This means $mgh = \left(mv_A^2\right)/2 \Rightarrow v_A = \sqrt{2gh}$ and $mgh = \left(mv_B^{\ 2}\right)/2 \Rightarrow v_B = \sqrt{2gh}$. Therefore, $v_A = v_B$. The billiard balls have the same speed at the end.

(b) Ball B undergoes an acceleration of *a* and a deceleration of –*a* due to the dip in the track. The effects of the acceleration and deceleration ultimately cancel. However, the ball rolling on track B will have a greater speed over of the lowest section of track. Therefore, ball B will win the race.

6.19. Because the girl/swing system swings out, then returns to the same point, the girl/swing system has moved over a closed path and the work done is zero. Therefore the forces acting on the girl/swing system are conservative. Assuming no friction, the only forces acting on the girl/swing system are the tension in the ropes holding up the girl/swing system and the force of gravity. Assume that the ropes cannot be stretched so that the tension in the ropes is conservative. Gravity is a conservative force, so it is expected that all forces are conservative for the girl/swing system.

6.20. No. Friction is a dissipative force (non-conservative). The work done by friction cannot be stored in a potential form.

6.21. No. The mathematical expression for the potential energy of a spring is $U=\left(kx^2\right)/2.$ The spring constant, k is a positive constant. The square of the displacement of the spring, x, will always be non-negative. Hence, the potential energy of a spring will always be non-negative.

6.22. The elastic force is given by $\vec{F}=-k\vec{r}$, where $\vec{r}$ is the displacement of the spring. The force is therefore a function of displacement, so denote that the force by $\vec{F}\left(\vec{r}\right).$ The sum of the inner product between $\vec{F}\left(\vec{r}\right)$ and the local displacements Δr can be expressed as $\sum_{i=1}^{n}\vec{F}\left(\vec{r}_i\right)\cdot\Delta r.$ If the local displacements are chosen so they are infinitesimally small, the sum can be expressed as an integral:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\vec{F}\left(\vec{r}_i\right)\cdot\Delta r=\int_a^b\vec{F}\left(\vec{r}\right)\cdot d\vec{r}.$$

If the trajectory is a closed loop, then $a = b$ and the integral becomes $\int_a^b\vec{F}\left(\vec{r}\right)\cdot d\vec{r}=0$ because

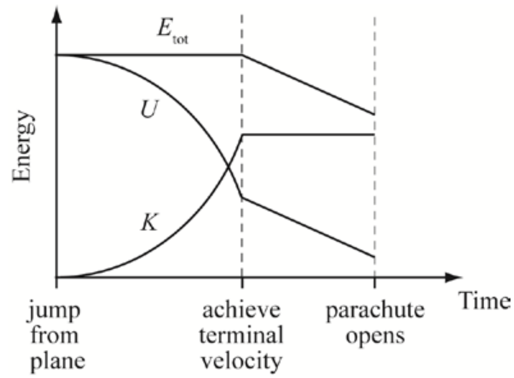
$$\int_a^b\vec{F}\left(\vec{r}\right)\cdot d\vec{r}=f\left(b\right)-f\left(a\right)=f(a)-f(a)=0.$$

It should be noted that work, $W=\int\vec{F}\cdot d\vec{r}$, is independent of a path because the force is conservative. If there was a dissipative force, such as friction present, the force would be non-conservative and therefore be path-dependent.

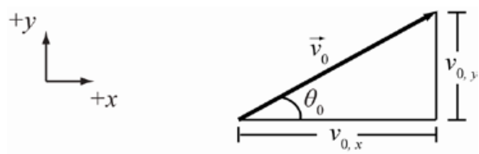
6.23. No. There is not a 1-1 correspondence between potential energy functions and conservative forces. The conservative force is the negative gradient of the potential energy. Therefore, two conservative forces will have the same potential energy function U if they differ by a constant. For example, consider the force $F=0.$ The corresponding potential function is a constant, but it could be any constant depending on the situation. Therefore there is not necessarily a unique potential function corresponding to a conservative force.

6.24. When the person first steps out of the plane, all of the energy is potential energy and as they fall, the potential energy is converted to kinetic energy. In the first stage, before they reach the terminal velocity, they accelerate at a constant rate, so their velocity increases at a linear rate, and so $K=\frac{1}{2}mv^2$ increases at a quadratic rate. On the other hand, their height decreases at a quadratic rate, so $U=mgh$ decreases at a quadratic rate. Because there is no air resistance in the first stage of the model, the total energy, $E_{\text{tot}}=K+U$, remains constant. In the second stage, their acceleration becomes zero, and their velocity becomes constant. This means that $K=\frac{1}{2}mv^2$ is constant, and $U=mgh$ decreases at a linear rate. The sum of the energies is no longer constant. The lost energy is due to the air resistance that counter-balances the acceleration due to gravity.

The rate of decrease of energy in the system is equal to the rate of decrease of potential energy.



6.25.



The lengths of the component vectors of v_0 are $v_{0,x} = v_0 \cos\theta_0$ and $v_{0,y} = v_0 \sin\theta_0$. Velocity is a vector quantity, so $\vec{v} = v_x \hat{x} + v_y \hat{y}$. Let $v = |\vec{v}|$. Then, $v^2 = v_x^{\ 2} + v_y^{\ 2}$. The velocity vector $\vec{v}$ has component vectors $v_x = v_0 \cos\theta_0$ (horizontal component is constant) and $v_y = v_0 \sin\theta_0 - gt$ (which changes relative to time). To compute the kinetic energy, use the formula $K = mv^2/2$. First, compute

$$\begin{aligned} v^2 &= v_0^{\ 2} \cos\theta_0^{\ 2} + v_0^{\ 2} \sin^2\theta_0 - 2v_0 \sin\theta_0 gt + g^2t^2 \\ &= v_0^{\ 2}\left(\cos^2\theta_0 + \sin^2\theta_0\right) - 2v_0 \sin\theta_0 gt + g^2t^2 \\ &= v_0^2 - 2v_0 \sin\theta_0 gt + g^2t^2. \end{aligned}$$

So, $K(t) = \left[m\left(v_0^{\ 2} - 2v_0 \sin\theta_0 gt + g^2t^2\right)\right]/2$. The potential energy only changes with displacement in the vertical direction. The gravitational potential energy is given by $U = mgy$. From kinematics equations, $y = y_0 + v_{0,y}t - \left(gt^2\right)/2$. Because the projectile was launched from the ground, $y_0 = 0$. Substitute $v_{0,y} = v_0 \sin\theta_0$ into the equation to get $y = v_0 \sin\theta_0 t - \left(gt^2\right)/2$. Substituting this into the expression for U yields $U(t) = mg\left(v_0 \sin\theta_0 t - \left(\mathrm{g}t^2\right)/2\right)$. The total energy of the projectile is $E(t) = K(t) + U(t)$. This equation can be written as

$$E(t) = \frac{m\left(v_0^{\ 2} - 2v_0 \sin\theta_0 gt + g^2t^2\right)}{2} + mg\left(v_0 \sin\theta_0 t - \frac{1}{2}gt^2\right).$$

Grouping like terms, the equation can be simplified:

$$E(t) = \frac{m}{2}\left(g^2t^2 - g^2t^2\right) - mgv_0 \sin\theta_0 t + mgv_0 \sin\theta_0 t + \frac{1}{2}mv_0^{\ 2} \;\Rightarrow\; E(t) = \frac{1}{2}mv_0^{\ 2}.$$

Notice that E is actually not time dependent. This is due to the conservation of energy.

6.26. (a) The total energy is given by the sum of the kinetic energy, $K = mv^2/2$, and potential energy, $U = mgh$. This gives the formula $E = \frac{1}{2}mv^2 + mgh$ for total energy. Therefore,

$$H(m,h,v) = \frac{\frac{1}{2}mv^2 + mgh}{mg} = \frac{\frac{1}{2}v^2 + gh}{g} = \frac{v^2}{2g} + h.$$

(b) The aircraft has a mass of $m = 3.5\cdot 10^5$ kg, a velocity of v = 250.0 m/s and a height of $h = 1.00\cdot 10^4$ m Substituting these values gives

$$H = \frac{(250.0\text{ m/s})^2}{2(9.81\text{ m/s}^2)} + 1.00\cdot 10^4\text{ m} = 13{,}185.5\text{ m} \approx 13{,}200\text{ m}.$$

6.27. (a) The energy in the system is the sum of the kinetic energy and the gravitational potential energy, $E = K + U$. For motion in the x-direction, $U = 0$ and $K = mv^2/2$. So, $E = mv^2/2$. Newton's second law is $\vec{F} = m\vec{a}$, which can also be written as $\vec{F} = m\,d\vec{v}/dt$. By the work-kinetic energy theorem, $W = \Delta K$ and $W = \vec{F}\bullet\vec{x}$. If the work on the body as a function of position is to be determined,

$$W = \sum_{i=1}^{n} \vec{F}(x_i)\Delta\vec{x}.$$

If the motion is continuous, let the intervals become infinitesimal so that the sum becomes an integral, $W = \int_a^b \vec{F}\bullet d\vec{x}$. Since $\vec{F} = m\,d\vec{v}/dt$ and $\vec{v} = d\vec{x}/dt$, it must be that $d\vec{x} = \vec{v}\,dt$. Substituting these values into the equation:

$$W = \int m\frac{d\vec{v}}{dt}\bullet\vec{v}\,dt = \int m\vec{v}\bullet d\vec{v}.$$

Work is also equal to the change in kinetic energy, therefore,

$$\Delta K = \int_{v_i}^{v_f} m\vec{v}\bullet d\vec{v} = \frac{1}{2}mv^2\,\Big|_{v_i}^{v_f} = \frac{1}{2}m\left(v_f^2 - v_i^2\right).$$

(b) Newton's second law, expressed as $\vec{F} = m\vec{a}$, does not hold for objects on the subatomic scale or for objects approaching the speed of light. The law of conservation of energy holds under all known circumstances.

6.28. (a) The force function is $F(x) = -\dfrac{dU(x)}{dx} = 4U_0\left[\dfrac{12x_0^{12}}{x^{13}} - \dfrac{6x_0^{6}}{x^{7}}\right]$.

(b) The two atoms experience zero force from each other when F = 0, which is when $\left[12\dfrac{x_0^{12}}{x^{13}} - \dfrac{6x_0^6}{x^7}\right] = 0.$

Solving for x yields $\dfrac{6x_0^6}{x^7} = \dfrac{12x_0^{12}}{x^{13}} \Rightarrow x^6 = 2x_0^6$ or $x = \pm\sqrt[6]{2}x_0$. Since x is the separation, $x = \sqrt[6]{2}x_0$.

(c) For separations larger than $x = \sqrt[6]{2}x_0$, let $x = 3x_0$:

$$U(3x_0) = 4U_0\left[\left(\frac{x_0}{3x_0}\right)^{12} - \left(\frac{x_0}{3x_0}\right)^{6}\right] = 4U_0\left[\left(\frac{1}{3}\right)^{12} - \left(\frac{1}{3}\right)^{6}\right].$$

The factor $\left[(1/3^{12}) - (1/3^6)\right]$ is negative and the potential is negative. Therefore, for $x > \sqrt[6]{2}x_0$, the nuclei attract. For separations smaller than $x = \sqrt[6]{2}x_0$, let $x = x_0/2$:

$$U(x_0/2) = 4U_0\left[\left(\frac{2x_0}{x_0}\right)^{12} - \left(\frac{2x_0}{x_0}\right)^{6}\right] = 4U_0\left[2^{12} - 2^6\right].$$

The term $\left[2^{12} - 2^6\right]$ is positive and the potential is positive. So, when $x < \sqrt[6]{2}x_0$, the potential is positive and the nuclei repel.

6.29. (a) In two-dimensional situations, the force components can be obtained from the potential energy using the equations $F_x = -\dfrac{\partial U(x,y)}{\partial x}$ and $W_a = (10.0 \text{ N/cm})\left((5.00 \text{ cm})^2 - (-5.00 \text{ cm})^2\right)/2 = 0 \text{ J}$ The net force is given by:

$$\vec{F} = F_x\hat{x} + F_y\hat{y} = -\frac{\partial U(x,y)}{\partial x}\hat{x} - \frac{\partial U(x,y)}{\partial y}\hat{y} = -\frac{1}{2}k\left(\frac{\partial}{\partial x}\left(x^2+y^2\right)\hat{x} + \frac{\partial}{\partial y}\left(x^2+y^2\right)\hat{y}\right)$$

$$= -\frac{1}{2}k\left(2x\hat{x} + 2y\hat{y}\right) = -k(x\hat{x} + y\hat{y}).$$

(b) The equilibrium point will be where $\vec{F} = 0$. This occurs if and only if *x* and *y* are both zero.

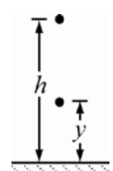
(c) These forces will accelerate the mass in the $-\hat{x}$ and $-\hat{y}$ directions for positive values of *x* and *y* and vice versa for negative values of *x* and *y*.

(d) $\left|\vec{F}\right| = \left[\left(F_x\right)^2 + \left(F_y\right)^2\right]^{\frac{1}{2}}$. For *x* = 3.00 cm, *y* = 4.00 cm and *k* = 10.0 N/cm:

$$\left|\vec{F}\right| = \left[\left(-(10.0 \text{ N/cm})(3.00 \text{ cm})\right)^2 + \left(-(10.0 \text{ N/cm})(4.00 \text{ cm})\right)^2\right]^{\frac{1}{2}} = 50.0 \text{ N}.$$

(e) A turning point is a place where the kinetic energy, *K* is zero. Since *K* = *E* – *U*, the turning point will occur when *U* = *E*, so the turning points occurs when *U* = 10 J. Solve $U(x,y) = 10 \text{ J} = \frac{1}{2}k\left(x^2+y^2\right)$. This gives $20.0 \text{ J} = \frac{10.0 \text{ N}}{\text{cm}} \cdot \frac{100 \text{ cm}}{\text{m}}\left(x^2+y^2\right)$, or $x^2 + y^2 = 0.0200 \text{ m}^2$. The turning points are the points on the circle centered at the origin of radius 0.141 m.

6.30. Setting the kinetic energy equal to the potential energy will normally not yield useful information. To use the example in the problem, if the rock is dropped from a height, *h* above the ground, then solving for the speed at two different locations:

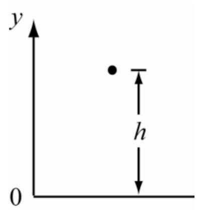


Point 1: $U_1 = mgh$ and $K_1 = \left(mv_1^{\ 2}\right)/2$. If $mgh = mv^2/2$, then solving for v_1: $v_1 = \sqrt{2gh}$. But the rock has not been dropped yet so in fact v_1 is really zero. Point 2: just before the rock hits the ground. In this case, the rock's height above the ground, *y*, is almost zero. If $U_2 = K_2$, then $mgy = mv_2^{\ 2}/2$ or $v_2 = \sqrt{2gy}$. But if *y* is about 0 m, then $v_2 \approx 0$ m/s. At point 2, the rock's velocity is reaching its maximum value, so by setting the potential and kinetic energy equal to one another at this point, the wrong value is calculated for the rock's speed.

Exercises

6.31. **THINK:** The mass of the book is m = 2.00 kg and its height above the floor is h = 1.50 m. Determine the gravitational potential energy, U_g.

SKETCH:



RESEARCH: Taking the floor's height as $U_g = 0$, U_g for the book can be determined from the formula $U_g = mgh$.

SIMPLIFY: It is not necessary to simplify.

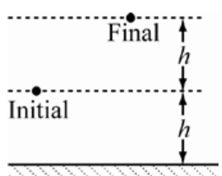
CALCULATE: $U_g = (2.00 \text{ kg})(9.81 \text{ m/s}^2)(1.50 \text{ m}) = 29.43 \text{ J}$

ROUND: The given initial values have three significant figures, so the result should be rounded to $U_g = 29.4$ J.

DOUBLE-CHECK: This is a reasonable value for a small mass held a small distance above the floor.

6.32. **THINK:** The rock's mass is m = 40.0 kg and the gravitational potential energy is $U_g = 500.$ J. Determine:
(a) the height of the rock, h, and
(b) the change, ΔU_g if the rock is raised to twice its original height, $2h$.

SKETCH:



RESEARCH: Use the equation $U_g = mgh$. Note: $\Delta U_g = U_g - U_{g,0}$.

SIMPLIFY:

(a) $U_g = mgh \Rightarrow h = \dfrac{U_g}{mg}$

(b) $$\begin{aligned}\Delta U_g &= U_g - U_{g,0} \\ &= mg(2h) - mgh \\ &= mgh \\ &= U_g\end{aligned}$$

CALCULATE:

(a) $h = \dfrac{500. \text{ J}}{40.0 \text{ kg}(9.81 \text{ m/s}^2)} = 1.274 \text{ m}$

(b) $\Delta U_g = 500.$ J

ROUND:

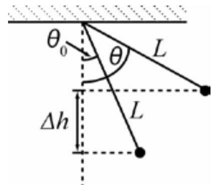
(a) $h = 1.27$ m

(b) $\Delta U_g = 500.$ J does not need to be rounded.

DOUBLE-CHECK: The initial height is reasonable for such a large mass, despite the large U_g. Since the potential energy is proportional to height, it should double when the height is doubled.

6.33. **THINK:** The rock's mass is m = 0.773 kg. The length of the string is L = 2.45 m. The gravitational acceleration on the Moon is $g_{\text{M}} = g/6.$ The initial and final angles are $\theta_0 = 3.31°$ and $\theta = 14.01°$, respectively. Determine the rock's change in gravitational potential energy, ΔU.

SKETCH:



RESEARCH: To determine ΔU , the change in height of the rock, Δh , is needed. This can be determined using trigonometry. Then $\Delta U = mg_{\text{M}}\Delta h.$

SIMPLIFY: To determine Δh: $\Delta h = L\cos\theta_0 - L\cos\theta = L(\cos\theta_0 - \cos\theta)$. Then

$$\Delta U = mg_{\text{M}}\Delta h = \frac{1}{6}mgL(\cos\theta_0 - \cos\theta).$$

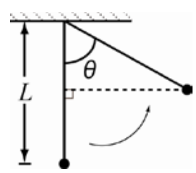
CALCULATE: $\Delta U = \frac{1}{6}(0.773 \text{ kg})(9.81 \text{ m/s}^2)(2.45 \text{ m})(\cos(3.31°) - \cos(14.01°)) = 0.08694 \text{ J}$

ROUND: With three significant figures in the values, the result should be rounded to $\Delta U = 0.0869$ J.

DOUBLE-CHECK: ΔU is small, as it should be considering the smaller gravitational acceleration and the small change in height.

6.34. **THINK:** The child's mass is m = 20.0 kg. Each rope has a length of L = 1.50 m. Determine (a) U_{g} at the lowest point of the swing's trajectory, (b) U_{g} when the ropes are $\theta = 45.0°$ from the vertical and (c) the position with the higher potential energy.

SKETCH:



RESEARCH: Use $U_{\text{g}} = mgh.$

SIMPLIFY:

(a) Relative to the point where $U_{\text{g}} = 0$, the height of the swing is $-L$. Then $U_{\text{g}} = -mgL.$

(b) Now, the height of the swing is $-L\cos\theta$. Then $U_{\text{g}} = -mgL\cos\theta.$

CALCULATE:

(a) $U_{\text{g}} = -(20.0 \text{ kg})(9.81 \text{ m/s}^2)(1.50 \text{ m}) = -294.3 \text{ J}$

(b) $U_{\text{g}} = -(20.0 \text{ kg})(9.81 \text{ m/s}^2)(1.50 \text{ m})\cos 45.0° = -208.1 \text{ J}$

(c) Relative to the point $U_{\text{g}} = 0$, the position in part (b) has greater potential energy.

ROUND: With three significant figures in m and L :

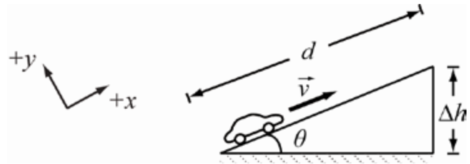
(a) $U_{\text{g}} = -294$ J

(b) $U_{\text{g}} = -208$ J

DOUBLE-CHECK: Had $U_{\text{g}} = 0$ been set at the lowest point of the swing's trajectory, the potential energy in part (b) would still be greater than the potential energy in part (a), as it should be.

6.35. **THINK:** The mass of the car is $m = 1.50 \cdot 10^3$ kg. The distance traveled is $d = 2.50$ km $= 2.50 \cdot 10^3$ m. The angle of inclination is $\theta = 3.00°$. The car travels at a constant velocity. Determine the change in the car's potential energy, ΔU and the net work done on the car, W_{net}.

SKETCH:



RESEARCH: To determine ΔU the change of height of the car Δh must be known. From trigonometry, the change in height is $\Delta h = d\sin\theta$. Then, $\Delta U = mg\Delta h$. To determine W_{net} use the work-kinetic energy theorem. Despite the fact that non-conservative forces are at work (friction force on the vehicle), it is true that $W_{\text{net}} = \Delta K$.

SIMPLIFY: $\Delta U = mg\Delta h = mgd\sin\theta$

$$W_{\text{net}} = \Delta K = \frac{1}{2}mv_{\text{f}}^2 - \frac{1}{2}mv_0^2 = \frac{1}{2}m\left(v_{\text{f}}^2 - v_0^2\right)$$

CALCULATE: $\Delta U = \left(1.50\cdot 10^3 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right)\left(2.50\cdot 10^3 \text{ m}\right)\sin\left(3.00°\right) = 1925309 \text{ J}$

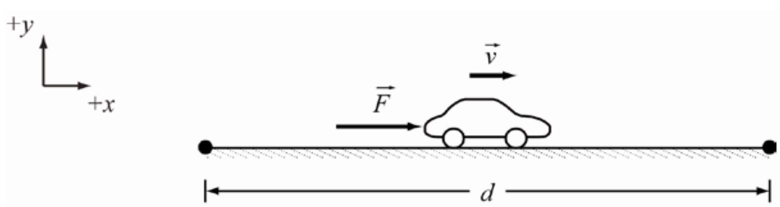
$$W_{\text{net}} = \frac{1}{2}m\left(v_{\text{f}}^2 - v_0^2\right) = \frac{1}{2}m(0) = 0$$

ROUND: Since θ has two significant figures, $\Delta U = 1.93\cdot 10^6$ J, and there is no net work done on the car.

DOUBLE-CHECK: The change in potential energy is large, as the car has a large mass and a large change in height, $\Delta h = \left(2.50\cdot 10^3 \text{ m}\right)\sin\left(3.00°\right) = 131$ m. The fact that the net work done is zero while there is a change in potential energy means that non-conservative forces did work on the car (friction, in this case).

6.36. **THINK:** The constant force is F = 40.0 N. The distance traveled is $d = 5.0\cdot 10^3$ m. Assume the force is parallel to the distance traveled. Determine how much work is done, and if it is done on or by the car. The car's speed is constant.

SKETCH:



RESEARCH: In general,

$$W = \int_{x_0}^{x} F(r)\,dr \text{ (in one dimension).}$$

Here the force is constant, so $F(r) = F$. Bearing in mind that $W_{\text{net}} = \Delta K = 0$, due to the constant speed, the work done by the constant force, F can still be calculated.

SIMPLIFY: $W = \int_{x_0}^{x_1} F\,dr = F\int_{x_0}^{x_1} dr = F\Delta x = Fd$

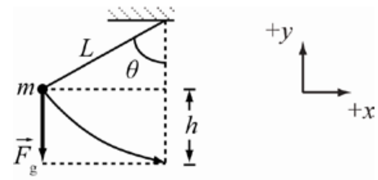
CALCULATE: $W = \left(40.0 \text{ N}\right)\left(5.0\cdot 10^3 \text{ m}\right) = 200{,}000 \text{ J}$. This is the work done on the car by the constant force, as it is a positive value.

ROUND: With two significant figures in d, $W = 2.0\cdot 10^5$ J.

DOUBLE-CHECK: This is a reasonable amount of work done by F, given the large distance the force acts over.

6.37. **THINK:** The piñata's mass is m = 3.27 kg. The string length is L = 0.810 m. Let h be the height of the piñata at its initial position, at an initial angle of $\theta = 56.5°$ to the vertical. Determine the work done by gravity, W_{g}, by the time the string reaches a vertical position for the first time.

SKETCH:



RESEARCH: Since the force of gravity is constant, the work is given by $W_g = \vec{F}_g h = mgh$.

SIMPLIFY: $W_g = mgh = mg(L - L\cos\theta) = mgL(1-\cos\theta)$

CALCULATE: $W_g = (3.27 \text{ kg})(9.81 \text{ m/s}^2)(0.810 \text{ m})(1-\cos(56.5°)) = 11.642 \text{ J}$

ROUND: With three significant figures in *L*, $W_g = 11.6$ J.

DOUBLE-CHECK: The work done by gravity should be positive because F_g pulls the piñata downward.

6.38. **THINK:** $U(x) = \dfrac{1}{x} + x^2 + x - 1$. Determine (a) a function which describes the force on the particle, and (b) a plot of the force and the potential functions and (c) the force on the particle when $x =$ 2.00 m.

SKETCH: A sketch will be provided when part (b) is completed.

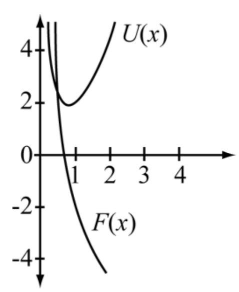
RESEARCH: The relationship between *F* and *U*, in one dimension, is $F(x) = -\dfrac{d}{dx}U(x)$.

SIMPLIFY: (a) $F(x) = -\dfrac{d}{dx}\left(\dfrac{1}{x} + x^2 + x - 1\right) = -\left(-x^{-2} + 2x + 1\right) = \dfrac{1}{x^2} - 2x - 1$

CALCULATE:

(a) Not necessary.

(b) Plotting yields:



(c) At $x = 2.00$ m, $F(2.00) = \dfrac{1}{(2.00)^2} - 2(2.00) - 1 = -4.75$ N (SI units are assumed).

ROUND: $F(2.00 \text{ m}) = -4.75$ N

DOUBLE-CHECK: $F(x)$ is the negative of the slope of $U(x)$. *F*(*x*) crosses the *x*-axis where *U*(*x*) has a local minimum, as would be expected.

6.39. **THINK:** The potential energy functions are (a) $U(y) = ay^3 - by^2$ and (b) $U(y) = U_0 \sin(cy)$. Determine $F(y)$ from $U(y)$.

SKETCH: A sketch is not necessary.

RESEARCH: $F(y) = -\dfrac{\partial U(y)}{\partial y}$

SIMPLIFY:

(a) $F(y) = -\dfrac{\partial\left(ay^3 - by^2\right)}{\partial y} = 2by - 3ay^2$

(b) $F(y) = -\frac{\partial\left(U_0 \sin(cy)\right)}{\partial y} = -cU_0\cos(cy)$

CALCULATE: There are no numerical calculations to perform.
ROUND: It is not necessary to round.
DOUBLE-CHECK: The derivative of a cubic polynomial should be a quadratic, so the answer obtained for (a) makes sense. The derivative of a sine function is a cosine function, so it makes sense that the answer obtained for (b) involves a cosine function.

6.40. **THINK:** The potential energy function is of the form $U(x,z) = ax^2 + bz^3$. Determine the force vector, $\vec{F}$, associated with U.
SKETCH: Not applicable.
RESEARCH: $\vec{F}(x,y,z) = -\vec{\nabla}U(x,y,z) = -\left(\frac{\partial}{\partial x}U\hat{x} + \frac{\partial}{\partial y}U\hat{y} + \frac{\partial}{\partial z}U\hat{z}\right)$
SIMPLIFY: The expression cannot be further simplified.
CALCULATE: $\vec{F} = -\frac{\partial\left(ax^2 + bz^3\right)\hat{x}}{\partial x} - \frac{\partial\left(ax^2 + bz^3\right)\hat{y}}{\partial y} - \frac{\partial\left(ax^2 + bz^3\right)\hat{z}}{\partial z}$

$$= -(2ax)\hat{x} - 0\hat{y} - \left(3bz^2\right)\hat{z}$$

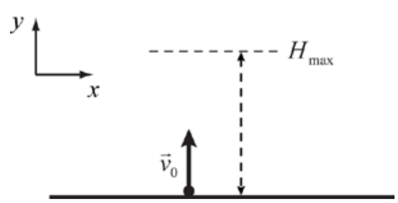
$$= -(2ax)\hat{x} - \left(3bz^2\right)\hat{z}$$

ROUND: Not applicable.
DOUBLE-CHECK: Notice that U is the sum of a function of x and a function of z, namely,

if $G(x) = ax^2$ and $H(z) = bz^3$ then $U(x,z) = G(x) + H(z)$.

Since $G(x)$ has a critical point at $x = 0$ and $H(z)$ has a critical point at $z = 0$, we may expect that $\vec{F} = 0$. And in fact, $\vec{F} = -(2a(0))\hat{x} - \left(3b(0)^2\right)\hat{z} = 0$. Therefore, the answer is reasonable.

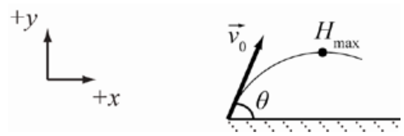
6.41. **THINK:** The maximum height achieved is $H_{max} = 5.00$ m, while the initial height h_0 is zero. The speed of the ball when it reaches its maximum height is $v = 0$. Determine the initial speed.
SKETCH:



RESEARCH: In an isolated system with only conservative forces, $\Delta E_{mec} = 0$. Then, $\Delta K = -\Delta U$. Use $U = mgH_{max}$ and $K = mv^2/2$.
SIMPLIFY: $K_f - K_i = -\left(U_f - U_i\right) = U_i - U_f$, so $\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = mgh_0 - mgH_{max}$.
Substituting $v = 0$ and $h_0 = 0$ gives the equation $-\frac{1}{2}mv_0^2 = -mgH_{max}$. Therefore, $v_0 = \sqrt{2gH_{max}}$.
CALCULATE: $v_0 = \sqrt{2\left(9.81 \text{ m/s}^2\right)(5.00 \text{ m})} = 9.9045$ m/s
ROUND: With three significant figures in H_{max}, $v_0 = 9.90$ m/s.
DOUBLE-CHECK: This is a reasonable speed to throw a ball that reaches a maximum height of 5 m.

6.42. **THINK:** The cannonball's mass is m = 5.99 kg. The launch angle is $\theta = 50.21°$ above the horizontal. The initial speed is $v_0 = 52.61$ m/s and the final vertical speed is $v_y = 0$. The initial height is zero. Determine the gain in potential energy, ΔU.

SKETCH:



RESEARCH: Neglecting air resistance, there are only conservative forces at work. Then, $\Delta K = -\Delta U$ or $\Delta U = -\Delta K$. Determine ΔK from $K = mv^2/2$. From trigonometry, $v_x = v_0 \cos\theta$.

SIMPLIFY: $\Delta U = -\Delta K = -(K_f - K_i) = K_i - K_f$. Note that initially the ball has a horizontal speed v_x (which is constant throughout the cannonball's motion) and a vertical speed v_{y0}. At its maximum height, $v_y = 0$. Then, $\Delta U = \frac{1}{2}mv_0{}^2 - \frac{1}{2}m(v_0 \cos\theta)^2 = \frac{1}{2}mv_0{}^2(1-\cos^2\theta)$.

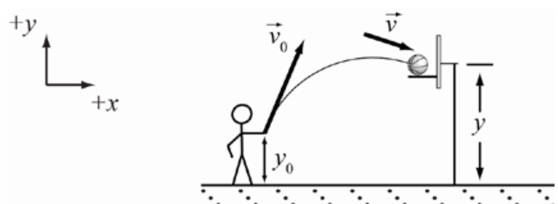
CALCULATE: $\Delta U = \frac{1}{2}(5.99 \text{ kg})(52.61 \text{ m/s})^2(1-\cos^2(50.21°)) = 4894.4 \text{ J}$

ROUND: With three significant figures in m, $\Delta U = 4890$ J.

DOUBLE-CHECK: The change in potential energy is positive, implying that the ball gained potential energy, which it would if raised any height above its initial point. Since the horizontal velocity of the cannonball is constant, it makes sense that the initial vertical velocity is converted entirely into potential energy when the cannonball reaches the highest point.

6.43. **THINK:** The initial height of the basketball is $y_0 = 1.20$ m. The initial speed of the basketball is $v_0 = 20.0$ m/s. The final height is $y = 3.05$ m. Determine the speed of the ball at this point.

SKETCH:



RESEARCH: Neglecting air resistance, there are only conservative forces, so $\Delta K = -\Delta U$. The kinetic energy K can be determined from $K = mv^2/2$ and U from $U = mgh$.

SIMPLIFY: $K_f - K_i = U_i - U_f$, so $(1/2)mv^2 - (1/2)mv_0^2 = mgy_0 - mgy$. Dividing through by the mass m yields the equation $(1/2)v^2 - (1/2)v_0{}^2 = gy_0 - gy$. Then solving for v gives

$$v = \sqrt{2\left(g(y_0 - y) + \frac{1}{2}v_0{}^2\right)}.$$

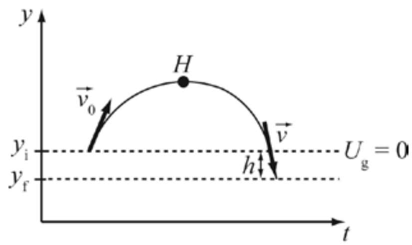
CALCULATE:
$$\begin{aligned} v &= \sqrt{2\left((9.81 \text{ m/s}^2)(1.20 \text{ m} - 3.05 \text{ m}) + \frac{1}{2}(20.0 \text{ m/s})^2\right)} \\ &= \sqrt{2(-18.1485 \text{ m}^2/\text{s}^2 + 200.0 \text{ m}^2/\text{s}^2)} \\ &= 19.071 \text{ m/s} \end{aligned}$$

ROUND: The initial height is given with the fewest number of significant figures. Since it has three significant figures the value of v needs to be rounded to three significant figures: v = 19.1 m/s.

DOUBLE-CHECK: The final speed should be less than the initial speed since the final height is greater than the initial one.

6.44. **THINK:** The book's mass is m = 1.0 kg. The initial height is $y_0 = 1.0$ m, where $U_g = 0$, the maximum height is H = 3.0 m, and the final height is $y_f = 0$ m. Determine (a) the potential energy of the book when it hits the ground, U_g, and (b) the velocity of the book as it hits the ground, v_f. The book is thrown straight up into the air, so the launch angle is vertical. The sketch is not a plot of the trajectory of the book, but a plot of height versus time.

SKETCH:



RESEARCH:

(a) Gravitational potential energy is given by $U_g = mgh$. To compute the final energy, consider the height relative to the height of zero potential, $y_i = 1.0$ m.

(b) To determine v_f, consider the initial point to be at $y = H$ (where $v = 0$), and the final point to be at the point of impact $y = y_f = 0$. Assume there are only conservative forces, so that $\Delta K = -\Delta U$. ΔU between H and y_f is unaffected by the choice of reference point.

SIMPLIFY:

(a) Relative to $U_g = 0$ at y_i, the potential energy of the book when it hits the ground is given by $U_g = mgh = mg(y_f - y_i)$.

(b) $\Delta K = -\Delta U \Rightarrow K_f - K_i = -(U_f - U_i)$. With $v = 0$ at the initial point, $K_f = U_i - U_f$ and $(1/2)mv^2 = mgH - mgy_f = mgH$. Solving for v_f gives the equation: $v_f = -\sqrt{2gH}$. The negative root is chosen because the book is falling.

CALCULATE:

(a) $U_g = (1.0 \text{ kg})(9.81 \text{ m/s}^2)(0 - 1.0 \text{ m}) = -9.81 \text{ J}$

(b) $v_f = -\sqrt{2(9.81 \text{ m/s}^2)(3.0 \text{ m})} = -7.6720 \text{ m/s}$

ROUND: With two significant figures in m, y_i and H:

(a) $U_g = -9.8$ J

(b) $v_f = -7.7$ m / s, or 7.7 m/s downward.

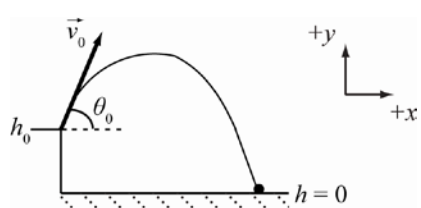
DOUBLE-CHECK: U_g should be negative at y_f, relative to $U_g = 0$ at y_0, because there should be a loss of potential energy. Also, it is sensible for the final velocity of the book to be directed downward.

6.45. **THINK:** The ball's mass is m = 0.0520 kg. The initial speed is $v_0 = 10.0$ m/s. The launch angle is $\theta_0 = 30.0°$. The initial height is $h_0 = 12.0$ m. Determine:

(a) kinetic energy of the ball when it hits the ground, K_f and

(b) the ball's speed when it hits the ground, v.

SKETCH:



RESEARCH: Assuming only conservative forces act on the ball (and neglecting air resistance), $\Delta K = -\Delta U$. K_f can be determined using the equations $\Delta K = -\Delta U$, $K = mv^2/2$ and $U = mgh$. Note that $U_f = 0$, as $h = 0$. With K_f known, v can be determined.

SIMPLIFY:

(a) $\Delta K = -\Delta U \Rightarrow K_f - K_i = U_i - U_f = U_i \Rightarrow K_f = U_i + K_i = mgh_0 + \frac{1}{2}mv_0^2$

(b) $K_f = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2K_f / m}$

CALCULATE:

(a) $K_f = (0.0520 \text{ kg})(9.81 \text{ m/s}^2)(12.0 \text{ m}) + \frac{1}{2}(0.0520 \text{ kg})(10.0 \text{ m/s})^2 = 6.121 \text{ J} + 2.60 \text{ J} = 8.721 \text{ J}$

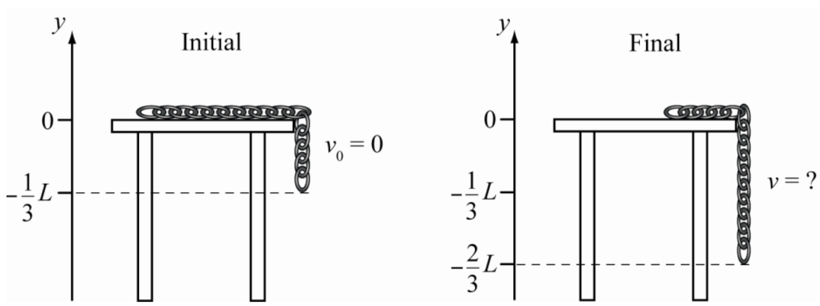
(b) $v = \sqrt{2(8.721 \text{ J})/(0.0520 \text{ kg})} = 18.32 \text{ m/s}$

ROUND: With m having three significant figures, $K_f = 8.72$ J and v = 18.3 m/s.

DOUBLE-CHECK: The amount of kinetic energy computed is a reasonable amount for a ball. The final speed should be greater than the initial speed because the mechanical energy has been completely transformed to kinetic energy. It is, so the calculated value is reasonable.

6.46. **THINK:** The chain's mass is m and has a length of L = 1.00 m. A third of the chain hangs over the edge of the table and held stationary. After the chain is released, determine its speed, v, when two thirds of the chain hangs over the edge.

SKETCH:



RESEARCH: Consider the center of mass (com) location for the part of the chain that hangs over the edge. Since the chain is a rigid body, and it is laid out straight (no slack in the chain), $v_{com} = v$. $\Delta K = -\Delta U$, $K = (mv^2)/2$ and $U = mgh$.

SIMPLIFY: Initially, 1/3 of the chain is hanging over the edge and then $m_{com,0} = m/3$, and $h_{com,i} = -L/6$. When 2/3 of the chain is hanging over the edge, the hanging mass is $m_{com} = 2m/3$. Then, $\Delta K = -\Delta U \Rightarrow K_f - K_i = U_i - U_f$ and $K_i = 0$, so $K_f = U_i - U_f$. Substituting gives $(1/2)mv_{com}^2 = m_{com,i}gh_{com,i} - m_{com}gh_{com}$, so $\frac{1}{2}mv^2 = \left(\frac{m}{3}\right)g\left(-\frac{L}{6}\right) - \left(\frac{2m}{3}\right)g\left(-\frac{L}{3}\right)$, and dividing through by m gives the equation $\frac{1}{2}v^2 = -\frac{1}{18}gL + \frac{2}{9}gL$. Solving for v yields $v = \sqrt{gL/3}$.

CALCULATE: $v = \sqrt{(9.81 \text{ m/s}^2)(1.00 \text{ m})/3} = 1.808 \text{ m/s}$

ROUND: With three significant figures in L, v = 1.81 m/s.

DOUBLE-CHECK: This is a reasonable speed for the chain to achieve while sliding off the table.

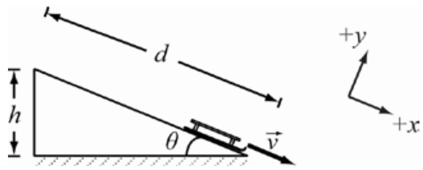
6.47. **THINK:** The initial height is h = 40.0 m. Determine:

(a) the speed v_f at the bottom, neglecting friction,

(b) if the steepness affects the final speed; and

(c) if the steepness affects the final speed when friction is considered.

SKETCH:



RESEARCH:

(a) With conservative forces, $\Delta K = -\Delta U$. v can be determined from $K = \left(mv_f^{\,2}\right)/2$ and $U = mgh$.

(b and c) Note that the change in the angle θ affects the distance, *d*, traveled by the toboggan: as θ gets larger (the incline steeper), d gets smaller.

(c) The change in thermal energy due to friction is proportional to the distance traveled: $\Delta E_{th} = \mu_k Nd$. The total change in energy of an isolated system is $\Delta E_{tot} = 0$, where $\Delta E_{tot} = \Delta K + \Delta U + \Delta E_{th}$, and ΔE_{th} denotes the non-conservative energy of the toboggan-hill system (in this case, friction).

SIMPLIFY:

(a) With $K_i = 0$ (assuming $v_0 = 0$) and $U_f = 0$ (taking the bottom to be $h = 0$):

$$K_f = U_i \;\Rightarrow\; \frac{1}{2}mv_f^{\,2} = mgh \;\Rightarrow\; v_f = \sqrt{2gh}$$

(b) The steepness does not affect the final speed, in a system with only conservative forces, the distance traveled is not used when conservation of mechanical energy is considered.

(c) With friction considered, then for the toboggan-hill system,

$$\Delta E = \Delta K + \Delta U + \Delta E_{th} = 0 \;\Rightarrow\; \Delta K = -\Delta U - \Delta E_{th} \;\Rightarrow\; K_f = U_i - \Delta E_{th} = mgh - \mu_k Nd$$

The normal force *N* is given by $N = mg\cos\theta$, while on the hill. With $d = h/\sin\theta$,

$$K_f = mgh - \mu_k\left(mg\cos\theta\right)\left(\frac{h}{\sin\theta}\right) = mgh\left(1 - \mu_k\cot\theta\right).$$

The steepness of the hill does affect K_f and therefore v at the bottom of the hill.

CALCULATE:

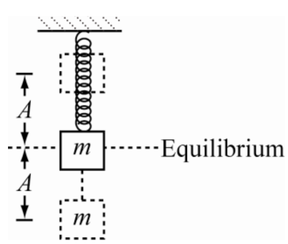
(a) $v_f = \sqrt{2\left(9.81\text{ m/s}^2\right)\left(40.0\text{ m}\right)} = 28.01\text{ m/s}$

ROUND: Since *h* has three significant figures, $v = 28.0$ m/s.

DOUBLE-CHECK: This is a very fast, but not unrealistic speed for the toboggan to achieve.

6.48. **THINK:** The block's mass is $m = 0.773$ kg, the spring constant is $k = 239.5$ N/m and the amplitude is $A = 0.551$ m. The block oscillates vertically. Determine the speed v of the block when it is at $x = 0.331$ m from equilibrium.

SKETCH:



RESEARCH: The force of gravity in this system displaces the equilibrium position of the hanging block by *mg*/*k*. Since the distance from equilibrium is given, the following equation can be used to determine *v*:
$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2.$

SIMPLIFY: $v=\sqrt{\frac{k}{m}\left(A^2-x^2\right)}$

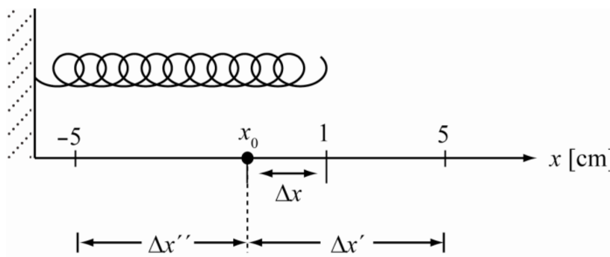
CALCULATE: $v=\sqrt{\frac{239.5\text{ N/m}}{0.773\text{ kg}}\left((0.551\text{ m})^2-(0.331\text{ m})^2\right)}=7.7537\text{ m/s}$

ROUND: The least precise value has three significant figures, so round the answer to three significant figures: v = 7.75 m/s.

DOUBLE-CHECK: This is a reasonable speed for the block on the spring.

6.49. **THINK:** It is known that $k=$ 10.0 N/cm and $\Delta x=1.00$ cm. Determine (a) the energy needed to further stretch the spring to $\Delta x'=5.00$ cm and (b) the energy needed to compress the spring from $\Delta x'=5.00$ cm to $\Delta x''=-5.00$ cm.

SKETCH:



RESEARCH: Assume the spring is stationary at all positions given above. The energy required to stretch the spring is the work applied to the spring, W_a, and $W_a=-W_s$ for $\Delta k=0$. It is known that $W_s=\left[\left(kx_i^2\right)/2\right]-\left[\left(kx_f^2\right)/2\right]$.

SIMPLIFY: $W_a=-W_s=-\left[\left(kx_i^2\right)/2\right]+\left[\left(kx_f^2\right)/2\right]=k\left(x_f^2-x_i^2\right)/2$

CALCULATE:

(a) $W_a=(10.0\text{ N/cm})\left((5.00\text{ cm})^2-(1.00\text{ cm})^2\right)/2=120.\text{ N cm}=1.20\text{ J}$

(b) $W_a=(10.0\text{ N/cm})\left((5.00\text{ cm})^2-(-5.00\text{ cm})^2\right)/2=0\text{J}$

ROUND: With three significant figures in each given value, (a) $W_a=1.20$ J and (b) $W_a=0.$ Take this zero to be precise.

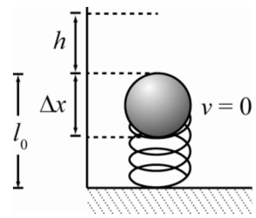
DOUBLE-CHECK:

(a) W_a should be positive because energy must be transferred to the spring to stretch it further from equilibrium.

(b) The spring is the same distance from the equilibrium point, so the net energy transferred to the spring must be zero.

6.50. **THINK:** The mass of the ball is m = 5.00 kg. The initial height is h = 3.00 m. The initial speed is $v_0=5.00$ m/s. The spring constant is k = 1600. N/m. The final speed of the ball is zero. Determine (a) the maximum compression Δx of the spring and (b) the total work done on the ball while the spring compresses. The spring is initially at equilibrium, so the height given is the height above the equilibrium point of the spring.

SKETCH:



RESEARCH:

(a) There are no non-conservative forces, so $\Delta K = -\Delta U$, $U_s = (kx^2)/2$, $U_g = mgh$ and $K = (mv^2)/2$.

(b) Use the work-kinetic energy theorem to find the net work done on the ball while the spring compresses Δx by $W_{net} = \Delta K$.

SIMPLIFY:

(a) $\Delta K = -\Delta U$ so $K_f - K_i = U_{si} - U_{sf} + U_{gi} - U_{gf}$. Note that the equilibrium position of the spring is l_0. Since K_f and U_{si} are zero, $0 - K_i = 0 - U_{sf} + U_{gi} - U_{gf}$, and $-\frac{1}{2}mv_0^2 = -\frac{1}{2}k(\Delta x)^2 + mg(l_0 + h) - mg(l_0 - \Delta x)$, which simplifies to $-\frac{1}{2}mv_0^2 = -\frac{1}{2}k(\Delta x)^2 + mgh + mg\Delta x$ and subsequently $\frac{1}{2}k(\Delta x)^2 - mg\Delta x - \frac{1}{2}mv_0^2 - mgh = 0$.

Solving the quadratic equation gives

$$\Delta x = \frac{mg \pm \sqrt{(-mg)^2 + 2k\left(\frac{1}{2}mv_0^2 + mgh\right)}}{k} = \frac{mg \pm \sqrt{(mg)^2 + mk(v_0^2 + 2gh)}}{k}.$$

(b) $W_{net} = \Delta K = -\Delta U = -\Delta U_s - \Delta U_g$

$$\begin{aligned} W_{net} &= U_{si} - U_{sf} + U_{gi} - U_{gf} \\ &= 0 - \frac{1}{2}k(\Delta x)^2 + mgl_0 - mg(l_0 - \Delta x) \\ &= -\frac{1}{2}k(\Delta x)^2 + mg\Delta x \end{aligned}$$

CALCULATE:

(a)
$$\Delta x = \frac{(5.00\text{ kg})(9.81\text{ m/s}^2)}{1600.\text{ N/m}} \pm \frac{\sqrt{(5.00\text{ kg})^2(9.81\text{ m/s}^2)^2 + (1600.\text{ N/m})(5.00\text{ kg})\left((5.00\text{ m/s})^2 + 2(9.81\text{ m/s}^2)(3.00\text{ m})\right)}}{1600.\text{ N/m}}$$

$$= 0.54349\text{ m},\ -0.48218\text{ m}$$

Since Δx is defined as a positive distance (not a displacement), the solution must be positive. Take $\Delta x = 0.54349$ m.

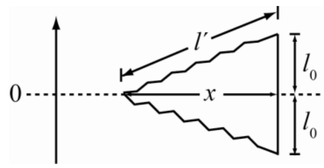
(b) $W_{net} = -\frac{1}{2}(1600.\text{ N/m})(0.54349\text{ m})^2 + (5.00\text{ kg})(9.81\text{ m/s}^2)(0.54349\text{ m}) = -209.6\text{ J}$

ROUND: Since the least precise value given in the question has three significant figures, both answers will have three significant figures: $\Delta x = 0.543$ m and $W_{net} = -210.$ J.

DOUBLE-CHECK: Δx should be positive. Relative to the height, *h*, the value of Δx is reasonable. Because the net work is negative, and since $|\Delta U_s| > |\Delta U_g|$ for the distance Δx, the clay ball does positive work on the spring and the spring does negative work on the clay ball. This makes sense for spring compression.

6.51. **THINK:** The spring constant for each spring is k = 30.0 N/m. The stone's mass is $m =$ 1.00 kg. The equilibrium length of the springs is $l_0 = 0.500$ m. The displacement to the left is $x =$ 0.700 m. Determine the system's total mechanical energy, E_{mec} and (b) the stone's speed, v, at x = 0.

SKETCH:



Note: The sketch is a side view. The word "vertical" means that the springs are oriented vertically above the ground. The path the stone takes while in the slingshot is completely horizontal so that gravity is neglected.

RESEARCH:

(a) In order to determine E_{mec}, consider all kinetic and potential energies in the system. Since the system is at rest, the only form of mechanical energy is spring potential energy, $U_s = \left(kx^2\right)/2$.

(b) By energy conservation, ΔE_{mec} (no non-conservative forces). v can be determined by considering $\Delta E_{\text{mec}} = 0$.

SIMPLIFY:

(a) $E_{\text{mec}} = K + U = U_s = U_{s1} + U_{s2} = \frac{1}{2}k_1\left(l_0 - l'\right)^2 + \frac{1}{2}k_2\left(l_0 - l'\right)^2 = k\left(l_0 - l'\right)^2$. To determine l', use the Pythagorean theorem, $l' = \sqrt{l_0^2 + x^2}$. Then, $E_{\text{mec}} = k\left(l_0 - \sqrt{l_0^2 + x^2}\right)^2$.

(b) As the mechanical energy is conserved, $E_{\text{mec f}} = E_{\text{mec i}}$ so $K_{\text{f}} + U_{s\text{f}} = E_{\text{mec}}$ (with $U_{\text{f}} = 0$), and therefore $K_{\text{f}} = E_{\text{mec}}$. Solving the equation for kinetic energy, $\frac{1}{2}mv^2 = E_{\text{mec}} \Rightarrow v = \sqrt{2E_{\text{mec}}/m}$.

CALCULATE:

(a) $E_{\text{mec}} = 30.0 \text{ N/m}\left(0.500 \text{ m} - \sqrt{\left(0.500 \text{ m}\right)^2 + \left(0.700 \text{ m}\right)^2}\right)^2 = 3.893 \text{ J}$

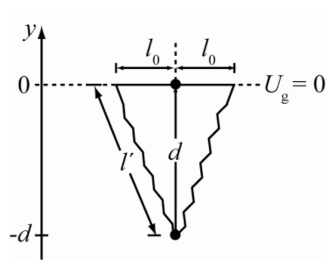
(b) $v = \sqrt{2\left(3.893 \text{ J}\right)/1.00 \text{ kg}} = 2.790 \text{ m/s}$

ROUND: Since all of the given values have three significant figures, the results should be rounded to $E_{\text{mec}} = 3.89$ J and v = 2.79 m/s.

DOUBLE-CHECK: The values are reasonable considering the small spring constant.

6.52. **THINK:** The spring constant for each spring is k = 30.0 N/m. The stone's mass is m = 0.100 kg. The equilibrium length of each spring is $l_0 = 0.500$ m. The initial vertical displacement is d = 0.700 m. Determine (a) the total mechanical energy, E_{mec} and (b) the stone's speed, v, when it passes the equilibrium point.

SKETCH:



RESEARCH:
(a) To determine E_{mec}, all forms of kinetic and potential energy must be calculated for the system. Note that initially $K = 0$. Use the equations $U_s = \left(kx^2\right)/2$ and $U_g = mgh$.
(b) As there are no non-conservative forces, E_{mec} is conserved. The speed, v, can be determined from $E_{\text{mec f}} = E_{\text{mec i}}$, using $K = \left(mv^2\right)/2$.
SIMPLIFY:
(a) $E_{\text{mec}} = K + U = U_g + U_{s1} + U_{s2} = mg(-d) + 2\left(\frac{1}{2}k\left(l' - l_0\right)^2\right)$. Note $l' = \sqrt{l_0^2 + d^2}$ from Pythagorean's theorem. Then, $E_{\text{mec}} = k\left(\sqrt{l_0^2 + d^2} - l_0^2\right)^2 - mgd$.
(b) $E_{\text{mec i}} = E_{\text{mec f}} = E_{\text{mec}}$ so $K_f = E_{\text{mec}}$ (as $U_{\text{gf}} = U_{\text{sf}} = 0$) and therefore $\frac{1}{2}mv^2 = E_{\text{mec}} \Rightarrow v = \sqrt{2E_{\text{mec}}/m}$.

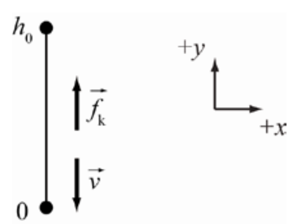
CALCULATE:
(a) $E_{\text{mec}} = (30.0 \text{ N/m})\left(\sqrt{(0.500 \text{ m})^2 + (0.700 \text{ m})^2} - 0.500 \text{ m}\right)^2 - (0.100 \text{ kg})\left(9.81 \text{ m/s}^2\right)(0.700 \text{ m})$

$= 3.893 \text{ J} - 0.6867 \text{ J}$

$= 3.206 \text{ J}$

(b) $v = \sqrt{2(3.206 \text{ J})/(0.100 \text{ kg})} = 8.0075 \text{ m/s}$
ROUND: As each given value has three significant figures, the results should be rounded to $E_{\text{mec}} = 3.21 \text{ J}$ and $v = 8.01$ m/s.
DOUBLE-CHECK: E_{mec} is decreased by the gravitational potential energy. The stone's speed is reasonable considering its small mass.

6.53. **THINK:** The mass of the man is $m = 80.0$ kg. His initial height is $h_0 = 3.00$ m. The applied frictional force is $f_k = 400.$ N. His initial speed is $v_0 = 0$. What is his final speed, v?
SKETCH:



RESEARCH: In an isolated system, the total energy is conserved. $\Delta E_{\text{tot}} = \Delta K + \Delta U + \Delta E_{\text{th}} = 0$. Using $K = \left(mv^2\right)/2$, $U_g = mgh_0$ and $\Delta E_{\text{th}} = f_k d$, v can be determined.
SIMPLIFY: Note $K_i = U_{\text{gf}} = 0$. Then, $\Delta K + \Delta U + \Delta E_{\text{th}} = 0 \Rightarrow K_f - U_{\text{gi}} + \Delta E_{\text{th}} = 0$. Note that the force of friction acts over the length of the pole, h_0. Then, $\frac{1}{2}mv^2 - mgh_0 + f_k h_0 = 0 \Rightarrow v = \sqrt{2\left(gh_0 - f_k h_0/m\right)} = \sqrt{2h_0\left(g - f_k/m\right)}$.
CALCULATE: $v = \sqrt{2\left(\left(9.81 \text{ m/s}^2\right)(3.00 \text{ m}) - (400. \text{ N})(3.00 \text{ m})/(80.0 \text{ kg})\right)}$

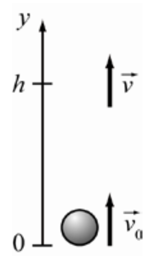
$= \sqrt{2\left(29.43 \text{ m}^2/\text{s}^2 - 15.0 \text{ m}^2/\text{s}^2\right)}$

$= 5.372 \text{ m/s}$
ROUND: With three significant figures in each given value, the result should be rounded to $v = 5.37$ m/s.

DOUBLE-CHECK: This velocity is less than it would be if the man had slid without friction, in which case v would be $\sqrt{2gh_0} \approx 8$ m/s.

6.54. **THINK:** The ball's mass is $m = 0.100$ kg. The initial speed is $v_0 = 10.0$ m/s. The final height is $h = 3.00$ m and the final speed is $v = 3.00$ m/s. Determine the fraction of the original energy lost to air friction. Note that the initial height is taken to be zero.

SKETCH:



RESEARCH: For an isolated system, $\Delta E_{\text{tot}} = \Delta K + \Delta U + \Delta E_{\text{other}} = 0$. The fraction that must be determined is as follows: $\dfrac{\Delta E_{\text{friction}}}{E_{\text{initial}}} = \dfrac{\Delta E_{\text{f}}}{E_{\text{i}}}$.

SIMPLIFY: $\Delta K + \Delta U + \Delta E_{\text{f}} = 0$. Note $U_i = 0$. This means that

$$\Delta E_{\text{f}} = -\Delta K - \Delta U = -\left(\frac{1}{2}m\left(v^2 - v_0^{\ 2}\right) + mgh\right) \text{ and } E_i = K_i + U_i = K_i = \frac{1}{2}mv_0^{\ 2}.$$

Then, $\dfrac{\Delta E_{\text{f}}}{E_{\text{i}}} = \dfrac{\frac{1}{2}m\left(v_0^{\ 2} - v^2 - 2gh\right)}{\frac{1}{2}mv_0^{\ 2}} = \dfrac{\left(v_0^{\ 2} - v^2 - 2gh\right)}{v_0^{\ 2}}$.

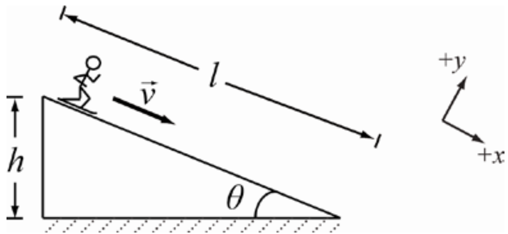
CALCULATE: $\dfrac{\Delta E_{\text{f}}}{E_{\text{i}}} = \dfrac{(10.0 \text{ m/s})^2 - (3.00 \text{ m/s})^2 - 2\left(9.81 \text{ m/s}^2\right)(3.00 \text{ m})}{(10.0 \text{ m/s})^2} = 0.3214$

ROUND: Each given value has three significant figures, so the result should be rounded as $\Delta E_{\text{f}} = 0.321E_{\text{i}}$. The final answer is 32.1% of E_i is lost to air friction.

DOUBLE-CHECK: If there were no friction and the ball started upward with an initial speed of $v_0 = 10$ m/s, its speed at a height of 3 m would be using kinematics $v = \sqrt{v_0^2 - 2gh} = \sqrt{(10 \text{ m/s})^2 - 2\left(9.81 \text{ m/s}^2\right)(3 \text{ m})} = 6.41$ m/s. This corresponds to a mechanical energy of $E = \frac{1}{2}(0.1 \text{ kg})(6.41 \text{ m/s})^2 + (0.1 \text{ kg})\left(9.81 \text{ m/s}^2\right)(3 \text{ m}) = 5.00$ J. The ball actually had a mechanical energy of $E = \frac{1}{2}(0.1 \text{ kg})(3 \text{ m/s})^2 + (0.1 \text{ kg})\left(9.81 \text{ m/s}^2\right)(3 \text{ m}) = 3.93$ J, which corresponds to a 32.1% loss, which agrees with the result using energy concepts.

6.55. **THINK:** The skier's mass is m = 55.0 kg. The constant speed is v = 14.4 m/s. The slope length is $l = 123.5$ m and the angle of the incline is $\theta = 14.7°$. Determine the mechanical energy lost to friction, ΔE_{th}.

SKETCH:



RESEARCH: The skier and the ski slope form an isolated system. This implies that $\Delta E_{\text{tot}} = \Delta K + \Delta U + \Delta E_{\text{th}} = 0$. Note that $\Delta K = 0$ since v is constant. Use the equation $U = mgh$, where the height of the ski slope can be found using trigonometry: $h = l\sin\theta$.

SIMPLIFY: At the bottom of the slope, $U_{\text{f}} = 0$. Then,

$$\Delta E_{\text{th}} = -\Delta U = -\left(U_{\text{f}} - U_{\text{i}}\right) = U_{\text{i}} = mgh = mgl\sin\theta.$$

CALCULATE: $\Delta E_{\text{th}} = (55.0 \text{ kg})\left(9.81 \text{ m/s}^2\right)(123.5 \text{ m})\sin 14.7° = 16909 \text{ J}$

ROUND: With three significant figures in m, g and θ, the result should be rounded to $\Delta E_{\text{th}} = 16.9 \text{ kJ}$.

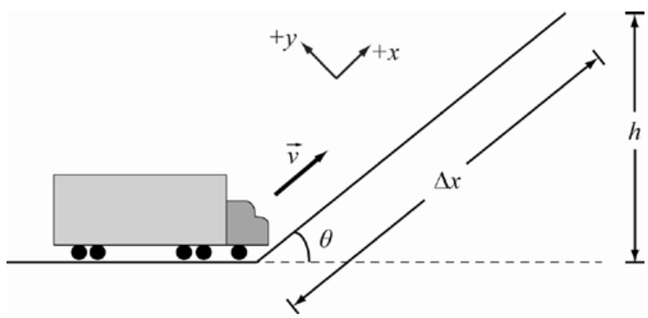
DOUBLE-CHECK: If this energy had been transformed completely to kinetic energy (no friction), and if the skier had started from rest, their final velocity would have been 24.8 m/s at the bottom of the slope. This is a reasonable amount of energy transferred to thermal energy generated by friction.

6.56. **THINK:** The truck's mass is m = 10,212 kg. The initial speed is

$$v_0 = 61.2 \text{ mph}\left(\frac{1609.3 \text{ m}}{1 \text{ mile}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right) = 27.3581 \text{ m/s}.$$

The incline angle is $\theta = 40.15°$ and the coefficient of friction is $\mu_{\text{k}} = 0.634$. Determine the distance traveled along the incline, Δx, before the truck stops (when $v = 0$).

SKETCH:



RESEARCH: The truck and the gravel incline form an isolated system. Use energy conservation to determine Δx. The initial energy is purely kinetic, $K = \left(mv^2\right)/2$. The final energies are thermal, $\Delta E_{\text{th}} = f_{\text{k}}d$ and gravitational potential, $U = mgh$.

SIMPLIFY:

$$\begin{aligned}
\Delta E_{\text{tot}} &= 0 \\
\Delta K + \Delta U + \Delta E_{\text{th}} &= 0 \\
-K_{\text{i}} + U_{\text{f}} + \Delta E_{\text{th}} &= 0 \\
-\frac{1}{2}mv_0^{\,2} + mgh + f_{\text{k}}\Delta x &= 0 \\
-\frac{1}{2}mv_0^{\,2} + mg\Delta x\sin\theta + \mu_{\text{k}}N\Delta x &= 0
\end{aligned}$$

Note that $N = mg\cos\theta$ on the incline. This gives:

$$-\frac{1}{2}mv_0^{\ 2} + mg\,\Delta x\sin\theta + \mu_k mg\cos\theta\,\Delta x = 0$$

$$\Delta x\left(g\sin\theta + \mu_k g\cos\theta\right) = \frac{1}{2}v_0^{\ 2}$$

$$\Delta x = \frac{v_0^{\ 2}}{2g\left(\sin\theta + \mu_k\cos\theta\right)}$$

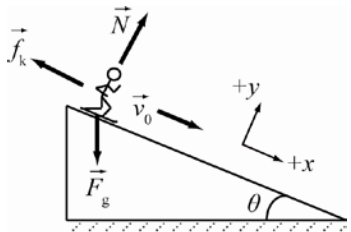
CALCULATE: $\Delta x = \dfrac{\left(27.3581\text{ m/s}\right)^2}{2\left(9.81\text{ m/s}^2\right)\left(\sin\left(40.15°\right) + 0.634\cos\left(40.15°\right)\right)} = 33.777\text{ m}$

ROUND: With three significant figures in v_0, the result should be rounded to $\Delta x = 33.8$ m.

DOUBLE-CHECK: This is a reasonable stopping distance given the incline angle and high coefficient of friction.

6.57. **THINK:** The snowboarder's mass is m = 70.1 kg. The initial speed is $v_0 = 5.10$ m/s. The slope angle is $\theta = 37.1°$. The coefficient of kinetic friction is $\mu_k = 0.116$. Determine the net work, W_{net} done on the snowboarder in the first t = 5.72 s.

SKETCH:



RESEARCH: It is known that $W_{net} = \Delta K$. By considering the forces acting on the skier, and assuming constant acceleration, v_f can be determined at t = 5.72 s. Use $f_k = \mu_k N$, $F_{x\text{ net}} = \sum F_x$ and $v = v_0 + at$.

SIMPLIFY: In the x-direction (along the slope), $F_{x\text{ net}} = F_{gx} - f_k$. Since $N = mg\cos\theta$, the force equation is expanded to

$$ma_{net} = mg\sin\theta - \mu_k mg\cos\theta \Rightarrow a_{net} = g\left(\sin\theta - \mu_k\cos\theta\right).$$

Then, the velocity is given by the formula $v = v_0 + at = v_0 + g\left(\sin\theta - \mu_k\cos\theta\right)t$, and $W_{net} = K_f - K_i = \frac{1}{2}m\left(\left(v_0 + g\left(\sin\theta - \mu_k\cos\theta\right)t\right)^2 - v_0^{\ 2}\right)$.

CALCULATE:

$$\begin{aligned}
W_{net} &= \frac{1}{2}(70.1\text{ kg})\left(\left(5.10\text{ m/s} + \left(9.81\text{ m/s}^2\right)\left(\sin\left(37.1°\right) - 0.116\cos\left(37.1°\right)\right)\left(5.72\text{ s}\right)\right)^2 - \left(5.10\text{ m/s}\right)^2\right) \\
&= \frac{1}{2}(70.1\text{ kg})\left(\left(5.10\text{ m/s} + 28.66\text{ m/s}\right)^2 - \left(5.10\text{ m/s}\right)^2\right) \\
&= \frac{1}{2}(70.1\text{ kg})\left(1139.5\text{ m}^2/\text{s}^2 - 26.01\text{ m}^2/\text{s}^2\right) \\
&= 39027.5\text{ J}
\end{aligned}$$

ROUND: Because the m and v_0 have three significant figures, the result should be rounded to $W_{net} = 39.0$ kJ.

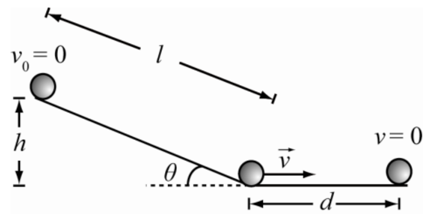
DOUBLE-CHECK: This is a reasonable energy required to change the snowboarder's speed.

6.58. **THINK:** The ball's mass is m = 0.0459 kg. The length of the bar is $l = 30.0\text{ in}\left(0.0254\text{ m/in}\right) = 0.762\text{ m}.$ The incline angle is $\theta = 20.0°.$ The distance traveled on the green is

$$d = 11.1\text{ ft}\left(\frac{0.3048\text{ m}}{1\text{ ft}}\right) = 3.38328\text{ m}.$$

Determine the coefficient of friction between the green and the ball. Assume the bar is frictionless.

SKETCH:



RESEARCH: The ball-bar-green system is isolated, so $\Delta E_{\text{tot}} = 0.$ Take the initial point to be when the ball starts to roll down the bar, and the final point where the ball has stopped rolling on the green after traveling a distance, d , on the green. $K_{\text{i}} = K_{\text{f}} = U_{\text{f}} = 0$. Then, $\Delta K + \Delta U + \Delta E_{\text{th}} = 0$, with $U = mgh$ and $\Delta E_{\text{th}} = f_{\text{k}}d$ can be used to determine μ_{k}.

SIMPLIFY: $\Delta K + \Delta U + \Delta E_{\text{th}} = 0 \Rightarrow -U_{\text{i}} + \Delta E_{\text{th}} = 0 \Rightarrow \Delta E_{\text{th}} = U_{\text{i}} \Rightarrow f_{\text{k}}d = mgh \Rightarrow \mu_{\text{k}}mgd = mgl\sin\theta$

$$\Rightarrow \mu_{\text{k}} = \frac{l\sin\theta}{d}$$

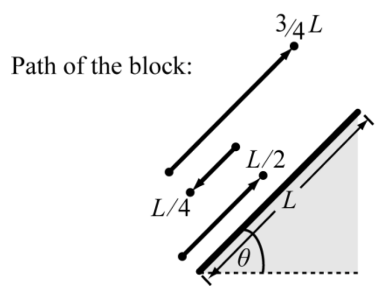
CALCULATE: $\mu_{\text{k}} = \dfrac{\left(0.762\text{ m}\right)\sin\left(20.0°\right)}{3.38328\text{ m}} = 0.0770316$

ROUND: With three significant figures in each given value, the result should be rounded to $\mu_{\text{k}} = 0.0770.$

DOUBLE-CHECK: μ_{k} has no units and has a small value, which is reasonable for golf greens.

6.59. **THINK:** The block's mass is m = 1.00 kg. The length of the plank is L = 2.00 m. The incline angle is $\theta = 30.0°.$ The coefficient of kinetic friction is $\mu_{\text{k}} = 0.300.$ The path taken by the block is $L/2$ upward, $L/4$ downward, then up to the top of the plank. Determine the work , W_b, done by the block against friction.

SKETCH:



RESEARCH: Friction is a non-conservative force. The work done by friction, W_{f}, is therefore dependent on the path. It is known that $W_{\text{f}} = -f_{\text{k}}d$, and with $W_{\text{b}} = -W_{\text{f}}$, the equation is $W_{\text{b}} = f_{\text{k}}d.$ The total path of the block is $d = L/2 + L/4 + 3L/4 = 1.5L.$

SIMPLIFY: $W_{\text{b}} = f_{\text{k}}d = \mu_{\text{k}}Nd = \mu_{\text{k}}mg\left(\cos\theta\right)d$ ($N = mg\cos\theta$ on the incline)

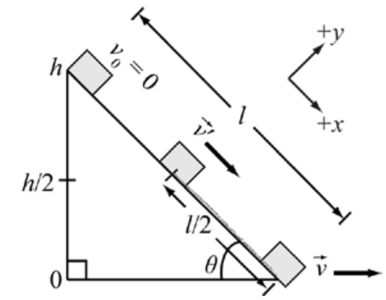
CALCULATE: $W_b = \left(0.300\right)\left(1.00\text{ kg}\right)\left(9.81\text{ m/s}^2\right)\cos\left(30.0°\right)\left(1.50\left(2.00\text{ m}\right)\right) = 7.646\text{ J}$

ROUND: Each given value has three significant figures, so the result should be rounded to $W_{\text{b}} = 7.65$ J.

DOUBLE-CHECK: This is a reasonable amount of work done against friction considering the short distance traveled.

6.60. **THINK:** The block's mass is m = 1.00 kg. The initial velocity is $v_0 = 0$ m/s. The incline's length is $l = 4.00$ m. The angle of the incline is $\theta = 45.0°$. The coefficient of friction is $\mu_k = 0.300$ for the lower half of the incline. Determine (a) the block's speed just before the rough section, v', and (b) the block's speed at the bottom, v.

SKETCH:



RESEARCH: Energy is conserved in the block/incline system. Recall $K = (mv^2)/2$, $U = mgh$ and $\Delta E_{th} = f_k d = \mu_k N d$.

(a) With no friction, $\Delta K + \Delta U = 0$.

(b) With friction, $\Delta K + \Delta U + \Delta E_{th} = 0$.

SIMPLIFY:

(a) With $v_0 = 0$ m/s and $K_i = 0$, $K_f - K_i + U_f - U_i = 0$ becomes $K_f = U_i - U_f$.

$$\frac{1}{2}mv'^2 = mgh - mg\left(\frac{h}{2}\right) = \frac{mgh}{2} \Rightarrow \frac{1}{2}v'^2 = \frac{1}{2}gl\sin\theta \Rightarrow v' = \sqrt{gl\sin\theta}$$

(b) Consider the initial point to be halfway down l (when the velocity is v'), and the final point where $U_f = 0$: $\Delta K + \Delta U + \Delta E_{th} = 0 \Rightarrow K_f - K_i + U_f - U_i + \Delta E_{th} = 0 \Rightarrow K_f = K_i + U_i - \Delta E_{th}$

$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + mg\left(\frac{h}{2}\right) - f_k d \Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{1}{2}mgl\sin\theta - \mu_k mg\cos\theta\left(\frac{l}{2}\right)$ since $N = mg\cos\theta$ and $f_k = \mu_k N$. So $v^2 = v'^2 + gl\sin\theta - \mu_k lg\cos\theta$. Since $v'^2 = gl\sin\theta$, $v = \sqrt{gl(2\sin\theta - \mu_k\cos\theta)}$.

CALCULATE:

(a) $v' = \sqrt{(9.81 \text{ m/s}^2)(4.00 \text{ m})\sin(45.0°)} = 5.2675$ m/s

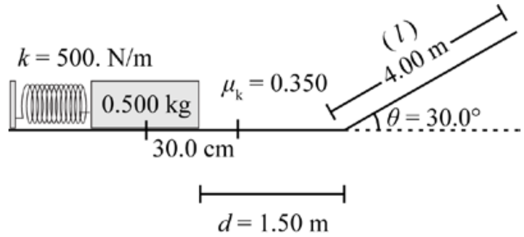
(b) $v = \sqrt{(9.81 \text{ m/s}^2)(4.00 \text{ m})(2\sin(45.0°) - 0.300\cos(45.0°))} = 6.868$ m/s

ROUND: With l having three significant figures, the results should be rounded to $v' = 5.27$ m/s and $v = 6.87$ m/s.

DOUBLE-CHECK: In the complete absence of friction, the speed at the bottom would be $v = \sqrt{2gh} = 7.45$ m/s. The velocity calculated in part (b) is less than this due to the thermal energy dissipated by friction.

6.61. **THINK:** The spring constant is k = 500. N/m. The mass is m = 0.500 kg. The spring compression is $x = 30.0$ cm. The length of the plane is $l = 4.00$ m. The incline angle is $\theta = 30.0°$. The coefficient of kinetic friction is $\mu_k = 0.350$. With the spring compressed, the mass is 1.50 m from the bottom of the inclined plane. Determine (a) the speed of the mass at the bottom of the inclined plane, (b) the speed of the mass at the top of the inclined plane, and (c) the total work done by friction from beginning to end.

SKETCH:



RESEARCH:

(a) The elastic potential energy is $U_{\text{spring}} = \left(kx^2\right)/2$. The mass loses energy $W_{\text{f}} = -F_{\text{f}}d = -\mu_k mgd$ due to friction. Therefore, the kinetic energy at the bottom is given by $K_{\text{b}} = \frac{1}{2}mv_{\text{b}}^2 = \frac{1}{2}kx^2 - \mu_{\text{k}}mgd.$

(b) To reach the top of the incline, the gravitational potential energy must also be considered: $\Delta U_{\text{gravity}} = U_{\text{top}} - U_{\text{bottom}}$. Since the plane has length, *l,* and incline angle, θ, $\Delta U_{\text{gravity}} = mgl\sin\theta.$ The kinetic energy at the top (and thus the speed) can then be calculated by subtracting the gravitational potential energy and work due to friction from the kinetic energy at the bottom: $K_{\text{top}} = K_{\text{b}} - \mu_{\text{k}}mgl\cos\theta - mgl\sin\theta.$

(c) The total work due to friction is given by $W_{\text{f}} = -F_{\text{f}}\left(d + l\right).$

SIMPLIFY:

(a) $K_{\text{b}} = \frac{1}{2}mv_{\text{b}}^2 = \frac{1}{2}kx^2 - \mu_{\text{k}}mgd \Rightarrow v_b = \sqrt{\frac{kx^2}{m} - 2\mu_{\text{k}}gd}$

(b) $K_{\text{top}} = \frac{1}{2}mv_{\text{top}}^2 = K_{\text{b}} - \mu_{\text{k}}mgl\cos\theta - mgl\sin\theta \Rightarrow v_{\text{top}} = \sqrt{\frac{2}{m}\left[K_{\text{b}} - mgl\left(\mu_{\text{k}}\cos\theta + \sin\theta\right)\right]}$

(c) $W_{\text{f}} = -\mu_{\text{k}}mgd - \mu_{\text{k}}mg\left(\cos\theta\right)l$

CALCULATE:

(a) $v_{\text{b}} = \sqrt{\frac{\left(500.\text{ N/m}\right)\left(0.300\text{ m}\right)^2}{0.500\text{ kg}} - 2\left(0.350\right)\left(9.81\text{ m/s}^2\right)\left(1.50\text{ m}\right)} = 8.927\text{ m/s}$

$K_{\text{b}} = \frac{1}{2}\left(0.500\text{ kg}\right)\left(8.927\text{ m/s}\right)^2 = 19.92\text{ J}$

(b) $v_{\text{top}} = \sqrt{\frac{2}{0.500\text{ kg}}\left[\left(19.92\text{ J}\right) - \left(0.500\text{ kg}\right)\left(9.81\text{ m/s}^2\right)\left(4.00\text{ m}\right)\left(0.350\cos 30.0° + \sin 30.0°\right)\right]} = 4.08\text{ m/s}$

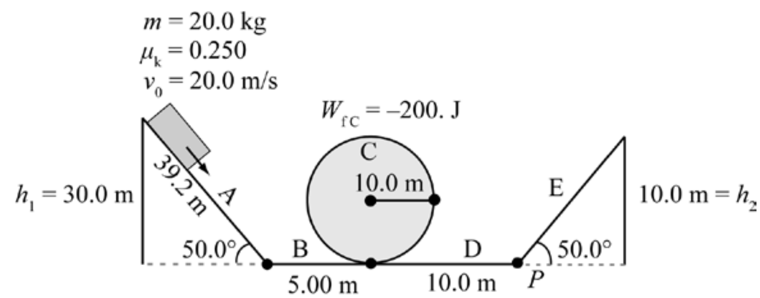
(c) $W_{\text{f}} = -\left(0.350\right)\left(0.500\text{ kg}\right)\left(9.81\text{ m/s}^2\right)\left(1.50\text{ m} + 4.00\cos\left(30.0°\right)\text{ m}\right) = -8.52\text{ J}$

ROUND: Rounding to three significant figures, $v_{\text{b}} = 8.93\text{ m/s}$, $v_{\text{top}} = 4.08\text{ m/s}$ and $W_{\text{f}} = -8.52\text{ J}.$

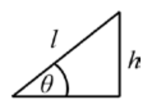
DOUBLE-CHECK: The results are reasonable for the given values.

6.62. **THINK:** Determine the speed of the sled at the end of the track or the maximum height it reaches if it stops before reaching the end. The initial velocity is $v_0 = 20.0$ m/s.

SKETCH:



RESEARCH: The total initial energy is given by $E_0 = K_0 - U_0$. When the sled reaches point *p* at the bottom of the second incline, it has lost energy due to friction given by $W_p = W_A + W_B + W_C + W_D$, where $W_A = A\mu_k mg\cos\theta$, $W_B = B\mu_k mg$, $W_C = 200.$ J, and $W_D = D\mu_k mg$. As the sled reaches point *p*, it has kinetic energy $K_p = E_0 - W_p$. In order for the sled to reach the end of the incline, it needs to have enough energy to cover the work due to friction as well as the gravitational potential energy at the top. Therefore, if $K_p > U_E + W_E$, then it does reach the top and the speed can be determined from the kinetic energy at the top: $K_{top} = K_p - P_E - W_E$. If $K_p < U_E + W_{fE}$, then it stops before reaching the top and the height the sled reaches can be determined by considering the gravitational potential energy equation: $U = mgh = K_p - W_{fE}$, where W_{fE} is the work due to friction for the section of the incline up to *h*. The height can be related to the distance covered on the incline by recalling that $h = l\sin\theta \Rightarrow l = h/\sin\theta$.



Therefore,

$$W_{fE} = \mu_k mg(\cos\theta)l = \mu_k mg\frac{(\cos\theta)h}{\sin\theta} = \mu_k mg(\cot\theta)h.$$

SIMPLIFY: It is convenient to evaluate the following terms separately:
E_0, W_A, W_B, W_C, W_D, U_E, W_E and W_{fE}.

$$E_0 = \frac{1}{2}mv_0^2 + mgh_1,\ U_E = mgh_2,\ W_E = E\mu_k mg\cos\theta = h_2\mu_k mg\cot\theta.$$

CALCULATE: $E_0 = \frac{1}{2}(20.0 \text{ kg})(20.0 \text{ m/s})^2 + (20.0 \text{ kg})(9.81 \text{ m/s}^2)(30.0 \text{ m}) = 9886 \text{ J}$

$W_A = (39.2 \text{ m})(0.250)(20.0 \text{ kg})(9.81 \text{ m/s}^2)\cos(50.0°) = 1236 \text{ J}$

$W_B = (5.00 \text{ m})(0.250)(20.0 \text{ kg})(9.81 \text{ m/s}^2) = 245 \text{ J}, \qquad W_C = 200. \text{ J}$

$W_D = (10.0 \text{ m})(0.250)(20.0 \text{ kg})(9.81 \text{ m/s}^2) = 491 \text{ J}, \quad U_E = (20.0 \text{ kg})(9.81 \text{ m/s}^2)(10.0 \text{ m}) = 1962 \text{ J}$

$W_E = (10.0 \text{ m})(0.250)(20.0 \text{ kg})(9.81 \text{ m/s}^2)\cot(50.0°) = 412 \text{ J}$

Therefore, $K_p = (9886 \text{ J}) - (1236 \text{ J}) - (245 \text{ J}) - 200. \text{ J} - (491 \text{ J}) = 7714 \text{ J}$, and
$U_E + W_E = (1962 \text{ J}) + (412 \text{ J}) = 2374 \text{ J}$. Therefore, since $K_p > U_E + W_E$, the sled will reach the top and have speed: $K_{top} = K_p - U_E - W_E \Rightarrow \frac{1}{2}mv_{top}^2 = K_p - U_E - W_E$

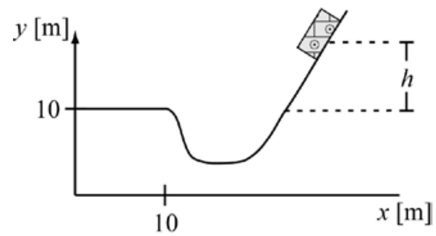
$$\Rightarrow v_{top} = \sqrt{\frac{2}{m}(K_p - U_E - W_E)} = \sqrt{\left(\frac{2}{20.0 \text{ kg}}\right)((7714 \text{ J}) - (2374 \text{ J}))} = 23.11 \text{ m/s}.$$

ROUND: Rounding to three significant figures, $v_{top} = 23.1$ m/s.

DOUBLE-CHECK: The fact that the sled reaches the top of the second ramp is reasonable given how much higher the second ramp is than the first. The value of the velocity is of the same order of magnitude as the initial velocity so it is reasonable.

6.63. **THINK:** The mass of the cart is 237.5 kg. The initial velocity is $v_0 = 16.5$ m/s. The surface is frictionless. Determine the turning point shown on the graph in the question, sketched below.

SKETCH:



RESEARCH: Since the system is conservative, $E_{tot} = \text{constant} = U_{max} = K_{max}$. Therefore, the kinetic energy at $x = 0$, $y = 10.$ m is the same as the kinetic energy whenever the track is at $y = 10.$ m again. Set $y = 10.$ m as the origin for gravitational potential energy. Therefore,

$$E_{tot} = K_{max} = \frac{mv_0^2}{2}.$$

This is the available energy to climb the track from $y = 10.$ m. The turning point is when $v = 0$ and

$$U_{max} = K_{max} \Rightarrow mgh = \frac{mv_0^2}{2}.$$

SIMPLIFY: $h = \dfrac{v_0^2}{2g}$, $y = 10.\text{ m} + h$

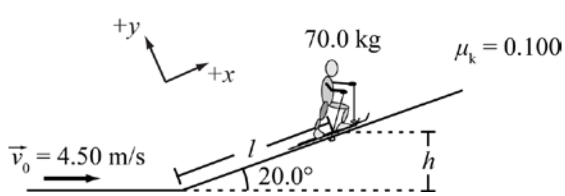
CALCULATE: $h = \dfrac{(16.5 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} = 13.9$ m, $y = 10.\text{ m} + 13.9\text{ m} = 23.9$ m

ROUND: Reading off the graph is accurate to the nearest integer, so round the value of y to 24 m.
Reading off the graph, the value of x at $y = 24$ m is $x = 42$ m.

DOUBLE-CHECK: It is reasonable that the cart will climb about 18 m with an initial velocity of $v_0 = 16.5$ m/s.

6.64. **THINK:** A 70.0 kg skier's initial velocity is $v_0 = 4.50$ m/s towards a $20.0°$ incline. Determine (a) the range up the incline if there is no friction and (b) the range up the incline if $\mu_k = 0.100$.

SKETCH:



RESEARCH:

(a) Since the system is conservative, $E_{tot} = K_{max} = U_{max} \Rightarrow (mv_0^2)/2 = mgh_1 = mgl_1 \sin\theta$.

(b) The work due to friction is determined by $W_f = F_f l_2 = \mu_k mgl_2 \cos\theta$. Therefore,

$$K_{bottom} = U_{top} - W_f.$$

SIMPLIFY:

(a) $\frac{1}{2}mv_0^2 = mgl_1 \sin\theta \Rightarrow l_1 = \frac{v_0^2}{2g\sin\theta}$

(b) $\frac{1}{2}mv_0^2 = mgl_2 \sin\theta + \mu_k mgl_2 \cos\theta \Rightarrow \frac{v_0^2}{2} = l_2 g(\sin\theta + \mu_k \cos\theta) \Rightarrow l_2 = \frac{v_0^2}{2g(\sin\theta + \mu_k \cos\theta)}$

CALCULATE:

(a) $l = \frac{(4.50 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)\sin(20.0°)} = 3.0177 \text{ m}$

(b) $l_2 = \frac{(4.50 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)(\sin(20.0°) + 0.100\cos(20.0°))} = 2.3672 \text{ m}$

ROUND: The final rounded answer should contain 3 significant figures:

(a) $l = 3.02$ m

(b) $l = 2.37$ m

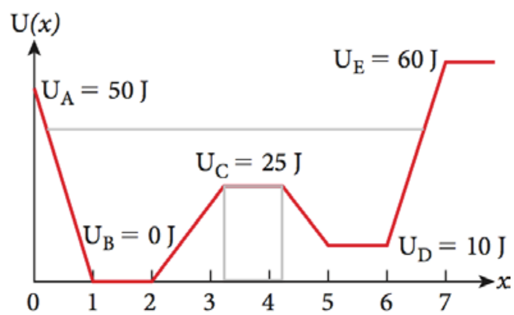
DOUBLE-CHECK: As expected, introducing friction into the system will decrease the available mechanical energy.

6.65. **THINK:** The particle has a total energy of $E_{\text{tot}} = 40.0$ J at its initial position and retains it everywhere. Thus we can draw a horizontal line (gray) for its total energy, approximately at 4/5 of the value of the potential energy at point A (= 50.0 J) for this value of the total energy. The locations of the turning points are here this horizontal line intersects the potential energy curve (red).

Further, we can determine the shape of the potential curve in a more analytical form. From the drawing we can clearly see that it is piecewise linear, falling from 50.0 J at x = 0 to 0 J at x = 1, rising from 0 J at x = 2 to 25.0 J at x = 3.25, falling again from 25.0 J at x = 4.25 to 10.0 J at x = 5, and finally rising from 10.0 J at x = 6 to 60.0 J at x = 7. (We have drawn in a gray rectangle; this way it is easier to see at what x-values the slopes change.)

The turning points are where v = 0, which is where the total energy is equal to the potential energy.

SKETCH:



RESEARCH: Assume a conservative system and $E_{\text{tot}} = K + U$.

(a) Consider the potential energy at the point x = 3 m and call it U_3:

$$E_{\text{tot}} = K + U \Rightarrow K_3 = E_{\text{tot}} - U_3, \text{ and } K_3 = \frac{mv_3^2}{2}.$$

(b) Similarly, $K_{4.5} = E_{\text{tot}} - U_{4.5}$, and $K_{4.5} = (mv_{4.5}^2)/2$.

(c) Since $E_{\text{tot}} = 40.0$ J at x = 4.00 and $U_C = 25.0$ J, then $E_{\text{tot}} - U_C = K_C$. This kinetic energy will become potential energy to reach the turning point.

SIMPLIFY:

(a) $\frac{1}{2}mv_3^2 = E_{\text{tot}} - U_3 \Rightarrow v_3 = \sqrt{\frac{2}{m}(E_{\text{tot}} - U_3)}$. U_3 is obtained from the graph.

(b) $v_{4.5} = \sqrt{\frac{2}{m}\left(E_{\text{tot}} - U_{4.5}\right)}$. $U_{4.5}$ is obtained from the graph.

(c) $E_{\text{tot}} - U_{\text{C}} = K_{\text{C}} = U_4$. Therefore, $U_{\text{turning}} = U_{\text{C}} + U_{\text{t}} = E_{\text{tot}}$.

CALCULATE:

(a) Interpolation between $x = 2$ and $x = 3.25$ yields

$$U(x) = U_{\text{C}}(x-2)/(3.25-2) \Rightarrow U_3 \equiv U(x{=}3) = (25.0 \text{ J})(3-2)/(3.25-2) = 20.0 \text{ J}$$

$$v_3 = \sqrt{\left(\frac{2}{0.200 \text{ kg}}\right)\left(40.0 \text{ J} - 20.0 \text{ J}\right)} = 14.14 \text{ m/s}$$

(b) Interpolation between $x = 4.25$ and $x = 5$ yields

$$U(x) = U_{\text{C}} - (U_{\text{C}} - U_{\text{D}})(x-4.25)/(5-4.25) \Rightarrow U_{4.5} \equiv U(x{=}4.5) = (25.0 \text{ J}) - (15.0 \text{ J})(4.5-4.25)/(0.75) = 20.0 \text{ J}$$

$$v_{4.5} = \sqrt{\left(\frac{2}{0.200 \text{ kg}}\right)\left(40.0 \text{ J} - 20.0 \text{ J}\right)} = 14.14 \text{ m/s}$$

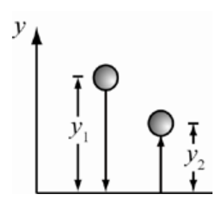
(c) Graphical interpolation between 0 and 1 and between 6 and 7 then results in turning points results in $x_L = 1 - E/U_A = 1 - (40.0 \text{ J})/(50.0 \text{ J}) = 0.2$ for the left turning point, and $x_R = 6 + (E - U_D)/(U_E - U_D) = 6 + (30.0 \text{ J})/(50.0 \text{ J}) = 6.6$ for the right one.

ROUND: Since we are reading data of a graph, we should probably round to two figures and state our results as $v_3 = v_{4.5} = 14$ m/s and $x_L = 0.2$ m and $x_R = 6.6$ m.

DOUBLE-CHECK: Our numerical findings for the turning points agree with our graphical estimation, within the uncertainties stated here.

6.66. **THINK:** The mass of the ball is m = 1.84 kg. The initial height is $y_1 = 1.49$ m and the second height is $y_2 = 0.87$ m. Determine the energy lost in the bounce.

SKETCH:



RESEARCH: Consider the changes in the potential energy from y_1 to y_2. The energy lost in the bounce is given by $U_1 - U_2$.

SIMPLIFY: $E_{\text{lost}} = mgy_1 - mgy_2 = mg\left(y_1 - y_2\right)$

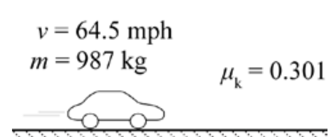
CALCULATE: $E_{\text{lost}} = \left(1.84 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right)\left(1.49 \text{ m} - 0.87 \text{ m}\right) = 11.2$ J

ROUND: Since the least precise value is given to two significant figures, the result is $E_{\text{lost}} = 11$ J.

DOUBLE-CHECK: The ball lost roughly half of its height, so it makes sense that it lost roughly half of its potential energy (which was about 27 J).

6.67. **THINK:** The mass of the car is m = 987 kg. The speed is v = 64.5 mph. The coefficient of kinetic friction is $\mu_{\text{k}} = 0.301$. Determine the mechanical energy lost.

SKETCH:



RESEARCH: Since all of the mechanical energy is considered in the form of kinetic energy, the energy lost is equal to the kinetic energy before applying the brakes. Using the conversion 1 mph is equal to 0.447 m/s, the speed can be converted to SI units. Convert the speed: $v=(64.5\text{ mph})\left(\frac{0.447\text{ m/s}}{1\text{ mph}}\right)=28.8\text{ m/s}.$

SIMPLIFY: $E_{\text{lost}}=\frac{1}{2}mv^2$

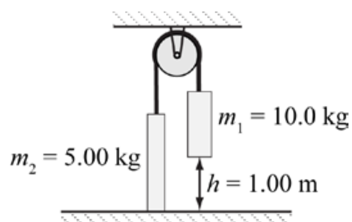
CALCULATE: $E_{\text{lost}}=\frac{1}{2}(987\text{ kg})(28.8\text{ m/s})^2=4.10\cdot 10^5\text{ J}$

ROUND: Rounding to three significant figures, $E_{\text{lost}}=4.10\cdot 10^5$ J.

DOUBLE-CHECK: For an object this massive, it is reasonable that it requires such a large amount of energy to stop it.

6.68. **THINK:** Two masses, $m_1=10.0\text{ kg}$ and $m_2=5.00\text{ kg}$ are attached to a frictionless pulley. The first mass drops h = 1.00 m. Determine (a) the speed of the 5.00 kg mass before the 10.0 kg mass hits the ground and (b) the maximum height of the 5.00 kg mass.

SKETCH:



RESEARCH:

(a) Since energy is conserved, $\Delta K=-\Delta U$. Since the masses are attached to each other, their speeds are the same before one touches the ground.

(b) When m_1 hits the ground, m_2 is at h = 1.00 m with a speed v. The kinetic energy for m_2 is then $(m_2v^2)/2$ and this is given to potential energy for a height above h = 1.00 m. Let h_t be the height where the potential and kinetic energies are equal. When the kinetic energies are equal, $U=K\ \Rightarrow\ m_2gh_t=(m_2v^2)/2\ \Rightarrow\ h_t=v^2/2g$. Therefore, the maximum height is $h_{\max}=h+h_t$.

SIMPLIFY:

(a)
$$K_{\text{f}}-K_{\text{i}}=U_{\text{i}}-U_{\text{f}}$$
$$\left(\frac{1}{2}m_1v^2+\frac{1}{2}m_2v^2\right)-0=(m_1gh+m_2gh)-0$$
$$(m_1+m_2)v^2=2gh(m_1-m_2)$$
$$v^2=2gh\left(\frac{m_1-m_2}{m_1+m_2}\right)$$
$$v=\pm\sqrt{2gh\left(\frac{m_1-m_2}{m_1+m_2}\right)}$$

(b) $h_{\max}=h+\frac{v^2}{2g}$

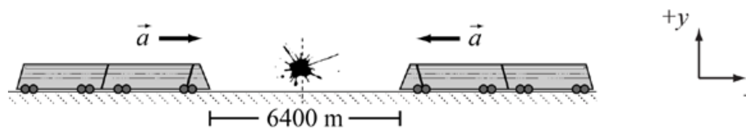
CALCULATE:

(a) $v=\sqrt{2(9.81\text{ m/s}^2)(1.00\text{ m})\left(\frac{10.0\text{ kg}-5.00\text{ kg}}{10.0\text{ kg}+5.00\text{ kg}}\right)}=2.557\text{ m/s}$

(b) $h_{max} = 1.00 \text{ m} + \dfrac{(2.557 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} = 1.333 \text{ m}$

ROUND: Rounding to three significant figures, $v = 2.56$ m/s and $h_{max} = 1.33$ m.
DOUBLE-CHECK: The calculated values have appropriate units and are of reasonable orders of magnitude for a system of this size.

6.69. **THINK:** The distance that each train covered is $\Delta x = 3200$ m. The weight of each train is $w = 1.2 \cdot 10^6$ N. Their accelerations have a magnitude of $a = 0.26 \text{ m/s}^2$, but are in opposite directions. Determine the total kinetic energy of the two trains just before the collision. The trains start from rest.
SKETCH:



RESEARCH: The total kinetic energy will be twice the kinetic energy for one train. With $K = (mv^2)/2$, m can be determined from $w = mg$ and v from $v^2 = v_0^2 + 2a\Delta x$.
SIMPLIFY: $m = w/g$. Then, $K_{tot} = 2K = 2((mv^2)/2) = w(2a\Delta x)/g$.

CALCULATE: $K_{tot} = \dfrac{2(1.2 \cdot 10^6 \text{ N})(0.26 \text{ m/s}^2)(3200 \text{ m})}{(9.81 \text{ m/s}^2)} = 2.035 \cdot 10^8 \text{ J}$

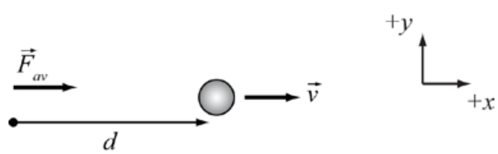
ROUND: With two significant figures in each given value, $K_{tot} = 2.0 \cdot 10^8$ J.
DOUBLE-CHECK: For such a horrific explosion, a very large kinetic energy is expected before impact.

6.70. **THINK:** The ball's mass is $m = 5.00 \text{ oz}(0.02835 \text{ kg/oz}) = 0.14175 \text{ kg}$. The final speed is

$$v = 90.0 \frac{\text{miles}}{\text{h}}\left(\frac{1609.3 \text{ m}}{1 \text{ mile}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right) = 40.2325 \text{ m/s}.$$

The distance traveled is $d = 2(28.0 \text{ in})\left(\dfrac{0.0254 \text{ m}}{1 \text{ in}}\right) = 1.4224$ m. Determine the average force, F_{av}.

SKETCH:



RESEARCH: There are no non-conservative forces in the system. So, $\Delta K = -\Delta U$. With F_{av} as a conservative force, the work it does is given by $W_c = -\Delta U$ and $W_c = \vec{F} \bullet \vec{d}$. From this, F_{av} can be determined.
SIMPLIFY: Note $\vec{F}_{av}$ and $\vec{d}$ are in the same direction, so $W_c = F_{av}d$ and $\Delta K = -\Delta U = W_c \Rightarrow K_f - K_i = W_c$. Since $K_i = 0$, $K_f = W_c$.

$$\frac{1}{2}mv^2 = F_{av}d \Rightarrow F_{av} = \frac{mv^2}{2d}$$

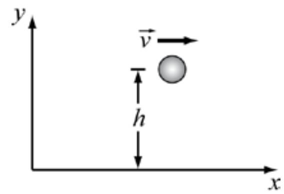
CALCULATE: $F_{av} = \dfrac{(0.14175 \text{ kg})(40.2325 \text{ m/s})^2}{2(1.4224 \text{ m})} = 80.654 \text{ N}$

ROUND: Since the values are given to three significant figures, $F_{av} = 80.7$ N.

DOUBLE-CHECK: This average force is equal to holding an object that has a mass of 14.8 kg ($m = F/g = (145\text{ N})/\left(9.81\text{ m/s}^2\right)$), so it is reasonable.

6.71. **THINK:** The mass of the ball is m = 1.50 kg. Its speed is v = 20.0 m/s and its height is h = 15.0 m. Determine the ball's total energy, E_{tot}.

SKETCH:



RESEARCH: Total energy is the sum of the mechanical energy and other forms of energy. As there are no non-conservative forces (neglecting air resistance), the total energy is the total mechanical energy. $E_{\text{tot}} = K + U$. Use $K = \left(mv^2\right)/2$ and $U = mgh$.

SIMPLIFY: $E_{\text{tot}} = m\left(\frac{1}{2}v^2 + gh\right)$

CALCULATE:

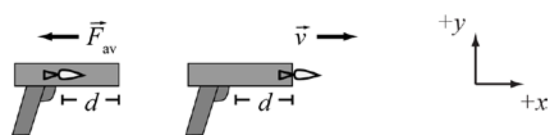
$$\begin{aligned} E_{\text{tot}} &= 1.50\text{ kg}\left(\frac{1}{2}(20.0\text{ m/s})^2 + \left(9.81\text{ m/s}^2\right)(15.0\text{ m})\right) \\ &= 1.50\text{ kg}\left(200\text{ m}^2/\text{s}^2 + 147.15\text{ m}^2/\text{s}^2\right) \\ &= 520.725\text{ J} \end{aligned}$$

ROUND: As the speed has three significant figures, the result should be rounded to $E_{\text{tot}} = 521$ J.

DOUBLE-CHECK: The energy is positive and has the correct unit of measurement. It is also on the right order of magnitude for the given values. This is a reasonable energy for a ball.

6.72. **THINK:** The average force used to load the dart gun is $F_{\text{av}} = 5.5$ N. The dart's mass is $m = 4.5 \cdot 10^{-3}$ kg and the distance the dart is inserted into the gun is d = 0.060 m. Determine the speed of the dart, v, as it exits the gun.

SKETCH:



RESEARCH: Assuming the barrel is frictionless, and neglecting air resistance, the conservation of mechanical energy can be used to determine v. Use $\Delta K = -\Delta U$, $K = \left(mv^2\right)/2$ and $W_c = -\Delta U = \vec{F} \bullet \vec{d}$ (W_c is work done by a conservative force).

SIMPLIFY: Note $\vec{F}$ and $\vec{d}$ are in the same direction so the equation can be reduced to $-\Delta U = W_c = F_{\text{av}}d$.

$$\Delta K = -\Delta U \Rightarrow K_f - K_i = F_{\text{av}}d \;\Rightarrow\; \frac{1}{2}mv^2 = F_{\text{av}}d \;\left(\text{as } v_0 = 0\right) \;\Rightarrow\; v = \sqrt{2F_{\text{av}}d/m}$$

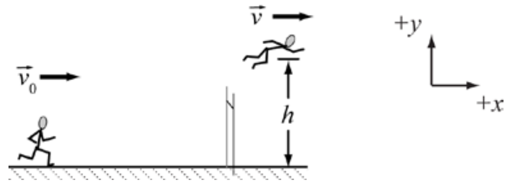
CALCULATE: $v = \sqrt{2(5.5\text{ N})(0.060\text{ m})/\left(4.5 \cdot 10^{-3}\text{ kg}\right)} = 12.111$ m/s

ROUND: All given values have two significant figures, so the result should be rounded to v = 12 m/s.

DOUBLE-CHECK: This is a reasonable velocity for a dart to exit a dart gun.

6.73. **THINK:** The jumper's initial speed is $v_0 = 9.00 \text{ m/s}$ and his final speed as he goes over the bar is v = 7.00 m/s. Determine his highest altitude, h.

SKETCH:



RESEARCH: As there are no non-conservative forces in the system, the conservation of mechanical energy can be used to solve for h as follows, $\Delta K = -\Delta U$.

SIMPLIFY: $K_\text{f} - K_\text{i} = U_\text{i} - U_\text{f} \;\Rightarrow\; \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -mgh$

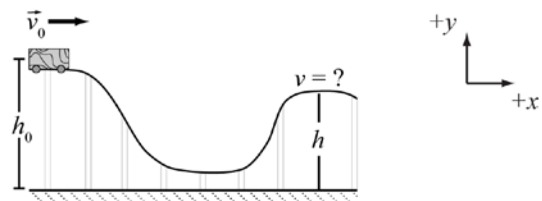
CALCULATE: $h = \dfrac{(9.00 \text{ m/s})^2 - (7.00 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} = 1.63099 \text{ m}$

ROUND: There are three significant figures in v_0 and v, so the result should be rounded to h = 1.63 m.

DOUBLE-CHECK: Since $v < v_0$, it is necessary that $h > h_0$ to conserve mechanical energy.

6.74. **THINK:** The initial speed of the roller coaster is $v_0 = 2.00 \text{ m/s}$ and its initial height is $h_0 = 40.0 \text{ m}$. Determine the speed, v at the top of the second peak at a height of h = 15.0 m.

SKETCH:



RESEARCH: As there are no non-conservative forces in this system, to solve for v, the conservation of mechanical energy can be used: $\Delta K = -\Delta U$, where $K = (mv^2)/2$ and $U = mgh$.

SIMPLIFY:

$$\Delta K = -\Delta U$$
$$K_\text{f} - K_\text{i} = U_\text{i} - U_\text{f}$$
$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = mgh_0 - mgh$$
$$v = \sqrt{2g(h_0 - h) + v_0^2}$$

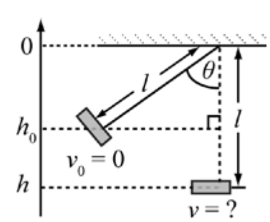
CALCULATE: $v = \sqrt{2(9.81 \text{ m/s}^2)(40.0 \text{ m} - 15.0 \text{ m}) + (2.00 \text{ m/s})^2} = 22.24 \text{ m/s}$

ROUND: As h_0 has three significant figures, the result should be rounded to v = 22.2 m/s.

DOUBLE-CHECK: The speed on the lower hill must be greater than the speed on the higher hill.

6.75. **THINK:** The length of the chain is l = 4.00 m and the maximum displacement angle is $\theta = 35°$. Determine the speed of the swing, v, at the bottom of the arc.

SKETCH:



RESEARCH: At the maximum displacement angle, the speed of the swing is zero. Assuming there are no non-conservative forces, to determine the speed, v, the conservation of mechanical energy can be used: $\Delta K = -\Delta U$. Use $K = \left(mv^2\right)/2$ and $U = mgh$. The initial height can be determined using trigonometry. Take the top of the swing to be $h = 0$.

SIMPLIFY: $v_0 = 0$ and $K_i = 0$. From the sketch, $h_0 = -l\cos\theta$ and $h = -l$. Then,

$$K_f = U_i - U_f \Rightarrow \frac{1}{2}mv^2 = mg\left(-l\cos\theta\right) - mg\left(-l\right) \Rightarrow \frac{1}{2}v^2 = g\left(l - l\cos\theta\right) \Rightarrow v = \sqrt{2g\left(l - l\cos\theta\right)}$$

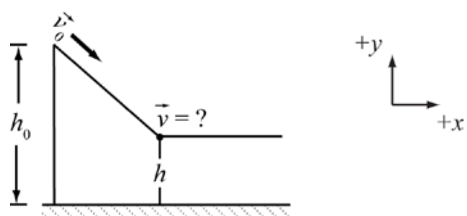
CALCULATE: $v = \sqrt{2\left(9.81 \text{ m/s}^2\right)\left(4.00 \text{ m} - \left(4.00 \text{ m}\right)\cos 35.0°\right)} = 3.767$ m/s

ROUND: l and θ have two significant figures, so the result should be rounded to v = 3.77 m/s.

DOUBLE-CHECK: This is a reasonable speed for a swing to achieve when initially displaced from the vertical by 35°.

6.76. **THINK:** The initial height of the truck is $h_0 = 680$ m. The initial speed is $v_0 = 15$ m/s and the final height is h = 550 m. Determine the maximum final speed, v.

SKETCH:



RESEARCH: The maximum final speed, v, can be determined by neglecting non-conservative forces and using the conservation of mechanical energy, $\Delta K = -\Delta U$. Use $K = \left(mv^2\right)/2$ and $U = mgh$.

SIMPLIFY: $K_f - K_i = U_i - U_f \;\Rightarrow\; \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = mgh_0 - mgh \;\Rightarrow\; v = \sqrt{2g\left(h_0 - h\right) + v_0^2}$

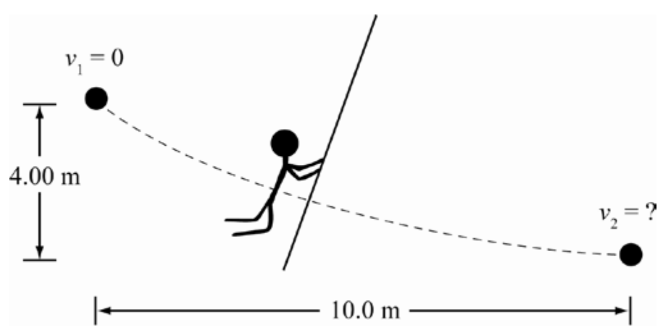
CALCULATE: $v = \sqrt{2\left(9.81 \text{ m/s}^2\right)\left(680 \text{ m} - 550 \text{ m}\right) + \left(15.0 \text{ m/s}\right)^2} = 52.68$ m/s

ROUND: Each initial value has two significant figures, so the result should be rounded to v = 53 m/s.

DOUBLE-CHECK: Since the truck is going downhill, its final speed must be greater than its initial speed in the absence of non-conservative forces.

6.77. **THINK:** Determine Tarzan's speed when he reaches a limb on a tree. He starts with a speed of $v_0 = 0$ and reaches a limb on a tree which is 10.0 m away and 4.00 m below his starting point. Consider the change in potential energy as he moves to the final point and relate this to the change in kinetic energy. The velocity can be determined from the kinetic energy.

SKETCH:



RESEARCH: Gravitational potential energy is given by $U = mgh$. The change in potential energy is given by $\Delta U = mgh_2 - mgh_1$. Kinetic energy is given by $K = \left(mv^2\right)/2$. The change in kinetic energy is given by $\Delta K = \left(mv_2^2\right)/2 - \left(mv_1^2\right)/2$.

SIMPLIFY: Assume the system is conservative. The change in potential energy must be equal to the negative of the change in kinetic energy:

$$\begin{aligned}\Delta U &= -\Delta K \\ mgh_2 - mgh_1 &= -\left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right) \\ g\left(h_1 - h_2\right) &= \frac{1}{2}\left(v_2^2 - v_1^2\right) \\ \left(v_2^2 - v_1^2\right) &= 2g\left(h_1 - h_2\right) \\ v_2 &= \sqrt{2g\left(h_1 - h_2\right) + v_1^2}.\end{aligned}$$

CALCULATE: $v_2 = \sqrt{2\left(9.81 \text{ m/s}^2\right)\left(4.00 \text{ m}\right)} = 8.86 \text{ m/s}$

ROUND: Since the values are given to three significant figures, the result remains $v_2 = 8.86$ m/s.

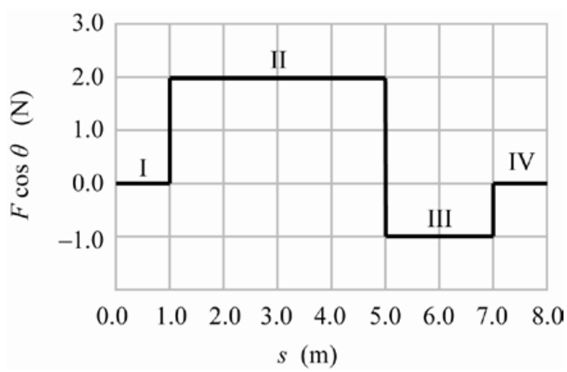
DOUBLE-CHECK: This speed is reasonable for swinging on a vine.

6.78. THINK:

(a) Determine the net work done on the block given a varying applied force, $F\cos\theta$. The mass of the block is 2.0 kg.

(b) Given an initial speed of zero at $s = 0$, determine the final speed at the end of the trajectory.

SKETCH:



RESEARCH:

(a) The net work is given by $W_{\text{net}} = \sum_i W_i$ and $W_{\text{i}} = F_{\text{i}}d_{\text{i}}$.

(b) By the work-energy theorem, $W_{\text{net}} = \Delta K$, where $\Delta K = \left(mv_2^2\right)/2 - \left(mv_1^2\right)/2$.

SIMPLIFY:

(a) $W_{\text{net}} = F_{\text{I}}d_{\text{I}} + F_{\text{II}}d_{\text{II}} + F_{\text{III}}d_{\text{III}} + F_{\text{IV}}d_{\text{IV}}$.

(b) $v_2 = \sqrt{\frac{2}{m}\left(W_{\text{net}} + \frac{1}{2}mv_1^2\right)}$

CALCULATE:

(a) $W_{net} = (0.0\text{ N})(1.0\text{ m}) + (2.0\text{ N})(4.0\text{ m}) + (-1.0\text{ N})(2.0\text{ m}) + (0.0\text{ N})(1.0\text{ m})$

$= 8.0\text{ N m} - 2.0\text{ N m}$

$= 6.0\text{ N m}$

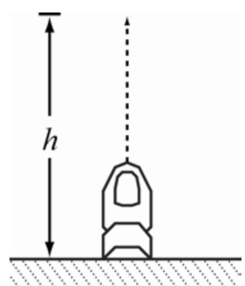
(b) $$v_2 = \sqrt{\left(\frac{2}{2.0\text{ kg}}\right)\left(6.0\text{ N m} + \frac{1}{2}(2.0\text{ kg})(0.0\text{ m/s})^2\right)} = \sqrt{6.0\ \frac{\text{N m}}{\text{kg}}} = \sqrt{6.0\ \frac{\text{m}^2}{\text{s}^2}} = 2.4\text{ m/s}$$

ROUND: Since all values are given to two significant figures, the results remain $W_{net} = 6.0\text{ N m}$ and $v_2 = 2.4\text{ m/s}$.

DOUBLE-CHECK: An increase of speed of 2.4 m/s after doing $6.0\text{ N}\cdot\text{m}$ of work is reasonable.

6.79. **THINK:** A rocket that has a mass of $m = 3.00$ kg reaches a height of $1.00\cdot 10^2$ m in the presence of air resistance which takes $8.00\cdot 10^2$ J of energy away from the rocket, so $W_{air} = -8.00\cdot 10^2$ J. Determine the height the rocket would reach if air resistance could be neglected.

SKETCH:



RESEARCH: Air resistance performs $-8.00\cdot 10^2$ J of work on the rocket. The absence of air resistance would then provide an extra $8.00\cdot 10^2$ J of energy to the system. If this energy is converted into potential energy, the increase in height of the rocket can be determined.

SIMPLIFY: $U_t = -W_{air} \Rightarrow mgh_t = -W_{air} \Rightarrow h_t = \frac{-W_{air}}{mg}$, where h_t is the added height.

CALCULATE: $h_t = \frac{-(-8.00\cdot 10^2\text{ J})}{(3.00\text{ kg})(9.81\text{ m/s}^2)} = 27.183\text{ J/kg}\cdot\text{m/s}^2 = 27.183\text{ m}$

Therefore, the total height reached by the rocket in the absence of air resistance is

$$h_{tot} = h_0 + h_t = 1.00\cdot 10^2\text{ m} + 0.27183\cdot 10^2\text{ m} = 1.27183\cdot 10^2\text{ m}.$$

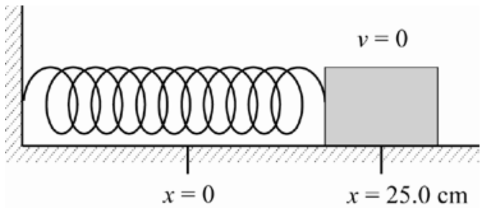
ROUND: Since the values are given to three significant figures, the result should be rounded to $h_{tot} = 1.27\cdot 10^2$ m.

DOUBLE-CHECK: It is reasonable that air resistance will decrease the total height by approximately a fifth.

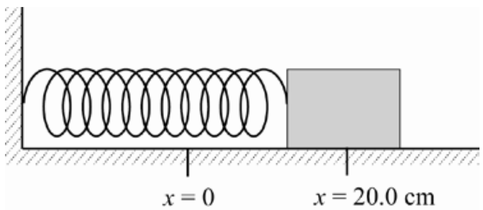
6.80. **THINK:** The mass-spring system is frictionless. The spring constant is $k = 100.$ N/m and the mass is 0.500 kg. For a stretch of 25.0 cm, determine (a) the total mechanical energy of the system, (b) the speed of the mass after it has moved 5.0 cm (at $x = 20.0$ cm) and (c) the maximum speed of the mass.

SKETCH:

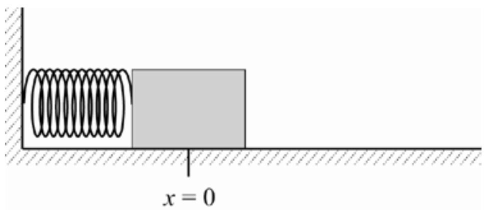
(a)



(b)



(c)



RESEARCH:

(a) The total mechanical energy of the system is given by $E_{\text{tot}} = K + U$. For a conservative system, it is known that $E_{\text{tot}} = \text{constant} = K_{\text{max}} = U_{\text{max}}$. The maximum potential energy can be calculated so the total mechanical energy can be determined:

$$E_{\text{tot}} = U_{\text{max}} = \frac{1}{2}kx_{\text{max}}^2.$$

(b) The speed at any point can be determined by considering the difference in potential energy and relating this to the kinetic energy. Kinetic energy at x is given by

$$K(x) = -\Delta U = \frac{kx_{\text{max}}^2}{2} - \frac{kx^2}{2}, \text{ and } K = \frac{mv^2}{2}.$$

(c) Speed, and therefore kinetic energy, is at its maximum when potential energy is zero, i.e., at the equilibrium position $x = 0$. Since $K_{\text{max}} = U_{\text{max}}$, $\left(mv_{\text{max}}^2\right)/2 = \left(kx_{\text{max}}^2\right)/2$.

SIMPLIFY:

(a) $E_{\text{tot}} = \frac{1}{2}kx_{\text{max}}^2$

(b) $K(x) = \frac{1}{2}mv_x^2 = \frac{1}{2}kx_{\text{max}}^2 - \frac{1}{2}kx^2 \Rightarrow v_x = \sqrt{\frac{k}{m}\left(x_{\text{max}}^2 - x^2\right)}$

(c) $\frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2}kx_{\text{max}}^2 \Rightarrow v_{\text{max}} = \sqrt{\frac{k}{m}x_{\text{max}}^2} = x_{\text{max}}\sqrt{\frac{k}{m}}$

CALCULATE:

(a) $E_{\text{tot}} = \frac{1}{2}(100.\text{ N/m})\left(2.50 \cdot 10^{-1}\text{ m}\right)^2 = 3.125\text{ J}$

(b) $v_x = \sqrt{\left(\frac{100.\text{ N/m}}{0.500\text{ kg}}\right)\left[\left(2.50\cdot 10^{-1}\text{ m}\right)^2 - \left(2.00\cdot 10^{-1}\text{ m}\right)^2\right]} = \sqrt{4.5\text{ m}^2/\text{s}^2} = 2.1213\text{ m/s}$

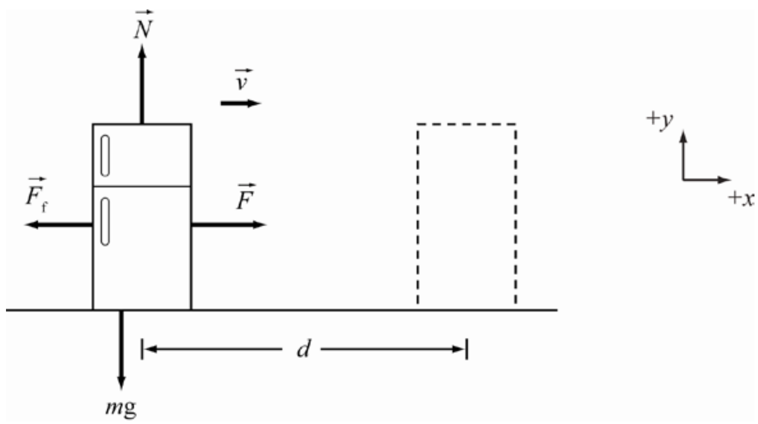
(c) $v_{\text{max}} = \left(2.50\cdot 10^{-1}\text{ m}\right)\sqrt{\frac{(100.\text{ N/m})}{0.500\text{ kg}}} = 3.5355\text{ m/s}$

ROUND: The results should be rounded to three significant figures: $E_{\text{tot}} = 3.13\text{ J}$, $v_x = 2.12\text{ m/s}$ and $v_{\text{max}} = 3.54\text{ m/s}$.

DOUBLE-CHECK: A total mechanical energy of 3 J is reasonable for this system, based on the given values. A speed anywhere other than at $x = 0$ must be less than at $x = 0$. In this case, v_x must be less than v_{max}. At $x = 0$ the potential energy is zero. Therefore, all of the energy is kinetic energy, so the velocity is maximized. This value is greater than the value found in part (b), as expected.

6.81. **THINK:** The mass of a refrigerator is m = 81.3 kg. The displacement is d = 6.35 m. The coefficient of kinetic friction is $\mu_{\text{k}} = 0.437$.

SKETCH:



RESEARCH: The force of friction is given by $F_{\text{f}} = \mu_{\text{k}}N$. Use Newton's second law and $W = Fd$. This net mechanical work is the work done by you. The net mechanical work done by the roommate is zero, since he/she lifts the refrigerator up and then puts it back down. Therefore, $\Delta E = 0$.

SIMPLIFY: $\sum F_y = 0 \Rightarrow N - mg = 0 \Rightarrow N = mg$, $\sum F_x = 0 \Rightarrow F - F_{\text{f}} = 0 \Rightarrow F = \mu_{\text{k}}N \Rightarrow F = \mu_{\text{k}}mg$

The work done is given by $W = Fd = \mu_{\text{k}}mgd$.

CALCULATE: $W = (0.437)(81.3\text{ kg})(9.81\text{ m/s}^2)(6.35\text{ m}) = 2213.17\text{ J}$

ROUND: Rounding to three significant figures, $W = 2.21$ kJ.

DOUBLE-CHECK: Joules are a usual unit for work. One kilogram is equivalent to about 10 Newtons on Earth, and the fridge weighs about 100 kilograms. The fridge is being moved about 5 meters with a coefficient of friction around a half, so the work should be roughly $0.5\cdot 100\cdot 10\cdot 5 = 2500$ J. The calculated value is reasonable close to this approximation, so the calculated value is reasonable.

6.82. **THINK:** A 1.00 kg block is moving between two springs with constants $k_1 = 100.$ N/m and $k_2 = 50.0$ N/m. If the block is compressed against spring 1 by 20.0 cm, determine

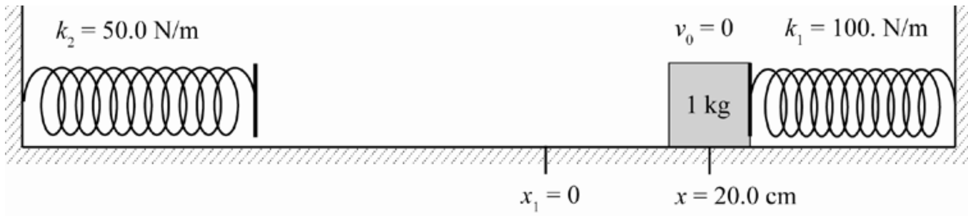
(a) the total energy in the system,

(b) the speed of the block as it moves from one spring to the other and

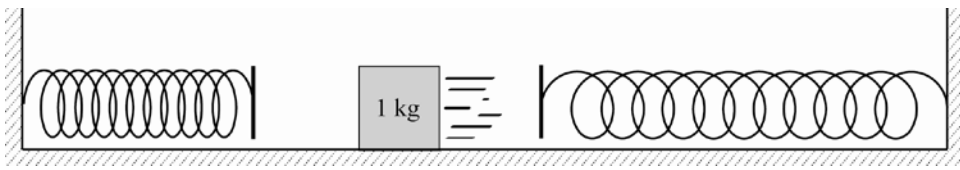
(c) the maximum compression on spring 2.

SKETCH:

(a)



(b)



(c)

RESEARCH:

(a) The total mechanical energy can be determined by recalling that in a conservative system $E_{\text{tot}} = \text{constant} = U_{\max} = K_{\max}$. $U_{\max}$ can be determined from spring 1: $U_{\max} = \frac{1}{2}k_1 x_{\max}^2 = E_{\text{tot}}$.

(b) $K_{\max} = U_{\max} \Rightarrow \left(mv_{\max}^2\right)/2 = \left(k_1 v_{\max,1}^2\right)/2$. Since the system is conservative, the speed of the block is $v_{\max}$ anytime it is not touching a spring.

(c) The compression on spring 2 can be determined by the following relation:

$$U_{\max,2} = K_{\max} \Rightarrow \frac{1}{2}k_2 v_{\max,\,2}^2 = K_{\max}.$$

SIMPLIFY:

(a) $E_{\text{tot}} = \frac{1}{2}k_1 x_{\max,1}^2$

(b) $v_{\max} = \sqrt{\frac{k_1}{m}x_{\max,1}^2} = x_{\max,1}\sqrt{\frac{k_1}{m}}$

(c) $x_{\max,2} = \sqrt{\frac{2K_{\max}}{k_2}}$

CALCULATE:

(a) $E_{\text{tot}} = \frac{1}{2}(100.\text{ N/m})\left(20.0\cdot 10^{-2}\text{ m}\right)^2 = 2.00\text{ J}$

(b) $v_{\max} = \left(20.0\cdot 10^{-2}\text{ m}\right)\sqrt{\frac{(100.\text{ N/m})}{1.00\text{ kg}}} = 2.00\text{ m/s}$

(c) $x_{\max,2} = \sqrt{\frac{2(2.00\text{ J})}{50.0\text{ N/m}}} = 2.83\cdot 10^{-1}\text{ m} = 28.3\text{ cm}$

ROUND: Since the least number of significant figures in the given values is three, so the results should be rounded to $E_{\text{tot}} = 2.00 \text{ J}$, $v_{\text{max}} = 2.00 \text{ m/s}$ and $x_{\text{max,2}} = 28.3 \text{ cm}$.

DOUBLE-CHECK: It can be seen that $U_{\text{max,1}} = U_{\text{max,2}} = K_{\text{max}}$

$$U_{\text{max,1}} = \frac{1}{2}k_1 x_{\text{max,1}}^2 = \frac{1}{2}(100.\text{ N/m})\left(20.0\cdot 10^{-2}\text{ m}\right)^2 = 2.00 \text{ J}$$

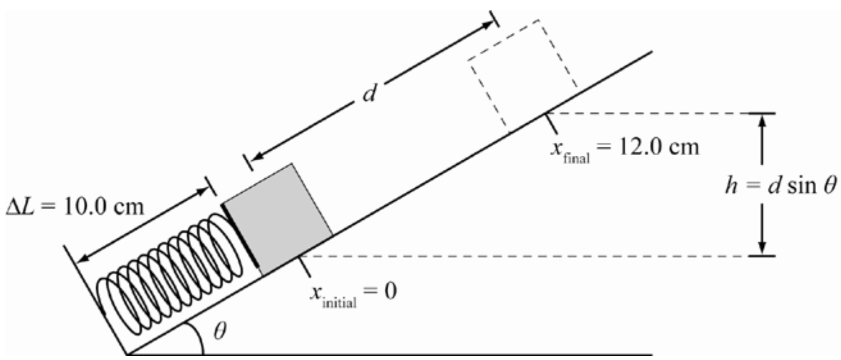
$$U_{\text{max,2}} = \frac{1}{2}k_1 x_{\text{max,1}}^2 = \frac{1}{2}(50.\text{ N/m})\left(28.3\cdot 10^{-2}\text{ m}\right)^2 = 2.00 \text{ J}$$

$$K_{\text{max}} = \frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2}(1.00 \text{ kg})(2.00 \text{ m/s})^2 = 2.00 \text{ J}$$

and all results are reasonable for the given values.

6.83. **THINK:** A block of mass, m = 1.00 kg is against a spring on an inclined plane of angle, $\theta = 30.0°$. The coefficient of kinetic friction is $\mu_{\text{k}} = 0.100$. The spring is initially compressed 10.0 cm and the block moves to 2.00 cm beyond the springs normal length after release (therefore the block moves d = 12.0 cm after it is released). Determine (a) the change in the total mechanical energy and (b) the spring constant.

SKETCH:



RESEARCH:

(a) Since this is not a conservative system, the change in the total mechanical energy can be related to the energy lost due to friction. This energy can be determined by calculating the work done by the force of friction: $W_{\text{friction}} = F_{\text{friction}}d = \mu_{\text{k}}mg(\cos\theta)d$, and $\Delta E_{\text{tot}} = -W_{\text{friction}} = -\mu_{\text{k}}mg(\cos\theta)d$.

(b) From conservation of energy, the change in total energy, ΔE_{tot} determined in (a), is equal to $\Delta K + \Delta U$. Since K = 0 at both the initial and final points it follows that

$$\Delta E_{\text{tot}} = U_{\text{final}} - U_{\text{initial}} = mgd\sin\theta - \frac{1}{2}k\Delta L^2.$$

SIMPLIFY:

(a) $\Delta E_{\text{tot}} = -\mu_k mg(\cos\theta)d$

(b) $k = 2\dfrac{(mgd\sin\theta - \Delta E_{\text{tot}})}{\Delta L^2}$

CALCULATE:

(a) $\Delta E_{\text{tot}} = -(0.100)(1.00 \text{ kg})\left(9.81 \text{ m/s}^2\right)\cos(30.0°)\left(12.0\cdot 10^{-2}\text{ m}\right) = -0.1019 \text{ J}$

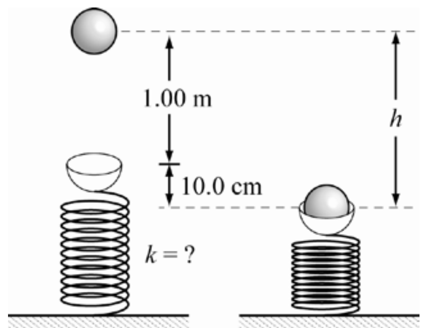
(b) $k = -2\dfrac{(1.00 \text{ kg})\left(9.81 \text{ m/s}^2\right)(0.120 \text{ m})\sin(30.0°) - (-0.1019 \text{ J})}{(0.100 \text{ m})^2} = 138.1 \text{ N/m}$

ROUND:

(a) Since the lowest number of significant figures is three, the result should be rounded to $\Delta E_{\text{tot}} = -1.02\cdot 10^{-1}\text{ J}$ (lost to friction).

(b) Since the mass is given to three significant figures, the result should be rounded to $k = 138$ N/m.
DOUBLE-CHECK:
(a) A change of about 0.1 J given away to friction for a distance of 12 cm and with this particular coefficient of friction is reasonable.
(b) The spring constant is in agreement with the expected values.

6.84. **THINK:** A 0.100 kg ball is dropped from a height of 1.00 m. If the spring compresses 10.0 cm, determine (a) the spring constant and (b) the percent difference between a spring constant calculated by neglecting a change in U_{gravity} while compressing the spring, and the result in part (a).
SKETCH:



RESEARCH:
(a) Determine the spring constant by relating the gravitational potential energy, given to the system, to the elastic potential energy stored by the spring: $U_{\text{gravity}} = U_{\text{spring}} \Rightarrow mgh = (1/2)kx^2$.
(b) If the change in gravitational potential energy is ignored during the compression:

$$mg(h-x) = \frac{1}{2}kx^2.$$

To calculate the percent difference, use $\%\text{ difference} = \dfrac{|k_1 - k_2|}{(k_1+k_2)/2}(100\%)$.

SIMPLIFY:
(a) $mgh = \dfrac{1}{2}k_1x^2 \Rightarrow k_1 = \dfrac{2mgh}{x^2}$

(b) $mg(h-x) = \dfrac{1}{2}k_2x^2 \Rightarrow k_2 = \dfrac{2mg(h-x)}{x^2}$

Therefore,

$$\%\text{ difference} = \frac{\left|\dfrac{2mgh}{x^2} - \dfrac{2mg(h-x)}{x^2}\right|}{\left(\dfrac{2mgh}{x^2} + \dfrac{2mg(h-x)}{x^2}\right)/2} = \frac{|h-(h-x)|}{(h+h-x)/2} = \frac{2x}{2h-x}.$$

CALCULATE:
(a) $k_1 = \dfrac{2(0.100\text{ kg})(9.81\text{ m/s}^2)(1.10\text{ m})}{(0.100\text{ m})^2} = 215.82$ N/m

(b) $\%\text{ difference} = \dfrac{2(0.100\text{ m})}{2(1.10\text{ m}) - 0.100\text{ m}} = 9.52\%$

ROUND: Rounding to three significant figures, $k_1 = 216$ N/m and the % difference is 9.52 %.
DOUBLE-CHECK: The percent difference is reasonable.

6.85. **THINK:** The mass is m = 1.00 kg, k = 100. N/m, the amplitude is A = 0.500 m and $x_1 = 0.250$ m. Determine:

(a) the total mechanical energy,

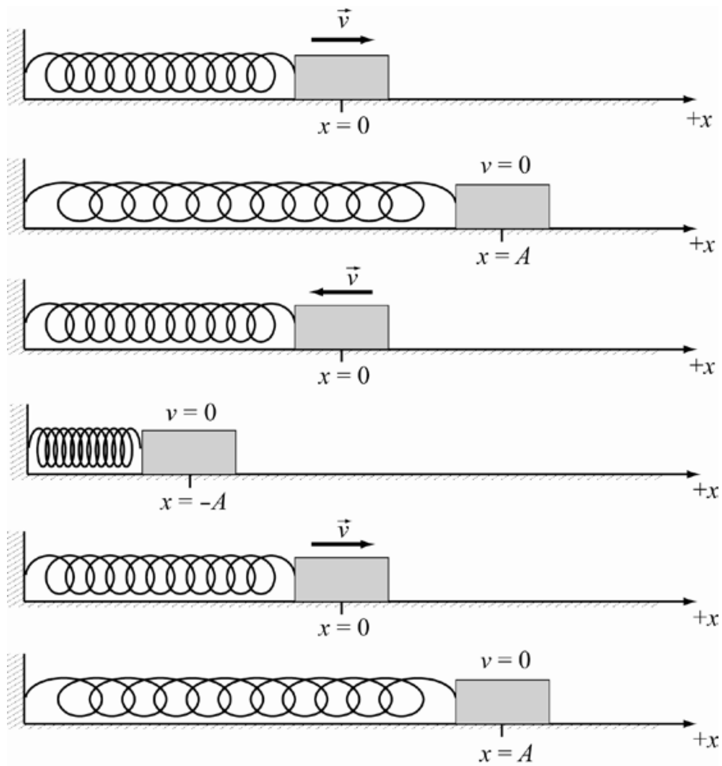
(b) the potential energy for the system and the kinetic energy of the mass at x_1,

(c) the kinetic energy of the mass at $x = 0$, that is K_{max},

(d) the change in kinetic energy of the mass if the amplitude is cut in half due to friction, and

(e) the change in potential energy if the amplitude is cut in half due to friction.

SKETCH:



RESEARCH:

(a) Assume a frictionless table and write $E_{tot} = U_{\text{max}} = K_{\text{max}}$ and calculate $U_{\text{max}} = \left(kA^2\right)/2.$

(b) At x_1, the potential energy is $U_{x_1} = \left(\text{k}x_1^{\,2}\right)/2$ and the kinetic energy will be given by:

$$K_{x_1} = U_{\text{max}} - U_{x_1}.$$

(c) At $x = 0$, all the energy is in the form of kinetic energy, therefore $K_{x=0} = K_{\text{max}} = U_{\text{max}}$.

(d) Let $K_{\text{max}}{}^*$ denote that the maximum kinetic energy of the mass if there was friction between the mass and the table. At the moment when the amplitude is cut in half, the maximum kinetic energy is obtained by the maximum potential energy:

$$K_{\text{max}} = U_{\text{max}} = \frac{1}{2}kA^2 \;\Rightarrow\; K_{\text{max}}{}^* = \frac{1}{2}k\left(\frac{A}{2}\right)^2 = \frac{1}{4}\left(\frac{1}{2}kA^2\right).$$

(e) As described in part (d), $U_{\text{max}}{}^* = \frac{1}{4}U_{\text{max}}$.

SIMPLIFY:

(a) $E_{\text{tot}} = \frac{1}{2}kA^2$

(b) $K_{x_1} = U_{\text{max}} - U_{x_1} = \frac{1}{2}kA^2 - \frac{1}{2}kx_1^{\,2} = \frac{1}{2}k\left(A^2 - x_1^{\,2}\right)$

(c) $K_{max} = U_{max} = \frac{1}{2}kA^2$

(d) $K_{max}* = \frac{1}{4}K_{max}$

(e) $U_{max}* = \frac{1}{4}U_{max}$

CALCULATE:

(a) $E_{tot} = \frac{1}{2}(100.\text{ N/m})(0.500\text{ m})^2 = 12.5\text{ J}$

(b) $U_{x_1} = \frac{1}{2}(100.\text{N/m})(0.250\text{m})^2 = 3.125\text{J}, \; K_{x_1} = \frac{1}{2}(100.\text{ N/m})\left[(0.500\text{ m})^2 - (0.250\text{m})^2\right] = 9.375\text{ J}$

(c) $K_{max} = E_{tot} = 12.5\text{ J}$

(d) A factor of ¼.

(e) A factor of ¼.

ROUND: Rounding to three significant figures:

(a) $E_{tot} = 12.5\text{ J}$

(b) $U_{x_1} = 3.13\text{J} \; K_{x_1} = 9.38\text{ J}$

(c) $K_{max} = 12.5\text{ J}$

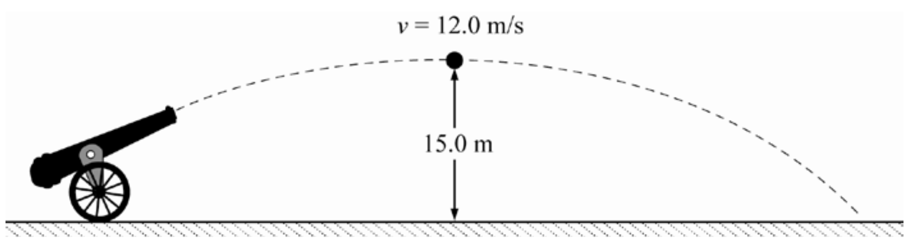
(d) K_{max} changes by a factor of ¼.

(e) U_{max} changes by a factor of ¼.

DOUBLE-CHECK: As expected, the kinetic energy at any point other than $x = 0$ is less than the maximum kinetic energy.

6.86. **THINK:** Bolo has a mass of 80.0 kg and is projected from a 3.50 m long barrel. Determine the average force exerted on him in the barrel in order to reach a speed of 12.0 m/s at the top of the trajectory at 15.0 m above the ground.

SKETCH:



RESEARCH: When Bolo is at the top of the trajectory, his total energy (neglecting air friction) is $E_{tot} = U + K$. This energy can be related to the force exerted by the cannon by means of the work done on Bolo by the cannon: $W = Fd \Rightarrow F = W/d$. Since all the energy was provided by the cannon, $W = E_{tot} \Rightarrow F = E_{tot}/d$.

SIMPLIFY: $F = \frac{E_{tot}}{d} = \frac{U+K}{d} = \frac{mgh + \frac{1}{2}mv^2}{d} = \frac{m}{d}\left(gh + \frac{v^2}{2}\right)$

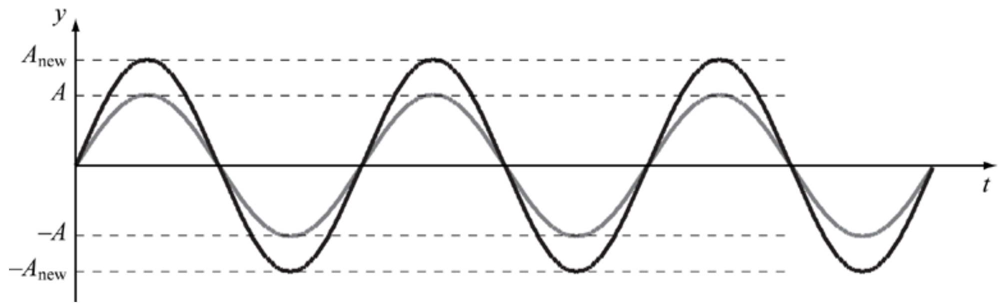
CALCULATE: $F = \left(\frac{80.0\text{ kg}}{3.50\text{ m}}\right)\left((9.81\text{ m/s}^2)(15.0\text{ m}) + \frac{(12.0\text{ m/s})^2}{2}\right) = 5009.14\text{ N}$

ROUND: Since the number of significant figures in the calculation is three, the result rounds to $F = 5010$ N.

DOUBLE-CHECK: That a force of about 5000 N is required to propel an 80 kg object through such a distance is reasonable.

6.87. **THINK:** The mass hanging vertically from a spring can be treated using a method that is independent of gravitational effects on the mass (see page 185 in the text). The mechanical energy of the mass on a spring is defined in terms of the amplitude of the oscillation and the spring constant. When the mass is pushed, the system gains mechanical energy. This new mechanical energy can be used to calculate the new velocity of the mass at the equilibrium position (b) and the new amplitude (c).

SKETCH: Before the mass is hit, the amplitude of the oscillation is A. After the mass is hit, the amplitude of the oscillation is A_{new}.



RESEARCH: The total mechanical energy before the hit is $E=\frac{1}{2}kA^2$. After the hit, the total mechanical energy is given by $E_{\text{new}}=\frac{1}{2}kA^2+\frac{1}{2}mv_{\text{push}}^2$ where v_{push} is the speed with which the mass is pushed. The new speed at equilibrium is given by $\frac{1}{2}mv_{\text{new}}^2=E_{\text{new}}$ and the new amplitude of oscillation is given by $\frac{1}{2}kA_{\text{new}}^2=E_{\text{new}}$.

SIMPLIFY:

(a) $E_{\text{new}}=\frac{1}{2}kA^2+\frac{1}{2}mv_{\text{push}}^2$

(b) $v_{\text{new}}=\sqrt{\frac{2E_{\text{new}}}{m}}$

(c) $A_{\text{new}}=\sqrt{\frac{2E_{\text{new}}}{k}}$

CALCULATE:

(a) $E_{\text{new}}=\frac{1}{2}kA^2+\frac{1}{2}mv_{\text{push}}^2=\frac{1}{2}(100.\text{ N/m})(0.200\text{ m})^2+\frac{1}{2}(1.00\text{ kg})(1.00\text{ m/s})^2=2.50\text{ J}$

(b) $v_{\text{new}}=\sqrt{\frac{2E_{\text{new}}}{m}}=\sqrt{\frac{2(2.50\text{ J})}{1.00\text{ kg}}}=2.236\text{ m/s}$

(c) $A_{\text{new}}=\sqrt{\frac{2E_{\text{new}}}{k}}=\sqrt{\frac{2(2.50\text{ J})}{100.\text{ N/m}}}=0.2236\text{ m}$

ROUND: Rounding to three significant figures: $E_{\text{new}}=2.50\text{ J}$, $v_{\text{max,2}}=2.24\text{ m/s}$ and $A_2=22.4\text{ cm}$.

DOUBLE-CHECK: The mechanical energy before the hit was

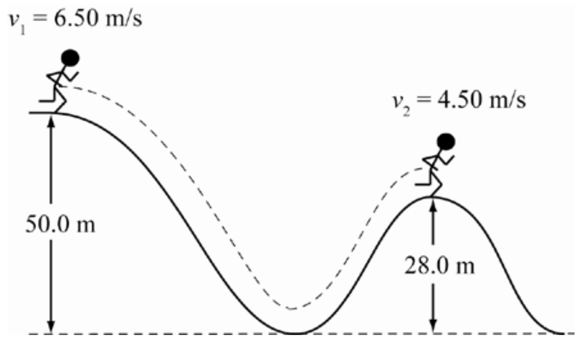
$$E=(1/2)kA^2=(1/2)(100.\text{ N/m})(0.200\text{ m})^2=2.00\text{ J}.$$

The speed of the mass passing the equilibrium point before the hit was $v=\sqrt{\frac{2E}{m}}=\sqrt{\frac{2(2.00\text{ J})}{1.00\text{ kg}}}=2.00\text{ m/s}$.

It is reasonable that adding 0.5 J to the total energy by means of a hit results in an increase of the speed of the mass at the equilibrium point of 0.24 m/s and an increase of about 2.4 cm to the amplitude.

6.88. **THINK:** Determine the total work done by a runner on a track where the initial speed is $v_1 = 6.50$ m/s at a height of 50.0 m and the final speed is $v_2 = 4.50$ m/s at a different hill with a height of 28.0 m. The runner has a mass of 83.0 kg, there is a constant resistance of 9.00 N and the total distance covered is 400. m.

SKETCH:



RESEARCH: Let the force of resistance be denoted F_r The total work done by the runner can be determined by considering the change in kinetic and potential energy and by considering the work done by the resistance force: $W_1 = \Delta K$, $W_2 = \Delta U$ and $W_3 = F_r d$.

SIMPLIFY: $W_1 = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 = \frac{1}{2}m\left(v_1^2 - v_2^2\right)$, $W_2 = mg\left(h_1 - h_2\right)$, and $W_3 = F_r d$. The total energy at point 1: $E_{\text{tot},1} = \frac{1}{2}mv_1^2 + mgh_1$. The total energy at point 2: $E_{\text{tot},2} = \frac{1}{2}mv_2^2 + mgh_2$.

CALCULATE: $E_{\text{tot},1} = \frac{1}{2}(83.0 \text{ kg})(6.50 \text{ m/s})^2 + (83.0 \text{ kg})(9.81 \text{ m/s}^2)(50.0 \text{ m}) = 4.25 \cdot 10^4 \text{ J}$

$E_{\text{tot},2} = \frac{1}{2}(83.0 \text{ kg})(4.50 \text{ m/s})^2 + (83.0 \text{ kg})(9.81 \text{ m/s}^2)(28.0 \text{ m}) = 2.36 \cdot 10^4 \text{ J}$

Therefore, $\Delta E_{\text{tot}} = 4.25 \cdot 10^4 \text{ J} - 2.36 \cdot 10^4 \text{ J} = 1.89 \cdot 10^4 \text{ J}$.

$W_{\text{friction}} = (9.00 \text{ N})(400. \text{ m}) = 3600 \text{ J}$.

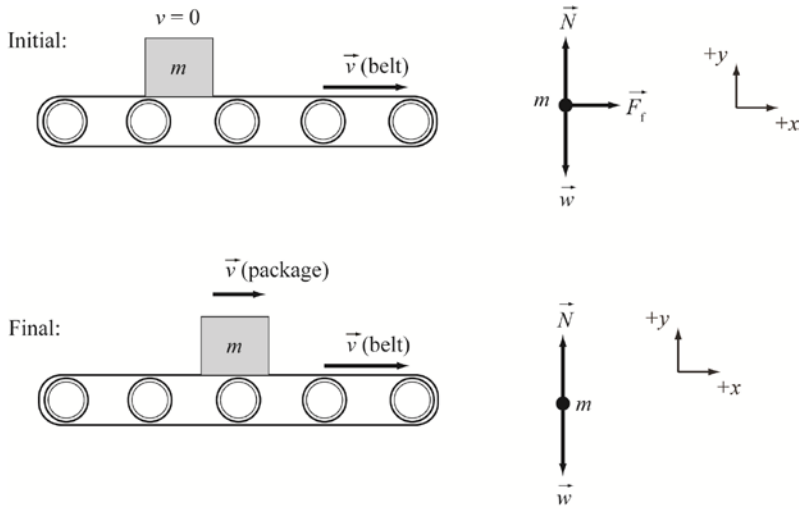
Therefore, $E_{\text{lost}} = \Delta E_{\text{tot}} + W_{\text{friction}} = 1.89 \cdot 10^4 \text{ J} + 3.60 \cdot 10^3 \text{ J} = 2.25 \cdot 10^4 \text{ J}$.

ROUND: Rounding to three significant figures, $E_{\text{lost}} = 2.25 \cdot 10^4$ J.

DOUBLE-CHECK: This is a reasonable value for the energy exerted by a runner with the given values.

6.89. **THINK:** Once the package is dropped on the left, the only horizontal force acting on the package is friction. The speed the package is moving relative to the belt is known, so the constant acceleration expressions can be used to determine the time taken for the package to stop sliding on the belt, i.e. the time it takes for the package to stop moving relative to the belt (part (a)). For the remaining problems, the principles of work and conservation of energy can be used to determine the required values. The known quantities are: v (the speed of the belt relative to the package), m (the mass of the package), μ_k (the coefficient of kinetic friction).

SKETCH:



RESEARCH: Work is given by $W = Fd$ ($\vec{F}$ is parallel to $\vec{d}$). Kinetic energy is given by $K = \left(mv^2\right)/2.$ The constant acceleration equations are: $v_{\text{f}} = v_{\text{i}} + at$ and $v_{\text{f}}^2 = v_{\text{i}}^2 + 2ax.$

SIMPLIFY:

(a) $v_{\text{f}} = v_{\text{i}} + at$, $v_{\text{i}} = 0 \Rightarrow t = \dfrac{v_{\text{f}}}{a}$, $v_{\text{f}} = v$, $ma = F_{\text{f}} = \mu_{\text{k}} mg \Rightarrow a = \mu_{\text{k}} g$, and $t = \dfrac{v_{\text{f}}}{a} = \dfrac{v}{\mu_{\text{k}} g}.$

(b) $v_{\text{f}}^2 = v_{\text{i}}^2 + 2ax$, $v_{\text{i}} = 0$, $v_{\text{f}} = v$, $a = \mu_{\text{k}} g$, and $x = \dfrac{v_{\text{f}}^2}{2a} = \dfrac{v^2}{2\mu_{\text{k}} g}.$

(c) The energy dissipated is equal to the work done by the belt minus the change in kinetic energy:
$W - \Delta E = Fd - \left(mv^2\right)/2 = \left(\mu_{\text{k}} mg\right)\left(vt\right) - \left(mv^2\right)/2 = \left(\mu_{\text{k}} mg\right)\left(v^2 / \mu_{\text{k}} g\right) - \left(mv^2\right)/2 = \left(mv^2\right)/2$

(d) The total work done by the belt is $W = Fd = \left(\mu_{\text{k}} mg\right)\left(vt\right) = \left(\mu_{\text{k}} mg\right)\left(v^2 / \mu_{\text{k}} g\right) = mv^2$

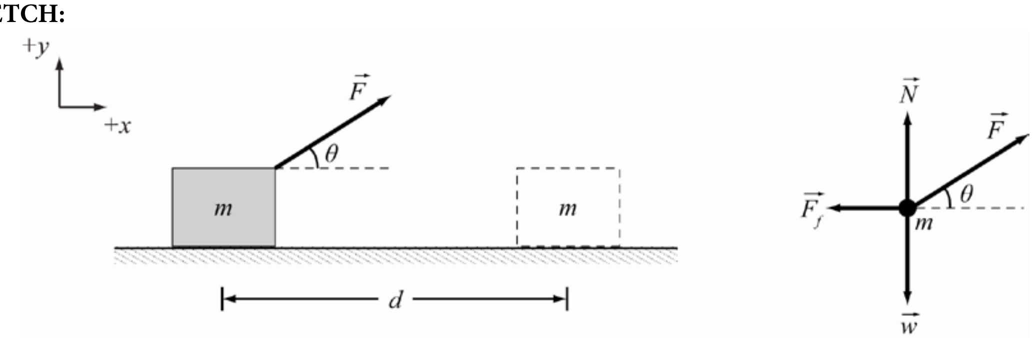
CALCULATE: It is not necessary to calculate any values.

ROUND: This step is not necessary.

DOUBLE-CHECK: Of the work done by the conveyor belt, half has ended up as kinetic energy of the package and the other half has been dissipated as friction heat. This seems reasonable, since the package transitioned steadily from a state ($v_{\text{i}} = 0$) where all the belt work was being dissipated as friction to a state ($v_{\text{f}} = v$) where none of it was.

6.90. **THINK:** There is enough information to determine all the forces. From the forces, the work can be determined. The given values are as follows: m = 85.0 kg, d = 8.00 m, $\theta = 20.0°$, $\left|\vec{F}\right| = 2.40 \cdot 10^2$ N and $\mu_{\text{k}} = 0.200.$

SKETCH:



RESEARCH: $W = \vec{F} \cdot \vec{d} = Fd\cos\theta \Rightarrow W_{\text{tot}} = F_{\text{net}} d\cos\theta$

SIMPLIFY:

(a) $W_{\text{father}} = F_{\text{father}} d\cos\theta$

(b) $W_{\text{friction}} = F_{\text{friction}} d$ (the force is parallel to the displacement), $F_{\text{friction}} = \mu_{\text{k}}(mg - F\sin\theta)$

(c) $W_{\text{total}} = W_{\text{father}} + W_{\text{friction}}$

CALCULATE:

(a) $W_{\text{father}} = \left(2.40 \cdot 10^2 \text{ N}\right)\left(8.00 \text{ m}\right)\cos\left(20.0^\circ\right) = 1.8042 \cdot 10^3 \text{ J}$

(b) $F_{\text{friction}} = \left(0.200\right)\left(\left(85.0 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right) - \left(2.40 \cdot 10^2 \text{ N}\right)\sin\left(20.0^\circ\right)\right) = 150.35 \text{ N}$

$W_{\text{friction}} = \left(1.5035 \cdot 10^2 \text{ N}\right)\left(8.00 \text{ m}\right)\cos\left(180^\circ\right) = -1.2028 \cdot 10^3 \text{ J}$

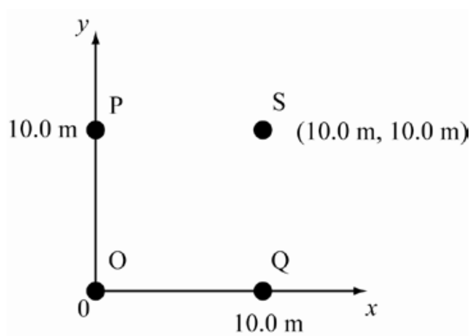
(c) $W_{\text{total}} = \left(1.8042 \cdot 10^3 \text{ J}\right) - \left(1.2028 \cdot 10^3 \text{ J}\right) = 6.014 \cdot 10^2 \text{ J}$

ROUND: The given quantities have three significant figures, so the results should be rounded to $W_{\text{father}} = 1.80 \cdot 10^3 \text{ J}$, $W_{\text{friction}} = -1.20 \cdot 10^3 \text{ J}$ and $W_{\text{total}} = 601 \text{ J}$.

DOUBLE-CHECK: Note also that the total work can be calculated using the net force, $W_{\text{tot}} = F_{\text{net}} d\cos\theta$, which gives the same result.

6.91. **THINK:** The total work can be determined if the path taken and the force applied are known. These are both given as follows: $\vec{F}(x,y) = \left(x^2\hat{x} + y^2\hat{y}\right)$ N and the points are S(10.0 m,10.0 m), P(0 m,10.0 m), Q(10.0 m,0 m) and O(0 m,0 m).

SKETCH:



RESEARCH: Work is given by:

$$W = \int_a^b d\vec{l} \cdot \vec{F} = \int_a^b \left(x^2 dx + y^2 dy\right).$$

The equations of the paths are: along OP, $x = 0$, $dx = 0$; along OQ, $y = 0$, $dy = 0$; along OS, $y = x$, $dy = dx$; along PS, $y = 10$, $dy = 0$; along QS, $x = 10$, $dx = 0$.

SIMPLIFY:

(a) OPS: $W = \int_O^P \left(x^2 dx + y^2 dy\right) + \int_P^S \left(x^2 dx + y^2 dy\right)$

$$= \int_0^{10} y^2 dy + \int_0^{10} x^2 dx = \frac{1}{3} y^3 \Big|_0^{10} + \frac{1}{3} x^3 \Big|_0^{10}$$

$$= \frac{1}{3}(10)^3 + \frac{1}{3}(10)^3 = \frac{2}{3}(10)^3$$

$$= W_{\text{OP}} + W_{\text{PS}}$$

(b) OQS: $W = \int_O^Q d\vec{l} \bullet \vec{F} + \int_Q^S d\vec{l} \bullet \vec{F}$

$$= \int_0^{10} x^2 dx + \int_0^{10} y^2 dy$$
$$= W_{OQ} + W_{QS}$$
$$= W_{PS} + W_{OP}$$
$$= \left(\frac{2}{3}\right)10^3$$

(c) OS: $W = W_{OS} = \int_O^S \left(x^2 dx + y^2 dy\right) \Rightarrow \int_O^S \left(x^2 dx + x^2 dx\right) = \int_0^{10} 2x^2 dx = 2W_{\text{PS}} = \frac{2}{3}\left(10^3\right)$

(d) OPSQO: $W = W_{\text{OP}} + W_{\text{PS}} + W_{\text{SQ}} + W_{\text{QO}} = \frac{10^3}{3} + \frac{10^3}{3} + \left(-W_{\text{QS}}\right) + \left(-W_{\text{OQ}}\right) = \frac{2}{3}\left(10^3\right) - \frac{2}{3}\left(10^3\right) = 0$

(e) OQSPO: $W = W_{\text{OQ}} + W_{\text{QS}} + W_{\text{SP}} + W_{\text{PO}} = \frac{10^3}{3} + \frac{10^3}{3} - \frac{10^3}{3} - \frac{10^3}{3} = 0$

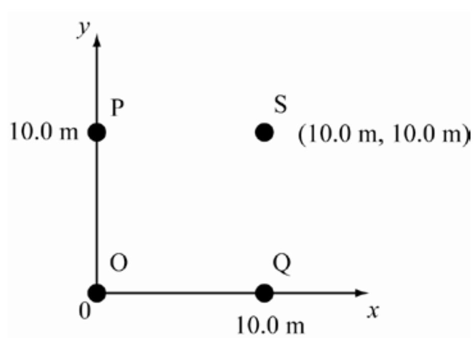
CALCULATE: $\frac{2}{3}\left(10.0^3\right) = 666.67.$ (a), (b) and (c): $W = 666.67.$ (d) and (e): $W = 0.$

ROUND: Rounding to three significant figures, (a), (b) and (c): W = 667 J, and (d), (e): W = 0 J.

DOUBLE-CHECK: The force is conservative and it should not depend on the path. It is expected that $W_{OS} = W_{OP} + W_{PS} = W_{OQ} + W_{QS}$, which is shown to be true in the calculation. It is also expected that the work along a closed path is zero, which is also shown to be true in the calculations.

6.92. **THINK:** The net work done is the sum of the work done by the applied force, calculated in the previous problem, and the work done by the frictional force.

SKETCH:



RESEARCH: The force of friction is constant, $F_{\text{f}} = \mu_k mg,$ and always points opposite to the direction of motion. First determine the work done by friction, W_{f}, and then calculate $W_{\text{net}} = W_{\text{applied}}$ (from 6.87) + W_{f}. Refer to the constraints on x, y, dx, and dy determined in 6.87.

Along OP: $\vec{F}_{\text{f}} = -F_{\text{f}}\hat{y}$, $\vec{F}_{\text{f}} \cdot dl = -\mu_{\text{k}} mg dy \Rightarrow W_{\text{f}} = -\mu_{\text{k}} mg \int_0^{10} dy = -10\mu_{\text{k}} mg$

Along OQ: $\vec{F}_{\text{f}} = -F_{\text{f}}\hat{x}$, $\vec{F}_{\text{f}} \cdot dl = -\mu_{\text{k}} mg dx \Rightarrow W_{\text{f}} = -10\mu_{\text{k}} mg$

Along OS: $\vec{F}_{\text{f}} = -F_{\text{f}} \frac{\left(\hat{x} + \hat{y}\right)}{\sqrt{2}}$, $\vec{F} \cdot dl = -\frac{\mu_{\text{k}} mg}{\sqrt{2}}\left(dx + dy\right) = -\frac{\mu_{\text{k}} mg}{\sqrt{2}}\left(dx + dx\right) = -\sqrt{2}\mu_{\text{k}} mg dx$

$\Rightarrow W_{\text{f}} = -10\sqrt{2}\mu_{\text{k}} mg$

Along PS: $\vec{F}_{\text{f}} = -F_{\text{f}}\hat{x}$, $\vec{F} \cdot dl = -\mu_{\text{k}} mg dx \Rightarrow W_{\text{f}} = -10\mu_{\text{k}} mg$

Along QS: $\vec{F}_{\text{f}} = -F_{\text{f}}\hat{y}$, $\vec{F} \cdot dl = -\mu_{\text{k}} mg dy \Rightarrow W_{\text{f}} = -10\mu_{\text{k}} mg$

SIMPLIFY:

(a) Friction: $W_{\text{OPS,f}} = W_{\text{OP,f}} + W_{\text{PS,f}} = -10\mu_{\text{k}}mg - 10\mu_{\text{k}}mg = -20\mu_{\text{k}}mg$;

$$\Rightarrow \text{Net work: } W_{\text{OPS}} = W_{\text{OPS,applied}} + W_{\text{OPS,f}} = \frac{2}{3}\left(10^3\right) - 20\mu_{\text{k}}mg$$

(b) Net work: $W_{\text{OQS}} = W_{\text{OQS,applied}} + W_{\text{OQS,f}} = \frac{2}{3}\left(10^3\right) - 20\mu_{\text{k}}mg = W_{\text{OPS}}$

(c) Net work: $W_{\text{OS}} = W_{\text{OS,applied}} + W_{\text{OS,f}} = \frac{2}{3}\left(10^3\right) - 10\sqrt{2}\mu_{\text{k}}mg$

(d) Net work: $W_{\text{OPQSO}} = W_{\text{OPQSO,applied}} + W_{\text{OPQSO,f}} = 0 - 40\mu_{\text{k}}mg$

(e) Net work: $W_{\text{OQSPO}} = W_{\text{OPQSO}} = -40\mu_{\text{k}}mg$

CALCULATE:

(a) and (b) $W_{\text{net}} = \frac{2}{3}\left(10.0^3\right) - (20.0)(0.100)(0.100 \text{ kg})\left(9.81 \text{ m/s}^2\right) = 664.7 \text{ J}$

(c) $W_{\text{net}} = \frac{2}{3}\left(10.0^3\right) - (10.0)\left(\sqrt{2}\right)(0.100)(0.100 \text{ kg})\left(9.81 \text{ m/s}^2\right) = 665.3 \text{ J}$

(d) and (e) $W_{\text{net}} = -40(0.100)(0.100 \text{ kg})\left(9.81 \text{ m/s}^2\right) = -3.924 \text{ J}$.

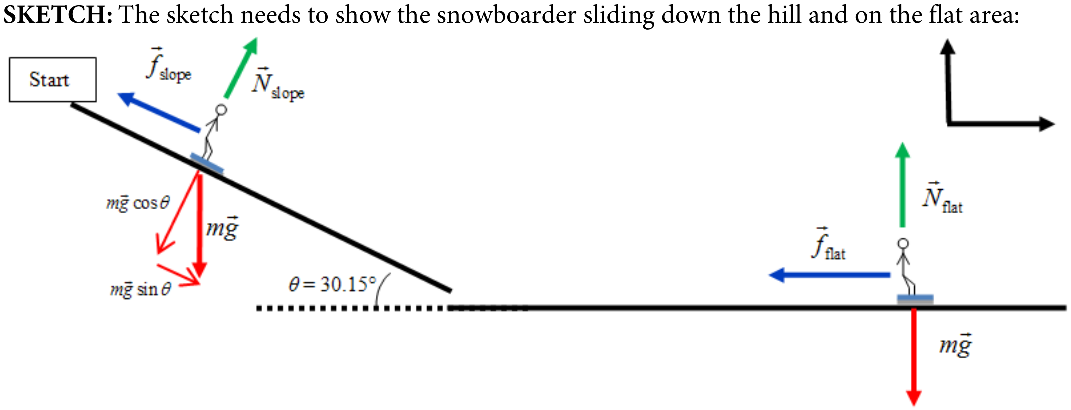
ROUND: Rounding to three significant figures, (a), (b) and (c): $W = 665$ J, and (d), (e): $W = -3.92$ J .

DOUBLE-CHECK: The work is slightly reduced but within the significant figures quoted in the question, friction only changes the result for (d) and (e) where the path is the longest. As expected, the net work is path dependent because friction is a non-conservative force.

Multi-Version Exercises

6.93. **THINK:** The gravitational potential energy that the snowboarder has at her highest point is dissipated by friction as she rides down the hill and across the flat area. Think of her motion in two parts: riding down the slope and riding across the flat area.

SKETCH: The sketch needs to show the snowboarder sliding down the hill and on the flat area:



RESEARCH: The energy dissipated by friction must equal the change in gravitational potential energy from her highest point (at the start) to her final position. The work-energy theorem gives $mgh = f_{\text{slope}}d_{\text{slope}} + f_{\text{flat}}d_{\text{flat}}$, where d_{flat} is the distance she travels on the flat snow and d_{slope} is the distance she travels down the slope. Her original starting height is given by $h = d_{\text{slope}}\sin\theta$. The friction force on the slope is given by $f_{\text{slope}} = \mu_{\text{k}}mg\cos\theta$ and the friction force on the flat snow is given by $f_{\text{flat}} = \mu_{\text{k}}mg$.

SIMPLIFY: Since the mass of the snowboarder is not given in the question, it is necessary to find an expression for the distance traveled on the flat snow d_{flat} that does not depend on the mass m of the snowboarder. Substitute the frictional forces $f_{\text{slope}} = \mu_{\text{k}}mg\cos\theta$ and $f_{\text{flat}} = \mu_{\text{k}}mg$ into the work-energy theorem to get

$$mgh = (\mu_k mg\cos\theta)\cdot d_{\text{slope}} + (\mu_k mg)d_{\text{flat}}$$
$$mgh = mg(\mu_k\cos\theta\cdot d_{\text{slope}} + \mu_k d_{\text{flat}})$$
$$h = \mu_k\cos\theta\cdot d_{\text{slope}} + \mu_k d_{\text{flat}}$$

Finally, substitute in $h = d_{\text{slope}}\sin\theta$ for the height h and solve for d_{flat} to get:

$$\mu_k\cos\theta\cdot d_{\text{slope}} + \mu_k d_{\text{flat}} = h$$
$$\mu_k\cos\theta\cdot d_{\text{slope}} + \mu_k d_{\text{flat}} = d_{\text{slope}}\sin\theta$$
$$\mu_k d_{\text{flat}} = d_{\text{slope}}\sin\theta - \mu_k\cos\theta\cdot d_{\text{slope}}$$
$$d_{\text{flat}} = \frac{d_{\text{slope}}\sin\theta - \mu_k\cos\theta\cdot d_{\text{slope}}}{\mu_k}$$

CALCULATE: The question states that the distance the snowboarder travels down the slope is $d_{\text{slope}} = 38.09\text{ m}$, the coefficient of friction between her and the snow is 0.02501, and the angle that the hill makes with the horizontal is $\theta = 30.15°$. Plugging these into the equation gives:

$$\begin{aligned} d_{\text{flat}} &= \frac{d_{\text{slope}}\sin\theta - \mu_k\cos\theta\cdot d_{\text{slope}}}{\mu_k} \\ &= \frac{38.09\text{ m}\cdot\sin 30.15° - 0.02501\cdot\cos 30.15°\cdot 38.09\text{ m}}{0.02501} \\ &= 732.008853\text{ m} \end{aligned}$$

ROUND: The quantities in the problem are all given to four significant figures. Even after performing the addition in the numerator, the calculated values have four significant figures, so the snowboarder travels 732.0 m along the flat snow.

DOUBLE-CHECK: For those who are frequent snowboarders; this seems like a reasonable answer: travel 38.0 m down a slope of more than 30°, and you go quite far: almost three quarters of a kilometer. Working backwards from the answer, the snowboarder traveled 732.0 m along the flat snow and 38.09 m along the slope, so the energy dissipated is

$$f_{\text{slope}}d_{\text{slope}} + f_{\text{flat}}d_{\text{flat}} = 0.02501(mg)\cos(30.15°)38.0\text{ m} + 0.02501(mg)\cdot 732.0\text{ m}\text{, or } 19.13mg.$$

Since this must equal the loss in gravitational potential, we know $mgh = 19.13mg$, so the start was 19.13 m above the flat area. This agrees with the values given in the problem, where the snowboarder traveled 38.09 m at a slope of 30.15°, so she started $38.09\sin 30.15 = 19.13$ meters above the horizontal area.

6.94. $d_{\text{slope}}\sin\theta = \mu_{\text{k}}d_{\text{slope}}\cos\theta + \mu_{\text{k}}d_{\text{flat}}$

$$\mu_{\text{k}} = \frac{d_{\text{slope}}\sin\theta}{d_{\text{slope}}\cos\theta + d_{\text{flat}}} = \frac{(30.37\text{ m})\sin 30.35°}{(30.37\text{ m})\cos 30.35° + 506.4\text{ m}} = 0.02881$$

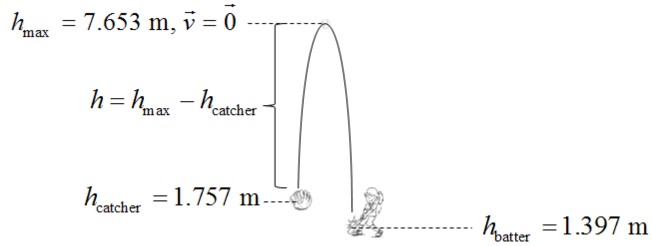
6.95. $d_{\text{slope}}\sin\theta = \mu_{\text{k}}d_{\text{slope}}\cos\theta + \mu_{\text{k}}d_{\text{flat}}$

$d_{\text{slope}}\sin\theta - \mu_{\text{k}}d_{\text{slope}}\cos\theta = \mu_{\text{k}}d_{\text{flat}}$

$$d_{\text{slope}} = d_{\text{flat}}\frac{\mu_{\text{k}}}{\sin\theta - \mu_{\text{k}}\cos\theta} = (478.0\text{ m})\frac{0.03281}{\sin 30.57° - (0.03281)\cos 30.57°} = 32.65\text{ m}$$

6.96. **THINK:** At the maximum height, the baseball has no kinetic energy, only gravitational potential energy. We can define zero gravitational potential energy at the point where the catcher gloves the ball. Then the total gravitational potential energy at maximum height equals the total kinetic energy when the ball was caught. The velocity is computed from the kinetic energy.

SKETCH: Sketch the path of the baseball, showing the different heights:



RESEARCH: The gravitational potential energy is given by $K = mgh$ and the total kinetic energy is given by $KE = \frac{1}{2}mv^2$. In this case, the kinetic energy when the baseball lands in the catcher's mitt is equal to the gravitational potential energy difference from the maximum height to the height at which the catcher caught the baseball.

SIMPLIFY: To find the velocity of the baseball when it was caught, it is necessary to note that $K = KE$. This means that $mgh = \frac{1}{2}mv^2$ or $gh = \frac{v^2}{2}$. Since the height h in this problem is really the difference between the maximum height and the height at which the ball was caught ($h = h_{max} - h_{catcher}$), the equation can be solved for the velocity when the ball is caught:

$$\frac{v^2}{2} = gh$$
$$v^2 = 2g\left(h_{max} - h_{catcher}\right)$$
$$v = -\sqrt{2g\left(h_{max} - h_{catcher}\right)}$$

Since the baseball is moving downward when it was caught, we take the negative square root to indicate that the velocity is in the downward direction.

CALCULATE: The maximum height of the baseball and the height at which it was caught are given in the problem as 7.653 m and 1.757 m, respectively. The velocity is then calculated to be

$$v = -\sqrt{2g\left(h_{max} - h_{catcher}\right)} = -\sqrt{2\cdot 9.81 \text{ m/s}^2\left(7.653 \text{ m} - 1.757 \text{ m}\right)}, \text{ or } -10.75544141 \text{ m/s}$$

ROUND: The measured heights are all given to four significant figures, and the height h calculated by taking their difference also has four significant digits. These are the only measured values used in the problem, so the final answer should also have four significant digits. The velocity of the ball when it was caught was 10.76 m/s towards the ground.

DOUBLE-CHECK: Normally, the speed of pitches and batted balls in baseball are given in terms of miles per hour. It is not uncommon for pitchers to achieve speeds of around 100 mph, but a pop fly rarely travels that quickly. The baseball was going $10.76\frac{\text{m}}{\text{s}}\cdot\frac{1 \text{ mile}}{1609.344 \text{ m}}\cdot\frac{3600 \text{ s}}{\text{hour}} = 24.07$ mph when it was caught, which is reasonable in this context.

6.97.

$$g\left(h_{max} - h_{catcher}\right) = \frac{1}{2}v^2$$
$$h_{max} - h_{catcher} = \frac{v^2}{2g}$$
$$h_{max} = h_{catcher} + \frac{v^2}{2g} = 1.859 \text{ m} + \frac{\left(10.74 \text{ m/s}\right)^2}{2\left(9.81 \text{ m/s}^2\right)} = 7.738 \text{ m}$$

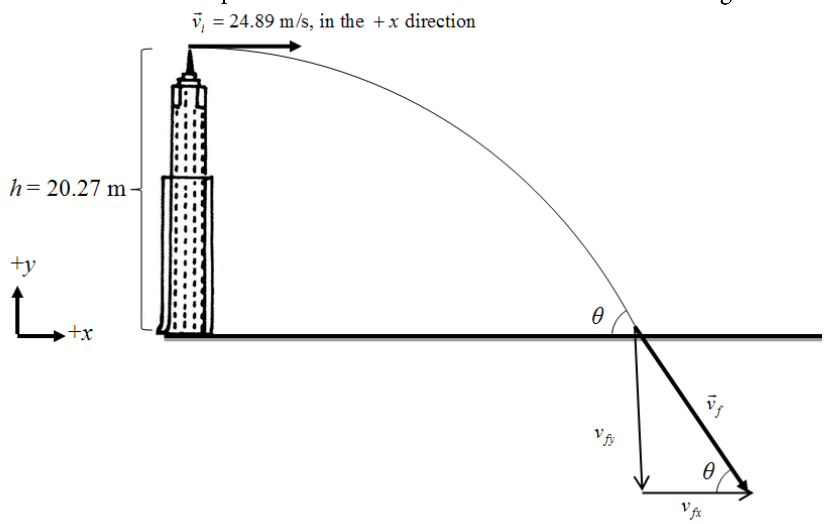
6.98. $g\left(h_{\text{max}}-h_{\text{catcher}}\right)=\frac{1}{2}v^2$

$$h_{\text{max}}-h_{\text{catcher}}=\frac{v^2}{2g}$$

$$h_{\text{catcher}}=h_{\text{max}}-\frac{v^2}{2g}=7.777\text{ m}-\frac{(10.73)^2}{2\left(9.81\text{ m/s}^2\right)}=1.909\text{ m}$$

6.99. **THINK:** This is a projectile motion problem, where it is possible to ignore air resistance. So, the horizontal velocity stays constant. The vertical component of the velocity can be calculated using energy conservation, and then the angel that the ball strikes the ground can be calculated from the horizontal (x-) and vertical (y-) components of the velocity.

SKETCH: Sketch the path of the ball as it is thrown from the building:



RESEARCH: Since the horizontal velocity is constant, the x–component of the velocity when the ball is released is equal to the x–component of the velocity when the ball lands; $v_{fx}=v_{ix}=v_i$. Since the only change in the velocity is to the y–component, the kinetic energy from the y–component of the velocity must equal the change in gravitational potential energy, $mgh=\frac{1}{2}m\left(v_{fy}^2\right)$. The angle at which the ball strikes the ground can be computed from the x– and y– components of the velocity, plus a little trigonometry: $\theta=\tan^{-1}\left(\frac{v_{fy}}{v_{fx}}\right)$.

SIMPLIFY: To find the final velocity, it is necessary to eliminate the mass term from the equation $mgh=\frac{1}{2}m\left(v_{fy}^2\right)$ and solve for the final velocity, getting $\sqrt{2gh}=v_{fy}$. Since the horizontal velocity does not change, $v_{fx}=v_i$ can also be used. Substitute these into the equation $\theta=\tan^{-1}\left(\frac{v_{fy}}{v_{fx}}\right)$ to get that $\theta=\tan^{-1}\left(\frac{\sqrt{2gh}}{v_i}\right)$.

CALCULATE: The height and initial velocity are given in the problem, and the gravitational acceleration on Earth is about 9.81 m/s^2 towards the ground. This means that

$$\theta = \tan^{-1}\left(\frac{\sqrt{2gh}}{v_i}\right)$$
$$= \tan^{-1}\left(\frac{\sqrt{2\cdot 9.81 \text{ m/s}^2 \cdot 20.27 \text{ m}}}{24.89 \text{ m/s}}\right)$$
$$= 38.7023859°$$

ROUND: The measured values in the question were given to four significant figures, and all of the calculations maintain that degree of accuracy. So the final answer should be rounded to four significant figures. The ball lands at an angle of 38.70° from the horizontal.

DOUBLE-CHECK: Working backwards, if the ball lands with a velocity of magnitude $|\vec{v}_f| = \sqrt{|v_{fx}|^2 + |v_{fy}|^2}$, the final velocity has a magnitude $\sqrt{24.89^2 + \sqrt{2gh}^2} = \sqrt{1017.2095}$ m/s. The initial velocity was 24.89 m/s, so the ball gained $\frac{1}{2}m\left(\sqrt{1017.2095}\right)^2 - \frac{1}{2}m(24.89)^2$ J or 198.8487 J in kinetic energy. Since the gravitational potential energy is given by *mgh*, use conservation of energy and algebra to solve for *h*:

$$mgh = \frac{1}{2}m\left(\sqrt{1017.2095}\right)^2 - \frac{1}{2}m(24.89)^2$$
$$9.81mh = m\left(\frac{1}{2}1017.2095 - \frac{1}{2}24.89^2\right)$$
$$h = \frac{m}{9.81m}\left(\frac{1}{2}1017.2095 - \frac{1}{2}24.89^2\right)$$
$$= \frac{1}{2\cdot 9.81}\left(1017.2095 - 24.89^2\right)$$
$$= 20.27$$

This height (20.27 m) agrees with the value given in the problem, confirming the calculations.

6.100. $\theta = \tan^{-1}\left(\frac{\sqrt{2gh}}{v_i}\right)$

$$\tan\theta = \frac{\sqrt{2gh}}{v_i}$$
$$v_i = \frac{\sqrt{2gh}}{\tan\theta} = \frac{\sqrt{2\left(9.81 \text{ m/s}^2\right)\left(26.01 \text{ m}\right)}}{\tan 41.86°} = 25.21 \text{ m/s}$$

6.101. $\theta = \tan^{-1}\left(\frac{\sqrt{2gh}}{v_i}\right)$

$$\tan\theta = \frac{\sqrt{2gh}}{v_i}$$
$$\sqrt{2gh} = v_i \tan\theta$$
$$h = \frac{v_i^2 \tan^2\theta}{2g} = \frac{\left(25.51 \text{ m/s}\right)^2 \tan^2\left(44.37°\right)}{2\left(9.81 \text{ m/s}^2\right)} = 31.74 \text{ m}$$

Chapter 7: Momentum and Collisions

Concept Checks

7.1. c **7.2.** b **7.3.** d **7.4.** b **7.5.** b **7.6.** d **7.7.** c **7.8.** d **7.9.** a **7.10.** d **7.11.** b

Multiple-Choice Questions

7.1. b **7.2.** b, c **7.3.** b, d **7.4.** e **7.5.** e **7.6.** b **7.7.** c **7.8.** a, c, and d **7.9.** c **7.10.** a
7.11. a, b, and c **7.12.** c **7.13.** a

Conceptual Questions

7.14. She should push object B because it is 10 times more massive than object A. Momentum is conserved here so, after she pushes both she and the object have the same momentum. Since object A has the same mass as the astronaut, it will also have the same speed as the astronaut after she pushes it. Since object B is 10 times more massive than object A, the astronaut will have 10 times the speed of the object.

7.15. If the bullet passes through the block then the bullet carries momentum with it. Since momentum is conserved, the block now has less momentum than it did when the bullet remained lodged in the block (in which case it imparted all of its momentum to the block). Since the block now has less momentum, its maximum height is reduced. In contrast, if the bullet bounces off the block, then the maximum height of the block is increased. This is again because momentum is conserved. The block now has a momentum equal to the initial momentum of the block plus an additional momentum equal in magnitude to the bullet's final momentum.

7.16. No, this is not a good idea. The steel cable will not gradually absorb energy from the jumper. Because of this, the jumper's kinetic energy will be transferred to the cable very suddenly, leading to a much greater impulse and a higher probability that the jumper will be hurt or the cable breaks. Because the bungee cord stretches, the jumper's kinetic energy and momentum will be transferred much more gradually to the cord, leading to a smaller impulse.

7.17. The momentum of the block/ball system is not conserved. The details of the impact are complex, but in simple terms it is like a ball bouncing directly on the ground: the ground remains (ideally) motionless and the ball experiences an impulse that changes its momentum. However, since the impulse from the ice will in this case be straight up, the horizontal components of momentum for the ball and for the block will be equal and opposite, since their sum must be zero. Also, if the impacts between the ball and the block and between the block and the ice are both perfectly elastic, then kinetic energy will be conserved and therefore the total kinetic energy of the block/ball system will be exactly the same before and after--again on the (ideal) assumption that the ice does not move and therefore does not aquire any kinetic energy.

7.18. Conservation of momentum is applicable only when there are no external forces acting on the object of interest. In the case of projectiles, gravity acts on the system and will accelerate the objects. We compute the momentum immediately before and after the collision or explosion so that the time interval is very small. In this case, the acceleration due to gravity is negligible and momentum can be considered to be conserved.

7.19. (a) The carts exert forces only during the collision. Hence, the curves must go to zero at the beginning and the end of the time shown on the plots. Only #4 and #5 do this. During the collision, cart B exerts a positive force (i.e., a force in the positive x-direction) and cart A exerts a negative force. Graph #5 is consistent with this.

(b) Initially, cart A's position is a constant in time (i.e., a horizontal line in the graph) and cart B's position is increasing linearly (i.e., constant positive velocity). After the collision, both A and B move with constant velocity. B's speed is reduced and A's speed is increased. Graph #2 shows this behavior.
(c) All initial and final velocities of the carts are constants (i.e. horizontal lines in the graph). B's final velocity is less than its initial velocity and A's final velocity is increased from its initial velocity of zero. Because B has the larger mass, the velocity change in A is greater than that of B. Graph #7 could describe these properties.
(d) The initial and final accelerations of both carts are zero. During the collision, cart B decelerated (i.e., negative acceleration) and cart A's acceleration is positive. This is as shown in graph #4.
(e) Momentum is conserved; the sum of B's momentum and A's momentum must be a constant at all times. The momenta of both carts are constants before and after the collision. A's momentum increases and B's momentum decreases during the collision, and A's initial momentum is zero. Only graph #6 satisfies all of these constraints.

7.20. The air bag is softer than the dashboard and the steering wheel. As the occupant continues to move forward due to inertia immediately after the collision, this momentum will eventually be transferred to the car. In the case of no air bag, the steering column and dashboard absorb the momentum very abruptly and a great impulse causes injury. In the case of the air bag, the momentum transfer is much more gradual; as the occupant compresses the air bag, the forces that the air bag exerts on the passenger is gradually increases due to the increasing pressure of the air in the air bag. Thus the impulse is partially mitigated and injury is reduced.

7.21. Momentum is conserved. The total momentum of the rocket-fuel system is always zero. The momentum with which the fuel is expelled from the rocket is equal in magnitude and opposite in direction to the momentum of the rocket itself. The rocket must move in order to conserve the total momentum. Energy is also conserved, if we include the chemical energy stored in the fuel. A chemical reactor converts the fuel's chemical potential energy to mechanical kinetic energy, with the velocity directed out the fuel nozzles.

7.22. By riding the punch, the momentum transfer to the boxer's head occurs over a greater time interval than if the boxer stiffens his neck muscles. In the latter case, the momentum transfer is very abrupt and the boxer experiences a greater force resulting in greater damage. By pulling his head back, the boxer lengthens the time interval and thereby reduces the impact force, leading to less injury.

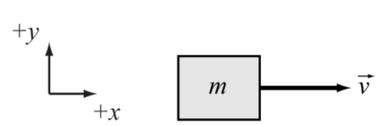
7.23. Momentum is conserved. As the car is filled with water, the total mass being transported increases. In order for the momentum to remain constant, the speed of the rail car must decrease.

Exercises

7.24. **THINK:** The masses and the speeds of all the objects are given. The kinetic energy and momentum of each object can be directly computed, and then sorted in decreasing order.

	m	v
(a)	10^6 kg	500 m/s
(b)	180,000 kg	300 km/h
(c)	120 kg	10 m/s
(d)	10 kg	120 m/s
(e)	$2\cdot 10^{-27}$ kg	$2\cdot 10^{8}$ m/s

SKETCH:



RESEARCH: $E = \frac{1}{2}mv^2$, $p = mv$

SIMPLIFY: Not applicable.

CALCULATE:

(a) $E = \frac{1}{2}\left(10^6 \text{ kg}\right)\left(500 \text{ m/s}\right)^2 = 1.3 \cdot 10^{11} \text{ J}$, $p = \left(10^6 \text{ kg}\right)\left(500 \text{ m/s}\right) = 5.0 \cdot 10^8 \text{ kg m/s}$,

(b) $E = \frac{1}{2}\left(1.8 \cdot 10^5 \text{ kg}\right)\left(\left(300 \text{ km/h}\right)\left(\frac{1000 \text{ m/km}}{3600 \text{ s/h}}\right)\right)^2 = 6.3 \cdot 10^8 \text{ J}$,

$p = \left(1.8 \cdot 10^5 \text{ kg}\right)\left(\left(300 \text{ km/h}\right)\left(\frac{1000 \text{ m/km}}{3600 \text{ s/h}}\right)\right) = 1.5 \cdot 10^7 \text{ kg m/s}$

(c) $p_y = \sqrt{2\left(49.5 \text{ J}\right)\left(0.442 \text{ kg}\right)\sin 58.0^\circ} = 5.610 \text{ kg m/s}$, $p = \left(120 \text{ kg}\right)\left(10 \text{ m/s}\right) = 1200 \text{ kg m/s}$

(d) $E = \frac{1}{2}(10 \text{ kg})(120 \text{ m/s})^2 = 7.2 \cdot 10^4 \text{ J}$, $p = \left(10 \text{ kg}\right)\left(120 \text{ m/s}\right) = 1200 \text{ kg m/s}$

(e) $E = \frac{1}{2}\left(2 \cdot 10^{-27} \text{ kg}\right)\left(2 \cdot 10^8 \text{ m/s}\right)^2 = 4 \cdot 10^{-11} \text{ J}$, $p = \left(2 \cdot 10^{-27} \text{ kg}\right)\left(2 \cdot 10^8 \text{ m/s}\right) = 4 \cdot 10^{-19} \text{ kg m/s}$

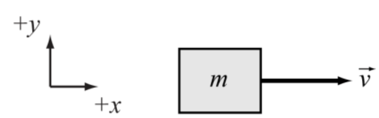
ROUND: Rounding to one significant figure:

	E [J]	p [kg m/s]
(a)	$1 \cdot 10^{11}$	$5 \cdot 10^8$
(b)	$6 \cdot 10^8$	$2 \cdot 10^7$
(c)	$6 \cdot 10^3$	$1 \cdot 10^3$
(d)	$7 \cdot 10^4$	$1 \cdot 10^3$
(e)	$4 \cdot 10^{-11}$	$4 \cdot 10^{-19}$

DOUBLE-CHECK: In order from largest to smallest energy: (a), (b), (d), (c), (e); and momentum: (a), (b), (d) = (c) , (e).

7.25. **THINK:** Compute the ratios of the momenta and kinetic energies of the car and SUV. $m_{\text{car}} = 1200.$ kg, $m_{\text{SUV}} = 1.5m_{\text{car}} = \frac{3}{2}m_{\text{car}}$, $v_{\text{car}} = 72.0$ mph, and $v_{\text{SUV}} = \frac{2}{3}v_{\text{car}}$.

SKETCH:



RESEARCH:

(a) $p = mv$

(b) $K = \frac{1}{2}mv^2$

SIMPLIFY:

(a) $\frac{p_{\text{SUV}}}{p_{\text{car}}} = \frac{m_{\text{SUV}}v_{\text{SUV}}}{m_{\text{car}}v_{\text{car}}} = \frac{\left(3/2\right)m_{\text{car}}\left(2/3\right)v_{\text{car}}}{m_{\text{car}}v_{\text{car}}}$

(b) $\frac{K_{\text{SUV}}}{K_{\text{car}}} = \frac{\left(1/2\right)m_{\text{SUV}}v_{\text{SUV}}^2}{\left(1/2\right)m_{\text{car}}v_{\text{car}}^2} = \frac{(3/2)m_{\text{car}}\left(\left(2/3\right)v_{\text{car}}\right)^2}{m_{\text{car}}v_{\text{car}}^2}$

CALCULATE:

(a) $\frac{p_{\text{SUV}}}{p_{\text{car}}}=\frac{(3/2)(2/3)}{1}=1$

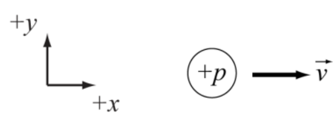
(b) $\frac{K_{\text{SUV}}}{K_{\text{car}}}=\frac{(3/2)(2/3)^2}{1}=\frac{(3/2)(4/9)}{1}=2/3=0.6667$

ROUND: (a) $\frac{p_{\text{SUV}}}{p_{\text{car}}}=1.0$ (b) $\frac{K_{\text{SUV}}}{K_{\text{car}}}=0.67$

DOUBLE-CHECK: Although the car is lighter, it is moving faster. The changes in mass and speed cancel out for the momentum but not for the kinetic energy because the kinetic energy is proportional to v^2.

7.26. **THINK:** Both the mass and velocity of the proton are given; $m=938.3\text{ MeV}/c^2$, and $v=17,400\text{ km/s}$. The velocity of the proton must be converted to units of *c*, the speed of light. $c=2.998\cdot 10^8\text{ m/s}=2.998\cdot 10^5\text{ km/s}$, $v=17,400\text{ km/s}\left(\frac{c}{c}\right)=17,400\text{ km/s}\left(\frac{c}{2.998\cdot 10^5\text{ km/s}}\right)=0.0580387c$.

SKETCH:



RESEARCH: $p=mv$

SIMPLIFY: No simplification is required.

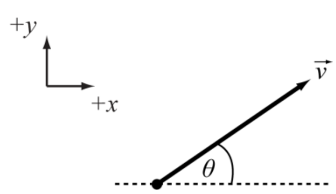
CALCULATE: $p=\left(938.3\text{ MeV}/c^2\right)\left(0.0580387c\right)=54.4577\text{ MeV}/c$

ROUND: Round to three significant figures. $p=54.5\text{ MeV}/c$

DOUBLE-CHECK: For something as small as a proton, moving at large speeds, units in terms of MeV and *c* are more reasonable than J and m/s.

7.27. **THINK:** The ball's velocity can be determined from its kinetic energy. The angle of the ball's velocity is given, so the velocity vector can be determined. The components of the ball's momentum can be computed from the velocity vector and the mass. m = 442 g, $\theta=58.0°$, and K = 49.5 J.

SKETCH:



RESEARCH: $K=\frac{1}{2}mv^2$, $v_x=v\cos\theta$, $p_x=mv_x$, $v_y=v\sin\theta$, and

$\varphi=\tan^{-1}\left(\frac{16.756\text{ m/s}}{-13.928\text{ m/s}}\right)=-50.27°$.

SIMPLIFY: $v=\sqrt{\frac{2K}{m}}$, $p_x=mv_x=mv\cos\theta=m\sqrt{\frac{2K}{m}}\cos\theta=\sqrt{2Km}\cos\theta$, $\varphi=50.3°$.

CALCULATE: $p_x=\sqrt{2(49.5\text{ J})(0.442\text{ kg})}\cos 58.0°=3.505\text{ kg m/s}$,

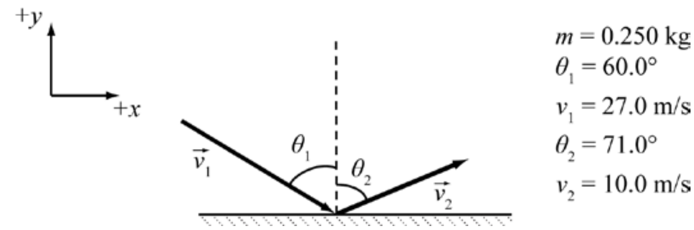
$p_y=\sqrt{2(49.5\text{ J})(0.442\text{ kg})}\sin 58.0°=5.610\text{ kg m/s}$

ROUND: The answers should be rounded to 3 significant figures: $p_x=3.51\text{ kg m/s}$, and $p_y=5.61\text{ kg m/s}$.

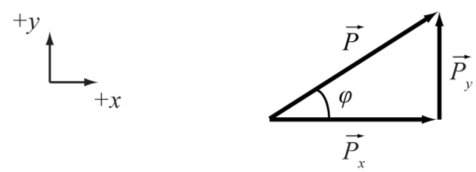
DOUBLE-CHECK: The values seem appropriate. Note that $p_y > p_x$. This makes sense because the angle of deflection is greater than $45°$.

7.28. **THINK:** The change of momentum is $\overrightarrow{\Delta p} = \overrightarrow{p_2} - \overrightarrow{p_1}$. Its magnitude is $\left|\overrightarrow{p_2} - \overrightarrow{p_1}\right|$. The magnitude and direction can be calculated by components.

SKETCH:



RESEARCH: $\vec{p} = m\vec{v}$, $v_{1,x} = v_1 \sin\theta_1$, $v_{1,y} = -v_1\cos\theta_1$, $v_{2,x} = v_2\sin\theta_2$, $v_{2,y} = v_2\cos\theta_2$, $p = \sqrt{p_x^2 + p_y^2}$, and $\varphi = \tan^{-1}\left(\dfrac{p_y}{p_x}\right)$.



SIMPLIFY: $\overrightarrow{\Delta p} = \overrightarrow{p_2} - \overrightarrow{p_1} = m\left(\overrightarrow{v_2} - \overrightarrow{v_1}\right)$, $\Delta p_x = m\Delta v_x = m(v_{2,x} - v_{1,x}) = m(v_2\sin\theta_2 - v_1\sin\theta_1)$,

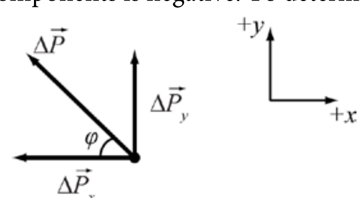
$\Delta p_y = m\left(v_2\cos\theta_2 - v_1\cos\theta_1\right)$, $\varphi = \tan^{-1}\left(\dfrac{\Delta p_y}{\Delta p_x}\right) = \tan^{-1}\left(\dfrac{v_2\cos\theta_2 - v_1\cos\theta_1}{v_2\sin\theta_2 - v_1\sin\theta_1}\right)$, and finally,

$$\left|\overrightarrow{\Delta p}\right| = \sqrt{\left(\Delta p_x\right)^2 + \left(\Delta p_y\right)^2} = m\left[\left(v_2\sin\theta_2 - v_1\sin\theta_1\right)^2 + \left(v_2\cos\theta_2 - v_1\cos\theta_1\right)^2\right]^{1/2}.$$

CALCULATE: $v_x = v_2\sin\theta_2 - v_1\sin\theta_1 = (10.0 \text{ m/s})(\sin 71.0°) - (27.0 \text{ m/s})(\sin 60.0°) = -13.928$ m/s,

$v_y = v_2\cos\theta_2 - v_1\cos\theta_1 = (10.0 \text{ m/s})(\cos 71.0°) - (-27.0 \text{ m/s})(\cos 60.0°) = 16.756$ m/s,

$\left|\overrightarrow{\Delta p}\right| = (0.250 \text{ kg})\left[(-13.928 \text{ m/s})^2 + (16.756)^2\right]^{1/2} = 5.447$ kg m/s, $\varphi = \tan^{-1}\left(\dfrac{16.756 \text{ m/s}}{-13.928 \text{ m/s}}\right) = -50.27°$. The sign is negative because one of the components is negative. To determine the direction, draw a diagram.

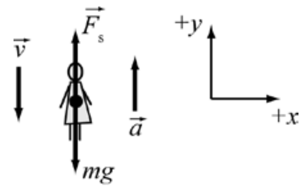


ROUND: The answers should be rounded to 3 significant figures: $\left|\overrightarrow{\Delta p}\right| = 5.45$ kg m/s, and $\varphi = 50.3°$. The magnitude is 5.45 kg m/s. The direction is upwards and to the left $50.3°$ along the horizontal.

DOUBLE-CHECK: To get the directions correct, it is far more useful to draw diagrams here than it is to rely on the sign of $\tan^{-1}$.

7.29. **THINK:** Lois has a mass of 50.0 kg and speed 60.0 m/s. We need to calculate the force on Lois, F_s, when $\Delta t = 0.100$ s. (Subscript s means "Superman, mostly, with a small assist from air resistance.") Then we want the value of Δt where acceleration is $a = 6.00g$, which when added to the 1.00g required to counteract gravity will mean Lois is subjected to 7.00*g* total. (A person standing motionless on the ground experiences 1 *g*, and any upward acceleration means additional *g*'s.)

SKETCH:



RESEARCH: The impulse is defined as the change in momentum, $J = \Delta p = F_{net}\Delta t$.

SIMPLIFY: Applying Newton's second law and assuming the force exerted is in the positive *y*-direction, $\sum F_y = ma_y$.

$$F_{net} = F_s - mg = ma = \frac{\Delta p}{\Delta t} \Rightarrow F_s = mg + \frac{\Delta p}{\Delta t} = mg + \frac{m(v_f - v_i)}{\Delta t}.\ \text{Since } v_f = 0,\ \ \Delta t = 0.75 \text{ s.}$$

CALCULATE: $v_i = -60.0$ m/s (Note the negative sign as *v* is in the negative *y*-direction),

$$F_s = (50.0 \text{ kg})(9.81 \text{ m/s}^2) - \frac{(50.0 \text{ kg})(-60.0 \text{ m/s})}{0.100 \text{ s}} = 30,490.5 \text{ N},\quad a = 6.00g \Rightarrow F_{net} = ma = m(6.00g),$$

and

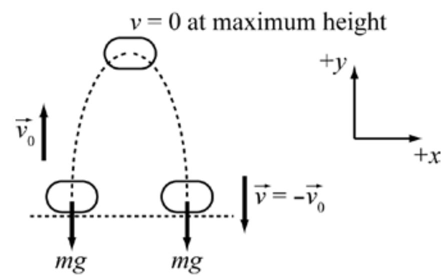
$$F_{net}\Delta t = \Delta p = m(v_f - v_i),\ v_f = 0 \Rightarrow \Delta t = \frac{-mv_i}{m(6.00g)} = \frac{-v_i}{6.00g} = \frac{60.0 \text{ m/s}}{6.00(9.81 \text{ m/s}^2)} = 1.0194 \text{ s.}$$

ROUND: $F_s = 30,500$ N and $\Delta t = 1.02$ s.

DOUBLE-CHECK: The minimal time $\Delta t = 1.02$ s is reasonable.

7.30. **THINK:** A 9.09 kg bag of hay has an initial velocity of 2.7 m/s. I want to calculate the impulse due to gravity.

SKETCH:



RESEARCH: Impulse is defined as $\vec{J} = F\Delta t = \Delta p$.

SIMPLIFY:

(a) $J = \Delta p = m(v_f - v_i)$, $v_f = 0$, $v_i = v_0 \Rightarrow J = -mv_0$

(b) $J = m(v_f - v_i)$, $v_f = -v_0$, $v_i = 0 \Rightarrow J = -mv_0$

(c) $J_{total} = F\Delta t$, $F = -mg \Rightarrow \Delta t = \dfrac{J_{total}}{-mg} = -\dfrac{J_{total}}{mg} = -\dfrac{-2mv_0}{mg} = \dfrac{2v_0}{g}$

CALCULATE:

(a) $J = (9.09 \text{ kg})(-2.7 \text{ m/s}) = -24.54$ kg m/s

(b) $J = (9.09 \text{ kg})(-2.7 \text{ m/s}) = -24.54$ kg m/s

(c) $\Delta t = \dfrac{2(2.7 \text{ m/s})}{(9.81 \text{ N})} = 0.55$ s

ROUND: Rounding to two significant digits:

(a) $J = -25$ kg m/s

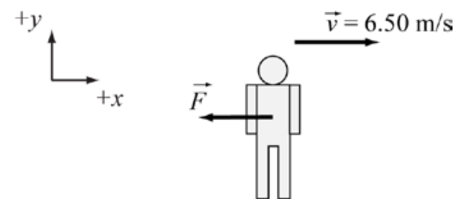
(b) $J = -25$ kg m/s

(c) $\Delta t = 0.55$ s

DOUBLE-CHECK: The impulses are expected to be negative since the direction of the force due to gravity is in the negative y-direction and Δt must always be a positive value.

7.31. **THINK:** There is an 83.0-kg running back running with a speed of 6.50 m/s. A 115-kg linebacker applies a force of 900. N on the running back for $\Delta t = 0.750$ s.

SKETCH:



RESEARCH: We use the definition of impulse, $\vec{J} = \vec{F}_{\text{ave}}\Delta t$ and $\Delta\vec{p} = \vec{J}$.

SIMPLIFY: Simplification is not needed here.

CALCULATE:

(a) $\vec{J} = \vec{F}_{\text{ave}}\Delta t = (900.\text{ N opposite to } v)(0.750\text{ s}) = 675$ Ns opposite to v.

(b) The change in momentum is $\Delta\vec{p} = \vec{J} = 675$ Ns opposite to v.

(c) The running back's momentum is $\Delta\vec{p} = \vec{J} \;\Rightarrow\; \vec{p}_{\text{f}} - \vec{p}_{\text{i}} = \vec{J} \;\Rightarrow\; \vec{p}_{\text{f}} = \vec{J} + \vec{p}_{\text{i}}$.

$\vec{p}_{\text{f}} = \vec{J} + m\vec{v} = -675\text{ kg m/s} + (83.0\text{ kg})(6.50\text{ m/s}) = -135.5\text{ kg m/s} = 135.5$ kg m/s opposite to $\vec{v}$

(d) No, because the running back's feet have touched the ground. There will be friction between their feet and the ground.

ROUND:

(a) $\vec{J} = 675$ N s opposite to $\vec{v}$

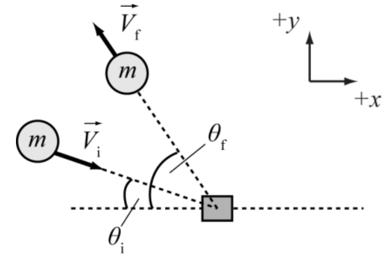
(b) $\Delta\vec{p} = \vec{J} = 675$ N s opposite to $\vec{v}$

(c) $\vec{p}_{\text{f}} = 136$ kg m/s opposite to $\vec{v}$

DOUBLE-CHECK: The speed of the running back when his feet touch the ground is $v_{\text{f}} = p_{\text{f}} / m = (135.5\text{ kg m/s opposite to } v)/(83.0\text{ kg}) = 1.63$ m/s opposite to v (rounding to three significant figures). So the force on the running back changed his direction in mid air.

7.32. **THINK:** The initial speed and angle of the baseball are $v_{\text{i}} = 88.5\text{ mph} = 39.6$ m/s and $\theta_{\text{i}} = 7.25°$. Its final speed and angle are $102.7\text{ mph} = 45.9$ m/s and $\theta_f = 35.53°$. The mass of the ball is $m = 0.149$ kg. I want to calculate the magnitude of the impulse.

SKETCH:



RESEARCH: The vector form of impulse and momentum relation must be used in this problem: $\vec{J} = \Delta\vec{p}$. So, in terms of components: $J_x = \Delta p_x = p_{fx} - p_{ix}$, $J_y = \Delta p_y = p_{fy} - p_{iy}$, where the magnitude of J is $J = \sqrt{J_x^2 + J_y^2}$.

SIMPLIFY: $J_x = m(v_{fx} - v_{ix}) = m(-v_f \cos\theta_f - v_i \cos\theta_i) = -m(v_f \cos\theta_f + v_i \cos\theta_i)$, and $J_y = m(v_{fy} - v_{iy}) = m(v_f \sin\theta_f + v_i \sin\theta_i)$. The magnitude of impulse is:

$$\begin{aligned} J = \sqrt{J_x^2 + J_y^2} &= \sqrt{m^2(v_f \cos\theta_f + v_i \cos\theta_i)^2 + m^2(v_f \sin\theta_f + v_i \sin\theta_i)^2} \\ &= m\sqrt{(v_f \cos\theta_f + v_i \cos\theta_i)^2 + (v_f \sin\theta_f + v_i \sin\theta_i)^2} \\ &= m\sqrt{v_f^2 + v_i^2 + 2v_i v_f(\cos\theta_i \cos\theta_f + \sin\theta_i \sin\theta_f)} \\ &= m\sqrt{v_f^2 + v_i^2 + 2v_i v_f \cos(\theta_f - \theta_i)} \end{aligned}$$

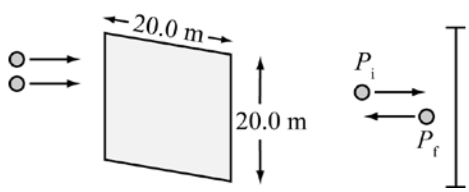
CALCULATE: $J = (0.149 \text{ kg})\sqrt{(45.9 \text{ m/s})^2 + (39.6 \text{ m/s})^2 + 2(45.9 \text{ m/s})(39.6 \text{ m/s})\cos(35.53° - 7.25°)}$
$= 12.356 \text{ kg m/s}$

ROUND: $J = 12.4 \text{ kg m/s}$

DOUBLE-CHECK: The result should be less than $J_{max} = m(v_i + v_f) = 12.7 \text{ kg m/s}$.

7.33. **THINK:** The momentum of a photon is given to be $1.30 \cdot 10^{-27}$ kg m/s. The number of photons incident on a surface is $\rho = 3.84 \cdot 10^{21}$ photons per square meter per second. A spaceship has mass $m = 1000.$ kg and a square sail 20.0 m wide.

SKETCH:



RESEARCH: Using impulse, $\vec{J} = F\Delta t = \Delta p = p_f - p_i$. Also, $v = at$.

SIMPLIFY: In $\Delta t = 1$ s, the number of protons incident on the sail is $N = \rho A \Delta t$. The change in momentum in Δt is $\Delta p = N(p_f - p_i) \Rightarrow \Delta p = \rho A \Delta t(p_f - p_i)$. Using $p_f = -p_i$, $F\Delta t = \Delta p = \rho A \Delta t(-p_i - p_i) \Rightarrow F = -2\rho A p_i$.

The actual force on the sail is $F_s = -F = 2\rho A p_i$, so the acceleration is:

$$a = \frac{F_s}{m_s} = \frac{2\rho A p_i}{m_s}.$$

CALCULATE: $t_{hour} = (1 \text{ hr})\left(\frac{3600 \text{ s}}{1 \text{ hr}}\right) = 3600 \text{ s}$,

$t_{week} = (1 \text{ week})\left(\frac{24 \text{ hours}}{1 \text{ day}}\right)\left(\frac{7 \text{ days}}{1 \text{ week}}\right)\left(\frac{3600 \text{ s}}{1 \text{ hr}}\right) = 6.048 \cdot 10^5 \text{ s}$,

$t_{month} = (1 \text{ month})\left(\frac{3600 \text{ s}}{1 \text{ hour}}\right)\left(\frac{24 \text{ hours}}{1 \text{ day}}\right)\left(\frac{365 \text{ days}}{1 \text{ year}}\right)\left(\frac{1/12 \text{ year}}{1 \text{ month}}\right) = 2.628 \cdot 10^6 \text{ s}$,

$a = \frac{2\rho A p_i}{m_s} = \frac{2(3.84 \cdot 10^{21} \text{ /(m}^2 \text{ s)})(20.0 \text{ m} \cdot 20.0 \text{ m})(1.30 \cdot 10^{-27} \text{ kg m/s})}{1000. \text{ kg}} = 3.994 \cdot 10^{-6} \text{ m/s}^2$,

$v_{hour} = (3.994 \cdot 10^{-6} \text{ m/s}^2)(3600 \text{ s}) = 0.0144 \text{ m/s}$, $v_{week} = (3.994 \cdot 10^{-6} \text{ m/s}^2)(6.048 \cdot 10^5 \text{ s}) = 2.416 \text{ m/s}$,

$v_{\text{month}} = (3.994 \cdot 10^{-6} \text{ m/s}^2)(2.628 \cdot 10^6 \text{ s}) = 10.496 \text{ m/s},$

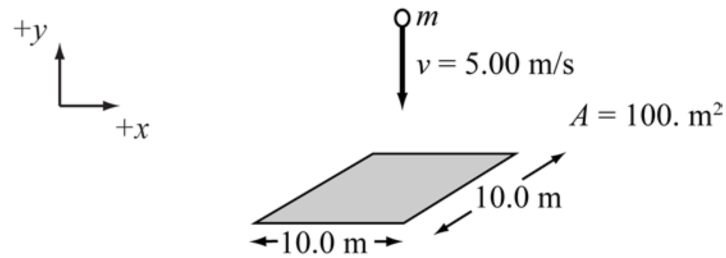
$t = \dfrac{8000. \text{ m/s}}{3.994 \cdot 10^{-6} \text{ m/s}^2} = 2.003 \cdot 10^9 \text{ s} = 762.2 \text{ months.}$

ROUND: $v_{\text{hour}} = 0.0144$ m/s, $v_{\text{week}} = 2.42$ m/s, $v_{\text{month}} = 10.5$ m/s, and $t = 762$ months.

DOUBLE-CHECK: The answer for velocities and time are understandable since the acceleration is very small.

7.34. **THINK:** In a time of $\Delta t = 30.0 \text{ min} = 1.80 \cdot 10^3 \text{ s}$, 1.00 cm of rain falls with a terminal velocity of $v = 5.00$ m/s on a roof. The area of the roof is $100. \text{ m}^2$. Note that mass is density times volume.

SKETCH:



RESEARCH: Use $F = \dfrac{\Delta p}{\Delta t} = \dfrac{m(v_{\text{f}} - v_{\text{i}})}{\Delta t}$, where $v_{\text{f}} = 0$.

SIMPLIFY: $F = -mv_{\text{i}}/\Delta t$. The mass of the rain is $\rho_w V$, where $V = Ah$ is the volume of the water for a depth h of rainfall. $F = -\rho_w A h v_{\text{i}} / \Delta t$.

CALCULATE: From a table in the textbook, $\rho_w = 1.00 \cdot 10^3 \text{ kg/m}^3$. $h = 1.00 \text{ cm} = 1.00 \cdot 10^{-2}$ m, $v = -5.00$ m/s, and $\Delta t = 1.80 \cdot 10^3$ s.

$$F = -\frac{(1.00 \cdot 10^3 \text{ kg/m}^3)(100. \text{ m}^2)(1.00 \cdot 10^{-2} \text{ m})(-5.00 \text{ m/s})}{1.80 \cdot 10^3 \text{ s}} = -2.777778 \text{ N}$$

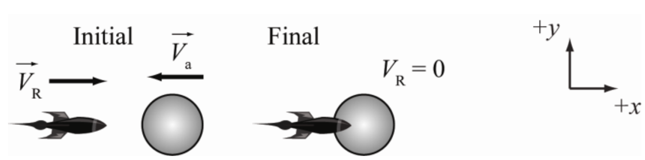
ROUND: Round to three significant figures: $F = -2.78$ N

DOUBLE-CHECK: This result looks reasonable. It is the equivalent of an approximately half-pound object sitting on the roof.

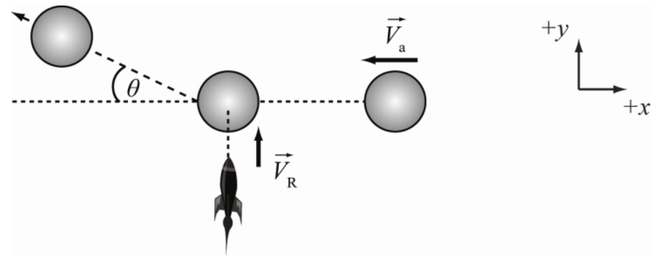
7.35. **THINK:** An asteroid has mass $m = 2.10 \cdot 10^{10}$ kg and speed $v_a = 12.0$ km/s, and a rocket has mass $8.00 \cdot 10^4$ kg. I want to calculate the speed of the rocket necessary to a. stop the asteroid, and b. divert it from its path by 1.00°.

SKETCH:

(a)



(b)

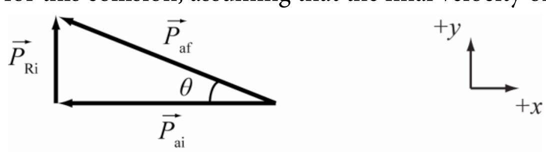


RESEARCH: Use conservation of momentum: $\vec{p}_i = \vec{p}_f$.

SIMPLIFY:

(a) The rocket and asteroid collide head on. $p_i = p_f \Rightarrow m_a v_{ai} + m_R v_{Ri} = m_a v_{af} + m_R v_{Rf}$. The final velocities of the rocket and the asteroid are $v_{Rf} = 0$ and $v_{af} = 0$. $m_R v_{Ri} = -m_a v_{ai} \Rightarrow v_{Ri} = -\frac{m_a}{m_R} v_{ai}$

(b) I draw a vector diagram for this collision, assuming that the final velocity of the rocket is $v_{Rf} = 0$.



Therefore, $\tan\theta = \frac{\left|\overrightarrow{p_{Ri}}\right|}{\left|\overrightarrow{p_{ai}}\right|} \Rightarrow \left|\overrightarrow{p_{Ri}}\right| = \left|\overrightarrow{p_{ai}}\right| \tan\theta.$

$p_{Ri} = p_{ai}\tan\theta \Rightarrow m_R v_{Ri} = m_a v_{ai}\tan\theta \Rightarrow v_{Ri} = \frac{m_a}{m_R} v_{ai}\tan\theta$

CALCULATE:

(a) $v_{ai} = -12.0 \cdot 10^3$ m/s, $v_{Ri} = \frac{\left(-2.10 \cdot 10^{10} \text{ kg}\right)\left(-12.0 \cdot 10^3 \text{ m/s}\right)}{8.00 \cdot 10^4 \text{ kg}} = 3.15 \cdot 10^9$ m/s

(b) $v_{Ri} = \frac{\left(2.10 \cdot 10^{10} \text{ kg}\right)\left(12.0 \cdot 10^3 \text{ m/s}\right)\tan 1.00°}{8.00 \cdot 10^4 \text{ kg}} = 5.498 \cdot 10^7$ m/s

ROUND:

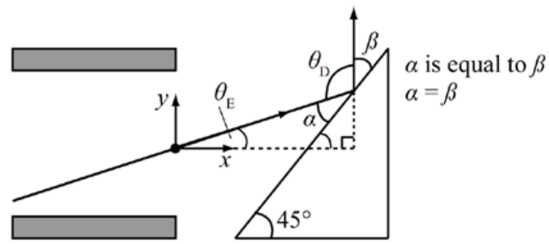
(a) $v_{Ri} = 3.15 \cdot 10^9$ m/s

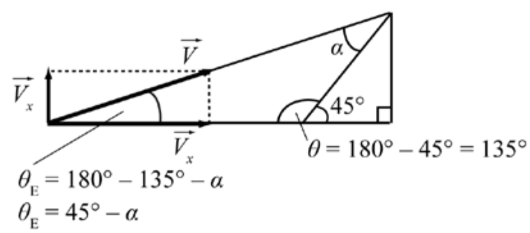
(b) $v_{Ri} = 5.50 \cdot 10^7$ m/s

DOUBLE-CHECK: For comparison, the speed of the rocket cannot exceed the speed of light, which is about $3.00 \cdot 10^8$ m/s. The speed of the rocket in (a) is greater than the speed of light, which would be impossible. This means the rocket could not stop the asteroid. The result in (b) is large but is still less than the speed of light.

7.36. **THINK:** An electron has velocity $v_x = 1.00 \cdot 10^5$ m/s. The vertical force is $8.0 \cdot 10^{-13}$ N. If $v_y = 0$ and the wall is at 45°, the deflection angle θ_D is 90°. I want to calculate Δt such that the deflection angle θ_D is 120.0°.

SKETCH:





RESEARCH: I first need to calculate the angle of the electron velocity after the vertical force has been applied. I need to calculate the angle θ_E in the above diagram. $\theta_E = 45° - \alpha$, $\alpha + \beta + \theta_D = 180°$, and $\alpha = \beta$.

SIMPLIFY: Since it is a reflection condition, the angle of incidence is equal to the angle of reflection. Thus, $2\alpha = 180° - \theta_D \Rightarrow \alpha = 90° - \theta_D/2$. $\theta_E = 45° - (90° - \theta_D/2) = \theta_D/2 - 45°$. Using impulse, $J = F\Delta t = \Delta p_y = m(v_{yf} - v_{yi})$, where $v_{yi} = 0$ and $v_{yf} = v_x \tan\theta_E$. $\Delta t = \dfrac{mv_x \tan\theta_E}{F} = \dfrac{mv_x \tan(\theta_D/2 - 45°)}{F}$.

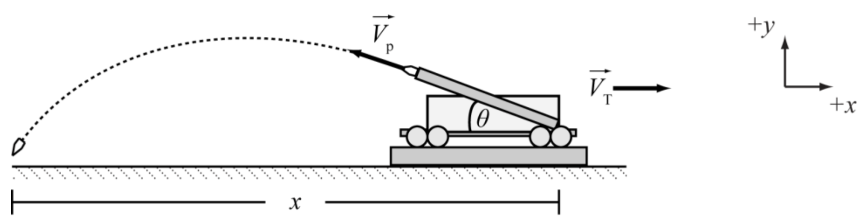
CALCULATE: $m_E = 9.1\cdot 10^{-31}$ kg, $\Delta t = \dfrac{(9.1\cdot 10^{-31}\text{ kg})(1.00\cdot 10^5\text{ m/s})\tan(120°/2 - 45°)}{8.0\cdot 10^{-13}\text{ N}} = 30.48$ fs

ROUND: Rounding to two significant figures: $\Delta t = 30.5$ fs.

DOUBLE-CHECK: As a comparison, compute the time taken for an electron with speed $v = 1.00\cdot 10^5$ m/s to move a distance of 1 nm $= 10^9$ m: $t = 10^{-9}\text{ m}/1.00\cdot 10^5\text{ m/s} = 1.0\cdot 10^{-14}\text{ s} = 10$ fs. The result $\Delta t = 30.$ fs is reasonable.

7.37. **THINK:** A projectile with mass 7502 kg is fired at an angle of 20.0°. The total mass of the gun, mount and train car is $1.22\cdot 10^6$ kg. The speed of the railway gun is initially zero and $v = 4.68$ m/s after finishing. I want to calculate the initial speed of the projectile and the distance it travels.

SKETCH:



RESEARCH: Use the conservation of momentum. $p_{xi} = p_{xf}$ and $p_{xi} = 0$, so $p_{xf} = 0$.

SIMPLIFY: $m_p v_p \cos\theta + m_T v_T = 0 \Rightarrow v_p = -\dfrac{m_T v_T}{m_p \cos\theta}$

$x = v_{px} t$, where t is twice the time it takes to reach the maximum height. $t_0 = v_{py}/g$, and $t = 2t_0$.

$$x = v_{px}(2t_0) = v_{px}\left(\frac{2v_{py}}{g}\right) = \frac{2v_p^2 \sin\theta\cos\theta}{g} = \frac{v_p^2 \sin 2\theta}{g}$$

CALCULATE: $v_p = -\dfrac{(1.22 \cdot 10^6 \text{ kg})(4.68 \text{ m/s})}{(7502 \text{ kg})\cos 20.0°} = -809.9 \text{ m/s}$

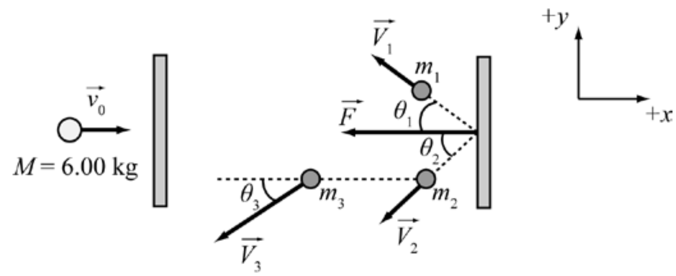
$$x = \frac{(-809.9 \text{ m/s})^2 \sin(2 \cdot 20.0°)}{9.81 \text{ m/s}^2} = 42979 \text{ m}$$

ROUND: $v_p = -810.$ m/s and $x = 43.0$ km.

DOUBLE-CHECK: The documented muzzle velocity for Gustav was 820 m/s, and its maximum range was approximately 48 km.

7.38. **THINK:** A 6.00-kg clay ball collides with a wall and then shatters into three pieces with masses $m_1 = 2.00$ kg, $m_2 = 1.00$ kg and $m_3 = 3.00$ kg, and velocities $v_1 = 10.0$ m/s at an angle of $32.0°$ above the horizontal, $v_2 = 8.00$ m/s at an angle of $28.0°$ below the horizontal and v_3. I need to calculate the velocity of the third mass. The wall exerts a force on the ball of 2640 N for 0.100 s.

SKETCH:



RESEARCH: To solve this problem, use the definition of impulse, $\vec{J} = \vec{F}\Delta t = \Delta\vec{p}$, or, in component form, $F_x\Delta t = p_{xf} - p_{xi}$ and $p_{yi} = p_{yf}$ since $F_y = 0$.

SIMPLIFY: $-F\Delta t = -m_1v_1\cos\theta_1 - m_2v_2\cos\theta_2 - m_3v_{3x} - Mv_0$ and $0 = m_1v_1\sin\theta_1 + m_2v_2\sin\theta_2 + m_3v_{3y}$. Rearranging these expressions gives:

$$v_{3x} = \frac{F\Delta t - m_1v_1\cos\theta_1 - m_2v_2\cos\theta_2 - Mv_0}{m_3}, \quad \text{and} \quad v_{3y} = \frac{-m_1v_1\sin\theta_1 - m_2v_2\sin\theta_2}{m_3}.$$

Use $v_3 = \sqrt{v_{3x}^2 + v_{3y}^2}$ and $\tan\theta_3 = v_{3y}/v_{3x}$ to get the speed and the angle.

CALCULATE: $v_{3y} = \dfrac{-(2.00 \text{ kg})(10.0 \text{ m/s})\sin 32.0° - (1.00 \text{ kg})(8.00 \text{ m/s})\sin(-28.0°)}{3.00 \text{ kg}} = -2.281 \text{ m/s},$

$$v_{3x} = \frac{(2640 \text{ N})(0.100 \text{ s}) - (2.00 \text{ kg})(10.0 \text{ m/s})\cos 32.0° - (1.00 \text{ kg})(8.00 \text{ m/s})\cos 28.0° - (6.00 \text{ kg})(22.0 \text{ m/s})}{3.00 \text{ kg}}$$

$= 35.992$ m/s,

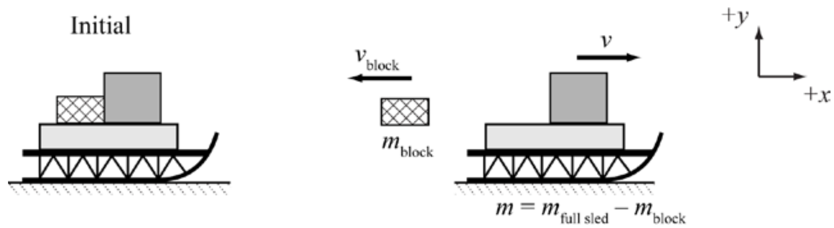
$v_3 = \sqrt{(35.992 \text{ m/s})^2 + (2.281 \text{ m/s})^2} = 36.064$ m/s, and $\theta_3 = \tan^{-1}\left(\dfrac{-2.281 \text{ m/s}}{35.992 \text{ m/s}}\right) = -3.6263°.$

ROUND: Rounding to three significant figures: $v_3 = 36.0$ m/s, $\theta_3 = 3.63°$ below the horizontal

DOUBLE-CHECK: The angle θ_3 is expected to be negative or below the horizontal.

7.39. **THINK:** The mass of a sled and its contents is $m_{\text{full sled}} = 52.0$ kg. A block of mass $m_{\text{block}} = 13.5$ kg is ejected to the left with velocity $v_{\text{block}} = -13.6$ m/s. I need to calculate the speed of the sled and remaining contents.

SKETCH:



RESEARCH: Use the conservation of momentum. $p_i = p_f$, and $p_i = 0$ since the sled and its contents are initially at rest.

SIMPLIFY: $p_f = m_{block}v_{block} + mv = 0 \;\Rightarrow\; v = -\dfrac{m_{block}v_{block}}{m_{full\ sled} - m_{block}}$

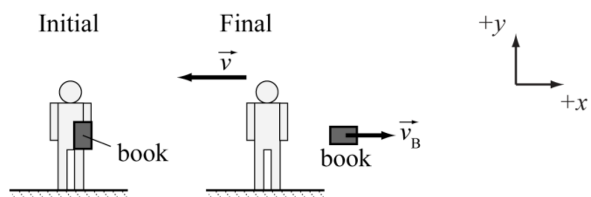
CALCULATE: $v = -\dfrac{(13.5\text{ kg})(-13.6\text{ m/s})}{52.0\text{ kg} - 13.5\text{ kg}} = 4.7688\text{ m/s}$

ROUND: $v = 4.77\text{ m/s}$

DOUBLE-CHECK: Because the sled and its remaining contents have a mass larger than the mass of the block, it is expected that the speed of the sled and the remaining contents is less than the block's speed, i.e. $v < v_{block}$.

7.40. **THINK:** The mass of the book is $m_B = 5.00\text{ kg}$ and the mass of the person is $m = 62.0\text{ kg}$. Initially the book and the person are at rest, and then the person throws the book at 13.0 m/s. I need to calculate speed of the person on the ice after throwing the book.

SKETCH:



RESEARCH: We use conservation of momentum. $p_i = p_f$ and $p_i = 0$ since the speed is initially zero.

SIMPLIFY: $p_f = 0 = mv + m_B v_B \;\Rightarrow\; v = -\dfrac{m_B v_B}{m}$

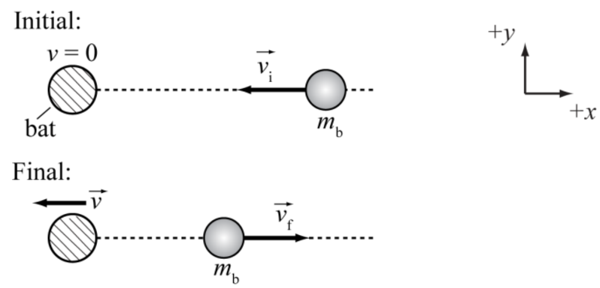
CALCULATE: $v = -\dfrac{(5.00\text{ kg})(13.0\text{ m/s})}{62.0\text{ kg}} = -1.0484\text{ m/s}$

ROUND: $v = -1.05\text{ m/s}$

DOUBLE-CHECK: The direction of the person's motion should be in the direction opposite to the direction of the book.

7.41. **THINK:** The astronaut's mass is $m_A = 50.0$ kg and the baseball's mass is $m_b = 0.140$ kg. The baseball has an initial speed of 35.0 m/s and a final speed of 45.0 m/s.

SKETCH:



RESEARCH: Use the conservation of momentum. $p_i = p_f$.

SIMPLIFY: $p_i = p_f \Rightarrow m_b v_i + 0 = m_b v_f + m_A v_A \Rightarrow v_A = \dfrac{m_b(v_i - v_f)}{m_A}$

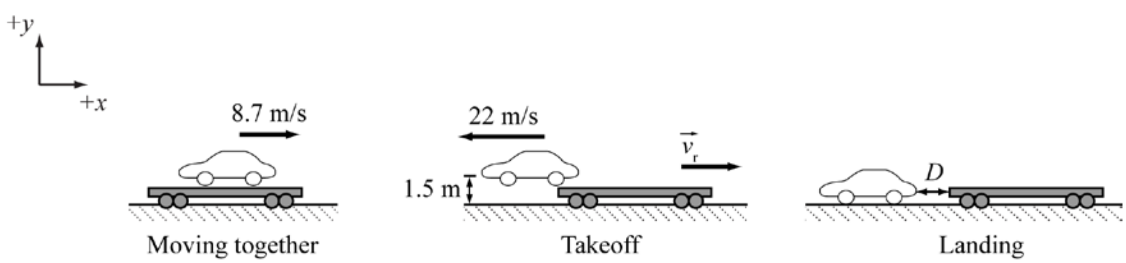
CALCULATE: $v_i = -35.0$ m/s, $v_f = 45.0$ m/s, and $v_A = \dfrac{(0.140 \text{ kg})(-35.0 \text{ m/s} - 45.0 \text{ m/s})}{50.0 \text{ kg}} = -0.224$ m/s.

ROUND: Three significant figures: $v_A = -0.224$ m/s.

DOUBLE-CHECK: The magnitude of v_A is proportional to m_b/m_A, which is about 10^{-3} so it would be expected to find the velocity of the astronaut as relatively small.

7.42. **THINK:** The mass of an automobile is $m_a = 1450$ kg and the mass of a railcar is $m_r = 38,500$ kg. Initially, both are moving at $v_i = +8.7$ m/s. The automobile leaves the railcar at a speed of $v_{af} = -22$ m/s. I need to determine the distance D between the spot where it lands and the left end of the railcar. Call the *x*-component of the velocity of the railcar v_r and that of the automobile v_a.

SKETCH:



RESEARCH: I need first to calculate the speed of the railcar just after the automobile leaves and then I need to find the amount of time it takes for the automobile to reach the ground. Conservation of momentum leads to the two equations $p_i = p_f \Rightarrow m_a v_{ai} + m_r v_{ri} = m_a v_{af} + m_r v_{rf}$ and $v_{ai} = v_{ri} = v_i \Rightarrow v_{rf} = \dfrac{(m_a + m_r)v_i - m_a v_{af}}{m_r}$. The final relative velocity between the automobile and the railcar is $\Delta v = v_{rf} - v_{af}$. The time to reach the ground is determined using $h = gt^2/2 \Rightarrow t = \sqrt{2h/g}$. The separation distance is the product of time and the relative velocity, $D = t\Delta v$.

SIMPLIFY: Insert the expression for the time and the relative velocity into the distance equation and obtain:

$$\begin{aligned} D = t\Delta v &= \sqrt{2h/g}\,(v_{\text{rf}} - v_{\text{af}}) \\ &= \sqrt{2h/g}\left(\frac{(m_{\text{a}} + m_{\text{r}})v_{\text{i}} - m_{\text{a}}v_{\text{af}}}{m_{\text{r}}} - v_{\text{af}}\right) \\ &= \sqrt{2h/g}\left(\frac{m_{\text{a}} + m_{\text{r}}}{m_{\text{r}}}\right)(v_{\text{i}} - v_{\text{af}}) \end{aligned}$$

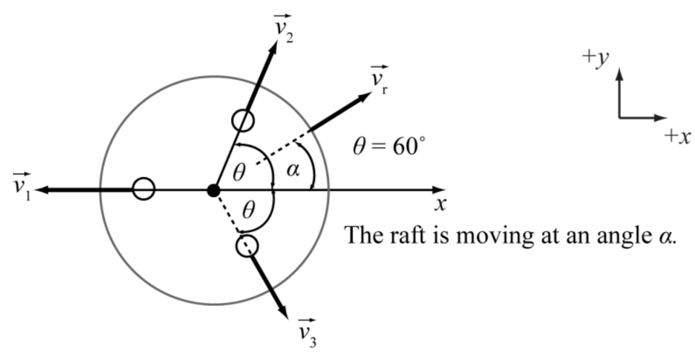
CALCULATE: $D = \sqrt{2(1.5\text{ m})/(9.81\text{ m/s}^2)}\left(\frac{1450\text{ kg} + 38{,}500\text{ kg}}{38{,}500\text{ kg}}\right)(8.7\text{ m/s} + 22\text{ m/s}) = 17.6165\text{ m}$

ROUND: Rounding to three significant figures, $D = 17.6$ m.

DOUBLE-CHECK: The value of D is reasonable. If the mass of the automobile relative to that of the railcar is neglected, then the railcar's velocity is not changed due to the recoil from the car, and the relative velocity between the two is simply $(8.7 + 22)$ m/s = 30.7 m/s. If something moves at a speed of 30.7 m/s for $\sqrt{2(1.5\text{ m})/(9.81\text{ m/s}^2)}$ = 0.553 s, then it moves a distance of 16.977 m. The actual answer is close to this estimate. The actual answer has to be slightly bigger than this estimate because the railcar receives a small velocity boost forward due to the car jumping off in the backwards direction.

7.43. **THINK:** The raft is given to be of mass $m_{\text{r}} = 120.\text{ kg}$ and the three people of masses $m_1 = 62.0\text{ kg}$, $m_2 = 73.0\text{ kg}$ and $m_3 = 55.0\text{ kg}$ have speeds $v_1 = 12.0\text{ m/s}$, $v_2 = 8.00\text{ m/s}$, and $v_3 = 11.0\text{ m/s}$. I need to calculate the speed of the raft.

SKETCH:



RESEARCH: Because of the conservation of momentum, $\vec{p}_{\text{i}} = \vec{p}_{\text{f}}$, or $p_{x\text{i}} = p_{x\text{f}}$ and $p_{y\text{i}} = p_{y\text{f}}$. The question provides the information that $\vec{p}_{\text{i}} = 0$, i.e. $p_{x\text{i}} = 0$ and $p_{y\text{i}} = 0$.

SIMPLIFY: $v_{\text{r}} = \sqrt{v_{\text{r}x}^2 + v_{\text{r}y}^2}$.

$$p_{x\text{f}} = 0 \Rightarrow -m_1v_1 + m_2v_2\cos\theta + m_3v_3\cos\theta + m_{\text{r}}v_{\text{r}x} = 0 \Rightarrow v_{\text{r}x} = \frac{m_1v_1 - m_2v_2\cos\theta - m_3v_3\cos\theta}{m_{\text{r}}}.$$

$$p_{y\text{f}} = 0 \Rightarrow 0 + m_2v_2\sin\theta - m_3v_3\sin\theta + m_{\text{r}}v_{\text{r}y} = 0 \Rightarrow v_{\text{r}y} = \frac{-m_2v_2\sin\theta + m_3v_3\sin\theta}{m_{\text{r}}}.$$

CALCULATE:

$$v_{\text{r}x} = \frac{(62.0\text{ kg})(12.0\text{ m/s}) - (73.0\text{ kg})(8.00\text{ m/s})\cos 60.0^\circ - (55.0\text{ kg})(11.0\text{ m/s})\cos 60.0^\circ}{120.\text{ kg}} = 1.2458\text{ m/s}$$

$$v_{\text{r}y} = \frac{-(73.0\text{ kg})(8.00\text{ m/s})\sin 60.0^\circ + (55.0\text{ kg})(11.0\text{ m/s})\sin 60.0^\circ}{120.\text{ kg}} = 0.1516\text{ m/s}$$

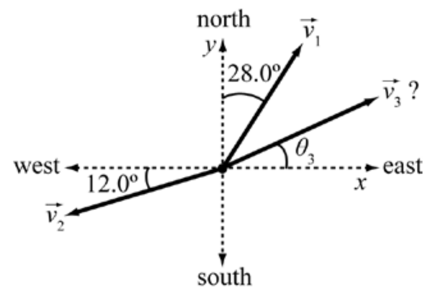
$v_r = \sqrt{(1.2458 \text{ m/s})^2 + (0.1516 \text{ m/s})^2} = 1.2550 \text{ m/s}$

ROUND: $v_r = 1.26$ m/s

DOUBLE-CHECK: Due to the large mass of the raft, v_r is expected to be small, and it is smaller than 8.00 m/s.

7.44. **THINK:** A missile that breaks into three pieces of equal mass $m_1 = m_2 = m_3 = m$. The first piece has a speed of 30.0 m/s in the direction 28.0° east of north. The second piece has a speed of 8.00 m/s and is in the direction 12.0° south of west. I want to calculate the speed and direction of the third piece.

SKETCH:



RESEARCH: Use the conservation of momentum. $\vec{p}_i = \vec{p}_f$, and in component form $p_{xi} = p_{xf}$ and $p_{yi} = p_{yf}$. Also, $\vec{p}_i = 0$.

SIMPLIFY: $v_3 = \sqrt{v_{3x}^2 + v_{3y}^2}$

$p_{xi} = 0 = p_{xf}$, $mv_1 \sin\theta_1 - mv_2 \cos\theta_2 + mv_{3x} = 0 \Rightarrow v_{3x} = -v_1 \sin\theta_1 + v_2 \cos\theta_2$

$p_{yi} = 0 = p_{yf}$, $mv_1 \cos\theta_1 - mv_2 \sin\theta_2 + mv_{3y} = 0 \Rightarrow v_{3y} = -v_1 \cos\theta_1 + v_2 \sin\theta_2$

CALCULATE: $v_{3x} = -(30.0 \text{ m/s})\sin 28.0° + (8.00 \text{ m/s})\cos 12.0° = -6.26$ m/s,

$v_{3y} = -(30.0 \text{ m/s})\cos 28.0° + (8.00 \text{ m/s})\sin 12.0° = -24.83$ m/s,

$v = \sqrt{(-6.26 \text{ m/s})^2 + (-24.83 \text{ m/s})^2} = 25.61$ m/s.

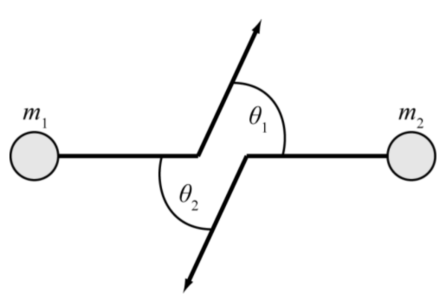
$\theta = \tan^{-1}\left(\frac{-24.83 \text{ m/s}}{-6.26 \text{ m/s}}\right) = 75.8498°$. Because both v_{3x} and v_{3y} are negative, we need to add 180° to the angle. Thus $\theta = 180° + 75.85° = 255.85°$.

ROUND: Round to three significant figures: $v = 25.6$ m/s, $\theta = 256°$, or 75.8° south of west.

DOUBLE-CHECK: Since v_1 is much larger than v_2, v_3 is roughly the same speed as v_1 but in the opposite direction.

7.45. **THINK:** A soccer ball and a basketball have masses $m_1 = 0.400$ kg and $m_2 = 0.600$ kg respectively. The soccer ball has an initial energy of 100. J and the basketball 112 J. After collision, the second ball flew off at an angle of 32.0° with 95.0 J of energy. I need to calculate the speed and angle of the first ball. Let subscript 1 denote the soccer ball, and subscript 2 denote the basketball.

SKETCH:



RESEARCH: I need to calculate the speed of the balls using $1/2mv^2 = K$, or $v = \sqrt{2K/m}$, and then apply the conservation of momentum to get $p_{xi} = p_{xf}$ and $p_{yi} = p_{yf}$. I also use $p_{yi} = 0$.

SIMPLIFY: $p_{xi} = p_{xf} \Rightarrow m_1v_{1i} - m_2v_{2i} = m_1v_{1f}\cos\theta_1 - m_2v_{2f}\cos\theta_2$,

$p_{yi} = p_{yf} = 0 \Rightarrow m_1v_{1f}\sin\theta_1 - m_2v_{2f}\sin\theta_2 = 0$,

$v_{1x} = v_{1f}\cos\theta_1$ and $v_{1y} = v_{1f}\sin\theta_1 \Rightarrow v_{1x} = \dfrac{m_1v_{1i} - m_2v_{2i} + m_2v_{2f}\cos\theta_2}{m_1}$ and $v_{1y} = \dfrac{m_2v_{2f}\sin\theta_2}{m_1}$,

$v_{1i} = \sqrt{\dfrac{2K_{1i}}{m_1}}$, $v_{2i} = \sqrt{\dfrac{2K_{2i}}{m_2}}$, $v_{2f} = \sqrt{\dfrac{2K_{2f}}{m_2}}$, and $v_{1f} = \sqrt{v_{1x}^2 + v_{1y}^2}$.

CALCULATE: $v_{1i} = \sqrt{\dfrac{2(100.\text{ J})}{0.400\text{ kg}}} = 22.36$ m/s, $v_{2i} = \sqrt{\dfrac{2(112\text{ J})}{0.600\text{ kg}}} = 19.32$ m/s, $v_{2f} = \sqrt{\dfrac{2(95.0\text{ J})}{0.600\text{ kg}}} = 17.80$ m/s,

$$v_{1x} = \frac{(0.400\text{ kg})(22.36\text{ m/s}) - (0.600\text{ kg})(19.32\text{ m/s}) + (0.600\text{ kg})(17.80\text{ m/s})\cos 32.0^\circ}{0.400\text{ kg}} = 16.02\text{ m/s},$$

$v_{1y} = \dfrac{(0.600\text{ kg})(17.80\text{ m/s})\sin 32.0^\circ}{0.400\text{ kg}} = 14.15$ m/s, $v_{1f} = \sqrt{(16.02\text{ m/s})^2 + (14.15\text{ m/s})^2} = 21.37$ m/s, and

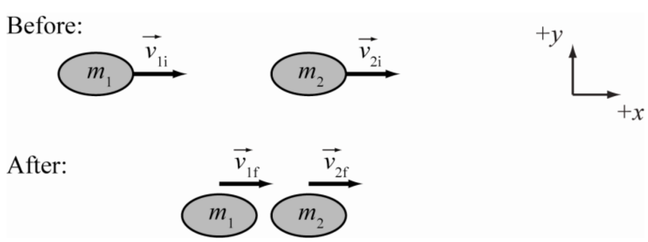
$\theta_1 = \tan^{-1}\left(\dfrac{14.15\text{ m/s}}{16.02\text{ m/s}}\right) = 41.5^\circ$.

ROUND: $v_{1f} = 21.4$ m/s and $\theta_1 = 41.5^\circ$.

DOUBLE-CHECK: The results for speed and angle are comparable to v_2 and θ_2, which is expected. From energy conservation (assuming elastic collision), the energy is $E_{1f} = E_{1i} + E_{2i} - E_{2f}$ = 100. J + 112 J – 95.0 J = 117 J, which corresponds to a speed of 24.2 m/s for v_{1f}. The result $v_{1f} = 21.4$ m/s is less than this because the energy is not conserved in this case.

7.46. **THINK:** Two bumper cars have masses $m_1 = 188$ kg and $m_2 = 143$ kg and speeds $v_1 = 20.4$ m/s and $v_2 = 9.00$ m/s respectively. I want to calculate v_1 after the elastic collision.

SKETCH:



RESEARCH: Use the conservation of momentum and the conservation of energy. $p_i = p_f$ and $E_i = E_f$.

SIMPLIFY: $p_i = p_f \Rightarrow m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f} \Rightarrow m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i})$ (1)

$$E_i = E_f$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_iv_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2)$$

$$m_1(v_{1i} + v_{1f})(v_{1i} - v_{1f}) = m_2(v_{2f} + v_{2i})(v_{2f} - v_{2i})$$

$$v_{1i} + v_{1f} = \frac{m_2(v_{2f} - v_{2i})}{m_1(v_{1i} - v_{1f})}(v_{2f} + v_{2i})$$

Using (1) above, $v_{1i} + v_{1f} = v_{2f} + v_{2i} \Rightarrow v_{2f} = v_{1i} + v_{1f} - v_{2i}$. Substituting back into the equation of conservation of momentum,

$$m_1v_{1i}+m_2v_{2i}=m_1v_{1f}+m_2\left(v_{1i}+v_{1f}-v_{2i}\right)$$
$$m_1v_{1i}+m_2v_{2i}=\left(m_1+m_2\right)v_{1f}+m_2v_{1i}-m_2v_{2i}$$
$$v_{1f}=\frac{m_1-m_2}{m_1+m_2}v_{1i}+\frac{2m_2}{m_1+m_2}v_{2i}$$

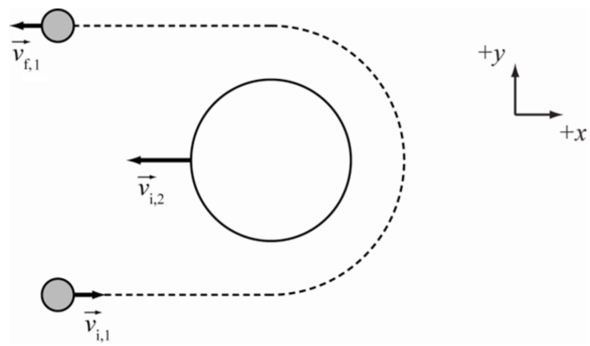
CALCULATE: $v_{1f}=\left(\frac{188\text{ kg}-143\text{ kg}}{188\text{ kg}+143\text{ kg}}\right)(20.4\text{ m/s})+\left(\frac{2\cdot 143\text{ kg}}{188\text{ kg}+143\text{ kg}}\right)(9.00\text{ m/s})=10.55\text{ m/s}$

ROUND: Rounding to three significant figures: $v_{1f}=10.6\text{ m/s}$

DOUBLE-CHECK: It is expected that some of the kinetic energy of m_1 is transferred to m_2. As a result, v_{1f} is smaller than v_{2i}. Since $m_1>m_2$, v_{1f} should be smaller than v_{2i}.

7.47. **THINK:** The mass of the satellite is $m_1=274\text{ kg}$ and its initial speed is $v_{i,1}=13.5\text{ km/s}$ The initial speed of the planet is $v_{i,2}=-10.5\text{ km/s}$. I want to calculate the speed of the satellite after collision. It is assumed that the mass of the planet is much larger than the mass of the satellite, i.e. $m_2\gg m_1$.

SKETCH:



RESEARCH: Use the conservation of energy and the conservation of momentum; $E_i=E_f$ and $p_i=p_f$.

SIMPLIFY: $p_i=p_f \;\Rightarrow\; m_1v_{i,1}+m_2v_{i,2}=m_1v_{f,1}+m_2v_{f,2} \;\Rightarrow\; m_1\left(v_{i,1}-v_{f,1}\right)=m_2\left(v_{f,2}-v_{i,2}\right)$ (1)

$$E_i=E_f$$
$$\frac{1}{2}m_1v_{i,1}^2+\frac{1}{2}m_2v_{i,2}^2=\frac{1}{2}m_1v_{f,1}^2+\frac{1}{2}m_2v_{f,2}^2$$
$$m_1\left(v_{i,1}^2-v_{f,1}^2\right)=m_2\left(v_{f,2}^2-v_{i,2}^2\right)$$
$$m_1\left(v_{i,1}+v_{f,1}\right)\left(v_{i,1}-v_{f,1}\right)=m_2\left(v_{f,2}+v_{i,2}\right)\left(v_{f,2}-v_{i,2}\right)$$
$$v_{i,1}+v_{f,1}=\frac{m_2\left(v_{f,2}-v_{i,2}\right)}{m_1\left(v_{i,1}-v_{f,1}\right)}\left(v_{f,2}+v_{i,2}\right)$$

Using (1), $v_{i,1}+v_{f,1}=v_{f,2}+v_{i,2} \;\Rightarrow\; v_{f,2}=v_{i,1}+v_{f,1}-v_{i,2}$. Substituting back into the conservation of momentum equation above,

$$m_1v_{i,1}+m_2v_{i,2}=m_1v_{f,1}+m_2\left(v_{i,1}+v_{f,1}-v_{i,2}\right)$$
$$m_1v_{i,1}+m_2v_{i,2}=\left(m_1+m_2\right)v_{f,1}+m_2v_{i,1}-m_2v_{i,2}$$
$$v_{f,1}=\frac{m_1-m_2}{m_1+m_2}v_{i,1}+\frac{2m_2}{m_1+m_2}v_{i,2}$$

Using the fact that $m_2\gg m_1$, $\left(m_1-m_2\right)/\left(m_1+m_2\right)\approx -1$ and $2m_2/\left(m_1+m_2\right)\approx 2$. Therefore, $v_{f,1}\approx -v_{i,1}+2v_{i,2}$.

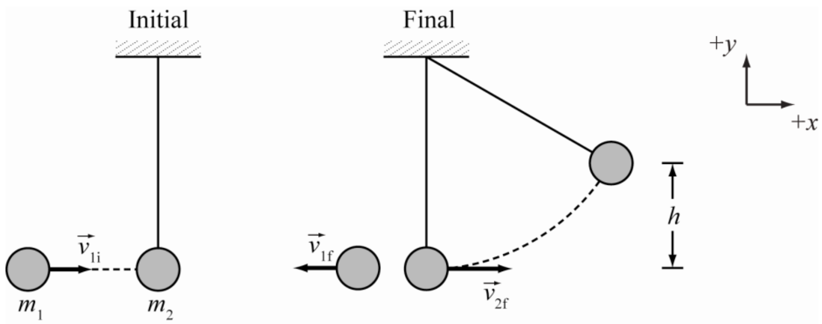
CALCULATE: $v_{i,1}=13.5\text{ km/s}$, $v_{i,2}=-10.5\text{ km/s}$, $v_{f,1}=-13.5\text{ km/s}+2\left(-10.5\text{ km/s}\right)=-34.5\text{ km/s}$

ROUND: $v_{f,1} = -34.5$ km/s

DOUBLE-CHECK: The result makes sense. $v_{f,1}$ should be negative since it is in the opposite direction.

7.48. **THINK:** A stone has mass of $m_1 = 0.250$ kg. The mass of one of the shoes is $m_2 = 0.370$ kg. I need to calculate the speed of the shoe after collision, and then the height of the shoe.

SKETCH:



RESEARCH: Use the conservation of momentum and energy, $p_i = p_f$ and $E_i = E_f$, as well as $K_2 = mgh$.

SIMPLIFY: $p_f = p_i \Rightarrow m_1v_{1i} + 0 = m_1v_{1f} + m_2v_{2f} \Rightarrow m_1(v_{1i} - v_{1f}) = m_2v_{2f}$, and

$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2v_{2f}^2 \Rightarrow m_2v_{2f}(v_{1i} + v_{1f}) = m_2v_{2f}^2 \Rightarrow v_{1i} + v_{1f} = v_{2f} \Rightarrow v_{1f} = v_{2f} - v_{1i}$.

Substituting back into the conservation of momentum equation,

$$m_1v_{1i} = m_1(v_{2f} - v_{1i}) + m_2v_{2f} \Rightarrow (m_1 + m_1)v_{1i} = (m_1 + m_2)v_{2f} \Rightarrow v_{2f} = \frac{2m_1}{m_1 + m_2}v_{1i}$$

Using $K = \frac{1}{2}mv_{2f}^2 = mgh, \quad h = \frac{v_{2f}^2}{2g} \Rightarrow \left(\frac{2m_1v_{1i}}{m_1 + m_2}\right)^2 \frac{1}{2g}$.

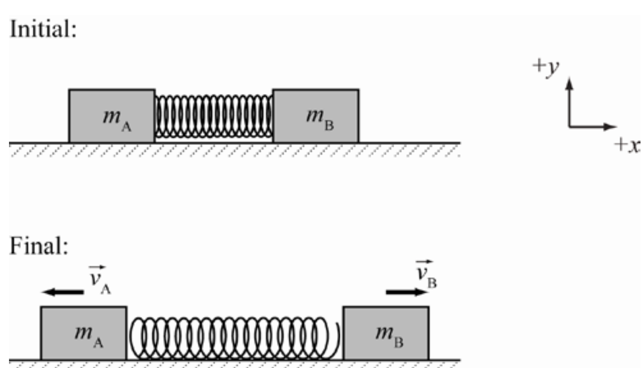
CALCULATE: $h = \left(\frac{2(0.250 \text{ kg})(2.30 \text{ m/s})}{0.250 \text{ kg} + 0.370 \text{ kg}}\right)^2 \left(\frac{1}{2(9.81 \text{ m/s}^2)}\right) = 0.1754$ m

ROUND: Rounding to three significant figures, $h = 0.175$ m.

DOUBLE-CHECK: This is a reasonable height. As a comparison, the length of a shoelace is between about 0.5 m and 1.8 m.

7.49. **THINK:** Two blocks with a spring between them sit on an essentially frictionless surface. The spring constant is k = 2500. N/m. The spring is compressed such that $\Delta x = 3.00 \text{ cm} = 3.00 \cdot 10^{-2}$ m. I need to calculate the speeds of the two blocks. $m_A = 1.00$ kg, and $m_B = 3.00$ kg.

SKETCH:



RESEARCH: I use the conservation of momentum and the conservation of energy. Thus $p_i = p_f$, and $E_i = E_f$. I also know that $p_i = 0$ and $E_i = E_s = (1/2)k\Delta x^2$.

SIMPLIFY: $p_i = p_f = 0 \Rightarrow m_A v_A + m_B v_B = 0 \Rightarrow m_A v_A = -m_B v_B$

$$E_i = E_f \Rightarrow \frac{1}{2}k\Delta x^2 = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 \Rightarrow k\Delta x^2 = m_A\left(-\frac{m_B}{m_A}v_B\right)^2 + m_B v_B^2 \Rightarrow k\Delta x^2 = \frac{m_B^2}{m_A}v_B^2 + m_B v_B^2$$

Simplifying further gives:

$$\left(\frac{m_B^2}{m_A} + m_B\right)v_B^2 = k\Delta x^2 \Rightarrow v_B = \sqrt{\frac{k\Delta x^2}{\frac{m_B^2}{m_A} + m_B}} = \sqrt{\frac{k\Delta x^2}{m_B\left(1 + \frac{m_B}{m_A}\right)}} \quad \text{and} \quad v_A = -\frac{m_B}{m_A}v_B.$$

CALCULATE: $v_B = \sqrt{\dfrac{(2500.\text{ N/m})\left(3.00 \cdot 10^{-2}\text{ m}\right)^2}{(3.00\text{ kg})\left(1 + \dfrac{3.00\text{ kg}}{1.00\text{ kg}}\right)}} = 0.4330\text{ m/s}$

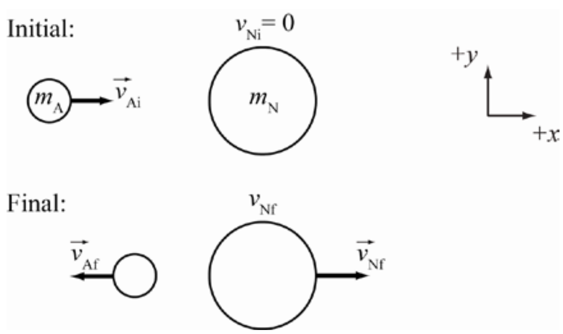
$$v_A = -\frac{3.00\text{ kg}}{1.00\text{ kg}}(0.4330\text{ m/s}) = -1.299\text{ m/s}$$

ROUND: $v_B = 0.433$ m/s and $v_A = -1.30$ m/s.

DOUBLE-CHECK: The speed of block A should be larger than the speed of block B since m_A is less than m_B.

7.50. **THINK:** An alpha particle has mass $m_A = 4.00$ u and speed v_{Ai}, and a nucleus has mass $m_N = 166$ u and is at rest. Conservation of momentum and energy can be used to calculate the kinetic energy of the nucleus after the elastic collision.

SKETCH:



RESEARCH: Conservation of momentum and energy are: $p_i = p_f$ and $E_i = E_f$.

SIMPLIFY: Conservation of momentum gives

$$p_i = p_f \Rightarrow m_A v_{Ai} + 0 = m_A v_{Af} + m_N v_{Nf} \Rightarrow m_A\left(v_{Ai} - v_{Af}\right) = m_N v_{Nf}$$

Conservation of energy gives:

$$E_i = E_f$$
$$\frac{1}{2}m_A v_{Ai}^2 + 0 = \frac{1}{2}m_A v_{Af}^2 + \frac{1}{2}m_N v_{Nf}^2$$
$$m_A\left(v_{Ai}^2 - v_{Af}^2\right) = m_N v_{Nf}^2$$
$$m_A\left(v_{Ai} - v_{Af}\right)\left(v_{Ai} + v_{Af}\right) = m_N v_{Nf}^2$$
$$m_N v_{Nf}\left(v_{Ai} + v_{Af}\right) = m_N v_{Nf}^2$$
$$v_{Ai} + v_{Af} = v_{Nf}$$
$$v_{Af} = v_{Nf} - v_{Ai}$$

Substituting this back into the equation of conservation of momentum gives:

$$m_{\text{A}}\left(v_{\text{Ai}}-\left(v_{\text{Nf}}-v_{\text{Ai}}\right)\right)=m_{\text{N}}v_{\text{Nf}}$$
$$-m_{\text{A}}v_{\text{Nf}}+2m_{\text{A}}v_{\text{Ai}}=m_{\text{N}}v_{\text{Nf}}$$
$$m_{\text{N}}v_{\text{Nf}}+m_{\text{A}}v_{\text{Nf}}=2m_{\text{A}}v_{\text{Ai}}$$
$$v_{\text{Nf}}=\frac{2m_{\text{A}}}{m_{\text{A}}+m_{\text{N}}}v_{\text{Ai}}$$

The kinetic energy of the nucleus is:

$$K_{\text{N}}=\frac{1}{2}m_{\text{N}}v_{\text{Nf}}^2=\frac{1}{2}m_{\text{N}}\left(\frac{2m_{\text{A}}}{m_{\text{A}}+m_{\text{N}}}\right)^2v_{\text{Ai}}^2=\frac{4m_{\text{A}}m_{\text{N}}}{\left(m_{\text{A}}+m_{\text{N}}\right)^2}\left(\frac{1}{2}m_{\text{A}}v_{\text{Ai}}^2\right)=\frac{4m_{\text{A}}m_{\text{N}}}{\left(m_{\text{A}}+m_{\text{N}}\right)^2}K_{\text{A}},$$

which gives

$$\frac{K_{\text{N}}}{K_{\text{A}}}=\frac{4m_{\text{A}}m_{\text{N}}}{\left(m_{\text{A}}+m_{\text{N}}\right)^2}.$$

CALCULATE: $\frac{K_{\text{N}}}{K_{\text{A}}}=\frac{4(4.00\text{ u})(166\text{ u})}{(4.00\text{ u}+166\text{ u})^2}=0.09190=9.190\%$

ROUND: To three significant figures, $\frac{K_{\text{N}}}{K_{\text{A}}}=9.19\ \%.$

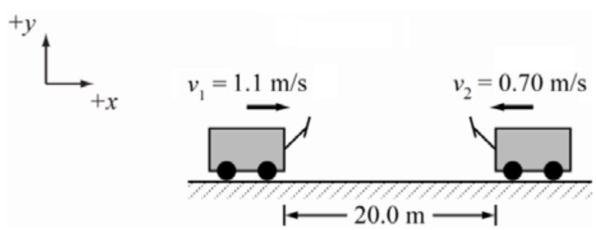
DOUBLE-CHECK: To check the equation, set the masses equal:

$$\frac{K_2}{K_1}=\frac{4m_1m_2}{\left(m_1+m_2\right)^2}=\frac{4m^2}{(2m)^2}=\frac{4m^2}{4m^2}=1.$$

This means that all of energy is transferred, which is expected for two equal masses (i.e. billiard balls). This confirms that the derived equation is correct. Here, since the mass of the nucleus is much larger than the mass of the alpha particle, it is reasonable that the ratio is small.

7.51. **THINK:** Two carts, separated by a distance $x_0=20.0\text{ m}$, are travelling towards each other with speeds $v_1=1.10\text{ m/s}$ and $v_2=0.700\text{ m/s}$. They collide for $\Delta t=0.200\text{ s}$. This is an elastic collision. I need to plot *x* vs. *t*, *v* vs. *t* and *F* vs. *t*.

SKETCH:



RESEARCH: Use the conservation of momentum and energy to get the speeds after collision. Then use the impulse $\vec{J}=\vec{F}\Delta t=\Delta\vec{p}$ to get the force.

SIMPLIFY: First, need the position of the collision. Using $x=x_0+v_0t\ \Rightarrow\ x_1=0+v_1t$ and $x_2=x_0-v_2t$, $x_1=x_2=v_1t=x_0-v_2t\ \Rightarrow\ t=x_0/\left(v_1+v_2\right)$. Conservation of momentum:

$$p_{\text{i}}=p_{\text{f}}\ \Rightarrow\ m_1v_{\text{i1}}+m_2v_{\text{i2}}=m_1v_{\text{f1}}+m_2v_{\text{f2}}$$

$m_1=m_2=m$ because they are both the same type of cart. Then

$$v_{\text{i1}}-v_{\text{f1}}=v_{\text{f2}}-v_{\text{i2}}. \qquad (1)$$

$$K_i = K_f$$
$$\frac{1}{2}m_1v_{i1}^2 + \frac{1}{2}m_2v_{i2}^2 = \frac{1}{2}m_1v_{f1}^2 + \frac{1}{2}m_2v_{f2}^2$$
$$v_{i1}^2 - v_{f1}^2 = v_{f2}^2 - v_{i2}^2$$
$$(v_{i1} - v_{f1})(v_{i1} + v_{f1}) = (v_{f2} - v_{i2})(v_{f2} + v_{i2})$$
$$v_{i1} + v_{f1} = v_{f2} + v_{i2}$$
$$v_{f2} = v_{i1} + v_{f1} - v_{i2}$$

Substituting back into (1):

$$v_{i1} - v_{f1} = v_{i1} + v_{f1} - v_{i2} - v_{i2} \;\Rightarrow\; 2v_{f1} = 2v_{i2} \;\Rightarrow\; v_{f1} = v_{i2} \text{ and } v_{f2} = v_{i1}.$$

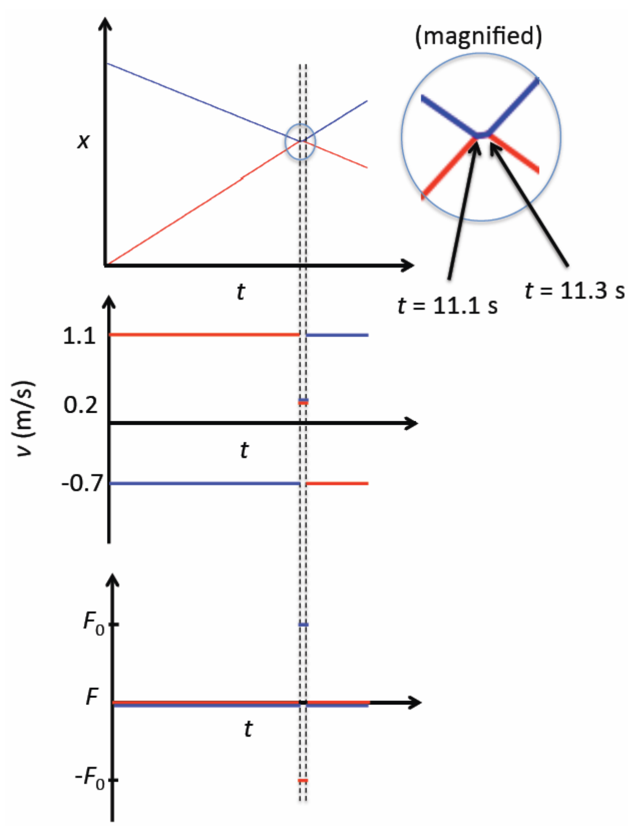
The change of momentum is $\Delta p_2 = m(v_{f2} - v_{i2}) = m(v_{i1} - v_{i2})$. The force on the other cart is

$$F_2\Delta t = \Delta p_2 \;\Rightarrow\; F_2 = \frac{\Delta p_2}{\Delta t} = \frac{m(v_{i1} - v_{i2})}{\Delta t}.$$

The force on your car is equal and opposite.

CALCULATE: The time for the collision to occur is $t = \dfrac{20.0 \text{ m}}{0.700 \text{ m/s} + 1.10 \text{ m/s}} = 11.11 \text{ s}$ and during this time the other cart has moved $x = (0.700 \text{ m/s})(11.11 \text{ s}) = 7.78 \text{ m}$.

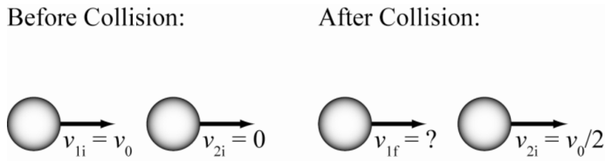
ROUND: For the two calculations shown above three significant figures are required: $t = 11.1$ s and $x = 7.78$ m.



DOUBLE-CHECK: Since $m_1 = m_2$ it makes sense that $v_{f1} = v_{i2}$ and $v_{f2} = v_{i1}$. This means that energy is transferred completely from one to the other.

7.52. **THINK:** There are two balls with masses $m_1 = 0.280$ kg and m_2. The initial speeds are $v_{1i} = v_0$ and $v_{2i} = 0$. After the collision, the speeds are v_{1f} and $v_{2f} = (1/2)v_0$. I want to calculate the mass of the second ball. This is an elastic collision.

SKETCH:



RESEARCH: I use the conservation of momentum and energy. $p_i = p_f$ and $E_i = E_f$.

SIMPLIFY:

(a) $p_i = p_f \Rightarrow m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f} \Rightarrow m_1v_{1i} + 0 = m_1v_{1f} + m_2v_{2f} \Rightarrow m_2v_{2f} = m_1(v_{1i} - v_{1f})$ (1)

$$E_i = E_f$$
$$\frac{1}{2}m_1v_{1i}^2 + 0 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$
$$m_1(v_{1i}^2 - v_{1f}^2) = m_2v_{2f}^2$$
$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2v_{2f}^2$$
$$v_{1i} + v_{1f} = v_{2f}$$
$$v_{1f} = v_{2f} - v_{1i}$$

Substituting back into (1), I have:

$$m_2v_{2f} = m_1(v_{1i} - (v_{2f} - v_{1i})) = m_1(2v_{1i} - v_{2f}) \Rightarrow m_2 = m_1\frac{2v_{1i} - v_{2f}}{v_{2f}} = m_1\left(2\frac{v_{1i}}{v_{2f}} - 1\right)$$

(b) The fraction of kinetic energy is $f = \frac{\Delta K}{K} = \frac{(1/2)m_2v_{2f}^2}{(1/2)m_1v_{1i}^2}$.

CALCULATE:

(a) $m_2 = (0.280 \text{ kg})(2(2) - 1) = 3(0.280 \text{ kg}) = 0.840 \text{ kg}$

(b) Using $m_2 = 3m_1$ and $v_{2f} = v_{1i}/2$, $f = \frac{3m_1(v_{1i}/2)^2}{m_1v_{1i}^2} = \frac{3}{4}$.

ROUND:

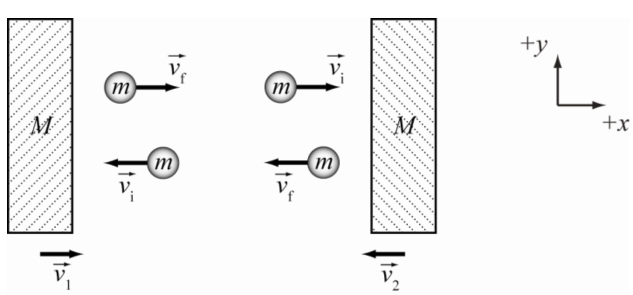
(a) $m_2 = 0.840$ kg

(b) $f = 3/4$

DOUBLE-CHECK: Because v_{2f} is less than v_{1i} it is expected that $m_2 > m_1$.

7.53. **THINK:** A particle has an initial velocity $v_i = -2.21 \cdot 10^3$ m/s. I want to calculate the speed after 6 collisions with the left wall (which has a speed of $v_1 = 1.01 \cdot 10^3$ m/s) and 5 collisions with the right wall (which has a speed of $v_2 = -2.51 \cdot 10^3$ m/s). The magnetic walls can be treated as walls of mass M.

SKETCH:



RESEARCH: Consider one wall with speed v_W. Using the conservation of momentum and energy, $p_i = p_f$ and $E_i = E_f$.

SIMPLIFY: $p_i = p_f \Rightarrow mv_i + Mv_{Wi} = mv_f + Mv_{Wf}$

$$m(v_i - v_f) = M(v_{Wf} - v_{Wi}) \quad (1)$$

$$E_i = E_f$$

$$\frac{1}{2}mv_i^2 + \frac{1}{2}Mv_{Wi}^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}Mv_{Wf}^2$$

$$m(v_i^2 - v_f^2) = M(v_{Wf}^2 - v_{Wi}^2)$$

$$m(v_i - v_f)(v_i + v_f) = M(v_{Wf} - v_{Wi})(v_{Wf} + v_{Wi})$$

$$v_i + v_f = v_{Wf} + v_{Wi}$$

$$v_{Wf} = v_i + v_f - v_{Wi}$$

Substituting back into (1):

$$mv_i + Mv_{Wi} = mv_f + M(v_i + v_f - v_{Wi}) \Rightarrow v_f = \frac{m-M}{m+M}v_i + \frac{2M}{m+M}v_{Wi}$$

If $m \ll M$ then $K_{Sf} = 121$ J.. This means that every collision results in an additional speed of $2v_W$. So after 6 collisions with the left wall and 5 collisions with the right wall, I get $v_f = -v_i + 6(2v_1) - 5(2v_2)$.

CALCULATE: $v_i = -2.21 \cdot 10^3$ m/s, $v_1 = 1.01 \cdot 10^3$ m/s, and $v_2 = -2.51 \cdot 10^3$ m/s.

$v_f = 2.21 \cdot 10^3 \text{ m/s} + 12(1.01 \cdot 10^3 \text{ m/s}) - 10(-2.51 \cdot 10^3 \text{ m/s}) = 3.943 \cdot 10^4 \text{ m/s}$

Since the last collision is with the left wall, the particle is moving to the right and the velocity is positive.

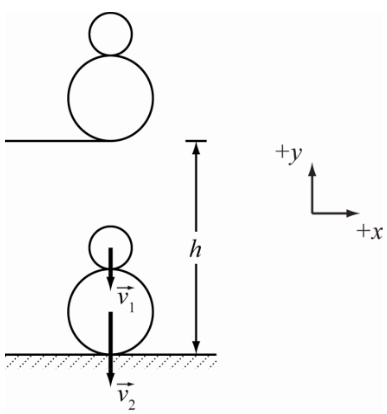
ROUND: $v_f = 3.94 \cdot 10^4$ m/s

DOUBLE-CHECK: Since there have been 11 collisions, it is expected that the resulting speed is about 10 times the original speed.

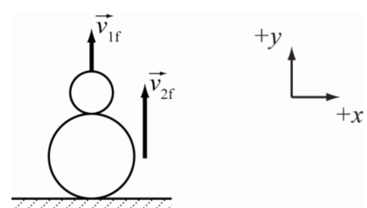
7.54. **THINK:** We have a golf ball with mass $m_1 = 0.0459$ kg and a basketball with mass $m_2 = 0.619$ kg. The balls are dropped from a height of 0.701 m.

SKETCH:

(a, b)



(c)



RESEARCH: Use the conservation of momentum and energy as well as $v^2 = 2gh \Rightarrow v = \sqrt{2gh}$.

SIMPLIFY:

(a) The momentum of the basketball is $p_2 = m_2 v_2 = m_2\sqrt{2gh}$.

(b) The momentum of the golf ball is $p_1 = m_1 v_1 = m_1\sqrt{2gh}$.

(c) The basketball collides with the floor first then collides with the golf ball. After the collision with the floor, the basketball's velocity is opposite the initial velocity. (See diagram (ii) above.) Using conservation of momentum, and conservation of energy: $p_i = p_f \Rightarrow m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$ and therefore

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}) \quad (1)$$
$$E_i = E_f$$
$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$
$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2)$$
$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2i} + v_{2f})$$
$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$
$$v_{2f} = v_{1i} + v_{1f} - v_{2i}$$

Substituting back into (1), we have:

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2(v_{1i} + v_{1f} - v_{2i}) \Rightarrow v_{1f} = \frac{m_1 - m_2}{m_1 + m_2}v_{1i} + \frac{2m_2}{m_1 + m_2}v_{2i}$$

The speed of the golf ball is calculated using $v_{1i} = -v_1 = -\sqrt{2gh}$ and $v_{2i} = v_2 = \sqrt{2gh}$.

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2}\left(-\sqrt{2gh}\right) + \frac{2m_2}{m_1 + m_2}\left(\sqrt{2gh}\right) = \frac{-m_1 + m_2 + 2m_2}{m_1 + m_2}\sqrt{2gh} = \frac{-m_1 + 3m_2}{m_1 + m_2}\sqrt{2gh}$$

(d) The height is calculated using $v^2 = 2gh \Rightarrow h = v^2/(2g)$.

CALCULATE:

(a) $p_2 = (0.619 \text{ kg})\sqrt{2(9.81 \text{ m/s}^2)(0.701 \text{ m})} = 2.296 \text{ kg m/s}$

(b) $p_1 = (0.459 \text{ kg})\sqrt{2(9.81 \text{ m/s}^2)(0.701 \text{ m})} = 0.1702 \text{ kg m/s}$

(c) $v_{1f} = \dfrac{-0.0459 \text{ kg} + 3(0.619 \text{ kg})}{0.0459 \text{ kg} + 0.619 \text{ kg}}\sqrt{2(9.81 \text{ m/s}^2)(0.701 \text{ m})} = 10.102 \text{ m/s}$

(d) $h = \dfrac{(10.102 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} = 5.201 \text{ m}$

ROUND: Rounding to three significant figures:

(a) $p_2 = 2.30 \text{ kg m/s}$

(b) $p_1 = 0.170 \text{ kg m/s}$

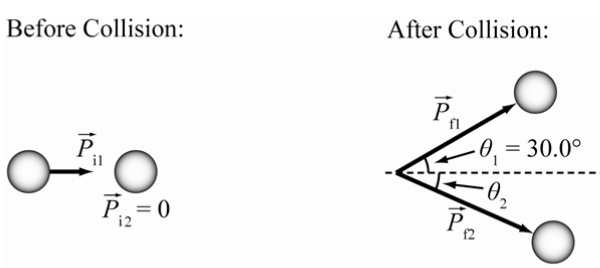
(c) $v_{1f} = 10.1 \text{ m/s}$

(d) $h = 5.20 \text{ m}$

DOUBLE-CHECK: We can see that $v_{1f} \approx 3v_{1i}$, so $h \approx (3v_{1i})^2/(2\text{g}) = 9(v_{1i}^2)/(2\text{g}) = 9h_0$. Our result in (d) should be about 9 times the original height.

7.55. **THINK:** There are two hockey pucks with equal mass $m_1 = m_2 = 0.170$ kg. The first puck has an initial speed of 1.50 m/s and a final speed after collision of 0.750 m/s at an angle of $30.0°$. We want to calculate the speed and direction of the second puck.

SKETCH:



RESEARCH: We need to use the conservation of momentum, i.e. $\vec{p}_i = \vec{p}_f$, or, in component form, $p_{ix} = p_{fx}$ and $p_{iy} = p_{fy}$.

SIMPLIFY: $p_{ix} = p_{fx} \Rightarrow m_1 v_{1i} + 0 = m_1 v_{1f}\cos\theta_1 + m_2 v_{2x}$. Since $m_1 = m_2 = m$, $v_{2x} = v_{1i} - v_{1f}\cos\theta_1$.
$p_{iy} = p_{fy} = 0 \Rightarrow 0 = m_1 v_{1f}\sin\theta_1 + m_2 v_{2y}$. Since $m_1 = m_2 = m$, $v_{2y} = -v_{1f}\sin\theta_1$.

The speed of the second puck is $v_2 = \sqrt{v_{2x}^2 + v_{2y}^2}$ and the angle is: $\theta = \tan^{-1}\left(\dfrac{-v_{1f}\sin\theta_1}{v_{1i} - v_{1f}\cos\theta_1}\right)$.

CALCULATE: $v_2 = \sqrt{\left(1.50 \text{ m/s} - (0.750 \text{ m/s})\cos 30.0^\circ\right)^2 + \left(-(0.750 \text{ m/s})\sin 30.0^\circ\right)^2} = 0.92949 \text{ m/s}$, and $\theta = \tan^{-1}\left(\dfrac{-(0.750 \text{ m/s})\sin 30.0^\circ}{1.50 \text{ m/s} - (0.750 \text{ m/s})\cos 30.0^\circ}\right) = -23.79^\circ$. Since $m_1 = m_2 = m$, the energy is conserved if $v_{1i}^2 = v_{1f}^2 + v_2^2$ we calculate these values to determine if the collision was elastic.

$$v_{1i}^2 = (1.50 \text{ m/s})^2 = 2.25 \text{ (m/s)}^2 \quad \text{and} \quad v_{1f}^2 + v_2^2 = (0.750 \text{ m/s})^2 + (0.930 \text{ m/s})^2 = 1.43 \text{ (m/s)}^2.$$

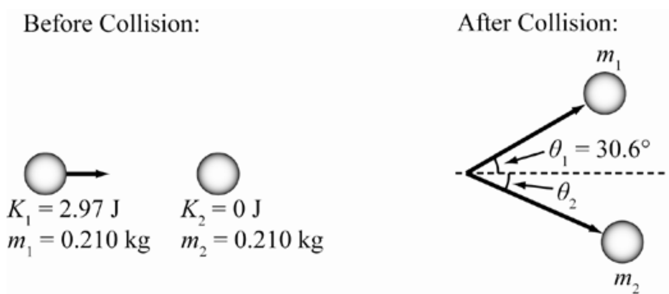
These values are not equal, thus $K_{1i} \neq K_{1f} + K_2$, and the collision is not elastic.

ROUND: $v_2 = 0.929$ m/s, and $\theta = -23.8^\circ$.

DOUBLE-CHECK: The angle and speed for the second puck are reasonable since they are comparable to the angle and speed of the first puck.

7.56. **THINK:** We want to find the kinetic energy of a ball after it collides elastically with a ball at rest. We know the energy and can easily calculate the momentum of the balls before collision. The kinetic energy of the two balls respectively are $K_1 = 2.97$ J and $K_2 = 0$ J. The masses of both balls are the same, $m = 0.210$ kg. After the collision we know only the angle the first ball makes with its own path, $\theta_1 = 30.6^\circ$. This means we have three unknowns: the velocities of the balls after collision and the angle of the second ball. Having three unknowns means we should have three equations.

SKETCH:



RESEARCH: The three equations we will use are the conservation of energy and one for each of the *x* and *y* components of the conservation of momentum: $K_{1i} + K_{2i} = K_{1f} + K_{2f}$, $p_{1ix} + p_{2ix} = p_{1fx} + p_{2fx}$, and $p_{1iy} + p_{2iy} = p_{1fy} + p_{2fy}$. Here, the kinetic energy is given by $(1/2)mv^2$ and the momentum by mv.

SIMPLIFY: $K_{1i} + K_{2i} = K_{1f} + K_{2f} \Rightarrow \frac{1}{2}mv^2 + 0 = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2 \Rightarrow v^2 = v_{1f}^2 + v_{2f}^2$ (1)

$$p_{1ix} + p_{2ix} = p_{1fx} + p_{2fx} \Rightarrow mv + 0 = mv_{1fx} + mv_{2fx} \Rightarrow v = v_{1f}\cos\theta_1 + v_{2f}\cos\theta_2 \quad (2)$$

$$p_{1iy} + p_{2iy} = p_{1fy} + p_{2fy} \Rightarrow 0 + 0 = mv_{1fy} + mv_{2fy} \Rightarrow 0 = v_{1f}\sin\theta_1 + v_{2f}\sin\theta_2 \quad (3)$$

Our goal is to solve for v_{1f} in equations (1), (2) and (3) so that we can calculate the kinetic energy. With this in mind, first rewrite equation (2) and (3) and then square them:

$$v = v_{1f}\cos\theta_1 + v_{2f}\cos\theta_2 \Rightarrow v_{2f}\cos\theta_2 = v - v_{1f}\cos\theta_1 \Rightarrow v_{2f}^2\cos^2\theta_2 = (v - v_{1f}\cos\theta_1)^2 \quad (4)$$

$$0 = v_{1f}\sin\theta_1 + v_{2f}\sin\theta_2 \Rightarrow v_{2f}\sin\theta_2 = -v_{1f}\sin\theta_1 \Rightarrow v_{2f}^2\sin^2\theta_2 = (-v_{1f}\sin\theta_1)^2 \quad (5)$$

Next we add equations (4) and (5) so that θ_2 can be removed from the equation

$$v_{2f}^2\cos^2\theta_2 + v_{2f}^2\sin^2\theta_2 = (-v_{1f}\sin\theta_1)^2 + (v - v_{1f}\cos\theta_1)^2$$

$$\Rightarrow v_{2f}^2 = (-v_{1f}\sin\theta_1)^2 + (v - v_{1f}\cos\theta_1)^2 = v^2 - 2vv_{1f}\cos\theta_1 + v_{1f}^2$$

Substituting $v^2{}_{2f}$ into equations (1), we obtain

$$v^2 = v_{1f}^2 + v_{2f}^2 = v_{1f}^2 + v^2 - 2vv_{1f}\cos\theta_1 + v^2{}_{1f} = v^2 - 2vv_{1f}\cos\theta_1 + 2v^2{}_{1f}$$

$$\Rightarrow\ 2v^2{}_{1f} = 2vv_{1f}\cos\theta_1$$

$$\Rightarrow\ v_{1f} = v\cos\theta_1$$

Note the kinetic energy of ball 1 after collision can now be represented in term of K_1

$$K_{1f} = \frac{1}{2}m_1 v^2{}_{1f} = \frac{1}{2}m_1 v^2\cos^2\theta_1 = K_1\cos^2\theta_1$$

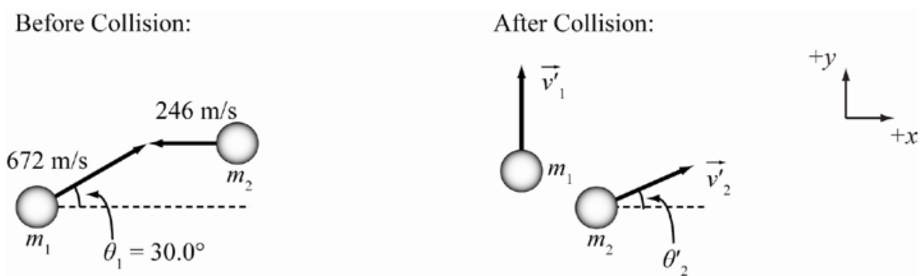
CALCULATE: $K_{1f} = 2.97\text{ J}\cdot\cos^2(30.6°) = 2.200404\text{ J}$

ROUND: The kinetic energy will be given to three significant digits since both K_1 and θ_1 are given to three significant figures. $K_{1f} = 2.20$ J.

DOUBLE-CHECK: The kinetic energy after the collision is less than the original kinetic energy, which makes sense.

7.57. **THINK:** I want to find the final velocity of the molecules after they collide elastically. The first molecule has a speed of $v_1 = 672$ m/s at an angle of $30.0°$ along the positive horizontal. The second has a speed of 246 m/s in the negative horizontal direction. After the collision, the first particle travels vertically. For simplicity, we ignore rotational effects and treat the molecules as simple spherical masses.

SKETCH:



RESEARCH: Since this is an elastic collision, there is conservation of momentum in the x and y components and conservation of energy. $p_{1ix} + p_{2ix} = p_{1fx} + p_{2fx}$, $p_{1iy} + p_{2iy} = p_{1fy} + p_{2fy}$ and $K_i = K_f$.

SIMPLIFY: In the x-direction, the momentum equation gives:

$$mv_{1i}\cos\theta_1 - mv_{2i} = mv_{2f}\cos\theta_2\ \Rightarrow\ v_{1i}\cos\theta_1 - v_{2i} = v_{2fx} \quad (1)$$

The y-component of the momentum gives:

$$mv_{1i}\sin\theta_1 = mv_{1f} + mv_2\sin\theta_2\ \Rightarrow\ v_{1i}\sin\theta_1 = v_{1f} + v_{2fy}\ \Rightarrow\ v_{1i}\sin\theta_1 - v_{1f} = v_{2fy} \quad (2)$$

The kinetic energy gives:

$$\frac{1}{2}mv_{1i}^2 + \frac{1}{2}mv_{2i}^2 = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2\ \Rightarrow\ v_{1i}^2 + v_{2i}^2 = v_{1f}^2 + v_{2fx}^2 + v_{2fy}^2 \quad (3)$$

Squaring and adding equations (1) and (2),

$$v^2{}_{2fx} + v^2{}_{2fy} = \left(v_{1i}\cos\theta_1 - v_{2i}\right)^2 + \left(v_{1i}\sin\theta_1 - v_{1f}\right)^2 \quad (4)$$

Substituting (4) into equation (3),

$$v_{1i}^2 + v_{2i}^2 = v_{1f}^2 + \left(v_{1i}\cos\theta_1 - v_{2i}\right)^2 + \left(v_{1i}\sin\theta_1 - v_{1f}\right)^2$$

$$\Rightarrow\ v_{1i}^2 + v_{2i}^2 = v_{1f}^2 + \left(v^2{}_{1i}\cos^2\theta_1 - 2v_{1i}v_{2i}\cos\theta_1 + v^2{}_{2i}\right) + \left(v^2{}_{1i}\sin^2\theta_1 - 2v_{1i}v_{1f}\sin\theta_1 + v^2{}_{1f}\right)$$

$$\Rightarrow\ v_{1i}^2 + v_{2i}^2 = 2v_{1f}^2 + v^2{}_{1i} + v^2{}_{2i} - 2v_{1i}v_{2i}\cos\theta_1 - 2v_{1i}v_{1f}\sin\theta_1$$

$$\Rightarrow\ v_{1f}^2 - v_{1i}v_{1f}\sin\theta_1 - v_{1i}v_{2i}\cos\theta_1 = 0$$

There is only one unknown in this equation, so I can solve for v_{1f}.

$$v_{1f} = \frac{v_{1i}\sin\theta_1 \pm \sqrt{\left(-v_{1i}\sin\theta_1\right)^2 + 4v_{1i}v_{2i}\cos\theta_1}}{2}$$

I can solve for v_{2fy} in terms of v_{1f} using $v_{2fy} = v_{1i}\sin\theta_1 - v_{1f}$. The angle $\theta_2' = \arctan\left(\frac{v_{2fy}}{v_{2fx}}\right)$.

CALCULATE: $v_{1f} = \dfrac{672\text{ m/s}\cdot\sin 30.0° \pm \sqrt{\left(-672\text{ m/s}\cdot\sin 30.0°\right)^2 + 4\cdot 672\text{ m/s}\cdot 246\text{ m/s}\cdot\cos 30.0°}}{2}$

$= 581.9908\text{ m/s or } -245.9908\text{ m/s}$

Since I know that the molecule travels in the positive y direction, $v_{1f} = 581.9908$ m/s .

$$v_{2fy} = v_{1i}\sin\theta_1 - v_{1f} = 672\text{ m/s}\cdot\sin 30° - 581.9908\text{ m/s} = -245.9908\text{ m/s}$$

$$v_{2fx} = v_{1i}\cos\theta_1 - v_{2i} = 672\text{ m/s}\cdot\cos 30° - 246\text{ m/s} = 335.9691\text{ m/s}$$

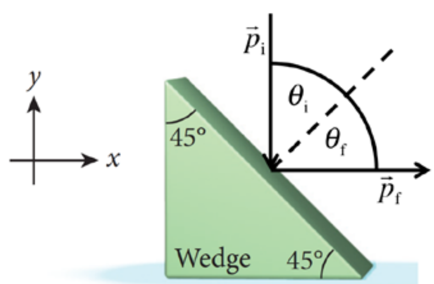
Therefore, $v_{2f} = \sqrt{\left(335.9691\text{ m/s}\right)^2 + \left(-245.9908\text{ m/s}\right)^2} = 416.3973\text{ m/s}$ at an angle of $\theta_2' = \arctan\left(\frac{-245.9908\text{ m/s}}{335.9691\text{ m/s}}\right) = -36.211°$.

ROUND: $v_{1f} = 582$ m/s in the positive y-direction and $v_{2f} = 416$ m/s at an angle of 36.2° below the positive x-axis.

DOUBLE-CHECK: The results show $v_{1f} < v_1$ and $v_{2f} > v_{2i}$ as expected, so the answers look reasonable.

7.58. **THINK:** Since the wedge is solidly attached to the ground, it will not move during the collision, because the Earth has, for all practical purposes, infinite mass. This means that we can consider the surface of the wedge as a rigid wall and the angle of deflection relative to the normal will be equal to the angle of incidence relative to the normal.

SKETCH: We can simply use the figure supplied in the problem as our sketch, where we indicate the surface normal to the wedge (dashed line), as well as the angle of incidence and the angle of reflection.



RESEARCH: In equation 7.19 we found that $\theta_f = \theta_i$. Since the normal to the wedge surface makes a 45°-angle with the x-axis, this implies that the final momentum of the ball after the collision points horizontally. Since the collision is totally elastic the kinetic energy is conserved, which means that the length of the ball's momentum vector does not change. Consequently, $\vec{p}_i = (0, -mv)$; $\vec{p}_f = (mv, 0)$. The momentum change of the ball in the collision is $\Delta\vec{p} = \vec{p}_f - \vec{p}_i$. Since the total momentum is conserved in all collisions, the recoil momentum that the Earth receives in this collision is then $\vec{p}_{recoil} = -\Delta\vec{p}$.

SIMPLIFY: $\vec{p}_{recoil} = -\Delta\vec{p} = -(\vec{p}_f - \vec{p}_i) = \vec{p}_i - \vec{p}_f = (0, -mv) - (mv, 0) = (-mv, -mv)$

The absolute value of the recoil momentum is $\left|\vec{p}_{recoil}\right| = \sqrt{(-mv)^2 + (-mv)^2} = \sqrt{2}mv$

CALCULATE: $\vec{p}_{recoil} = (-1,-1)(3.00\text{ kg})(4.50\text{ m/s}) = (-1,-1)(13.5\text{ kg m/s})$

$\left|\vec{p}_{recoil}\right| = \sqrt{2}(13.5\text{ kg m/s}) = 19.0919\text{ kg m/s}$

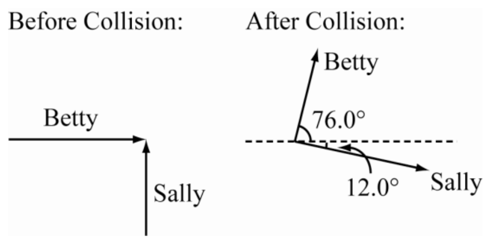
ROUND: We round the absolute value of the recoil momentum to three significant figures:

$\left|\vec{p}_{recoil}\right| = 19.1\text{ kg m/s}$

DOUBLE-CHECK: We have assumed that the wedge, which is attached to the Earth, does not move in the collision process. Is it reasonable then to find that the wedge+Earth system receives a finite recoil momentum in the collision process? The answer is yes, but only because we can assume that the mass of the Earth is practically infinitely large compared to the mass of the ball.

7.59. **THINK:** I want to find the kinetic energy of Betty and Sally after they collide together. Also, I would like to know if the collision is elastic. Betty and Sally have masses and velocities of $m_{\text{B}} = 55.0$ kg, $v_{\text{B}} = v_{\text{B},x} = 22.0 \text{ km/h} = 6.111 \text{ m/s}$, $m_{\text{S}} = 45.0$ kg and $v_{\text{S}} = v_{\text{S},y} = 28.0 \text{ km/h} = 7.778 \text{ m/s}$ respectively. After the collision, Betty is travelling $\theta_{\text{B}} = 76.0^\circ$ from the horizontal and Sally is moving $\theta_{\text{S}} = 12.0^\circ$ below the horizontal.

SKETCH:



RESEARCH: Use the conservation of momentum $\vec{p}_{\text{Bi}} + \vec{p}_{\text{Si}} = \vec{p}_{\text{Bf}} + \vec{p}_{\text{Sf}}$ to get the velocities after the collision. This information will allow calculation of the kinetic energy $mv^2/2$ for the skaters.

SIMPLIFY: The momentum gives the two following equations:

$$m_{\text{B}} v_{\text{Bi}} = m_{\text{S}} v_{\text{Sf}} \cos\theta_{\text{S}} + m_{\text{B}} v_{\text{Bf}} \cos\theta_{\text{B}} \quad (1)$$

$$m_{\text{S}} v_{\text{Si}} = -m_{\text{S}} v_{\text{Sf}} \sin\theta_{\text{S}} + m_{\text{B}} v_{\text{Bf}} \sin\theta_{\text{B}} \quad (2)$$

Solving equation (1) for v_{Sf},

$$v_{\text{Sf}} = \frac{m_{\text{B}} v_{\text{Bi}} - m_{\text{B}} v_{\text{Bf}} \cos\theta_{\text{B}}}{m_{\text{S}} \cos\theta_{\text{S}}}$$

Substituting into equation (2),

$$m_{\text{S}} v_{\text{Si}} = -m_{\text{S}} \left(\frac{m_{\text{B}} v_{\text{Bi}} - m_{\text{B}} v_{\text{Bf}} \cos\theta_{\text{B}}}{m_{\text{S}} \cos\theta_{\text{S}}} \right) \sin\theta_{\text{S}} + m_{\text{B}} v_{\text{Bf}} \sin\theta_{\text{B}}$$

$$m_{\text{S}} v_{\text{Si}} = -m_{\text{B}} v_{\text{Bi}} \tan\theta_{\text{S}} + m_{\text{B}} v_{\text{Bf}} \cos\theta_{\text{B}} \tan\theta_{\text{S}} + m_{\text{B}} v_{\text{Bf}} \sin\theta_{\text{B}}$$

$$v_{\text{Bf}} = \frac{m_{\text{S}} v_{\text{Si}} + m_{\text{B}} v_{\text{Bi}} \tan\theta_{\text{S}}}{m_{\text{B}} \left(\cos\theta_{\text{B}} \tan\theta_{\text{S}} + \sin\theta_{\text{B}} \right)}$$

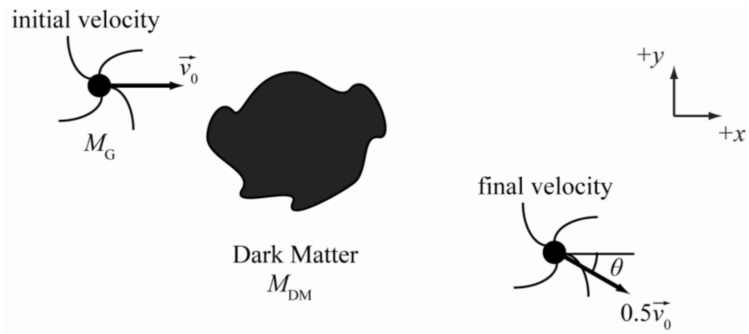
Similarly, $v_{\text{Sf}} = \dfrac{m_{\text{B}} v_{\text{Bi}} \tan\theta_{\text{B}} - m_{\text{S}} v_{\text{Si}}}{m_{\text{S}} \left(\sin\theta_{\text{S}} + \cos\theta_{\text{S}} \tan\theta_{\text{B}} \right)}$. To get the kinetic energy, we simply plug the result into the equation $K = \dfrac{1}{2} mv^2$.

CALCULATE: $v_{\text{Bf}} = \dfrac{(45.0 \text{ kg})(7.778 \text{ m/s}) + (55.0 \text{ kg})(6.111 \text{ m/s})\tan 12.0^\circ}{(55.0 \text{ kg})(\cos 76.0^\circ \tan 12.0^\circ + \sin 76^\circ)} = 7.49987 \text{ m/s}$ and

$v_{\text{Sf}} = \dfrac{(55.0 \text{ kg})(6.111 \text{ m/s})\tan 76.0^\circ - (45.0 \text{ kg})(7.778 \text{ m/s})}{(45.0 \text{ kg})(\sin 12.0^\circ + \cos 12.0^\circ \tan 76.0^\circ)} = 5.36874 \text{ m/s}$. Betty's final kinetic energy is then $\dfrac{1}{2} m_{\text{B}} v_{\text{Bf}}^2 = 1546.82 \text{ J}$. Sally's final kinetic energy is then $\dfrac{1}{2} m_{\text{S}} v_{\text{Sf}}^2 = 648.526 \text{ J}$. The ratio of the final and initial kinetic energy is $\dfrac{K_{\text{f}}}{K_{\text{i}}} = \dfrac{m_{\text{B}} v_{\text{Bf}}^2 + m_{\text{S}} v_{\text{Sf}}^2}{m_{\text{B}} v_{\text{Bi}}^2 + m_{\text{S}} v_{\text{Si}}^2} = 0.9193$.

ROUND: Our results will be reported to 3 significant figures, the same accuracy as the given values. $K_{\text{Bf}} = 1.55 \text{ kJ}$ and $K_{\text{Sf}} = 649 \text{ J}$. The ratio $K_{\text{f}}/K_{\text{i}}$ is not equal to one, so the collision is inelastic.
DOUBLE-CHECK: These are reasonable results.

7.60. **THINK:** I want to find the mass of dark matter in terms of M_{G}, v_0 and θ. The initial velocity of the galaxy is in the *x*-direction. After it interacts with the dark matter it travels at 50% of its original speed in the direction of θ below the *x*-axis.
SKETCH:



RESEARCH: Use the conservation of momentum in the *x* and *y* directions. Also use the conservation of energy. $p_{\text{Gi}} + p_{\text{DMi}} = p_{\text{Gf}} + p_{\text{DMf}}$, $K_{\text{i}} = K_{\text{f}}$.
SIMPLIFY: The momentum in the *x* and *y* direction gives:

$$M_{\text{G}}v_0 + 0 = M_{\text{G}}\left(0.50v_0\right)\cos\theta + p_{\text{DMf}x};\quad 0 = M_{\text{G}}\left(0.50v_0\right)\sin\theta + p_{\text{DMf}y}.$$

The conservation of energy gives:

$$\frac{1}{2}M_{\text{G}}v_0^2 = \frac{1}{2}M_{\text{G}}\left(0.50v_0\right)^2 + \frac{p_{\text{DMf}x}^2 + p_{\text{DMf}y}^2}{2M_{\text{DM}}} \Rightarrow M_{\text{G}}v_0^2 = M_{\text{G}}\left(0.50v_0\right)^2 + \frac{p_{\text{DMf}x}^2 + p_{\text{DMf}y}^2}{M_{\text{DM}}}.$$

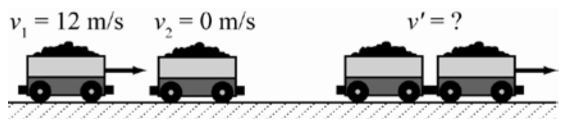
Use the conservation of energy to solve for M_{DM}.

$$\begin{aligned}
M_{\text{DM}} &= \frac{p_{\text{DMf}x}^2 + p_{\text{DMf}y}^2}{M_{\text{G}}v_0^2\left(1-0.50^2\right)} \\
&= \frac{\left(M_{\text{G}}v_0\left(1-0.50\cos\theta\right)\right)^2 + \left(-M_{\text{G}}v_0\left(0.50\sin\theta\right)\right)^2}{M_{\text{G}}v_0^2\left(1-0.50^2\right)} \\
&= \frac{M_{\text{G}}^2v_0^2}{M_{\text{G}}v_0^2}\frac{\left(1-0.50\cos\theta\right)^2 + \left(-0.50\sin\theta\right)^2}{1-0.50^2} \\
&= M_{\text{G}}\frac{1-\cos\theta+0.25\cos^2\theta+0.25\sin^2\theta}{0.75} \\
&= M_{\text{G}}\frac{1.25-\cos\theta}{3/4} \\
&= \frac{4}{3}M_{\text{G}}\left(1.25-\cos\theta\right)
\end{aligned}$$

CALCULATE: There are no values to calculate.
ROUND: There is no rounding to do.
DOUBLE-CHECK: This result is reasonable.

7.61. **THINK:** I want to know what the speed of the railroad car is after a perfectly inelastic collision occurs. Knowing $m_1 = m_2 = 1439$ kg, $v_1 = 12.0$ m/s and $v_2 = 0$ m/s.

SKETCH:



RESEARCH: The equation for a perfectly inelastic collision with identical masses is given by $v_{1i} + v_{2i} = 2v_f$.

SIMPLIFY: $v_f = (v_{1i} + v_{2i})/2$

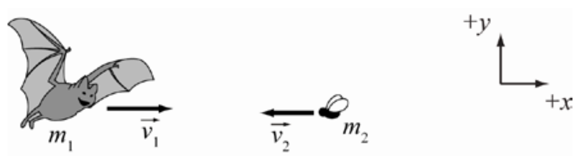
CALCULATE: $v_f = (12.0 \text{ m/s} + 0 \text{ m/s})/2 = 6.00 \text{ m/s}$

ROUND: Because the velocity before the collision is given to three significant figures, keep the result to three significant figures. The velocity of the cars after the collision is 6.00 m/s.

DOUBLE-CHECK: This is equivalent to a speed of 22 km/h, which is reasonable for railroad cars.

7.62. **THINK:** I want to know the speed of a 50.0 g bat after it catches a 5.00 g insect if they travel at 8.00 m/s and 6.00 m/s in opposite directions.

SKETCH:



RESEARCH: Since the bat catches the insect this is an elastic collision. Use the equation $m_{1i}v_{1i} + m_{2i}v_{2i} = (m_1 + m_2)v_f$.

SIMPLIFY: $v_f = \dfrac{m_{1i}v_{1i} + m_{2i}v_{2i}}{m_1 + m_2}$

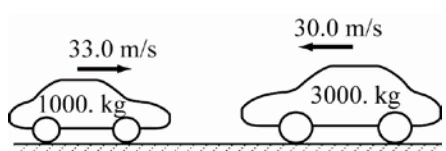
CALCULATE: $v_f = \dfrac{(50.0 \text{ g})(8.00 \text{ m/s}) + (5.00 \text{ g})(-6.00 \text{ m/s})}{50.0 \text{ g} + 5.00 \text{ g}} = 6.727 \text{ m/s}$

ROUND: To three significant figures, the speed of the bat after a tasty treat is 6.73 m/s.

DOUBLE-CHECK: I would expect a small loss in the speed of the bat since the insect is small compared to it.

7.63. **THINK:** I want to know the acceleration of the occupants of each car after a perfectly inelastic collision. The first car has mass $m_1 = 1000.$ kg and velocity $v_1 = 33.0$ m/s while the second has mass $m_2 = 3000.$ kg and velocity $v_2 = -30.0$ m/s. The collision lasts for 100. ms, or 0.100 s.

SKETCH:



RESEARCH: First to find the change of moment each car experiences using the equation of perfectly inelastic collision, $m_1v_1 + m_2v_2 = (m_1 + m_2)v_f$. Using this find the force experienced with the help of the equation $F\Delta t = \Delta p$.

SIMPLIFY: $v_f = \dfrac{m_1v_1 + m_2v_2}{m_1 + m_2}$. Then $\Delta p_1 = (m_1 + m_2)v_f - m_1v_1$ and $\Delta p_2 = (m_1 + m_2)v_f - m_2v_2$.

So $a_1 = \dfrac{m_2(v_2 - v_1)}{(m_1 + m_2)\Delta t}$ and $a_2 = \dfrac{m_1(v_1 - v_2)}{(m_1 + m_2)\Delta t}$.

CALCULATE: $a_1 = \dfrac{(3000.\text{ kg})(-30.0\text{ m/s} - 33.0\text{ m/s})}{(3000.\text{ kg} + 1000.\text{ kg})(0.100\text{ s})} = -472.5\text{ m/s}^2$, and

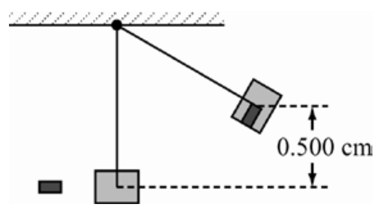
$$a_2 = \frac{(1000.\text{ kg})(33.0\text{ m/s} - (-30.0\text{ m/s}))}{(3000.\text{ kg} + 1000.\text{ kg})(0.100\text{ s})} = 157.5\text{ m/s}^2.$$

ROUND: The acceleration the occupants of the smaller car feel is $a_1 = -473\text{ m/s}^2$, or $-48.2g$. The acceleration the occupants of the larger car feel is $a_2 = 158\text{ m/s}^2$, or $16.1g$.

DOUBLE-CHECK: This makes sense since we often hear how the drivers of smaller cars fair worse than those in larger cars.

7.64. **THINK:** I am looking for the speed of the bullet of mass $m_{\text{bu}} = 2.00\text{ g}$ that moves the 2.00 kg block on a string. The kinetic energy of the block and bullet is converted to potential energy and attains a height of 0.500 cm. First start by converting the mass of the bullet to kilograms, $m_{\text{bu}} = 0.00200\text{ kg}$, and the height to meters; h = 0.00500 m.

SKETCH:



RESEARCH: First use the relation between the kinetic energy $T = \frac{1}{2}mv^2$ and the potential energy $U = mgh$. From these find the final velocity of the block and bullet. Then using the conservation of momentum for perfectly inelastic collisions, find the initial speed of the bullet. $m_1v_1 + m_2v_2 = (m_1 + m_2)v_{\text{f}}$

SIMPLIFY: Set the kinetic energy equal to the potential energy and solve for the velocity. Use this in the conservation of momentum equation: $\frac{1}{2}mv_{\text{f}}^2 = mgh \;\Rightarrow\; v_{\text{f}} = \sqrt{2gh}$. Note that $v_2 = v_{\text{bl}} = 0$.

$$m_1v_1 = (m_1 + m_2)v_{\text{f}} = (m_1 + m_2)\sqrt{2gh} \;\Rightarrow\; v_1 = \frac{m_1 + m_2}{m_1}\sqrt{2gh}$$

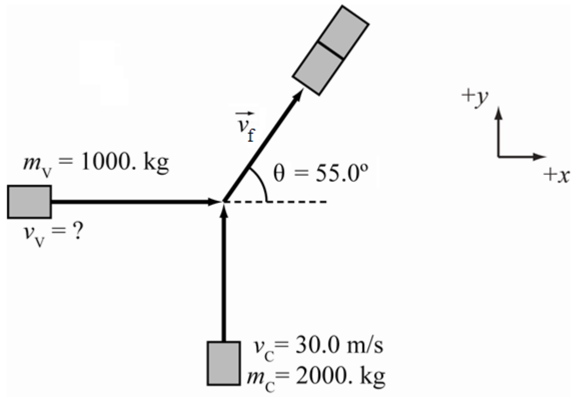
CALCULATE: $v_1 = \dfrac{2.00\text{ kg} + 0.00200\text{ kg}}{0.00200\text{ kg}}\sqrt{2(9.81\text{ m/s}^2)(0.00500\text{ m})} = 313.522\text{ m/s}$

ROUND: The height attained by the block and bullet was only given to three significant figures, thus the velocity of the bullet will be reported as 314 m/s.

DOUBLE-CHECK: The speed of a typical bullet is 1000 m/s, thus our answer is reasonable.

7.65. **THINK:** The Volkswagen of mass $m_{\text{V}} = 1000.\text{ kg}$ was going eastward before the collision and the Cadillac had mass $m_{\text{C}} = 2000.\text{ kg}$ and velocity $v_{\text{C}} = 30.0\text{ m/s}$ northward, and after the collision both cars stuck together travelling $\theta = 55.0°$ north of east.

SKETCH:



RESEARCH: The collision was perfectly inelastic so use the equation $m_C v_C + m_V v_V = (m_C + m_V) v_f$ for each component of the motion.

SIMPLIFY: In the east-west direction:

$$m_V v_V = (m_C + m_V) v_f \cos\theta \;\Rightarrow\; v_f = \frac{m_V v_V}{m_C + m_V}\frac{1}{\cos\theta},$$

and in the north-south direction:

$$m_C v_C = (m_C + m_V) v_f \sin\theta \;\Rightarrow\; v_f = \frac{m_C v_C}{m_C + m_V}\frac{1}{\sin\theta}$$

Equating these two expressions for the final velocity gives:

$$\frac{m_V v_V}{m_C + m_V}\frac{1}{\cos\theta} = \frac{m_C v_C}{m_C + m_V}\frac{1}{\sin\theta} \;\Rightarrow\; v_V = \frac{m_C}{m_V} v_C \frac{\cos\theta}{\sin\theta} = \frac{m_C}{m_V} v_C \cot\theta.$$

CALCULATE: $v_V = \dfrac{2000.\text{ kg}}{1000.\text{ kg}}(30.0\text{ m/s})\cot 55.0° = 42.01245\text{ m/s}$

ROUND: The Volkswagen's velocity is 42.0 m/s.

DOUBLE-CHECK: This is a reasonable result. It's in the same order as the Cadillac.

7.66. **THINK:** There are three things to calculate:

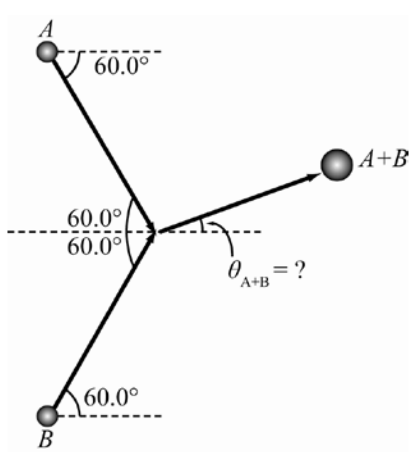
(a) the angle above the horizontal the mass *A*+*B* makes;

(b) the ratio v_f / v_A;

(c) the ratio E_f / E_i.

It is known that $m_A = m_B = m$ and that $v_B = 2v_A = 2v$. By inspection, $\theta_A = 60.0°$ and $\theta_B = 60.0°$.

SKETCH:



RESEARCH: The relevant equations are those for conservation of momentum for a perfectly inelastic collision for the x and y components, and for the kinetic energy.

$$p_{Ax} + p_{Bx} = p_{(A+B)x}, \quad p_{Ay} + p_{By} = p_{(A+B)y}, \text{ and } K = \frac{1}{2}mv^2.$$

Also, $m_A = m_B = m;\ m_A + m_B = 2m;\ v_B = 2v_A$

SIMPLIFY: In the x -direction:

$$m_A v_A \cos\theta_A + m_B v_B \cos\theta_B = (m_A + m_B) v_{AB} \cos\theta_{A+B}$$

$$m v_A \cos(60.0°) + m(2v_A)\cos 60.0° = 2m v_{AB} \cos\theta_{A+B}$$

$$v_A \cos(60.0°) + (2v_A)\cos 60.0° = 2v_{AB} \cos\theta_{A+B}$$

$$\frac{v_A}{2} + \frac{2v_A}{2} = 2v_{AB} \cos\theta_{A+B}$$

$$\frac{3}{4}v_A = v_{AB} \cos\theta_{A+B}$$

In the y-direction:

$$-m_A v_A \sin\theta_A + m_B v_B \sin\theta_B = (m_A + m_B) v_{AB} \sin\theta_{A+B}$$

$$-m v_A \sin(60.0°) + m(2v_A)\sin 60.0° = 2m v_{AB} \sin\theta_{A+B}$$

$$-v_A \sin(60.0°) + (2v_A)\sin 60.0° = 2v_{AB} \sin\theta_{A+B}$$

$$\frac{-\sqrt{3}v_A}{2} + \frac{2\sqrt{3}v_A}{2} = 2v_{AB} \sin\theta_{A+B}$$

$$\frac{\sqrt{3}}{2}v_A = 2v_{AB} \sin\theta_{A+B}$$

$$\frac{\sqrt{3}v_A}{4} = v_{AB} \sin\theta_{A+B}$$

To find the angle θ_{A+B}, divide the equation found for the y-component by the one for the x-component.

$$\frac{\frac{\sqrt{3}v_A}{4}}{\frac{3}{4}v_A} = \frac{v_{AB}\sin\theta_{A+B}}{v_{AB}\cos\theta_{A+B}} \Rightarrow \frac{1}{\sqrt{3}} = \tan\theta_{AB} \Rightarrow \theta_{AB} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

To find the ratio v_{AB} / v_A rearrange the y-component equation.

$$\frac{\sqrt{3}v_A}{4} = v_{AB}\sin\theta_{A+B} \Rightarrow \frac{v_{AB}}{v_A} = \frac{\sqrt{3}}{4\sin\theta_{A+B}}$$

The ratio K_f / K_i is:

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}(m_A + m_B)v_{AB}^2}{\frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2} = \frac{(m+m)v_{AB}^2}{mv^2 + m(2v_A)^2} = \frac{2v_{AB}^2}{v^2 + 4v_A^{\ 2}} = \frac{2}{5}\left(\frac{v_{AB}}{v_A}\right)^2 = \frac{2}{5}\left[\frac{\sqrt{3}}{4\sin\theta_{A+B}}\right]^2$$

CALCULATE:

(a) $\theta_{A+B} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30.0°$

(b) $\frac{v_{AB}}{v_A} = \frac{\sqrt{3}}{4\sin 30.0°} = \frac{\sqrt{3}}{2}$

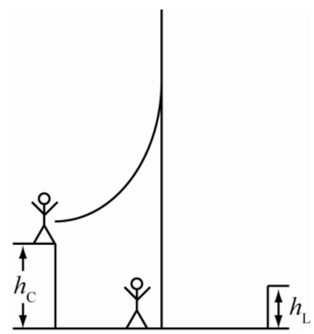
(c) $\frac{K_f}{K_i} = \frac{2}{5}\left[\frac{\sqrt{3}}{4\sin 30.0°}\right]^2 = \left(\frac{2}{5}\right)\left(\frac{3}{4}\right) = \frac{3}{10}$

ROUND:
(a) $\theta_{\text{A+B}} = 30.0°$
(b) $\dfrac{v_{\text{f}}}{v_{\text{A}}} = \dfrac{\sqrt{3}}{2} = 0.866$
(c) $\dfrac{K_{\text{f}}}{K_{\text{i}}} = \dfrac{3}{10} = 0.300$

DOUBLE-CHECK: These results are reasonable. When the objects collide and stick together is known as perfectly inelastic so we would expect the ratio $K_{\text{f}}/K_{\text{i}}$ to be less than one.

7.67. **THINK:** This is essentially an inelastic collision. Since Jane is standing still, she has no initial momentum. Tarzan must have initial kinetic energy such that when his and Jane's mass are combined, their momentum is sufficient to make it to the tree. Because this is an inelastic collision, energy is not conserved when Tarzan catches Jane, but momentum is conserved. Tarzan's mass is 80.0 kg, and Jane's mass is 40.0 kg. The vine Tarzan swings from is 30.0 m long. The cliff from which Tarzan jumps is 20.0 meters high, and the tree limb Tarzan and Jane must reach is 10.0 m high.

SKETCH:



RESEARCH: The problem is most easily solved by working backwards. The potential energy of Tarzan and Jane when they reach the tree is $U_{\text{TJ}} = (m_{\text{T}} + m_{\text{J}})gh_{\text{tree}}$. This potential energy must be equal to the kinetic energy just after the "collision": $K_{\text{TJ}} = \frac{1}{2}(m_{\text{T}} + m_{\text{J}})v_{\text{TJ}}^2$. The combined momentum of Tarzan and Jane after the collision, $P_{\text{TJ}} = (m_{\text{T}} + m_{\text{J}})v_{\text{TJ}}$, must be equal to the sum of their momenta before the collision, $m_{\text{T}}v_{\text{T}} + m_{\text{J}}v_{\text{J}} = m_{\text{T}}v_{\text{T}}$ (since Jane's initial momentum is zero). Tarzan's kinetic energy just before he catches Jane is $K_{\text{T}} = \frac{1}{2}m_{\text{T}}v_{\text{T}}^2$, which must be equal to his initial total energy, $U_{\text{T}} + K_{\text{T,0}} = m_{\text{T}}gh_{\text{cliff}} + \frac{1}{2}m_{\text{T}}v_{\text{T,0}}^2$. Tarzan's initial velocity, $v_{\text{T,0}}$, is the desired quantity.

SIMPLIFY:

$U_{\text{TJ}} = (m_{\text{T}} + m_{\text{J}})gh_{\text{tree}} = \frac{1}{2}(m_{\text{T}} + m_{\text{J}})v_{\text{TJ}}^2$. Solving for v_{TJ} : $\text{v}_{\text{TJ}} = \sqrt{2gh_{\text{tree}}}$.

$P_{\text{TJ}} = (m_{\text{T}} + m_{\text{J}})v_{\text{TJ}} = (m_{\text{T}} + m_{\text{J}})\sqrt{2gh_{\text{tree}}} = m_{\text{T}}v_{\text{T}}$. Solving for v_{T} : $\text{v}_{\text{T}} = \dfrac{(m_{\text{T}} + m_{\text{J}})\sqrt{2gh_{\text{tree}}}}{m_{\text{T}}}$.

$$K_{\text{T}} = \frac{1}{2}m_{\text{T}}v_{\text{T}}^2 = \frac{1}{2}m_{\text{T}}\left(\frac{(m_{\text{T}} + m_{\text{J}})\sqrt{2gh_{\text{tree}}}}{m_{\text{T}}}\right)^2 = \frac{(m_{\text{T}} + m_{\text{J}})^2 gh_{\text{tree}}}{m_{\text{T}}}$$

$U_T + K_{T,0} = m_T g h_{cliff} + \frac{1}{2} m_T v_{T,0}^2 = K_T = \frac{(m_T + m_J)^2 g h_{tree}}{m_T}$. Solving for $v_{T,0}$:

$$v_{T,0} = \sqrt{\frac{2}{m_T}\left(\frac{(m_T + m_J)^2 g h_{tree}}{m_T} - m_T g h_{cliff}\right)}.$$

CALCULATE:

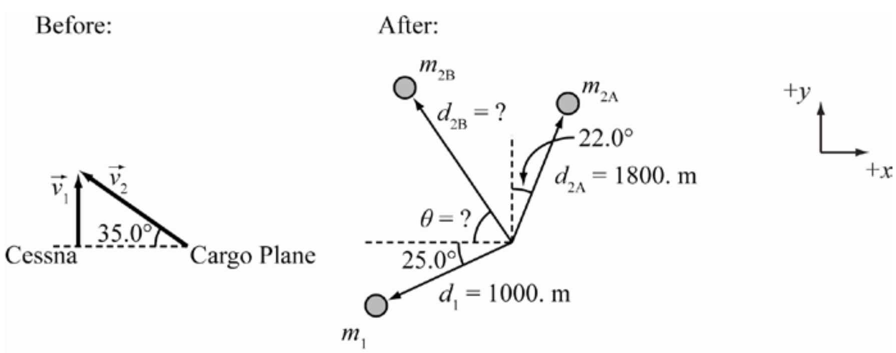
$$v_{T,0} = \sqrt{\frac{2}{80.0 \text{ kg}}\left(\frac{(80.0 \text{ kg} + 40.0 \text{ kg})^2 (9.81 \text{ m/s}^2)(10.0 \text{ m})}{80.0 \text{ kg}} - (80.0 \text{ kg})(9.81 \text{ m/s}^2)(20.0 \text{ m})\right)} = 7.00 \text{ m/s}.$$

ROUND: Tarzan must jump from the ledge at a speed of 7.00 m/s.

DOUBLE-CHECK: 7.00 m/s is a fast but reasonable speed for a fit person to achieve with a running jump.

7.68. **THINK:** I hope to find the region the second part of the cargo plane lands after the collision. Knowing the initial speed and mass of the Cessna to be $m_1 = 3000.0$ kg and $v_1 = 75.0$ m/s northward and the initial speed and mass of the cargo plane to be $m_2 = 7000.$ kg and $v_2 = 100.$ m/s $35.0°$ north of west. After the collision the plane drops $z = 1600.$ m to the ground. The Cessna is $d_1 = 1000.$ m at $25.0°$ south of west and one piece of the cargo plane of mass $m_{2A} = 4000.$ kg is 1800. m $22.0°$ east of north.

SKETCH:



RESEARCH: In order to calculate the position of the second piece of the cargo plane I need the conservation of momentum in the *x* (east-west) component and *y* (north-south) component.

$p_{1x} + p_{2x} = p'_{1x} + p_{2Ax} + p_{2Bx}$.

To find the speed of the planes after impact, use $z = gt^2/2$ and $d = vt$. The time it takes the pieces of the planes to fall to the ground is $t = \sqrt{2z/g}$, and the velocity of each piece is $v = d/t = d\sqrt{g/(2z)}$, where *d* is the horizontal distance traveled by a given piece. Therefore the distance traveled by each piece of debris is $d = vt = v\sqrt{(2z)/g}$.

SIMPLIFY: Now solve for the *x* and *y* components of the missing piece of debris.

$$p_{2Bx} = p_{1x} + p_{2x} - p'_{1x} - p_{2Ax} \quad \text{and} \quad p_{2By} = p_{1y} + p_{2y} - p'_{1y} - p_{2Ay}$$

Being careful with directions, these become:

$$p_{2Bx} = -m_2 v_2 \cos\theta_2 + m_1 v_1' \cos\theta_1' - m_{2A} v_{2A} \sin\theta_{2A}$$

$$m_{2B} v_{2Bx} = -m_2 v_2 \cos\theta_2 + m_1 d_1 \sqrt{\frac{g}{2z}} \cos\theta_1' - m_{2A} d_{2A} \sqrt{\frac{g}{2z}} \sin\theta_{2A}$$

$$m_{2B} d_{2Bx} \sqrt{\frac{g}{2z}} = -m_2 v_2 \cos\theta_2 + m_1 d_1 \sqrt{\frac{g}{2z}} \cos\theta_1' - m_{2A} d_{2A} \sqrt{\frac{g}{2z}} \sin\theta_{2A}$$

$$d_{2Bx} = -\frac{m_2 v_2 \cos\theta_2}{m_{2B}\sqrt{g/(2z)}} + \frac{m_1}{m_{2B}} \cos\theta_1' - \frac{m_{2A}}{m_{2B}} d_{2A} \sin\theta_{2A}$$

The y-component is:

$$p_{2By} = m_1 v_1 + m_2 v_2 \sin\theta_2 + m_1 v_1' \sin\theta_1' - m_{2A} v_{2A} \cos\theta_{2A}$$

$$m_{2B} d_{2By} \sqrt{g/(2z)} = m_1 v_1 + m_2 v_2 \sin\theta_2 + m_1 d_1 \sqrt{g/(2z)} \sin\theta_1' - m_{2A} d_{2A} \sqrt{g/(2z)} \cos\theta_{2A}$$

$$d_{2By} = \frac{m_1 v_1 + m_2 v_2 \sin\theta_2}{\sqrt{m_{2B}\sqrt{g/(2z)}}} + \frac{m_1}{m_{2B}} d_1 \sin\theta_1' - \frac{m_{2A}}{m_{2B}} d_{2A} \cos\theta_{2A}$$

The total distance is: $d_{2B} = \sqrt{(d_{2Bx})^2 + (d_{2By})^2}$. The direction of the missing piece of wreckage is: $\theta = \tan^{-1}(d_{2By}/d_{2Bx})$.

CALCULATE:

$$d_{2bx} = \frac{-(7000.\text{ kg})(100.\text{ m/s})\cos 35.0°}{(3000.\text{ kg})\sqrt{(9.81\text{ m/s}^2)/(2(1600.\text{ m}))}} + \left(\frac{3000.\text{ kg}}{3000.\text{ kg}}\right)(1000.\text{ m})\cos 25.0° - \left(\frac{4000.\text{ kg}}{3000.\text{ kg}}\right)(1800.\text{ m})\sin 22.0° = -3444.84\text{ m}$$

$$d_{2by} = \frac{(3000.\text{ kg})(75.0\text{ m/s}) + (7000.\text{ kg})(100.\text{ m/s})\sin 35.0°}{(3000.\text{ kg})\sqrt{(9.81\text{ m/s}^2)/(2(1600.\text{ m}))}} + \left(\frac{3000.\text{ kg}}{3000.\text{ kg}}\right)(1000.\text{ m})\sin 25.0° - \left(\frac{4000.\text{ kg}}{3000.\text{ kg}}\right)(1800.\text{ m})\cos 22.0° = 1969.126\text{ m}$$

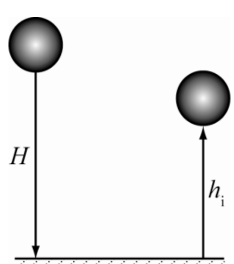
$$d_{2B} = \sqrt{(-3444.84\text{ m}) + (1969.126\text{ m})} = 3967.92\text{ m}, \quad \theta = \tan^{-1}\left(\frac{1969.126\text{ m}}{-3444.84\text{ m}}\right) = 29.75°$$

ROUND: 3970 m from the point of the collision, at an angle of $29.8°$ clockwise from the negative x-axis.

DOUBLE-CHECK: This is a reasonable answer. The distance is of the same order as the other crash sites.

7.69. **THINK:** I want to find the coefficient of restitution for a variety of balls.

SKETCH:



RESEARCH: Using the equation for the coefficient of restitution for heights. $\varepsilon = \sqrt{H/h_1}$.

SIMPLIFY: There is no need to simplify.

CALCULATE: An example calculation: A range golf ball has an initial height $H = 85.0$ cm and a final height $h_1 = 62.6$ cm.

$$\varepsilon = \sqrt{\frac{62.6 \text{ cm}}{85.0 \text{ cm}}} = 0.85818$$

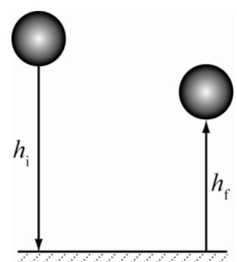
ROUND: All of the coefficients of restitution will be given to 3 significant figures because all the heights are given to 3 significant figures.

Object	H [cm]	h_1 [cm]	ε
Range golf ball	85.0	62.6	0.858
Tennis ball	85.0	43.1	0.712
Billiard ball	85.0	54.9	0.804
Hand ball	85.0	48.1	0.752
Wooden ball	85.0	30.9	0.603
Steel ball bearing	85.0	30.3	0.597
Glass marble	85.0	36.8	0.658
Ball of rubber bands	85.0	58.3	0.828
Hollow, hard plastic balls	85.0	40.2	0.688

DOUBLE-CHECK: All these values are less than one, which is reasonable.

7.70. **THINK:** I want to find the maximum height a ball reaches if it is started at 0.811 m and has a coefficient of restitution of 0.601.

SKETCH:



RESEARCH: Using the equation $\varepsilon = \sqrt{h_f / h_i}$.

SIMPLIFY: $\varepsilon = \sqrt{h_f / h_i} \Rightarrow \varepsilon^2 = h_f / h_i \Rightarrow h_f = h_i \varepsilon^2$

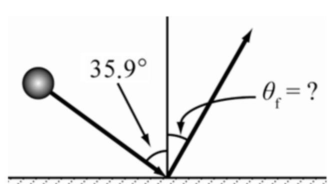
CALCULATE: $h_f = (0.811 \text{ m})(0.601)^2 = 0.292934011 \text{ m}$

ROUND: The values are given to 3 significant figures so the final height is $h_f = 0.293$ m.

DOUBLE-CHECK: This is a reasonable answer since $h_f < h_i$.

7.71. **THINK:** I want to know the angle relative to the wall after the ball hits the wall. The ball has mass m = 0.162 kg, a speed of v = 1.91 m/s and collides at an angle $\theta_i = 35.9^\circ$ with a coefficient of restitution $\varepsilon = 0.841$.

SKETCH:



RESEARCH: We will use $\theta_{\text{f}} = \cot^{-1}\left(\dfrac{\varepsilon p_{\text{i}\perp}}{p_{\text{i}\parallel}}\right)$.

SIMPLIFY: $\theta_{\text{f}} = \cot^{-1}\left(\dfrac{\varepsilon mv\cos\theta_{\text{i}}}{mv\sin\theta_{\text{i}}}\right) = \cot^{-1}\left(\dfrac{\varepsilon\cos\theta_{\text{i}}}{\sin\theta_{\text{i}}}\right)$

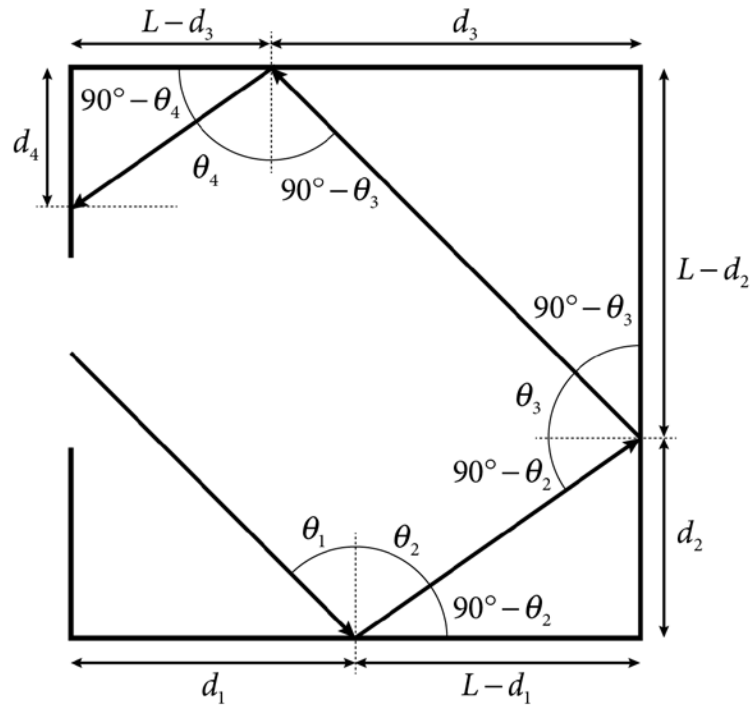
CALCULATE: $\theta_{\text{f}} = \cot^{-1}\left(\dfrac{(0.841)\cos 35.9^\circ}{\sin 35.9^\circ}\right) = 40.719775^\circ$

ROUND: All values are given to 3 significant figures. The final answer is 40.7°.

DOUBLE-CHECK: This is reasonable since $\theta_{\text{f}} > \theta_{\text{i}}$.

7.72. **THINK:** We want to find out if the ball will escape the room. The room is L = 6.00 m by 6.00 m with a 2.00 m wide doorway located in the center of the wall. The coefficient of restitution for the ball is 0.850.

SKETCH:



RESEARCH: The angle will be given by $\theta_{\text{f}} = \cot^{-1}\left(\varepsilon p_{\text{i}\perp}/p_{\text{i}\parallel}\right) = \cot^{-1}\left(\varepsilon\cot\theta_{\text{i}}\right)$ and this will be used through trigonometry to find the distances.

SIMPLIFY: The angles are:

$\theta_2 = \cot^{-1}\left(\varepsilon\cot\theta_1\right)$,

$\theta_3 = \cot^{-1}\left(\varepsilon\cot\left(90^\circ - \theta_2\right)\right)$, and

$\theta_4 = \cot^{-1}\left(\varepsilon\cot\left(90.0^\circ - \theta_3\right)\right)$.

The distances are:

$d_1 = L/2$,

$d_2 = \left(L - d_1\right)\tan\left(90^\circ - \theta_2\right)$,

$d_3 = \left(L - d_2\right)\tan\left(90^\circ - \theta_3\right)$, and

$d_4 = \left(L - d_3\right)\tan\left(90^\circ - \theta_4\right)$.

CALCULATE: First calculate the angles

$\theta_2 = \cot^{-1}\left(0.850\cot 45.0°\right) = 49.64°,$

$\theta_3 = \cot^{-1}\left(0.850\cot\left(90.00° - 49.64°\right)\right) = 45.00°,$ and

$\theta_4 = \cot^{-1}\left(0.850\cot^{-1}\left(90.00° - 45.00°\right)\right) = 49.64°.$

Now calculate the distances

$d_1 = 6.00 \text{ m}/2 = 3.00 \text{ m},$

$d_2 = \left(6.00 \text{ m} - 3.00 \text{ m}\right)\tan\left(90.00° - 49.64°\right) = 2.550 \text{ m},$

$d_3 = \left(6.00 \text{ m} - 2.550 \text{ m}\right)\tan\left(90.00° - 45.00°\right) = 3.450 \text{ m},$ and

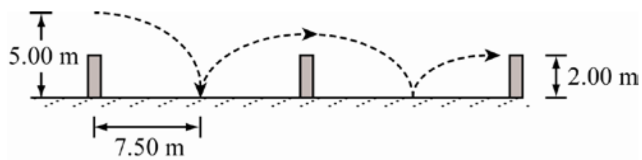
$d_4 = \left(6.00 \text{ m} - 3.450 \text{ m}\right)\tan\left(90.00° - 49.64°\right) = 2.168 \text{ m}.$

ROUND: The last distance d_4 is 2.17 m, which is more than 2.00 m from the wall where the door begins. Thus, the soccer ball does bounce back out of the room on the first trip around the room.

DOUBLE-CHECK: What would we expect if the coefficient of restitution were 1? We would have $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 45°$ and $d_1 = d_2 = d_3 = d_4 = L/2.$ The soccer ball would return to the same place it entered the room and would exit the room. By calculating d_4 for a given $\varepsilon,$ you can show that for $\varepsilon > 0.817,$ the soccer ball will exit the room on its first trip around the room.

7.73. **THINK:** I want to know if Jerry will make it over the second fence. Each yard begins and ends with a 2.00 m fence. The range and maximum height of Jerry's initial trajectory are 15.0 m and 5.00 m respectively. Jerry is 7.50 m away from the next fence and he has a coefficient of restitution of 0.80.

SKETCH:



RESEARCH: From the range and maximum height the initial velocity can be found, along with the angle of Jerry's trajectory. $R = \left(v_0^2 / g\right)\sin\left(2\theta\right)$ and $H = \left(v_0^2 / \left(2g\right)\right)\sin^2\theta$. With this I can find the x- and y-components of the velocity. Since the coefficient of restitution only acts on the momentum perpendicular to the ground, $v_{yf} = v_{yi}$ and v_x remains constant. With this information the height Jerry attains after travelling another 7.50 m can be found by using $x = v_x t$ and $y = v_y t - \frac{1}{2}gt^2$.

SIMPLIFY: $v_0^2 = \dfrac{Rg}{\sin\left(2\theta\right)} = \dfrac{2Hg}{\sin^2\theta} \Rightarrow \dfrac{R}{2\cos\theta} = \dfrac{2H}{\sin\theta} \Rightarrow \tan\theta = \dfrac{4H}{R},\qquad v_0 = \sqrt{\dfrac{Rg}{\sin 2\theta}} = \sqrt{\dfrac{2Hg}{\sin^2\theta}}$

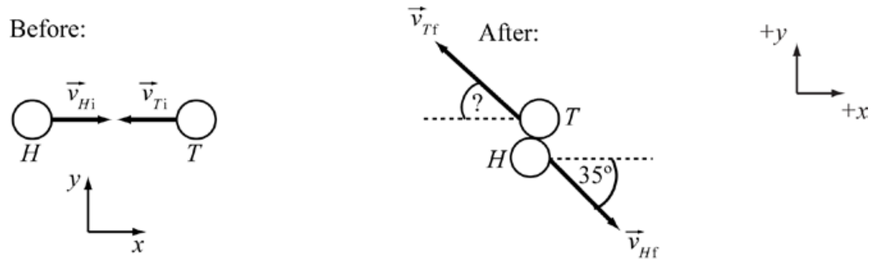
$v_{xi} = v_{xf} = v_0\cos\theta = \sqrt{\dfrac{2Hg}{\sin^2\theta}}\cos\theta = \dfrac{\sqrt{2Hg}}{\tan\theta} = \dfrac{\sqrt{2Hg}}{4H/R} = \dfrac{R\sqrt{2Hg}}{4H},$ and $v_{yf} = \varepsilon v_{yi} = \varepsilon v_0\sin\theta = \varepsilon\sqrt{2Hg}$. The time it takes to reach the fence is given by $x = v_x t$, or $t = x/v_x = \left(4Hx\right)/\left(R\sqrt{2Hg}\right)$, where x = 7.5 m. The height it attains in this time is:

$$y = v_y t - \frac{1}{2}gt^2 = \varepsilon\sqrt{2Hg}\frac{4Hx}{R\sqrt{2Hg}} - \frac{1}{2}g\left(\frac{4Hx}{R\sqrt{2Hg}}\right)^2 = \frac{4Hx\varepsilon}{R} - \frac{4Hx^2}{R^2}$$

CALCULATE: $y = \dfrac{4\left(5 \text{ m}\right)\left(7.50 \text{ m}\right)\left(0.800\right)}{15.0 \text{ m}} - \dfrac{4\left(5 \text{ m}\right)\left(7.50 \text{ m}\right)^2}{\left(15.0 \text{ m}\right)^2} = 3 \text{ m}$

ROUND: Jerry is at a height of 3 m when he reaches the fence, which means that he does make it over the next fence, with exactly 1 meter to spare.
DOUBLE-CHECK: This is a reasonable answer for the world of cartoon characters.

7.74. **THINK:** I want to find the angle θ_{T} at which Toyohibiki moves after collision. Hakurazan and Toyohibiki have masses and speeds of $m_{\text{H}} = 135$ kg , $v_{\text{Hi}} = 3.5$ m/s , $m_{\text{T}} = 173$ kg , and $v_{\text{Ti}} = 3.0$ m/s . After collision, there is a loss of 10% of the kinetic energy, and $\theta_{\text{H}} = 35°$.
SKETCH:



RESEARCH: I can use the conservation of momentum along the *x* and *y* axes. $\vec{p}_{\text{Hi}} + \vec{p}_{\text{Ti}} = \vec{p}_{\text{Hf}} + \vec{p}_{\text{Tf}}$. Since the relation between the initial and final kinetic energy is known, I can also use the equation $K_{\text{f}} = 0.90K_{\text{i}}$.
SIMPLIFY: First set up the three equations starting with momentum along the *x* axis:

$$m_{\text{H}}v_{\text{Hi}} - m_{\text{T}}v_{\text{Ti}} = m_{\text{H}}v_{\text{Hf}}\cos\theta_{\text{H}} - m_{\text{T}}v_{\text{Tf}}\cos\theta_{\text{T}} \quad (1)$$

Along the *y*-axis:

$$0 = m_{\text{T}}v_{\text{Tf}}\sin\theta_{\text{T}} - m_{\text{H}}v_{\text{Hf}}\sin\theta_{\text{H}} \quad (2)$$

The energy gives:

$$0.90\left(m_{\text{H}}v_{\text{Hi}}^2 + m_{\text{T}}v_{\text{Ti}}^2\right) = m_{\text{H}}v_{\text{Hf}}^2 + m_{\text{T}}v_{\text{Tf}}^2 \quad (3)$$

Use the first two equations to find $\left(m_{\text{T}}v_{\text{Tf}}\sin\theta_{\text{T}}\right)^2$ and $\left(m_{\text{T}}v_{\text{Tf}}\cos\theta_{\text{T}}\right)^2$.

$$\left(m_{\text{T}}v_{\text{Tf}}\cos\theta_{\text{T}}\right)^2 = \left(m_{\text{H}}v_{\text{Hf}}\cos\theta_{\text{H}} + m_{\text{T}}v_{\text{Ti}} - m_{\text{H}}v_{\text{Hi}}\right)^2 \text{ and } \left(m_{\text{T}}v_{\text{Tf}}\sin\theta_{\text{T}}\right)^2 = m_{\text{H}}^2v_{\text{Hf}}^2\sin^2\theta_{\text{H}}$$

$$\left(m_{\text{T}}v_{\text{Tf}}\cos\theta_{\text{T}}\right)^2 + \left(m_{\text{T}}v_{\text{Tf}}\sin\theta_{\text{T}}\right)^2 = \left(m_{\text{H}}v_{\text{Hf}}\cos\theta_{\text{H}} + m_{\text{T}}v_{\text{Ti}} - m_{\text{H}}v_{\text{Hi}}\right)^2 + m_{\text{H}}^2v_{\text{Hf}}^2\sin^2\theta_{\text{H}}$$

$$\left(m_{\text{T}}v_{\text{Tf}}\right)^2\left(\sin^2\theta_{\text{T}} + \cos^2\theta_{\text{T}}\right) = \left(m_{\text{H}}v_{\text{Hf}}\cos\theta_{\text{H}} + m_{\text{T}}v_{\text{Ti}} - m_{\text{H}}v_{\text{Hi}}\right)^2 + m_{\text{H}}^2v_{\text{Hf}}^2\sin^2\theta_{\text{H}}$$

$$m_{\text{T}}^2v_{\text{Tf}}^2 = \left(m_{\text{H}}v_{\text{Hf}}\cos\theta_{\text{H}} + m_{\text{T}}v_{\text{Ti}} - m_{\text{H}}v_{\text{Hi}}\right)^2 + m_{\text{H}}^2v_{\text{Hf}}^2\sin^2\theta_{\text{H}} \quad (4)$$

Substituting this into the third equation gives:

$$0.90\left(m_{\text{H}}v_{\text{Hi}}^2 + m_{\text{T}}v_{\text{Ti}}^2\right) = m_{\text{H}}v_{\text{Hf}}^2 + \frac{\left(m_{\text{H}}v_{\text{Hf}}\cos\theta_{\text{H}} + m_{\text{T}}v_{\text{Ti}} - m_{\text{H}}v_{\text{Hi}}\right)^2}{m_{\text{T}}} + \frac{m_{\text{H}}^2v_{\text{Hf}}^2}{m_{\text{T}}}\sin^2\theta_{\text{H}}.$$

This quadratic equation in v_{Hf} which simplifies to: $0.90\left((135\text{ kg})(3.5\text{ m/s})^2 + (173\text{ kg})(3\text{ m/s})^2\right)$

$$= (135\text{ kg})v_{\text{Hf}}^2 + \frac{\left((135\text{ kg})v_{\text{Hf}}\cos 35 + (173\text{ kg})(3\text{ m/s}) - (135\text{ kg})(3.5\text{ m/s})\right)^2}{(173\text{ kg})} + \frac{(135\text{ kg})^2v_{\text{Hf}}^2}{(173\text{ kg})}\sin^2 35°$$

$$\Rightarrow 240.3468v_{\text{Hf}}^2 + 59.4477v_{\text{Hf}} - 2877.176 = 0 \quad (5)$$

Solving equation (4) for v_{Tf} gives the equation: $v_{Tf}=\sqrt{\dfrac{\left(m_H v_{Hf}\cos\theta_H+m_T v_{Ti}-m_H v_{Hi}\right)^2+m_H^2 v_{Hf}^2\sin^2\theta_H}{m_T^2}}$.

CALCULATE: Solving equation (5), gives $v_{Hf}=3.3384$ m/s. Using this in equation (4), gives:

$$v_{Tf}=\sqrt{\frac{\left[(135)(3.3384)\cos 35^\circ+(173)(3.0)-(135)(3.5)\right]^2+(135)^2(3.3384)^2\sin^2 35^\circ}{(173)^2}}$$

$=2.829$, with units of:

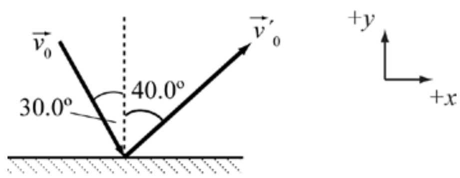
$\sqrt{\dfrac{\left[(\text{kg})(\text{m/s})+(\text{kg})(\text{m/s})-(\text{kg})(\text{m/s})\right]^2+(\text{kg})^2(\text{m/s})^2}{(\text{kg})^2}}=$ m/s. Therefore, $v_{Tf}=2.829$ m/s. Use equation (2) to find θ_T: $\theta_T=\sin^{-1}\left(\dfrac{(135\text{ kg})(3.3384\text{ m/s})}{(173\text{ kg})(2.829\text{ m/s})}\sin 35^\circ\right)=31.88^\circ$.

ROUND: The angle θ_H is given to two significant figures and limits our answer to two significant figures. $\theta_T=32^\circ$.

DOUBLE-CHECK: The sumo wrestlers' masses and initial speeds and directions are similar, so in is reasonable that their final speeds and directions would be similar as well.

7.75. **THINK:** I want to find the coefficient of restitution and the ratio of the final and initial kinetic energies. The puck initially has a mass, velocity and angle of m = 170. g, $v_0=2.00$ m/s, and $\theta_i=30.0^\circ$ respectively. The puck bounces off the board with an angle of $\theta_f=40.0^\circ$.

SKETCH:



RESEARCH: To find the coefficient of restitution we will use $\theta_f=\cot^{-1}\left(\varepsilon p_{i\wedge}/p_{iP}\right)$. To find the ratio for the initial kinetic energy we will use $p_{f\wedge}=p_{i\wedge}$ and $p_{fP}=p_{iP}$.

SIMPLIFY: The coefficient of restitution is given by:

$$\cot\theta_f=\frac{\varepsilon p_{i\perp}}{p_{i\parallel}}=\frac{\varepsilon v_0\cos\theta_i}{v_0\sin\theta_i}=\varepsilon\cot\theta_i\ \Rightarrow\ \varepsilon=\frac{\cot\theta_f}{\cot\theta_i}$$

Now we use $p_{f\parallel}=p_{i\parallel}$ to find v_0'.

$$p_{i\parallel}=p_{f\parallel}\ \Rightarrow\ mv_{xi}=mv_{xf}\ \Rightarrow\ v_0\sin\theta_i=v_0'\sin\theta_f\ \Rightarrow\ v_0'=v_0\frac{\sin\theta_i}{\sin\theta_f}$$

The ratio $K_f/K_i=\dfrac{(1/2)mv_0'^2}{(1/2)mv_0^2}=\dfrac{\sin^2\theta_i}{\sin^2\theta_f}$.

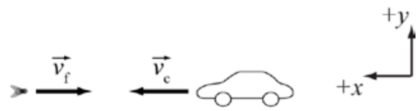
CALCULATE: $\varepsilon=\dfrac{\cot 40.0^\circ}{\cot 30.0^\circ}=0.688059$, and $\dfrac{K_f}{K_i}=\dfrac{\sin^2 30.0^\circ}{\sin^2 40.0^\circ}=0.6051$.

ROUND: The coefficient of restitution is $\varepsilon=0.688$ and the kinetic energy ratio $K_f/K_i=0.605$.

DOUBLE-CHECK: These numbers seem reasonable.

7.76. **THINK:** I want to know the speed a 5.00 g fly must have to slow a 1900. kg car by 5.00 mph. The car is travelling at an initial speed of 55.0 mph.

SKETCH:



RESEARCH: Using the conservation of momentum: $m_F\vec{v}_F + m_C\vec{v}_C = (m_F + m_C)\vec{v}'$.

SIMPLIFY: $-m_F v_F + m_C v_C = (m_F + m_C)(v_C - \Delta v)$

$$m_F v_F = m_C v_C - (m_F + m_C)(v_C - \Delta v)$$

$$v_F = \frac{m_C v_C - (m_F + m_C)(v_C - \Delta v)}{m_F}$$

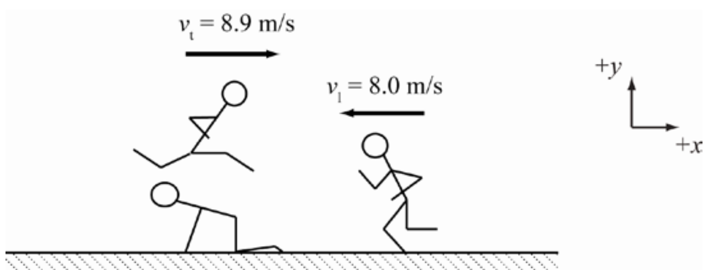
CALCULATE: $v_F = \dfrac{(1900.\text{ kg})(55.0\text{ mph}) - (1900.\text{ kg} + 0.00500\text{ kg})(50.0\text{ mph})}{0.00500\text{ kg}} = 1899950\text{ mph}$

ROUND: The fly must travel $1.90 \cdot 10^6$ mph to change the speed of the car by 5.00 mph.

DOUBLE-CHECK: This is a crazy speed for a fly to attain. It is about 800000 m/s. This value is extreme, but the notion of a fly being able to slow a car from 55 mph to 50 mph is absurd, and this is verified by the very high speed required of the fly.

7.77. **THINK:** I want to find the speed of the tailback and linebacker and if the tailback will score a touchdown. The tailback has mass and velocity $m_t = 85.0$ kg and $v_t = 8.90$ m/s, and the linebacker has mass and velocity $m_l = 110.$ kg and $v_l = -8.00$ m/s.

SKETCH:



RESEARCH: The conservation of momentum for this perfectly inelastic collision is

$$m_t v_t + m_l v_l = (m_t + m_l)v.$$

SIMPLIFY: Rearranging the equation to solve for the final velocity, $v = \dfrac{m_t v_t + m_l v_l}{m_t + m_l}$.

CALCULATE: $v = \dfrac{(85.0\text{ kg})(8.90\text{ m/s}) - (110.\text{ kg})(8.00\text{ m/s})}{85.0\text{ kg} + 110.\text{ kg}} = -0.633\text{ m/s}$

ROUND:

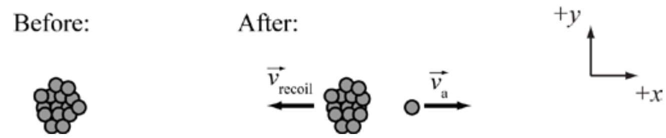
(a) The values are given to 3 significant figures so the final speed is 0.633 m/s.

(b) Since the velocity is negative, the two go in the direction of the linebacker and the tailback does not score a touchdown.

DOUBLE-CHECK: The speed is quite small, as would be expected of two people opposing each other's motion. Since the initial momentum of the linebacker is greater than the initial momentum of the tailback, the tailback should not be able to score. This is consistent with the calculated result.

7.78. **THINK:** I want to know the recoil speed of the remaining nucleus. The thorium-228 nucleus starts at rest with a mass of $m_t = 3.8 \cdot 10^{-25}$ kg and the emitted alpha particle has mass $m_a = 6.64 \cdot 10^{-27}$ kg and velocity $v_a = 1.8 \cdot 10^7$ m/s.

SKETCH:



RESEARCH: Using the conservation of momentum: $0 = (m_t - m_a)v_{recoil} - m_a v_a$.

SIMPLIFY: $(m_t - m_a)v_{recoil} = m_a v_a \Rightarrow v_{recoil} = \dfrac{m_a v_a}{m_t - m_a}$

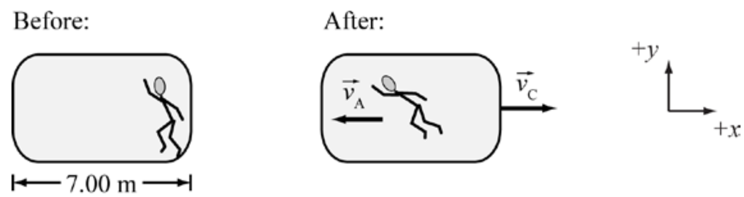
CALCULATE: $v_{recoil} = \dfrac{(6.64 \cdot 10^{-27} \text{ kg})(1.8 \cdot 10^7 \text{ m/s})}{(3.8 \cdot 10^{-25} \text{ kg}) - (6.64 \cdot 10^{-27} \text{ kg})} = 320{,}120 \text{ m/s}$

ROUND: The given values have two significant figures. Therefore, the recoil velocity is $3.2 \cdot 10^5$ m/s.

DOUBLE-CHECK: This value seems like a reasonable speed because it is less than the speed of the alpha particle.

7.79. **THINK:** I want to know the time it takes the astronaut to reach the other side of the 7.00 m long capsule. The astronaut and the capsule both start at rest. The astronaut and capsule have masses of 60.0 kg and 500. kg respectively. After the astronaut's kick, he reaches a velocity is 3.50 m/s.

SKETCH:



RESEARCH: I can find the speed of the capsule by using the conservation of momentum. The time it takes is the distance divided by the sum of the velocities.

SIMPLIFY: $m_A v_A = m_C v_C \Rightarrow v_C = \dfrac{m_A}{m_C} v_A$. Using this in our distance equation:

$$d = (v_A + v_C)t = \left(v_A + \frac{m_A}{m_C} v_A\right)t = v_A\left(1 + \frac{m_A}{m_C}\right)t \Rightarrow t = \frac{d}{v_A\left(1 + \frac{m_A}{m_C}\right)}$$

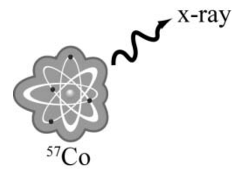
CALCULATE: $t = \dfrac{7.00 \text{ m}}{(3.50 \text{ m/s})(1 + (60.0 \text{ kg})/(500. \text{ kg}))} = 1.7857 \text{ s}$

ROUND: The time to cross the capsule is reported as 1.79 s.

DOUBLE-CHECK: This is a reasonable time.

7.80. **THINK:** The conservation of momentum and the definition of kinetic energy and momentum can be used to find the momentum and kinetic energy of a ^{57}Co nucleus that emits an x-ray. The nucleus has a mass of $m_{Co} = 9.52 \cdot 10^{-26}$ kg and the x-ray has a momentum and kinetic energy of 14 keV/c and 14 keV, respectively.

SKETCH:



RESEARCH: $p_{\text{Co}} = -p_{\text{x-ray}}$, $K = \frac{1}{2}mv^2$ and $p = mv$.

SIMPLIFY: $K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{(mv)^2}{m} = \frac{p^2}{2m}$

CALCULATE:

$$p_{\text{Co}} = -p_{\text{x-ray}} = -14 \text{ keV/c} = \frac{\left(-14 \cdot 10^3 \text{ eV/c}\right)\left(1.602 \cdot 10^{-19} \text{ J/eV}\right)}{\left(2.998 \cdot 10^8 \text{ (m/s)/c}\right)} = -7.4810 \cdot 10^{-24} \text{ kg m/s}$$

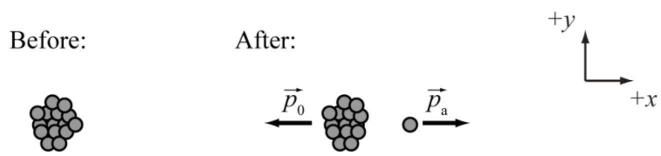
$$K = \frac{\left(-7.4810 \cdot 10^{-24} \text{ kg m/s}\right)^2}{2\left(9.52 \cdot 10^{-26} \text{ kg}\right)} = 2.939 \cdot 10^{-22} \text{ J} = 1.83 \cdot 10^{-3} \text{ eV}$$

ROUND: The momentum is given to two significant figures so the answer can be reported to two significant figures: $p_{\text{Co}} = -14$ keV/c and $K = 1.8 \cdot 10^{-3}$ eV. The negative sign means that the ^{57}Co nucleus is in the opposite direction of the x-ray.

DOUBLE-CHECK: These are reasonable values.

7.81. **THINK:** I am looking for the velocity of the nucleus after the decay. The atom starts at rest, i.e. $v_0 = 0$ m/s, and its nucleus has mass $m_0 = 3.68 \cdot 10^{-25}$ kg. The alpha particle has mass $m_{\text{a}} = 6.64 \cdot 10^{-27}$ kg and energy $8.79 \cdot 10^{-13}$ J.

SKETCH:



RESEARCH: I can find the velocity of the alpha particle with the equation $K = \frac{1}{2}m_{\text{a}}v_{\text{a}}^2$. The conservation of momentum gives $p_{\text{a}} = p_0$.

SIMPLIFY: $v_{\text{a}} = \sqrt{\frac{2K}{m_{\text{a}}}}$, $p_{\text{a}} = p_0 \Rightarrow m_{\text{a}}v_{\text{a}} = (m_0 - m_{\text{a}})v_0 \Rightarrow v_0 = \frac{m_{\text{a}}}{(m_0 - m_{\text{a}})}v_{\text{a}} = \frac{m_{\text{a}}}{(m_0 - m_{\text{a}})}\sqrt{\frac{2K}{m_{\text{a}}}}$

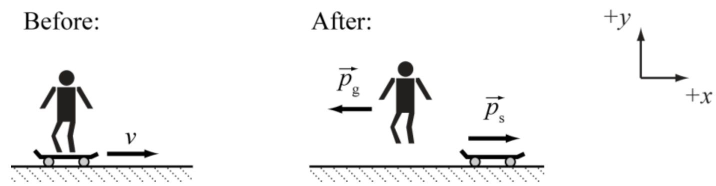
CALCULATE: $v_0 = \frac{6.64 \cdot 10^{-27} \text{ kg}}{\left(3.68 \cdot 10^{-25} \text{ kg} - 6.64 \cdot 10^{-27} \text{ kg}\right)}\sqrt{\frac{2\left(8.79 \cdot 10^{-13} \text{ J}\right)}{6.64 \cdot 10^{-27} \text{ kg}}} = 298988$ m/s

ROUND: The values are given to three significant figures, so $v_0 = 2.99 \cdot 10^5$ m/s.

DOUBLE-CHECK: Such a high speed is reasonable for such small masses.

7.82. **THINK:** I am looking for the speed of the skateboarder after she jumps off her skateboard. She has a mass of $m_{\text{g}} = 35.0$ kg and the skateboard has mass $m_{\text{s}} = 3.50$ kg. They initially travel at $v = 5.00$ m/s in the same direction.

SKETCH:



RESEARCH: To find the speed we can use the conservation of momentum.

SIMPLIFY: $\left(m_g + m_s\right)v = m_s v_s - m_g v_g \;\Rightarrow\; m_g v_g = m_s v_s - \left(m_g + m_s\right)v \;\Rightarrow\; v_g = \left|\dfrac{m_s v_s - \left(m_g + m_s\right)v}{m_g}\right|$

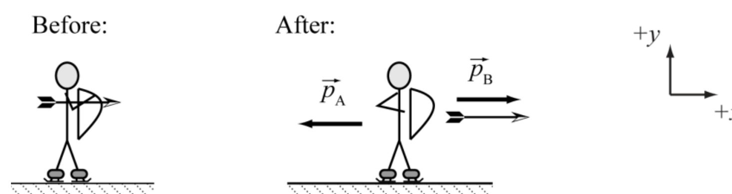
CALCULATE: $v_g = \left|\dfrac{(3.50 \text{ kg})(8.50 \text{ m/s}) - (35.0 \text{ kg} + 3.50 \text{ kg})(5.00 \text{ m/s})}{35.0 \text{ kg}}\right| = 4.65 \text{ m/s}$

ROUND: The speed is accurate to three significant figures since all of our values are given to three significant figures. The speed of the girl is $v_g = 4.65$ m/s.

DOUBLE-CHECK: This is a reasonable speed.

7.83. **THINK:** I am looking for the recoil the archer experiences. The mass of the archer and the arrow are $m_A = 50.0$ kg and $m_B = 0.100$ kg respectively. The initial velocity is 0 and the arrow has a velocity $v_B = 95.0$ m/s.

SKETCH:



RESEARCH: I use the conservation of momentum to find the recoil velocity: $p_A = -p_B$.

SIMPLIFY: $m_A v_A = -m_B v_B \;\Rightarrow\; v_A = -\dfrac{m_B}{m_A} v_B$

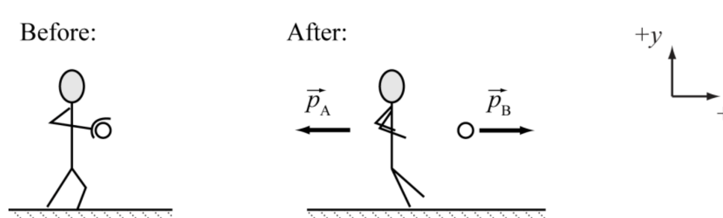
CALCULATE: $v_A = -\dfrac{0.100 \text{ kg}}{50.0 \text{ kg}}(95.0 \text{ m/s}) = -0.190 \text{ m/s}$

ROUND: The three significant figures of the values limit the answers to three significant figures. The recoil speed of the archer is 0.190 m/s.

DOUBLE-CHECK: This is reasonable recoil.

7.84. **THINK:** I want to find the recoil of an astronaut starting at rest after he throws a baseball. The astronaut and baseball have masses $m_A = 55.0$ kg and $m_B = 0.145$ kg respectively. The ball is thrown with a speed of 31.3 m/s.

SKETCH:



RESEARCH: I can find the recoil speed of the astronaut with the conservation of momentum: $p_A = -p_B$.

SIMPLIFY: $m_A v_A = -m_B v_B \Rightarrow v_A = -\frac{m_B}{m_A} v_B$

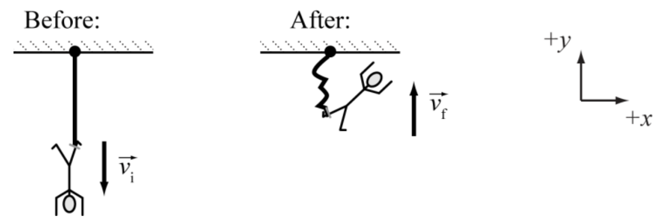
CALCULATE: $v_A = -\frac{0.145 \text{ kg}}{55.0 \text{ kg}}(31.3 \text{ m/s}) = -0.082518 \text{ m/s}$

ROUND: The values are given to three significant figures so the recoil speed will be reported to three significant figures. The recoil speed is $v_A = 0.0825$ m/s.

DOUBLE-CHECK: This is a reasonable speed.

7.85. **THINK:** I want to find the average force exerted on the jumper and the number of *g*'s she experiences. She has a mass of $m_j = 55.0$ kg and reaches a speed of $v_i = 13.3$ m/s downwards then goes $v_f = 10.5$ m/s upwards after the cord pulls her back up in $\Delta t = 1.25$ s.

SKETCH:



RESEARCH: I use the impulse equation, $F\Delta t = \Delta p$, to find the net force acting on the jumper. I can then use $F = ma$ to find the net force (cord pulling up plus gravity pulling down) and then the number of *g*'s experienced. Number of *g*'s is determined by the action of forces *other* than gravity, so in this case the cord tension. (A person standing motionless on the ground experiences 1 *g* from the upward normal force.)

SIMPLIFY: $F = \frac{\Delta p}{\Delta t} = \frac{m_j(v_f - v_i)}{\Delta t}$ and $a = \frac{F}{m_j}$.

CALCULATE: $F_{net} = \frac{(55.0 \text{ kg})(10.5 \text{ m/s} - (-13.3 \text{ m/s}))}{1.25 \text{ s}} = 1047.2 \text{ N}$

$F_{net} = F_{cord} - mg \Rightarrow F_{cord} = F_{net} + mg = 1047.2 \text{ N} + (55.0 \text{ kg})(9.81 \text{ m/s}^2) = 1586.75 \text{ N}$

Acceleration due to cord: $a = \frac{F_{cord}}{m} = \frac{1586.75 \text{ N}}{55.0 \text{ kg}} = 28.85 \text{ m/s}^2$.

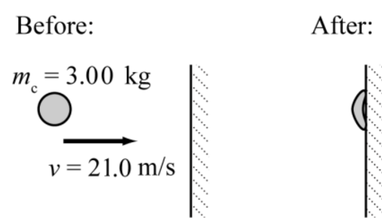
Dividing 28.85 by 9.81, the cord subjects the number to 2.9408 *g*'s.

ROUND: The values are given to three significant figures, so the average force is 1590 N and the jumper experiences 2.94 *g*'s.

DOUBLE-CHECK: These numbers are within reasonable levels. A person can experience a few *g*'s without harm and without losing consciousness.

7.86. **THINK:** I want to find the impulse exerted on the ball of clay when it sticks to a wall. The ball has a mass of $m_c = 3.00$ kg and speed $v = 21.0$ m/s.

SKETCH:



RESEARCH: I use the impulse equation. $J = \Delta p = F\Delta t$.

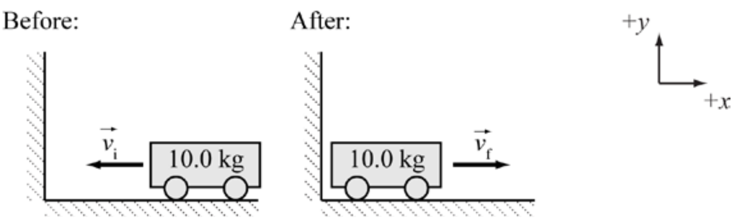
SIMPLIFY: $J = m_c(v_f - v_i)$

CALCULATE: $|J| = |(3.00\text{ kg})(0\text{ m/s} - 21.0\text{ m/s})| = 63.0\text{ kg m/s}$

ROUND: Our result will have threesignificant figures since our values are accurate to three significant figures. The impulse exerted on the ball is 63.0 kg m/s.

DOUBLE-CHECK: This is a reasonable value.

7.87. **THINK:** I want to find the change in the momentum of the cart. (This is the same as the impulse.) The cart has a mass of 10.0 kg and initially travels at $v_{\text{i}} = 2.00\text{ m/s}$ to the left then travels at $v_{\text{f}} = 1.00\text{ m/s}$ after it hits the wall.
SKETCH:



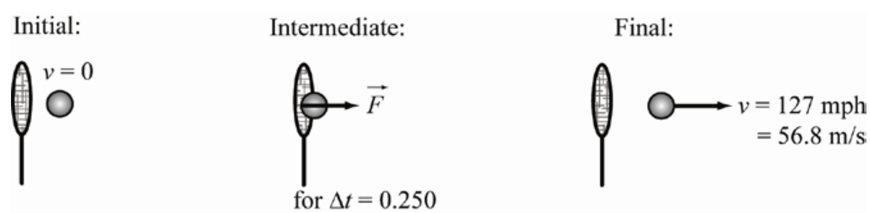
RESEARCH: All I need to do is find the momentum in each case then subtract them to find the change in momentum. $\Delta p = p_{\text{f}} - p_{\text{i}}$.

SIMPLIFY: $\Delta p = p_{\text{f}} - p_{\text{i}} = mv_{\text{f}} - mv_{\text{i}} = m(v_{\text{f}} - v_{\text{i}})$

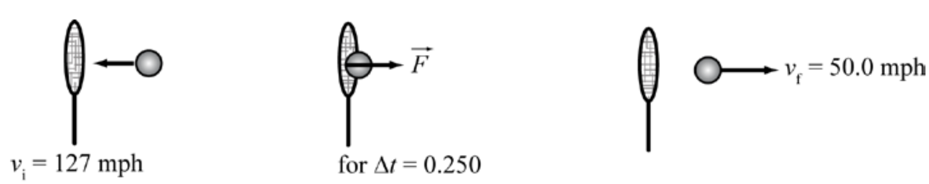
CALCULATE: $\Delta p = (10.0\text{ kg})(1.00\text{ m/s} - (-2.00\text{ m/s})) = 30.0\text{ kg m/s}$

ROUND: The change in momentum is 30.0 kg m/s.
DOUBLE-CHECK: This is a reasonable value.

7.88. **THINK:** I have a tennis ball with mass $57.0\text{ g} = 5.70 \cdot 10^{-2}\text{ kg}$ and speed 127 mph $= (127\text{ mph})(0.447\text{ (m/s)/mph}) = 56.8\text{ m/s}.$ I want to calculate impulse. I am given that $\Delta t = 0.250\text{ s}.$
SKETCH:
(a)



(b)



RESEARCH: I use the definition of impulse. $J = F_{ave}\Delta t = \Delta p.$
SIMPLIFY:
(a) The tennis ball is initially at rest before the serve. $v_{\text{i}} = 0\text{ m/s}.$

$$J = F\Delta t = \Delta p = m(v_{\text{f}} - v_{\text{i}}) = mv_{\text{f}} \quad \Rightarrow \quad F = \frac{mv_{\text{f}}}{\Delta t}$$

(b) The tennis ball has an initial speed $v_{\text{i}} = -127 \text{ mph} = -56.8 \text{ m/s}$ and a final speed $v_{\text{f}} = 50.0 \text{ mph} = 22.4 \text{ m/s}$.

$$J = F\Delta t = \Delta p = m(v_{\text{f}} - v_{\text{i}}) \;\Rightarrow\; F = \frac{m(v_{\text{f}} - v_{\text{i}})}{\Delta t}$$

CALCULATE:

(a) $F = \dfrac{(5.70 \cdot 10^{-2} \text{ kg})(56.8 \text{ m/s})}{0.250 \text{ s}} = 12.95 \text{ N}$

(b) $F = \dfrac{(5.70 \cdot 10^{-2} \text{ kg})(22.4 \text{ m/s} - (-56.8 \text{ m/s}))}{0.250 \text{ s}} = 18.06 \text{ N}$

ROUND:

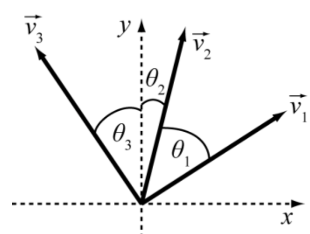
(a) $F = 13.0$ N

(b) $F = 18.1$ N

DOUBLE-CHECK: I expect the answer for (a) to be less than that of (b) because $v_{\text{f}} - v_{\text{i}}$ in (a) is less than in (b).

7.89. **THINK:** I have three birds with masses $m_1 = 0.100 \text{ kg}$, $m_2 = 0.123 \text{ kg}$, and $m_3 = 0.112 \text{ kg}$ and speeds $v_1 = 8.00 \text{ m/s}$, $v_2 = 11.0 \text{ m/s}$, and $v_3 = 10.0 \text{ m/s}$. They are flying in directions $\theta_1 = 35.0°$ east of north, $\theta_2 = 2.00°$ east of north, and $\theta_3 = 22.0°$ west of north, respectively. I want to calculate the net momentum.

SKETCH:



RESEARCH: $\vec{p} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3$, or in component form, $p_x = p_{1x} + p_{2x} + p_{3x}$ and $p_y = p_{1y} + p_{2y} + p_{3y}$.

SIMPLIFY: $p_x = m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2 - m_3 v_3 \sin\theta_3$, $p_y = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 + m_3 v_3 \cos\theta_3$,

$\vec{p} = p_x \hat{x} + p_y \hat{y}$, and $\vec{v} = \dfrac{\vec{p}}{m}$.

CALCULATE:

$$p_x = (0.100 \text{ kg})(8.00 \text{ m/s})\sin 35.0° + (0.123 \text{ kg})(11.0 \text{ m/s})\sin 2.00° - (0.112 \text{ kg})(10.0 \text{ m/s})\sin 22.0°$$
$$= 0.0865 \text{ kg m/s}$$

$$p_y = (0.100 \text{ kg})(8.00 \text{ m/s})\cos 35.0° + (0.123 \text{ kg})(11.0 \text{ m/s})\cos 2.00° + (0.112 \text{ kg})(10.0 \text{ m/s})\cos 22.0°$$
$$= 3.0459 \text{ kg m/s}$$

The speed of a 0.115 kg bird is: $\vec{v} = \dfrac{0.0865 \text{ kg m/s } \hat{x} + 3.0459 \text{ kg m/s } \hat{y}}{0.115 \text{ kg}} = 0.752 \text{ m/s } \hat{x} + 26.486 \text{ m/s } \hat{y}.$

$\left|\vec{v}\right| = \sqrt{(0.752 \text{ m/s})^2 + (26.486 \text{ m/s})^2} = 26.497 \text{ m/s},$

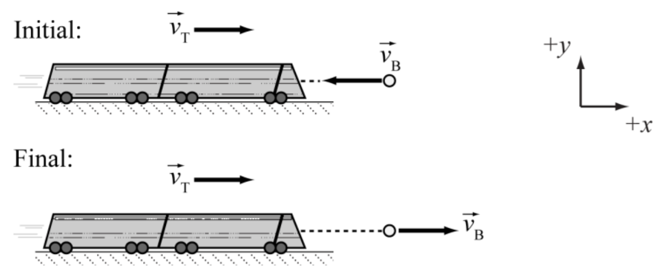
$\tan\theta = \dfrac{0.752 \text{ m/s}}{26.486 \text{ m/s}} = 0.02839 \;\Rightarrow\; \theta = \tan^{-1}(0.02839) = 1.626°$ east of north

ROUND: $p_x = 0.0865 \text{ kg m/s}$, $p_y = 3.05 \text{ kg m/s}$, $\vec{p} = 0.0865 \text{ kg m/s } \hat{x} + 3.05 \text{ kg m/s } \hat{y}$, $\left|\vec{v}\right| = 26.5 \text{ m/s}$, $\theta = 1.63°$ east of north

DOUBLE-CHECK: The speed of the fourth bird must be less than the sum of the speeds of the three birds. $v = v_1 + v_2 + v_3 = 8.00 \text{ m/s} + 11.0 \text{ m/s} + 10.0 \text{ m/s} = 29.0 \text{ m/s}$.

7.90. **THINK:** I have a golf ball with mass $m_{\text{B}} = 45.0\text{ g} = 0.0450\text{ kg}$ and speed $v_{\text{B}} = 120.\text{ km/h} = 33.3\text{ m/s}$. A train has mass $m_{\text{T}} = 3.80 \cdot 10^5\text{ kg}$ with speed $v_T = 300.\text{ km/h} = 83.3\text{ m/s}$. I want to calculate the speed of the golf ball after collision.

SKETCH:



RESEARCH: I use the conservation of momentum and energy. $p_{\text{i}} = p_{\text{f}}$ and $K_{\text{i}} = K_{\text{f}}$.

SIMPLIFY: $p_{\text{i}} = p_{\text{f}} \Rightarrow m_{\text{B}}v_{\text{Bi}} + m_{\text{T}}v_{\text{Ti}} = m_{\text{B}}v_{\text{Bf}} + m_{\text{T}}v_{\text{Tf}} \Rightarrow m_{\text{B}}\left(v_{\text{Bi}} - v_{\text{Bf}}\right) = m_{\text{T}}\left(v_{\text{Tf}} - v_{\text{Ti}}\right)$ (1)

$$K_{\text{i}} = K_{\text{f}}$$

$$\frac{1}{2}m_{\text{B}}v_{\text{Bi}}^2 + \frac{1}{2}m_{\text{T}}v_{\text{Ti}}^2 = \frac{1}{2}m_{\text{B}}v_{\text{Bf}}^2 + \frac{1}{2}m_{\text{T}}v_{\text{Tf}}^2$$

$$m_{\text{B}}\left(v_{\text{Bi}}^2 - v_{\text{Bf}}^2\right) = m_{\text{T}}\left(v_{\text{Ti}}^2 - v_{\text{Tf}}^2\right)$$

$$m_{\text{B}}\left(v_{\text{Bi}} - v_{\text{Bf}}\right)\left(v_{\text{Bi}} + v_{\text{Bf}}\right) = m_{\text{T}}\left(v_{\text{Ti}} - v_{\text{Tf}}\right)\left(v_{\text{Ti}} + v_{\text{Tf}}\right)$$

Using equation (1) above, I have $v_{\text{Bi}} + v_{\text{Bf}} = v_{\text{Tf}} + v_{\text{Ti}}$, or $v_{\text{Tf}} = v_{\text{Bi}} + v_{\text{Bf}} - v_{\text{Ti}}$. Therefore,

$$m_{\text{B}}v_{\text{Bi}} + m_{\text{T}}v_{\text{Ti}} = m_{\text{B}}v_{\text{Bf}} + m_{\text{T}}\left(v_{\text{Bi}} + v_{\text{Bf}} - v_{\text{Ti}}\right)$$

$$\left(m_{\text{B}} - m_{\text{T}}\right)v_{\text{Bi}} + 2m_{\text{T}}v_{\text{Ti}} = \left(m_{\text{B}} + m_{\text{T}}\right)v_{\text{Bf}}$$

$$v_{\text{Bf}} = \left(\frac{m_{\text{B}} - m_{\text{T}}}{m_{\text{B}} + m_{\text{T}}}\right)v_{\text{Bi}} + \left(\frac{2m_{\text{T}}}{m_{\text{B}} + m_{\text{T}}}\right)v_{\text{Ti}}$$

Since m_{B} is much smaller than m_{T}, i.e. $m_{\text{B}} \ll m_{\text{T}}$, I can approximate: $\dfrac{m_{\text{B}} - m_{\text{T}}}{m_{\text{B}} + m_{\text{T}}} \approx -1$, , $\dfrac{2m_{\text{T}}}{m_{\text{B}} + m_{\text{T}}} \approx 2$, and

$v_{\text{Bf}} \approx -v_{\text{Bi}} + 2v_{\text{Ti}}$

CALCULATE: $v_{\text{Bi}} = -33.3\text{ m/s}$, $v_{\text{Ti}} = 83.3\text{ m/s}$, and $v_{\text{Bf}} = -\left(-33.3\text{ m/s}\right) + 2\left(83.3\text{ m/s}\right) = 199.9\text{ m/s}$.

ROUND: Rounding to three significant figures: $v_{\text{Bf}} = 200.\text{ m/s}$

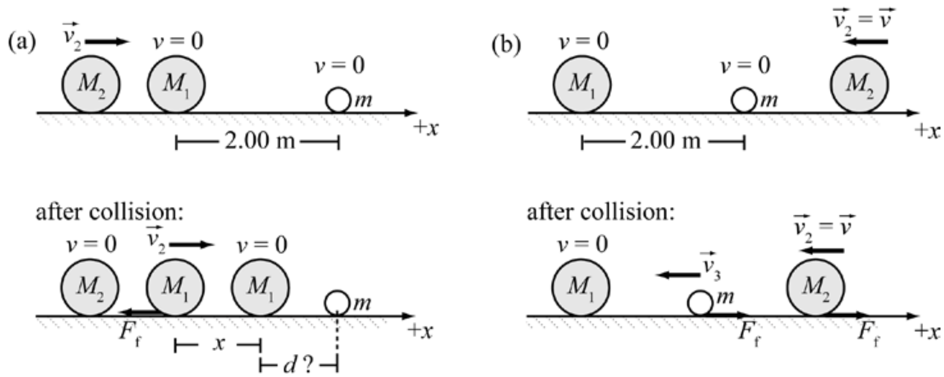
DOUBLE-CHECK: Let us compute $\left(m_{\text{B}} - m_{\text{T}}\right)/\left(m_{\text{B}} + m_{\text{T}}\right)$.

$$\frac{m_{\text{B}} - m_{\text{T}}}{m_{\text{B}} + m_{\text{T}}} = \frac{0.0450\text{ kg} - 3.80 \cdot 10^5\text{ kg}}{0.0450\text{ kg} + 3.80 \cdot 10^5\text{ kg}} = -0.9999997...$$

The approximation is correct.

7.91. **THINK:** I have two balls with masses $M_1 = 1.00\text{ kg}$ and $m = 0.0450\text{ kg}$ which are a distance of $d_0 = 2.00\text{ m}$ apart. I threw a third ball with mass M_2 at a speed $v = 1.00\text{ m/s}$. Calculate the distance between the balls after the collision. $M_1 = M_2 = M$.

SKETCH:



RESEARCH:

(a) Since $M_1 = M_2 = M$ and there is an elastic collision, the final speeds after collision are $v_2 = 0$ m/s and $v_1 = v = 1.00$ m/s.

(b) Since $m \ll M$ and there is an elastic collision, the final speeds after the collision are $v_2 = v$ and $v_3 = 2v_2 = 2v = 2.00$ m/s.

SIMPLIFY: Using Newton's second law, $F_f = \mu_k N = \mu_k m g = ma$. Therefore the acceleration is $a = \mu_k g$. The distance travelled by a ball is $x = v_0^2/(2a) \Rightarrow x = v_0^2/(2\mu_k g)$. Therefore the distance between the two balls is $d = d_0 - x = d_0 - v_0^2/(2\mu_k g)$.

CALCULATE:

(a) $v_0 = 1.00$ m/s. The distance between the first ball and the pallina is:

$$d = 2.00\text{ m} - \frac{(1.00\text{ m/s})^2}{2(0.200)(9.81\text{ m/s}^2)} = 1.745\text{ m}.$$

The distance between the second ball and the pallina is 2.00 m because it stops after the collision.

(b) (i) $2v_0 = v_3 = 2.00$ m/s. The distance between the first ball and the pallina is:

$$d_1 = d_0 - \frac{v_0^2}{2\mu_k g} = 2.00\text{ m} - \frac{(2.00\text{ m/s})^2}{2(0.200)(9.81\text{ m/s}^2)} = 0.9806\text{ m}.$$

(ii) $v_0 = v_2 = 1.00$ m/s. The distance between the first ball and second ball is:

$$d_2 = d_0 - \frac{v_0^2}{2\mu_k g} = 2.00\text{ m} - \frac{(1.00\text{ m/s})^2}{2(0.200)(9.81\text{ m/s}^2)} = 1.745\text{ m}$$

The distance between the second ball and the pallina is $d = d_2 - d_1 = 1.745\text{ m} - 0.9806\text{ m} = 0.764\text{ m}$.

ROUND:

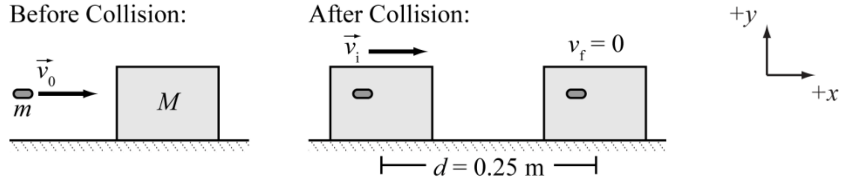
(a) Three significant figures: the distance between the first ball and the pallina is 1.75 m and the distance between the second ball and the pallina is 2.00 m.

(b) Two significant figures, because of subtraction: The distance between the first ball and the pallina is 0.98 m and the distance between the second ball and the pallina is 0.76 m.

DOUBLE-CHECK: Only the first ball is in motion after the collision in part (a) and in part (b) the second ball and the pallina are in motion. It makes sense that the distances in part (b) are shorter than the distances in part (a).

7.92. **THINK:** I have a soft pellet with mass $m = 1.2\text{ g} = 1.2 \cdot 10^{-3}$ kg and an initial speed $v_0 = 65$ m/s. The pellet gets stuck in a piece of cheese with mass $M = 0.25$ kg. The cheese slides 25 cm before coming to a stop. I want to calculate the coefficient of friction between the cheese and the surface of the ice.

SKETCH:



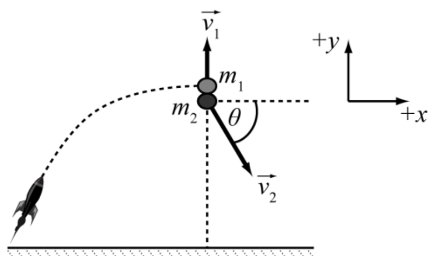
RESEARCH: I apply the conservation of momentum to calculate the speed of the cheese and the pellet after collision and then use $v_f^2 = v_i^2 - 2ad$ and $F_f = \mu_k N$ to obtain the coefficient of friction.

SIMPLIFY: $p_i = p_f \Rightarrow mv_0 + 0 = (m+M)v_i \Rightarrow v_i = \dfrac{mv_0}{m+M}$. Since $v_f = 0$, we have $a = v_i^2/(2d)$. Using Newton's second law, we get: $N = (m+M)g$ and $F_f = (m+M)a = \mu_k N = \mu_k(m+M)g$. The coefficient of friction is $\mu_k g = a$, or $\mu_k = a/g = v_i^2/(2gd)$.

CALCULATE: $v_i = \dfrac{(1.2\cdot 10^{-3}\text{ kg})(65\text{ m/s})}{0.25\text{ kg} + 1.2\cdot 10^{-3}\text{ kg}} = 0.311\text{ m/s}$ and $\mu_k = \dfrac{(0.311\text{ m/s})^2}{2(9.81\text{ m/s}^2)(0.25\text{ m})} = 0.01966.$

ROUND: To three significant figures, $\mu_k = 0.0200$.
DOUBLE-CHECK: This is reasonable since the initial speed is small.

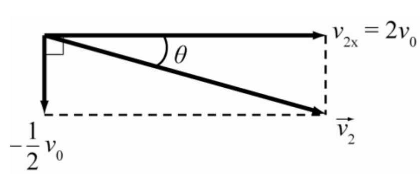
7.93. **THINK:** I have a rocket which at the top of the trajectory breaks into two equal pieces. One piece has half the speed of the rocket travelling upward. I want to calculate the speed and angle of the second piece.
SKETCH:



RESEARCH: Use the conservation of momentum. $\vec{p}_i = \vec{p}_f$, or in component form, $p_{xi} = p_{xf}$ and $p_{yi} = p_{yf}$. I also know that $p_{yi} = 0$. Let us assume that the speed of the rocket before it breaks is v_0 and mass m_0.

SIMPLIFY: $p_{xi} = p_{xf} \Rightarrow m_0 v_0 = m_1 v_{1x} + m_2 v_{2x}$. Since $v_{1x} = 0$ and $m_2 = \frac{1}{2}m_0$, $m_0 v_0 = \frac{1}{2}m_0 v_{2x}$ $\Rightarrow v_{2x} = 2v_0$. $p_{yi} = p_{yf} = 0 \Rightarrow 0 = m_1 v_{1y} + m_2 v_{2y}$. Since $m_1 = m_2 = \frac{1}{2}m_0$ and $v_{1y} = \frac{1}{2}v_0$, $v_{2y} = -v_{1y} \Rightarrow v_{2y} = -\frac{1}{2}v_0$. $v_2 = \sqrt{v_{2x}^2 + v_{2y}^2} = \sqrt{2^2 v_0^2 + (-1/2)^2 v_0^2}$; $\theta = \tan^{-1}\left(\dfrac{(-1/2)v_0}{2v_0}\right)$.

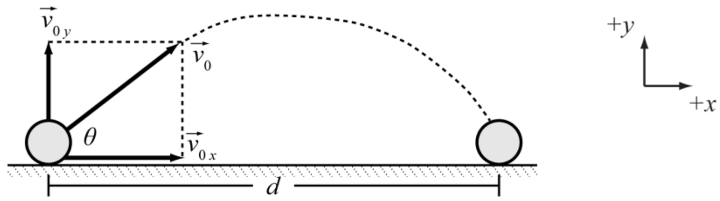
Drawing the vector $\vec{v}_2$:



CALCULATE: $v_2 = v_0\sqrt{4 + 1/4} = \dfrac{\sqrt{17}}{2}v_0$, $\theta = \tan^{-1}\left(-\dfrac{1}{4}\right) = -14.04°$

ROUND: Rounding is not needed here.
DOUBLE-CHECK: It makes sense that θ is negative since the first piece is travelling upwards. The y component of v_2 must be in the negative y-direction.

7.94. **THINK:** A soccer ball has mass 0.265 kg. The ball is kicked at an angle of 20.8° with respect to the horizontal. It travels a distance of 52.8 m. Calculation of the impulse received by the ball is needed.
SKETCH:



RESEARCH: I use the definition of impulse. $J = \Delta p = p_f - p_i$, and $p_i = 0$ since the ball is initially at rest. Thus $J = mv_0$. I need to determine v_0.
SIMPLIFY: I can determine the time to reach the maximum height by: $v = v_{0y} - gt = 0 \Rightarrow t = v_{0y} / g$. The time to reach a distance d is twice the time taken to reach the maximum height. So, $t_d = 2v_{0y} / g = 2v_0 \sin\theta / g$. I can also use:

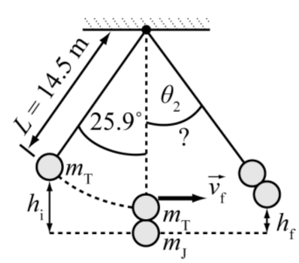
$$d = v_{0x}t_d = v_0 \cos\theta \frac{2v_0 \sin\theta}{g} = \frac{v_0^2 \sin 2\theta}{g} \Rightarrow v_0 = \sqrt{\frac{dg}{\sin 2\theta}}$$

The impulse is $J = m\sqrt{\dfrac{dg}{\sin 2\theta}}$.

CALCULATE: $J = (0.265 \text{ kg})\sqrt{\dfrac{(52.8 \text{ m})(9.81 \text{ m/s}^2)}{\sin(2\cdot 20.8^\circ)}} = 7.402 \text{ kg m/s}$

ROUND: $J = 7.40$ kg m/s
DOUBLE-CHECK: This is a reasonable value.

7.95. **THINK:** Tarzan swinging on a vine 14.5 m long picks Jane up at the bottom of his trajectory. At the beginning of his swing, the vine was at an angle of 25.9° to the vertical. What will be the maximum angle relative to the vertical Tarzan and Jane will reach? Tarzan and Jane have masses $m_T = 70.4$ kg and $m_J = 43.4$ kg.
SKETCH:



RESEARCH: Use conservation of energy to calculate the speed of Tarzan just before he picks up Jane. Use conservation of momentum to find the speed just after Tarzan picks up Jane. Then use conservation of energy again to find the final height. Relate the initial and final heights to the angles and L.
SIMPLIFY:
By conservation of energy, noting that Tarzan starts with $v = 0$ at h_i,

$$m_T g h_i = \frac{1}{2} m_T v_i^2 \Rightarrow v_i = \sqrt{2gh_i}$$

By the conservation of momentum,

$$p_i = p_f \;\Rightarrow\; m_T v_i = (m_T + m_J)v_f \;\Rightarrow\; v_f = \left(\frac{m_T}{m_T + m_J}\right)v_i = \left(\frac{m_T}{m_T + m_J}\right)\sqrt{2gh_i}$$

The final height is determined using conservation of energy, noting that $v = 0$ at the maximum height, $\sqrt{}$

$$\frac{1}{2}(m_T + m_J)v_f^2 = (m_T + m_J)gh_f \Rightarrow h_f = \frac{v_f^2}{2g} = \left(\frac{m_T}{m_T + m_J}\right)^2 \frac{2gh_i}{2g} = \left(\frac{m_T}{m_T + m_J}\right)^2 h_i.$$

Knowing that $h_f = L - L\cos\theta_2$ and $h_i = L - L\cos\theta_1$.

$$L - L\cos\theta_2 = \left(\frac{m_T}{m_T + m_J}\right)^2 (L - L\cos\theta_1)$$

$$1 - \cos\theta_2 = \left(\frac{m_T}{m_T + m_J}\right)^2 (1 - \cos\theta_1)$$

$$\cos\theta_2 = 1 - \left(\frac{m_T}{m_T + m_J}\right)^2 (1 - \cos\theta_1)$$

CALCULATE:

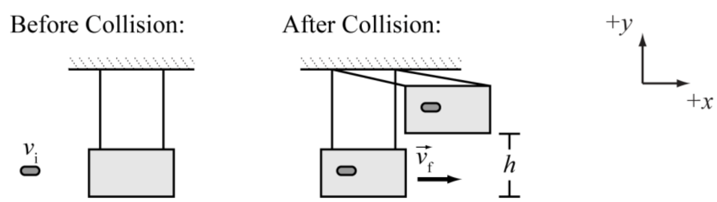
$$\cos\theta_2 = 1 - \left(\frac{70.4 \text{ kg}}{70.4 \text{ kg} + 43.4 \text{ kg}}\right)^2 (1 - \cos 25.9°) \;\Rightarrow\; \cos\theta_2 = 0.9616 \;\Rightarrow\; \theta_2 = \cos^{-1}(0.9616) = 15.929°$$

ROUND: $\theta_2 = 15.9°$

DOUBLE-CHECK: θ_2 must be less than θ_1 because v_f is less than v_i. This is the case.

7.96. **THINK:** Since the bullet has a mass $m = 35.5 \text{ g} = 0.0355 \text{ kg}$ and a block of wood with mass $M = 5.90 \text{ kg}$. The height is $h = 12.85 \text{ cm} = 0.1285 \text{ m}$. Determine the speed of the bullet.

SKETCH:



RESEARCH: First determine v_f using $v = \sqrt{2gh}$ and then determine the speed of the bullet using the conservation of momentum.

SIMPLIFY: $\frac{1}{2}mv_f^2 = mgh \Rightarrow v_f = \sqrt{2gh}$,

$$p_i = p_f \;\Rightarrow\; mv_i = (m + M)v_f \;\Rightarrow\; v_i = \left(\frac{m + M}{m}\right)v_f = \left(\frac{m + M}{m}\right)\sqrt{2gh}$$

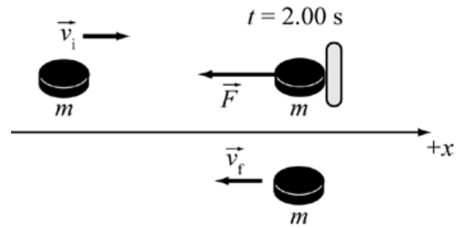
CALCULATE: $v_i = \left(\frac{0.0355 \text{ kg} + 5.90 \text{ kg}}{0.0355 \text{ kg}}\right)\sqrt{2(9.81 \text{ m/s}^2)(0.1285 \text{ m})} = 265.48 \text{ m/s}$

ROUND: $v_i = 265 \text{ m/s}$

DOUBLE-CHECK: The result is reasonable for a bullet.

7.97. **THINK:** I have a 170. g hockey puck with initial velocity $v_{\text{i}} = 30.0$ m/s and final velocity $v_{\text{f}} = -25.0$ m/s, changing over a time interval of $\Delta t = 0.200$ s.

SKETCH:



RESEARCH: The initial and final momentums are calculated by $p_{\text{i}} = mv_{\text{i}}$ and $p_{\text{f}} = mv_{\text{f}}$. The force is calculated using $J = F\Delta t = \Delta p = m(v_{\text{f}} - v_{\text{i}})$.

SIMPLIFY: Simplification is not necessary.

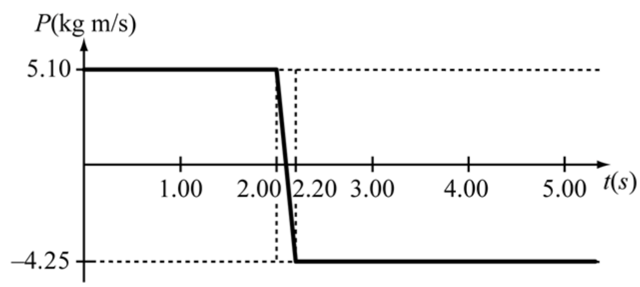
CALCULATE: $p_{\text{i}} = (0.170 \text{ kg})(30.0 \text{ m/s}) = 5.10$ kg m/s, $p_{\text{f}} = (0.170 \text{ kg})(-25.0 \text{ m/s}) = -4.25$ kg m/s, and

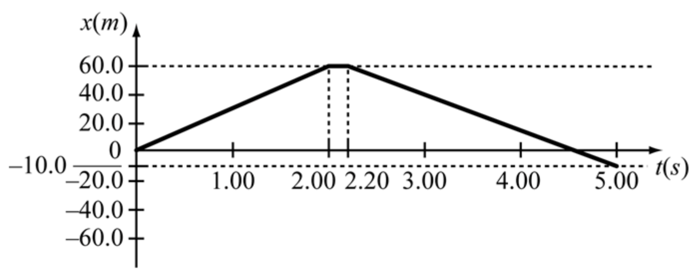
$F = \dfrac{p_{\text{f}} - p_{\text{i}}}{\Delta t} = \dfrac{(-4.25 \text{ kg m/s} - 5.10 \text{ kg m/s})}{0.200 \text{ s}} = -46.75$ N. The position of the puck at t = 2.00 s is:

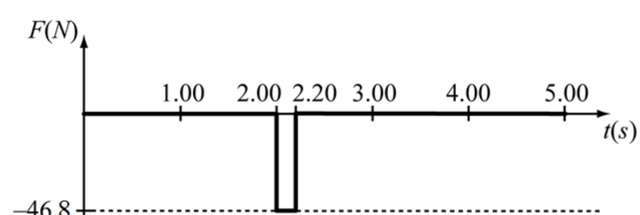
$x_2 = v_i t = (30.0 \text{ m/s})(2.00 \text{ s}) = 60.0$ m. The position of the puck at t = 5.00 s is:

$$x_5 = x_2 + v_{\text{f}}(5.00 \text{ s} - 2.20 \text{ s}) = 60.0 \text{ m} + (-25.0 \text{ m/s})(2.80 \text{ s}) = -10.0 \text{ m}.$$

With all this information I can plot p vs. t, x vs. t and F vs. t.





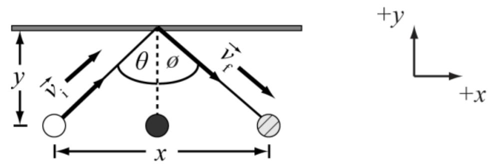


ROUND: $p_{\text{i}} = 5.10$ kg m/s , $p_{\text{f}} = -4.25$ kg m/s , $F = -46.8$ N , $x_2 = 60.0$ m , and $x_5 = -10.0$ m.

DOUBLE-CHECK: The force F is applied only during the interval of 0.200 s. At other times F = 0, or $a = 0$.

7.98. **THINK:** I know the distance between the cue ball and the stripe ball is x = 30.0 cm, and the distance between the cue ball and the bumper is y = 15.0 cm. I want (a) the angle of incidence θ_1 for the cue ball given an elastic collision between the ball and the bumper and (b) the angle θ_2 given a coefficient of restitution of $c_r = 0.600$.

SKETCH:



RESEARCH:

(a) To conserve momentum in a purely elastic collision, the incidence and reflection angles are equal. I can use basic trigonometry to find θ.

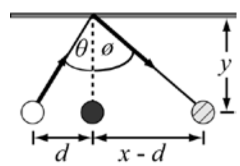
(b) When $c_r = 0.600$ I know how the speed of the ball changes after colliding with the bumper. Since there are no horizontal (x-direction) forces, only the vertical (y-direction) speed changes, and does so by a factor of c_r. That is, $v_{fx} = v_{ix}$ while $v_{fy} = c_r v_{iy}$

SIMPLIFY:

(a) From the sketch we see that $\tan\theta = \frac{(1/2)x}{y}$. Then $\theta = \tan^{-1}\left(\frac{x}{2y}\right)$.

(b) I know $v_{fx} = v_{ix} \Rightarrow v_f \sin\phi = v_i \sin\theta$, and $v_{fy} = c_r v_{iy} \Rightarrow v_f \cos\phi = c_r v_i \cos\theta$. Dividing these two equations gives:

$$\frac{v_f \sin\phi}{v_f \cos\phi} = \frac{v_i \sin\theta}{c_r v_i \cos\theta} \Rightarrow \tan\theta = c_r \tan\phi.$$



I know that $\tan\theta = d/y$ and that $\tan\phi = (x-d)/y$. Then $\tan\phi = x/y - d/y = x/y - \tan\theta$.

Now $\tan\theta = c_r \tan\phi$ becomes:

$$\tan\theta = c_r\left(\frac{x}{y} - \tan\theta\right) \Rightarrow \tan\theta(1+c_r) = c_r\frac{x}{y} \Rightarrow \theta = \tan^{-1}\left(\frac{c_r x}{y(1+c_r)}\right)$$

CALCULATE:

(a) $\theta = \tan^{-1}\left(\frac{30.0 \text{ cm}}{2(15.0 \text{ cm})}\right) = 45.0°$

(b) $\theta = \tan^{-1}\left(\frac{0.600(30.0 \text{ cm})}{(15.0 \text{ cm})(0.600+1)}\right) = 36.87°$

ROUND:

Rounding to three significant figures:

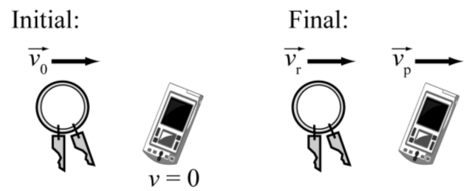
(a) $\theta = 45.0°$

(b) $\theta = 36.9°$

DOUBLE-CHECK: When c_r decreases from 1 (a perfectly inelastic collision), θ should become smaller (steeper).

7.99. **THINK:** I know that the phone's mass $m_p = 0.111$ kg, the key ring's mass $m_r = 0.020$ kg and the mass per key $m_k = 0.023$ kg. I want to find the minimum number of keys, *n*, to make the keys and the phone come out on the same side of the bookcase, and the final velocities of the phone, v_{2f} and the key ring, v_{1f}, if the key ring has five keys and an initial velocity of $v_{1i} = 1.21$ m/s.

SKETCH:



RESEARCH: Note that this is an elastic collision, therefore kinetic energy is conserved. Also, the phone is initially stationary and the collision is one dimensional. I can use the following equations:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \text{ and } v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$

When determining the minimum number of keys, keep in mind that the final key velocity must be positive, as negative would imply that the keys and the phone come out on opposite sides.

SIMPLIFY: Find *n* given the condition that v_{1f} is positive. Note the condition that ensures $v_{1f} > 0$ is $m_1 > m_2$. Then

$$(m_r + nm_k) > m_p \Rightarrow n > \frac{1}{m_k}(m_p - m_r)$$

$$v_{1f} = \frac{(m_r + nm_k) - m_p}{(m_r + nm_k) + m_p} v_{1i} \text{ and } v_{2f} = \frac{2(m_r + nm_k)}{(m_r + nm_k) + m_p} v_{1i}.$$

CALCULATE: $n > \dfrac{1}{0.023 \text{ kg}}(0.111 \text{ kg} - 0.020 \text{ kg}) = 3.96$

$$v_{1f} = \frac{(0.020 \text{ kg} + 5(0.023 \text{ kg})) - 0.111 \text{ kg}}{0.020 \text{ kg} + 5(0.023 \text{ kg}) + 0.111 \text{ kg}}(1.21 \text{ m/s}) = 0.118 \text{ m/s}$$

$$v_{2f} = \frac{2(0.020 \text{ kg} + 5(0.023 \text{ kg}))}{0.020 \text{ kg} + 5(0.023 \text{ kg}) + 0.111 \text{ kg}}(1.21 \text{ m/s}) = 1.3280 \text{ m/s}$$

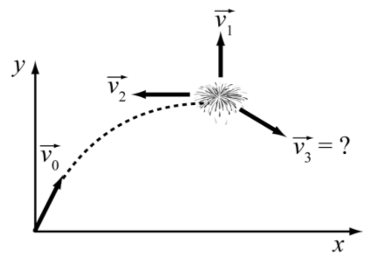
ROUND: m_r and m_k have two significant figures. As *n* is an integer, the minimum *n* is 4, $v_{1f} = 0.12$ m/s and $v_{2f} = 1.33$ m/s.

DOUBLE-CHECK: The combined mass of four keys and the key ring is just slightly more than the mass of the cell phone. The result is reasonable.

7.100. **THINK:** I know the ball's initial mass *M* = 7.00 kg, initial speed $v_0 = 10.0$ m/s, launch angle $\theta_0 = 40.0°$ and that it explodes at the peak of its trajectory. By choosing "straight up" to be along the positive *y* axis and "straight back" to be along the negative *x*-axis, I know one piece of the mass travels with $\vec{v}_1 = 3.00 \text{ m/s } \hat{y}$ and the other travels with $\vec{v}_2 = -2.00 \text{ m/s } \hat{x}$. Calculate the velocity of the third piece, $\vec{v}_3$.

Note that all three pieces have the same mass, $m = \frac{1}{3}M$.

SKETCH:



RESEARCH: Note that at the peak height, v_y for the ball (before exploding) is zero. Then the initial momentum of the ball prior to exploding is $\vec{p} = M(v_0 \cos\theta)\,\hat{x}$. Find $\vec{v}_3$ by conservation of momentum. Specifically, $p_{\text{ix}} = p_{\text{fx}}$ and $p_{\text{iy}} = p_{\text{fy}}$.

SIMPLIFY: Along *x*, $p_{\text{ix}} = p_{\text{fx}}$ so:

$$Mv_0 \cos\theta_0 = mv_{1x} + mv_{2x} + mv_{3x} = m(-v_2 + v_{3x}) \Rightarrow v_{3x} = \frac{M}{m}v_0 \cos\theta_0 + v_2 = 3v_0 \cos\theta_0 + v_2$$

Along y, $p_{\text{iy}} = p_{\text{fy}} \Rightarrow 0 = mv_{1y} + mv_{2y} + mv_{3y} \Rightarrow v_{3y} = -v_{1y}$. Then $v_3 = \sqrt{v_{3x}^2 + v_{3y}^2}$ and $\theta = \tan^{-1}(v_{3y} / v_{3x})$ with respect to the horizontal.

CALCULATE: $v_{3x} = 3(10.0 \text{ m/s})\cos 40.0° + 2.00 \text{ m/s} = 24.98 \text{ m/s}$,

$v_{3y} = -v_{1y} = -3.00 \text{ m/s}$, $v_3 = \sqrt{(24.98 \text{ m/s})^2 + (3.00 \text{ m/s})^2} = 25.16 \text{ m/s}$, and

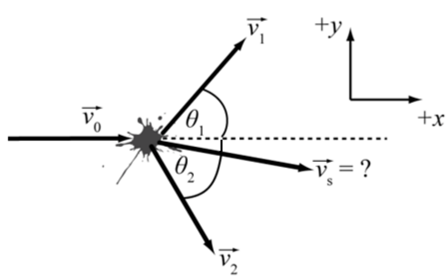
$\theta = \tan^{-1}\left(\frac{3.00 \text{ m/s}}{24.98 \text{ m/s}}\right) = 6.848°$.

ROUND: $v_3 = 25.2 \text{ m/s}$ and $\theta = 6.85°$ below the horizontal.

DOUBLE-CHECK: The speeds of the first two fragments are small; it makes sense that the third fragment should have a larger speed to conserve the total momentum.

7.101. **THINK:** I know the skier's initial speed $v_0 = 22.0 \text{ m/s}$, the skier's mass $M = 61.0 \text{ kg}$, the mass of each ski $m = 1.50 \text{ kg}$ and the final velocity of each ski: $\vec{v}_1 = 25.0 \text{ m/s}$ at $\theta_1 = 12.0°$ to the left of the initial direction, and $\vec{v}_2 = 21.0 \text{ m/s}$ at $\theta_2 = 5.00°$ to the right of the initial direction. Calculate the magnitude and direction with respect to the initial direction of the skier's final velocity, $\vec{v}_\text{s}$.

SKETCH:



RESEARCH: The conservation of momentum requires $\sum_j (p_{\text{fx}})_j = \sum_j (p_{\text{ix}})_j$ and $\sum_j (p_{\text{fy}})_j = \sum_j (p_{\text{iy}})_j$. By conserving momentum in each direction, find $\vec{v}_\text{s}$. Take the initial direction to be along the *x*-axis.

SIMPLIFY: Then, $p_{ix} = m_{\text{total}}v_0 = (M + 2m)v_0$, and take $p_{ix} = p_{fx}$ in the equation $p_{fx} = Mv_{\text{sx}} + mv_{1x} + mv_{2x} = Mv_{\text{sx}} + mv_1 \cos\theta_1 + mv_2 \cos\theta_2$.

$$(M + 2m)v_0 = Mv_{\text{sx}} + m(v_1 \cos\theta_1 + v_2 \cos\theta_2) \Rightarrow v_{\text{sx}} = \frac{1}{M}\left((M + 2m)v_0 - m(v_1 \cos\theta_1 + v_2 \cos\theta_2)\right).$$

Similarly,

$$p_{iy} = 0 = p_{fy} = Mv_{sy} + mv_{1y} - mv_{2y} = Mv_{sy} + m(v_1 \sin\theta_1 - v_2 \sin\theta_2) \Rightarrow v_{sy} = \frac{m}{M}(v_2 \sin\theta_2 - v_1 \sin\theta_1)$$

With v_{sx} and v_{sy} known, get the direction with respect to the initial direction from $\theta_s = \tan^{-1}(v_{sy}/v_{sx})$. The magnitude of the velocity is $v_s = \sqrt{v_{sx}^2 + v_{sy}^2}$.

CALCULATE:

$$v_{sx} = \frac{1}{61.0 \text{ kg}}\Big(\big(61.0 \text{ kg} + 2(1.50 \text{ kg})\big)(22.0 \text{ m/s}) - (1.50 \text{ kg})\big((25.0 \text{ m/s})\cos 12.0° + (21.0 \text{ m/s})\cos 5.00°\big)\Big)$$
$$= 21.9662 \text{ m/s}$$

$$v_{sy} = \frac{1.50 \text{ kg}}{61.0 \text{ kg}}\big((21.0 \text{ m/s})\sin 5.00° - (25.0 \text{ m/s})\sin 12.0°\big) = -0.08281 \text{ m/s}$$

$$v_s = \sqrt{(21.9662 \text{ m/s})^2 + (-0.08281 \text{ m/s})^2} = 21.9664 \text{ m/s}$$

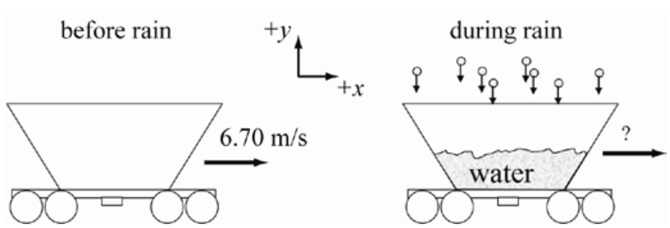
$\theta_s = \tan^{-1}\left(\frac{-0.08281 \text{ m/s}}{21.9662 \text{ m/s}}\right) = -0.2160°$, where the negative indicates that θ_s lies below the x-axis, or to the right of the initial direction.

ROUND: $\vec{v}_s = 22.0$ m/s at $0.216°$ to the right of the initial direction.

DOUBLE-CHECK: As the skier's mass is much greater than the mass of the two skis, it is reasonable that the skier carries the majority of the final momentum.

7.102. **THINK:** I know the car's initial speed $v_0 = 6.70$ m/s and mass $m_c = 1.18 \cdot 10^5$ kg . There is no friction or air resistance. I want (a) the speed of the car v_1 after collecting $m_w = 1.62 \cdot 10^4$ kg of water, and (b) the speed of the car v_2 after all the water has drained out, assuming an initial speed of $v_0 = 6.70$ m/s .

SKETCH:



RESEARCH: (a) Because the water enters the car completely in the vertical direction, it contributes mass but no horizontal momentum. Use conservation of momentum to determine the car's subsequent speed. $\Delta p = 0$. (b) The water drains out vertically in the moving frame of the car, which means that right after leaving the car the water has the same speed as the car. Therefore the speed of the car does not get changed at all by the draining water. No further calculation is necessary for part (b); the final speed of the car is the initial speed of 6.70 m/s.

SIMPLIFY:

(a) The initial mass is that of just the car. If I think of the water colliding perfectly inelastically with the car, the final mass is $m_c + m_w$.

$$p_f = p_i \Rightarrow (m_c + m_w)v_1 = m_c v_0 \Rightarrow v_1 = m_c v_0 / (m_c + m_w)$$

CALCULATE:

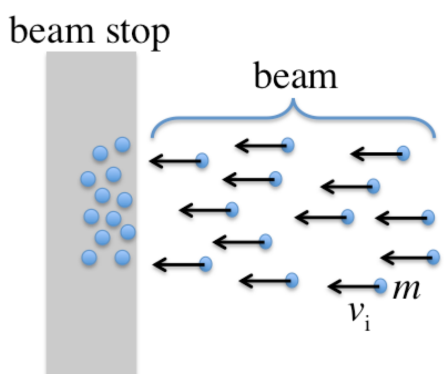
(a) $v_1 = \frac{(1.18 \cdot 10^5 \text{ kg})(6.70 \text{ m/s})}{1.18 \cdot 10^5 \text{ kg} + 1.62 \cdot 10^4 \text{ kg}} = 5.8912 \text{ m/s}$

ROUND: With three significant figures in v_0, (a) $v_1 = 5.89$ m/s and (b) $v_2 = 6.70$ m/s

DOUBLE-CHECK: The car slows down when the water is added. So why does the car not speed up when the water is drained? In the first case v decreased when mass was added to the car, because the water had no initial horizontal velocity component. But when the water was drained from the car the water did have the same initial velocity component as the car, which is where the essential difference lies.

7.103. **THINK:** At first this looks like a complicated problem involving nuclear physics, because it describes beams, nuclei, rare isotopes, and beam stops. However, all that the beam stop does is to stop the nuclei hitting it, i.e. it sets the final speed to 0. Since we are given the mass and initial speed of each nucleus, we can find its initial momentum. Since the final momentum is zero, we therefore know the impulse that the beam stop receives from the collision with an individual nucleus.

SKETCH:



RESEARCH: The magnitude of the impulse a given nucleus receives from the beam stop is $J = |mv_f - mv_i|$, where the final speed is zero. From momentum conservation we know that the magnitude of the impulse that the beam stop receives form the nucleus is the same. The average force is defined as $F_{ave} = J_{total} / \Delta t$, where the total impulse is the combined impulse of all nuclei hitting the beam stop in a given time interval: $J_{total} = J\dfrac{dn}{dt}\Delta t$, and $\dfrac{dn}{dt}$ is the rate of nuclei per second given in the problem text.

SIMPLIFY: The average force is

$$F_{ave} = J_{total} / \Delta t = \left(J\frac{dn}{dt}\Delta t \right) / \Delta t = J\frac{dn}{dt} = |mv_f - mv_i|\frac{dn}{dt} = mv_i\frac{dn}{dt}$$

CALCULATE:

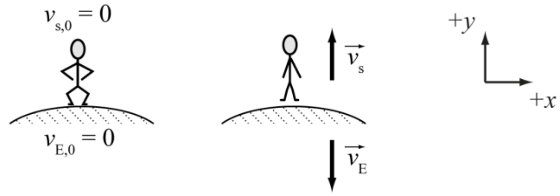
$$F_{ave} = (8.91 \cdot 10^{-26}\text{ kg})(0.247 \cdot 2.998 \cdot 10^{8}\text{ m/s})(7.25 \cdot 10^{5}\text{ / s}) = 4.78348 \cdot 10^{-12}\text{ kg m/s}^2$$

ROUND: We round to three significant figures and state $F_{ave} = 4.78\text{ pN}$ as our final answer.

DOUBLE-CHECK: Our first check should almost always by to make sure that the units and order of magnitude of our answer work out. The units we found, kg m/s^2, are indeed the same as the force-unit, N. The magnitude, however, may at first seem surprising, because a pico-Newton is an incredibly small force. The reason why this force is so small is that the mass of the individual nuclei is so incredibly small.

7.104. **THINK:** I know the student's mass $m_s = 60.0\text{ kg}$, her average force $F_{av} = 770.\text{ N}$, the time $\Delta t = 0.250\text{ s}$, and the Earth's mass $m_E = 5.98 \cdot 10^{24}\text{ kg}$. I want to know the student's momentum after the impulse, p_s, the Earth's momentum after the impulse, p_E, the speed of the Earth after the impulse, v_E, the fraction of the total kinetic energy produced by the student's legs that goes to the Earth, $K_{\text{E}}/K_{\text{s}}$, and the maximum height of the student, h.

SKETCH: Consider the student-Earth system:



RESEARCH: In the student-Earth system, momentum is conserved; $\Delta p = 0$. Find the change in the student's momentum from $\Delta p = F_{\text{av}}\Delta t$ and then find the Earth's momentum and speed from momentum conservation. To find $K_{\text{E}}/K_{\text{s}}$ calculate K_{E} and K_{s} using $K = \frac{1}{2}mv^2$. Using energy conservation I can find h for the student from $\Delta K_{\text{s}} + \Delta U = 0$.

SIMPLIFY: To find p_{s}: $\Delta p = F_{\text{av}}\Delta t$, with $p_{\text{s},0} = 0$ ($v_i = 0$). $p_{\text{s}} = F_{\text{av}}\Delta t$. To find p_{E}: $\Delta p_{\text{system}} = 0 \Rightarrow p_{\text{s}} - p_{\text{s},0} + p_{\text{E}} - p_{\text{E},0} = 0$. Then $p_{\text{E}} = -p_{\text{s}}$. To find v_{E}: $p = mv \Rightarrow v_E = \dfrac{|p_{\text{E}}|}{m_{\text{E}}}$. (only want the speed, not the velocity) To find $K_{\text{E}}/K_{\text{s}}$: $K_{\text{E}} = \frac{1}{2}m_E v_E^2 = \frac{1}{2}\frac{p_E^2}{m_{\text{E}}} = \frac{1}{2}\frac{p_{\text{s}}^2}{m_{\text{E}}}$, and $K_{\text{s}} = \frac{1}{2}m_{\text{s}}v_{\text{s}}^2 = \frac{1}{2}\frac{p_{\text{s}}^2}{m_{\text{s}}}$. Then

$\dfrac{K_{\text{E}}}{K_{\text{s}}} = \dfrac{(1/2)\left(p_{\text{s}}^2/m_{\text{E}}\right)}{(1/2)\left(p_{\text{s}}^2/m_{\text{s}}\right)} = \dfrac{m_{\text{s}}}{m_{\text{E}}}$. To find h: Note the kinetic energy of the Earth is negligible. Then $\Delta K + \Delta u = 0$ becomes $K_{\text{s,f}} - K_{\text{s,i}} + U_{\text{s,f}} - U_{\text{s,i}} = 0 \Rightarrow U_{\text{s,f}} = K_{\text{s,i}} \Rightarrow m_{\text{s}}gh = \dfrac{1}{2}\dfrac{p_{\text{s}}^2}{m_{\text{s}}} \Rightarrow h = \dfrac{p_{\text{s}}^2}{2gm_{\text{s}}^2}$.

CALCULATE: $p_{\text{s}} = F_{\text{av}}\Delta t = (770.\text{ N})(0.250\text{ s}) = 192.5\text{ kg m/s}$, $p_{\text{E}} = -p_{\text{S}} = -192.5\text{ kg m/s}$,

$v_{\text{E}} = \dfrac{|p_{\text{E}}|}{m_{\text{E}}} = \dfrac{192.5\text{ kg m/s}}{5.98\cdot 10^{24}\text{ kg}} = 3.2191\cdot 10^{-23}\text{ m/s}$, $\dfrac{K_{\text{E}}}{K_{\text{s}}} = \dfrac{m_{\text{s}}}{m_{\text{E}}} = \dfrac{60.0\text{ kg}}{5.98\cdot 10^{24}\text{ kg}} = 1.003\cdot 10^{-23}$, and

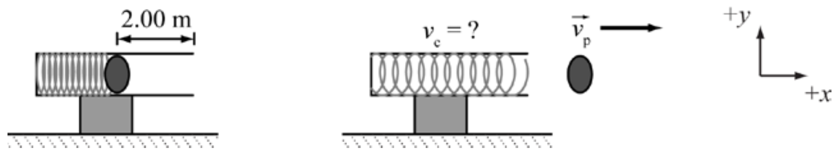
$$h = \frac{p_s^2}{2gm_{\text{s}}^2} = \frac{1}{2\left(9.81\text{ m/s}^2\right)}\left(\frac{192.5\text{ kg m/s}}{60.0\text{ kg}}\right) = 0.1635\text{ m}.$$

ROUND: To three significant figures, $p_{\text{s}} = 193\text{ kg m/s}$, $p_{\text{E}} = -193\text{ kg m/s}$, $v_E = 3.22\cdot 10^{-23}\text{ m/s}$, $K_{\text{E}}/K_{\text{s}} = 1.00\cdot 10^{-23}$ and $h = 0.164$ m.

DOUBLE-CHECK: Because the mass of the Earth is so large, its resulting velocity due to momentum conservation, and therefore its kinetic energy, should be negligible compared to the student's. The height h is reasonable considering the time Δt the student's F_{av} acts over.

7.105. **THINK:** I want (a) the cannon's velocity $\vec{v}_{\text{c}}$ when the potato has been launched, and (b) the initial and final mechanical energy. There is no friction in the potato-cannon-ice system. Let the cannon's mass be $m_{\text{c}} = 10.0$ kg, the potato's mass be $m_{\text{p}} = 0.850$ kg, the cannon's spring constant be $k_{\text{c}} = 7.06\cdot 10^3$ N/m, the spring's compression be $\Delta x = 2.00$ m, the cannon and the potato's initial velocities be $\vec{v}_{c,0} = \vec{v}_{p,0} = 0$, and the potato's launch velocity be $\vec{v}_p = 175\text{ m/s}\,\hat{x}$. Take "horizontally to the right" to be the positive $\hat{x}$ direction.

SKETCH:



RESEARCH:

(a) Use the conservation of momentum $\Delta\vec{p}=0$ to determine $\vec{v}_c$ when the potato, cannon and ice are considered as a system. Since the ice does not move, we can neglect the ice in the system and only consider the momenta of the potato and cannon.

(b) The total mechanical energy, E_{mec}, is conserved since the potato-cannon-ice system is isolated. That is, $E_{mec,f}=E_{mec,i}$. The value of $E_{mec,i}$ can be found by considering the spring potential energy of the cannon.

SIMPLIFY:

(a) $\Delta\vec{p}=0 \Rightarrow \vec{p}_p-\vec{p}_{p,0}+\vec{p}_c-\vec{p}_{c,0}=0 \Rightarrow \vec{p}_c=-\vec{p}_p \Rightarrow m_c\vec{v}_c=-m_p\vec{v}_p \Rightarrow \vec{v}_c=-\dfrac{m_p}{m_c}\vec{v}_p$

(b) $E_{mec,f}=E_{mec,i}=u_{s,i}=\dfrac{1}{2}k_c\left(\Delta x\right)^2$

CALCULATE:

(a) $\vec{v}_c=-\left(\dfrac{0.850\text{ kg}}{10.0\text{ kg}}\right)\left(175\text{ m/s}\right)\hat{x}=-14.875\text{ m/s }\hat{x}$

(b) $E_{mec,f}=E_{mec,i}=\dfrac{1}{2}\left(7.06\cdot 10^3\text{ N/m}\right)\left(2.00\text{ m}\right)^2=14120\text{ J}$

ROUND: With three significant figures for all given values,

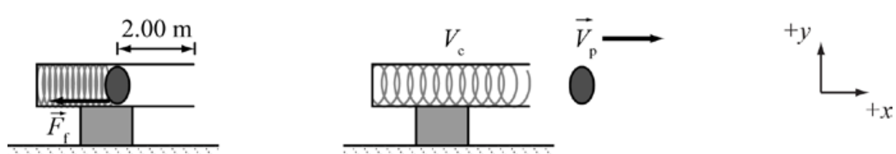
(a) $\vec{v}_c=-14.9\text{ m/s }\hat{x}$, or $\vec{v}_c=14.9\text{ m/s}$ horizontally to the left.

(b) $E_{mec,f}=E_{mec,i}=14.1\text{ kJ}$.

DOUBLE-CHECK: Note $\vec{v}_c$ and $\vec{v}_p$ are directed opposite of each other, and $v_c<v_p$, as expected. Also, if $E_{mec,f}$ had been determined by considering the kinetic energies of the potato and the cannon, the same value would have been found.

7.106. **THINK:** The cannon has mass $m_c=10.0\text{ kg}$ and the potato has mass $m_p=0.850\text{ kg}$. The cannon's spring constant $k_c=7.06\cdot 10^3\text{ N/m}$ and $\Delta x=2.00\text{ m}$. The initial and final speeds of the potato are $v_i=0$ and $v_f=165\text{ m/s}$ respectively. In this case there is friction between the potato and the cannon.

SKETCH:



RESEARCH:

(a) I will use the conservation of momentum: $p_i=p_f$.

(b) I will use $K=\dfrac{1}{2}mv^2$ and $U_s=\dfrac{1}{2}k\left(\Delta x\right)^2$.

(c) I will use $W=\Delta E$.

SIMPLIFY:

(a) $p_i=p_f=0 \Rightarrow 0=m_cv_c+m_pv_p \Rightarrow v_c=-\dfrac{m_p}{m_c}v_p$

(b) The total mechanical energy is $E_{\text{mec}} = \frac{1}{2}mv^2 + \frac{1}{2}k(\Delta x)^2$. Before firing $\Delta x = 2.00$ m and $v = 0$ so $E_{\text{mec,i}} = \frac{1}{2}k(\Delta x)^2$. After firing v_{c} and v_{p} are non-zero and $\Delta x = 0$, so $E_{\text{mec,f}} = \frac{1}{2}m_{\text{c}}v_{\text{c}}^2 + \frac{1}{2}m_{\text{p}}v_{\text{p}}^2$.

(c) The work done by friction is $W_{\text{f}} = \Delta E = E_{\text{mec,f}} - E_{\text{mec,i}}$.

CALCULATE:

(a) $v_{\text{c}} = -\frac{0.850\text{ kg}}{10.0\text{ kg}}(165\text{ m/s}) = -14.025\text{ m/s}$

(b) $E_{\text{mec,i}} = \frac{1}{2}k_{\text{c}}(\Delta x)^2 = \frac{1}{2}(7.06 \cdot 10^3\text{ N/m})(2.00\text{ m})^2 = 14120\text{ J}$,

$E_{\text{mec,f}} = \frac{1}{2}(10.0\text{ kg})(-14.025\text{ m/s})^2 + \frac{1}{2}(0.850\text{ kg})(165\text{ m/s})^2 = 12554.13\text{ J}$

(c) $W_{\text{f}} = 12554.13\text{ J} - 14120\text{ J} = -1565.87\text{ J}$

ROUND:

Round to three significant figures:

(a) $v_{\text{c}} = 14.0$ m/s, directed opposite to the direction of the potato.

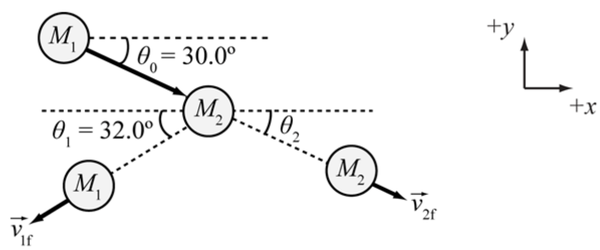
(b) $E_{\text{mec,i}} = 14.1$ kJ, $E_{\text{mec,f}} = 12.6$ KJ

(c) $W_{\text{f}} = -1.56$ KJ

DOUBLE-CHECK: The final mechanical energy is approximately 10% lower that the initial one. From the previous problem we know that the muzzle velocity in the absence of friction is 175 m/s. The muzzle velocity in this case is 165 m/s, about 5% lower. Since the kinetic energy is proportional to the square of the velocity, we should then expect a lowering of 10%. It also makes sense that W_{f} is negative as it is due to a frictional force

7.107. **THINK:** There are two masses $M_1 = 1.00$ kg and $M_2 = 2.00$ kg and the initial and final speeds of M_1; $v_{1\text{i}} = 2.50$ m/s and $v_{1\text{f}} = 0.500$ m/s.

SKETCH:



RESEARCH: Use the conservation of momentum, $p_{\text{i}} = p_{\text{f}}$, in the *x*- and *y*-directions.

SIMPLIFY:

$p_{\text{ix}} = p_{\text{fx}} \Rightarrow M_1v_{1\text{i}}\cos\theta_0 = M_2v_{2\text{f}}\cos\theta_2 - M_1v_{1\text{f}}\cos\theta_1$

$p_{\text{iy}} = p_{\text{fy}} \Rightarrow -M_1v_{1\text{i}}\sin\theta_0 = -M_2v_{2\text{f}}\sin\theta_2 - M_1v_{1\text{f}}\sin\theta_1$

$$v_{2x} = v_{2\text{f}}\cos\theta_2 = \frac{M_1v_{1\text{i}}\cos\theta_0 + M_1v_{1\text{f}}\cos\theta_1}{M_2} = \frac{M_1}{M_2}(v_{1\text{i}}\cos\theta_0 + v_{1\text{f}}\cos\theta_1)$$

$$v_{2y} = v_{2\text{f}}\sin\theta_2 = \frac{M_1v_{1\text{i}}\sin\theta_0 - M_1v_{1\text{f}}\sin\theta_1}{M_2} = \frac{M_1}{M_2}(v_{1\text{i}}\sin\theta_0 - v_{1\text{f}}\sin\theta_1)$$

$$v_{2\text{f}} = \sqrt{v_{2x}^2 + v_{2y}^2} = \frac{M_1}{M_2}\sqrt{(v_{1\text{i}}\cos\theta_0 + v_{1\text{f}}\cos\theta_1)^2 + (v_{1\text{i}}\sin\theta_0 - v_{1\text{f}}\sin\theta_1)^2}$$

CALCULATE:

$$v_{2f} = \frac{1.00 \text{ kg}}{2.00 \text{ kg}}\sqrt{\left[(2.50 \text{ m/s})\cos 30.0° + (0.50 \text{ m/s})\cos 32.0°\right]^2 + \left[(2.50 \text{ m/s})\sin 30.0° - (0.50 \text{ m/s})\sin 32.0°\right]^2}$$
$$= 1.3851 \text{ m/s}$$

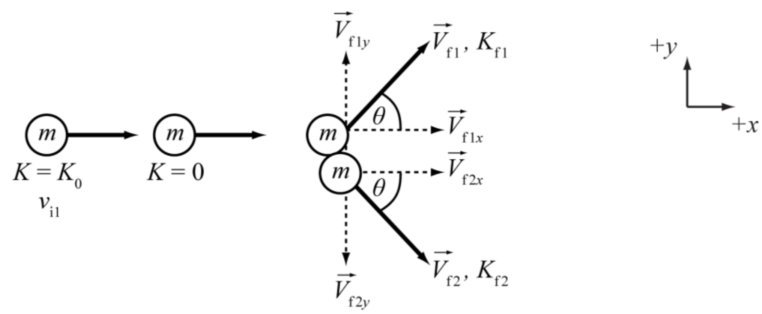
ROUND:

$v_{2f} = 1.39$ m/s

DOUBLE-CHECK: Initial kinetic energy is $\frac{1}{2}(1.00 \text{ kg})(2.50 \text{ m/s})^2 = 3.13$ J . Final kinetic energy is $\frac{1}{2}(1.00 \text{ kg})(0.500 \text{ m/s})^2 + \frac{1}{2}(2.00 \text{ kg})(1.39 \text{ m/s})^2 = 2.06$ J . Energy has been lost during the collision, as expected.

7.108. **THINK:** Since it is an elastic collision, kinetic energy and momentum are both conserved. Both particles are protons and therefore have equal masses. The first particle is deflected $\theta = 25^\circ$ from its path while the particle initially at rest is deflected by an angle ϕ. The initial kinetic energy of the first particle is K_0.

SKETCH:



RESEARCH: $v_x = v\cos\theta$, $v_y = v\sin\theta$, $K_0 = K_{f1} + K_{f2}$, $K_i = \frac{1}{2}mv_i^2$, $\sum p_i = \sum p_f$, $\cos^2\theta + \sin^2\theta = 1$, $\cos(A+B) = \cos A\cos B - \sin A \sin B$

SIMPLIFY: $K_0 = K_{f1} + K_{f2}$, $K_{f1} = \frac{1}{2}mv_{f1}^2$, and $K_{f2} = \frac{1}{2}mv_{f2}^2$. So:

$$K_0 = \frac{1}{2}mv_{f1}^2 + \frac{1}{2}mv_{f2}^2 \Rightarrow v_{f1}^2 + v_{f2}^2 = \frac{2K_0}{m}. \quad (1)$$

$p_{xi} = p_{xf}$, $p_{yi} = p_{yf}$, and $K_0 = \frac{1}{2}mv_{i1}^2$. So

$$mv_{i1} = mv_{f1x} + mv_{f2x} \Rightarrow v_{i1} = v_{f1x} + v_{f2x} \Rightarrow \sqrt{\frac{2K_0}{m}} = v_{f1}\cos 25° + v_{f2}\cos\phi. \quad (2)$$

$$0 = mv_{f1y} - mv_{f2y} \Rightarrow v_{f1y} - v_{f2y} = 0 \Rightarrow v_{f1}\sin 25° - v_{f2}\sin\phi = 0 \quad (3)$$

Squaring equations (2) and (3) and taking the sum,

$$v_{f1}^2\left(\cos^2 25° + \sin^2 25°\right) + v_{f2}^2\left(\cos^2\phi + \sin^2\phi\right) + 2v_{1f}v_{2f}\left(\cos 25°\cos\phi - \sin 25°\sin\phi\right) = \frac{2K_0}{m}$$
$$v_{f1}^2 + v_{f2}^2 + 2v_{1f}v_{2f}\cos(25° + \phi) = \frac{2K_0}{m}$$

Subtracting equation (1),

$$2v_{1f}v_{2f}\cos(25° + \phi) = 0 \Rightarrow \cos(25° + \phi) = 0 \Rightarrow 25° + \phi = 90° \Rightarrow \phi = 65°$$

Therefore $v_{f1}\cos 25° + v_{f2}\cos 65° = \sqrt{2K_0/m}$ and $v_{f1}\sin 25° - v_{f2}\sin 65° = 0$. $v_{f1} = \dfrac{\sin 65°}{\sin 25°}v_{f2}$, and $v_{f2} = v_{f1}\dfrac{\sin 25°}{\sin 65°}$:

$$v_{f2}\left(\frac{\sin 65°\cos 25°}{\sin 25°}\right) + v_{f2}\cos 65° = v_{f2}\left(\frac{\sin 65°}{\tan 25°} + \cos 65°\right) = \sqrt{\frac{2K_0}{m}} \Rightarrow \frac{2K_0}{m\left(\dfrac{\sin 65°}{\tan 25°} + \cos 65°\right)^2} = v_{f2}^2$$

$$v_{f1}\cos 25° + v_{f1}\left(\frac{\sin 25°\cos 65°}{\sin 65°}\right) = v_{f1}\left(\frac{\sin 25°}{\tan 65°} + \cos 25°\right) = \sqrt{\frac{2K_0}{m}} \Rightarrow \frac{2K_0}{m\left(\dfrac{\sin 25°}{\tan 65°} + \cos 25°\right)^2} = v_{f1}^2$$

Therefore, $K_{f1} = \dfrac{1}{2}mv_{f1}^2 = \dfrac{K_0}{\left(\dfrac{\sin 25°}{\tan 65°} + \cos 25°\right)^2}$ and $K_{f2} = \dfrac{1}{2}mv_{f2}^2 = \dfrac{K_0}{\left(\dfrac{\sin 65°}{\tan 25°} + \cos 65°\right)^2}$.

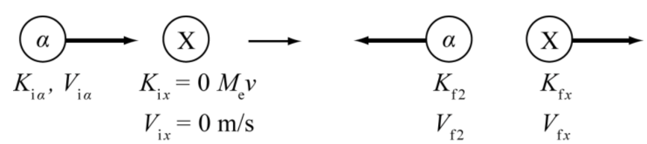
CALCULATE: $K_{f1} = \dfrac{K_0}{1.2174}$, and $K_{f2} = \dfrac{K_0}{5.5989}$.

ROUND: $K_{f1} = 0.821K_0$ and $K_{f2} = 0.179K_0$.

DOUBLE-CHECK: The sum of the coefficients is 0.18+0.82 = 1, which means $K_0 = K_{f1} + K_{f2}$ which means that energy is conserved, so it makes sense.

7.109. **THINK:** Since the collision is elastic, momentum and kinetic energy are conserved. Also, since the alpha particle is backscattered, that means that is reflected 180° back and therefore the collision can be treated as acting in one dimension. The initial and final energies of the alpha particle are given in units of MeV and not J, $K_{i\alpha} = 2.00$ MeV and $K_{f\alpha} = 1.59$ MeV. I can leave the energy in these units and not convert to Joules. $m_\alpha = 6.65 \cdot 10^{-27}$ kg.

SKETCH:



RESEARCH: $v_{f1} = \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right)v_{1i}$, $v_{f2} = \left(\dfrac{2m_1}{m_1 + m_2}\right)v_{1i}$, and $K_i = K_f$, $E = (1/2)mv^2$.

SIMPLIFY: $K_{i\alpha} + K_{iX} = K_{f\alpha} + K_{fx} \Rightarrow K_{fX} = K_{i\alpha} - K_{f\alpha}$, $E_{i\alpha} = \dfrac{1}{2}m_\alpha v_{i\alpha}^2 \Rightarrow v_{i\alpha} = \sqrt{\dfrac{2E_{i\alpha}}{m_\alpha}}$, and

$v_{fX\,\alpha} = \left(\dfrac{2m_\alpha}{m_\alpha + m_X}\right)v_i = \left(\dfrac{2m_\alpha}{m_\alpha + m_X}\right)\sqrt{\dfrac{2K_{i\alpha}}{m_\alpha}}$. Since $K_{fX} = \dfrac{1}{2}m_X v_{fX}^2$:

$$K_{i\alpha} - K_{f\alpha} = \frac{1}{2}m_X v_{fX}^2 = \frac{1}{2}m_X\left(\left(\frac{2m_\alpha}{m_\alpha + m_X}\right)\sqrt{\frac{2K_{i\alpha}}{m_\alpha}}\right)^2 = \frac{1}{2}m_X\left(\frac{4m_\alpha^2}{(m_\alpha + m_X)^2}\right)\left(\frac{2K_{i\alpha}}{m_\alpha}\right) = \frac{4m_X m_\alpha K_{i\alpha}}{m_\alpha^2 + 2m_\alpha m_X + m_X^2}.$$

This simplifies to $0 = \left(\dfrac{K_{i\alpha} - K_{f\alpha}}{K_{i\alpha}}\right)m_x^2 + \left(2\left(\dfrac{K_{i\alpha} - K_{f\alpha}}{K_{i\alpha}}\right)m_\alpha - 4m_\alpha\right)m_x + \left(\dfrac{K_{i\alpha} - K_{f\alpha}}{K_{i\alpha}}\right)m_\alpha^2$.

So, $m_X = \dfrac{\left(4 - 2\left(\dfrac{K_{i\alpha} - K_{f\alpha}}{K_{i\alpha}}\right)\right)m_\alpha \pm m_\alpha\sqrt{\left(2\left(\dfrac{K_{i\alpha} - K_{f\alpha}}{K_{i\alpha}}\right) - 4\right)^2 - 4\left(\dfrac{K_{i\alpha} - K_{f\alpha}}{K_{i\alpha}}\right)^2}}{2\left(\dfrac{K_{i\alpha} - K_{f\alpha}}{K_{i\alpha}}\right)}$ by the quadratic formula.

CALCULATE:

$$m_X = \frac{\left(4-2\left(\frac{2.00 \text{ MeV} - 1.59 \text{ MeV}}{2.00 \text{ MeV}}\right)\right)\left(6.65\cdot 10^{-27} \text{ kg}\right)}{2\left(\frac{2.00 \text{ MeV} - 1.59 \text{ MeV}}{2.00 \text{ MeV}}\right)} \pm \frac{\left(6.65\cdot 10^{-27} \text{ kg}\right)\sqrt{\left(2\left(\frac{2.00 \text{ MeV} - 1.59 \text{ MeV}}{2.00 \text{ MeV}}\right)-4\right)^2 - 4\left(\frac{2.00 \text{ MeV} - 1.59 \text{ MeV}}{2.00 \text{ MeV}}\right)^2}}{2\left(\frac{2.00 \text{ MeV} - 1.59 \text{ MeV}}{2.00 \text{ MeV}}\right)}$$

$$= 1.1608\cdot 10^{-25} \text{ kg},\ 3.8098\cdot 10^{-28} \text{ kg}$$

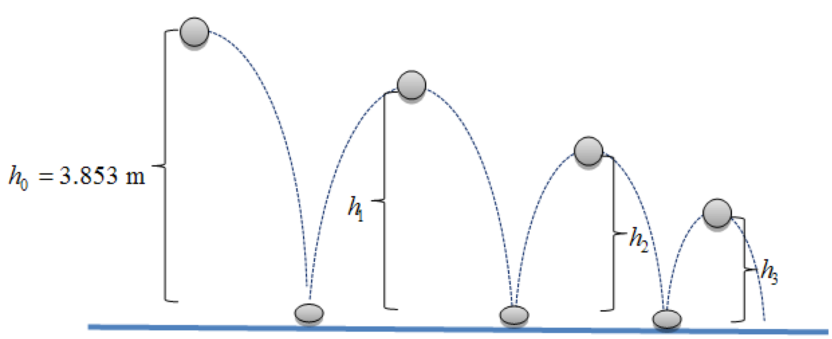
ROUND: $m_X = 1.16\cdot 10^{-25}$ kg. Since the ratio of the masses of atom X and the alpha particle is $m_X / m_\alpha = (1.16\cdot 10^{-25} \text{ kg})/(6.65\cdot 10^{-27} \text{ kg}) = 17.5$, and since the alpha particle has 4 nucleons, expect atom X to have 70 nucleons. Consulting the table in Appendix B, germanium is a good guess. Another possibility would be zinc, but zinc-70 is a relatively rare isotope of zinc, whereas germanium-70 is a common isotope of germanium.

DOUBLE-CHECK: Since the alpha particle is reflected back, the mass of atom X must be greater than the mass of the alpha particle, so this value makes sense.

Multi-Version Exercises

7.110. **THINK:** The coefficient of restitution is used to compute the height of the next bounce from the peak of the previous bounce. Since the ball was dropped (not thrown), assume that it started with no velocity, exactly as it would at the peak of a bounce.

SKETCH: The ball hits the floor three times:



RESEARCH: The coefficient of restitution is defined to be $\varepsilon = \sqrt{\frac{h_f}{h_i}}$. In this case, the ball bounces three times; it is necessary to find expressions relating h_0, h_1, h_2, and h_3.

SIMPLIFY: For the first bounce, $\varepsilon = \sqrt{\frac{h_1}{h_0}}$. For the second bounce, $\varepsilon = \sqrt{\frac{h_2}{h_1}}$, and for the third bounce, $\varepsilon = \sqrt{\frac{h_3}{h_2}}$. Squaring all three equations gives: $\varepsilon^2 = \frac{h_1}{h_0}$, $\varepsilon^2 = \frac{h_2}{h_1}$, and $\varepsilon^2 = \frac{h_3}{h_2}$. Now, solve for h_3 in terms of h_2 and ε: $h_3 = \varepsilon^2 h_2$. Similarly, solve for h_2 in terms of h_1 and ε, then for h_1 in terms of h_0 and ε, to get:

$$h_3 = \varepsilon^2 h_2$$
$$h_2 = \varepsilon^2 h_1$$
$$h_1 = \varepsilon^2 h_0$$

Finally, combine these three equations to get an expression for h_3 in terms of the values given in the problem, h_1 and ε.

$$\begin{aligned} h_3 &= \varepsilon^2 h_2 \\ &= \varepsilon^2\left(\varepsilon^2 h_1\right) \\ &= \varepsilon^2\left(\varepsilon^2\left(\varepsilon^2 h_0\right)\right) \\ &= \varepsilon^6 h_0. \end{aligned}$$

CALCULATE: The coefficient of restitution of the Super Ball is 0.8887 and the ball is dropped from a height of 3.853 m above the floor. So the height of the third bounce is:

$$\begin{aligned} h_3 &= \varepsilon^6 h_0 \\ &= (0.8887)^6\, 3.853 \text{ m} \\ &= 1.89814808 \text{ m} \end{aligned}$$

ROUND: The only numbers used here were the coefficient of restitution and the height. They were multiplied together and were given to four significant figures. Thus the answer should have four figures; the ball reached a maximum height of 1.898 m above the floor.

DOUBLE-CHECK: From experience with Super Balls, this seems reasonable. Double check by working backwards to find the maximum height of each bounce. If the ball bounced 1.898 m on the third bounce, then it reached a height of 1.898 m / 0.8887^2 = 2.403177492 m on the second bounce and 2.403177492 m / 0.8887^2 = 3.042814572 m on the first bounce. From there, the height at which the ball was dropped is computed to be 3.042814572 m / 0.8887^2 = 3.852699416 m. When rounded to four decimal places, this gives an initial height of 3.853 m, which agrees with the number given in the problems and confirms that the calculations were correct.

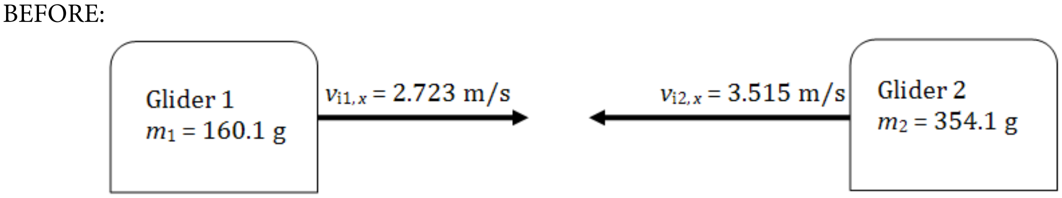
7.111. $h_0 = \dfrac{h_3}{\varepsilon^6} = \dfrac{2.234 \text{ m}}{(0.9115)^6} = 3.895 \text{ m}$

7.112. $\varepsilon = \sqrt[6]{\dfrac{h_3}{h_0}} = \sqrt[6]{\dfrac{2.621 \text{ m}}{3.935 \text{ m}}} = 0.9345$

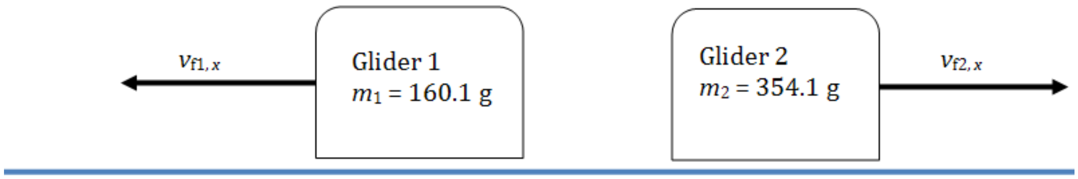
7.113. **THINK:** This problem uses the properties of conservation of energy. Since the masses and initial speeds of the gliders are given, it is possible to use the fact that the collision is totally elastic and the initial conditions to find the velocity of the glider after the collision.

SKETCH: The sketch shows the gliders before and after the collision. Note that the velocities are all in the x – direction. Define the positive x – direction to be to the right.

BEFORE:



AFTER:



RESEARCH: Since this is a one-dimensional, totally elastic collision, we know that the speed of the first glider after the collision is given by the equation:

$$v_{f1,x} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{i1\,x} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{i2\,x}$$

SIMPLIFY: Since the masses are given in grams and the velocities in meters per second, there is no need to convert any of the units in this problem. All of the values needed to compute the final velocity of Glider 1.

CALCULATE: The masses and velocities are given in the problem. Substitute them into the equation to get $v_{f1,x} = \left(\frac{160.1\text{ g} - 354.1\text{ g}}{160.1\text{ g} + 354.1\text{ g}}\right)(2.723\text{ m/s}) + \left(\frac{2\cdot 354.1\text{ g}}{160.1\text{ g} + 354.1\text{ g}}\right)\cdot(-3.515\text{ m/s})$, so the velocity of Glider 1 after the collision is −5.868504473 m/s. The velocity is negative to indicate that the glider is moving to the left.

ROUND: The measured numbers in this problem all have four significant figures, so the final answer should also have four figures. This means that the final velocity of Glider 1 is 5.869 m/s to the left.

DOUBLE-CHECK: Though the speed of Glider 1 is greater after the collision than it was before the collision, which makes sense because Glider 2 was more massive and had a faster speed going into the collision. The problem can also be checked by calculating the speed of Glider 2 after the collision using the equation $v_{f2,x} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{i1\,x} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{i2\,x}$ and confirming that the energy before the collision is equal to the energy after the collision, $\frac{p_{f1}^2}{2m_1} + \frac{p_{f2}^2}{2m_2} = \frac{p_{i1}^2}{2m_1} + \frac{p_{i2}^2}{2m_2}$. Before the collision,

$$\begin{aligned}\frac{p_{i1}^2}{2m_1} + \frac{p_{i2}^2}{2m_2} &= \frac{(m_1 v_{i1})^2}{2m_1} + \frac{(m_2 v_{i2})^2}{2m_2} \\ &= \frac{(160.1\text{ g}\cdot 2.723\text{ m/s})^2}{2\cdot 160.1\text{ g}} + \frac{(354.1\text{ g}\cdot 3.515\text{ m/s})^2}{2\cdot 354.1\text{ g}} \\ &= 2781.041643\text{ g}\cdot\text{m}^2/\text{s}^2\end{aligned}$$

After the collision,

$$\frac{p_{f1}^2}{2m_1}+\frac{p_{f2}^2}{2m_2}$$
$$=\frac{(m_1v_{f1})^2}{2m_1}+\frac{(m_2v_{f2})^2}{2m_2}$$
$$=\frac{(m_1v_{f1})^2}{2m_1}+\frac{\left(m_2\left[\left(\frac{2m_1}{m_1+m_2}\right)v_{i1x}+\left(\frac{m_2-m_1}{m_1+m_2}\right)v_{i2x}\right]\right)^2}{2m_2}$$
$$=\frac{(160.1\text{ g}\cdot 5.869\text{ m/s})^2}{2\cdot 160.1\text{ g}}$$
$$+\frac{\left(354.1\text{ g}\cdot\left[\left(\frac{2\cdot 160.1\text{ g}}{160.1\text{ g}+354.1\text{ g}}\right)\cdot 2.723\text{ m/s}+\left(\frac{354.1\text{ g}-160.1\text{ g}}{160.1\text{ g}+354.1\text{ g}}\right)\cdot(-3.515\text{ m/s})\right]\right)^2}{2\cdot 354.1\text{ g}}$$
$$=2781.507234\text{ g}\cdot\text{m}^2/\text{s}^2$$

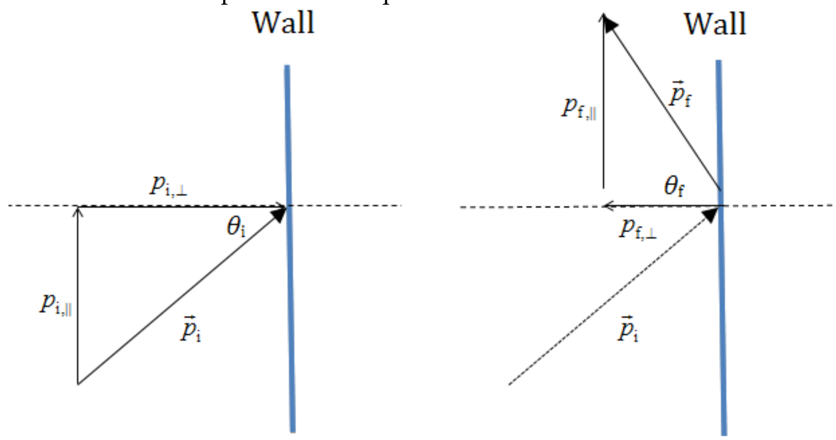
The energies before and after the collision are both close to 2781 $\text{g}\cdot\text{m}^2/\text{s}^2$, confirming that the values calculated for the speeds of the gliders were correct.

7.114. $$v_{f1}=\left(\frac{m_1-m_2}{m_1+m_2}\right)v_{i1}+\left(\frac{2m_2}{m_1+m_2}\right)v_{i2}$$
$$(m_1+m_2)v_{f1}=(m_1-m_2)v_{i1}+2m_2v_{i2}$$
$$m_1v_{f1}+m_2v_{f1}=m_1v_{i1}-m_2v_{i1}+2m_2v_{i2}$$
$$m_2v_{f1}+m_2v_{i1}-2m_2v_{i2}=m_1v_{i1}-m_1v_{f1}$$
$$m_2=\frac{m_1v_{i1}-m_1v_{f1}}{v_{f1}+v_{i1}-2v_{i2}}=\frac{m_1(v_{i1}-v_{f1})}{v_{i1}+v_{f1}-2v_{i2}}$$
$$m_2=\frac{(176.3\text{ g})(2.199\text{ m/s}-(-4.511\text{ m/s}))}{2.199\text{ m/s}-4.511\text{ m/s}-2(-3.301\text{ m/s})}=275.8\text{ g}$$

7.115. $$v_{f1}=\left(\frac{m_1-m_2}{m_1+m_2}\right)v_{i1}+\left(\frac{2m_2}{m_1+m_2}\right)v_{i2}$$
$$(m_1+m_2)v_{f1}=(m_1-m_2)v_{i1}+2m_2v_{i2}$$
$$m_1v_{f1}+m_2v_{f1}=m_1v_{i1}-m_2v_{i1}+2m_2v_{i2}$$
$$m_2v_{f1}+m_2v_{i1}-2m_2v_{i2}=m_1v_{i1}-m_1v_{f1}$$
$$m_1=\frac{m_2v_{f1}+m_2v_{i1}-2m_2v_{i2}}{v_{i1}-v_{f1}}=\frac{m_2(v_{f1}+v_{i1}-2v_{i2})}{v_{i1}-v_{f1}}$$
$$m_1=\frac{(277.3\text{ g})(-4.887\text{ m/s}+2.277\text{ m/s}-2(-3.789\text{ m/s}))}{2.277\text{ m/s}-(-4.887\text{ m/s})}=192.3\text{ g}$$

7.116. **THINK:** For this problem, it will help to think about the components of the momentum that are perpendicular to and parallel to the wall. After the collision, the momentum parallel to the wall is unchanged The perpendicular component is in the opposite direction and is multiplied by the coefficient of restitution after the collision with the wall.

SKETCH: Show the path of the racquetball before and after it hits the wall.



RESEARCH: The mass and initial speed can be used to calculate the initial momentum $\vec{p}_i = m\vec{v}$. The angle at which the racquetball hits the wall is used to calculate the parallel and perpendicular components from the initial momentum: $p_{i,\perp} = p_i \cos\theta_i$ and $p_{i,\|} = p_i \sin\theta_i$. The component of the momentum parallel to the wall is unchanged in the collision, so $p_{f,\|} = p_{i,\|}$. The component of the final momentum perpendicular to the wall has a magnitude equal to the coefficient of restitution times the component of the initial momentum parallel to the wall: $p_{f,\perp} = \varepsilon p_{i,\perp}$ in the opposite direction from $p_{i,\perp}$. With a little trigonometry, the final angle can be calculated from the perpendicular and parallel components of the final momentum: $\tan\theta_f = \dfrac{p_{f,\|}}{p_{f,\perp}}$.

SIMPLIFY: Since $\tan\theta_f = \dfrac{p_{f,\|}}{p_{f,\perp}}$, take the inverse tangent to find $\theta_f = \tan^{-1}\left(\dfrac{p_{f,\|}}{p_{f,\perp}}\right)$. Substitute $p_{f,\|} = p_{i,\|} = p_i \sin\theta_i$ and $p_{f,\perp} = \varepsilon p_i \cos\theta_i$ into the equation to get:

$$\theta_f = \tan^{-1}\left(\frac{p_i \sin\theta_i}{\varepsilon p_i \cos\theta_i}\right) = \tan^{-1}\left(\frac{1}{\varepsilon}\cdot\frac{\sin\theta_i}{\cos\theta_i}\right) = \tan^{-1}\left(\frac{1}{\varepsilon}\cdot\tan\theta_i\right)$$

CALCULATE: The exercise states that the initial angle is 43.53° and the coefficient of restitution is 0.8199. Using those values, the final angle is:

$$\theta_f = \tan^{-1}\left(\frac{\tan\theta_i}{\varepsilon}\right) = \tan^{-1}\left(\frac{\tan 43.53^\circ}{0.8199}\right) = 49.20289058^\circ$$

ROUND: The angle and coefficient of restitution are the only measured values used in these calculations, and both are given to four significant figures, so the final answer should also have four significant figures. The racquetball rebounds at an angle of 49.20° from the normal.

DOUBLE-CHECK: This answer is physically realistic. The component of the momentum does not change, but the perpendicular component is reduced by about one fifth, so the angle should increase. To double check the calculations, use the speed and mass of the racquetball to find the initial and final momentum: $p_{i,\|} = mv_{i,\|} = m\vec{v}_i \sin\theta_i$ and $p_{i,\perp} = mv_{i,\perp} = m\vec{v}_i \cos\theta_i$. Thus $p_{i,\|} = 437.9416827 \text{ g}\cdot\text{m/s}$ and

$p_{\text{i},\perp} = 461.0105692 \text{ g}\cdot\text{m/s}$. The parallel portion of the momentum is unchanged, and the perpendicular portion is the coefficient of restitution times the initial perpendicular momentum, giving a final parallel component of $p_{\text{f},\parallel} = 437.9416827 \text{ g}\cdot\text{m/s}$ and $p_{\text{f},\perp} = 377.9825657 \text{ g}\cdot\text{m/s}$. The final angle can be computed as $\theta_\text{f} = \tan^{-1}\left(\dfrac{p_{\text{f},\parallel}}{p_{\text{f},\perp}}\right)$ or 49.20°, which confirms the calculations.

7.117. $\tan\theta_\text{f} = \dfrac{1}{\varepsilon}\tan\theta_\text{i}$

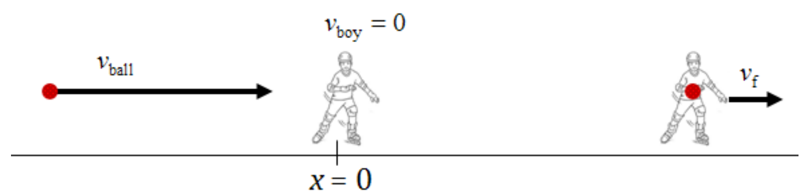
$$\varepsilon = \frac{\tan\theta_\text{i}}{\tan\theta_\text{f}} = \frac{\tan 48.67°}{\tan 55.75°} = 0.7742$$

7.118. $\tan\theta_\text{f} = \dfrac{1}{\varepsilon}\tan\theta_\text{i}$

$$\theta_\text{i} = \tan^{-1}\left(\varepsilon\tan\theta_\text{f}\right) = \tan^{-1}\left(0.8787\tan 57.24°\right) = 53.78°$$

7.119. **THINK:** When the boy catches the dodgeball, he holds on to it and does not let go. The boy and the ball stick together and have the same velocity after the collision, so this is a totally inelastic collision. This means that the final velocity of the boy and ball can be calculated from the initial velocities and masses of the boy and dodgeball.

SKETCH: Choose the x – axis to run in the same direction as the dodgeball, with the origin at the boy's initial location.



RESEARCH: In a totally inelastic collision, the final velocity of both objects is given by $v_\text{f} = \dfrac{m_{\text{ball}}v_{\text{ball}} + m_{\text{boy}}v_{\text{boy}}}{m_{\text{ball}} + m_{\text{boy}}}$.

SIMPLIFY: Because the initial velocity of the boy $v_{\text{boy}} = 0$, the equation can be simplified to $v_\text{f} = \dfrac{m_{\text{ball}}v_{\text{ball}} + m_{\text{boy}}\cdot 0}{m_{\text{ball}} + m_{\text{boy}}} = \dfrac{m_{\text{ball}}v_{\text{ball}}}{m_{\text{ball}} + m_{\text{boy}}}$. Since the mass of the ball is given in grams and the mass of the boy is given in kilograms, it is necessary to multiply the mass of the ball by a conversion factor of $\dfrac{1 \text{ kg}}{1000 \text{ g}}$.

CALCULATE: The mass of the ball is 511.1 g, or $511.1 \text{ g}\cdot\dfrac{1 \text{ kg}}{1000 \text{ g}} = 0.5111 \text{ kg}$. The mass of the boy is 48.95 kg and the initial velocity of the dodgeball is 23.63 m/s. The final velocity is

$$\begin{aligned} v_\text{f} &= \frac{m_{\text{ball}}v_{\text{ball}}}{m_{\text{ball}} + m_{\text{boy}}} \\ &= \frac{0.5111 \text{ kg}\cdot 23.63 \text{ m/s}}{48.95 \text{ kg} + 0.5111 \text{ kg}} \\ &= 0.2441776062 \text{ m/s}. \end{aligned}$$

ROUND: The measured values in this problem are given to four significant figures, and the sum of the masses also has four significant figures, so the final answer should also have four significant figures. The final velocity of the boy and dodg ball is 0.2442 m/s in the same direction that the dodgeball was traveling initially.

DOUBLE-CHECK: This answer makes sense. The mass of the boy is much greater than the mass of the dodgeball, so a smaller speed of this massive system (boy plus dodgeball) will have the same momentum as the ball traveling much faster. To confirm that the answer is correct, check that the momentum after the collision is equal to the momentum before the collision. Before the collision, the boy is not moving so he has no momentum, and the dodgeball has a momentum of $p_x = mv_x = 0.511 \text{ kg} \cdot 23.63 \text{ m/s}$ or 12.075 kg · m/s. After the collision, the total momentum is $mv_x = (0.511 \text{ kg} + 48.95 \text{ kg}) \cdot 0.2442 \text{ m/s}$ or 12.078 kg · m/s. These agree within rounding error, so this confirms that the original calculation was correct.

7.120.

$$v_\text{f} = \frac{m_\text{ball} v_\text{ball}}{m_\text{ball} + m_\text{boy}}$$

$$v_\text{f}\left(m_\text{ball} + m_\text{boy}\right) = m_\text{ball} v_\text{ball}$$

$$v_\text{ball} = \frac{v_\text{f}\left(m_\text{ball} + m_\text{boy}\right)}{m_\text{ball}} = \frac{(0.2304 \text{ m/s})(0.5131 \text{ kg} + 53.53 \text{ kg})}{0.5131 \text{ kg}} = 24.27 \text{ m/s}$$

7.121.

$$v_\text{f} = \frac{m_\text{ball} v_\text{ball}}{m_\text{ball} + m_\text{boy}}$$

$$v_\text{f}\left(m_\text{ball} + m_\text{boy}\right) = m_\text{ball} v_\text{ball}$$

$$v_\text{f} m_\text{ball} + v_\text{f} m_\text{boy} = m_\text{ball} v_\text{ball}$$

$$m_\text{boy} = \frac{m_\text{ball} v_\text{ball} - v_\text{f} m_\text{ball}}{v_\text{f}} = m_\text{ball} \frac{v_\text{ball} - v_\text{f}}{v_\text{f}} = (0.5151 \text{ kg}) \frac{24.91 \text{ m/s} - 0.2188 \text{ m/s}}{0.2188 \text{ m/s}} = 58.13 \text{ kg}$$

Chapter 8: Systems of Particles and Extended Objects

Concept Checks

8.1. b **8.2.** a **8.3.** d **8.4.** b **8.5.** a

Multiple-Choice Questions

8.1. d **8.2.** b **8.3.** d **8.4.** b and d **8.5.** e **8.6.** a **8.7.** b **8.8.** d **8.9.** b **8.10.** e **8.11.** a **8.12.** c **8.13.** a **8.14.** b **8.15.** b **8.16.** b

Conceptual Questions

8.17. It is reasonable to assume the explosion is entirely an internal force. This means the momentum, and hence the velocity of the center of mass remains unchanged. Therefore, the motion of the center of mass remains the same.

8.18. The length of the side of the cube is given as *d*. If the cubes have a uniform mass distribution, then the center of mass of each cube is at its geometric center. Let *m* be the mass of a cube. The coordinates of the center of mass of the structure are given by:

$$X_{\text{cm}} = \frac{m\left(\frac{d}{2}+\frac{d}{2}+\frac{d}{2}+\frac{3d}{2}\right)}{4m} = \frac{3d}{4},\quad Y_{\text{cm}} = \frac{m\left(\frac{d}{2}+\frac{d}{2}+\frac{d}{2}+\frac{3d}{2}\right)}{4m} = \frac{3d}{4}\quad \text{and}\quad Z_{\text{cm}} = \frac{m\left(\frac{d}{2}+\frac{d}{2}+\frac{d}{2}+\frac{3d}{2}\right)}{4m} = \frac{3d}{4}.$$

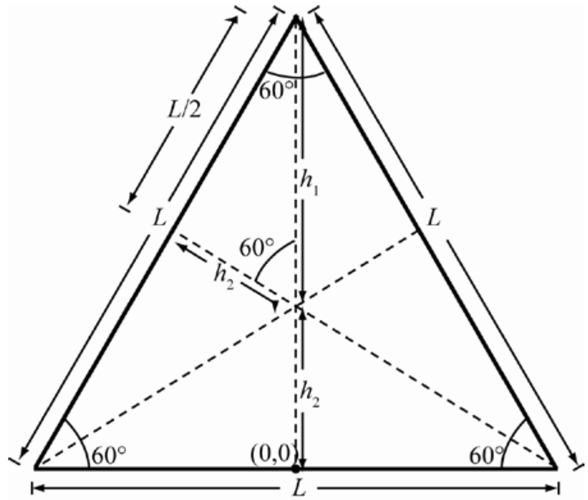
Therefore, the center of mass of the structure is located at $\vec{R} = \left(X_{\text{cm}},\ Y_{\text{cm}},\ Z_{\text{cm}}\right) = \left(\frac{3d}{4},\frac{3d}{4},\frac{3d}{4}\right).$

8.19. After the explosion, the motion of the center of mass should remain unchanged. Since both masses are equal, they must be equidistant from the center of mass. If the first piece has *x*-coordinate x_1 and the second piece has *x*-coordinate x_2, then $\left|X_{\text{cm}} - x_1\right| = \left|X_{\text{cm}} - x_2\right|$. For example, since the position of the center of mass is still 100 m, one piece could be at 90 m and the other at 110 m: $\left|100-90\right| = \left|100-110\right|$.

8.20. Yes, the center of mass can be located outside the object. Take a donut for example. If the donut has a uniform mass density, then the center of mass is located at its geometric center, which would be the center of a circle. However, at the donut's center, there is no mass, there is a hole. This means the center of mass can lie outside the object.

8.21. It is possible if, for example, there are outside forces involved. The kinetic energy of an object is proportional to the momentum squared $\left(K \propto p^2\right)$. So if *p* increases, *K* increases.

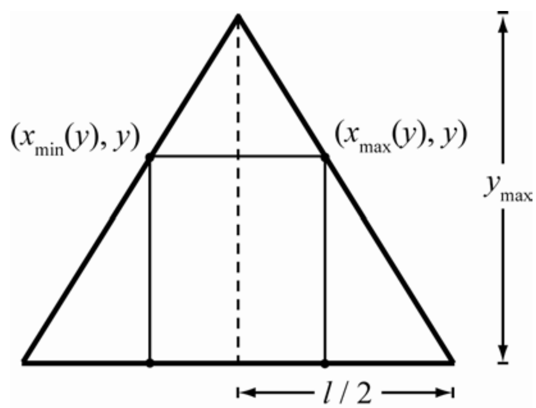
8.22. The intersection of the triangle's altitudes implies the triangle has a uniform mass density, meaning the center of mass is at the geometric center. To show this point by physical reasoning means using geometry to show where it is.



It can be seen that $h_1 \sin 60° = L/2$ and $h_1 \cos 60° = h_2$. Therefore,

$$h_1 = \frac{L}{2\sin 60°} = \frac{L}{2\left(\sqrt{3}/2\right)} = \frac{L}{\sqrt{3}} = \frac{L\sqrt{3}}{3} \Rightarrow h_2 = \frac{L\sqrt{3}}{3}\cos 60° = \frac{L\sqrt{3}}{3}\left(\frac{1}{2}\right) = \frac{L\sqrt{3}}{6}.$$

If the center of the bottom side of the triangle is (0, 0), then the center of mass is located at $\left(0, h_2\right) = \left(0, L\sqrt{3}/6\right)$. To calculate by direct measurement, note that due to symmetry by the choice of origin, the *x* coordinate of the center of mass is in the middle of the *x* axis. Therefore, $X_{\text{cm}} = 0$, which means only Y_{cm} must be determined.



Clearly, the *x* value of a point along the side of the triangle is dependent on the value of *y* for that point, meaning *x* is a function of *y*. When *y* is zero, *x* is *L*/2 and when *x* is zero, *y* is $y_{\max} = h_1 + h_2 = L\sqrt{3}/2$. The change in *x* should be linear with change in *y*, so $x = my + b$, where $m = \frac{\Delta x}{\Delta y} = \frac{\left(L/2\right) - 0}{0 - \left(L\sqrt{3}/2\right)} = -\frac{1}{\sqrt{3}}$.

Therefore, $\frac{L}{2} = -\frac{0}{\sqrt{3}} + b = 0 + b \Rightarrow b = \frac{L}{2}$ and $0 = -\frac{L\sqrt{3}}{2\sqrt{3}} + b = -\frac{L}{2} + b \Rightarrow b = \frac{L}{2}$. The equation for *x* is then given by $x\left(y\right) = -\frac{y}{\sqrt{3}} + \frac{L}{2}$. Since the mass density is uniform, the geometry of the triangle can be considered. $Y_{\text{cm}} = \frac{1}{A}\iint y\,dA$, where $A = \frac{L^2\sqrt{3}}{4}$ and $dA = dxdy$.

The integral then becomes:

$Y_{cm} = \frac{4}{L^2\sqrt{3}} \int_{y_{min}}^{y_{max}} y dy \int_{x_{min}(y)}^{x_{max}(y)} dx = \frac{4}{L^2\sqrt{3}} \int_0^{\frac{L\sqrt{3}}{2}} y\left(x_{max}(y) - x_{min}(y)\right)dy.$ Due to symmetry, $x_{max}(y) = -x_{min}(y)$ and $x_{max}(y) = x(y)$. Therefore,

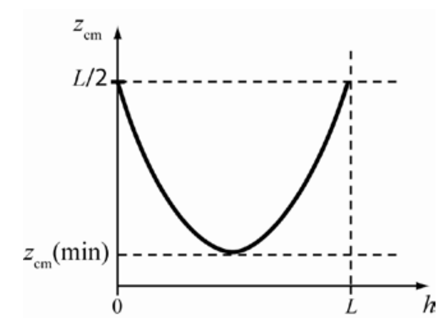
$$Y_{cm} = \frac{8}{L^2\sqrt{3}} \int_0^{\frac{L\sqrt{3}}{2}} yx(y)dy = \frac{8}{L^2\sqrt{3}} \int_0^{\frac{L\sqrt{3}}{2}} \left(\frac{-y^2}{\sqrt{3}} + \frac{yL}{2}\right)dy$$

$$= \frac{8}{L^2\sqrt{3}} \left[\frac{y^2L}{4} - \frac{y^3}{3\sqrt{3}}\right]_0^{\frac{L\sqrt{3}}{2}} = \frac{8}{L^2\sqrt{3}} \left[\frac{3L^3}{16} - \frac{L^3}{8}\right] = \frac{8}{L^2\sqrt{3}} \left[\frac{L^3}{16}\right]$$

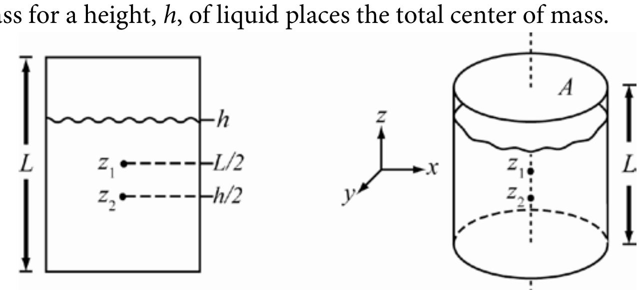
$$= \frac{L}{2\sqrt{3}} = \frac{L\sqrt{3}}{6}.$$

The center of mass is located at $R = (X_{cm}, Y_{cm}) = \left(0, \frac{L\sqrt{3}}{6}\right)$. This is consistent with reasoning by geometry.

8.23. (a) The empty can and the liquid should each have their centers of mass at their geometric centers, so initially the center of mass of both is at the center of the can (assuming the can is filled completely with soda). Assuming the liquid drains out uniformly, only the height changes and the cross sectional area remains constant, so the center of mass is initially at $L/2$ and changes only in height. As liquid drains, its mass M will drop by ΔM but the mass of the can, m, remains the same. As liquid drains, its center of mass will also fall such that if the liquid is at a height h, $0 < h < L$, its center of mass is at $h/2$. As long as $M - \Delta M > m$, the center of mass of both will also fall to some height h', $h/2 < h' < L$. Once $M - \Delta M < m$, the center of mass of both will begin to increase again until $M - \Delta M = 0$ and the center of mass is that of just the can at $L/2$. A sketch of the height of the center of mass of both as a function of liquid height is shown below.



(b) In order to determine the minimum value of the center of mass in terms of L, M and m, first consider where the center of mass for a height, h, of liquid places the total center of mass.



Z_1 is the center of mass of the can. Z_2 is the center of mass of the liquid. Notice the center of mass moves along the z axis only. A is the cross sectional area of the can in the xy plane. ρ_M is the density of the liquid. h is the height of the liquid.

The coordinate of the center of mass is given by

$$Z_{\text{cm}} = \frac{\frac{mL}{2} + \frac{Mh}{2}}{m+M}.$$

When $h = L$, $Z_{\text{cm}} = L/2$. When $h < L$, $h = \alpha L$, where $0 \le \alpha < 1$. In other words, the height of the liquid is a fraction, α, of the initial height, L. Initially the mass of the liquid is $M = \rho V = \rho AL$. When $h(\alpha) = \alpha L$, the mass of the liquid is $M(\alpha) = \rho A h(\alpha) = \alpha \rho AL = \alpha M$. This means the center of mass for some value of α is

$$Z_{\text{cm}}(\alpha) = \frac{\frac{mL}{2} + \frac{M(\alpha)h(\alpha)}{2}}{m + M(\alpha)} = \frac{\frac{mL}{2} + \frac{\alpha^2 ML}{2}}{m + \alpha M} = \frac{L}{2}\left(\frac{1+b\alpha^2}{1+b\alpha}\right).$$

where $b = M/m$ and M is the initial mass of the liquid. In order to determine the minimum value of Z_{cm}, $Z_{\text{cm}}(\alpha)$ must be minimized in terms of α to determine where α_{min} occurs and then determine $Z_{\text{cm}}(\alpha_{\text{min}})$.

$$\frac{dZ_{\text{cm}}(\alpha)}{d\alpha} = a\frac{d}{d\alpha}\left(\frac{1+b\alpha^2}{1+b\alpha}\right) = a\left[\frac{b^2\alpha^2 + 2b\alpha - b}{(1+b\alpha)^2}\right], \text{ where } a = L/2.$$

When $dZ_{\text{cm}}(\alpha)/d\alpha = 0 \Rightarrow b^2\alpha^2 + 2b\alpha - b = 0$. Using the quadratic equation, $\alpha = \dfrac{-1 \pm \sqrt{1+b}}{b}$. Since $b > 0$ and $\alpha > 0$, $\alpha_{\text{min}} = \dfrac{-1+\sqrt{1+b}}{b}$. Therefore, $Z_{\text{cm}}(\alpha_{\text{min}}) = a\left(\dfrac{1+b\alpha_{\text{min}}{}^2}{1+b\alpha_{\text{min}}}\right) = 2a\left(\dfrac{1+b-\sqrt{1+b}}{b\sqrt{1+b}}\right)$.

$$Z_{\text{cm}}(\alpha_{\text{min}}) = \frac{L\left(M + m - m\sqrt{1+\frac{M}{m}}\right)}{M\sqrt{1+\frac{M}{m}}}$$

If it is assumed that soda has a similar density to water and the can is made of aluminum, then the ratio of $M/m \approx 30$, giving a minimum Z_{cm} of about $L/6$.

8.24. (a) If the astronaut throws both at the same time, he gains their momentum of them moving at a velocity, v. If he throws one first at a velocity, v, he will recoil back at a velocity, v'. So when he throws the second item, he will gain its momentum at a velocity of $v - v'$, which is less than v. So he gains less momentum from throwing the second item after the first than if he throws both items at the same time. Therefore, he obtains maximum speed when he throws both at the same time.

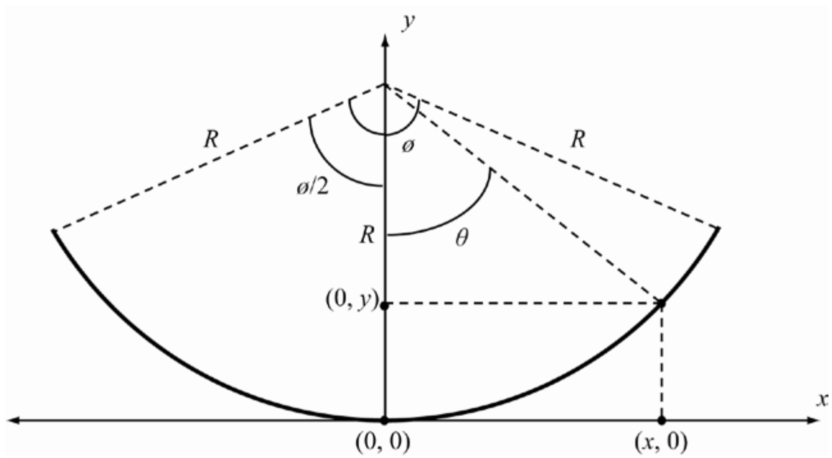
(b) If the astronaut throws the heavier object (tool box) first, it will give the astronaut a large velocity, v', so when he throws the lighter object (hammer), it will have a small velocity of $v - v'$. So its momentum contribution will be very small. However, if he throws the lighter item first, v' will be smaller in this scenario, so the momentum of the box will be dependent on $v - v'$, which is greater and contributes a large amount of momentum to the astronaut, giving him a larger velocity. Therefore, throwing the lighter object first will maximize his velocity.

(c) The absolute maximum velocity is when both items are thrown at the same time. Initially the momentum is zero and after the toss, the astronaut travels with velocity, v' and the box and hammer travel with velocity, v in the opposite direction.

$$\vec{p}_{\text{i}} = \vec{p}_{\text{f}} \Rightarrow 0 = Mv' - \left(\frac{M}{2} + \frac{M}{4}\right)v \Rightarrow v' = \frac{3}{4}v$$

Therefore, the maximum velocity is ¾ of the velocity at which he throws the two items.

8.25. Let the angle θ sweep through from $-\phi/2$ to $\phi/2$. Keeping R constant as θ increases, the length of the rod, $l = R\theta$, increases and in turn the mass, $m = \lambda l$, increases. Since the mass is uniformly distributed, the center of mass should be in the same location. So rather than bending a rod of constant length where θ and R change, keep R constant and change θ and l. Use Cartesian coordinates to determine the center of mass. Since the center of mass is a function of θ, it must be determined how the coordinates change with the angle θ.



$y = R - R\cos\theta,\ \ x = R\sin\theta,\ \ m = \lambda R\phi,\ \ dm = \lambda R\,d\theta$

$$X_{\text{cm}} = \frac{1}{m}\int x\,dm = \frac{1}{\lambda R\phi}\int_{-\phi/2}^{\phi/2} R\sin\theta\lambda R\,d\theta = \frac{R}{\phi}\int_{-\phi/2}^{\phi/2}\sin\theta\,d\theta = \left[-\frac{R}{\phi}\cos\theta\right]_{-\frac{\phi}{2}}^{\frac{\phi}{2}} = -\frac{R}{\phi}\left(\cos\frac{\phi}{2} - \cos\left(-\frac{\phi}{2}\right)\right) = 0$$

$$Y_{\text{cm}} = \frac{1}{m}\int y\,dm = \frac{1}{\lambda R\phi}\int_{-\phi/2}^{\phi/2}(R - R\cos\theta)\lambda R\,d\theta = \frac{R}{\phi}\int_{-\phi/2}^{\phi/2}(1-\cos\theta)d\theta = \left[\frac{R}{\phi}(\theta - \sin\theta)\right]_{-\frac{\phi}{2}}^{\frac{\phi}{2}}$$

$$= \frac{R}{\phi}\left(\frac{\phi}{2} - \left(-\frac{\phi}{2}\right)\right) - \frac{R}{\phi}\left(\sin\left(\frac{\phi}{2}\right) - \sin\left(-\frac{\phi}{2}\right)\right) = R - \frac{2R\sin\left(\frac{\phi}{2}\right)}{\phi}$$

$$\vec{R}_{\text{cm}} = (X_{\text{cm}}, Y_{\text{cm}}) = \left(0, R - \frac{2R\sin(\phi/2)}{\phi}\right)$$

8.26. As eggs *A*, *B* and/or *C* are removed, the center of mass will shift down and to the left. To determine the overall center of mass, use the center of the eggs as their center position, such that eggs *A*, *B* and *C* are located respectively at

$$\left(\frac{d}{2},\frac{d}{2}\right),\ \left(\frac{3d}{2},\frac{d}{2}\right),\ \left(\frac{5d}{2},\frac{d}{2}\right).$$

Since all of the eggs are of the same mass, *m,* and proportional to *d*, *m* and *d* can be factored out of the equations for X_{cm} and Y_{cm}.

(a) $X_{\text{cm}} = \dfrac{md}{11m}\left(2\left(-\dfrac{5}{2}\right) + 2\left(-\dfrac{3}{2}\right) + 2\left(-\dfrac{1}{2}\right) + \dfrac{1}{2} + 2\left(\dfrac{3}{2}\right) + 2\left(\dfrac{5}{2}\right)\right) = -\dfrac{d}{22},\ Y_{\text{cm}} = \dfrac{md}{11m}\left(6\left(-\dfrac{1}{2}\right) + 5\left(\dfrac{1}{2}\right)\right) = -\dfrac{d}{22}$

$$\vec{R}_{\text{cm}} = \left(-\frac{d}{22}, -\frac{d}{22}\right)$$

(b) $X_{\text{cm}} = \frac{md}{11m}\left(2\left(-\frac{5}{2}\right)+2\left(-\frac{3}{2}\right)+2\left(-\frac{1}{2}\right)+2\left(\frac{1}{2}\right)+\frac{3}{2}+2\left(\frac{5}{2}\right)\right)=-\frac{3d}{22}$, $Y_{\text{cm}} = \frac{md}{11m}\left(6\left(-\frac{1}{2}\right)+5\left(\frac{1}{2}\right)\right)=-\frac{d}{22}$

$\vec{R}_{\text{cm}} = \left(-\frac{3d}{22},-\frac{d}{22}\right)$

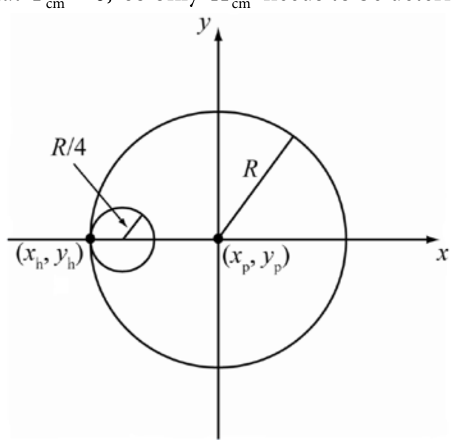
(c) $X_{\text{cm}} = \frac{md}{11m}\left(2\left(-\frac{5}{2}\right)+2\left(-\frac{3}{2}\right)+2\left(-\frac{1}{2}\right)+2\left(\frac{1}{2}\right)+2\left(\frac{3}{2}\right)+\frac{5}{2}\right)=-\frac{5d}{22}$, $Y_{\text{cm}} = \frac{md}{11m}\left(6\left(-\frac{1}{2}\right)+5\left(\frac{1}{2}\right)\right)=-\frac{d}{22}$

$\vec{R}_{\text{cm}} = \left(-\frac{5d}{22},-\frac{d}{22}\right)$

(d) $X_{\text{cm}} = \frac{md}{9m}\left(2\left(-\frac{5}{2}\right)+2\left(-\frac{3}{2}\right)+2\left(-\frac{1}{2}\right)+\frac{1}{2}+\frac{3}{2}+\frac{5}{2}\right)=-\frac{d}{2}$, $Y_{\text{cm}} = \frac{md}{9m}\left(6\left(-\frac{1}{2}\right)+3\left(\frac{1}{2}\right)\right)=-\frac{d}{6}$

$\vec{R}_{\text{cm}} = \left(-\frac{d}{2},-\frac{d}{6}\right)$

8.27. The center of the pizza is at $(0,0)$ and the center of the piece cut out is at $(-3R/4,0)$. Assume the pizza and the hole have a uniform mass density (though the hole is considered to have a negative mass). Then the center of mass can be determined from geometry. Also, because of symmetry of the two circles and their *y* position, it can be said that $Y_{\text{cm}} = 0$, so only X_{cm} needs to be determined.



$A_{\text{p}} = \pi R^2$, $A_{\text{h}} = \pi\left(\frac{R}{4}\right)^2 = \pi\frac{R^2}{16}$, $(x_{\text{p}},y_{\text{p}}) = (0,0)$, $(x_{\text{h}},y_{\text{h}}) = \left(-\frac{3}{4}R,0\right)$

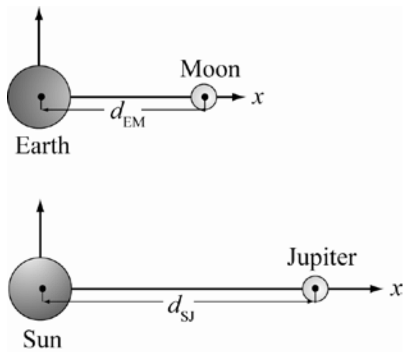
$$X_{\text{cm}} = \frac{x_{\text{p}}A_{\text{p}} - x_{\text{h}}A_{\text{h}}}{A_{\text{p}} - A_{\text{h}}} = \frac{0-\left(-\frac{3}{4}R\right)\left(\frac{\pi R^2}{16}\right)}{\pi R^2 - \frac{\pi R^2}{16}} = \frac{R}{20}, \quad \vec{R}_{\text{cm}} = \left(\frac{R}{20},0\right)$$

8.28. Since the overall mass of the hourglass does not change and the center of mass must move from the top half to the bottom half, then the center of mass velocity, v_{cm}, must be non-zero and pointing down. As the sand flows from the top part of the hourglass to the lower part, v_{cm} changes with time. The magnitude of v_{cm} is larger when the sand has just started to flow than just before all the sand has flowed through. Thus $dv_{\text{cm}}/dt = a_{\text{cm}}$ must be in the opposite direction from v_{cm}, which is the upward direction. The scale must supply the force required to produce this upward acceleration, so the hourglass weighs more when the sand is flowing than when the sand is stationary. You can find a published solution to a similar version of this problem at the following reference: K.Y. Shen and Bruce L. Scott, American Journal of Physics, **53**, 787 (1985).

Exercises

8.29. **THINK:** Determine (a) the distance, d_1, from the center of mass of the Earth-Moon system to the geometric center of the Earth and (b) the distance, d_2, from the center of mass of the Sun-Jupiter system to the geometric center of the Sun. The mass of the Earth is approximately $m_{\text{E}} = 5.9742 \cdot 10^{24}$ kg and the mass of the Moon is approximately $m_{\text{M}} = 7.3477 \cdot 10^{22}$ kg. The distance between the center of the Earth to the center of the Moon is $d_{\text{EM}} = 384,400$ km. Also, the mass of the Sun is approximately $m_{\text{S}} = 1.98892 \cdot 10^{30}$ kg and the mass of Jupiter is approximately $m_{\text{J}} = 1.8986 \cdot 10^{27}$ kg. The distance between the center of the Sun and the center of Jupiter is $d_{\text{SJ}} = 778,300,000$ km.

SKETCH:



RESEARCH: Determine the center of mass of the two object system from $\vec{R} = \dfrac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2}$. By considering the masses on the *x*-axis (as sketched), the one dimensional equation can be used for *x*. Assuming a uniform, spherically symmetric distribution of each planet's mass, they can be modeled as point particles. Finally, by placing the Earth (Sun) at the origin of the coordinate system, the center of mass will be determined with respect to the center of the Earth (Sun), i.e. $d_1\left(d_2\right) = x$.

SIMPLIFY:

(a) $d_1 = x = \dfrac{x_1 m_{\text{E}} + x_2 m_{\text{M}}}{m_{\text{E}} + m_{\text{M}}} = \dfrac{d_{\text{EM}} m_{\text{M}}}{m_{\text{E}} + m_{\text{M}}}$

(b) $d_2 = x = \dfrac{x_1 m_{\text{S}} + x_2 m_{\text{J}}}{m_{\text{S}} + m_{\text{J}}} = \dfrac{d_{\text{SJ}} m_{\text{J}}}{m_{\text{S}} + m_{\text{J}}}$

CALCULATE:

(a) $d_1 = \dfrac{\left(384,400 \text{ km}\right)\left(7.3477 \cdot 10^{22} \text{ kg}\right)}{\left(5.9742 \cdot 10^{24} \text{ kg}\right) + \left(7.3477 \cdot 10^{22} \text{ kg}\right)} = \dfrac{2.8244559 \cdot 10^{28} \text{ km} \cdot \text{kg}}{6.047677 \cdot 10^{24} \text{ kg}} = 4670.3 \text{ km}$

(b) $d_2 = \dfrac{\left(7.783 \cdot 10^{8} \text{ km}\right)\left(1.8986 \cdot 10^{27} \text{ kg}\right)}{\left(1.98892 \cdot 10^{30} \text{ kg}\right) + \left(1.8986 \cdot 10^{27} \text{ kg}\right)} = 742247.6 \text{ km}$

ROUND:

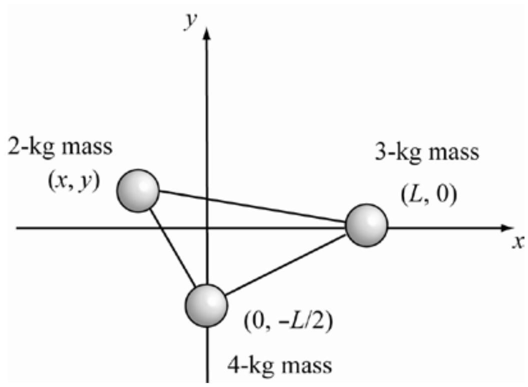
(a) d_{EM} has four significant figures, so $d_1 = 4670.$ km.

(b) d_{SJ} has four significant figures, so $d_2 = 742,200$ km.

DOUBLE-CHECK: In each part, the distance d_1 / d_2 is much less than half the separation distance $d_{\text{EM}} / d_{\text{SJ}}$. This makes sense as the center of mass should be closer to the more massive object in the two body system.

8.30. **THINK:** The center of mass coordinates for the system are $(L/4,-L/5)$. The masses are $m_1 = 2$ kg, $m_2 = 3$ kg and $m_3 = 4$ kg. The coordinates for m_2 are $(L,0)$ and the coordinates for m_3 are $(0,-L/2)$. Determine the coordinates for m_1.

SKETCH:



RESEARCH: The *x* and *y* coordinates for m_1 can be determined from the equations for the center of mass in each dimension:

$$X = \frac{1}{M}\sum_{i=1}^{n} x_i m_i \text{ and } Y = \frac{1}{M}\sum_{i=1}^{n} y_i m_i.$$

SIMPLIFY: $X = \dfrac{x_1 m_1 + x_2 m_2 + x_3 m_3}{m_1 + m_2 + m_3} \Rightarrow x_1 = \dfrac{1}{m_1}\left(X\left(m_1 + m_2 + m_3\right) - x_2 m_2 - x_3 m_3\right)$

Similarly, $y_1 = \dfrac{1}{m_1}\left(Y\left(m_1 + m_2 + m_3\right) - y_2 m_2 - y_3 m_3\right).$

CALCULATE: $x_1 = \left(\dfrac{1}{2\text{ kg}}\right)\left(\dfrac{L}{4}\left(2\text{ kg} + 3\text{ kg} + 4\text{ kg}\right) - L\left(3\text{ kg}\right) - 0\left(4\text{ kg}\right)\right) = -\dfrac{3}{8}L$

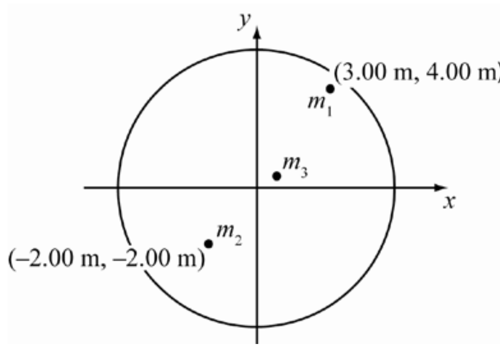
$y_1 = \left(\dfrac{1}{2\text{ kg}}\right)\left(-\dfrac{L}{5}\left(2\text{ kg} + 3\text{ kg} + 4\text{ kg}\right) - 0\left(3\text{ kg}\right) - \left(-\dfrac{L}{2}\right)\left(4\text{ kg}\right)\right) = \dfrac{1}{10}L$

ROUND: Rounding is not necessary since the initial values and the results are fractions, so m_1 is located at $(-3L/8, L/10)$.

DOUBLE-CHECK: The coordinates for m_1 are reasonable: since X_{cm} is positive and Y_{cm} is negative and both coordinates have comparatively small values (and thus the center of mass is close to the origin), it makes sense that *x* will be negative to balance the 3-kg mass and *y* will be positive to balance the 4-kg mass.

8.31. **THINK:** The mass and location of the first acrobat are known to be $m_1 = 30.0$ kg and $\vec{r}_1 = (3.00\text{ m}, 4.00\text{ m})$. The mass and location of the second acrobat are $m_2 = 40.0$ kg and $\vec{r}_2 = (-2.00\text{ m}, -2.00\text{ m})$. The mass of the third acrobat is $m_3 = 20.0$ kg. Determine the position of the third acrobat, $\vec{r}_3$, when the center of mass (com) is at the origin.

SKETCH:



RESEARCH: Let *M* be the sum of the three masses. The coordinates of m_3 can be determined from the center of mass equations for each dimension,

$$X = \frac{1}{M}\sum_{i=1}^{n} x_i m_i \text{ and } Y = \frac{1}{M}\sum_{i=1}^{n} y_i m_i.$$

SIMPLIFY: Since *X* = 0, $X = \frac{1}{M}(x_1 m_1 + x_2 m_2 + x_3 m_3) = 0 \Rightarrow x_3 = \frac{(-x_1 m_1 - x_2 m_2)}{m_3}$. Similarly, with *Y* = 0,

$$y_3 = \frac{(-y_1 m_1 - y_2 m_2)}{m_3}.$$

CALCULATE: $x_3 = \frac{(-(3.00 \text{ m})(30.0 \text{ kg}) - (-2.00 \text{ m})(40.0 \text{ kg}))}{20.0 \text{ kg}} = -0.500 \text{ m},$

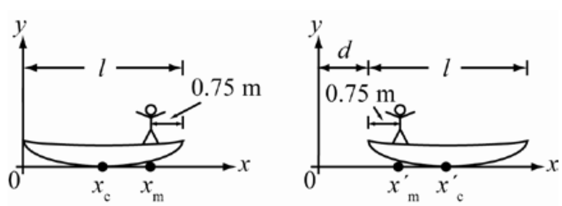
$$y_3 = \frac{(-(4.00 \text{ m})(30.0 \text{ kg}) - (-2.00 \text{ m})(40.0 \text{ kg}))}{20.0 \text{ kg}} = -2.00 \text{ m}$$

ROUND: $\vec{r}_3 = (-0.500 \text{ m}, -2.00 \text{ m})$

DOUBLE-CHECK: The resulting location is similar to the locations of the other acrobats.

8.32. **THINK:** The man's mass is $m_{\text{m}} = 55$ kg and the canoe's mass is $m_{\text{c}} = 65$ kg. The canoe's length is *l* = 4.0 m. The man moves from 0.75 m from the back of the canoe to 0.75 m from the front of the canoe. Determine how far the canoe moves, *d*.

SKETCH:



RESEARCH: The center of mass position for the man and canoe system does not change in our external reference frame. To determine *d*, the center of mass location must be determined before the canoe moves. Then the new location for the canoe after the man moves can be determined given the man's new position and the center of mass position. Assume the canoe has a uniform density such that its center of mass location is at the center of the canoe, $x_{\text{c}} = 2.0$ m. The man's initial position is $x_{\text{m}} = l - 0.75 \text{ m} = 3.25$ m. After moving, the canoe is located at x'_{c} and the man is located at $x'_{\text{m}} = x'_{\text{c}} + a$. *a* is the relative position of the man with respect to the canoe's center of mass and $a = -l/2 + 0.75 \text{ m} = -1.25$ m. Then the distance the canoe moves is $d = x'_{\text{c}} - x_{\text{c}}$.

SIMPLIFY:

$$X = \frac{1}{M}\sum_{i=1}^{n} x_i m_i.$$

The center of mass is $X = \frac{1}{M}\left(x_\text{m} m_\text{m} + x_\text{c} m_\text{c}\right)$. After moving,

$X = \frac{1}{M}\left(x'_\text{m} m_\text{m} + x'_\text{c} m_\text{c}\right) = \frac{1}{M}\left(\left(x'_\text{c} + a\right) m_\text{m} + x'_\text{c} m_\text{c}\right)$. Since X does not change, the equations can be equated:

$$\frac{1}{M}\left(\left(x'_\text{c} + a\right) m_\text{m} + x'_\text{c} m_\text{c}\right) = \frac{1}{M}\left(x_\text{m} m_\text{m} + x_\text{c} m_\text{c}\right)$$

This implies $x_\text{m} m_\text{m} + x_\text{c} m_\text{c} = x'_\text{c} m_\text{m} + x'_\text{c} m_\text{c} + a m_\text{m} \;\Rightarrow\; x'_\text{c} = \dfrac{x_\text{m} m_\text{m} + x_\text{c} m_\text{c} - a m_\text{m}}{m_\text{m} + m_\text{c}}$.

CALCULATE: $x'_c = \dfrac{(3.25 \text{ m})(55.0 \text{ kg}) + (2.00 \text{ m})(65.0 \text{ kg}) - (-1.25 \text{ m})(55.0 \text{ kg})}{55.0 \text{ kg} + 65.0 \text{ kg}} = 3.1458 \text{ m}$

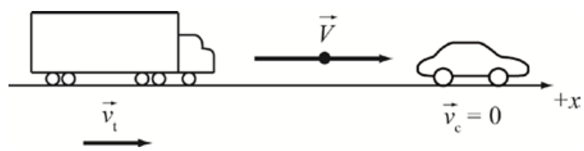
Then $d = 3.1458 \text{ m} - 2.00 \text{ m} = 1.1458 \text{ m}$.

ROUND: As each given value has three significant figures, d = 1.15 m.

DOUBLE-CHECK: This distance is less than the distance traveled by the man (2.5 m), as it should be to preserve the center of mass location.

8.33. **THINK:** The mass of the car is $m_\text{c} = 2.00 \text{ kg}$ and its initial speed is $v_\text{c} = 0$. The mass of the truck is $m_\text{t} = 3.50 \text{ kg}$ and its initial speed is $v_\text{t} = 4.00 \text{ m/s}$ toward the car. Determine (a) the velocity of the center of mass, $\vec{V}$, and (b) the velocities of the truck, $\vec{v}'_\text{t}$ and the car, $\vec{v}'_\text{c}$ with respect to the center of mass.

SKETCH:



RESEARCH:

(a) The velocity of the center of mass can be determined from $\vec{V} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{v}_i$.

Take $\vec{v}_\text{t}$ to be in the positive x-direction.

(b) Generally, the relative velocity, $\vec{v}\,'$, of an object with velocity, $\vec{v}$, in the lab frame is given by $\vec{v}\,' = \vec{v} - \vec{V}$, where $\vec{V}$ is the velocity of the relative reference frame. Note the speeds of the car and the truck relative to the center of mass do not change after their collision, but the relative velocities change direction; that is, $\vec{v}'_\text{t}(\text{before collision}) = -\vec{v}'_\text{t}(\text{after collision})$ and similarly for the car's relative velocity.

SIMPLIFY:

(a) Substituting $\vec{v}_\text{c} = 0$ and $M = m_\text{c} + m_\text{t}$, $\vec{V} = \frac{1}{M}\left(m_\text{c}\vec{v}_\text{c} + m_\text{t}\vec{v}_\text{t}\right)$ becomes $\vec{V} = \dfrac{\left(m_\text{t}\vec{v}_\text{t}\right)}{\left(m_\text{c} + m_\text{t}\right)}$.

(b) $\vec{v}'_\text{t}$ and $\vec{v}'_\text{c}$ before the collision are $\vec{v}'_\text{t} = \vec{v}_\text{t} - \vec{V}$ and $\vec{v}'_\text{c} = \vec{v}_\text{c} - \vec{V} = -\vec{V}$.

CALCULATE:

(a) $\vec{V} = \dfrac{(3.50 \text{ kg})(4.00\hat{x} \text{ m/s})}{(3.50 \text{ kg} + 2.00 \text{ kg})} = 2.545\hat{x} \text{ m/s}$

(b) $\vec{v}'_\text{t} = (4.00\hat{x} \text{ m/s}) - (2.545\hat{x} \text{ m/s}) = 1.4545\hat{x} \text{ m/s}$, $\vec{v}'_\text{c} = -2.545\hat{x} \text{ m/s}$

ROUND: There are three significant figures for each given value, so the results should be rounded to the same number of significant figures.

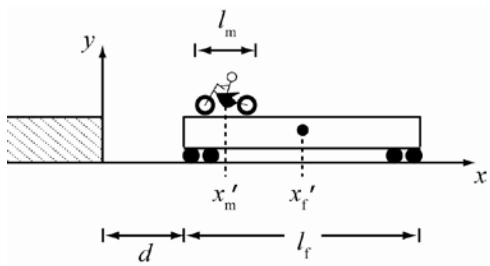
(a) $\vec{V} = 2.55\hat{x} \text{ m/s}$

(b) Before the collision, $\vec{v}_\mathrm{t}' = 1.45\hat{x}$ m/s and $\vec{v}_\mathrm{c}' = -2.55\hat{x}$ m/s. This means that after the collision, the velocities with respect to the center of mass become $\vec{v}_\mathrm{t}' = -1.45\hat{x}$ m/s and $\vec{v}_\mathrm{c}' = 2.55\hat{x}$ m/s.

DOUBLE-CHECK: $\vec{V}$ is between the initial velocity of the truck and the initial velocity of the car, as it should be.

8.34. **THINK:** The motorcycle with rider has a mass of $m_\mathrm{m} = 350.$ kg. The flatcar's mass is $m_\mathrm{f} = 1500.$ kg. The length of the motorcycle is $l_\mathrm{m} = 2.00$ m and the length of the flatcar is $l_\mathrm{f} = 20.0$ m. The motorcycle starts at one of end of the flatcar. Determine the distance, *d*, that the flatcar will be from the platform when the motorcycle reaches the end of the flatcar.

SKETCH: After the motorcycle and rider drive down the platform:



RESEARCH: The flatcar-motorcycle center of mass stays in the same position while the motorcycle moves. First, the center of mass must be determined before the motorcycle moves. Then the new location of the flatcar's center of mass can be determined given the center of mass for the system and the motorcycle's final position. Then the distance, *d*, can be determined. Assume that the motorcycle and rider's center of mass and the flatcar's center of mass are located at their geometric centers. Take the initial center of mass position for the motorcycle to be $x_\mathrm{m} = l_\mathrm{f} - l_\mathrm{m}/2$, and the initial center of mass for the flatcar to be $x_\mathrm{f} = l_\mathrm{f}/2$. The final position of the center of mass for the motorcycle will be $x_\mathrm{m}' = d + l_\mathrm{m}/2$, and the final position for the flatcar will be $x_\mathrm{f}' = d + l_\mathrm{f}/2$. Then *d* can be determined from

$$X = \frac{1}{M}\sum_{i=1}^{n} x_i m_i.$$

SIMPLIFY: Originally, $X = \frac{1}{M}\left(x_\mathrm{m}m_\mathrm{m} + x_\mathrm{f}m_\mathrm{f}\right)$. After the motorcycle moves, $X = \frac{1}{M}\left(x_\mathrm{m}'m_\mathrm{m} + x_\mathrm{f}'m_\mathrm{f}\right)$.

As the center of mass remains constant, the two expressions can be equated:

$$\frac{1}{M}\left(x_\mathrm{m}m_\mathrm{m} + x_\mathrm{f}m_\mathrm{f}\right) = \frac{1}{M}\left(x_\mathrm{m}'m_\mathrm{m} + x_\mathrm{f}'m_\mathrm{f}\right)$$

$$x_\mathrm{m}m_\mathrm{m} + x_\mathrm{f}m_\mathrm{f} = \left(d + \frac{1}{2}l_\mathrm{m}\right)m_\mathrm{m} + \left(d + \frac{1}{2}l_\mathrm{f}\right)m_\mathrm{f}$$

$$x_\mathrm{m}m_\mathrm{m} + x_\mathrm{f}m_\mathrm{f} = d\left(m_\mathrm{m} + m_\mathrm{f}\right) + \frac{1}{2}l_\mathrm{m}m_\mathrm{m} + \frac{1}{2}l_\mathrm{f}m_\mathrm{f}$$

$$d = \frac{\left(x_\mathrm{m} - \frac{1}{2}l_\mathrm{m}\right)m_\mathrm{m} + \left(x_\mathrm{f} - \frac{1}{2}l_\mathrm{f}\right)m_\mathrm{f}}{m_\mathrm{m} + m_\mathrm{f}}$$

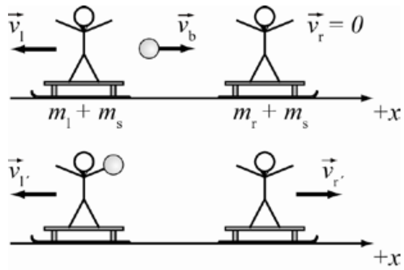
$x_\mathrm{m} = l_\mathrm{f} - \frac{l_\mathrm{m}}{2}$ and $x_\mathrm{f} = \frac{l_\mathrm{f}}{2}$, therefore $d = \frac{\left(l_\mathrm{f} - l_\mathrm{m}\right)m_\mathrm{m}}{m_\mathrm{m} + m_\mathrm{f}}$.

CALCULATE: $d = \frac{(20.0\text{ m} - 2.00\text{ m})(350.\text{ kg})}{350.\text{ kg} + 1500.\text{ kg}} = 3.4054\text{ m}$

ROUND: m_m has three significant figures, so the result should be rounded to d = 3.41 m.
DOUBLE-CHECK: It is reasonable that the distance moved is less than length of the flatcar.

8.35. **THINK:** The mass of the sled is $m_s = 10.0$ kg, the mass of the ball is $m_b = 5.00$ kg, and the mass of the student on the left is $m_l = 50.0$ kg. His relative ball-throwing speed is $v_{bl} = 10.0$ m/s. The mass of the student on the right is $m_r = 45.0$ kg and his relative ball-throwing speed is $v_{br} = 12.0$ m/s. Determine (a) the speed of the student on the left, v_l, after first throwing the ball, (b) the speed of the student on the right, v_r, after catching the ball, (c) the speed of the student on the left after catching the pass, v_l', and (d) the speed of the student on the right after throwing the pass, v_r'.

SKETCH:



RESEARCH: Momentum is conserved between each student and ball system. For each step, use $\vec{P}_i = \vec{P}_f$. In addition, the relative velocity of the ball is the difference between its velocity in the lab frame and the velocity of the student in the lab frame who has thrown it. That is, $\vec{v}_{bl} = \vec{v}_b - \vec{v}_l$ and $\vec{v}_{br} = \vec{v}_b - \vec{v}_r$. Recall each student begins at rest.

SIMPLIFY:

(a) Determine v_l after the ball is first thrown:

$$\vec{P}_i = \vec{P}_f \Rightarrow 0 = (m_s + m_l)\vec{v}_l + m_b\vec{v}_b \Rightarrow 0 = (m_s + m_l)\vec{v}_l + m_b(\vec{v}_{bl} + \vec{v}_l) \Rightarrow \vec{v}_l = -\frac{m_b\vec{v}_{bl}}{m_s + m_l + m_b}.$$

(b) Determine $\vec{v}_r$ after the student catches the ball. The velocity of the ball, $\vec{v}_b$, in the lab frame is needed. From part (a), $\vec{v}_l$ is known. Then $\vec{v}_b = \vec{v}_{bl} + \vec{v}_l$. So, $\vec{v}_b$ is known before it is caught. Now, for the student on the right catching the ball,

$$\vec{P}_i = \vec{P}_f \Rightarrow m_b\vec{v}_b = (m_b + m_r + m_s)\vec{v}_r \Rightarrow \vec{v}_r = \frac{m_b\vec{v}_b}{m_b + m_r + m_s}.$$

(c) Now the student on the right throws the ball and the student on the left catches it. To determine $\vec{v}_l'$, the velocity of the ball after it is thrown, $\vec{v}_b'$, is needed. It is known that $\vec{v}_{br} = \vec{v}_b - \vec{v}_r$. Then to determine $\vec{v}_b'$, consider the situation when the student on the right throws the ball. For the student on the right: $P_i = P_f \Rightarrow (m_s + m_r + m_b)\vec{v}_r = m_b\vec{v}_b' + (m_r + m_s)\vec{v}_r'$, where $\vec{v}_r$ is known from part (b) and $\vec{v}_{br} = \vec{v}_b' - \vec{v}_r' \Rightarrow \vec{v}_r' = \vec{v}_b' - \vec{v}_{br}$. Then, the fact that $(m_s + m_r + m_b)\vec{v}_r = m_b\vec{v}_b' + (m_r + m_s)(\vec{v}_b' - \vec{v}_{br})$ implies $\vec{v}_b' = \frac{(m_s + m_r + m_b)\vec{v}_r + (m_r + m_s)\vec{v}_{br}}{m_b + m_r + m_s}$. With $\vec{v}_b'$ known, consider the student on the left catching this ball: $P_i = P_f \Rightarrow m_b\vec{v}_b' + (m_l + m_s)\vec{v}_l = (m_b + m_l + m_s)\vec{v}_l'$. $\vec{v}_l$ is known from part (a) and $\vec{v}_b'$ has just been determined, so $\vec{v}_l' = \frac{m_b\vec{v}_b' + (m_l + m_s)\vec{v}_l}{m_b + m_l + m_s}$.

(d) $\vec{v}_{br} = \vec{v}_b' - \vec{v}_r' \Rightarrow \vec{v}_r' = \vec{v}_b' - \vec{v}_{br}$ and $\vec{v}_b'$ has been determined in part (c).

CALCULATE:

(a) $\vec{v}_l = -\frac{(5.00 \text{ kg})(10.0 \text{ m/s})}{10.0 \text{ kg} + 50.0 \text{ kg} + 5.00 \text{ kg}} = -0.76923$ m/s

(b) $\vec{v}_b = 10.0 \text{ m/s} - 0.769 \text{ m/s} = 9.231$ m/s, $\vec{v}_r = \frac{(5.00 \text{ kg})(9.23077 \text{ m/s})}{5.00 \text{ kg} + 45.0 \text{ kg} + 10.0 \text{ kg}} = 0.76923$ m/s

(c) The ball is thrown to the left, or along the $-\hat{x}$ axis by the student on the right. That is, $\vec{v}_{\text{br}} = -12.0$ m/s.

$$\vec{v}_{\text{b}}' = \frac{(10.0 \text{ kg} + 45.0 \text{ kg} + 5.00 \text{ kg})(0.769 \text{ m/s}) + (45.0 \text{ kg} + 10.0 \text{ kg})(-12.0 \text{ m/s})}{5.00 \text{ kg} + 45.0 \text{ kg} + 10.0 \text{ kg}} = -10.23100 \text{ m/s}$$

$$\vec{v}_{\text{l}}' = \frac{(5.00 \text{ kg})(-10.2310 \text{ m/s}) + (50.0 \text{ kg} + 10.0 \text{ kg})(-0.769 \text{ m/s})}{5.00 \text{ kg} + 50.0 \text{ kg} + 10.0 \text{ kg}} = -1.49685 \text{ m/s}$$

(d) $\vec{v}_{\text{r}}' = (-10.231 \text{ m/s}) - (-12.0 \text{ m/s}) = 1.769 \text{ m/s}$

ROUND:

(a) $\vec{v}_{\text{l}} = -0.769$ m/s (to the left)

(b) $\vec{v}_{\text{r}} = 0.769$ m/s (to the right)

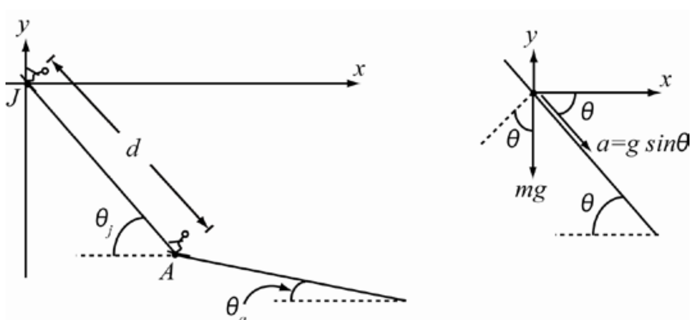
(c) $\vec{v}_{\text{l}}' = -1.50$ m/s (to the left)

(d) $\vec{v}_{\text{r}}' = 1.77$ m/s (to the right)

DOUBLE-CHECK: Before rounding, $|\vec{v}_{\text{l}}'| > |\vec{v}_{\text{l}}| > 0$ (where the initial speed was zero) and $|\vec{v}_{\text{r}}'| > |\vec{v}_{\text{r}}| > 0$, as expected.

8.36. **THINK:** Jack's mass is $m_{\text{J}} = 88.0$ kg. Jack's initial position is taken as $(0,0)$ and the angle of his slope is $\theta_{\text{J}} = 35.0°$. The distance of his slope is d = 100. m. Annie's mass is $m_{\text{A}} = 64.0$ kg. Her slope angle is $\theta_{\text{A}} = 20.0°$. Take her initial position to be $(d\cos\theta_{\text{J}}, -d\sin\theta_{\text{J}})$. Determine the acceleration, velocity and position vectors of their center of mass as functions of time, before Jack reaches the less steep section.

SKETCH:



RESEARCH: To determine the acceleration, velocity and position vectors for the center of mass, the vectors must be determined in each direction. Assuming a constant acceleration, the familiar constant acceleration equations can be used. In addition,

$$\vec{R} = \frac{1}{M}\sum_{i=1}^{n} \vec{r}_i m_i, \quad \vec{V} = \frac{d\vec{R}}{dt} = \frac{1}{M}\sum_{i=1}^{n} \vec{v}_i m_i, \quad \vec{A} = \frac{d\vec{V}}{dt} = \frac{1}{M}\sum_{i=1}^{n} \vec{a}_i m_i,$$

where each equation can be broken into its vector components.

SIMPLIFY: The magnitude of the net acceleration of each skier is $a = g\sin\theta$ down the incline of angle, θ. In the x-direction, $a_{\text{J}x} = (g\sin\theta_{\text{J}})\cos\theta_{\text{J}}$ and $a_{\text{A}x} = (g\sin\theta_{\text{A}})\cos\theta_{\text{A}}$. In the y-direction, $a_{\text{J}y} = -(g\sin\theta_{\text{J}})\sin\theta_{\text{J}} = -g\sin^2\theta_{\text{J}}$ and $a_{\text{A}y} = -(a\sin\theta_{\text{A}})\sin\theta_{\text{A}} = -g\sin^2\theta_{\text{A}}$. Then,

$$A_x = \frac{1}{M}(m_{\text{J}}a_{\text{J}x} + m_{\text{A}}a_{\text{A}x}) = \frac{g}{M}(m_{\text{J}}\sin\theta_{\text{J}}\cos\theta_{\text{J}} + m_{\text{A}}\sin\theta_{\text{A}}\cos\theta_{\text{A}}), \text{ where } M = m_{\text{J}} + m_{\text{A}} \text{ and}$$

$$A_y = \frac{1}{M}(m_{\text{J}}a_{\text{J}y} + m_{\text{A}}a_{\text{A}y}) = -\frac{g}{M}(m_{\text{J}}\sin^2\theta_{\text{J}} + m_{\text{A}}\sin^2\theta_{\text{A}}).$$

Each skier starts from rest. In the x-direction, $v_{\text{J}x} = a_{\text{J}x}t = g\sin\theta_{\text{J}}\cos\theta_{\text{J}}t$ and $v_{\text{A}x} = a_{\text{A}x}t = g\sin\theta_{\text{A}}\cos\theta_{\text{A}}t$. In the y-direction, $v_{\text{J}y} = a_{\text{J}y}t = -g\sin^2\theta_{\text{J}}t$ and $v_{\text{A}y} = a_{\text{A}y}t = -g\sin^2\theta_{\text{A}}t$.

Then,

$$V_x = \frac{1}{M}\left(m_{\rm J}v_{{\rm J}x} + m_{\rm A}v_{{\rm A}x}\right) = \frac{g}{M}\left(m_{\rm J}\sin\theta_{\rm J}\cos\theta_{\rm J} + m_{\rm A}\sin\theta_{\rm A}\cos\theta_{\rm A}\right)t = A_x t \text{ and}$$

$$V_y = \frac{1}{M}\left(m_{\rm J}v_{{\rm J}y} + m_{\rm A}v_{{\rm A}y}\right) = -\frac{g}{M}\left(m_{\rm J}\sin^2\theta_{\rm J} + m_{\rm A}\sin^2\theta_{\rm A}\right)t = A_y t.$$

The position in the x-direction is given by:

$$x_{\rm J} = \frac{1}{2}a_{{\rm J}x}t^2 + x_{\rm J0} = \frac{1}{2}g\sin\theta_{\rm J}\cos\theta_{\rm J}t^2 \text{ and } x_{\rm A} = \frac{1}{2}a_{{\rm A}x}t^2 + x_{\rm A0} = \frac{1}{2}g\sin\theta_{\rm A}\cos\theta_{\rm A}t^2 + d\cos\theta_{\rm J}.$$

In the y-direction,

$$y_{\rm J} = \frac{1}{2}a_{{\rm J}y}t^2 + y_{\rm J0} = -\frac{1}{2}g\sin^2\theta_{\rm J}t^2 \text{ and } y_{\rm A} = \frac{1}{2}a_{{\rm A}y}t^2 + y_{\rm A0} = -\frac{1}{2}g\sin^2\theta_{\rm A}t^2 - d\sin\theta_{\rm J}.$$

Then,

$$X = \frac{1}{M}\left(m_{\rm J}x_{\rm J} + m_{\rm A}x_{\rm A}\right) = \frac{1}{M}\left(\frac{1}{2}m_{\rm J}g\sin\theta_{\rm J}\cos\theta_{\rm J}t^2 + \frac{1}{2}m_{\rm A}g\sin\theta_{\rm A}\cos\theta_{\rm A}t^2 + m_{\rm A}d\cos\theta_{\rm J}\right) = \frac{1}{2}A_x t^2 + \frac{m_{\rm A}}{M}d\cos\theta_{\rm J}$$

$$Y = \frac{1}{M}\left(m_{\rm J}y_{\rm J} + m_{\rm A}y_{\rm A}\right) = -\frac{1}{M}\left(\frac{1}{2}m_{\rm J}g\sin^2\theta_{\rm J}t^2 + \frac{1}{2}m_{\rm A}g\sin^2\theta_{\rm A}t^2 + m_{\rm A}d\sin\theta_{\rm J}\right) = \frac{1}{2}A_y t^2 - \frac{m_{\rm A}}{M}d\sin\theta_{\rm J}.$$

CALCULATE:

$$A_x = \frac{\left(9.81 \text{ m/s}^2\right)}{88.0 \text{ kg} + 64.0 \text{ kg}}\left(\left(88.0 \text{ kg}\right)\sin 35.0°\cos 35.0° + \left(64.0 \text{ kg}\right)\sin 20.0°\cos 20.0°\right) = 3.996 \text{ m/s}^2$$

$$A_y = -\frac{\left(9.81 \text{ m/s}^2\right)}{88.0 \text{ kg} + 64.0 \text{ kg}}\left(\left(88.0 \text{ kg}\right)\sin^2\left(35.0°\right) + \left(64.0 \text{ kg}\right)\sin^2\left(20.0°\right)\right) = -2.352 \text{ m/s}^2$$

$$V_x = \left(3.996 \text{ m/s}^2\right)t,\quad V_y = \left(-2.352 \text{ m/s}^2\right)t$$

$$X = \frac{1}{2}\left(3.996 \text{ m/s}^2\right)t^2 + \frac{64.0 \text{ kg}}{\left(88.0 \text{ kg} + 64.0 \text{ kg}\right)}\left(100. \text{ m}\right)\cos\left(35.0°\right) = \left(1.998 \text{ m/s}^2\right)t^2 + 34.49 \text{ m}$$

$$Y = \frac{1}{2}\left(-2.352 \text{ m/s}^2\right)t^2 - \frac{64.0 \text{ kg}}{\left(88.0 \text{ kg} + 64.0 \text{ kg}\right)}\left(100. \text{ m}\right)\sin\left(35.0°\right) = \left(-1.176 \text{ m/s}^2\right)t^2 - 24.1506 \text{ m}$$

ROUND: Rounding to three significant figures, $A_x = 4.00 \text{ m/s}^2$, $A_y = -2.35 \text{ m/s}^2$, $V_x = \left(4.00 \text{ m/s}^2\right)t$ and $V_y = \left(-2.35 \text{ m/s}^2\right)t$, $X = \left(2.00 \text{ m/s}^2\right)t^2 + 34.5 \text{ m}$ and $Y = \left(-1.18 \text{ m/s}^2\right)t^2 - 24.2 \text{ m}$.

DOUBLE-CHECK: The acceleration of the center of mass is not time dependent.

8.37. **THINK:** The proton's mass is $m_{\rm p} = 1.6726 \cdot 10^{-27}$ kg and its initial speed is $v_{\rm p} = 0.700c$ (assumed to be in the lab frame). The mass of the tin nucleus is $m_{\rm sn} = 1.9240 \cdot 10^{-25}$ kg (assumed to be at rest). Determine the speed of the center of mass, v, with respect to the lab frame.

SKETCH: A sketch is not necessary.

RESEARCH: The given speeds are in the lab frame. To determine the speed of the center of mass use $V = \frac{1}{M}\sum_{i=1}^{n} m_i v_i$.

SIMPLIFY: $V = \frac{1}{m_{\rm p} + m_{\rm sn}}\left(m_{\rm p}v_{\rm p} + m_{\rm sn}v_{\rm sn}\right) = \frac{m_{\rm p}v_{\rm p}}{m_{\rm p} + m_{\rm sn}}$

CALCULATE: $V = \frac{\left(1.6726 \cdot 10^{-27} \text{ kg}\right)\left(0.700c\right)}{\left(1.6726 \cdot 10^{-27} \text{ kg}\right) + \left(1.9240 \cdot 10^{-25} \text{ kg}\right)} = 0.0060329c$

ROUND: Since $v_{\rm p}$ has three significant figures, the result should be rounded to $V = 0.00603c$.

DOUBLE-CHECK: Since $m_{\rm sn}$ is at rest and $m_{\rm sn} >> m_{\rm p}$, it is expected that $V << v_{\rm p}$.

8.38. **THINK:** Particle 1 has a mass of $m_1 = 2.0$ kg, a position of $\vec{r}_1 = (2.0 \text{ m}, 6.0 \text{ m})$ and a velocity of $\vec{v}_1 = (4.0 \text{ m/s}, 2.0 \text{ m/s})$. Particle 2 has a mass of $m_2 = 3.0$ kg, a position of $\vec{r}_2 = (4.0 \text{ m}, 1.0 \text{ m})$ and a velocity of $\vec{v}_2 = (0, 4.0 \text{ m/s})$. Determine (a) the position $\vec{R}$ and the velocity $\vec{V}$ for the system's center of mass and (b) a sketch of the position and velocity vectors for each particle and for the center of mass.

SKETCH: To be provided in the calculate step, part (b).

RESEARCH: To determine $\vec{R}$, use $X = \frac{1}{M}(x_1 m_1 + x_2 m_2)$ and $Y = \frac{1}{M}(y_1 m_1 + y_2 m_2)$. To determine $\vec{V}$, use $V_x = \frac{1}{M}(v_{1x} m_1 + v_{2x} m_2)$ and $V_y = \frac{1}{M}(v_{1y} m_1 + v_{2y} m_2)$.

SIMPLIFY: It is not necessary to simplify.

CALCULATE:

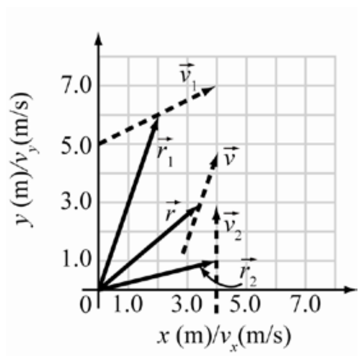
(a) $X = \frac{1}{2.00 \text{ kg} + 3.00 \text{ kg}}((2.00 \text{ m})(2.00 \text{ kg}) + (4.00 \text{ m})(3.00 \text{ kg})) = 3.20 \text{ m}$

$Y = \frac{1}{2.00 \text{ kg} + 3.00 \text{ kg}}((6.00 \text{ m})(2.00 \text{ kg}) + (1.00 \text{ m})(3.00 \text{ kg})) = 3.00 \text{ m}$

$V_x = \frac{1}{2.00 \text{ kg} + 3.00 \text{ kg}}((4.00 \text{ m/s})(2.00 \text{ kg}) + 0(3.00 \text{ kg})) = 1.60 \text{ m/s}$

$V_y = \frac{1}{2.00 \text{ kg} + 3.00 \text{ kg}}((2.00 \text{ m/s})(2.00 \text{ kg}) + (4.00 \text{ m/s})(3.00 \text{ kg})) = 3.20 \text{ m/s}$

(b)

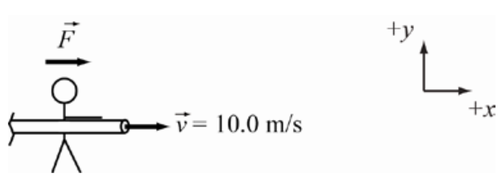


ROUND: Each given value has three significant figures, so the results should be rounded to X = 3.20 m, Y = 3.00 m, $V_x = 1.60$ m/s and $V_y = 3.20$ m/s.

DOUBLE-CHECK: $\vec{R}$ should point between $\vec{r}_1$ and $\vec{r}_2$, and $\vec{V}$ should point between $\vec{v}_1$ and $\vec{v}_2$.

8.39. **THINK:** The radius of the hose is r = 0.0200 m and the velocity of the spray is v = 10.0 m/s. Determine the horizontal force, $\vec{F}_\text{f}$, required of the fireman to hold the hose stationary.

SKETCH:



RESEARCH: By Newton's third law, the force exerted by the fireman is equal in magnitude to the force exerted by the hose. The thrust force of the hose can be determined from $\vec{F}_\text{thrust} = -\vec{v}_\text{c} dm/dt$. To determine dm/dt, consider the mass of water exiting the hose per unit time.

SIMPLIFY: The volume of water leaving the hose is this velocity times the area of the hose's end. That is,

$$\frac{dV_{\text{w}}}{dt} = Av = \pi r^2 v.$$

With $\rho_{\text{w}} = m / V_{\text{w}}$, $\frac{dm}{dt} = \rho_{\text{w}} \frac{dV_{\text{w}}}{dt} = \rho_{\text{w}} \pi r^2 v$. Now, by Newton's third law, $\vec{F}_{\text{f}} = -\vec{F}_{\text{thrust}}$, so

$\vec{F}_{\text{f}} = \vec{v}_{\text{c}} \frac{dm}{dt} = \vec{v}_{\text{c}} \rho_{\text{w}} \pi r^2 v$. Since v_{c} is in fact v, $F_{\text{f}} = \rho_{\text{w}} \pi r^2 v^2$.

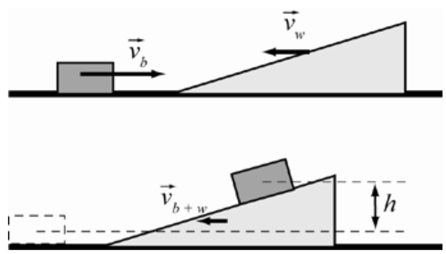
CALCULATE: $F_{\text{f}} = \pi \left(1000 \text{ kg/m}^3\right)\left(0.0200 \text{ m}\right)^2 \left(10.0 \text{ m/s}\right)^2 = 125.7 \text{ N}$

ROUND: Since v has three significant figures, $\vec{F}_{\text{f}} = 126 \text{ N}$ in the direction of the water's velocity.

DOUBLE-CHECK: The result has units of force. Also, this is a reasonable force with which to hold a fire hose.

8.40. **THINK:** The block's mass is $m_{\text{b}} = 1.2 \text{ kg}$. It has an initial velocity is $\vec{v}_{\text{b}} = 2.5 \text{ m/s}$ (with the positive x axis being the right direction). The wedge's mass is m_{w} and its initial velocity is $\vec{v}_{\text{w}} = -1.1 \text{ m/s}$. Their final velocity when the wedge stops moving is $\vec{v}_{\text{b+w}}$. Determine (a) m_{w}, if the block's center of mass rises by $h = 0.37$ m and (b) $\vec{v}_{\text{b+w}}$.

SKETCH:



RESEARCH: Momentum is conserved. As this is an elastic collision, and there are only conservative forces, mechanical energy is also conserved. Use $P_{\text{i}} = P_{\text{f}}$, $\Delta K + \Delta U = 0$, $K = mv^2/2$ and $U = mgh$ to determine m_{w} and ultimately $\vec{v}_{\text{b+w}}$.

SIMPLIFY: It will be useful to determine an expression for $\vec{v}_{\text{b+w}}$ first:

$$\vec{P}_{\text{i}} = \vec{P}_{\text{f}} \Rightarrow m_{\text{b}}\vec{v}_{\text{b}} + m_{\text{w}}\vec{v}_{\text{w}} = \left(m_{\text{b}} + m_{\text{w}}\right)\vec{v}_{\text{b+w}} \Rightarrow \vec{v}_{\text{b+w}} = \frac{m_{\text{b}}\vec{v}_{\text{b}} + m_{\text{w}}\vec{v}_{\text{w}}}{m_{\text{b}} + m_{\text{w}}}.$$

(a) From the conservation of mechanical energy:

$$\Delta K + \Delta U = K_{\text{f}} - K_{\text{i}} + U_{\text{f}} - U_{\text{i}} = 0 \Rightarrow \frac{1}{2}\left(m_{\text{b}} + m_{\text{w}}\right)\vec{v}_{\text{b+w}}^{\ 2} - \frac{1}{2}m_{\text{b}}\vec{v}_{\text{b}}^{\ 2} - \frac{1}{2}m_{\text{w}}\vec{v}_{\text{w}}^{\ 2} + m_{\text{b}}gh = 0$$

$$\Rightarrow \frac{1}{2}\left(m_{\text{b}} + m_{\text{w}}\right)\frac{\left(m_{\text{b}}\vec{v}_{\text{b}} + m_{\text{w}}\vec{v}_{\text{w}}\right)^2}{\left(m_{\text{b}} + m_{\text{w}}\right)^2} - \frac{1}{2}m_{\text{b}}\vec{v}_{\text{b}}^{\ 2} - \frac{1}{2}m_{\text{w}}\vec{v}_{\text{w}}^{\ 2} + m_{\text{b}}gh = 0$$

$$\Rightarrow \frac{\left(m_{\text{b}}^{\ 2}\vec{v}_{\text{b}}^{\ 2} + 2m_{\text{b}}m_{\text{w}}\vec{v}_{\text{b}}\vec{v}_{\text{w}} + m_{\text{w}}^{\ 2}\vec{v}_{\text{w}}^{\ 2}\right)}{2\left(m_{\text{b}} + m_{\text{w}}\right)} - \frac{1}{2}m_{\text{b}}\vec{v}_{\text{b}}^{\ 2} - \frac{1}{2}m_{\text{w}}\vec{v}_{\text{w}}^{\ 2} + m_{\text{b}}gh = 0$$

Multiply the expression by $2\left(m_{\text{b}} + m_{\text{w}}\right)$:

$$m_{\text{b}}^{\ 2}\vec{v}_{\text{b}}^{\ 2} + 2m_{\text{b}}m_{\text{w}}\vec{v}_{\text{b}}\vec{v}_{\text{w}} + m_{\text{w}}^{\ 2}\vec{v}_{\text{w}}^{\ 2} - m_{\text{b}}\vec{v}_{\text{b}}^{\ 2}\left(m_{\text{b}} + m_{\text{w}}\right) - m_{\text{w}}\vec{v}_{\text{w}}^{\ 2}\left(m_{\text{b}} + m_{\text{w}}\right) + 2m_{\text{b}}gh\left(m_{\text{b}} + m_{\text{w}}\right) = 0$$

$$\Rightarrow m_{\text{b}}^{\ 2}\vec{v}_{\text{b}}^{\ 2} + 2m_{\text{b}}m_{\text{w}}\vec{v}_{\text{b}}\vec{v}_{\text{w}} + m_{\text{w}}^{\ 2}\vec{v}_{\text{w}}^{\ 2} - m_{\text{b}}^{\ 2}\vec{v}_{\text{b}}^{\ 2} - m_{\text{b}}m_{\text{w}}\vec{v}_{\text{b}}^{\ 2} - m_{\text{w}}m_{\text{b}}\vec{v}_{\text{w}}^{\ 2} + m_{\text{w}}^{\ 2}\vec{v}_{\text{w}}^{\ 2} + 2m_{\text{b}}^{\ 2}gh + 2m_{\text{b}}m_{\text{w}}gh = 0$$

$$\Rightarrow 2m_{\text{b}}m_{\text{w}}\vec{v}_{\text{b}}\vec{v}_{\text{w}} - m_{\text{b}}m_{\text{w}}\vec{v}_{\text{b}}^{\ 2} - m_{\text{b}}m_{\text{w}}\vec{v}_{\text{w}}^{\ 2} + 2m_{\text{b}}^{\ 2}gh + 2m_{\text{b}}m_{\text{w}}gh = 0$$

$$\Rightarrow m_{\text{w}} = -\frac{2m_{\text{b}}^{\ 2}gh}{2m_{\text{b}}\vec{v}_{\text{b}}\vec{v}_{\text{w}} - m_{\text{b}}\vec{v}_{\text{b}}^{\ 2} - m_{\text{b}}\vec{v}_{\text{w}}^{\ 2} + 2m_{\text{b}}gh} = \frac{2m_{\text{b}}gh}{\vec{v}_{\text{b}}^{\ 2} + \vec{v}_{\text{w}}^{\ 2} - 2\vec{v}_{\text{b}}\vec{v}_{\text{w}} - 2gh}.$$

(b) With m_{w} known, $\vec{v}_{\text{b+w}} = \dfrac{m_{\text{b}}\vec{v}_{\text{b}} + m_{\text{w}}\vec{v}_{\text{w}}}{m_{\text{b}} + m_{\text{w}}}.$

CALCULATE:

(a) $m_{\text{w}} = \dfrac{2(1.20\text{ kg})(9.81\text{ m/s}^2)0.370\text{ m}}{(2.5\text{ m/s})^2 + (-1.10\text{ m/s})^2 - 2(2.50\text{ m/s})(-1.10\text{ m/s}) - 2(9.81\text{ m/s}^2)(0.370\text{ m})}$

$= \dfrac{8.712\text{ kg}\cdot\text{m}^2/\text{s}^2}{6.25\text{ m}^2/\text{s}^2 + 1.21\text{ m}^2/\text{s}^2 + 5.5\text{ m}^2/\text{s}^2 - 7.2594\text{ m}^2/\text{s}^2} = 1.528\text{ kg}$

(b) $\vec{v}_{\text{b+w}} = \dfrac{(1.20\text{ kg})(2.50\text{ m/s}) + (1.528\text{ kg})(-1.10\text{ m/s})}{1.20\text{ kg} + 1.528\text{ kg}} = 0.4835\text{ m/s}$

ROUND: Each given value has three significant figures, so the results should be rounded to: $m_w = 1.53\text{ kg}$ and $\vec{v}_{\text{b+w}} = 0.484\text{ m/s}$ to the right.

DOUBLE-CHECK: These results are reasonable given the initial values.

8.41. **THINK:** For rocket engines, the specific impulse is $J_{\text{spec}} = \dfrac{J_{\text{tot}}}{W_{\text{expended fuel}}} = \dfrac{1}{W_{\text{expended fuel}}}\int_{t_0}^{t} F_{\text{thrust}}(t')dt'.$

(a) Determine J_{spec} with an exhaust nozzle speed of v.

(b) Evaluate and compare J_{spec} for a toy rocket with $v_{\text{toy}} = 800.\text{ m/s}$ and a chemical rocket with $v_{\text{chem}} = 4.00\text{ km/s}.$

SKETCH: Not applicable.

RESEARCH: It is known that $\vec{F}_{\text{thrust}} = -v_{\text{c}}dm/dt$. Rewrite $W_{\text{expended fuel}}$ as $m_{\text{expended}}g$. With the given definition, J_{spec} can be determined for a general v, and for v_{toy} and v_{chem}.

SIMPLIFY: $J_{\text{spec}} = \dfrac{1}{m_{\text{expended}}g}\int_{m_0}^{m} -vdm = -\dfrac{v}{m_{\text{expended}}g}(m - m_0)$. Now, $m - m_0 = -m_{\text{expended}}$, so $J_{\text{spec}} = \dfrac{v}{g}.$

CALCULATE: $J_{\text{spec, toy}} = \dfrac{v_{\text{toy}}}{g} = \dfrac{800.\text{ m/s}}{(9.81\text{ m/s}^2)} = 81.55\text{ s},\ J_{\text{spec, chem}} = \dfrac{v_{\text{chem}}}{g} = \dfrac{4.00\cdot 10^3\text{ m/s}}{(9.81\text{ m/s}^2)} = 407.75\text{ s}$

$\dfrac{J_{\text{spec, toy}}}{J_{\text{spec, chem}}} = \dfrac{v_{\text{toy}}}{v_{\text{chem}}} = \dfrac{800.\text{ m/s}}{4.00\cdot 10^3\text{ m/s}} = 0.200$

ROUND:

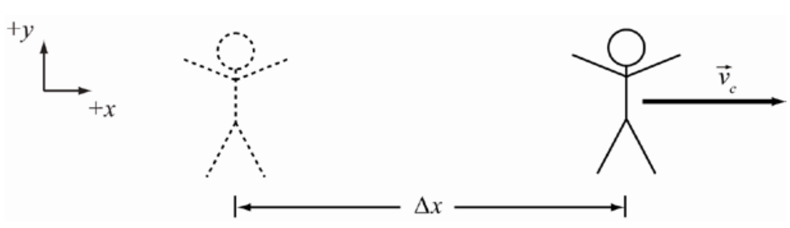
(a) $J_{\text{spec, toy}} = 81.6\text{ s}$

(b) $J_{\text{spec, chem}} = 408\text{ s}$ and $J_{\text{spec, toy}} = 0.200 J_{\text{spec, chem}}$.

DOUBLE-CHECK: The units of the results are units of specific impulse. Also, as expected $J_{\text{spec, toy}} < J_{\text{spec, chem}}$.

8.42. **THINK:** The astronaut's total mass is m = 115 kg. The rate of gas ejection is $dm/dt = 7.00\text{ g/s} = 0.00700\text{ kg/s}$ and the leak speed is $v_{\text{c}} = 800.\text{ m/s}$. After $\Delta t = 6.00\text{ s}$, how far has the astronaut moved from her original position, Δx?

SKETCH:



RESEARCH: Assume that the astronaut starts from rest and the acceleration is constant. Δx can be determined from $\Delta x = (v_\text{i} + v_\text{f})\Delta t / 2$. To determine v_f, use the rocket-velocity equation $v_\text{f} - v_\text{i} = v_\text{c} \ln(m_\text{i} / m_\text{f})$. The loss of mass can be determined from $\Delta m = \frac{dm}{dt}\Delta t$.

SIMPLIFY: Since $v_\text{i} = 0$, $v_\text{f} = v_\text{c} \ln(m_\text{i} / m_\text{f})$, where $m_\text{i} = m$ and $m_\text{f} = m - \Delta m = m - \frac{dm}{dt}\Delta t$. Then,

$$v_\text{f} = v_\text{c} \ln\left(\frac{m}{m - \frac{dm}{dt}\Delta t}\right) \text{ and } \Delta x = \frac{1}{2} v_\text{f} \Delta t.$$

CALCULATE: $v_\text{f} = (800.\text{ m/s})\ln\left(\frac{115\text{ kg}}{115\text{ kg} - (0.00700\text{ kg/s})(6.00\text{ s})}\right) = 0.29223\text{ m/s}$

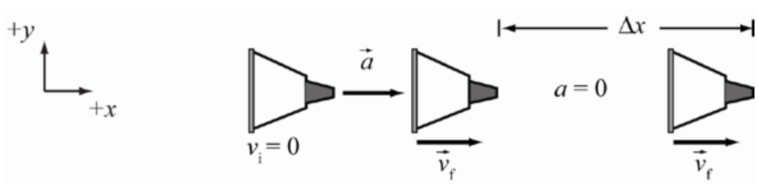
$$\Delta x = \frac{1}{2}(0.29223\text{ m/s})(6.00\text{ s}) = 0.87669\text{ m}$$

ROUND: The problem values have three significant figures, so the results should be rounded to $v_\text{f} = 0.292\text{ m/s}$ $\Delta x = 0.877\text{ m}$.

DOUBLE-CHECK: Considering how such a small amount of the total mass has escaped, this is a reasonable distance to have moved.

8.43. **THINK:** The mass of the payload is $m_\text{p} = 5190.0\text{ kg}$, and the fuel mass is $m_\text{f} = 1.551 \cdot 10^5\text{ kg}$. The fuel exhaust speed is $v_\text{c} = 5.600 \cdot 10^3\text{ m/s}$. How long will it take the rocket to travel a distance $\Delta x = 3.82 \cdot 10^8\text{ m}$ after achieving its final velocity, v_f? The rocket starts accelerating from rest.

SKETCH:



RESEARCH: The rocket's travel speed, v_f, can be determined from $v_\text{f} - v_\text{i} = v_\text{c} \ln(m_\text{i} / m_\text{f})$. Then Δt can be determined from $\Delta x = v \Delta t$.

SIMPLIFY: $v_\text{f} = v_\text{c} \ln\left(\frac{m_\text{p} + m_\text{f}}{m_\text{p}}\right)$, and $\Delta t = \Delta x / v_\text{f}$.

CALCULATE: $v_\text{f} = (5.600 \cdot 10^3\text{ m/s})\ln\left(\frac{5190.0\text{ kg} + 1.551 \cdot 10^5\text{ kg}}{5190.0\text{ kg}}\right) = 19209\text{ m/s}$,

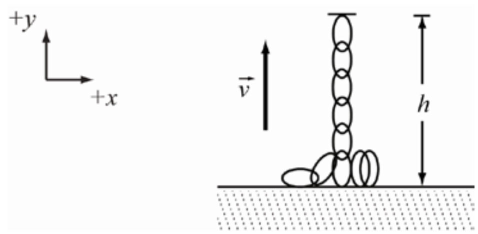
$$\Delta t = \frac{3.82 \cdot 10^8\text{ m}}{19209\text{ m/s}} = 19886\text{ s}$$

ROUND: Δx has three significant figures, so the result should be rounded to $\Delta t = 19{,}886\text{ s} = 5.52\text{ h}$.

DOUBLE-CHECK: This is a reasonable time for a rocket with such a large initial velocity to reach the Moon from the Earth.

8.44. **THINK:** The linear density of the chain is $\lambda = 1.32\text{ kg/m}$, and the speed at which one end of the chain is lifted is v = 0.47 m/s. Determine (a) the net force acting on the chain, F_net and (b) the force, F, applied to the end of the chain when h = 0.15 m has been lifted off the table.

SKETCH:



RESEARCH:

(a) Since the chain is raised at a constant rate, *v*, the net force is the thrust force, $F_{\text{thrust}} = v_{\text{c}} dm/dt$. Since the chain's mass in the air is increasing, $F_{net} = v\,dm/dt$.

(b) The applied force can be determined by considering the forces acting on the chain and the net force determined in part (a): $F_{\text{net}} = \sum F_i$.

SIMPLIFY:

(a) $F_{\text{net}} = v\dfrac{dm}{dt} = v\lambda\dfrac{dh}{dt} = v\lambda v = v^2\lambda$

(b) $F_{\text{net}} = F_{\text{applied}} - mg \Rightarrow F_{\text{applied}} = F_{\text{net}} + mg = v^2\lambda + mg = v^2\lambda + \lambda hg$

CALCULATE:

(a) $F_{\text{net}} = (0.470 \text{ m/s})^2(1.32 \text{ kg/m}) = 0.2916 \text{ N}$

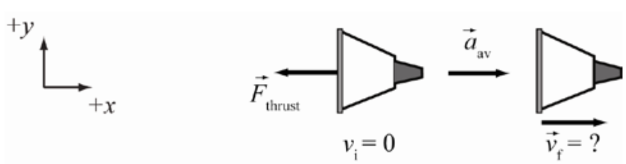
(b) $F_{\text{applied}} = 0.2916 \text{ N} + (1.32 \text{ kg/m})(0.150 \text{ m})(9.81 \text{ m/s}^2) = 0.2916 \text{ N} + 1.942 \text{ N} = 2.234 \text{ N}$

ROUND: *v* and *h* each have three significant figures, so the results should be rounded to $F_{\text{net}} = 0.292$ N and $F_{\text{applied}} = 2.23$ N.

DOUBLE-CHECK: These forces are reasonable to determine for this system. Also, $F_{\text{net}} < F_{\text{applied}}$.

8.45. **THINK:** The thrust force is $\vec{F}_{\text{thrust}} = 53.2 \cdot 10^6$ N and the propellant velocity is $v = 4.78 \cdot 10^3$ m/s. Determine (a) *dm*/*dt*, (b) the final speed of the spacecraft, v_{s}, given $v_{\text{i}} = 0$, $m_{\text{i}} = 2.12 \cdot 10^6$ kg and $m_{\text{f}} = 7.04 \cdot 10^4$ kg and (c) the average acceleration, a_{av} until burnout.

SKETCH:



RESEARCH:

(a) To determine *dm*/*dt*, use $\vec{F}_{\text{thrust}} = -v_{\text{c}} dm/dt$.

(b) To determine v_{f}, use $v_{\text{f}} - v_{\text{i}} = v_{\text{c}} \ln(m_{\text{i}} / m_{\text{f}})$.

(c) Δv is known from part (b). Δt can be determined from the equivalent ratios,

$\dfrac{dm}{dt} = \dfrac{\Delta m}{\Delta t}$, where $\Delta m = m_{\text{i}} - m_{\text{f}}$.

SIMPLIFY:

(a) Since $\vec{F}_{\text{thrust}}$ and $\vec{v}_{\text{c}}$ are in the same direction, the equation can be rewritten as:

$$F_{\text{thrust}} = v_{\text{c}}\frac{dm}{dt} \Rightarrow \frac{dm}{dt} = \frac{F_{\text{thrust}}}{v_{\text{c}}}.$$

(b) $v_i = 0 \;\Rightarrow\; v_f = v_c \ln\left(\dfrac{m_i}{m_f}\right)$

(c) $\dfrac{dm}{dt} = \dfrac{\Delta m}{\Delta t} \;\Rightarrow\; \Delta t = \dfrac{\Delta m}{dm/dt},\;\; a_{av} = \dfrac{\Delta v}{\Delta t} = \dfrac{v_f}{\Delta m}\left(\dfrac{dm}{dt}\right) \quad (v_i = 0)$

CALCULATE:

(a) $\dfrac{dm}{dt} = \dfrac{\left(53.2\cdot 10^6 \text{ N}\right)}{\left(4.78\cdot 10^3 \text{ m/s}\right)} = 11129.7 \text{ kg/s}$

(b) $v_f = \left(4.78\cdot 10^3 \text{ m/s}\right)\ln\left(\dfrac{2.12\cdot 10^6 \text{ kg}}{7.04\cdot 10^4 \text{ kg}}\right) = 1.6276\cdot 10^4 \text{ m/s}$

(c) $a_{av} = \dfrac{\left(1.6276\cdot 10^4 \text{ m/s}\right)}{\left(2.12\cdot 10^6 \text{ kg} - 7.04\cdot 10^4 \text{ kg}\right)}\left(11129.7 \text{ kg/s}\right) = 88.38 \text{ m/s}^2$

ROUND: Each given value has three significant figures, so the results should be rounded to $dm/dt = 11100 \text{ kg/s}$, $v_f = 1.63\cdot 10^4 \text{ m/s}$ and $a_{av} = 88.4 \text{ m/s}^2$.

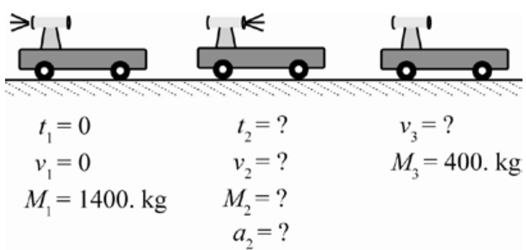
DOUBLE-CHECK: The results all have the correct units. Also, the results are reasonable for a spaceship with such a large thrust force.

8.46. **THINK:** The mass of the cart with an empty water tank is $m_c = 400.$ kg. The volume of the water tank is $V = 1.00 \text{ m}^3$. The rate at which water is ejected in SI units is

$$dV/dt = \left(200. \frac{\text{L}}{\text{min}}\right)\left(\frac{1 \text{ m}^3}{1000 \text{ L}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 0.003333 \text{ m}^3/\text{s}.$$

The muzzle velocity is $v_c = 25.0$ m/s. Determine (a) the time, t_2, to switch from backward to forward so the cart ends up at rest (it starts from rest), (b) the mass of the cart, M_2, and the velocity, v_2, at the time, t_2, (c) the thrust, F_{thrust}, of the rocket and (d) the acceleration, a_2, of the cart just before the valve is switched. Note the mass of the cart increases by 1000. kg when the water tank is full, as $m_w = \rho V = \left(1000. \text{ kg/m}^3\right)\left(1.00 \text{ m}^3\right)$. That is, the initial mass is $M_1 = 1400.$ kg.

SKETCH:



RESEARCH:

(a) t_2 can be determined from the ratio, $\dfrac{M_1 - M_2}{t_2 - t_1} = \dfrac{dm}{dt}$, with $t_1 = 0$. Note that, $dm/dt = \rho dV/dt$. M_2 can be determined from $v_f - v_i = v_c \ln\left(m_i/m_f\right)$. When the cart stops moving, the water tank is empty and the total mass is $M_3 = 400$ kg.

(b) Using the mass determined in part (a), v_2 can be determined from $v_f - v_i = v_c \ln\left(m_i/m_f\right)$.

(c) Use $\vec{F}_{\text{thrust}} = -\vec{v}_c dm/dt$.

(d) Since $\vec{F}_{\text{thrust}} = M\vec{a}_{\text{net}}$, a_2 can be determined from this equation.

SIMPLIFY:

(a) Consider the first leg of the trip before the valve is switched:

$$v_2 - v_1 = v_c \ln\left(M_1 / M_2\right) \Rightarrow v_2 = v_c \ln\left(M_1 / M_2\right).$$

In the second leg, v_c changes direction, and the similar equation is

$$v_3 - v_2 = -v_c \ln\left(M_2 / M_3\right) \Rightarrow v_2 = v_c \ln\left(M_2 / M_3\right).$$

Then it must be that $\ln\left(M_2 / M_3\right) = \ln\left(M_2 / M_3\right)$, or $M_1 / M_2 = M_2 / M_3$. Then $M_2 = \sqrt{M_3 M_1}$. Now,

$$\frac{M_1 - M_2}{t_2} = \frac{dm}{dt} = \rho\frac{dV}{dt} \Rightarrow t_2 = \frac{M_1 - M_2}{\rho\frac{dV}{dt}} = \frac{M_1 - \sqrt{M_3 M_1}}{\rho\frac{dV}{dt}}.$$

(b) From above, $M_2 = \sqrt{M_3 M_1}$, $v_2 = v_c \ln\left(\frac{M_1}{M_2}\right)$.

(c) $\vec{F}_{\text{thrust}} = -\vec{v}_c \frac{dm}{dt} = -\vec{v}_c \rho \frac{dV}{dt}$

(d) $\vec{a}_2 = \frac{\vec{F}_{\text{thrust}}}{M_2}$

CALCULATE:

(a) $t_2 = \frac{1400.\text{ kg} - \sqrt{(400.\text{ kg})(1400.\text{ kg})}}{\left(1000.\text{ kg/m}^3\right)\left(0.003333\text{ m}^3/\text{s}\right)} = 195.5\text{ s}$

(b) $M_2 = \sqrt{(400.\text{ kg})(1400.\text{ kg})} = 748.33\text{ kg}$, $v_2 = (25.0\text{ m/s})\ln\left(\frac{(1400.\text{ kg})}{(748.33\text{ kg})}\right) = 15.66\text{ m/s}$

(c) Before the valve is switched, v_c is directed backward, i.e. $\vec{v}_c = -25.0\text{ m/s}$. Then $\vec{F}_{\text{thrust}} = -(-25.0\text{ m/s})\left(1000.\text{ kg/m}^3\right)\left(0.003333\text{ m}^3/\text{s}\right) = 83.33\text{ N}$ forward. After the valve is switched, $\vec{F}_{\text{thrust}}$ is directed backward, i.e. $\vec{F}_{\text{thrust}} = -83.33\text{ N}$.

(d) Before the valve is switched, $\vec{a}_2 = \frac{83.33\text{ N}}{748.33\text{ kg}} = 0.111355\text{ m/s}^2$.

ROUND:

Rounding to three significant figures:

(a) $t_2 = 196\text{ s}$

(b) $M_2 = 748\text{ kg}$ and $v_2 = 15.7\text{ m/s}$.

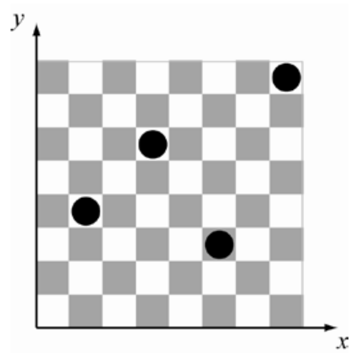
(c) $\vec{F}_{\text{thrust}} = -83.3\text{ N}$

(d) $\vec{a}_2 = 0.111\text{ m/s}^2$

DOUBLE-CHECK: All the units for the results are appropriate. Also, the results are of reasonable orders of magnitude.

8.47. **THINK:** The checkerboard has dimensions 32.0 cm by 32.0 cm. Its mass is $m_b = 100.\text{ g}$ and the mass of each of the four checkers is $m_c = 20.0\text{ g}$. Determine the center of mass of the system. Note the checkerboard is 8 by 8 squares, thus the length of the side of each square is 32.0 cm/8 = 4.00 cm. From the figure provided, the following *x-y* coordinates can be associated with each checker's center of mass: $m_1 : (22.0\text{ cm}, 10.0\text{ cm})$, $m_2 : (6.00\text{ cm}, 14.0\text{ cm})$, $m_3 : (14.0\text{ cm}, 22.0\text{ cm})$, $m_4 : (30.0\text{ cm}, 30.0\text{ cm})$. Assuming a uniform density distribution, the checkerboard's center of mass is located at $(x_b, y_b) = (16.0\text{ cm}, 16.0\text{ cm})$.

SKETCH:



RESEARCH: To determine the system's center of mass, use the following equations: $X = \frac{1}{M}\sum_{i=1}^{n} x_i m_i$ and $Y = \frac{1}{M}\sum_{i=1}^{n} y_i m_i$.

SIMPLIFY: $M = m_\text{b} + 4m_\text{c}$

$X = \frac{1}{M}\left(x_\text{b}m_\text{b} + m_\text{c}\left(x_1 + x_2 + x_3 + x_4\right)\right),\ Y = \frac{1}{M}\left(y_\text{b}m_\text{b} + m_\text{c}\left(y_1 + y_2 + y_3 + y_4\right)\right)$

CALCULATE: $M = 100.\text{ g} + 4(20.0\text{ g}) = 180.\text{ g}$

$$X = \frac{1}{180.\text{ g}}\left(16.0\text{ cm}(100.0\text{ g}) + 20.0\text{ g}(22.0\text{ cm} + 6.00\text{ cm} + 14.0\text{ cm} + 30.0\text{ cm})\right) = 16.889\text{ cm}$$

$$Y = \frac{1}{180.\text{ g}}\left(16.0\text{ cm}(100.\text{ g}) + 20.0\text{ g}(10.0\text{ cm} + 14.0\text{ cm} + 22.0\text{ cm} + 30.0\text{ cm})\right) = 17.33\text{ cm}$$

ROUND: $X = 16.9$ cm and $Y = 17.3$ cm. The answer is (16.9 cm,17.3 cm).

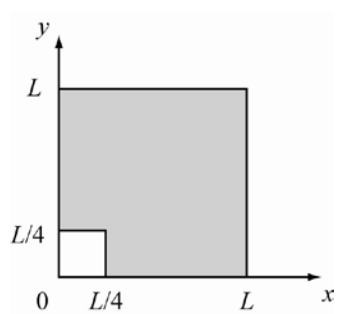
DOUBLE-CHECK: $m_\text{b} > m_\text{c}$, so it is reasonable to expect the system's center of mass to be near the board's center of mass.

8.48. **THINK:** The total mass of the plate is $M_\text{tot} = 0.205$ kg. The dimensions of the plate are L by L, $L = 5.70$ cm. The dimensions of the smaller removed plate are $L/4$ by $L/4$. The mass of the smaller removed plate is

$$\frac{M_\text{tot}}{A_\text{tot}} = \frac{m_\text{s}}{A_\text{s}} \Rightarrow m_\text{s} = A_\text{s}\frac{M_\text{tot}}{A_\text{tot}} = \left(\frac{L}{4}\right)^2 \frac{M_\text{tot}}{L^2} = \frac{1}{16}M_\text{tot}.$$

Determine the distance from the bottom left corner of the plate to the center of mass after the smaller plate is removed. Note the mass of the plate with the void is $m_\text{p} = M_\text{tot} - m_\text{s} = 15M_\text{tot}/16$.

SKETCH:



RESEARCH: The center of mass in each dimension is $X = \frac{1}{M}\sum_{i=1}^{n} x_i m_i$ and $y = \frac{1}{M}\sum_{i=1}^{n} y_i m_i$. The center of mass of the plate with the void, (X_p, Y_p), can be determined by considering the center of mass of the total system as composed of the smaller plate of mass m_s and the plate with the void of mass m_p. Note the center of mass of the total system is at the total plate's geometric center, $(X,Y) = (L/2, L/2)$, assuming uniform density. Similarly, the center of mass of the smaller plate is at its center $(X_\text{s}, Y_\text{s}) = (L/8, L/8)$. The distance of the center of mass of the plate from the origin is then $d = \sqrt{X_\text{p}^2 + Y_\text{p}^2}$.

SIMPLIFY: $X = \frac{1}{M_\text{tot}}\left(X_\text{p} m_\text{p} + X_\text{s} m_\text{s}\right)$, and $X_\text{p} = \frac{\left(XM_\text{tot} - X_\text{s} m_\text{s}\right)}{M_\text{tot} - \frac{1}{16}M_\text{tot}} = \frac{L\left(\frac{1}{2}M_\text{tot} - \frac{1}{8}\left(\frac{1}{16}M_\text{tot}\right)\right)}{\frac{15}{16}M_\text{tot}} = \frac{21}{40}L.$

Similarly, $Y_\text{p} = \frac{\left(YM_\text{tot} - Y_\text{s} m_\text{s}\right)}{m_\text{p}} = \frac{L\left(\frac{1}{2}M_\text{tot} - \frac{1}{8}\left(\frac{1}{16}M_\text{tot}\right)\right)}{\frac{15}{16}M_\text{tot}} = \frac{21}{40}L.$

CALCULATE: $X_\text{p} = Y_\text{p} = \frac{21}{40}(5.70 \text{ cm}) = 2.9925 \text{ cm}$

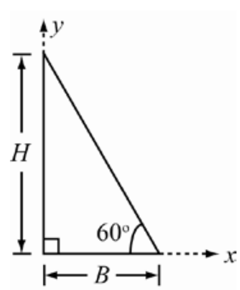
$d = \sqrt{(2.9925 \text{ cm})^2 + (2.9925 \text{ cm})^2} = 4.232 \text{ cm}$

ROUND: Since L has three significant figures, the result should be rounded to $d = 4.23$ cm.

DOUBLE-CHECK: It is expected that the center of mass for the plate with the void would be further from the origin than the center of mass for the total plate.

8.49. **THINK:** The height is H = 17.3 cm and the base is B = 10.0 cm for a flat triangular plate. Determine the x and y-coordinates of its center of mass. Since it is not stated otherwise, we assume that the mass density of this plate is constant.

SKETCH:



RESEARCH: Assuming the mass density is constant throughout the object, the center of mass is given by $\vec{R} = \frac{1}{A}\int_A \vec{r}\,dA$, where A is the area of the object. The center of mass can be determined in each dimension.

The x coordinate and the y coordinate of the center of mass are given by $X = \frac{1}{A}\int_A x\,dA$ and $Y = \frac{1}{A}\int_A y\,dA$, respectively. The area of the triangle is $A = HB/2$.

SIMPLIFY: The expression for the area of the triangle can be substituted into the formulae for the center of mass to get

$$X = \frac{2}{HB}\int_A x\,dA \text{ and } Y = \frac{2}{HB}\int_A y\,dA.$$

In the x-direction we have to solve the integral:

$$\int_A x dA = \int_0^B \int_0^{y_m(x)} x dy dx = \int_0^B x dx \int_0^{y_m(x)} dy = \int_0^B x y_m(x) dx = \int_0^B x H(1 - x/B) dx = H \int_0^B x - \left(x^2/B\right) dx$$
$$= H\left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3/B\right)\Big|_0^B = \tfrac{1}{2}HB^2 - \tfrac{1}{3}HB^2 = \tfrac{1}{6}HB^2$$

Note that in this integration procedure the maximum for the y–integration depends on the value of x: $y_m(x) = H(1 - x/B)$. Therefore we arrive at

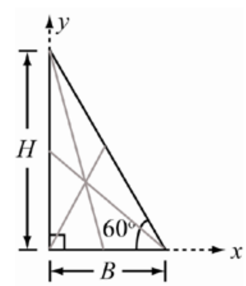
$$X = \frac{2}{HB}\int_A x\, dA = \frac{2}{HB} \cdot \frac{HB^2}{6} = \tfrac{1}{3}B$$

In the same way we can find that $Y = \tfrac{1}{3}H$.

CALCULATE: $X_{\text{com}} = \tfrac{1}{3}(10.0 \text{ cm}) = 3.33333 \text{ cm}$, $Y_{\text{com}} = \tfrac{1}{3}(17.3 \text{ cm}) = 5.76667 \text{ cm}$

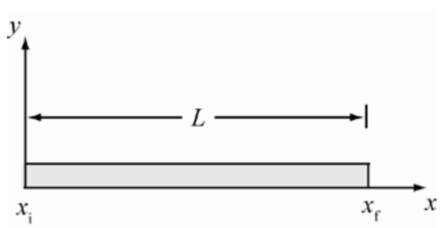
ROUND: Three significant figures were provided in the question, so the results should be written $X = 3.33 \text{ cm}$ and $Y = 5.77$ cm.

DOUBLE-CHECK: Units of length were calculated for both X and Y, which is dimensionally correct. We also find that the center of mass coordinates are inside the triangle, which always has to be true for simple geometrical shape without holes in it. Finally, we can determine the location of the center of mass for a triangle geometrically by connecting the center of each side to the opposite corner with a straight line (see drawing). The point at which these three lines intersect is the location of the center of mass. You can see from the graph that this point has to be very close to our calculated result of $(\tfrac{1}{3}B, \tfrac{1}{3}H)$.



8.50. **THINK:** The linear density function for a 1.00 m long rod is $\lambda(x) = 100. \text{ g/m} + 10.0x \text{ g/m}^2$. One end of the rod is at $x_\text{i} = 0$ m and the other end is situated at $x_\text{f} = 1.00$ m. The total mass, M of the rod and the center of mass coordinate are to be determined.

SKETCH:



RESEARCH:

(a) The linear density of the rod is given by $\lambda(x) = dm/dx$. This expression can be rearranged to get $\lambda(x)dx = dm$. An expression for $\lambda(x)$ was given so both sides can be integrated to solve for M.

(b) The center of mass coordinate is given by $X_{\text{com}} = \dfrac{1}{M}\int x dm$.

SIMPLIFY:

(a) Integrate both of sides of the linear density function to get:

$$\int_{x_\text{i}}^{x_\text{f}} \left(100. \text{ g/m} + 10.0x \text{ g/m}^2\right) dx = \int_0^M dm \Rightarrow \left[100.x \text{ g/m} + 5.0x^2 \text{ g/m}^2\right]_{x_\text{i}}^{x_\text{f}} = M.$$

(b) Substitute $dm = \lambda(x)dx$ into the expression for X_{com} to get

$$X_{\text{com}} = \frac{1}{M}\int_{x_\text{i}}^{x_\text{f}} x\lambda(x)dx.$$

The value calculated in part (a) for M can later be substituted. Substitute $\lambda(x) = 100 \text{ g/m} + 10.0x \text{ g/m}^2$ into the expression for X_{com} to get

$$X_{\text{com}} = \frac{1}{M}\int_{x_i}^{x_f}\left(100.x \text{ g/m} + 10.0x^2 \text{ g/m}^2\right)dx \Rightarrow \left[\frac{1}{M}\left(50.0x^2 \text{ g/m} + \frac{10.0}{3}x^3 \text{ g/m}^2\right)\right]_{x_i}^{x_f}.$$

CALCULATE:

(a) $M = 100. \text{ g/m}(1 \text{ m}) + 5.0 \text{ g}\dfrac{(1 \text{ m})^2}{\text{m}^2} = 105 \text{ g}$

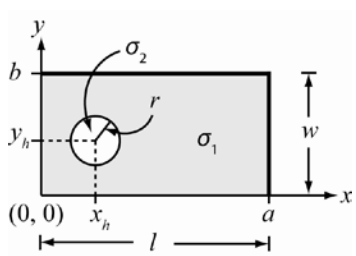
(b) $X_{\text{com}} = \dfrac{1}{105 \text{ g}}\left(50.0(1 \text{ m})^2 \text{ g/m} + \dfrac{10.0}{3}(1 \text{ m})^3 \text{ g/m}^2\right) = 0.50793651 \text{ m}$

ROUND:
Rounding to three significant figures
(a) $M = 105 \text{ g}$
(b) $X_{\text{com}} = 0.508 \text{ m}$

DOUBLE-CHECK: The correct units were calculated for the mass and the center of mass so the results are dimensionally correct. Our result for the location of the center of mass of the rod, 50.8 cm, is just larger than the geometric center of the rod, 50.0 cm. This makes sense because the density of the rod increases slightly with increasing distance.

8.51. **THINK:** The area density for a thin, rectangular plate is given as $\sigma_1 = 1.05 \text{ kg/m}^2$. Its length is a = 0.600 m and its width is b = 0.250 m. The lower left corner of the plate is at the origin. A circular hole of radius, r = 0.0480 m is cut out of the plate. The hole is centered at the coordinates $x_{\text{h}} = 0.068 \text{ m}$ and $y_{\text{h}} = 0.068 \text{ m}$. A round disk of radius, r is used to plug the hole. The disk, D, has a uniform area density of $\sigma_2 = 5.32 \text{ kg/m}^2$. The distance from the origin to the modified plate's center of mass, R, is to be determined.

SKETCH:



RESEARCH: The center of mass, R, of an object can be defined mathematically as $R = \frac{1}{M}\sum_{i=1}^{n}\vec{r}_i m_i$ (1). In this equation, M is the total mass of the system. The vector $\vec{r}_i$ denotes the position of the i^{th} object's center of mass and m_{i} is the mass of that object. To solve this problem, the center of mass of the plate, R_{p}, and the center of mass of the disk, $R_{\text{D,}}$ must be determined. Then equation (1) can be used to determine the distance from the origin to the modified center of mass, R. First, consider the rectangular plate, P, which has the hole cut in it. The position of the center of mass, R_{p}, is not known. The mass of P can be denoted m_{p}. Consider the disk of material, d, that was removed (which has a uniform area density of σ_1), and denote its center of mass as R_{d} and its mass as m_{d}. Next, define S as the system of the rectangular plate, P, and the disc of removed material, d. The mass of S can be denoted $m_{\text{s}} = m_{\text{p}} + m_{\text{d}}$. The center of mass of S is $R_{\text{s}} = (a/2)\hat{x} + (b/2)\hat{y}$. m_{p} and m_{d} are not known but it is known that they have uniform area density of σ_1. The uniform area density is given by $\sigma = m/A$. Therefore, $m_{\text{p}} = \sigma_1 A_{\text{p}}$ and

$m_{\text{d}} = \sigma_1 A_{\text{d}}$, where A_{p} is the area of the plate minus the area of the hole and A_{d} is the area of the disk, *d*. The expressions for these areas are $A_{\text{p}} = ab - \pi r^2$ and $A_{\text{d}} = \pi r^2$. Substituting these area expressions into the expressions for m_{p} and m_{d} gives $m_{\text{p}} = \sigma_1\left(ab - \pi r^2\right)$ and $m_{\text{d}} = \sigma_1 \pi r^2$. So the center of mass of the system is given by:

$$\vec{R}_{\text{s}} = \frac{\left(x_{\text{h}}\hat{x} + y_{\text{h}}\hat{y}\right)m_{\text{d}} + \vec{R}_{\text{p}}m_{\text{p}}}{\sigma_1\left(ab - \pi r^2\right) + \sigma_1 \pi r^2} \quad (2).$$

Now, consider the disk, *D*, that is made of the material of uniform area density, σ_2. Define its center of mass as $\vec{R}_{\text{D}} = x_{\text{h}}\hat{x} + y_{\text{h}}\hat{y}$. Also, define its mass as $m_{\text{D}} = \sigma_2 \pi r^2$.

SIMPLIFY: Rearrange equation (2) to solve for $\vec{R}_{\text{p}}$:

$$\vec{R}_{\text{p}}m_{\text{p}} = \vec{R}_{\text{s}}\sigma_1 ab - \left(x_{\text{h}}\hat{x} + y_{\text{h}}\hat{y}\right)m_{\text{d}} \Rightarrow \vec{R}_{\text{p}} = \frac{\vec{R}_{\text{s}}\sigma_1 ab - \left(x_{\text{h}}\hat{x} + y_{\text{h}}\hat{y}\right)m_{\text{d}}}{m_{\text{p}}}.$$

Now, substitute the values for $\vec{R}_{\text{s}}$, m_{d} and m_{p} into the above equation to get:

$$\vec{R}_{\text{p}} = \frac{\left(\frac{a}{2}\hat{x} + \frac{b}{2}\hat{y}\right)\sigma_1 ab - \left(x_{\text{h}}\hat{x} + y_{\text{h}}\hat{y}\right)\sigma_1 \pi r^2}{\sigma_1\left(ab - \pi r^2\right)}.$$

Once $\vec{R}_{\text{p}}$ is solved, it can be substituted into the expression for $\vec{R}$ to get $\vec{R} = \dfrac{\vec{R}_{\text{p}}m_{\text{p}} + \vec{R}_{\text{D}}m_{\text{D}}}{m_{\text{p}} + m_{\text{D}}}$. Use the distance formula $R = \sqrt{R_x^{\ 2} + R_y^{\ 2}}$.

CALCULATE:

$$\vec{R}_{\text{p}} = \frac{\left(\frac{0.600}{2}\hat{x} + \frac{0.250}{2}\hat{y}\right)\left(\left(1.05\text{ kg/m}^3\right)\left(0.600\text{ m}\right)\left(0.250\text{ m}\right)\right) - \left(0.068\hat{x} + 0.068\hat{y}\right)\left(1.05\text{ kg/m}^3\right)\pi\left(0.0480\text{ m}\right)^2}{\left(1.05\text{ kg/m}^3\right)\left(\left(0.600\text{ m}\right)\left(0.250\text{ m}\right) - \pi\left(0.0480\text{ m}\right)^2\right)}$$

$$= \left(0.31176\hat{x} + 0.12789\hat{y}\right)\text{ m}$$

$$\vec{R} = \frac{\left(0.31176\hat{x} + 0.12789\hat{y}\right)\text{ m}\left(0.1499\text{ kg}\right) + \left(0.068\hat{x} + 0.068\hat{y}\right)\text{ m}\left(0.038507\text{ kg}\right)}{0.1499\text{ kg} + 0.038507\text{ kg}}$$

$$= \left(0.26194\hat{x} + 0.11565\hat{y}\right)\text{ m}$$

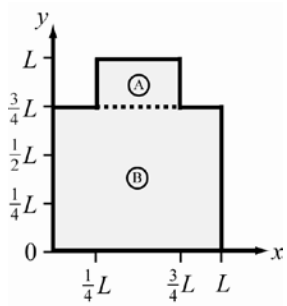
Then, the distance to the origin is given by $R = \sqrt{\left(0.26194\text{ m}\right)^2 + \left(0.11565\text{ m}\right)^2} = 0.28633\text{ m}$.

ROUND: Densities are given to three significant figures. For dimensions the subtraction rule applies, where all dimensions are known to three decimal places. The result should be rounded to $R = 0.286$ m.

DOUBLE-CHECK: The position of the center of mass for the modified system is shifted closer to the position of the disk, *D*, which has an area density of 5.32 kg/m^2. This is reasonable because the disk has a much higher area density than the rest of the plate. Also, the results are reasonable considering the given values.

8.52. **THINK:** The object of interest is a uniform square metal plate with sides of length, *L* = 5.70 cm and mass, *m* = 0.205 kg. The lower left corner of the plate sits at the origin. Two squares with side length, *L*/4 are removed from each side at the top of the square. Determine the *x*-coordinate and the *y*-coordinate of the center of mass, denoted X_{com} and Y_{com}, respectively.

SKETCH:



RESEARCH: Because the square is uniform, the equations for X_{com} and Y_{com} can be expressed by

$$X_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \text{ and } Y_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i.$$

M is the total mass of the system. In this problem it will be useful to treat the system as if it were made up of two uniform metal rectangles, R_A and R_B.

(a) The center of mass x-coordinate for rectangle A is $x_{\text{A}} = (L/2)\hat{x}$. The center of mass x-coordinate for rectangle B is $x_{\text{B}} = (L/2)\hat{x}$.

(b) The center of mass y-coordinate for rectangle A is $y_{\text{A}} = (7L/8)\hat{y}$. The center of mass y-coordinate for rectangle B is $y_{\text{B}} = (3L/8)\hat{y}$. Both rectangles have the same uniform area density, σ. The uniform area density is given by $\sigma = m_{\text{A}}/A_{\text{A}} = m_{\text{B}}/A_{\text{B}}$. Therefore, $m_{\text{A}} = m_{\text{B}}A_{\text{A}}/A_{\text{B}}$. The areas are given by the following expressions:

$$A_{\text{A}} = \left(\frac{L}{4}\right)\left(\frac{L}{2}\right) = \frac{L^2}{8} \text{ and } A_{\text{B}} = \left(\frac{3L}{4}\right)L = \frac{3L^2}{4}.$$

SIMPLIFY:

(a) $X_{\text{com}} = \dfrac{x_{\text{A}}m_{\text{A}} + x_{\text{B}}m_{\text{B}}}{m_{\text{A}} + m_{\text{B}}}$

Substitute the expression for m_{A} into the above equation to get:

$$X_{\text{com}} = \frac{x_{\text{A}}m_{\text{B}}\dfrac{A_{\text{A}}}{A_{\text{B}}} + x_{\text{B}}m_{\text{B}}}{m_{\text{B}}\dfrac{A_{\text{A}}}{A_{\text{B}}} + m_{\text{B}}} = \frac{x_{\text{A}}\left(\dfrac{A_{\text{A}}}{A_{\text{B}}}\right) + x_{\text{B}}}{\dfrac{A_{\text{A}}}{A_{\text{B}}} + 1}.$$

Then substitute the expressions for x_{A}, x_{B}, A_{A} and A_{B} to get:

$$X_{\text{com}} = \frac{\dfrac{L}{2}\left(\dfrac{L^2/8}{3L^2/4}\right) + \dfrac{L}{2}}{\dfrac{L^2/8}{3L^2/4} + 1} = \frac{\dfrac{L}{12} + \dfrac{L}{2}}{\dfrac{1}{6} + 1} = \frac{\dfrac{7L}{12}}{\dfrac{7}{6}} = \frac{1}{2}L.$$

(b) The same procedure can be used to solve for the y-coordinate of the center of mass:

$$Y_{\text{com}} = \frac{y_{\text{A}}\left(\dfrac{A_{\text{A}}}{A_{\text{B}}}\right) + y_{\text{B}}}{\dfrac{A_{\text{A}}}{A_{\text{B}}} + 1} = \frac{\dfrac{7L}{8}\left(\dfrac{1}{6}\right) + \dfrac{3L}{8}}{\dfrac{7}{6}} = \frac{\dfrac{7L}{48} + \dfrac{18L}{48}}{\dfrac{7}{6}} = \frac{25L}{48}\left(\frac{6}{7}\right) = \frac{25L}{56}.$$

CALCULATE:

(a) $X_{\text{com}} = \dfrac{1}{2}(5.70 \text{ cm}) = 2.85 \text{ cm}$

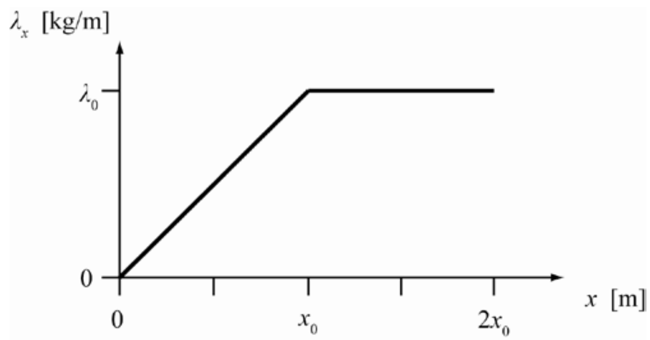
(b) $Y_{\text{com}} = \frac{25}{56}(5.70 \text{ cm}) = 2.5446 \text{ cm}$

ROUND: Three significant figures were provided in the question, so the results should be rounded to $X_{\text{com}} = 2.85 \text{ cm}$ and $Y_{\text{com}} = 2.54 \text{ cm}$.

DOUBLE-CHECK: Units of distance were calculated, which is expected when calculating the center of mass coordinates. The squares were removed uniformly at the top of the large square, so it makes sense that the *x*-coordinate of the center of mass stays at *L*/2 by symmetry and the *y*-coordinate of the center of mass is shifted slightly lower.

8.53. **THINK:** The linear mass density, $\lambda(x)$, is provided in the graph. Determine the location for the center of mass, X_{com}, of the object. From the graph, it can be seen that $\lambda(x) = \begin{Bmatrix} \frac{\lambda_0}{x_0}x, \ 0 \le x < x_0 \\ \lambda_0, \ x_0 \le x \le 2x_0 \end{Bmatrix}$.

SKETCH:



RESEARCH: The linear mass density, $\lambda(x)$, depends on *x*. To determine the center of mass, use the equation $X_{\text{com}} = \frac{1}{M}\int_L x\lambda(x)dx$. The mass of the system, M, can be determined using the equation $M = \int_L \lambda(x)dx$. In order to evaluate the center of mass of the system, two separate regions must be considered; the region from *x* = 0 to $x = x_0$ and the region from $x = x_0$ to $x = 2x_0$. The equation for X_{com} can be expanded to $X_{\text{com}} = \frac{1}{M}\int_0^{x_0} x\frac{\lambda_0}{x_0}xdx + \frac{1}{M}\int_{x_0}^{2x_0} \lambda_0 xdx$. The equation for *M* is $M = \int_0^{x_0} \frac{\lambda_0}{x_0}xdx + \int_{x_0}^{2x_0} \lambda_0 dx$.

SIMPLIFY: Simplify the expression for *M* first and then substitute it into the expression for X_{com}.

$$M = \int_0^{x_0} \frac{\lambda_0}{x_0}xdx + \int_{x_0}^{2x_0} \lambda_0 dx = \left[\frac{1}{2}\frac{\lambda_0}{x_0}x^2\right]_0^{x_0} + \left[x\lambda_0\right]_{x_0}^{2x_0} = \frac{1}{2}\lambda_0 x_0 + 2x_0\lambda_0 - x_0\lambda_0 = \frac{3}{2}x_0\lambda_0.$$

Substitute the above expression into the equation for X_{com} to get:

$$X_{\text{com}} = \frac{2}{3x_0\lambda_0}\left[\int_0^{x_0} x^2\frac{\lambda_0}{x_0}dx + \int_{x_0}^{2x_0}\lambda_0 xdx\right] = \frac{2}{3x_0\lambda_0}\left[\frac{1}{3}\lambda_0 x_0^2 + 2\lambda_0 x_0^2 - \frac{1}{2}\lambda_0 x_0^2\right] = \frac{2}{3x_0\lambda_0}\left[\lambda_0 x_0^2\left(\frac{2}{6}+\frac{12}{6}-\frac{3}{6}\right)\right]$$

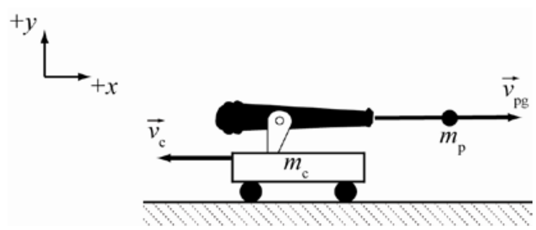
$$= \frac{2}{3x_0\lambda_0}\left(\frac{11}{6}\lambda_0 x_0^2\right) = \frac{11x_0}{9}.$$

CALCULATE: This step does not apply.

ROUND: This step does not apply.

DOUBLE-CHECK: The units for the result are units of length, so the answer is dimensionally correct. It is reasonable that the calculated value is closer to the denser end of the object.

8.54. **THINK:** The mass of the cannon is $m_c = 750$ kg and the mass of the projectile is $m_p = 15$ kg. The total mass of the cannon and projectile system is $M = m_c + m_p$. The speed of the projectile is $v_p = 250$ m/s with respect to the muzzle just after the cannon has fired. The cannon is on wheels and can recoil with negligible friction. Determine the speed of the projectile with respect to the ground, v_{pg}.
SKETCH:



RESEARCH: The problem can be solved by considering the conservation of linear momentum. The initial momentum is $\vec{P}_i = 0$ because the cannon and projectile are both initially at rest. The final momentum is $\vec{P}_f = m_c\vec{v}_c + m_p\vec{v}_{pg}$. The velocity of the recoiling cannon is v_c. The equation for the conservation of momentum is $\vec{P}_i = \vec{P}_f$. The velocity of the projectile with respect to the cannon's muzzle can be represented as $\vec{v}_p = \vec{v}_{pg} - \vec{v}_c$. Take $\vec{v}_{pg}$ to be in the positive x-direction.

SIMPLIFY: Rearrange the above equation so that it becomes $\vec{v}_c = \vec{v}_{pg} - \vec{v}_p$. Then substitute this expression into the conservation of momentum equation:

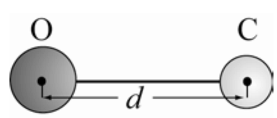
$$P_i = P_f \Rightarrow 0 = m_c v_c + m_p v_{pg} \Rightarrow 0 = m_c\left(v_{pg} - v_p\right) + m_p v_{pg} \Rightarrow v_{pg}\left(m_c + m_p\right) = m_c v_p \Rightarrow v_{pg} = \frac{m_c v_p}{\left(m_c + m_p\right)}.$$

CALCULATE: $v_{pg} = \dfrac{(750 \text{ kg})(250 \text{ m/s})}{(750 \text{ kg} + 15 \text{ kg})} = 245.098$ m/s

ROUND: The least number of significant figures provided in the question is three, so the result should be rounded to $v_{pg} = 245$ m/s.

DOUBLE-CHECK: The units of speed are correct for the result. The velocity calculated for the projectile with respect to the ground is slower than its velocity with respect to the cannon's muzzle, which is what is expected.

8.55. **THINK:** The mass of a carbon atom is $m_C = 12.0$ u and the mass of an oxygen atom is $m_O = 16.0$ u. The distance between the atoms in a CO molecule is $d = 1.13 \cdot 10^{-10}$ m. Determine how far the center of mass, X_{com}, is from the carbon atom. Denote the position of the carbon atoms as X_C and the position of the oxygen atom as X_O.
SKETCH:



RESEARCH: The center of mass of a system is given by $X_{com} = \dfrac{1}{M}\sum_{i=1}^{n} m_i x_i$.

The total mass of the system is $M = m_C + m_O$. It is convenient to assign the position of the oxygen atom to be at the origin, $X_O = 0$. Then the center of mass becomes

$$X_{com} = \frac{(0)m_O + m_C d}{m_O + m_C} = \frac{m_C d}{m_O + m_C}.$$

Once X_{com} is determined, then the distance from it to the carbon atom can be determined using the equation $X_{\text{dC}} = X_{\text{C}} - X_{\text{com}}$, where X_{dC} is the distance from the center of mass to the carbon atom.

SIMPLIFY: Substitute the expression $X_{\text{com}} = (m_{\text{C}}d)/(m_{\text{O}} + m_{\text{C}})$ into the expression for X_{dC} to get $X_{\text{dC}} = X_{\text{C}} - \dfrac{m_{\text{C}}d}{m_{\text{O}} + m_{\text{C}}}$. Substitute $X_C = d$ to get $X_{\text{dC}} = d - \dfrac{m_{\text{C}}d}{m_{\text{O}} + m_{\text{C}}}$.

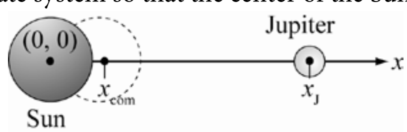
CALCULATE: $X_{\text{dC}} = (1.13 \cdot 10^{-10} \text{ m}) - \left(\dfrac{12.0 \text{ u}}{28.0 \text{ u}}\right)(1.13 \cdot 10^{-10} \text{ m}) = 6.4571 \cdot 10^{-11} \text{ m}$

ROUND: Three significant figures were provided in the problem so the answer should be rounded to $X_{\text{dC}} = 6.46 \cdot 10^{-11}$ m.

DOUBLE-CHECK: The center of mass of the system is closer to the more massive oxygen atom, as it should be.

8.56. **THINK:** The system to be considered consists of the Sun and Jupiter. Denote the position of the Sun's center of mass as X_{S} and the mass as m_{S}. Denote the position of Jupiter's center of mass as X_{J} and its mass as m_{J}. Determine the distance that the Sun wobbles due to its rotation about the center of mass. Also, determine how far the system's center of mass, X_{com}, is from the center of the Sun. The mass of the Sun is $m_{\text{S}} = 1.98892 \cdot 10^{30}$ kg. The mass of Jupiter is $m_{\text{J}} = 1.8986 \cdot 10^{27}$ kg. The distance from the center of the Sun to the center of Jupiter is $X_{\text{J}} = 7.78 \cdot 10^{8}$ km.

SKETCH: Construct the coordinate system so that the center of the Sun is positioned at the origin.



RESEARCH: The system's center of mass is given by $X_{\text{com}} = \dfrac{1}{M}\sum_{i=1}^{n} m_i x_i$.

The total mass of the system is $M = m_{\text{S}} + m_{\text{J}}$. The dashed line in the sketch denotes the Sun's orbit about the system's center of mass. From the sketch it can be seen that the distance the sun wobbles is twice the distance from the Sun's center to the system's center of mass.

SIMPLIFY: $X_{\text{com}} = \dfrac{m_{\text{S}}X_{\text{S}} + m_{\text{J}}X_{\text{J}}}{m_{\text{S}} + m_{\text{J}}}$. The coordinate system was chosen in such a way that $X_{\text{S}} = 0$. The center of mass equation can be simplified to $X_{\text{com}} = \dfrac{m_{\text{J}}X_{\text{J}}}{m_{\text{S}} + m_{\text{J}}}$. Once X_{com} is determined, it can be doubled to get the Sun's wobble.

CALCULATE: $X_{\text{com}} = \dfrac{(1.8986 \cdot 10^{27} \text{ kg})(7.78 \cdot 10^{8} \text{ km})}{1.98892 \cdot 10^{30} \text{ kg} + 1.8986 \cdot 10^{27} \text{ kg}} = 741961.5228 \text{ km}$

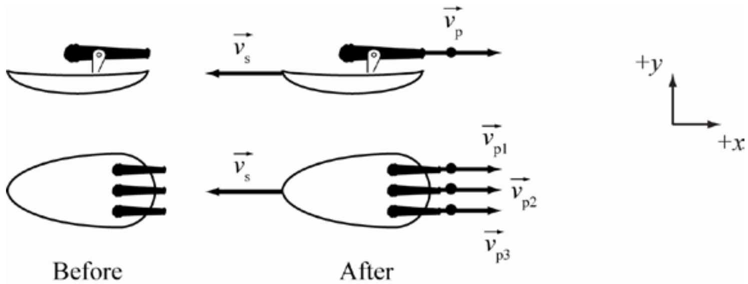
The Sun's wobble is $2(741961.5228 \text{ km}) = 1483923.046$ km.

ROUND: Rounding the results to three figures, $X_{\text{com}} = 7.42 \cdot 10^{5}$ km and the Sun's wobble is $1.49 \cdot 10^{6}$ km.

DOUBLE-CHECK: It is expected that the system's center of mass is much closer to the Sun than it is to Jupiter, and the results are consistent with this.

8.57. **THINK:** The mass of the battleship is $m_{\text{s}} = 136{,}634{,}000$ lbs. The ship has twelve 16-inch guns and each gun is capable of firing projectiles of mass, $m_{\text{p}} = 2700.$ lb, at a speed of $v_{\text{p}} = 2300.$ ft/s. Three of the guns fire projectiles in the same direction. Determine the recoil velocity, v_{s}, of the ship. Assume the ship is initially stationary.

SKETCH:



RESEARCH: The total mass of the ship and projectile system is $M = m_s + \sum_{i=1}^{n} m_{pi}$.

All of the projectiles have the same mass and same speed when they are shot from the guns. This problem can be solved considering the conservation of momentum. The equation for the conservation of momentum is $\vec{P}_i = \vec{P}_f$. $\vec{P}_i$ is the initial momentum of the system and $\vec{P}_f$ is the final momentum of the system. Assume the ship carries one projectile per gun. $\vec{P}_i = 0$ because the battleship is initially at rest and $\vec{P}_f = -\left(m_s + 9m_p\right)v_s + 3m_p v_p$.

SIMPLIFY: $\vec{P}_i = \vec{P}_f \Rightarrow 0 = -\left(m_s + 9m_p\right)v_s + 3m_p v_p \Rightarrow v_s = \dfrac{3m_p v_p}{\left(m_s + 9m_p\right)}$

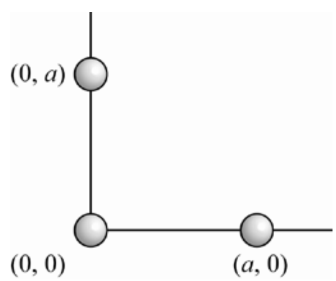
CALCULATE: $v_s = \dfrac{3(2700.\text{ lb})(2300.\text{ ft/s})}{\left(136{,}634{,}000\text{ lb} + 9(2700.\text{ lb})\right)} = 0.136325\text{ ft/s}$

ROUND: The values for the mass and speed of the projectile that are given in the question have four significant figures, so the result should be rounded to $v_s = 0.1363$ ft/s. The recoil velocity is in opposite direction than the cannons fire.

DOUBLE-CHECK: The mass of the ship is much greater than the masses of the projectiles, so it reasonable that the recoil velocity is small because momentum depends on mass and velocity.

8.58. **THINK:** The system has three identical balls of mass *m*. The *x* and *y* coordinates of the balls are $\vec{r}_1 = \left(0\hat{x}, 0\hat{y}\right)$, $\vec{r}_2 = \left(a\hat{x}, 0\hat{y}\right)$ and $\vec{r}_3 = \left(0\hat{x}, a\hat{y}\right)$. Determine the location of the system's center of mass, *R*.

SKETCH:



RESEARCH: The center of mass is a vector quantity, so the *x* and *y* components must be considered separately. The *x*- and *y*-components of the center of mass are given by

$$X_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \text{ and } Y_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i.$$

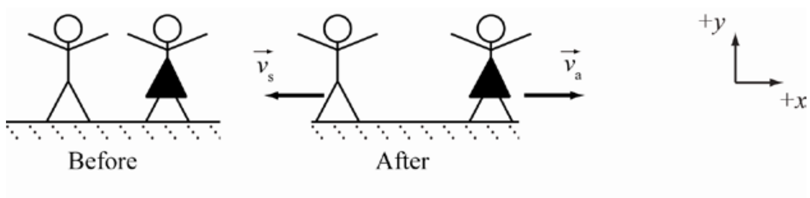
For this system, the equations can be rewritten as

$$X_{com} = \frac{m(0) + ma\hat{x} + m(0)}{3m} = \frac{a}{3}\hat{x} \text{ and } Y_{com} = \frac{m(0) + m(0) + ma\hat{y}}{3m} = \frac{a}{3}\hat{y}.$$

SIMPLIFY: The x and y components of the center of mass are known, so $\vec{R}_{com} = \frac{a}{3}\hat{x} + \frac{a}{3}\hat{y}.$

CALCULATE: This step is not necessary.
ROUND: This step is not necessary.
DOUBLE-CHECK: Considering the geometry of the system, the results are reasonable. In the *x*-direction we would expect the center of mass to be twice as far from the mass on the right as from the two on the left, and in the *y*-direction we would expect the center of mass to be twice as far from the upper mass as from the two lower ones.

8.59. **THINK:** Sam's mass is $m_S = 61.0$ kg and Alice's mass is $m_A = 44.0$ kg. They are standing on an ice rink with negligible friction. After Sam pushes Alice, she is moving away from him with a speed of $v_A = 1.20$ m/s with respect to the rink. Determine the speed of Sam's recoil, v_S. Also, determine the change in kinetic energy, ΔK, of the Sam-Alice system.
SKETCH:



RESEARCH:
(a) To solve the problem, consider the conservation of momentum. The equation for conservation of momentum can be written $\vec{P}_i = \vec{P}_f$. $\vec{P}_i$ is the initial momentum of the system and $\vec{P}_f$ is the final momentum of the system. $\vec{P}_i = 0$ because Sam and Alice are initially stationary and $\vec{P}_f = -m_S\vec{v}_S + m_A\vec{v}_A$.
(b) The change in kinetic energy is $\Delta K = K_f - K_i = \left(m_S v_S^2\right)/2 + \left(m_A v_A^2\right)/2.$
SIMPLIFY:
(a) $\vec{P}_i = \vec{P}_f \Rightarrow 0 = -m_S\vec{v}_S + m_A\vec{v}_A \Rightarrow v_S = \dfrac{m_A\vec{v}_A}{m_S}$
(b) The expression determined for v_S in part (a) can be substituted into the equation for ΔK to get

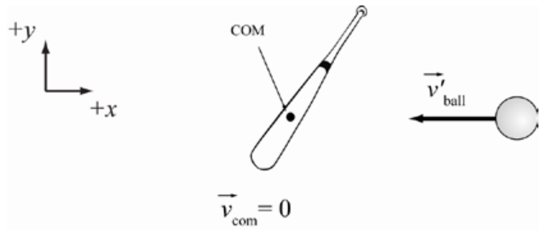
$$\Delta K = \frac{1}{2}m_S\left(\frac{m_A\vec{v}_A}{m_S}\right)^2 + \frac{1}{2}m_A v_A^2.$$

CALCULATE:
(a) $v_S = \dfrac{(44.0 \text{ kg})(1.20 \text{ m/s})}{61.0 \text{ kg}} = 0.8656$ m/s

(b) $\Delta K = \frac{1}{2}(61.0 \text{ kg})\left(\dfrac{(44.0 \text{ kg})(1.20 \text{ m/s})}{61.0 \text{ kg}}\right)^2 + \frac{1}{2}(44.0 \text{ kg})(1.20 \text{ m/s})^2 = 54.53$ J
(c) Sam did work on Alice when he pushed her. The work that Sam did was the source of the kinetic energy. Sam was able to do this work by converting chemical energy that was stored in his body into mechanical energy. The energy stored in Sam's body was provided by food that he ate and his body processed.
ROUND: Three significant figures were provided in the problem so the results should be rounded accordingly to $v_S = 0.866$ m/s and $\Delta K = 55$ J.
DOUBLE-CHECK: Sam's mass is greater than Alice's so it reasonable that his recoil speed is slower than her sliding speed. The change in kinetic energy is reasonable considering the masses and velocities given.

8.60. **THINK:** The mass of the bat is m_{bat} and the mass of the ball is m_{ball}. Assume that the center of mass of the ball and bat system is essentially at the bat. The initial velocity of the ball is $\vec{v}_{\text{ball,i}} = -30.0$ m/s and the initial velocity of the bat is $\vec{v}_{\text{bat}} = 35.0$ m/s. The bat and ball undergo a one-dimensional elastic collision. Determine the speed of the ball after the collision.

SKETCH:



RESEARCH: In the center of mass frame, $\vec{v}_{\text{com}} = 0$. Since the collision is elastic, in the center of mass frame the final velocity of the ball, $\vec{v}_{\text{ball,f}}$, will be equal to the negative of the ball's initial velocity, $\vec{v}_{\text{ball,i}}$. This statement can be written mathematically as $\vec{v}_{\text{ball,i}} = -\vec{v}_{\text{ball,f}}$. Since the center of mass is in the bat, the $\vec{v}_{\text{com}}$ in the lab reference frame equals $\vec{v}_{\text{bat}}$. The following relationships can be written for this system: $\vec{v}'_{\text{ball,i}} = \vec{v}_{\text{ball,i}} - \vec{v}_{\text{com}}$ (1) and $\vec{v}'_{\text{ball,f}} = \vec{v}_{\text{ball,f}} - \vec{v}_{\text{com}}$ (2).

SIMPLIFY: Recall that $\vec{v}'_{\text{ball,i}} = -\vec{v}'_{\text{ball,f}}$. Therefore, the following equality can be written:

$\vec{v}_{\text{ball,i}} - \vec{v}_{\text{com}} = -\left(\vec{v}_{\text{ball,f}} - \vec{v}_{\text{com}}\right) \Rightarrow \vec{v}_{\text{ball,f}} = 2\vec{v}_{\text{com}} - \vec{v}_{\text{ball,i}}$.

Recall that $\vec{v}_{\text{com}}$ is equal to $\vec{v}_{\text{bat}}$, so the above expression can be rewritten as $\vec{v}_{\text{ball,f}} = 2\vec{v}_{\text{bat}} - \vec{v}_{\text{ball,i}}$.

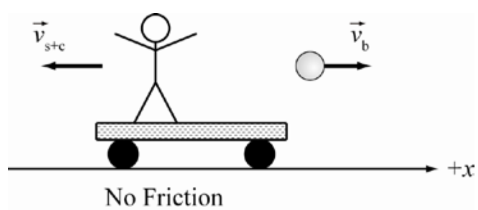
CALCULATE: $\vec{v}_{\text{ball,f}} = 2(35.0 \text{ m/s}) - (-30.0 \text{ m/s}) = 100.0 \text{ m/s}$

ROUND: Rounding to three significant figures: $\vec{v}_{\text{ball,f}} = 100.$ m/s

DOUBLE-CHECK: The initial velocities of the bat and ball are similar, but the bat is much more massive than the ball, so the speed of the ball after the collision is expected to be high.

8.61. **THINK:** The student's mass is $m_{\text{s}} = 40.0$ kg, the ball's mass is $m_{\text{b}} = 5.00$ kg and the cart's mass is $m_{\text{c}} = 10.0$ kg. The ball's relative speed is $v'_{\text{b}} = 10.0$ m/s and the student's initial speed is $v_{\text{si}} = 0$. Determine the ball's velocity with respect to the ground, $\vec{v}_{\text{b}}$, after it is thrown.

SKETCH:



RESEARCH: $\vec{v}_{\text{b}}$ can be determined by considering the conservation of momentum, $\vec{P}_{\text{i}} = \vec{P}_{\text{f}}$, where $p = mv$. Note the ball's relative speed is $\vec{v}'_{\text{b}} = \vec{v}_{\text{b}} - \vec{v}_{\text{s+c}}$, where $\vec{v}_{\text{b}}$ and $\vec{v}_{\text{s+c}}$ are measured relative to the ground.

SIMPLIFY: $\vec{P}_{\text{i}} = \vec{P}_{\text{f}} \Rightarrow 0 = \left(m_{\text{s}} + m_{\text{c}}\right)\vec{v}_{\text{s+c}} + m_{\text{b}}\vec{v}_{\text{b}} \Rightarrow 0 = \left(m_{\text{s}} + m_{\text{c}}\right)\left(\vec{v}_{\text{b}} - \vec{v}'_{\text{b}}\right) + m_{\text{b}}\vec{v}_{\text{b}} \Rightarrow \vec{v}_{\text{b}} = \dfrac{\vec{v}'_{\text{b}}\left(m_{\text{s}} + m_{\text{c}}\right)}{m_{\text{s}} + m_{\text{c}} + m_{\text{b}}}$

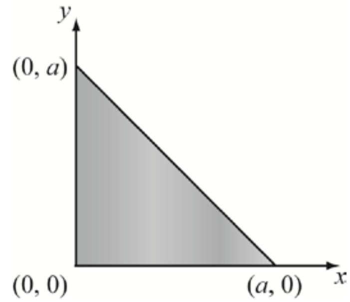
CALCULATE: $\vec{v}_{\text{b}} = \dfrac{(10.0 \text{ m/s})(40.0 \text{ kg} + 10.0 \text{ kg})}{(40.0 \text{ kg} + 10.0 \text{ kg} + 5.00 \text{ kg})} = 9.0909 \text{ m/s}$

ROUND: $\vec{v}_{\text{b}} = 9.09$ m/s in the direction of $\vec{v}'_{\text{b}}$ (horizontal)

DOUBLE-CHECK: It is expected that $v_{\text{b}} < v'_{\text{b}}$ since the student and cart move away from the ball when it is thrown.

8.62. **THINK:** Determine the center of mass of an isosceles triangle of constant density σ.

SKETCH:



RESEARCH: To determine the center of mass of a two-dimensional object of constant density σ, use $X = \frac{1}{A}\int_A \sigma x dA$ and $Y = \frac{1}{A}\int_A \sigma y dA.$

SIMPLIFY: Note the boundary condition on the hypotenuse of the triangle, $x + y = a.$ First, determine X. As x varies, take $dA = ydx.$ Then the equation becomes $X = \frac{\sigma}{A}\int_0^a xydx.$ From the boundary condition, $y = a - x.$ Then the equation can be rewritten as $X = \frac{\sigma}{A}\int_0^a x(a-x)dx = \left[\frac{\sigma}{A}\left(\frac{1}{2}ax^2 - \frac{1}{3}x^3\right)\right]_0^a = \frac{a^3\sigma}{6A}.$ Similarly for Y, take $dA = xdy$ and $x = a - y$ to get $Y = \frac{\sigma}{A}\int_0^a y(a-y)dy = \left[\frac{\sigma}{A}\left(\frac{1}{2}ay^2 - \frac{1}{3}y^3\right)\right]_0^a = \frac{a^3\sigma}{6A},$ with $A = \int \sigma dA = \sigma \cdot \frac{bh}{2} = \frac{a^2\sigma}{2}$ we get $X = Y = \frac{2}{a^2\sigma}\cdot\frac{a^3\sigma}{6A} = \frac{a}{3}.$

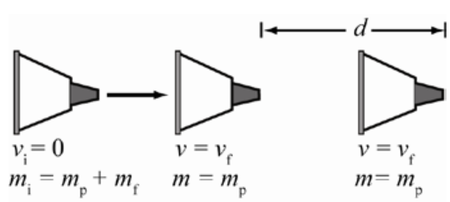
CALCULATE: This step is not applicable.

ROUND: This step is not applicable.

DOUBLE-CHECK: The center of mass coordinates that we obtained are contained within the isosceles triangle, as expected for a solid object.

8.63. **THINK:** The payload's mass is $m_\text{p} = 4390.0 \text{ kg}$ and the fuel mass is $m_\text{f} = 1.761\cdot 10^5 \text{ kg}.$ The initial velocity is $v_\text{i} = 0.$ The distance traveled after achieving v_f is $d = 3.82\cdot 10^8 \text{ m}.$ The trip time is $t = 7.00 \text{ h} = 2.52\cdot 10^4 \text{ s}.$ Determine the propellant expulsion speed, $v_\text{c}.$

SKETCH:



RESEARCH: v_c can be determined from $v_\text{f} - v_\text{i} = v_\text{c}\ln(m_\text{i}/m_\text{f}).$ First, v_f must be determined from the relationship $v = d/t.$

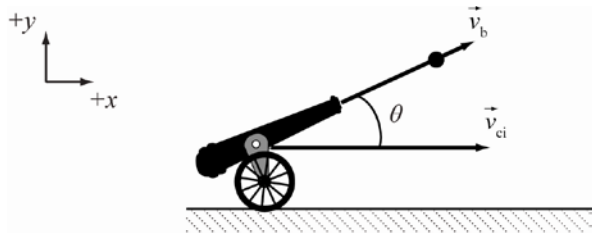
SIMPLIFY: First, determine v_f from $v_\text{f} = d/t.$ Substitute this expression and $v_\text{i} = 0$ into the above equation to determine v_c:

$$v_\text{c} = \frac{v_\text{f}}{\ln\left(\frac{m_\text{i}}{m_\text{f}}\right)} = \frac{d}{t\ln\left(\frac{m_\text{i}}{m_\text{f}}\right)} = \frac{d}{t\ln\left(\frac{m_\text{p}+m_\text{f}}{m_\text{p}}\right)}.$$

CALCULATE: $v_c = \dfrac{3.82\cdot 10^8\text{ m}}{\left(2.52\cdot 10^4\text{ s}\right)\ln\left(\dfrac{4390.0\text{ kg}+1.761\cdot 10^5\text{ kg}}{4390.0\text{ kg}}\right)} = 4.079\cdot 10^3\text{ m/s}$

ROUND: Since t has three significant figures, the result should be rounded to $v_c = 4.08$ km/s.
DOUBLE-CHECK: This expulsion velocity is reasonable.

8.64. **THINK:** The cannon's mass is M = 350 kg. The cannon's initial speed is $v_{ci} = 7.5$ m/s. The ball's mass is m = 15 kg and the launch angle is $\theta = 55°$. The cannon's final velocity after the shot is $v_{cf} = 0$. Determine the velocity of the ball relative to the cannon, $\vec{v}\,'_b$.
SKETCH:



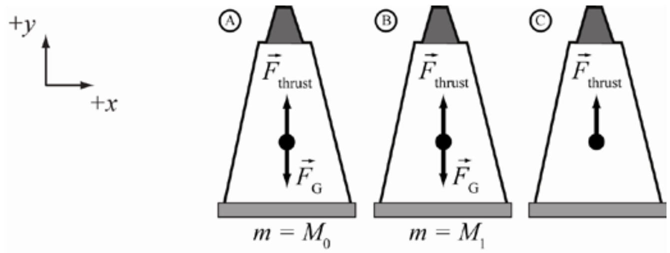
RESEARCH: Use conservation of momentum, $\vec{P}_i = \vec{P}_f$, where $\vec{P} = m\vec{v}$. To determine the relative velocity, $\vec{v}\,'_b$, with respect to the cannon, use $\vec{v}\,'_b = \vec{v}_b - \vec{v}_c$, where $\vec{v}_b$ is the ball's velocity in the lab frame. Finally, since the cannon moves only in the horizontal (x) direction, consider only momentum conservation in this dimension. Take $\vec{v}_{ci}$ to be along the positive x-direction, that is $v_{ci} = +7.5$ m/s. With v_{bx} known, find v_b from the expression $v_{bx} = v_b\cos\theta$ and then v'_b can be determined.
SIMPLIFY: $P_{xi} = P_{xf} \Rightarrow \left(m_b + m_c\right)v_{ci} = m_c v_{cf} + m_b v_{bx}$. Note since v_{cf} is zero, $v_{bx} = v'_{bx}$, that is, the ball's speed relative to the cannon is the same as its speed in the lab frame since the cannon has stopped moving.
Rearranging the above equation gives $v_{bx} = \dfrac{\left(m_b + m_c\right)v_{ci}}{m_b} \Rightarrow v_b = \dfrac{\left(m_b + m_c\right)v_{ci}}{m_b\cos\theta}$.

CALCULATE: $v_b = \dfrac{\left(15.0\text{ kg}+350\text{ kg}\right)\left(7.50\text{ m/s}\right)}{\left(15.0\text{ kg}\right)\cos\left(55.0°\right)} = 318.2\text{ m/s}$

ROUND: Each given value has three significant figures, so the result should be rounded to $v_b = 318$ m/s.
DOUBLE-CHECK: This is a reasonable speed at which to launch a cannonball. The component of the momentum of the cannon/cannon ball system in the x-direction before the ball is shot is $p_{x,\text{before}} = \left(350\text{ kg}+15\text{ kg}\right)\left(7.5\text{ m/s}\right) = 2737.5$ kg m/s. The component of the momentum of the cannon/cannon ball system in the x-direction after the ball is shot is $p_{x,\text{after}} = \left(15\text{ kg}\right)\left(318.2\text{ m/s}\right)\cos\left(55°\right) = 2737.68$ kg m/s. These components agree to within three significant figures.

8.65. **THINK:** The rocket's initial mass is $M_0 = 2.80\cdot 10^6$ kg. Its final mass is $M_1 = 8.00\cdot 10^5$ kg. The time to burn all the fuel is $\Delta t = 160.$ s. The exhaust speed is $v = v_c = 2700.$ m/s. Determine (a) the upward acceleration, a_0, of the rocket as it lifts off, (b) its upward acceleration, a_1, when all the fuel has burned and (c) the net change in speed, Δv in time Δt in the absence of a gravitational force.

SKETCH:



RESEARCH: To determine the upward acceleration, all the vertical forces on the rocket must be balanced. Use the following equations: $\vec{F}_{\text{thrust}} = -\vec{v}_{\text{c}}\dfrac{dm}{dt}$, $\vec{F}_{\text{g}} = m\vec{g}$, $\dfrac{dm}{dt} = \dfrac{\Delta m}{\Delta t}$. The mass of the fuel used is $\Delta m = M_0 - M_1$. To determine Δv in the absence of other forces (other than $\vec{F}_{\text{thrust}}$), use $v_{\text{f}} - v_{\text{i}} = v_{\text{c}} \ln\left(m_{\text{i}} / m_{\text{f}}\right)$.

SIMPLIFY:

(a) $\dfrac{dm}{dt} = \dfrac{M_0 - M_1}{\Delta t}$

Balancing the vertical forces on the rocket gives

$$F_{\text{net}} = F_{\text{thrust}} - F_{\text{g}} = ma \Rightarrow M_0 a_0 = v_{\text{c}}\frac{dm}{dt} - M_0 g \Rightarrow a_0 = \frac{v_{\text{c}}}{M_0}\left(\frac{M_0 - M_1}{\Delta t}\right) - g \Rightarrow a_0 = \frac{v_{\text{c}}}{\Delta t}\left(1 - \frac{M_1}{M_0}\right) - g.$$

(b) Similarly to part (a):

$$F_{\text{net}} = F_{\text{thrust}} - F_{\text{g}} = ma \Rightarrow M_1 a_1 = v_{\text{c}}\frac{dm}{dt} - M_1 g \Rightarrow a_1 = \frac{v_{\text{c}}}{M_1}\left(\frac{M_0 - M_1}{\Delta t}\right) - g \Rightarrow a_1 = \frac{v_{\text{c}}}{\Delta t}\left(\frac{M_0}{M_1} - 1\right) - g.$$

(c) In the absence of gravity, $F_{\text{net}} = F_{\text{thrust}}$. The change in velocity due to this thrust force is $\Delta v = v_{\text{c}} \ln\left(M_0 / M_1\right)$.

CALCULATE:

(a) $a_0 = \left(\dfrac{2700.\text{ m/s}}{160\text{ s}}\right)\left(1 - \dfrac{8.00 \cdot 10^5\text{ kg}}{2.80 \cdot 10^6\text{ kg}}\right) - 9.81\text{ m/s}^2 = 2.244\text{ m/s}^2$

(b) $a_1 = \left(\dfrac{2700.\text{ m/s}}{160.\text{ s}}\right)\left(\dfrac{2.80 \cdot 10^6\text{ kg}}{8.00 \cdot 10^5\text{ kg}} - 1\right) - 9.81\text{ m/s}^2 = 32.38\text{ m/s}^2$

(c) $\Delta v = \left(2700.\text{ m/s}\right)\ln\left(\dfrac{2.80 \cdot 10^6\text{ kg}}{8.00 \cdot 10^5\text{ kg}}\right) = 3382\text{ m/s}$

ROUND:

(a) $a_0 = 2.24\text{ m/s}^2$

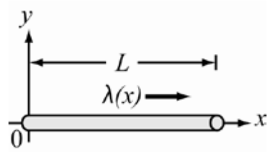
(b) $a_1 = 32.4\text{ m/s}^2$

(c) $\Delta v = 3380\text{ m/s}$

DOUBLE-CHECK: It can be seen that $a_1 > a_0$, as it should be since $M_1 < M_0$. It is not unusual for Δv to be greater than v_{c}.

8.66. **THINK:** The rod has a length of L and its linear density is $\lambda(x)=cx$, where c is a constant. Determine the rod's center of mass.

SKETCH:



RESEARCH: To determine the center of mass, take a differentially small element of mass: $dm=\lambda dx$ and use $X=\frac{1}{M}\int_L x\cdot dm=\frac{1}{M}\int_L x\lambda(x)dx,$ where $M=\int_L dm=\int_L \lambda(x)dx.$

SIMPLIFY: First, determine M from $M=\int_0^L cxdx=\left[c\frac{1}{2}x^2\right]_0^L=\frac{1}{2}cL^2.$ Then, the equation for the center of mass becomes:

$$X=\frac{1}{M}\int_0^L x(cx)dx=\frac{1}{M}\int_0^L cx^2dx=\frac{1}{M}c\left[\frac{1}{3}x^3\right]_0^L=\frac{1}{3M}cL^3.$$

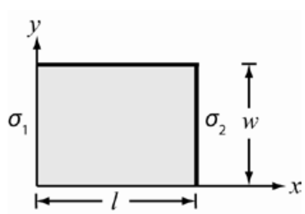
Substituting the expression for M into the above equation gives:

$$X=\frac{cL^3}{3\left(\frac{1}{2}cL^2\right)}=\frac{2}{3}L.$$

CALCULATE: This step is not applicable.
ROUND: This step is not applicable.
DOUBLE-CHECK: X is a function of L. Also, as expected, X is closer to the denser end of the rod.

8.67. **THINK:** The length and width of the plate are l = 20.0 cm and w = 10.0 cm, respectively. The mass density, σ, varies linearly along the length; at one end it is $\sigma_1=5.00\text{ g/cm}^2$ and at the other it is $\sigma_2=20.0\text{ g/cm}^2$. Determine the center of mass.

SKETCH:



RESEARCH: The mass density does not vary in width, i.e. along the y-axis. Therefore, the Y coordinate is simply $w/2$. To determine the X coordinate, use

$$X=\frac{1}{M}\int_A x\sigma(\vec{r})dA,\text{ where } M=\int_A \sigma(\vec{r})dA.$$

To obtain a functional form for $\sigma(\vec{r})$, consider that it varies linearly with x, and when the bottom left corner of the plate is at the origin of the coordinate system, σ must be σ_1 when x = 0 and σ_2 when $x = l$. Then, the conditions are satisfied by $\sigma(\vec{r})=\sigma(x)=\frac{(\sigma_2-\sigma_1)}{l}x+\sigma_1.$

SIMPLIFY: First determine M from $M=\int_A \sigma(\vec{r})dA=\int_0^l\int_0^w \sigma(x)dydx=\int_0^w dy\int_0^l\left(\frac{(\sigma_2-\sigma_1)}{l}x+\sigma_1\right)dx.$ y is not dependent on x in this case, so

$$M=\left[y\right]_0^w\left[\frac{1}{2}\frac{(\sigma_2-\sigma_1)}{l}x^2+\sigma_1 x\right]_0^l=w\left(\frac{1}{2}\frac{(\sigma_2-\sigma_1)}{l}l^2+\sigma_1 l\right)=wl\left(\frac{1}{2}(\sigma_2-\sigma_1)+\sigma_1\right)=\frac{wl}{2}(\sigma_2+\sigma_1).$$

Now, reduce the equation for the center of mass:

$$X=\frac{1}{M}\int_A x\sigma(x)dA=\frac{1}{M}\int_0^l\int_0^w x\left(\frac{(\sigma_2-\sigma_1)}{l}x+\sigma_1\right)dydx=\frac{1}{M}\int_0^w dy\int_0^l\left(\frac{(\sigma_2-\sigma_1)}{l}x^2+\sigma_1 x\right)dx$$

$$=\frac{1}{M}\left[y\right]_0^w\left[\frac{(\sigma_2-\sigma_1)}{3l}x^3+\frac{1}{2}\sigma_1 x^2\right]_0^l=\frac{1}{M}w\left(\frac{(\sigma_2-\sigma_1)}{3l}l^3+\frac{1}{2}\sigma_1 l^2\right)=\frac{1}{M}wl^2\left(\frac{1}{3}(\sigma_2-\sigma_1)+\frac{1}{2}\sigma_1\right)$$

$$=\frac{1}{M}wl^2\left(\frac{1}{3}\sigma_2+\frac{1}{6}\sigma_1\right)$$

Substitute the expression for M into the above equation to get

$$X=\frac{l\left(\frac{1}{3}\sigma_2+\frac{1}{6}\sigma_1\right)}{\frac{1}{2}(\sigma_2+\sigma_1)}=\frac{2l\left(\frac{1}{3}\sigma_2+\frac{1}{6}\sigma_1\right)}{\sigma_2+\sigma_1}.$$

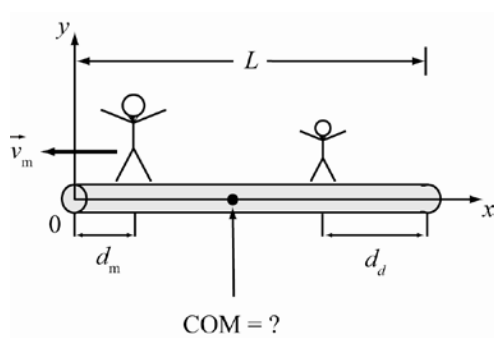
CALCULATE: $X=\dfrac{2(20.0\text{ cm})\left(\frac{1}{3}(20.0\text{ g/cm}^2)+\frac{1}{6}(5.00\text{ g/cm}^2)\right)}{20.0\text{ g/cm}^2+5.00\text{ g/cm}^2}=12.00\text{ cm},\ \ Y=\frac{1}{2}(10.0\text{ cm})=5.00\text{ cm}$

ROUND: The results should be written to three significant figures: $X=12.0\text{ cm}$ and $Y=5.00\text{ cm}$. The center of mass is at (12.0 cm, 5.00 cm).

DOUBLE-CHECK: It is expected that the center of mass for the *x* coordinate is closer to the denser end of the rectangle (before rounding).

8.68. **THINK:** The log's length and mass are $L=2.50\text{ m}$ and $m_l=91.0\text{ kg}$, respectively. The man's mass is $m_\text{m}=72\text{ kg}$ and his location is $d_\text{m}=0.220\text{ m}$ from one end of the log. His daughter's mass is $m_\text{d}=20.0\text{ kg}$ and her location is $d_\text{d}=1.00\text{ m}$ from the other end of the log. Determine (a) the system's center of mass and (b) the initial speed of the log and daughter, $v_{\text{l+d}}$, when the man jumps off the log at a speed of $v_\text{m}=3.14\text{ m/s}$.

SKETCH:



RESEARCH: In one dimension, the center of mass location is given by $X=\frac{1}{M}\sum_{i=1}^{n}x_i m_i$. Take the origin of the coordinate system to be at the end of log near the father. To determine the initial velocity of the log and girl system, consider the conservation of momentum, $\vec{p}_\text{i}=\vec{p}_\text{f}$, where $\vec{p}=m\vec{v}$. Note that the man's velocity is away from the daughter. Take this direction to be along the $-\hat{x}$ direction, so that $\vec{v}_\text{m}=-3.14\text{ m/s}\,\hat{x}$.

SIMPLIFY:

(a) $X=\frac{1}{M}\left(x_{\mathrm{m}}m_{\mathrm{m}}+x_{\mathrm{d}}m_{\mathrm{d}}+x_{\mathrm{l}}m_{\mathrm{l}}\right)=\frac{\left(d_{\mathrm{m}}m_{\mathrm{m}}+\left(L-d_{\mathrm{d}}\right)m_{\mathrm{d}}+\frac{1}{2}Lm_{\mathrm{l}}\right)}{m_{\mathrm{m}}+m_{\mathrm{d}}+m_{\mathrm{l}}}$

(b) $\vec{p}_{\mathrm{i}}=\vec{p}_{\mathrm{f}}\ \Rightarrow\ 0=m_{\mathrm{m}}\vec{v}_{\mathrm{m}}+\left(m_{\mathrm{d}}+m_{\mathrm{l}}\right)\vec{v}_{\mathrm{d+l}}\ \Rightarrow\ \vec{v}_{\mathrm{d+l}}=-\frac{m_{\mathrm{m}}\vec{v}_{\mathrm{m}}}{\left(m_{\mathrm{d}}+m_{\mathrm{l}}\right)}$

CALCULATE:

(a) $X=\frac{\left(\left(0.220\text{ m}\right)\left(72.0\text{ kg}\right)+\left(2.50\text{ m}-1.00\text{ m}\right)\left(20.0\text{ kg}\right)+\frac{1}{2}\left(2.50\text{ m}\right)\left(91.0\text{ kg}\right)\right)}{72.0\text{ kg}+20.0\text{ kg}+91.0\text{ kg}}=0.8721\text{ m}$

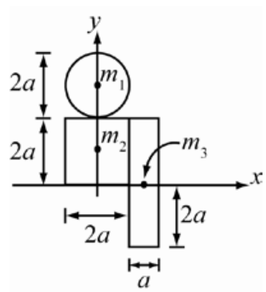
(b) $\vec{v}_{\mathrm{d+l}}=-\frac{\left(72.0\text{ kg}\right)\left(-3.14\text{ m/s }\hat{x}\right)}{\left(20.0\text{ kg}+91.0\text{ kg}\right)}=2.0368\text{ m/s }\hat{x}$

ROUND: To three significant figures, the center of mass of the system is X = 0.872 m from the end of the log near the man, and the speed of the log and child is $v_{\mathrm{d+l}}=2.04$ m/s.

DOUBLE-CHECK: As it should be, the center of mass is between the man and his daughter, and $v_{\mathrm{d+l}}$ is less than v_{m} (since the mass of the log and child is larger than the mass of the man).

8.69. **THINK:** Determine the center of mass of an object which consists of regularly shaped metal of uniform thickness and density. Assume that the density of the object is ρ.

SKETCH:



RESEARCH: First, as shown in the figure above, divide the object into three parts, m_1, m_2 and m_3. Determine the center of mass by using $\vec{R}=\frac{1}{M}\sum_{i=1}^{3}m_i\vec{r}_i$, or in component form $X=\frac{1}{M}\sum_{i=1}^{3}m_ix_i$ and $Y=\frac{1}{M}\sum_{i=1}^{3}m_iy_i$. Also, use $m=\rho At$ for the mass, where A is the area and t is the thickness.

SIMPLIFY: The center of mass components are given by:

$$X=\frac{m_1x_1+m_2x_2+m_3x_3}{M}\text{ and }Y=\frac{m_1y_1+m_2y_2+m_3y_3}{M}$$

The masses of the three parts are $m_1=\rho\pi a^2t$, $m_2=\rho\left(2a\right)^2t$ and $m_3=\rho 4a^2t$. The center of mass of the three parts are $x_1=0$, $y_1=3a$, $x_2=0$, $y_2=a$, $x_3=3a/2$ and $y_3=0$. The total mass of the object is $M=m_1+m_2+m_3=\rho\pi a^2t+4\rho a^2t+4\rho a^2t=\rho a^2t\left(8+\pi\right)$.

CALCULATE: The center of mass of the object is given by the following equations:

$$X=\frac{0+0+4\rho a^2t\left(3a/2\right)}{\rho a^2t\left(8+\pi\right)}=\left(\frac{6}{8+\pi}\right)a;$$

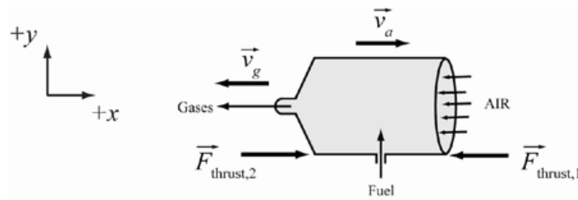
$$Y=\frac{\rho\pi a^2t\left(3a\right)+4\rho a^2t\left(a\right)+0}{\rho a^2t\left(8+\pi\right)}=\left(\frac{4+3\pi}{8+\pi}\right)a.$$

ROUND: Rounding is not required.

DOUBLE-CHECK: The center of mass of the object is located in the area of m_2. By inspection of the figure this is reasonable.

8.70. **THINK:** A jet aircraft has a speed of 223 m/s. The rate of change of the mass of the aircraft is $(dM/dt)_{\text{air}} = 80.0$ kg/s (due to the engine taking in air) and $(dM/dt)_{\text{fuel}} = 3.00$ kg/s (due to the engine taking in and burning fuel). The speed of the exhaust gases is 600. m/s. Determine the thrust of the jet engine.

SKETCH:



RESEARCH: The thrust is calculated by using $\vec{F}_{\text{thrust}} = -\vec{v}\, dM/dt$, where $\vec{v}$ is the velocity of the gases or air, relative to the engine. There are two forces on the engine. The first force, $F_{\text{thrust,1}}$, is the thrust due to the engine taking in air and the second force, $F_{\text{thrust,2}}$, is the thrust due to the engine ejecting gases.

$$\vec{F}_{\text{thrust,1}} = -\vec{v}_{\text{a}}\left(\frac{dM}{dt}\right)_{\text{air}}, \quad \vec{F}_{\text{thrust,2}} = -\vec{v}_{\text{g}}\left[\left(\frac{dM}{dt}\right)_{\text{air}} + \left(\frac{dM}{dt}\right)_{\text{fuel}}\right]$$

The net thrust is given by $\vec{F}_{\text{thrust}} = \vec{F}_{\text{thrust,1}} + \vec{F}_{\text{thrust,2}}$.

SIMPLIFY: Simplification is not required.

CALCULATE: $\vec{F}_{\text{thrust,1}} = -(223 \text{ m/s } \hat{x})(80.0 \text{ kg/s}) = -17840 \text{ N } \hat{x}$,

$\vec{F}_{\text{thrust,2}} = -(600. \text{ m/s } (-\hat{x}))[80.0 \text{ kg/s} + 3.00 \text{ kg/s}] = 49800 \text{ N } \hat{x}$,

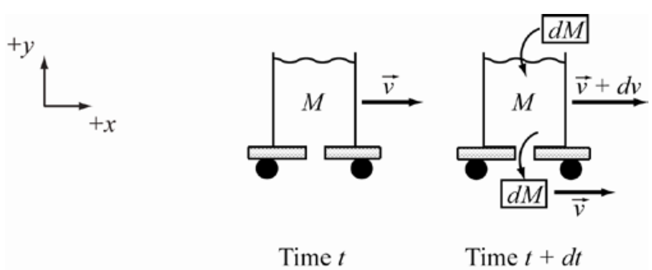
$\vec{F}_{\text{thrust}} = -17840 \text{ N } \hat{x} + 49800 \text{ N } \hat{x} = 31960 \text{ N } \hat{x}$

ROUND: To three significant figures, the thrust of the jet engine is $\vec{F}_{\text{thrust}} = 32.0 \text{ kN } \hat{x}$.

DOUBLE-CHECK: Since the $\hat{x}$ direction is in the forward direction of the aircraft, the plane moves forward, which it must. A jet engine is very powerful, so the large magnitude of the result is reasonable.

8.71. **THINK:** The solution to this problem is similar to a rocket system. Here the system consists of a bucket, a skateboard and water. The total mass of the system is M = 10.0 kg. The total mass of the bucket, skateboard and water remains constant at $\lambda = dM/dt = 0.100$ kg/s since rain water enters the top of the bucket at the same rate that it exits the bottom. Determine the time required for the bucket and the skateboard to reach a speed of half the initial speed.

SKETCH:



RESEARCH: To solve this problem, consider the conservation of momentum, $\vec{p}_{\text{i}} = \vec{p}_{\text{f}}$. The initial momentum of the system at time t is $p_{\text{i}} = Mv$. After time $t + dt$, the momentum of the system is $p_{\text{f}} = vdM + M(v + dv)$.

SIMPLIFY: $p_{\text{i}} = p_{\text{f}} \Rightarrow Mv = vdM + Mv + Mdv \Rightarrow Mdv = -vdM$

Dividing both sides by dt gives

$$M\frac{dv}{dt}=-v\frac{dM}{dt}=-v\lambda \text{ or } \frac{1}{v}\frac{dv}{dt}=-\frac{\lambda}{M} \Rightarrow \frac{1}{v}dv=-\frac{\lambda}{M}dt.$$

Integrate both sides to get

$$\int_{v=v_0}^{v}\frac{1}{v}dv=\int_{t=0}^{t}-\frac{\lambda}{M}dt \Rightarrow \ln v-\ln v_0=-\frac{\lambda}{M}t \Rightarrow \ln\left(\frac{v}{v_0}\right)=-\frac{\lambda}{M}t.$$

Determine the time such that $v=v_0/2$. Substituting $v=v_0/2$ into the above equation gives

$$\ln\left(\frac{v_0/2}{v_0}\right)=-\frac{\lambda}{M}t \Rightarrow t=-\frac{M}{\lambda}\ln\left(\frac{1}{2}\right)=\frac{M}{\lambda}\ln(2).$$

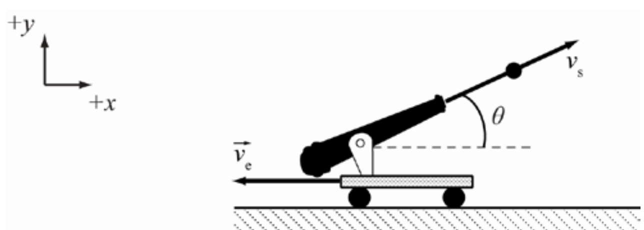
CALCULATE: $t=\frac{(10.0\text{ kg})\ln(2)}{0.100\text{ kg/s}}=69.3147\text{ s}$

ROUND: To three significant figures, the time for the system to reach half of its initial speed is t = 69.3 s.

DOUBLE-CHECK: It is reasonable that the time required to reduce the speed of the system to half its original value is near one minute.

8.72. **THINK:** The mass of a cannon is M = 1000. kg and the mass of a shell is m = 30.0 kg. The shell is shot at an angle of $\theta=25.0°$ above the horizontal with a speed of $v_s=500.$ m/s. Determine the recoil velocity of the cannon.

SKETCH:



RESEARCH: The momentum of the system is conserved, $p_i=p_f$, or in component form, $p_{xi}=p_{xf}$ and $p_{yi}=p_{yf}$. Use only the x component of the momentum.

SIMPLIFY: p_{xi} is equal to zero since both the cannon and the shell are initially at rest. Therefore,

$$p_{xi}=p_{xf} \Rightarrow mv_s\cos\theta+Mv_c=0 \Rightarrow v_c=-\frac{m}{M}v_s\cos\theta.$$

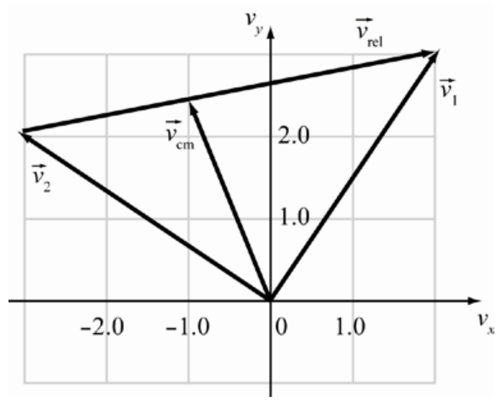
CALCULATE: $v_c=-\frac{(30.0\text{ kg})(500.\text{ m/s})\cos(25.0°)}{1000.\text{ kg}}=-13.595\text{ m/s}$

ROUND: To three significant figures: $v_c=-13.6$ m/s

DOUBLE-CHECK: The direction of the recoil is expected to be in the opposite direction to the horizontal component of the velocity of the shell. This is why the result is negative.

8.73. **THINK:** There are two masses, $m_1=2.0$ kg and $m_2=3.0$ kg. The velocity of their center of mass and the velocity of mass 1 relative to mass 2 are $\vec{v}_{cm}=(-1.00\hat{x}+2.40\hat{y})$ m/s and $\vec{v}_{rel}=(5.00\hat{x}+1.00\hat{y})$ m/s. Determine the total momentum of the system and the momenta of mass 1 and mass 2.

SKETCH:



RESEARCH: The total momentum of the system is $\vec{p}_{cm} = M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2$. The velocity of mass 1 relative to mass 2 is $\vec{v}_{rel} = \vec{v}_1 - \vec{v}_2$.

SIMPLIFY: The total mass *M* of the system is $M = m_1 + m_2$. The total momentum of the system is given by $\vec{p}_{cm} = M\vec{v}_{cm} = (m_1 + m_2)\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2$. Substitute $\vec{v}_2 = \vec{v}_1 - \vec{v}_{rel}$ into the equation for the total momentum of the system to get $M\vec{v}_{cm} = m_1\vec{v}_1 + m_2(\vec{v}_1 - \vec{v}_{rel}) = (m_1 + m_2)\vec{v}_1 - m_2\vec{v}_{rel}$. Therefore, $\vec{v}_1 = \vec{v}_{cm} + \frac{m_2}{M}\vec{v}_{rel}$. Similarly, substitute $\vec{v}_1 = \vec{v}_2 + \vec{v}_{rel}$ into the equation for the total momentum of the system to get $M\vec{v}_{cm} = m_1\vec{v}_{rel} + (m_1 + m_2)\vec{v}_2$ or $\vec{v}_2 = \vec{v}_{cm} - \frac{m_1}{M}\vec{v}_{rel}$. Therefore, the momentums of mass 1 and mass 2 are $\vec{p}_1 = m_1\vec{v}_1 = m_1\vec{v}_{cm} + \frac{m_1 m_2}{M}\vec{v}_{rel}$ and $\vec{p}_2 = m_2\vec{v}_2 = m_2\vec{v}_{cm} - \frac{m_1 m_2}{M}\vec{v}_{rel}$.

CALCULATE:

(a)

$$\vec{p}_{cm} = (2.00 \text{ kg} + 3.00 \text{ kg})(-1.00\hat{x} + 2.40\hat{y}) \text{ m/s} = (-5.00\hat{x} + 12.0\hat{y}) \text{ kg m/s}$$

$$\vec{p}_{cm} = (2.0 \text{ kg} + 3.0 \text{ kg})(-1.0\hat{x} + 2.4\hat{y}) \text{ m/s} = (-5.0\hat{x} + 12\hat{y}) \text{ kg m/s}$$

(b) $$\vec{p}_1 = (2.00 \text{ kg})(-1.00\hat{x} + 2.40\hat{y}) \text{ m/s} + \frac{(2.00 \text{ kg})(3.00 \text{ kg})}{2.00 \text{ kg} + 3.00 \text{ kg}}(5.00\hat{x} + 1.00\hat{y}) \text{ m/s}$$
$$= (-2.00\hat{x} + 4.80\hat{y}) \text{ kg m/s} + (6.00\hat{x} + 1.20\hat{y}) \text{ kg m/s} = (4.00\hat{x} + 6.00\hat{y}) \text{ kg m/s}$$

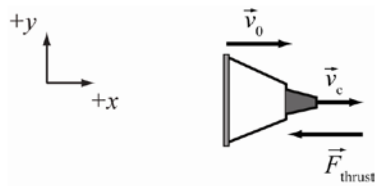
(c) $$\vec{p}_2 = (3.00 \text{ kg})(-1.00\hat{x} + 2.40\hat{y}) \text{ m/s} - \frac{(2.00 \text{ kg})(3.00 \text{ kg})}{2.00 \text{ kg} + 3.00 \text{ kg}}(5.00\hat{x} + 1.00\hat{y}) \text{ m/s}$$
$$= (-3.00\hat{x} + 7.20\hat{y}) \text{ kg m/s} - (6.00\hat{x} + 1.20\hat{y}) \text{ kg m/s} = (-9.00\hat{x} + 6.00\hat{y}) \text{ kg m/s}$$

ROUND: The answers have already been rounded to three significant figures.

DOUBLE-CHECK: It is clear from the results of (a), (b) and (c) that $\vec{p}_{cm} = \vec{p}_1 + \vec{p}_2$.

8.74. **THINK:** A spacecraft with a total initial mass of $m_s = 1000.$ kg and an initial speed of $v_0 = 1.00$ m/s must be docked. The mass of the fuel decreases from 20.0 kg. Since the mass of the fuel is small compared to the mass of the spacecraft, we can ignore it. To reduce the speed of the spacecraft, a small retro-rocket is used which can burn fuel at a rate of $dM/dt = 1.00$ kg/s and with an exhaust speed of $v_E = 100.$ m/s.

SKETCH:



RESEARCH:

(a) The thrust of the retro-rocket is determined using $F_{thrust} = v_c dM / dt$.

(b) In order to determine the amount of fuel needed, first determine the time to reach a speed of $v = 0.0200$ m/s. Use $v = v_0 - at$. By Newton's Second Law the thrust is also given by $\vec{F}_{thrust} = m_s\vec{a}$.

(c) The burn of the retro-rocket must be sustained for a time sufficient to reduce the speed to 0.0200 m/s, found in part (b).

(d) Use the conservation of momentum, $\vec{p}_i = \vec{p}_f$.

SIMPLIFY:

(a) $\vec{F}_{thrust} = -\vec{v}_c \dfrac{dM}{dt}$

(b) $t = \dfrac{v_0 - v}{a}$

The acceleration is given by $a = F_{thrust} / m_s$. Substitute this expression into the equation for t above to get $t = \dfrac{(v_0 - v)m_s}{F_{thrust}}$. Therefore, the mass of fuel needed is $m_F = \left(\dfrac{dM}{dt}\right)t = \left(\dfrac{dM}{dt}\right)\dfrac{(v_0 - v)m_s}{F_{thrust}}$.

(c) $t = \dfrac{(v_0 - v)m_s}{F_{thrust}}$

(d) $m_s\vec{v} = (M + m_s)\vec{v}_f \Rightarrow \vec{v}_f = \dfrac{m_s}{M + m_s}\vec{v}$, where M is the mass of the space station.

CALCULATE:

(a) The thrust is $\vec{F}_{thrust} = -(100.\text{ m/s})(1.00\text{ kg/s})\hat{v}_E = -100.0\text{ N }\hat{v}_c$, or 100.0 N in the opposite direction to the velocity of the spacecraft.

(b) $m_F = (1.00\text{ kg/s})\dfrac{(1.00\text{ m/s} - 0.0200\text{ m/s})1000.\text{ kg}}{100.0\text{ N}} = 9.800\text{ kg}$

(c) $t = \dfrac{(1.00\text{ m/s} - 0.0200\text{ m/s})1000.\text{ kg}}{100.0\text{ N}} = 9.800\text{ s}$

(d) $\vec{v}_f = \dfrac{1000.\text{ kg}(0.0200\text{ m/s})}{5.00 \cdot 10^5\text{ kg} + 1000.\text{ kg}}\hat{v} = 3.992 \cdot 10^{-5}\text{ m/s }\hat{v}$; that is, in the same direction as the spacecraft is moving.

ROUND: The answers should be expressed to three significant figures:

(a) $\vec{F}_{thrust} = -100.\text{ N }\hat{v}_c$

(b) $m_F = 9.80\text{ kg}$

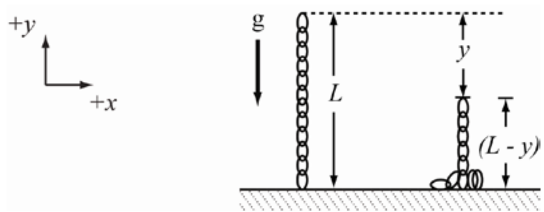
(c) $t = 9.80\text{ s}$

(d) $\vec{v}_f = 3.99 \cdot 10^{-5}\text{ m/s }\hat{v}$

DOUBLE-CHECK: It is expected that the speed of the combined mass will be very small since its mass is very large.

8.75. **THINK:** A chain has a mass of 3.00 kg and a length of 5.00 m. Determine the force exerted by the chain on the floor. Assume that each link in the chain comes to rest when it reaches the floor.

SKETCH:



RESEARCH: Assume the mass per unit length of the chain is $\rho = M/L$. A small length of the chain, dy has a mass of dm, where $dm = Mdy/L$. At an interval of time dt, the small element of mass dm has reached the floor. The impulse caused by the chain is given by $J = F_J dt = \Delta p = vdm$. Therefore, the force F_J is given by $F_J = v\dfrac{dm}{dt} = v\dfrac{dm}{dy}\dfrac{dy}{dt}$.

SIMPLIFY: Using $dm/dy = M/L$ and $v = dy/dt$, the expression for force, F_J is

$$F_J = v^2\frac{M}{L}.$$

For a body in free fall motion, $v^2 = 2gy$. Thus, $F_J = 2Mgy/L$. There is another force which is due to gravity. The gravitational force exerted by the chain on the floor when the chain has fallen a distance y is given by $F_g = Mgy/L$ (the links of length y are on the floor). The total force is given by

$$F = F_J + F_g = \frac{2Mgy}{L} + \frac{Mgy}{L} = \frac{3Mgy}{L}.$$

When the last link of the chain lands on the floor, the force exerted by the chain is obtained by substituting $y = L$, that is, $F = \dfrac{3Mgy}{L} = 3Mg$.

CALCULATE: $F = 3(3.0 \text{ kg})(9.81 \text{ m/s}^2) = 88.29 \text{ N}$

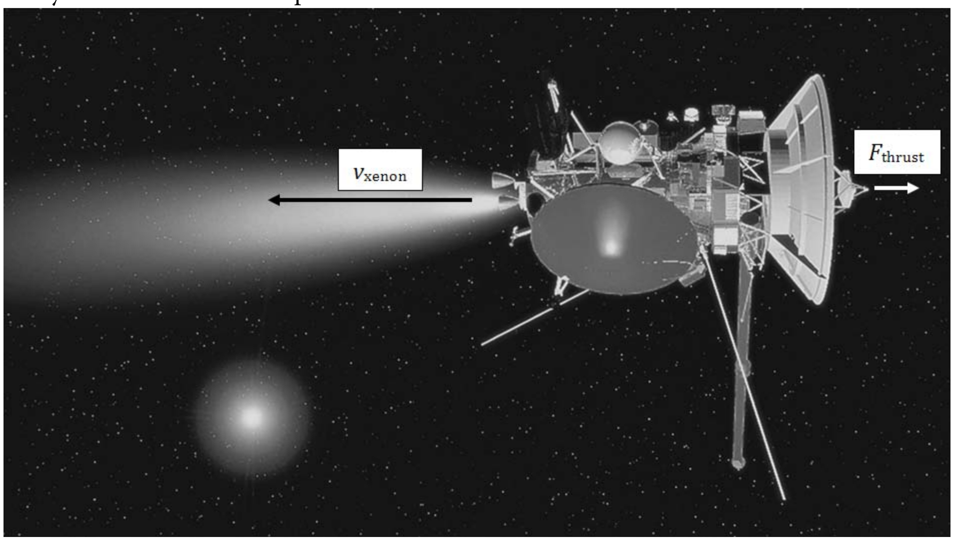
ROUND: To three significant figures, the force exerted by the chain on the floor as the last link of chain lands on the floor is $F = 88.3$ N.

DOUBLE-CHECK: F is expected to be larger than Mg due to the impulse caused by the chain as it falls.

Multi-Version Exercises

8.76. **THINK:** This question asks about the fuel consumption of a satellite. This is an example of rocket motion, where the mass of the satellite (including thruster) decreases as the fuel is ejected.

SKETCH: The direction in which the xenon ions are ejected is opposite to the direction of the thrust. The velocity of the xenon with respect to the satellite and the thrust force are shown.



RESEARCH: The equation of motion for a rocket in interstellar space is given by $\vec{F}_{\text{thrust}} = -\vec{v}_c \dfrac{dm}{dt}$. The velocity of the xenon ions with respect to the shuttle is given in km/s and the force is given in Newtons, or kg · m / s^2. The conversion factor for the velocity is given by $\dfrac{1000 \text{ m/s}}{1 \text{ km/s}}$.

SIMPLIFY: Since the thrust and velocity act along a single axis, it is possible to use the scalar form of the equation, $F_{\text{thrust}} = -v_c \dfrac{dm}{dt}$. The rate of fuel consumption equals the change in mass (the loss of mass is due to xenon ejected from the satellite), so solve for $\dfrac{dm}{dt}$ to get $\dfrac{dm}{dt} = -\dfrac{F_{\text{thrust}}}{v_c}$.

CALCULATE: The question states that the speed of the xenon ions with respect to the rocket is $v_c = v_{\text{xenon}}$ = 21.45 km/s. The thrust produced is F_{thrust} = 1.187·10^{-2} N. Thus the rate of fuel consumption is:

$$\begin{aligned}
\frac{dm}{dt} &= -\frac{F_{\text{thrust}}}{v_c} \\
&= -\frac{1.187\cdot 10^{-2} \text{ N}}{21.45 \text{ km/s}\cdot \dfrac{1000 \text{ m/s}}{1 \text{ km/s}}} \\
&= -5.533799534\cdot 10^{-7} \text{ kg/s} \\
&= -1.992167832 \text{ g/hr}
\end{aligned}$$

ROUND: The measured values are all given to four significant figures, and the final answer should also have four significant figures. The thruster consumes fuel at a rate of 5.534·10^{-7} kg/s or 1.992 g/hr.

DOUBLE-CHECK: Because of the cost of sending a satellite into space, the weight of the fuel consumed per hour should be pretty small; a fuel consumption rate of 1.992 g/hr is reasonable for a satellite launched from earth. Working backwards, if the rocket consumes fuel at a rate of 5.534·10^{-4} g/s, then the thrust is

$$-21.45 \text{ km/s}\cdot\left(-5.534\cdot 10^{-4} \text{ g/s}\right) = 0.01187 \text{ km}\cdot\text{g/s}^2 = 1.187\cdot 10^{-2} \text{ N}$$

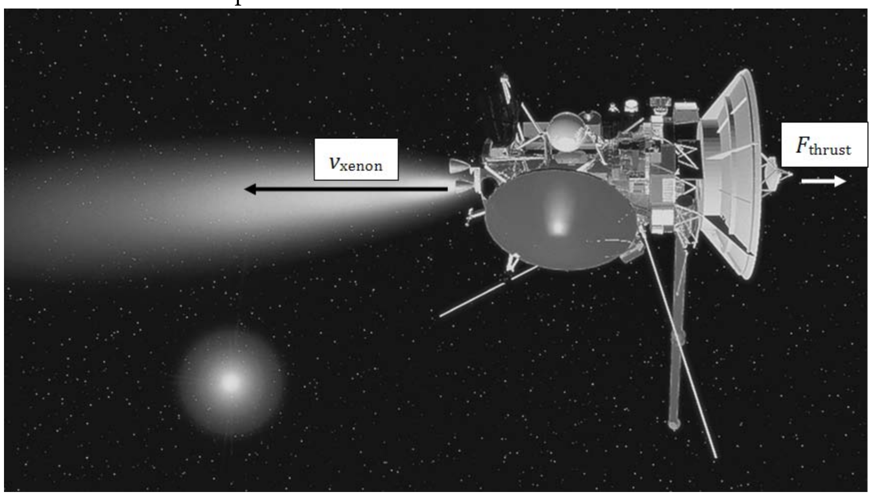
(the conversion factor is 1 km·g/s^2 = 1 kg·m/s^2). So, this agrees with the given thrust force of 1.187·10^{-2} N.

8.77. $F = v_c \frac{dm}{dt} = \left(23.75 \cdot 10^3 \text{ m/s}\right)\left(5.082 \cdot 10^{-7} \text{ kg/s}\right) = 1.207 \cdot 10^{-2} \text{ N}$

8.78. $v_c = \frac{F}{dm/dt} = \frac{1.299 \cdot 10^{-2} \text{ N}}{4.718 \cdot 10^{-7} \text{ kg/s}} = 26.05 \text{ km/s}$

8.79. **THINK:** This question asks about the speed of a satellite. This is an example of rocket motion, where the mass of the satellite (including thruster) decreases as the fuel is ejected.

SKETCH: The direction in which the xenon ions are ejected is opposite to the direction of the thrust. The velocity of the xenon with respect to the satellite and the thrust force are shown.



RESEARCH: Initially, the mass of the system is the total mass of the satellite, including the mass of the fuel: $m_i = m_{satellite}$ After all of the fuel is consumed, the mass of the system is equal to the mass of the satellite minus the mass of the fuel consumed: $m_f = m_{satellite} - m_{fuel}$. The change in speed of the satellite is given by the equation $v_f - v_i = v_c \ln\left(m_i / m_f\right)$, where v_c is the speed of the xenon with relative to the satellite.

SIMPLIFY: To make the problem easier, choose a reference frame where the initial speed of the satellite equals zero. Then $v_f - v_i = v_f - 0 = v_f$, so it is necessary to find $v_f = v_c \ln\left(m_i / m_f\right)$. Substituting in the masses of the satellite and fuel, this becomes $v_f = v_c \ln\left(m_{satellite} / \left[m_{satellite} - m_{fuel}\right]\right)$.

CALCULATE: The initial mass of the satellite (including fuel) is 2149 kg, and the mass of the fuel consumed is 23.37 kg. The speed of the ions with respect to the satellite is 28.33 km/s, so the final velocity of the satellite is:

$$\begin{aligned} v_f &= v_c \ln\left(m_{satellite} / \left[m_{satellite} - m_{fuel}\right]\right) \\ &= \left(28.33 \text{ km/s}\right)\ln\left(\frac{2149 \text{ kg}}{2149 \text{ kg} - 23.37 \text{ kg}}\right) \\ &= 3.0977123 \cdot 10^{-1} \text{ km/s} \end{aligned}$$

ROUND: The measured values are all given to four significant figures, and the weight of the satellite minus the weight of the fuel consumed also has four significant figures, so the final answer will have four figures. The change in the speed of the satellite is $3.098 \cdot 10^{-1}$ km/s or 309.8 m/s.

DOUBLE-CHECK: Alhough the satellite is moving quickly after burning all of its fuel, this is not an unreasonable speed for space travel. Working backwards, if the change in speed was $3.098 \cdot 10^{-1}$ km/s, then the velocity of the xenon particles was $v_c = \frac{\Delta v_{satellite}}{\ln(m_i / m_f)}$, or

$$v_c = \frac{3.098 \cdot 10^{-1} \text{km /s}}{\ln(2149 \text{ kg} / \left[2149 \text{ kg} - 23.37 \text{ kg}\right])} = 28.33 \text{ km /s}.$$

This agrees with the number given in the question, confirming that the calculations are correct.

8.80. $\Delta v = v_{\text{c}} \ln\left(\frac{m_{\text{i}}}{m_{\text{f}}}\right)$

$$\frac{\Delta v}{v_{\text{c}}} = \ln\left(\frac{m_{\text{i}}}{m_{\text{f}}}\right)$$

$$e^{\frac{\Delta v}{v_{\text{c}}}} = \frac{m_{\text{i}}}{m_{\text{f}}}$$

$$m_{\text{f}} = m_{\text{i}} e^{-\frac{\Delta v}{v_{\text{c}}}}$$

$$m_{\text{fuel}} = m_{\text{i}} - m_{\text{f}} = m_{\text{i}} - m_{\text{i}} e^{-\frac{\Delta v}{v_{\text{c}}}} = m_{\text{i}}\left(1 - e^{-\frac{\Delta v}{v_{\text{c}}}}\right)$$

$$m_{\text{fuel}} = (2161 \text{ kg})\left(1 - e^{-\frac{236.4 \text{ m/s}}{20.61 \cdot 10^3 \text{ m/s}}}\right) = 24.65 \text{ kg}$$

8.81. $\Delta v = v_{\text{c}} \ln\left(\frac{m_{\text{i}}}{m_{\text{f}}}\right)$

$$\frac{\Delta v}{v_{\text{c}}} = \ln\left(\frac{m_{\text{i}}}{m_{\text{f}}}\right)$$

$$e^{\frac{\Delta v}{v_{\text{c}}}} = \frac{m_{\text{i}}}{m_{\text{f}}}$$

$$m_{\text{i}} = m_{\text{f}} e^{\frac{\Delta v}{v_{\text{c}}}}$$

$$m_{\text{f}} = m_{\text{i}} - m_{\text{fuel}}$$

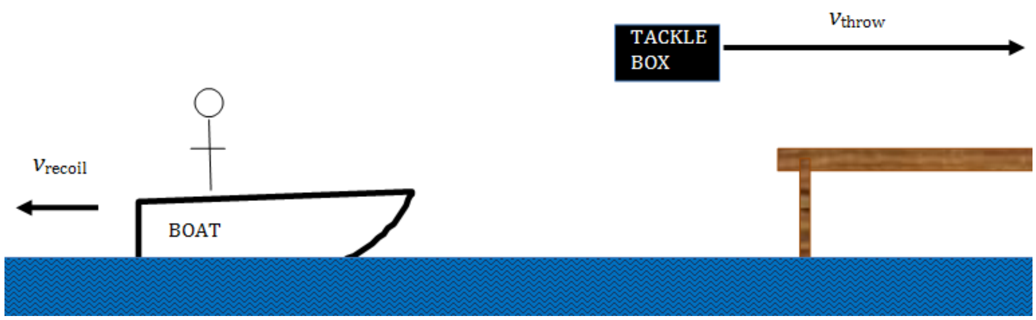
$$m_{\text{i}} = (m_{\text{i}} - m_{\text{fuel}}) e^{\frac{\Delta v}{v_{\text{c}}}} = m_{\text{i}} e^{\frac{\Delta v}{v_{\text{c}}}} - m_{\text{fuel}} e^{\frac{\Delta v}{v_{\text{c}}}}$$

$$m_{\text{i}} e^{\frac{\Delta v}{v_{\text{c}}}} - m_{\text{i}} = m_{\text{fuel}} e^{\frac{\Delta v}{v_{\text{c}}}}$$

$$m_{\text{i}} = \frac{m_{\text{fuel}} e^{\frac{\Delta v}{v_{\text{c}}}}}{e^{\frac{\Delta v}{v_{\text{c}}}} - 1} = m_{\text{fuel}} \frac{1}{1 - e^{-\frac{\Delta v}{v_{\text{c}}}}} = (25.95 \text{ kg}) \frac{1}{1 - e^{-\frac{275.0 \text{ m/s}}{22.91 \cdot 10^3 \text{ m/s}}}} = 2175 \text{ kg}$$

8.82. **THINK:** The fisherman, boat, and tackle box are at rest at the beginning of this problem, so the total momentum of the fisherman, boat, and tackle box before and after the fisherman throws the tackle box must be zero. Using the principle of conservation of momentum and the fact that the momentum of the tackle box must cancel out the momentum of the fisherman and boat, it is possible to find the speed of the fisherman and boat after the tackle box has been thrown.

SKETCH: The sketch shows the motion of the tackle box, boat, and fisherman after the throw:



RESEARCH: The total initial momentum is zero, because there is no motion with respect to the dock. After the fisherman throws the tackle box, the momentum of the tackle box is $p_{\text{box}} = m_{\text{box}} v_{\text{box}} = m_{\text{box}} v_{\text{throw}}$ towards the dock. The total momentum after the throw must equal the total momentum before the throw, so the sum of the momentum of the box, the momentum of the boat, and the momentum of the fisherman must be zero: $p_{\text{box}} + p_{\text{fisherman}} + p_{\text{boat}} = 0$. The fisherman and boat both have the same velocity, so $p_{\text{fisherman}} = m_{\text{fisherman}} v_{\text{fisherman}} = m_{\text{fisherman}} v_{\text{recoil}}$ away from the dock and $p_{\text{boat}} = m_{\text{boat}} v_{\text{boat}} = m_{\text{boat}} v_{\text{recoil}}$ away from the dock.

SIMPLIFY: The goal is to find the recoil velocity of the fisherman and boat. Using the equation for momentum after the tackle box has been thrown, $p_{\text{box}} + p_{\text{fisherman}} + p_{\text{boat}} = 0$, substitute in the formula for the momenta of the tackle box, boat, and fisherman: $0 = m_{\text{box}} v_{\text{throw}} + m_{\text{fisherman}} v_{\text{recoil}} + m_{\text{boat}} v_{\text{recoil}}$. Solve for the recoil velocity:

$$
\begin{aligned}
m_{\text{box}} v_{\text{throw}} + m_{\text{fisherman}} v_{\text{recoil}} + m_{\text{boat}} v_{\text{recoil}} &= 0 \\
m_{\text{fisherman}} v_{\text{recoil}} + m_{\text{boat}} v_{\text{recoil}} &= -m_{\text{box}} v_{\text{throw}} \\
v_{\text{recoil}} \left(m_{\text{fisherman}} + m_{\text{boat}} \right) &= -m_{\text{box}} v_{\text{throw}} \\
v_{\text{recoil}} &= -\frac{m_{\text{box}} v_{\text{throw}}}{m_{\text{fisherman}} + m_{\text{boat}}}
\end{aligned}
$$

CALCULATE: The mass of the tackle box, fisherman, and boat, as well as the velocity of the throw (with respect to the dock) are given in the question. Using these values gives:

$$
\begin{aligned}
v_{\text{recoil}} &= -\frac{m_{\text{box}} v_{\text{throw}}}{m_{\text{fisherman}} + m_{\text{boat}}} \\
&= -\frac{13.63 \text{ kg} \cdot 2.911 \text{ m/s}}{75.19 \text{ kg} + 28.09 \text{ kg}} \\
&= -0.3841685709 \text{ m/s}
\end{aligned}
$$

ROUND: The masses and velocity given in the question all have four significant figures, and the sum of the mass of the fisherman and the mass of the boat has five significant figures, so the final answer should have four significant figures. The final speed of the fisherman and boat is –0.3842 m/s towards the dock, or 0.3842 m/s away from the dock.

DOUBLE-CHECK: It makes intuitive sense that the much more massive boat and fisherman will have a lower speed than the less massive tackle box. Their momenta should be equal and opposite, so a quick way to check this problem is to see if the magnitude of the tackle box's momentum equals the magnitude of the man and boat. The tackle box has a momentum of magnitude 13.63 kg · 2.911 m/s = 39.68 kg·m/s after it is thrown. The boat and fisherman have a combined mass of 103.28 kg, so their final momentum has a magnitude of 103.28 kg · 0.3842 m/s = 39.68 kg·m/s. This confirms that the calculations were correct.

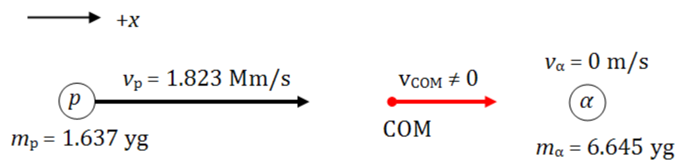
8.83. $$v_{\text{box}} = \frac{m_{\text{man}} + m_{\text{boat}}}{m_{\text{box}}} v_{\text{boat}} = \frac{77.49 \text{ kg} + 28.31 \text{ kg}}{14.27 \text{ kg}} \left(0.3516 \text{ m/s}\right) = 2.607 \text{ m/s}$$

8.84.

$$
\begin{aligned}
&\left(m_{\text{man}} + m_{\text{boat}}\right) v_{\text{boat}} = m_{\text{box}} v_{\text{box}} \\
&m_{\text{man}} v_{\text{boat}} + m_{\text{boat}} v_{\text{boat}} = m_{\text{box}} v_{\text{box}} \\
&m_{\text{man}} = \frac{m_{\text{box}} v_{\text{box}} - m_{\text{boat}} v_{\text{boat}}}{v_{\text{boat}}} = m_{\text{box}} \frac{v_{\text{box}}}{v_{\text{boat}}} - m_{\text{boat}} \\
&m_{\text{man}} = \left(14.91 \text{ kg}\right) \frac{3.303 \text{ m/s}}{0.4547 \text{ m/s}} - 28.51 \text{ kg} = 79.80 \text{ kg}
\end{aligned}
$$

8.85. **THINK:** The masses and initial speeds of both particles are known, so the momentum of the center of mass can be calculated. The total mass of the system is known, so the momentum can be used to find the speed of the center of mass.

SKETCH: To simplify the problem, choose the location of the particle at rest to be the origin, with the proton moving in the +*x* direction. All of the motion is along a single axis, with the center of mass (COM) between the proton and the alpha particle.



RESEARCH: The masses and velocities of the particles are given, so the momenta of the particles can be calculated as the product of the mass and the speed $p_\alpha = m_\alpha v_\alpha$ and $p_p = m_p v_p$ towards the alpha particle. The center-of-mass momentum can be calculated in two ways, either by taking the sum of the momenta of each particle ($P_{COM} = \sum_{i=0}^{n} p_i$) or as the product of the total mass of the system times the speed of the center of mass ($P_{COM} = M \cdot v_{COM}$).

SIMPLIFY: The masses of both particles are given in the problem, and the total mass of the system M is the sum of the masses of each particle, $M = m_p + m_\alpha$. The total momentum $P_{COM} = \sum_{i=0}^{n} p_i = p_\alpha + p_p$ and $P_{COM} = M \cdot v_{COM}$, so $M \cdot v_{COM} = p_\alpha + p_p$. Substitute for the momenta of the proton and alpha particle (since the alpha particle is not moving, it has zero momentum), substitute for the total mass, and solve for the velocity of the center of mass:

$$\begin{aligned} M \cdot v_{COM} &= p_\alpha + p_p \Rightarrow \\ v_{COM} &= \frac{p_\alpha + p_p}{M} \\ &= \frac{m_\alpha v_\alpha + m_p v_p}{m_\alpha + m_p} \\ &= \frac{m_\alpha \cdot 0 + m_p v_p}{m_\alpha + m_p} \\ &= \frac{m_p v_p}{m_\alpha + m_p} \end{aligned}$$

CALCULATE: The problem states that the proton has a mass of $1.673 \cdot 10^{-27}$ kg and moves at a speed of $1.823 \cdot 10^6$ m/s towards the alpha particle, which is at rest and has a mass of $6.645 \cdot 10^{-27}$ kg. So the center of mass has a speed of

$$\begin{aligned} v_{\text{COM}} &= \frac{m_{\text{p}} v_{\text{p}}}{m_{\text{p}\alpha} + m} \\ &= \frac{\left(1.823 \cdot 10^6 \text{ m/s}\right)\left(1.673 \cdot 10^{-27} \text{ kg}\right)}{1.673 \cdot 10^{-27} \text{ kg} + 6.645 \cdot 10^{-27} \text{ kg}} \\ &= 3.666601346 \cdot 10^5 \text{ m/s} \end{aligned}$$

ROUND: The masses of the proton and alpha particle, as well as their sum, have four significant figures. The speed of the proton also has four significant figures. The alpha particle is at rest, so its speed is not a calculated value, and the zero speed does not change the number of figures in the answer. Thus, the speed of the center of mass is $3.667 \cdot 10^5$ m/s, and the center of mass is moving towards the alpha particle.

DOUBLE-CHECK: To double check, find the location of the center of mass as a function of time, and take the time derivative to find the velocity. The distance between the particles is not given in the problem, so call the distance between the particles at an arbitrary starting time $t = 0$ to be d_0. The positions of each particle can be described by their location along the axis of motion, $r_\alpha = 0$ and $r_p = d_0 + v_p t$.

Using this, the location of the center of mass is

$$R_{\text{COM}} = \frac{1}{m_{\text{p}} + m_\alpha}\left(r_{\text{p}} m_{\text{p}} + r_\alpha m_\alpha\right).$$

Take the time derivative to find the velocity:

$$\begin{aligned}
\frac{d}{dt}R_{\text{COM}} &= \frac{d}{dt}\left[\frac{1}{m_{\text{p}} + m_\alpha}\left(r_{\text{p}} m_{\text{p}} + r_\alpha m_\alpha\right)\right] \\
&= \frac{1}{m_{\text{p}} + m_\alpha}\frac{d}{dt}\left[\left(d_0 + v_{\text{p}} t\right) m_{\text{p}} + 0 \cdot m_\alpha\right] \\
&= \frac{1}{m_{\text{p}} + m_\alpha}\frac{d}{dt}\left(d_0 m_{\text{p}} + v_{\text{p}} m_{\text{p}} t + 0\right) \\
&= \frac{1}{m_{\text{p}} + m_\alpha}\frac{d}{dt}\left(d_0 m_{\text{p}} + v_{\text{p}} m_{\text{p}} t\right) \\
&= \frac{1}{m_{\text{p}} + m_\alpha}\left(0 + v_{\text{p}} m_{\text{p}}\right) \\
&= \frac{v_{\text{p}} m_{\text{p}}}{m_{\text{p}} + m_\alpha} \\
&= \frac{\left(1.823 \cdot 10^{6} \text{ m/s}\right)\left(1.673 \cdot 10^{-27} \text{ kg}\right)}{1.673 \cdot 10^{-27} \text{ kg} + 6.645 \cdot 10^{-27} \text{ kg}} \\
&= 3.666601346 \cdot 10^{5} \text{ m/s}
\end{aligned}$$

This agrees with the earlier result.

8.86. $(m_p + m_\alpha)v_{\text{cm}} = m_p v_p + m_\alpha v_\alpha$

Since $v_\alpha = 0$,

$$v_p = \frac{m_p + m_\alpha}{m_p} v_{\text{cm}} = \frac{1.673 \cdot 10^{-27} \text{ kg} + 6.645 \cdot 10^{-27} \text{ kg}}{\left(1.673 \cdot 10^{-27} \text{ kg}\right)}\left(5.509 \cdot 10^{5} \text{ m/s}\right) = 2.739 \cdot 10^{6} \text{ m/s}$$

Chapter 9: Circular Motion

Concept Checks

9.1. e **9.2.** b **9.3.** a **9.4.** d **9.5.** b **9.6.** d

Multiple-Choice Questions

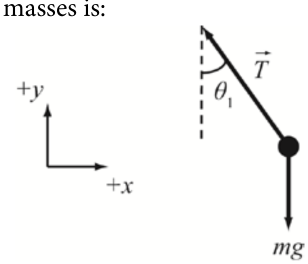
9.1. d **9.2.** c **9.3.** b **9.4.** d **9.5.** c **9.6.** a **9.7.** c **9.8.** a **9.9.** a **9.10.** c **9.11.** a **9.12.** a **9.13.** d **9.14.** d **9.15.** d **9.16.** a

Conceptual Questions

9.17. A ceiling fan is rotating in the clockwise direction, as viewed from below. This also means that the direction of angular velocity of the fan is in the clockwise direction. The angular velocity is decreasing or slowing down. This indicates that the angular acceleration is negative or in the opposite direction of the angular velocity. Therefore, the angular acceleration is in the counter-clockwise direction.

9.18. No, it will not. This is because when the actor swings across the stage there will be an additional tension on the rope needed to hold the action in circular motion. Note that the total mass of the rope and the actor is 3 lb + 147 lb = 150 lb. This is the maximum mass that can be supported by the hook. Therefore, the additional tension on the rope will break the hook.

9.19. The force body diagram for one of the masses is:



The force of tension in the x-axis is equal to the centripetal force, $T\sin\theta_1 = m\omega^2 r$. The force of the tension along the y-axis must be equal to the force of gravity, $T\cos\theta_1 = mg$. This means $\frac{T\sin\theta_1}{T\cos\theta_1} = \tan\theta_1 = \frac{\omega^2 r}{g}$; therefore, both θ_1 and θ_2 are the same, since they don't depend on the mass.

9.20. For the two points of interest, there are two forces acting on the person; the force of gravity and the normal force. These two forces combine to create the centripetal force. In case A: $F_{\text{c,A}} = F_{\text{N,A}} - F_{\text{g}}$ and case B: $-F_{\text{c,A}} = F_{\text{N,B}} - F_{\text{g}}$. This means that the normal force is $F_{\text{N,A}} = F_{\text{c,A}} + F_{\text{g}} = m\omega^2 r + mg$ and $F_{\text{N,B}} = F_{\text{g}} - F_{\text{c,A}} = mg - m\omega^2 r.$ Therefore, $F_{\text{N,A}}$ is greater than $F_{\text{N,B}}$.

9.21. The linear speed of the bicycle is given by $v = r\omega$. The smaller the diameter, D, the lower the linear speed for the same angular speed because $r = D/2$ so tires with a lower diameter than 25 cm will have a velocity too slow to be practical transportation.

9.22. Both the angular velocity and acceleration are independent of the radius. This means they are the same at the edge and halfway between the edge and center. The linear velocity and acceleration, however, do change with radius, r. At the edge $v_{\text{e}} = r\omega$ and $a_{\text{e}} = R\omega^2$. The halfway point gives $v_{1/2} = \frac{r\omega}{2}$ and $a_{1/2} = \frac{1}{2}R\omega^2$. Comparing the two points, it can be seen that $v_{\text{e}} = 2v_{1/2}$ and $a_{\text{e}} = 2a_{1/2}$.

9.23. For the car to stay on the road there must be a force in the direction of the centripetal force. In this case, the road is not banked, leaving the force of friction as the only possible choice. Since the car travels with constant speed, the force of friction holding the car on a circular path points in radial inward direction towards the center of the circle.

9.24. As the car makes the turn, both strings have a new angular position, θ. From the discussion of the conical pendulum on page 290 and 291, you can see that this angle is given by $\tan\theta = r\omega^2/g$, where r is the distance to the center of the circle. This means that the pendulum that is further away from the center has a larger angle. A larger angle means a larger sideways deflection of the pendulum, and thus the distance between the two pendula increase during the turn, both for a right turn and for a left turn. If the distance d between the two pendula is small compared to the turning radius r, however, this effect is hard to measure or see.

9.25. The kinetic energy when the point mass gets to the top of the loop is equal to the difference in potential energy between the height h and $2R$. Rearranging $\frac{mv^2}{2} = mgh - 2mgR$ for v^2 gives $v^2 = 2g(h-2R)$. For the particle to stay connected to the loop, the centripetal force has to be greater than or equal to the force of gravity. This requirement means $v^2 = Rg$. Using these two equations, the height, h, can be determined:

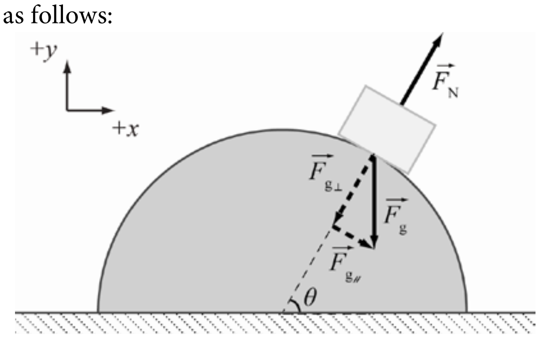
$$2g(h-2R) = Rg \Rightarrow h = 2R + \frac{1}{2}R = \frac{5}{2}R.$$

The height should be $5R/2$ or greater for the point to complete the loop.

9.26. The bob is moving in a horizontal circle at constant speed. This means that the bob experiences a net force equal to the centripetal force inwards. This force is equal to the horizontal component of the tension. The vertical component of the tension must be balanced by the force of gravity. The two forces acting on the bob are the tension and the force of gravity.

9.27. From our discussion of the conical pendulum on page 271, you can see that this angle is given by $\tan\theta = r\omega^2/g$. As the angular speed assumes larger and larger values, the angle *approaches* a value of $90°$, which is the condition that the string is parallel to the ground. However, the *exact* value of $90°$ cannot be reached, because it would correspond to an infinitely high value of the angular speed, which cannot be achieved.

9.28. A picture of the situation is as follows:



This picture tells us that the normal force can be related to the force of gravity by $F_N = F_{g\perp} = F_g \sin\theta = mg\sin\theta.$ In this situation, the normal force provides the centripetal force, so $F_c = mg\sin\theta$ and $a_c = g\sin\theta$. As θ decreases, $\sin\theta$ decreases, and therefore a_c decreases. The acceleration vector for circular motion has two components; the centripetal acceleration, a_c, and the tangential acceleration, $a_t = \cos\theta$, which increases as θ decreases to zero. This satisfies the requirement that $\vec{a} = a_t\hat{t} - a_c\hat{r}$.

9.29. The forces that you feel at the top and bottom of the loop are equal to the normal force. At the top and bottom of the loop, the normal force are along the vertical direction and are $N_{\text{T}} = mg - \frac{mv_{\text{T}}^2}{R}$ and $N_{\text{B}} = mg + \frac{mv_B^2}{R}$. If you experience weightlessness at the top, then $N_{\text{T}} = 0 \Rightarrow mg = mv_{\text{T}}^2 / R$. Energy conservation tells us that $\frac{1}{2}mv_{\text{B}}^2 = 2mgR + \frac{1}{2}mv_{\text{T}}^2$. Insert both of these results into the expression for the normal force at the bottom and find:

$$N_{\text{B}} = mg + mv_{\text{B}}^2 / R = mg + (4mgR + mv_{\text{T}}^2) / R = mg + 4mg + mg = 6mg$$

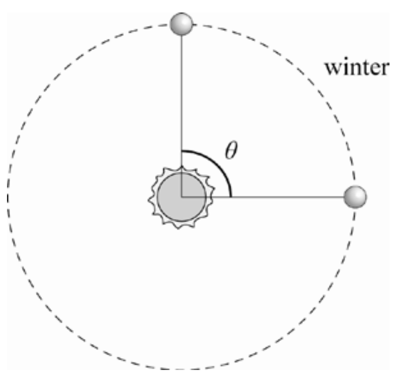
This means that the normal force exerted by the seat on you, your apparent weight, is indeed 6 times your weight at the top of the loop.

9.30. The combined weight of the five daredevils is W. To determine the strength of the rope needed, the tension at the bottom of the arc must be determined. At this point the centripetal force is equal to the difference between the tension and the force of gravity. The tension is equal to $T = \frac{mv^2}{R} + mg$ $= \frac{Wv^2}{Rg} + W$. The kinetic energy at the bottom of the arc is equal to the potential energy at the level of the bridge, $\frac{1}{2}mv^2 = mgR$ or $v^2 = 2gR$. Using this, $T = \frac{W2gR}{Rg} + W = 3W$. The rope must be able to withstand a tension equal to three times the combined weight of the daredevils.

Exercises

9.31. **THINK:** Determine the change in the angular position in radians. Winter lasts roughly a fourth of a year. There are 2π radians in a circle. Consider the orbit of Earth to be circular.

SKETCH:



RESEARCH: The angular velocity of the earth is $\omega = 2\pi / \text{yr}$. The angular position is given by $\theta = \theta_0 + \omega_0 t$.

SIMPLIFY: $\Delta\theta = \theta - \theta_0 = \omega_0 t$

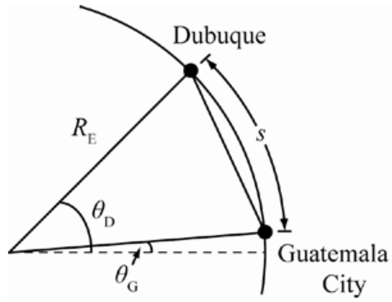
CALCULATE: $\Delta\theta = \frac{2\pi \text{ rad}}{\text{yr}}\left(\frac{1}{4}\text{ yr}\right) = \frac{\pi \text{ rad}}{2} = \frac{3.14 \text{ rad}}{2} = 1.57 \text{ rad}$

ROUND: Since π is used to three significant figures, the angle the Earth sweeps over winter is 1.57 rad. It would also be entirely reasonable to leave the answer as $\pi / 2$ radians.

DOUBLE-CHECK: This value makes sense, since there are four seasons of about equal length, so the angle should be a quarter of a circle.

9.32. **THINK:** Determine the arc length between Dubuque and Guatemala City. The angular positions of Dubuque and Guatemala City are $\theta_{\text{D}} = 42.50°$ and $\theta_{\text{G}} = 14.62°$, respectively. The radius of the Earth is $R_{\text{E}} = 6.37 \cdot 10^6$ m.

SKETCH:



RESEARCH: The length of an arc is given by $s = r\theta$, where r is the radius of the circle, and θ is the arc angle given in radians.

SIMPLIFY: $s = R_{\text{E}}(\theta_{\text{D}} - \theta_{\text{G}})\dfrac{2\pi}{360°}$, where the units of the angles are degrees.

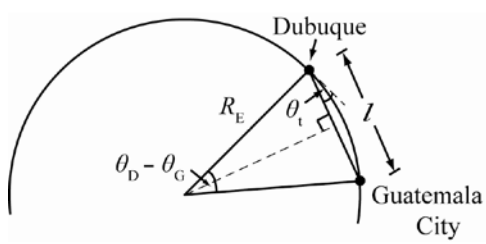
CALCULATE: $s = \left(6.37 \cdot 10^6 \text{ m}\right)\left(42.50° - 14.62°\right)\dfrac{2\pi}{360°} = 3.0996 \cdot 10^6 \text{ m}$

ROUND: The arc length's accuracy is given by the least accurate value used to determine it. In this case, the least accurate value is the radius of Earth, given to three significant figures, so the arc length is $3.10 \cdot 10^6$ m.

DOUBLE-CHECK: This is equal to 3100 km, a reasonable distance between the northern United States and Central America.

9.33. **THINK:** Determine the linear distance between Dubuque and Guatemala city. Also, determine the angle below the horizontal for a tunnel that connects the two. The angular positions of Dubuque and Guatemala City are $\theta_{\text{D}} = 42.50°$ and $\theta_{\text{G}} = 14.62°$, respectively. The radius of the Earth is $R_{\text{E}} = 6.37 \cdot 10^6$ m.

SKETCH:



RESEARCH: Use the triangle of the drawing to relate $\theta_{\text{D}} - \theta_{\text{G}}$, R_{E} and $l/2$. The right triangle gives rise to the equation $\sin\dfrac{\theta_{\text{D}} - \theta_{\text{G}}}{2} = \dfrac{l/2}{R_{\text{E}}}$. The angle of the tunnel is $\theta_{\text{t}} = \left(\dfrac{\theta_{\text{D}} - \theta_{\text{G}}}{2}\right)$.

SIMPLIFY: $l = 2R_{\text{E}}\sin\left(\dfrac{\theta_{\text{D}} - \theta_{\text{G}}}{2}\right)$

CALCULATE: $l = 2\left(6.37 \cdot 10^6 \text{ m}\right)\sin\left(\dfrac{42.50° - 14.62°}{2}\right) = 3.06914 \cdot 10^6 \text{ m}$ $\theta_{\text{t}} = \left(\dfrac{42.50° - 14.62°}{2}\right) = 13.94°$

ROUND: The length will have the same accuracy as the radius of Earth. The angle of the tunnel will be as accurate as the latitude of the cities. Therefore, the length of the tunnel is $3.07 \cdot 10^6$ m, with an angle of 13.94° below the surface of the Earth.

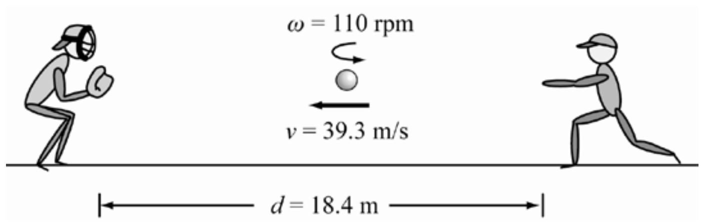
DOUBLE-CHECK: The length of the tunnel is a bit shorter than the arc length, which is expected. See the solution to Problem 9.32.

9.34. **THINK:** Determine the number of rotations the ball will make as it travels to the catcher's glove. The linear and angular speeds of the ball are $v = 88$ mph and $\omega = 110$ rpm . In SI units, these are

$$v = 88\text{ mph}\left(\frac{0.447\text{m / s}}{\text{mph}}\right) = 39.3\text{ m/s} \quad \text{and} \quad \omega = 110\text{ rpm}\left(\frac{2\pi\text{ rad}}{1\text{ rot}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) = 11.52\text{ rad/s.}$$

The ball travels a distance, $d = 60.5$ ft or 18.4 m.

SKETCH:



RESEARCH: The time it takes for the ball to reach the catcher is given by $t = v / d.$ This time will then be used to calculate the number of rotations, given by $n = \omega t$. This number n will be in radians which will then have to be converted to rotations, where $1\text{ rad}\left(\frac{1\text{ rev}}{2\pi}\right) = 0.16$ revolution.

SIMPLIFY: $n = \omega\left(\frac{d}{v}\right)$

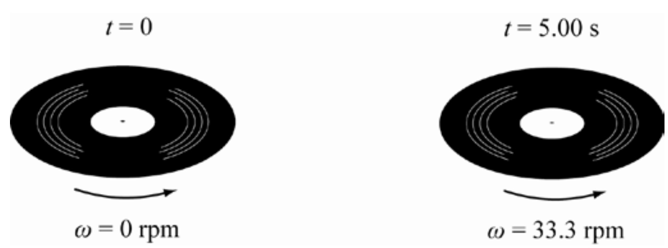
CALCULATE: $n = \frac{18.4\text{ m}(11.52\text{ rad/s})}{39.3\text{ m/s}} = 5.394\text{ rad}\left(\frac{1\text{ rev}}{2\pi}\right) = 0.8586$ rotations

ROUND: The linear speed of the ball, the distance traveled, and the angular speed of the ball are all given to three significant figures, so the number of rotations should be 0.859.

DOUBLE-CHECK: Dimensional analysis: $[n] = \text{rad/s}\cdot\frac{\text{m}}{\text{m/s}}\cdot\frac{\text{revolution}}{2\pi\text{ rad}}.$ All units cancel giving a dimensionless quantity, as expected.

9.35. **THINK:** Determine the average angular acceleration of the record and its angular position after reaching full speed. The initial and final angular speeds are 0 rpm to 33.3 rpm. The time of acceleration is 5.00 s.

SKETCH:



RESEARCH: The equation for angular acceleration is $\alpha = \left(\omega_\text{f} - \omega_\text{i}\right) / \Delta t$. The angular position of an object under constant angular acceleration is given by $\theta = \frac{1}{2}\alpha t^2$.

SIMPLIFY: There is no need to simplify the equation.

CALCULATE: $\alpha = \frac{33.3\text{ rpm} - 0\text{ rpm}}{5.00\text{ s}(60\text{ s/min})} = 0.111\text{ rev/s}^2\cdot\frac{2\pi\text{ rad}}{1\text{ rev}} = 0.6974\text{ rad/s}^2$

$$\theta = \frac{1}{2}(0.111\text{ rev/s})(5.00\text{ s})^2 = 1.3875\text{ rev}\cdot\frac{2\pi\text{ rad}}{1\text{ rev}} = 8.718\text{ rad}$$

ROUND: To three significant figures, the angular acceleration and position are:

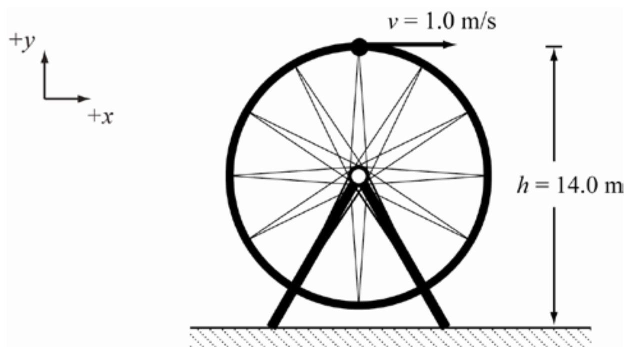
(a) $\alpha = 0.697 \text{ rad/s}^2$

(b) $\theta = 8.72 \text{ rad}$

DOUBLE-CHECK: The calculations yield the correct units of radians and rad/s^2.

9.36. **THINK:** Determine the horizontal distance the teddy bear travels during its fall. In order to do this, the height and the horizontal speed of the bear must be determined. The diameter of the wheel is 12.0 m, the bottom of which is 2.0 m above the ground. The rim of the wheel travels at a speed of $v = 1.0 \text{ m/s}$. The height of the bear is 14.0 m from the ground and is traveling at a speed of 1.0 m/s in the horizontal direction when it falls.

SKETCH:



RESEARCH: The horizontal distance is given by $x = vt$. The time is not yet known but can be determined from $h = \frac{1}{2}gt^2$.

SIMPLIFY: The time it takes the bear to fall is $t^2 = \frac{2h}{g}$ or $t = \sqrt{\frac{2h}{g}}$. The horizontal distance traveled is $x = vt = v\sqrt{\frac{2h}{g}}$.

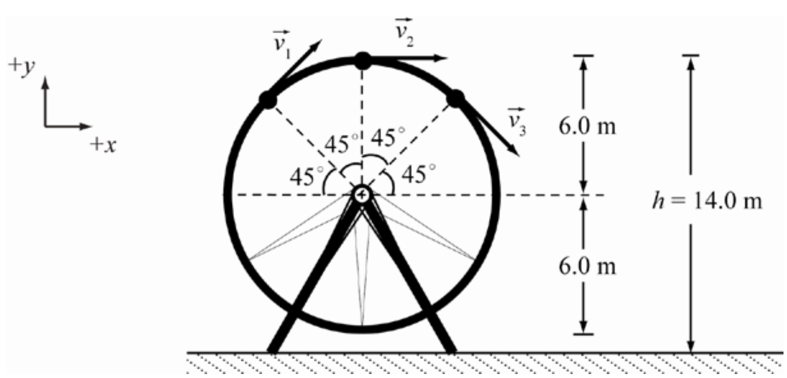
CALCULATE: $x = (1.00 \text{ m/s})\sqrt{\frac{2(14.0 \text{ m})}{9.81 \text{ m/s}^2}} = 1.6894 \text{ m}$

ROUND: The velocity is given to three significant figures, so round the distance to 1.69 m.

DOUBLE-CHECK: The bear lands a short distance from the base of the wheel, as one would expect given its small initial velocity.

9.37. **THINK:** Determine the distance between the three teddy bears. The bears will be traveling at 1.00 m/s but will have different directions and distances from the ground. The angle between adjacent bears is $45.0°$. The diameter of the wheel is 12.0 m and the bottom of the wheel is 2.00 m above the ground.

SKETCH:



RESEARCH: The height of bear 1 and 3 is the same and is $h_1 = \left(8.00 + 6.00\sin\left(45.0°\right)\right)$ m. The second bear is $h_2 = 14.0$ m above the ground. The velocities of each bear in the horizontal and vertical directions are $v_x = v\cos\theta$ and $v_y = v\sin\theta$, where $\theta_1 = 45.0°$, $\theta_2 = 0°$ and $\theta_3 = -45.0°$. The distance between each bear before they are dropped is $\Delta d = 6.00\sin\left(45.0°\right)$ m. Use the regular equations for projectile motion: $\Delta x = v_x t$ and $\Delta y = v_y t - \frac{1}{2}gt^2$.

SIMPLIFY: For different initial heights, *H*, the time of the fall can be determined from $h(t) = v\sin\left(\theta\right)t - \frac{1}{2}gt^2 + H$. This is a quadratic equation with solution $t = \frac{v\sin\theta}{g} \pm \sqrt{\frac{v^2\sin^2\theta}{g^2} + \frac{2H}{g}}$. Choose the positive root, that is, $t = \frac{v\sin\theta}{g} + \sqrt{\frac{v^2\sin^2\theta}{g^2} + \frac{2H}{g}}$. The change in distance is $\Delta x = v\cos\left(\theta\right)t = v\cos\theta\left(\frac{v\sin\theta}{g} + \sqrt{\frac{v^2\sin^2\theta}{g^2} + \frac{2H}{g}}\right)$, which means that the value of *x* is given by the equation $x = x_0 + v\cos\theta\left(\frac{v\sin\theta}{g} + \sqrt{\frac{v^2\sin^2\theta}{g^2} + \frac{2H}{g}}\right)$.

CALCULATE: For the first bear $x_0 = 0$, $H = h_1$, and $\theta_1 = 45.0°$. Recall $\cos 45.0° = \sin 45.0° = 1/\sqrt{2}$.

$$x_1 = 0 + \left(\left(1.00 \text{ m/s}\right)\left(\frac{1}{\sqrt{2}}\right)\right)\left[\frac{1.00 \text{ m/s}}{9.81 \text{ m/s}^2}\left(\frac{1}{\sqrt{2}}\right) + \sqrt{\frac{\left(1.00 \text{ m/s}\right)^2}{2\left(9.81 \text{ m/s}^2\right)^2} + \frac{2\left(8.00 + 6.00/\sqrt{2}\right) \text{ m}}{9.81 \text{ m/s}^2}}\right] = 1.1693 \text{ m}$$

The initial velocity of the second bear is horizontal, so the bear travels a horizontal distance of $\Delta x = 1.6894$ m (see solution to question 9.36). The second bear's position is $x_2 = x_0 + \Delta x = \Delta d + \Delta x = \frac{6.00 \text{ m}}{\sqrt{2}} + 1.6894 \text{ m} = 5.9320$ m from the origin. For the third bear, $x_0 = 2\Delta d$, $H = h_1$, and $\theta_2 = -45.0°$.

$$x_3 = 2\left(\frac{6.00 \text{ m}}{\sqrt{2}}\right) + \frac{\left(1.00 \text{ m/s}\right)}{\sqrt{2}}\left[\frac{-1.00 \text{ m/s}}{\sqrt{2}\left(9.81 \text{ m/s}^2\right)} + \sqrt{\frac{\left(1.00 \text{ m/s}\right)^2}{2\left(9.81 \text{ m/s}^2\right)^2} + \frac{2\left(8.00 + 6.00/\sqrt{2}\right) \text{ m}}{9.81 \text{ m/s}^2}}\right] = 9.5526 \text{ m}$$

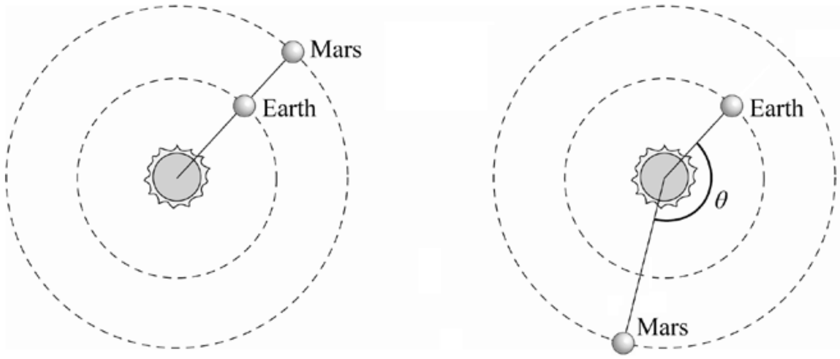
The distance between the first 2 bears is $\Delta d_{12} = 4.7627$ m. The distance between the last two bears is $\Delta d_{23} = 3.6206$ m.

ROUND: The velocity has three significant figures, so the results should also have three significant figures. The distances between the bears once they hit the ground are $\Delta d_{12} = 4.76$ m and $\Delta d_{23} = 3.62$ m.

DOUBLE-CHECK: The result is reasonable since $\Delta d_{12} > \Delta d_{23}$. This must be so since the third bear is in the air for a shorter time because the original horizontal velocity points towards the ground.

9.38. **THINK:** Determine (a) the angular distance between the two planets a year later, (b) the time it takes the two planets to align again and (c) the angular position the alignment occurs at. The radius and period of each planet's orbit are $r_M = 228 \cdot 10^6$ km, $T_M = 687$ days, $r_E = 149.6 \cdot 10^6$ km and $T_E = 365.26$ days.

SKETCH:



RESEARCH: The questions can be answered using $\theta = \omega t$ and $\omega = 2\pi / T$.

SIMPLIFY: The angular distance is

$$\Delta\theta = \theta_{\text{E}} - \theta_{\text{M}} = \omega_{\text{E}} T_{\text{E}} - \omega_{\text{M}} T_{\text{E}} = 2\pi\left(\frac{T_{\text{E}}}{T_{\text{E}}} - \frac{T_{\text{E}}}{T_{\text{M}}}\right) = 2\pi\left(1 - \frac{T_{\text{E}}}{T_{\text{M}}}\right).$$

The time it takes the planets to realign occurs when $\theta_{\text{E}} = \theta_{\text{M}} + 2\pi$ or $\omega_{\text{E}}\Delta t = \omega_{\text{M}}\Delta t + 2\pi$, so

$$\Delta t = \frac{2\pi}{\omega_{\text{E}} - \omega_{\text{M}}} = \frac{2\pi}{\frac{2\pi}{T_{\text{E}}} - \frac{2\pi}{T_{\text{M}}}} = \frac{T_{\text{E}} T_{\text{M}}}{T_{\text{M}} - T_{\text{E}}}.$$

The angular position is found by solving for the angle instead of the time. $\theta_{\text{M}} = \omega_{\text{M}}\Delta t \;\Rightarrow\; \Delta t = \theta_{\text{M}} / \omega_{\text{M}}$,

so: $\theta_{\text{E}} = \omega_{\text{E}}\Delta t = \dfrac{\omega_{\text{E}}\theta_{\text{M}}}{\omega_{\text{M}}} = \theta_{\text{M}} + 2\pi \;\Rightarrow\; \theta_{\text{M}} = \dfrac{2\pi}{\dfrac{\omega_{\text{E}}}{\omega_{\text{M}}} - 1} - 2\pi = \dfrac{2\pi}{\dfrac{T_{\text{M}}}{T_{\text{E}}} - 1} - 2\pi = \dfrac{2\pi T_{\text{E}}}{T_{\text{M}} - T_{\text{E}}} - 2\pi.$ Subtract 2π from the answer, so that $\theta \le 2\pi$.

CALCULATE: $\Delta\theta = 2\pi\left(1 - \dfrac{365.26}{687}\right) = 2.9426 \text{ rad}$, $\Delta t = \dfrac{687(365.26)}{687 - 365.26} = 779.93 \text{ days}$,

$\theta = \dfrac{2\pi(365.26)}{687 - 365.26} - 2\pi = 0.84989 \text{ rad}$

ROUND: The periods of Mars and Earth have three significant figures, so the results should be rounded accordingly.

(a) $\Delta\theta = 2.94 \text{ rad}$

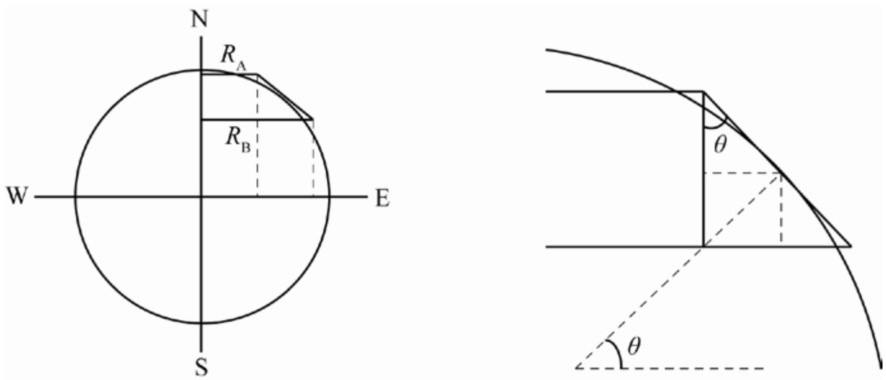
(b) $\Delta t = 780. \text{ days}$

(c) $\theta = 0.850 \text{ rad}$

DOUBLE-CHECK: The numbers are of the correct order for this solar system.

9.39. **THINK:** Determine (a) the magnitude and direction of the velocities of the pendulum at position *A* and *B*, (b) the angular speed of the pendulum motion, (c) the period of the rotation and (d) the effects of moving the pendulum to the equator. The latitude of the pendulum is $55.0°$ above the equator. The pendulum swings over a distance of d = 20.0 m. The period of the Earth's rotation is $T_{\text{E}} = 23 \text{ hr} + 56 \text{ min} = 86160 \text{ s}$ and the Earth's radius is $R_{\text{E}} = 6.37 \cdot 10^{6}$ m.

SKETCH:



RESEARCH: The following equations can be used: $\omega = \frac{2\pi}{T}$, $v = r\omega$, $R_A = R_E \cos\theta - \left(\frac{d}{2}\sin\theta\right)$ and $R_B = R_E \cos\theta + \left(\frac{d}{2}\sin\theta\right)$.

SIMPLIFY: The magnitudes of the velocities are:

$$v_A = R_A\omega_A = \frac{2\pi R_A}{T_E} = \frac{2\pi\left(R_E\cos\theta - \left(\frac{d}{2}\sin\theta\right)\right)}{T_E} \text{ and } v_B = \frac{2\pi\left(R_E\cos\theta + \left(\frac{d}{2}\sin\theta\right)\right)}{T_E}.$$

The angular speed of the rotation is related to the linear speed by $\Delta v = \omega_R d$. Rearranging gives:

$$\omega_R = \frac{\Delta v}{d} = \left(\frac{1}{d}\right)\frac{2\pi}{T_E}\left(\left(R_E\cos\theta + \left(\frac{d}{2}\sin\theta\right)\right) - \left(R_E\cos\theta - \left(\frac{d}{2}\sin\theta\right)\right)\right) = \frac{2\pi}{dT_E}d\sin\theta = \frac{2\pi}{T_E}\sin\theta.$$

The period is then $T_R = \frac{2\pi}{w_R} = \frac{2\pi}{\frac{2\pi}{T_E}\sin\theta} = \frac{T_E}{\sin\theta}$. At the equator, $\theta = 0°$.

CALCULATE:

(a) $$v_A = 2\pi\left(\frac{\left(6.37\cdot 10^6 \text{ m}\right)\cos(55.0°) - (10.0 \text{ m})\sin(55.0°)}{86{,}160 \text{ s}}\right) = 266.44277 \text{ m/s}$$

$$v_B = 2\pi\left(\frac{\left(6.37\cdot 10^6 \text{ m}\right)\cos(55.0°) + (10.0 \text{ m})\sin(55.0°)}{86{,}160 \text{ s}}\right) = 266.44396 \text{ m/s}$$

$\Delta v = v_B - v_A = 266.44396 \text{ m/s} - 266.44277 \text{ m/s} = 0.00119 \text{ m/s}$ or 1.19 mm/s

(b) $$\omega_R = \frac{2\pi\sin(55.0°)}{86{,}160 \text{ s}} = 5.97\cdot 10^{-5} \text{ rad/s}$$

(c) $$T_R = \frac{86{,}160 \text{ s}}{\sin(55.0°)} = 105{,}182 \text{ s or about 29.2 hours}$$

(d) At the equator, $T_R = \lim_{\theta\to 0}\frac{T_E}{\sin\theta} = \infty$.

ROUND: The values given in the question have three significant figures, so the answers should also be rounded to three significant figures:

(a) The velocities are $v_A = 266.44277$ m/s and $v_B = 266.44396$ m/s, are in the direction of the Earth's rotation eastward. This means the difference between the velocities is $\Delta v = 1.19$ mm/s.

(b) The angular speed of rotation is $w_R = 1.19\cdot 10^{-4}$ rad/s.

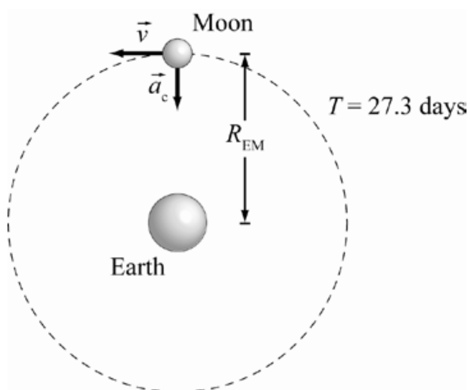
(c) The period of rotation is about 29.2 hours.

(d) At the equator there is no difference between the velocities at *A* and *B*, so the period is $T_R = \infty$. This means the pendulum does not rotate.

DOUBLE-CHECK: These are reasonable answers. If the difference in velocities was larger, these effects would be seen in everyday life but they are not. These are things pilots deal with when planning a flight path.

9.40. **THINK:** Determine the centripetal acceleration of the Moon around the Earth. The period of the orbit is $T = 27.3$ days and the orbit radius is $R = 3.85 \cdot 10^8$ m.

SKETCH:



RESEARCH: The centripetal acceleration is given by $a_c = \dfrac{v^2}{R}$. The radius, *R*, is known, so the speed, *v*, can be determined by making use of the period and noting that in this period the moon travels a distance equal to the circumference of a circle of radius *R*. Therefore,

$$v = \frac{\text{circumference}}{\text{period}} = \frac{2\pi R_{EM}}{T}.$$

SIMPLIFY: $a_c = \dfrac{v^2}{R} = \left(\dfrac{2\pi R_{EM}}{T}\right)^2 \left(\dfrac{1}{R_{EM}}\right) = \dfrac{4\pi^2 R_{EM}}{T^2}$

CALCULATE: Convert the period to seconds: 27.3 days = $2.3587 \cdot 10^6$ seconds. Therefore,

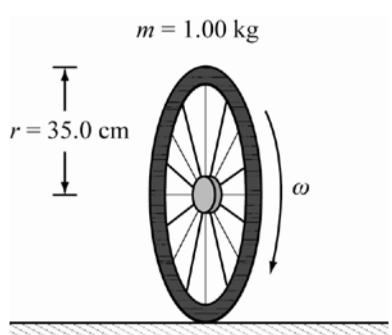
$$a_c = \frac{4\pi^2 \left(3.85 \cdot 10^8 \text{ m}\right)}{\left(23.587 \cdot 10^5 \text{ s}\right)^2} = 2.732 \cdot 10^{-3} \text{ m/s}^2.$$

ROUND: Since the values are given to three significant figures, $a_c = 2.73 \cdot 10^{-3}$ m/s^2.

DOUBLE-CHECK: This is reasonable for a body in uniform circular motion with the given values.

9.41. **THINK:** Determine the angular acceleration of a wheel given that it takes 1.20 seconds to stop when put in contact with the ground after rotating at 75.0 rpm. The wheel has a radius 35.0 cm and a mass of 1.00 kg.

SKETCH:



RESEARCH: Consider the angular speed of the wheel, and the necessary acceleration to bring that speed to zero in the given time. The angular acceleration is given by $\alpha = \frac{\Delta\omega}{\Delta t} = \frac{(\omega - \omega_0)}{t}$ and the rotational speed is given by:

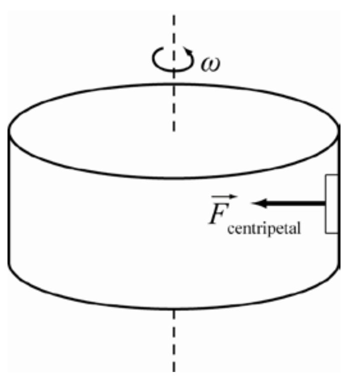
$$\omega = (\text{rpm})\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right).$$

SIMPLIFY: Since the final rotational speed is zero, $\alpha = \frac{-\omega_0}{t} = -\frac{(\text{rpm})\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{\text{min}}{60 \text{ s}}\right)}{t}.$

CALCULATE: $\alpha = -\frac{(75.0 \text{ rpm})(2\pi \text{ rad/rev})(1 \text{ min}/60 \text{ s})}{(1.20 \text{ s})} = -6.54 \text{ rad/s}^2$

ROUND: Since the values are given to three significant figures, the result is $\alpha = -6.54 \text{ rad/s}^2$.
DOUBLE-CHECK: It is important that the acceleration is negative since it is slowing down the wheel. The magnitude seems reasonable based on the given values.

9.42. **THINK:** Determine the frequency of rotation required to produce an acceleration of $1.00 \cdot 10^5 g$. The radius is R = 10.0 cm.
SKETCH:



RESEARCH: Recall that the centripetal acceleration is given by $a_c = \omega^2 R$. Also, $\omega = 2\pi f$. Therefore, $a_c = (2\pi f)^2 R = 4\pi^2 f^2 R$.

SIMPLIFY: Solving for *f*, $f = \frac{1}{2\pi}\sqrt{\frac{a_c}{R}}$.

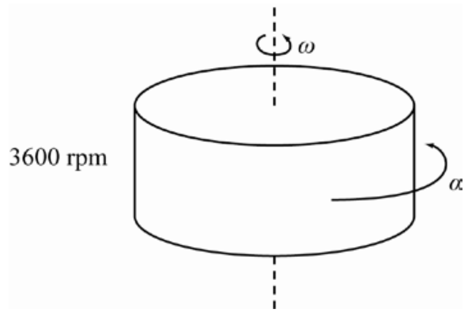
CALCULATE: $f = \frac{1}{2\pi}\sqrt{\frac{(1.00 \cdot 10^5)(9.81 \text{ m/s}^2)}{0.100 \text{ m}}} = 498.49 \text{ Hz}$

ROUND: Since all values are given to three significant figures, the result is $f = 498$ Hz.

DOUBLE-CHECK: A frequency of about 500 Hz seems reasonable to try to obtain an acceleration five orders of magnitude greater than *g*.

9.43. **THINK:** The initial angular speed is $\omega_0 = 3600. \text{ rpm} = 3600. \text{ rpm} \cdot \frac{2\pi \text{ rad}}{1 \text{ rotation}} \cdot \frac{1 \text{ min}}{60 \text{ s}} = 120\pi \text{ rad/s}.$ Calculate the time, t_1, it takes for the centrifuge to come to a stop $(\omega_1 = \omega(t_1) = 0)$ by using the average angular speed, $\bar{\omega}$, and the fact that that it completes $n = 60.0$ rotations. Use the time taken to stop to find the angular acceleration.

SKETCH:



RESEARCH: The average angular speed is given by $\bar{\omega} = \frac{1}{2}(\omega_f + \omega_0)$. Since the centrifuge completes 60.0 turns while decelerating, it turns through an angle of $\Delta\theta = 60.0 \text{ turns} \cdot \frac{2\pi \text{ rad}}{\text{turn}} = 120\pi$ rad. Use the two previous calculated values in the formula $\Delta\theta = \bar{\omega} t_1$ to obtain the time taken to come to a stop. Then, use the equation $\omega(t) = \omega_0 + \alpha t$ to compute the angular acceleration, α.

SIMPLIFY: The time to decelerate is given by, $t_1 = \frac{\Delta\theta}{\bar{\omega}} = \frac{\Delta\theta}{(\omega_1 + \omega_0)/2}$. Substituting this into the last equation given in the research step gives the equation, $\omega(t_1) = \omega_1 = \omega_0 + \alpha \frac{2\Delta\theta}{(\omega_1 + \omega_0)}$. Solving for α yields the equation: $\alpha = \frac{(\omega_1 - \omega_0)(\omega_1 + \omega_0)}{2\Delta\theta}$.

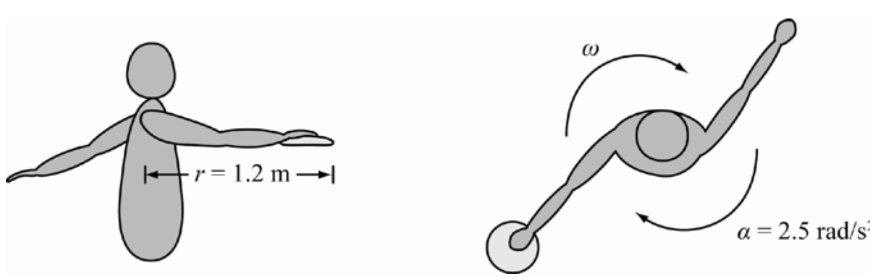
CALCULATE: $\alpha = \frac{(0 - 120\pi \text{ rad/s})(0 + 120\pi \text{ rad/s})}{2(120\pi \text{ rad})} = -60\pi \text{ rad/s}^2 = -188.496 \text{ rad/s}^2$

ROUND: Since the number of rotations is given to three significant figures, the final result should be also rounded to three significant figures: $\alpha = -188 \text{ rad/s}^2$.

DOUBLE-CHECK: The negative sign of α indicates deceleration, which is appropriate since the centrifuge is coming to a stop. The centrifuge decelerates from 120π rad/s to rest in $t_1 = \frac{2\Delta\theta}{(\omega_1 + \omega_0)} = \frac{2(120\pi \text{ rad})}{(0 + 120\pi \text{ rad/s})} = 2$ s, and since the angular deceleration is constant, it must be the case that the deceleration is $60\pi \text{ rad/s}^2$. The answer is therefore reasonable.

9.44. **THINK:** A circular motion has a constant angular acceleration of $\alpha = 2.5 \text{ rad/s}^2$ and a radius of $r = 1.2$ m. Determine (a) the time required for the angular speed to reach 4.7 rad/s, (b) the number of revolutions to reach this angular speed of 4.7 rad/s, (c) the linear speed when the angular speed is 4.7 rad/s, (d) the linear acceleration when the angular speed is 4.7 rad/s, (e) the magnitude of the centripetal acceleration when the angular speed is 4.7 rad/s and (f) the magnitude of the discus' total acceleration.

SKETCH:



RESEARCH:

(a) Since the angular acceleration is constant, the time required to reach the final angular speed can be determined by means of the kinematic equation, $\omega = \omega_0 + \alpha t$, where $\omega_0 = 0.0$ rad/s.

(b) Once the time required to reach the angular speed, ω, is determined, the number of revolutions can be determined by setting $1 \text{ rev} = 2\pi$ rad, where the number of radians is obtained from $d[\text{rad}] = \frac{1}{2}(\omega + \omega_0)t.$

(c) The linear speed, v, can be determined from the angular speed, ω, by the relation $v = \omega r$.

(d) The linear acceleration, a_t, can be obtained from the angular acceleration, α, by the relation $a_t = \alpha r$.

(e) The magnitude of the centripetal acceleration can be determined from the linear speed by the relation $a_c = \frac{v^2}{r}.$

(f) The total acceleration, a_T, can be found as the hypotenuse of a right angle triangle where the sides are the linear (tangential) acceleration, a_t, and the angular acceleration, α. The relationship is $a_T = \sqrt{a_t^2 + \alpha^2}$.

SIMPLIFY:

(a) $\omega = \omega_0 + \alpha t = \alpha t \Rightarrow t = \frac{\omega}{\alpha}$

(b) $d[\text{rad}] = \frac{1}{2}(\omega + \omega_0)t = \frac{1}{2}(\omega t)$, and to convert to the number of revolutions, $\text{rev} = \frac{d}{(2\pi)}$.

(c) $v = \omega r$

(d) $a_t = \alpha r$

(e) $a_c = \frac{v^2}{r}$

(f) $a_T = \sqrt{a_t^2 + \alpha^2}$

CALCULATE:

(a) $t = \frac{4.70 \text{ rad/s}}{2.50 \text{ rad/s}^2} = 1.88 \text{ s}$

(b) $d[\text{rad}] = \frac{(4.70 \text{ rad/s})(1.88 \text{ s})}{2} = 4.42 \text{ rad} \text{ or } 4.42 \text{ rad}\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right) = 0.70314 \text{ rev}$

(c) $v = (4.70 \text{ rad/s})(1.20 \text{ m}) = 5.64 \text{ m/s}$

(d) $a_t = (2.50 \text{ rad/s}^2)(1.20 \text{ m}) = 3.00 \text{ m/s}^2$

(e) $a_c = \frac{(5.64 \text{ m/s})^2}{1.20 \text{ m}} = 26.5 \text{ m/s}^2$

(f) $a_T = \sqrt{(2.88 \text{ m/s}^2)^2 + (26.5 \text{ m/s}^2)^2} = 26.656 \text{ m/s}^2$

ROUND: Rounding to three significant figures:

(a) $t = 1.88$ s

(b) 0.703 revolutions

(c) $v = 5.64$ m/s

(d) $a_t = 3.00 \text{ m/s}^2$

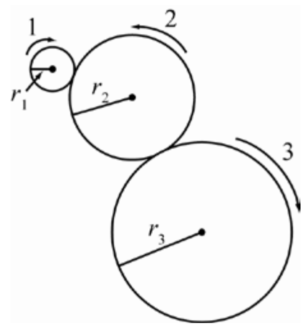
(e) $a_c = 26.5 \text{ m/s}^2$

(f) $a_T = 26.7 \text{ m/s}^2$

DOUBLE-CHECK: Based on the given values, these results are reasonable.

9.45. **THINK:** Three coupled disks have radii $r_1 = 0.100$ m, $r_2 = 0.500$ m, $r_3 = 1.00$ m. The rotation rate of disk 3 is one revolution every 30.0 seconds. Determine (a) the angular speed of disk 3, (b) the tangential velocities of the three disks, (c) the angular speeds of disks 1 and 2 and (d) if the angular acceleration of disk 1 is $\alpha_1 = 0.100 \text{ rad/s}^2$, what are the angular accelerations of disks 2 and 3?

SKETCH:



RESEARCH:

(a) To obtain the angular speed of disk 3, use its rotation rate, $T = 30.0$ s, and the relationship between revolutions and radians, 2π rad/rev . Therefore, $\omega_3 = 2\pi / T$.

(b) Since the three disks are touching each other and there is no slipping, they all have the same tangential speed. Therefore, only one tangential speed must be determined. Since the angular speed of disk 3 is known, the tangential speed can be determined from $v = \omega_3 r_3$.

(c) Calculate the angular speed for disks 1 and 2 from the tangential speeds and the radii. That is, $\omega_1 = v / r_1$, and $\omega_2 = v / r_2$.

(d) Since the angular acceleration of disk 1 is known, its tangential acceleration can be determined. Since the disks are touching each other, and no slipping occurs, this tangential acceleration is common to all disks. The angular acceleration for disks 2 and 3 can be determined from this tangential acceleration and the radii. Therefore, $\alpha_1 = a_1 / r_1$, implies $a_1 = \alpha_1 r_1$. Since $a_1 = a_2 = a_3 = a$, $\alpha_2 = a / r_2$ and $\alpha_3 = a / r_3$.

SIMPLIFY:

(a) $\omega_3 = 2\pi / T$

(b) $v = \omega_3 r_3$

(c) $\omega_1 = v / r_1$, and $\omega_2 = v / r_2$.

(d) $\alpha_2 = a / r_2$ and $\alpha_3 = a / r_3$, where $a = \alpha_1 r_1$.

CALCULATE:

(a) $\omega_3 = \dfrac{(2\pi \text{ rad/rev})}{30.0 \text{ s}} = 0.209 \text{ rad/s}$

(b) $v = (0.209 \text{ rad/s})(1.00 \text{ m}) = 0.209 \text{ m/s}$ for all three disks.

(c) $\omega_1 = \dfrac{0.209 \text{ m/s}}{0.100 \text{ m}} = 2.09 \text{ rad/s}$ and $\omega_2 = \dfrac{0.209 \text{ m/s}}{0.500 \text{ m}} = 0.419 \text{ rad/s}.$

(d) $a = (0.100 \text{ rad/s}^2)(0.100 \text{ m}) = 1.00 \cdot 10^{-2} \text{ m/s}^2.$

Therefore, $\alpha_2 = \dfrac{1.00 \cdot 10^{-2} \text{ m/s}^2}{0.500 \text{ m}} = 2.00 \cdot 10^{-2} \text{rad/s}^2$ and $\alpha_3 = \dfrac{1.00 \cdot 10^{-2} \text{ m/s}^2}{1.00 \text{ m}} = 1.00 \cdot 10^{-2} \text{ rad/s}^2.$

ROUND: Keeping three significant figures, the results are:

(a) $\omega_3 = 0.209 \text{ rad/s}$

(b) $v = 0.209 \text{ m/s}$ for all three disks

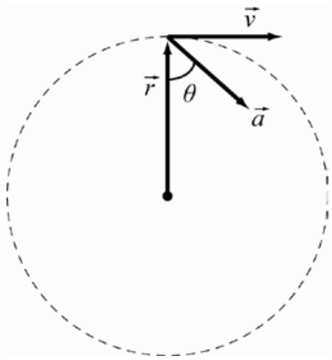
(c) $\omega_1 = 2.09$ rad/s and $\omega_2 = 0.419$ rad/s

(d) $\alpha_2 = 2.00 \cdot 10^{-2}$ rad/s^2 and $\alpha_3 = 1.00 \cdot 10^{-2}$ rad/s^2

DOUBLE-CHECK: Based on the given values, all the results are reasonable.

9.46. **THINK:** Determine the speed of a particle whose acceleration has a magnitude of $a = 25.0$ m/s^2 and makes an angle of $\theta = 50.0°$ with the radial vector.

SKETCH:



RESEARCH: To determine the tangential speed, v, recall that the centripetal acceleration is given by $a_c = \frac{v^2}{r}$. The centripetal acceleration is the projection of the total acceleration on the radial axis, i.e. $a_c = a_T \cos\theta$.

SIMPLIFY: Therefore, the tangential speed is given by $v = \sqrt{a_c r} = \sqrt{a_T r \cos\theta}$.

CALCULATE: $v = \sqrt{(25.0 \text{ m/s}^2)\cos(50.0°)(1.00 \text{ m})} = 4.01$ m/s

ROUND: The values are given to three significant figures, therefore the result is $v = 4.01$ m/s.

DOUBLE-CHECK: This result is reasonable based on the magnitudes of the given values.

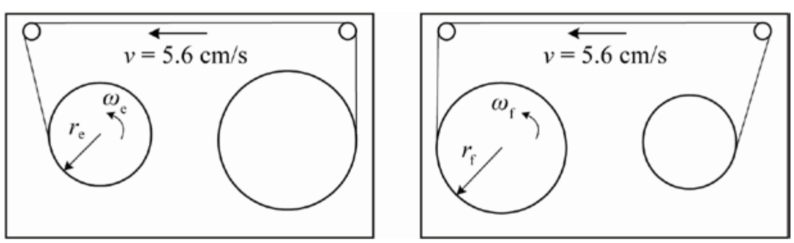
9.47. **THINK:** Determine the angular speed of the take-up spool in a tape recorder in the following cases:

(a) When the take-up spool is empty with radius, $r_e = 0.800$ cm.

(b) When the take-up spool is full with radius, $r_f = 2.20$ cm.

(c) Determine the average angular acceleration of the take-up spool if the length of the tape is $l = 100.80$ m. The magnetic tape has a constant linear speed of $v = 5.60$ cm/s.

SKETCH:



RESEARCH:

(a) & (b) To determine the angular speed, make use of the relationship $v = \omega r \Rightarrow \omega = \frac{v}{r}$.

(c) To determine an average angular acceleration, use the definition, $\alpha = \Delta w / \Delta t$, where the time is determined from $\Delta t = \frac{(\text{distance})}{(\text{speed})} = \frac{l}{v}$.

SIMPLIFY:

(a) $\omega_{\text{e}} = \dfrac{v}{r_{\text{e}}}$

(b) $\omega_{\text{f}} = \dfrac{v}{r_{\text{f}}}$

(c) $\alpha = \dfrac{\Delta\omega}{\Delta T} = \dfrac{\omega_{\text{f}} - \omega_{\text{e}}}{l/v} = \dfrac{v\left(\omega_{\text{f}} - \omega_{\text{e}}\right)}{l}$

CALCULATE:

(a) $\omega_e = \dfrac{5.60 \cdot 10^{-2} \text{ m/s}}{8.00 \cdot 10^{-3} \text{ m}} = 7.00 \text{ rad/s}$

(b) $\omega_{\text{f}} = \dfrac{5.60 \cdot 10^{-2} \text{ m/s}}{2.20 \cdot 10^{-2} \text{ m}} = 2.54 \text{ rad/s}$

(c) $\alpha = \dfrac{\left(5.60 \cdot 10^{-2} \text{ m/s}\right)\left(2.54 - 7.00\right)}{100.80 \text{ m}} = -2.48 \cdot 10^{-3} \text{ rad/s}^2$

ROUND: Keep three significant figures:

(a) $\omega_e = 7.00$ rad/s

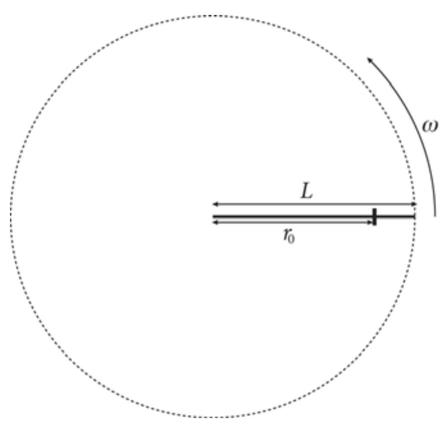
(b) $\omega_{\text{f}} = 2.54$ rad/s

(c) $\alpha = -2.48 \cdot 10^{-3}$ rad/s^2

DOUBLE-CHECK: It is reasonable that the angular speed of the spool when it's empty is greater than when it's full. Also, it is expected that the angular acceleration is negative since the angular speed is decreasing as the spool gets full.

9.48. **THINK:** Determine the radial velocity of a ring fitted around a rod as it reaches the end of the rod. The rod is spun in a horizontal circle at a constant angular velocity. The given values are the length of the rod, l = 0.50 m, the initial distance of the ring from the fixed end of the rod, $r_0 = 0.30$ m, and the constant angular velocity, $\omega = 4.0$ rad/s.

SKETCH:



RESEARCH: For the ring to move in a circular path at a fixed distance along the rod, it would require a centripetal acceleration of $a_{\text{c}} = \omega^2 r$ directed toward the center of the path. However, there is no force on the ring that will supply this acceleration, thus the inertia of the ring will tend to pull it outward along the rod. The resulting radial acceleration is equal to the missing centripetal acceleration, $a_{\text{c}} = \omega^2 r.$ Since this radial acceleration depends on the radial position, the differential kinematic relations must be used:

$$\frac{dv_{\text{r}}}{dt} = \omega^2 r \;\Rightarrow\; \left(\frac{dv_{\text{r}}}{dr}\right)\left(\frac{dr}{dt}\right) = \omega^2 r,$$

where the second equation follows from using the chain rule of calculus.

SIMPLIFY: Since $\frac{dr}{dt} = v_r$, use separation of variables to set up the integral:

$$v_r dv_r = \omega^2 r dr \Rightarrow \int_0^v v_r' dv_r' = \omega^2 \int_{r_0}^l r' dr'$$

$$\left.\frac{(v_r')^2}{2}\right|_0^{v_r} = \omega^2 \left.\frac{(r')^2}{2}\right|_{r_0}^{l}$$

$$\frac{v_r^2}{2} = \frac{\omega^2 (l^2 - r_0^2)}{2} \rightarrow v_r = \omega\sqrt{l^2 - r_0^2}.$$

CALCULATE: The speed is therefore, $v_r = (4.00 \text{ rad/s})\sqrt{(0.500 \text{ m})^2 - (0.300 \text{ m})^2} = 1.60 \text{ m/s}.$

ROUND: Since the angular velocity is given to three significant figures, the result remains $v_r = 1.60$ m/s.

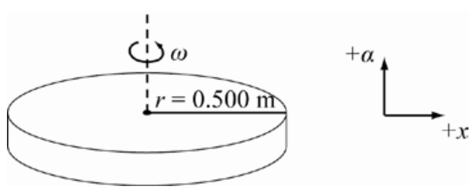
DOUBLE-CHECK: Based on the given values, the resulting radial velocity is reasonable.

9.49. **THINK:** A flywheel with a diameter of 1.00 m is initially at rest, and has an angular acceleration in terms of time as $\alpha(t) = 0.1t^2$, and has units of rad/s^2. Determine:

(a) The angular separation between the initial and final positions of a point on the rim 8.00 seconds after the rotations begin.

(b) Find the linear position, velocity, and acceleration of a point 8.00 seconds after the wheel starts rotating, where the starting position of the point is at $\theta = 0$. Use the known equations relating the position and velocity to the acceleration.

SKETCH:



RESEARCH:

(a) The angular separation can be determined by first considering the change in angular speed through the time period:

$$\Delta\omega = \int_i^f \alpha(t)\, dt.$$

Since the initial angular speed is zero, $\Delta\omega = \omega.$ Then consider the change in the angle through the time period:

$$\Delta\theta = \int_i^f \omega\, dt.$$

(b) The angular acceleration and angular velocity are known and can be related to the tangential component of the linear acceleration and to the velocity through the equations $a_t = \alpha(t)r$ and $v = \omega r.$ The radial component of the acceleration vector is the centripetal acceleration, $a_r = v^2/r.$ The position will be on the circumference, given by $\vec{r} = r(\cos\theta)\hat{x} + r(\sin\theta)\hat{y}$ where the angle is known from (a). Note that in this case, the question indicates that $v_0 = 0$ and $\theta_0 = 0.$ By convention, θ is measured counterclockwise from the positive x-axis.

SIMPLIFY:

(a) $\Delta\omega = \int_i^f \alpha\, dt$ and $\Delta\theta = \int_i^f \omega\, dt.$

(b) $a = \alpha(t)r$, $v = \omega(t)r,$ $\vec{r} = r(\cos\theta)\hat{x} + r(\sin\theta)\hat{y}$ do not need simplifying.

CALCULATE:

(a) $\Delta\omega = \int_0^t (0.1)t^2\ dt = \dfrac{0.1t^3}{3},\ \Delta\theta = \int_{\mathrm{i}}^{\mathrm{f}} \omega\ dt = \int_0^8 \dfrac{0.1t^3}{3}\ dt = \left.\dfrac{0.1t^4}{12}\right|_0^8 = 34.13333 \text{ rad} = 5.43249 \text{ rev}$

This is 5 complete revolutions plus an additional 0.43249 of a revolution. Therefore the angular separation is given by $(0.43249)(2\pi) = 2.717$ rad.

(b) For the linear velocity, $v = \dfrac{0.1t^3 r}{3} = \dfrac{(0.1)(8.00 \text{ s})^3(0.500 \text{ m})}{3} = 8.53333$ m/s. The linear position,

$\vec{r} = (0.500 \text{ m})\left[\cos(2.717 \text{ rad})\right]\hat{x} + (0.500 \text{ m})\left[\sin(2.717 \text{ rad})\right]\hat{y} = -(0.4556 \text{ m})\hat{x} + (0.20597 \text{ m})\hat{y}$

For the tangential acceleration, $a_t = 0.1t^2 r = (0.1)(8.00 \text{ s})^2(0.500 \text{ m}) = 3.200 \text{ m/s}^2$.

For the radial acceleration: $a_r = (8.53333 \text{ m/s})^2 / (0.500 \text{ m}) = 145.635 \text{ m/s}^2$.

Therefore the magnitude of the total acceleration is dominated by the centripetal acceleration:

$a = \sqrt{a_r^2 + a_t^2} = \sqrt{145.635^2 + 3.2^2} \text{ m/s}^2 = 145.671 \text{ m/s}^2$.

ROUND: The constant 0.1 in the function for α is treated as precise.

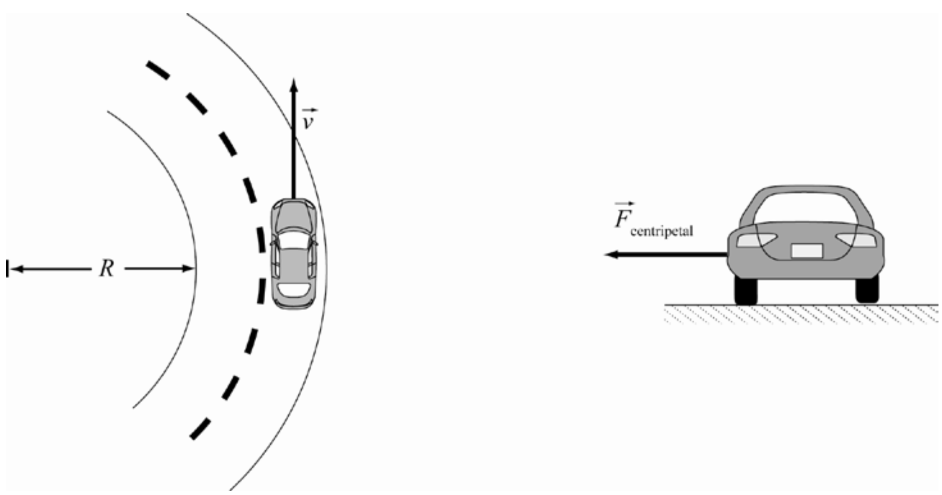
(a) Since all values are given to three significant figures, the result is $\Delta\theta = 2.72$ rad.

(b) To three significant figures, the results are $a_t = 3.20 \text{ m/s}^2$, $a_r = 146 \text{ m/s}^2$, and $v = 8.53$ m/s. The position of the point is $-0.456\hat{x} + 0.206\hat{y}$ (or 0.500 m from the center at an angle of $+2.72$ rad from its initial position).

DOUBLE-CHECK: Based on the given values, these results are reasonable. The magnitude of the linear position vector is $|\vec{r}| = \sqrt{(-0.4556 \text{ m})^2 + (0.20597 \text{ m})^2} = 0.500$ m, which is consistent with the requirement that the point is at the edge of the wheel.

9.50. **THINK:** Determine the force that plays the role of and has the value of the centripetal force on a vehicle of mass m = 1500. kg, with speed v = 15.0 m/s around a curve of radius R = 400. m.

SKETCH:



RESEARCH: The force that keeps the vehicle from slipping out of the curve is the force of static friction. The force can be calculated by recalling the form of the centripetal force,

$$F_{\mathrm{c}} = m\frac{v^2}{R}.$$

SIMPLIFY: The equation is in its simplest form.

CALCULATE: $F_c = (1500.\text{ kg})\dfrac{(15.0\text{ m})^2}{400.\text{ m}} = 843.75\text{ N}$

ROUND: To three significant figures: $F_c = 844\text{ N}$

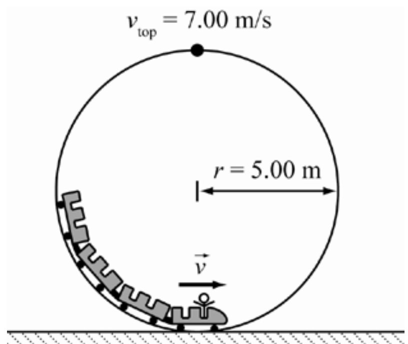
DOUBLE-CHECK: The coefficient of static friction can be determined from the equation $F_c = \mu_s mg$:

$$\Rightarrow \mu_s = \frac{F_c}{mg} = \frac{800\text{ N}}{1500\text{ kg}(9.81\text{ m/s}^2)} = 0.05.$$

This is within the expected values before slipping occurs. Therefore this is a reasonable force to obtain for the centripetal force.

9.51. **THINK:** The apparent weight of a rider on a roller coaster at the bottom of the loop is to be determined. From Solved Problem 9.1, the radius is r = 5.00 m, and the speed at the top of the loop is 7.00 m/s.

SKETCH:



RESEARCH: The apparent weight is the normal force from the seat acting on the rider. At the bottom of the loop the normal force is the force of gravity plus the centripetal force:

$$N = F_g + \frac{mv^2}{r} = mg + \frac{mv^2}{r}.$$

The velocity at the bottom of the loop can be determined by considering energy conservation between the configuration at the top and that at the bottom:

$$\frac{1}{2}mv^2 = mgh + \frac{1}{2}mv_t^2$$

where $h = 2r$. In Solved Problem 9.1 it as determined that the feeling of weightlessness at the top is achieved if $\dfrac{mv_t^2}{r} = mg$.

SIMPLIFY: Multiply the equation for energy conservation by a factor of $2/r$ and find:

$$\frac{mv^2}{r} = \frac{2mgh}{r} + \frac{mv_t^2}{r}.$$

Since $h = 2r$, this results in:

$$\frac{mv^2}{r} = 4mg + \frac{mv_t^2}{r}.$$

Insert this for the normal force and see

$$N = mg + \frac{mv^2}{r} = mg + 4mg + \frac{mv_t^2}{r} = mg + 4mg + mg = 6mg.$$

CALCULATE: Not needed.

ROUND: Not needed.

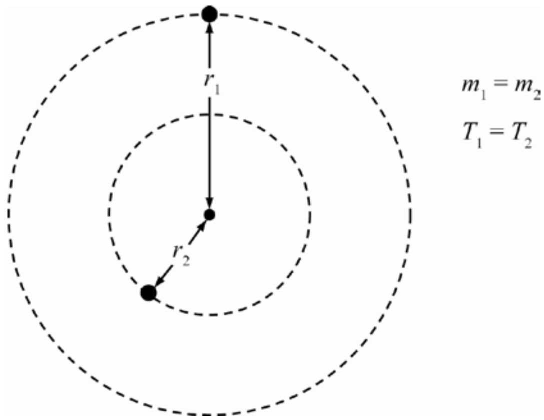
DOUBLE-CHECK: Our result means that you experience 6g of acceleration at the bottom of the loop, which seems like a large number, if you consider that the maximum acceleration during the launch of a Space Shuttle is kept to 3g. However, if you have ever had the opportunity to ride on such a roller coaster, then our result does not seem unreasonable.

9.52. **THINK:** Two skaters have equal masses and periods of rotation but the radius of one is half of the other. Determine:

(a) The ratio of their speeds.

(b) The ratio of the magnitudes of the forces on each skater.

SKETCH:



RESEARCH:

(a) The ratio of the speeds, v_1 / v_2, can be determined by considering the period of rotation, given by $T = 2\pi r / v$. Since the two skaters have the same period, $T = 2\pi r_1 / v_1 = 2\pi r_2 / v_2$.

(b) The force acting on each skater has only a centripetal component whose magnitude is mv^2 / r. Therefore the ratio of the magnitudes is simply $\frac{F_2}{F_1}$, and $\frac{F_2}{F_1} = \frac{m_2\left(v_2^2 / r_2\right)}{m_2\left(v_1^2 / r_1\right)}$.

SIMPLIFY:

(a) $\frac{2\pi r_1}{v_1} = \frac{2\pi r_2}{v_2} \Rightarrow \frac{r_1}{v_1} = \frac{r_2}{v_2} \Rightarrow \frac{r_1}{r_2} = \frac{v_1}{v_2}$

(b) $\frac{F_2}{F_1} = \frac{m_2\left(v_2^2 / r_2\right)}{m_2\left(v_1^2 / r_1\right)} = \frac{\left(v_2^2 / r_2\right)}{\left(v_1^2 / r_1\right)} = \frac{\left(r_2 / T_2^2\right)}{\left(r_1 / T_1^2\right)} = \frac{r_2}{r_1}$

CALCULATE:

(a) Since $r_2 = \frac{r_1}{2}$, $\frac{r_2}{r_1} = \frac{v_2}{v_1} = \frac{1}{2}$.

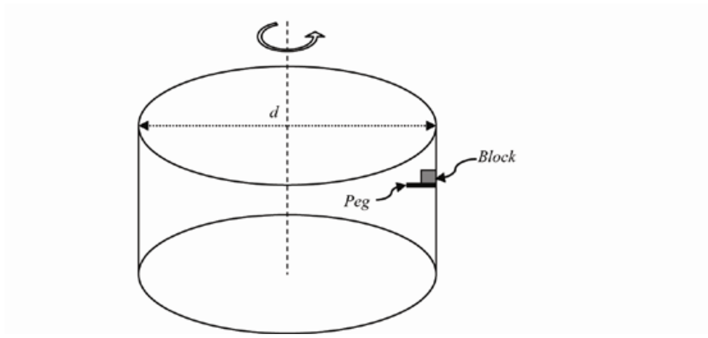
(b) $\frac{F_2}{F_1} = \frac{r_2}{r_1} = \frac{v_2}{v_1} = \frac{1}{2}$

ROUND: It is not necessary to round. The result for both parts (a) and (b) is a ratio of 1/2.

DOUBLE-CHECK: It is reasonable that by doubling the radius, both the speed of rotation and centripetal force also double.

9.53. **THINK:** Determine the minimum time required for a block held by a peg inside a cylinder to stay in place once the cylinder starts rotating with angular acceleration, α. The coefficient of static friction is given as μ. To avoid slipping in the vertical direction, balance the force due to gravity with the force due to friction between the block and the cylinder. For large values of the angular acceleration, we also obtain a significant force in tangential direction. However, we restrict our considerations to the case of small angular acceleration and neglect the tangential force.

SKETCH:



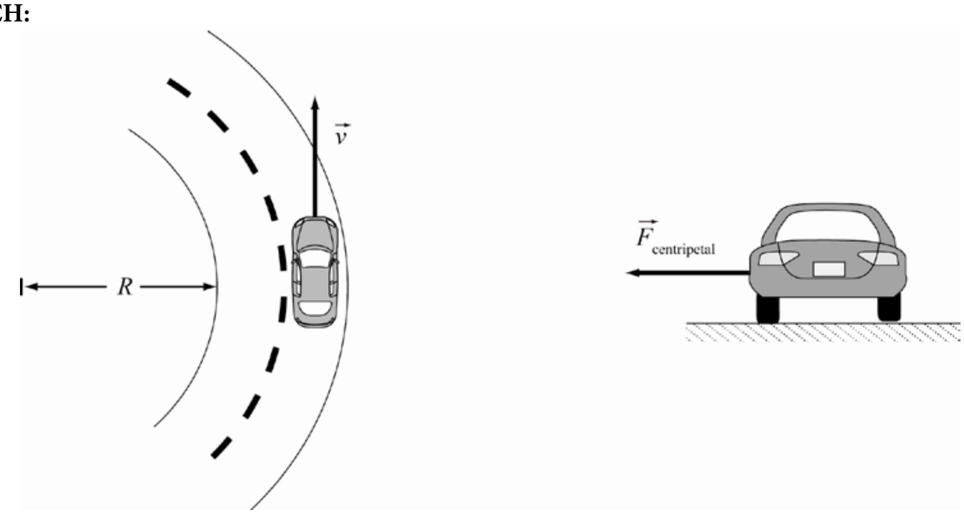
RESEARCH: The force due to friction is given by $f = \mu N,$ and in this case N is simply the centripetal force, $F_c = mv^2/(d/2).$ The time required to reach a suitable centripetal force can be determined by means of the angular speed, $\omega = v/r$, and the angular acceleration, $\alpha = \omega/t$.

SIMPLIFY: The centripetal force can be rewritten as $F_c = mv^2/r = m(\omega r)^2/(d/2) = m\omega^2 d/2.$ Thus, the force of static friction is given by $f_f = \mu m(d/2)\omega^2.$ Therefore, from the balancing of the vertical forces: $f = F_g,$ or $\mu m\omega^2 d/2 = mg \Rightarrow \omega^2 = 2g/(\mu d).$ Since $t = \omega/\alpha$, the time interval is:

$$t = \sqrt{2g/\mu d}/\alpha = \sqrt{\frac{2g}{\mu d\alpha^2}}.$$

CALCULATE: There are no numbers to insert in this problem.
ROUND: There is nothing to round since there are no numerical values.
DOUBLE-CHECK: An easy check we can perform right away is to make sure that the units on the right-hand side of our formula indeed work out to be seconds.

9.54. **THINK:** The maximum velocity such that the car performs uniform circular motion without slipping must be determined. The coefficient of static friction is $\mu_s = 1.20$ and the radius of the circular path is r = 10.0 m.
SKETCH:



RESEARCH: Consider which force is providing the centripetal force. Since the car is not sliding, it is the force of static friction. Those two forces must be related to determine the maximum velocity. That is,

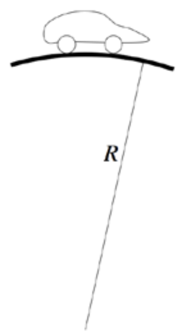
$$F_{\text{friction}} = F_{\text{centripetal}} \Rightarrow \mu_s mg = \frac{mv^2}{r}.$$

SIMPLIFY: $\mu_s mg = \dfrac{mv^2}{r} \Rightarrow v = \sqrt{\mu_s gr}$

CALCULATE: $v = \sqrt{\mu_s gr} = \sqrt{(1.20)(9.81 \text{ m/s}^2)(10.0 \text{ m})} = 10.84988 \text{ m/s}$

ROUND: Since the values given have three significant figures, the result is then $v = 10.8$ m/s.
DOUBLE-CHECK: This result may seem quite small for a racecar. But, consider that 10.8 m/s are ~24 mph, and that this is a very tight curve with a diameter of less than the length of a basketball court. It then seems reasonable that a car cannot go very fast through such a tight curve. Also, note that as expected, the maximum velocity is independent of the mass of the car.

9.55. **THINK:** Determine the maximum speed of a car as it goes over the top of a hill such that the car always touches the ground. The radius of curvature of the hill is 9.00 m. As the car travels over the top of the hill it undergoes circular motion in the vertical plane. The only force that can provide the centripetal force for this motion is gravity. Clearly, for small speeds the car remains in contact with the road due to gravity. But the car will lose contact if the centripetal acceleration exceeds gravity.
SKETCH:



RESEARCH: In the limiting case of the maximum speed we can set the centripetal acceleration equal to g:
$g = v_{max}^2 / r$.

SIMPLIFY: Solve for the maximum speed and find $v_{max} = \sqrt{gr}$.

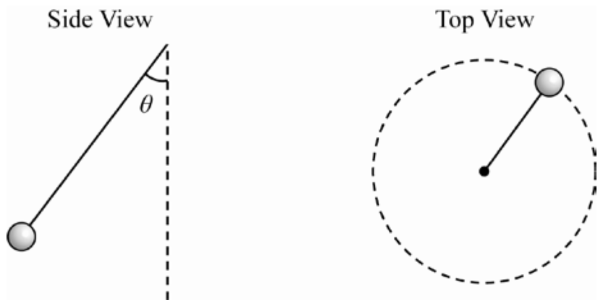
CALCULATE: $v_{max} = \sqrt{gr} = \sqrt{(9.81 \text{ m/s}^2)(9.00 \text{ m})} = 9.40 \text{ m/s}$

ROUND: Since the radius is given to three significant figures, the result is $v_{max} = 9.40$ m/s.

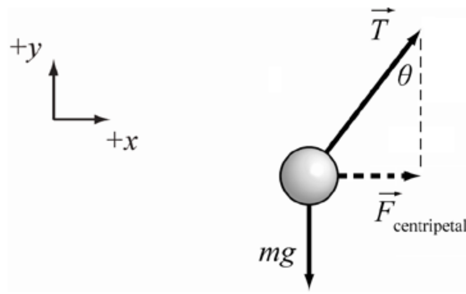
DOUBLE-CHECK: This speed of 9.40 m/s, which is approximately 21.0 mph, seems very small. But on the other hand, this is a very significant curvature at the top of the hill, equivalent to a good-sized speed bump. Going over this type of bump at more than 21 mph makes it likely that your car will lose contact with the road surface.

9.56. **THINK:** A ball attached to a string is in circular motion as described by the sketch. Determine:
(a) The free-body diagram for the ball.
(b) The force acting as the centripetal force.
(c) The required speed of the ball such that $\theta = 45.0°$.
(d) The tension on the string.

SKETCH:



RESEARCH:
(a)



(b) As shown in the sketch, the projection of the tension onto the horizontal plane provides the centripetal force. Therefore, $mv^2/r = T\sin\theta.$
(c) From the sketch, the force due to gravity is balanced by the projection of the tension on the vertical axis, i.e. $mg = T\cos\theta.$ From part (b), the centripetal force is given by $mv^2/r = T\sin\theta.$ By solving both equations for T and then equating them, the speed for the given angle can be determined.
(d) The tension on the string can most easily be found from $mg = T\cos\theta,$ for the given angle, $\theta.$

SIMPLIFY:
(a) Not applicable.
(b) Not applicable.
(c) $mg = T\cos\theta \Rightarrow T = \dfrac{mg}{\cos\theta}$, and $\dfrac{mv^2}{r} = T\sin\theta \Rightarrow T = \dfrac{mv^2}{r\sin\theta}.$

Equating the above equations gives $\dfrac{mg}{\cos\theta} = \dfrac{mv^2}{r\sin\theta} \Rightarrow v = \sqrt{gr\tan\theta},$ where $r = L\sin\theta.$

(d) $mg = T\cos\theta \Rightarrow T = \dfrac{mg}{\cos\theta}$

CALCULATE:
(a) Not applicable.
(b) Not applicable.
(c) $v = \sqrt{\left(9.81 \text{ m/s}^2\right)\left(\left(1.00 \text{ m}\right)\sin\left(45.0°\right)\right)\tan 45.0°} = 2.63376 \text{ m/s}$

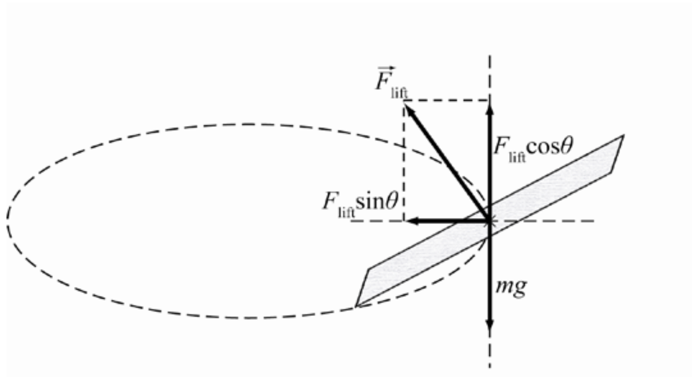
(d) $T = \dfrac{\left(0.200 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right)}{\cos\left(45.0°\right)} = 2.7747 \text{ N}$

ROUND:
(a) Not applicable.
(b) Not applicable.
(c) Since values are given to three significant figures, the result is $v = 2.63 \text{ m/s}.$
(d) Keeping three significant figures, $T = 2.77 \text{ N}.$

DOUBLE-CHECK: All results are reasonable based on the given values. It is expected that the tension on the string will be greater than the weight of the ball.

9.57. **THINK:** Determine the banking angle for a plane performing uniform circular motion. The radius is 7.00 miles ($1.12654\cdot 10^4$ meters), the speed is 360. mph (160.93 m/s), the height is $2.00\cdot 10^4$ ft (6096 meters) and the plane length is 275 ft (83.82 meters).

SKETCH:



RESEARCH: Suppose *F* is the lift force, which makes an angle, θ, with the vertical as shown in the sketch. Also, suppose the weight of the plane is *mg*. Now, $F\cos\theta$ balances the weight of the plane when the plane is banked with the horizontal and $F\sin\theta$ provides the necessary centripetal force for the circular motion. Therefore,

$$F\sin\theta = \frac{mv^2}{r} \text{ and } F\cos\theta = mg.$$

SIMPLIFY: Dividing the two equations gives $\tan\theta = \dfrac{v^2}{rg} \Rightarrow \theta = \tan^{-1}\left(\dfrac{v^2}{rg}\right)$.

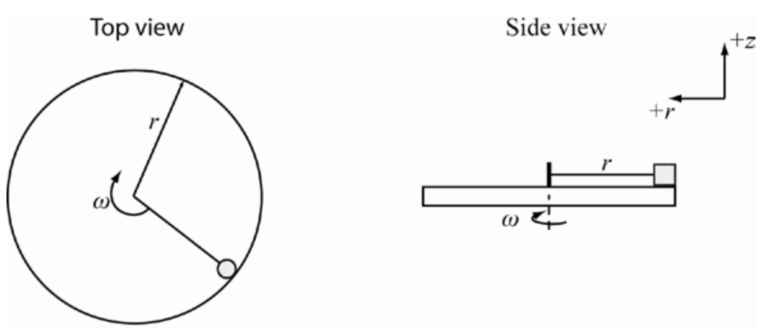
CALCULATE: $\theta = \tan^{-1}\left(\dfrac{(160.93 \text{ m/s})^2}{(1.12654\cdot 10^4 \text{ m})(9.81 \text{ m/s}^2)}\right) = 13.189°$

ROUND: Rounding to three significant figures, the result is an angle of approximately $13.2°$.

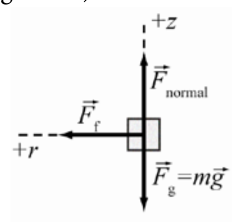
DOUBLE-CHECK: Based on the given values, the result is reasonable.

9.58. **THINK:** Determine the tension on the string attaching a cylinder (*m* = 20.0 g) to the center of a turntable as the angular velocity increases up to 60.0 rpm. The coefficient of static friction is $\mu_s = 0.800$ and the distance between the center of the turntable and the cylinder of *l* = 80.0 cm.

SKETCH:



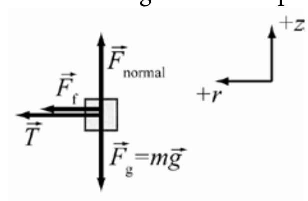
RESEARCH: As the turntable speeds up from the rest, the static friction force provides the centripetal force and no tension is built into the string for a while. The corresponding free body diagram for the cylinder under these conditions is presented. (Since the turntable speeds up very slowly, the tangential static friction force that acts on the cylinder from the turntable and keeps it moving with the turntable is important physically, but negligible in magnitude).



At a critical value, ω_1, of the angular velocity, the static friction force reaches its maximum value, so $F_{\text{f}} = \mu_{\text{s}} mg$ becomes

$$\frac{mv^2}{r} = m\omega_1^2 r = \mu_{\text{s}} mg \Rightarrow \omega_1 = \sqrt{\frac{\mu_{\text{s}} g}{r}}.$$

Once the angular velocity exceeds ω_1, static friction alone is not enough to provide the required centripetal force, and a tension is built into the string. The corresponding free body diagram is presented.



SIMPLIFY: The tension in the string when the angular velocity of the turntable is $\omega_2 = 60.0 \text{ rpm}\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ sec}}\right) = 2.00\pi \text{ rad/s}$ is calculated from the centripetal force at this velocity, $F_{\text{c2}} = m\omega_2^2 r,$ and the tension is given by $T = F_{\text{c2}} - F_{\text{f}} = m\omega_2^2 r - \mu_{\text{s}} mg = m\left(\omega_2^2 r - \mu_{\text{s}} g\right).$

CALCULATE: $T = (0.0200 \text{ kg})\left[(2.00\pi \text{ rad/s})^2 (0.800 \text{ m}) - (0.800)(9.81 \text{ m/s}^2)\right] = 0.475 \text{ N}$

ROUND: To three significant figures, $T = 0.475$ N.

DOUBLE-CHECK: This is a reasonable tension for the small system described.

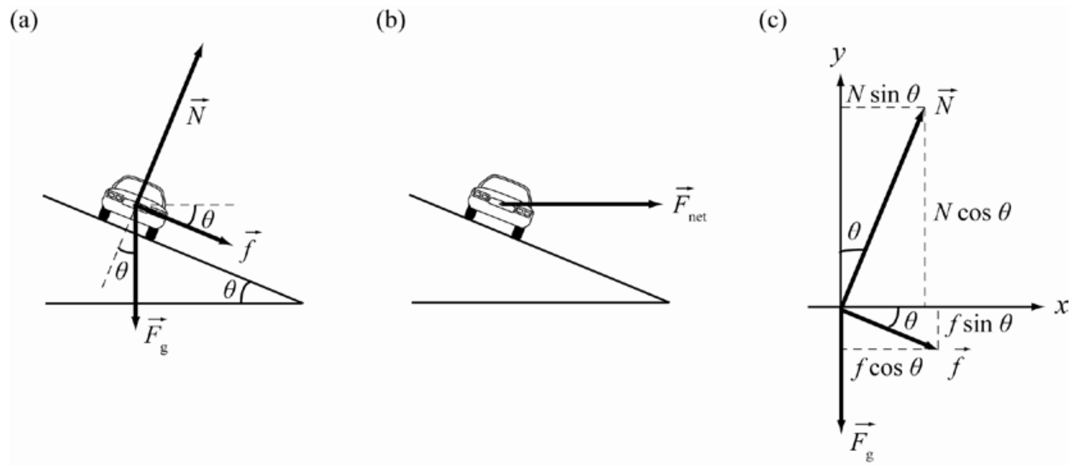
9.59. **THINK:** A speedway turn has a radius, R, and is banked at an angle of θ above the horizontal. This problem is a special case of Solved Problem 9.4, and the results of that solved problem will be used to obtain a solution to this problem. Determine:

(a) The optimal speed to take the turn when there is little friction present.

(b) The maximum and minimum speeds at which to take the turn if there is now a coefficient of static friction, μ_{s}.

(c) The value for parts (a) and (b) if $R = 400.$ m, $\theta = 45.0°$, and $\mu_s = 0.700$.

SKETCH:



RESEARCH: It was found in Solved Problem 9.4 that the maximum speed a car can go through the banked curve is given by

$$v_{\text{max}} = \sqrt{\frac{Rg(\sin\theta + \mu_s \cos\theta)}{\cos\theta - \mu_s \sin\theta}}.$$

SIMPLIFY:

(a) For the case of zero friction the case above approaches the limit of $v_{\text{zero friction}} = \sqrt{\dfrac{Rg\sin\theta}{\cos\theta}} = \sqrt{Rg\tan\theta}$.

(b) For the maximum speed we can use the formula already quoted above. The minimum speed that the car can travel through the curve is given by reversing the direction of the friction force. In this case the friction force points up the bank, because it needs to prevent the car from sliding down. Reversing the sign of the friction force leads to $v_{\text{min}} = \sqrt{\dfrac{Rg(\sin\theta - \mu_s \cos\theta)}{\cos\theta + \mu_s \sin\theta}}$.

CALCULATE:

(c) For the results from part (a):

$$v_{\text{zero friction}} = \sqrt{(400.\text{ m})(9.81\text{ m/s}^2)\tan 45.0^\circ} = 62.64184\text{ m/s}.$$

For the results from part (b), the minimum speed is:

$$v_{\text{min}} = \sqrt{\frac{(400.\text{ m})(9.81\text{ m/s}^2)(\sin 45.0^\circ - 0.700\cos 45.0^\circ)}{\cos 45.0^\circ + 0.700\sin 45.0^\circ}} = 26.31484\text{ m/s}.$$

and the maximum speed is:

$$v_{\text{max}} = \sqrt{\frac{(400.\text{ m})(9.81\text{ m/s}^2)(\sin 45.0^\circ + 0.700\cos 45.0^\circ)}{\cos 45.0^\circ - 0.700\sin 45.0^\circ}} = 149.1174\text{ m/s}.$$

ROUND:

(a) Not applicable.

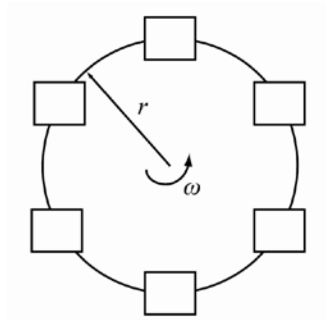
(b) Not applicable.

(c) $v_{\text{zero friction}} = 62.6\text{ m/s}$, $v_{\text{min}} = 26.3\text{ m/s}$ and $v_{\text{max}} = 149\text{ m/s}$.

DOUBLE-CHECK: The results are reasonable considering that the friction-free speed should be within the minimum and maximum speed. The values for the given parameters are consistent with experiment.

9.60. **THINK:** A Ferris wheel has a radius of 9.00 m, and a period of revolution of T = 12.0 s. Let's start with part (a) and solve it all the way.

SKETCH:



RESEARCH: The constant speed of the riders can be determined by the equation for the speed, v = distance/time, where the distance is calculated from the circumference of the path.

SIMPLIFY: $v = \dfrac{d}{T} = \dfrac{2\pi r}{T}$

CALCULATE: $v = \dfrac{2\pi(9.00\text{ m})}{12.0\text{ s}} = 4.712\text{ m/s}$

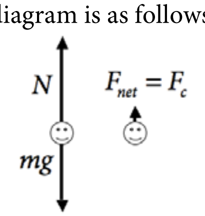
ROUND: Since the input values are given to two significant figures, the result for the linear speed is: $v = 4.7\text{ m/s}$.

DOUBLE-CHECK: For part (b) and part (c), realize that there is an essential difference between a Ferris wheel and a loop in a roller coaster: the speed of the Ferris wheel is gentle enough so that the riders do not get lifted out of their seats at the top. However, we need to check that the speed is actually sufficiently small so that this does not happen. In Solved Problem 9.1 we found that the minimum speed to experience weightlessness (i.e. zero normal force from the seat) at the top of the loop is $v_{N=0} = \sqrt{Rg}$. For the given value of R this speed works out to 9.4 m/s. Since our result is below this value, it is at least possible that a Ferris wheel could exist, which uses the values given here. Note that the centripetal acceleration from the speed use here is:

$$a_c = \frac{v^2}{R} = \frac{(4.71\text{ m/s})^2}{9\text{ m}} = 2.47\text{ m/s}^2 = 0.25g.$$

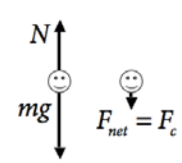
With the above information from our double-check we can solve parts (b) and (c):

(b) At the bottom of the ride the normal force has to balance gravity and in addition provide the centripetal force of 0.25 *mg*. The free-body diagram is as follows:



The normal force at the bottom of the path is thus: $N = mg + 0.25mg = 1.25mg$.

(c) At the top of the Ferris wheel gravity points in the direction of the centripetal force. The free-body diagram at the top is therefore:

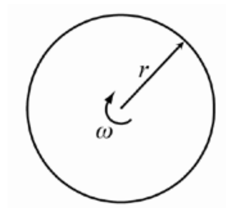


The normal force is in this case: $N = mg - 0.25mg = 0.75mg$.

Note the essential difference in parts (b) and (c): in part (b) the magnitude of the vector for the normal force is greater than that of the gravitational force, and in part (c) it is smaller.

9.61. **THINK:** The radius of the Ferris wheel is r = 9.00 m and its period is T= 12.0 s. Use these values to calculate ω. $\Delta\omega$ and $\Delta\theta$ are known when stopping at a uniform rate, which is sufficient to determine α. Also, the time it takes to stop, Δt, can be determined and with this, the tangential acceleration, a_t, can be determined.

SKETCH:



RESEARCH:

(a) $\omega = \dfrac{2\pi \text{ rad}}{T}$

(b) $\Delta\theta = \omega_i \Delta t + \dfrac{1}{2}\alpha\Delta t^2$, $\Delta\omega = \alpha\Delta t$

(c) $a_t = r\alpha$

SIMPLIFY:

(a) $\omega = \dfrac{2\pi}{T}$

(b) $\Delta\theta = \omega_i \Delta t + \dfrac{1}{2}\alpha\Delta t^2$ and $\Delta t = \dfrac{\Delta\omega}{\alpha} = \dfrac{\omega_f - \omega_i}{\alpha} = \dfrac{-\omega_i}{\alpha}$, since $\omega_f = 0$.

$$\Rightarrow \Delta\theta = \omega_i\left(\frac{-\omega_i}{\alpha}\right) + \frac{1}{2}\alpha\left(\frac{\omega_i^2}{\alpha^2}\right) = -\frac{\omega_i^2}{\alpha} + \frac{\omega_i^2}{2\alpha} = -\frac{\omega_i^2}{2\alpha} \Rightarrow \alpha = \frac{-\omega_i^2}{2\Delta\theta}$$

(c) $a_t = r\alpha$

CALCULATE:

(a) $\omega = \dfrac{2\pi \text{ rad}}{12.0 \text{ s}} = 0.5236 \text{ rad/s}$

(b) $\alpha = \dfrac{-(0.5236 \text{ rad/s})^2}{2(\pi/2) \text{ rad}} = -0.08727 \text{ rad/s}^2$

(c) $a_t = (-0.08727 \text{ rad/s}^2)(9.00 \text{ m}) = -0.785 \text{ m/s}^2$

ROUND: The given values have three significant figures, so the results should be rounded accordingly.

(a) $\omega = 0.524$ rad/s

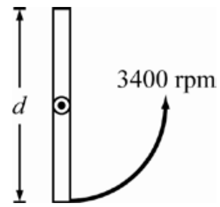
(b) $\alpha = -0.0873 \text{ rad/s}^2$

(c) $a_t = -0.785 \text{ m/s}^2$

DOUBLE-CHECK: These numbers are reasonable for a Ferris wheel. Note that the radius is only required for part (c). As expected, the value for the tangential acceleration is small compared to the gravitational acceleration g.

9.62. **THINK:** Determine the linear speed, given the blade's rotation speed and its diameter. To help determine the constant (negative) acceleration, it is given that it takes a time interval of 3.00 s for the blade to stop. The known values are ω = 3400. rpm, d = 53.0 cm.

SKETCH:



RESEARCH: $1\text{ rpm} = \dfrac{2\pi \text{ rad}}{60 \text{ s}}$

(a) $v = \omega r = \dfrac{1}{2}\omega d$

(b) $\alpha = \dfrac{\Delta w}{\Delta t} = \dfrac{w_f - w_i}{\Delta t}$

SIMPLIFY: It is not necessary to simplify.

CALCULATE:

(a) $v = \left(3400.\left(\dfrac{2\pi}{60}\right)\right)\text{ rad/s}\left(\dfrac{0.530 \text{ m}}{2}\right) = 94.35 \text{ m/s}$

(b) $\omega_f = 0,\ \omega_i = 3400.\left(\dfrac{2\pi}{60}\text{ rad/s}\right) = 356 \text{ rad/s}$ and $\Delta t = 3$ s, so $\alpha = \dfrac{-356 \text{ rad/s}}{3.00 \text{ s}} = -118.7 \text{ rad/s}^2.$

ROUND: The results should be rounded to three significant figures.

(a) $v = 94.3$ m/s

(b) $\alpha = -119 \text{ rad/s}^2$

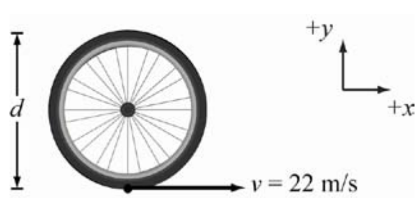
DOUBLE-CHECK: For lawn mower blades, these are reasonable values.

9.63. **THINK:**

(a) If the distance traveled can be determined, then the number of revolutions the tires made can be determined, since the diameter of the tires is known.

(b) The linear speed of the tires and the diameter of the tires are known, so the angular speed can be determined. The known variables are $v_i = 0,\ v_f = 22.0 \text{ m/s},\ \Delta t = 9.00 \text{ s},\ d = 58.0 \text{ cm}.$ Use $1\ \dfrac{\text{rev}}{\text{s}} = 2\pi\ \dfrac{\text{rad}}{\text{s}} \Rightarrow 1\ \dfrac{\text{rad}}{\text{s}} = \dfrac{1}{2\pi}\dfrac{\text{rev}}{\text{s}}.$

SKETCH:



RESEARCH: The circumference of a circle is given by $C = 2\pi r = \pi d.$ The displacement at constant acceleration is $\Delta x = v_i \Delta t + \dfrac{1}{2}a\Delta t^2$, where $v = \omega r.$

SIMPLIFY:

(a) $v_i = 0 \Rightarrow \Delta x = \dfrac{1}{2}a\Delta t^2,\ a = \dfrac{\Delta v}{\Delta t} \Rightarrow \Delta x = \dfrac{1}{2}\dfrac{\Delta v}{\Delta t}\Delta t^2 = \dfrac{1}{2}\Delta v \Delta t$

Let N = number of revolutions and the displacement is given by $\Delta x = \left(\dfrac{\text{displacement}}{\text{revolution}}\right)N.$ The displacement per revolution is simply the circumference, C, so

$$\Delta x = CN \Rightarrow N = \frac{\Delta x}{C} = \frac{1}{\pi d}\left(\frac{1}{2}\Delta v \Delta t\right) = \frac{\Delta v \Delta t}{2\pi d}.$$

(b) $\omega = \frac{v}{r} = \frac{v}{d/2} = \frac{2v}{d}$

CALCULATE:

(a) $N = \frac{(22.0 \text{ m/s})(9.00 \text{ s})}{2\pi(0.58 \text{ m})} = 54.33$ revolutions

(b) $\omega = \frac{2(22.0 \text{ m/s})}{0.58 \text{ m}} = 75.86 \text{ rad/s} = \frac{75.86}{2\pi} \text{ rev/s} = 12.07 \text{ rev/s}$

ROUND: The results should be rounded to three significant figures.

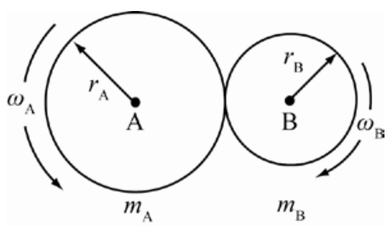
(a) $N = 54.3$ revolutions

(b) $\omega = 12.1$ rev/s

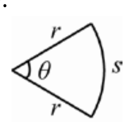
DOUBLE-CHECK: For the given values, these results are reasonable.

9.64. **THINK:** First, determine the number of revolutions gear A undergoes while slowing down. From this, determine the total arc-length of gear A. Gear B must have the same arc-length, from which the number of rotations undergone by gear B can be determined. The following values are given: $\omega_{\text{i,A}} = 120.$ rpm, $\omega_{\text{f,A}} = 60.0$ rpm, $\Delta t = 3.00$ s, $r_{\text{A}} = 55.0$ cm, $r_{\text{B}} = 30.0$ cm, $m_{\text{A}} = 1.00$ kg, $m_{\text{B}} = 0.500$ kg and $\Delta\omega_{\text{A}} = \omega_{\text{f,A}} - \omega_{\text{i,A}} = -60.0$ rpm. Use the conversion factor $1 \text{ rpm} = \frac{2\pi \text{ rad}}{60 \text{ s}}$.

SKETCH:



RESEARCH: The arc-length is given by $s = r\theta$.



The angular displacement is $\Delta\theta = \omega_{\text{i}}\Delta t + \frac{1}{2}\alpha t^2$ The angular acceleration is $\alpha = \frac{\Delta\omega}{\Delta t}$.

SIMPLIFY: The arc-length of gear A is given by $s_{\text{A}} = r_{\text{A}}\Delta\theta_{\text{A}} = r_{\text{A}}\left(\omega_{\text{i,A}}\Delta t + \frac{1}{2}\alpha_{\text{A}} t^2\right)$.

$$\alpha_{\text{A}} = \frac{\Delta\omega_{\text{A}}}{\Delta t} \Rightarrow s_{\text{A}} = r_{\text{A}}\left(\omega_{\text{i,A}}\Delta t + \frac{\Delta\omega_{\text{A}}\Delta t}{2}\right)$$

Gear B has the same arc-length, $s_{\text{A}} = s_{\text{B}}$. The angular displacement of gear B is $s_{\text{B}} = r_{\text{B}}\Delta\theta_{\text{B}}$, so $\Delta\theta_{\text{B}} = \frac{s_{\text{B}}}{r_{\text{B}}} = \frac{s_{\text{A}}}{r_{\text{B}}}$ (since $s_{\text{A}} = s_{\text{B}}$).

The number of rotations, *n*, of gear *B* is $n = \frac{\Delta\theta_{\text{B}}}{2\pi}$, so $n = \frac{s_{\text{A}}}{2\pi r_{\text{B}}} = \frac{1}{2\pi}\frac{r_{\text{A}}}{r_{\text{B}}}\left(\omega_{\text{i,A}}\Delta t + \frac{1}{2}\Delta\omega_{\text{A}}\Delta t\right)$

$$n = \frac{1}{2\pi}\frac{r_{\text{A}}}{r_{\text{B}}}\Delta t\left(\omega_{\text{i,A}} + \frac{1}{2}\Delta\omega_{\text{A}}\right)$$

CALCULATE: $n=\frac{1}{2\pi}\left(\frac{0.550\text{ m}}{0.300\text{ m}}\right)\left[\left(120.\left(\frac{2\pi}{60}\right)\right)\text{ s}^{-1}+\frac{1}{2}\left(-60.0\left(\frac{2\pi}{60}\right)\right)\text{ s}^{-1}\right](3.00\text{ s})$

$$=\frac{1}{2\pi}\left(\frac{0.550\text{ m}}{0.300\text{ m}}\right)\left[4\pi-\pi\right](3.00)=\frac{9.00}{2}\left(\frac{0.550}{0.300}\right)=8.250$$

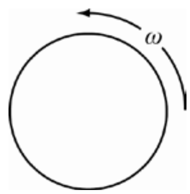
ROUND: Rounding the result to three significant figures gives n = 8.25 rotations.

DOUBLE-CHECK: There is an alternate solution. The average angular speed of A during the slowing down is $(120+60)/2\text{ rpm}=90\text{ rpm}.$ In 3 s, A undergoes $90(3/60)=4.5$ rotations. Since B has a smaller radius, it undergoes a proportionally greater number of rotations. The proportionality is the ratio of the radii:

$$n=4.5\left(\frac{0.55\text{ m}}{0.30\text{ m}}\right)=8.25,\text{ as before.}$$

9.65. **THINK:** The angular acceleration is constant, so the uniform angular acceleration equations can be used directly. The known quantities are $\omega_\text{i}=10.0$ rev/s, $\omega_\text{f}=0$ and $\Delta t=10.0$ min.

SKETCH:



RESEARCH: $\alpha=\frac{\Delta\omega}{\Delta t},\quad \Delta\theta=\omega_\text{i}\Delta t+\frac{1}{2}\alpha t^2$

SIMPLIFY: Simplification is not necessary.

CALCULATE: $\alpha=\frac{-10.0\text{ rev/s}(2\pi\text{ rad/rev})}{10.0\text{ min}(60\text{ s/min})}=-\frac{20\pi}{600}\text{ rad/s}^2=-\frac{\pi}{30}\text{ rad/s}^2=-0.1047\text{ rad/s}^2$

$$\Delta\theta=(10.0\text{ rev/s}(2\pi\text{ rad/rev}))(10.0\text{ min}(60\text{ s/min}))-\frac{1}{2}\left(\frac{\pi}{30}\text{rad/s}^2\right)(10.0\text{ min}(60\text{ s/min}))^2$$

$$=(20\pi\text{ rad/s})(600\text{ s})-\frac{\pi}{60}\text{ rad/s}^2(600\text{ s})^2=3.77\cdot10^4\text{ rad}-1.885\cdot10^4\text{ rad}=1.885\cdot10^4\text{ rad}$$

ROUND: Rounding each result to three significant figures gives $\alpha=-0.105\text{ rad/s}^2$ and $\Delta\theta=1.88\cdot10^4$ rad.

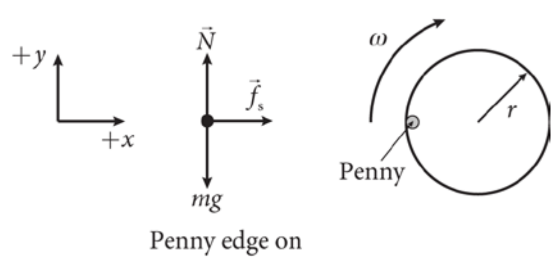
DOUBLE-CHECK: The average angular speed is

$$\frac{0+10.0\text{ rev/s}}{2}=5\text{ rev/s}=5(2\pi\text{ rad/s}).$$

The displacement during this time interval for the average speed is $\Delta\theta=\omega_\text{avg}\Delta t$ $=(10\pi\text{ rad/s})(600\text{ s})=1.885\cdot10^4$ rad, as above. The results are consistent and reasonable.

9.66. **THINK:** The force of static friction between the penny and the phonograph disk provides the centripetal force to keep the penny moving in a circle.

SKETCH:



RESEARCH: The maximum force of static friction between the penny and the photograph disk is $f_s = \mu_s mg$. The centripetal force required to keep the penny moving in a circle is $F_c = mr\omega^2$. Frequency is related to angular frequency by $\omega = 2\pi f$.

SIMPLIFY: $mr\omega^2 = \mu_s mg \Rightarrow \mu_s = \frac{\omega^2 r}{g} \Rightarrow \mu_s = \frac{(2\pi f)^2 r}{g}$.

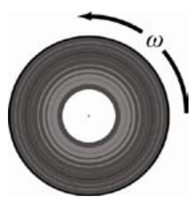
CALCULATE: $f = 33 \frac{\text{rev}}{\text{min}} \frac{\text{min}}{60 \text{ s}} = 0.5500 \text{ s}^{-1}$, $r = \frac{12 \text{ in}}{2} \frac{2.54 \text{ cm}}{\text{in}} \frac{1 \text{ m}}{100 \text{ cm}} = 0.1524 \text{ m}$,

$$\mu_s = \frac{\left[2\pi\left(0.5500 \text{ s}^{-1}\right)\right]^2 (0.1524 \text{ m})}{9.81 \text{ m/s}^2} = 0.1855.$$

ROUND: Rounding the result to two significant figures gives $\mu_s = 0.19$.

DOUBLE-CHECK: The results are reasonable for the given values.

9.67. **THINK:** The acceleration is uniform during the given time interval. The average angular speed during this time interval can be determined and from this, the angular displacement can be determined.
SKETCH:



RESEARCH: $\omega_{\text{avg}} = \frac{\omega_f + \omega_i}{2}$, $\Delta\theta = \omega_{\text{avg}} \Delta t$

SIMPLIFY: Simplification is not necessary.

CALCULATE: $\omega_i = 33.33 \text{ rpm} = 33.33 \text{ rpm}\left(\frac{2\pi}{60 \text{ s}}\right) = 3.491 \text{ rad/s}$, $\omega_f = 0$

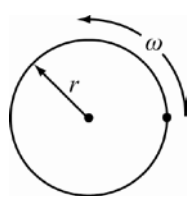
$$\Delta\theta = \left(\frac{3.491}{2} \text{ rad s}^{-1}\right)(15.0 \text{ s}) = 26.18 \text{ rad}$$

$$\text{number of rotations} = \frac{\Delta\theta}{2\pi} = 4.167$$

ROUND: Rounding the result to three significant figures gives the number of rotations to be 4.17 rotations.

DOUBLE-CHECK: These are reasonable results for a turntable.

9.68. **THINK:** Given the radius (2.0 cm) and rotation speed (250 rpm), the linear and angular speeds and acceleration can be determined.
SKETCH:



RESEARCH: $\omega = \text{rpm}\left(\frac{2\pi}{60} \text{ rad/s}\right)$, $v = \omega r$, $a = \omega^2 r$, $\alpha = 0$

SIMPLIFY: Simplification is not necessary.

CALCULATE: $\omega = 250.\left(\frac{2\pi}{60} \text{ rad/s}\right) = 26.18 \text{ rad/s}$, $v = \omega r = (26.18)(0.0200) \text{ m/s} = 0.5236 \text{ m/s}$

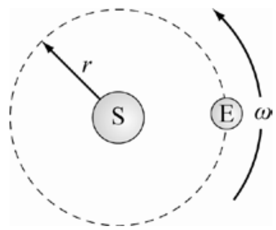
$a=\omega^2 r=(26.18)^2(0.020)\text{ m/s}^2=13.71\text{ m/s}^2,\ \alpha=0$

ROUND: Rounding the results to three significant figures gives $\omega=26.2$ rad/s, $v=0.524$ m/s, $a=14\text{ m/s}^2$, and $\alpha=0$.

DOUBLE-CHECK: The rotation speed is constant, so $\alpha=0$. The other values are likewise reasonable.

9.69. **THINK:** The angular acceleration of the Earth is zero. The linear acceleration is simply the centripetal acceleration. $r=1$ AU or $r=1.50\cdot 10^{11}$ m and $\omega=2\pi$ rad / year.

SKETCH:



RESEARCH: $a=\omega^2 r,\ 1\text{ yr}=3.16\cdot 10^7\text{ s}$

SIMPLIFY: Simplification is not necessary.

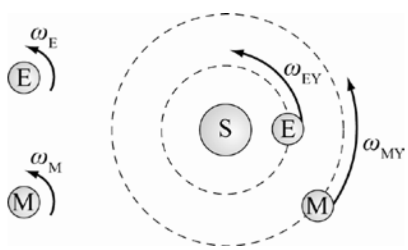
CALCULATE: $a=\left(\dfrac{2\pi}{3.16\cdot 10^7\text{ s}}\right)^2\left(1.50\cdot 10^{11}\text{ m}\right)=5.93\cdot 10^{-3}\text{ m/s}^2$

ROUND: Keeping three significant figures, $a=5.93\cdot 10^{-3}\text{ m/s}^2$.

DOUBLE-CHECK: The linear acceleration is rather small because the distance to the Sun is so great.

9.70. **THINK:** From the given data, the ratio of the angular accelerations of Mars and Earth can be determined.

SKETCH:



RESEARCH: $\omega_{\text{day}}=\dfrac{2\pi\text{ rad}}{1\text{ day}},\ \omega_{\text{yr}}=\dfrac{2\pi\text{ rad}}{1\text{ yr}}$

SIMPLIFY: $\omega_{\text{M}}=\dfrac{2\pi\text{ rad}}{24.6\text{ hr}},\ \omega_{\text{E}}=\dfrac{2\pi\text{ rad}}{24\text{ hr}},\ \omega_{\text{My}}=\dfrac{2\pi\text{ rad}}{687\text{ Earth-days}},\ \omega_{\text{Ey}}=\dfrac{2\pi\text{ rad}}{365\text{ Earth-days}}$

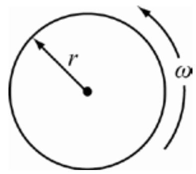
CALCULATE: $\dfrac{\omega_{\text{M}}}{\omega_{\text{E}}}=\dfrac{24.0\text{ hr}}{24.6}=0.9756,\ \dfrac{\omega_{\text{My}}}{\omega_{\text{Ey}}}=\dfrac{365}{687}=0.5319$

ROUND: Rounding the results to three significant figures gives $\dfrac{\omega_{\text{M}}}{\omega_{\text{E}}}=0.976$ and $\dfrac{\omega_{\text{My}}}{\omega_{\text{Ey}}}=0.532.$

DOUBLE-CHECK: The angular speed of Mars' orbit is 0.532 that of Earth. The latter is reasonable given that Mars is further from the Sun than Earth, as we will learn in Chapter 12.

9.71. **THINK:** Parts (a) and (b) can be solved using the constant angular acceleration equations. For part (c), calculate the angular displacement and, from this, compute the total arc-length, which is equal to the distance traveled.

SKETCH:



RESEARCH:

(a) $v = \omega r$

(b) $\alpha = \frac{\Delta\omega}{\Delta t}, \ \Delta\theta = \omega_i \Delta t + \frac{1}{2}\alpha \Delta t^2$

(c) $s = r\Delta\theta, \ \Delta\theta = 2\pi(\text{total revs.})$

SIMPLIFY:

(a) $\omega_\text{i} = \frac{v_\text{i}}{r}$

(b) $\Delta\omega = \omega_\text{f} - \omega_\text{i} = 0 - \frac{v_\text{i}}{r} = -\omega_\text{i}, \ \Delta t = \frac{\Delta\omega}{\alpha} = -\frac{\omega_\text{i}}{\alpha}$

$$\Delta\theta = \omega_i \Delta t + \frac{1}{2}\alpha\Delta t^2 = \omega_i\left(\frac{-\omega_i}{\alpha}\right) + \frac{\alpha}{2}\left(\frac{-\omega_i}{\alpha}\right)^2 = \frac{-\omega_i^2}{\alpha} + \frac{\omega_i^2}{2\alpha} = -\frac{\omega_i^2}{2\alpha} \Rightarrow \alpha = \frac{-\omega_i^2}{2\Delta\theta}$$

(c) $s = r\Delta\theta$

CALCULATE:

(a) $\omega_i = \frac{35.8 \text{ m/s}}{0.550 \text{ m}} = 65.09 \text{ s}^{-1}$

(b) $\alpha = \frac{-\left(65.09 \text{ s}^{-1}\right)^2}{2\left(2\pi(40.2)\right)} = -8.387 \ \text{s}^{-2}$

(c) $s = (0.550 \text{ m})\left(2\pi(40.2)\right) = 138.92 \text{ m}$

ROUND: Rounding the results to three significant figures:

(a) $\omega_i = 65.1 \text{ s}^{-1}$

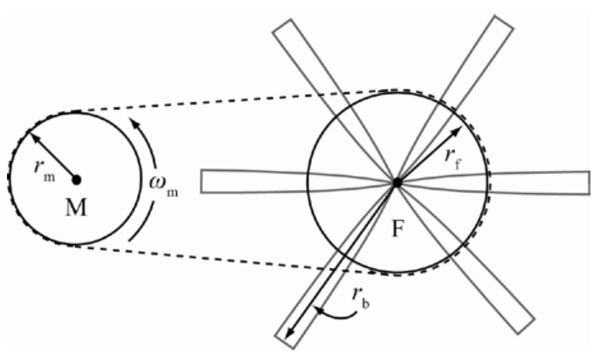
(b) $\alpha = -8.39 \text{ s}^{-2}$

(c) $s = 139 \text{ m}$

DOUBLE-CHECK: For the parameters given, these are reasonable results.

9.72. **THINK:** Everything in the problem rotates at constant angular speed. The two wheels have radii of $r_\text{m} = 2.00$ cm and $r_\text{f} = 3.00$ cm and rotate at the same linear speed.

SKETCH:



RESEARCH: $v = \omega r$

SIMPLIFY: $v_\text{m} = \omega_\text{m} r_\text{m}, \ v_\text{f} = \omega_\text{f} r_\text{f}, \ v_\text{b} = \omega_\text{b} r_\text{b}$

The wheels are attached by a belt, so $v_{\text{m}} = v_{\text{f}} \;\Rightarrow\; \omega_{\text{m}} r_{\text{m}} = \omega_{\text{f}} r_{\text{f}} \;\Rightarrow\; \omega_{\text{f}} = \dfrac{\omega_{\text{m}} r_{\text{m}}}{r_{\text{f}}}$. The blades are attached to wheel *F*, so $\omega_{\text{b}} = \omega_{\text{f}} \;\Rightarrow\; v_{\text{b}} = \omega_{\text{f}} r_{\text{b}} = \dfrac{\omega_{\text{m}} r_{\text{m}}}{r_{\text{f}}} r_{\text{b}}$.

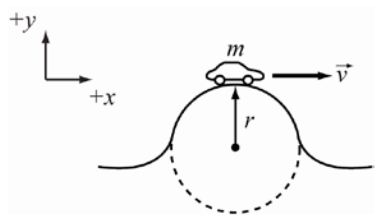
CALCULATE: $v_{\text{b}} = \dfrac{1}{0.03 \text{ m}}\left(1200\left(\dfrac{2\pi}{60 \text{ s}}\right)\right)(0.02 \text{ m})(0.15 \text{ m}) = 12.57 \text{ m/s}$

ROUND: Rounding the result to three significant figures gives $v_{\text{b}} = 12.6$ m/s.

DOUBLE-CHECK: From $v_{\text{b}} = \dfrac{\omega_{\text{m}} r_{\text{m}} r_{\text{b}}}{r_{\text{f}}}$, it can be seen that v_{b} grows with ω_{m}, r_{m}, r_{b}, and v_{b} decreases as r_{f} grows. All these relations are reasonable.

9.73. **THINK:** The net force due to gravity (down) and normal force from the hill (upward) equals the centripetal force determined by the car's speed and the path's radius of curvature. The force the car exerts on the hill is equal and opposite to the force of the hill on the car.

SKETCH:



RESEARCH: $F_{\text{g}} = mg$ acts downward, and let *N* be the upward force of the hill on the car. The net force, F_{net}, which is the centripetal force $F_{\text{c}} = m\dfrac{v^2}{r}$, acts downward.

SIMPLIFY: Taking upward force as positive and downward force as negative,

$$-F_{\text{net}} = N - F_{\text{g}} = N - mg = -F_{\text{c}} = -m\frac{v^2}{r} \;\Rightarrow\; N = m\left(g - \frac{v^2}{r}\right)$$

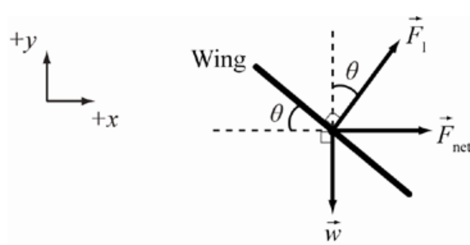
CALCULATE: $N = (1000.\text{ kg})\left(9.81 \text{ m/s}^2 - \dfrac{(60.0 \text{ m/s})^2}{370. \text{ m}} - \right) = 80.270 \text{ N}$

ROUND: To three significant figures, $N = 80.3$ N.

DOUBLE-CHECK: The equation confirms what we know from observation, namely that if v is large enough, then the normal force will go to zero and the car will lose contact with the ground.

9.74. **THINK:** A free body diagram will show all the forces acting on the plane. The net force is horizontal, directed towards the center of the radius of curvature. The speed is v = 4800 km/h and the turning radius is r = 290 km. The banking angle, θ, must be determined.

SKETCH:



RESEARCH: $\vec{F}_{\text{net}} = m\vec{a},\ \vec{F}_{\text{net}} = \vec{w} + \vec{F}_1$

SIMPLIFY: $F_{\text{net}} = ma = m\dfrac{v^2}{r},\ w = mg$

$$\sum F_y = mg - F_1\cos\theta = 0 \Rightarrow mg = F_1\cos\theta \Rightarrow \frac{F_1}{m} = \frac{g}{\cos\theta}, \qquad \sum F_x = F_1\sin\theta = m\frac{v^2}{r} \Rightarrow \frac{F_1}{m} = \frac{v^2}{r\sin\theta}$$

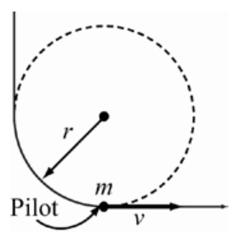
Equating the two above equations gives $\dfrac{g}{\cos\theta} = \dfrac{v^2}{r\sin\theta} \Rightarrow \tan\theta = \dfrac{v^2}{gr} \Rightarrow \theta = \tan^{-1}\left(\dfrac{v^2}{gr}\right).$

CALCULATE: $v = 4800 \text{ km/h} = 4800\left(\dfrac{10^3 \text{ m}}{3600 \text{ s}}\right) = 1.333\cdot 10^3 \text{ m/s},\ \ r = 290 \text{ km} = 2.9\cdot 10^5 \text{ m}$

$$\theta = \tan^{-1}\frac{\left(1.333\cdot 10^3 \text{ m/s}\right)^2}{\left(9.81 \text{ m/s}^2\right)\left(2.9\cdot 10^5 \text{ m}\right)} = 32.00°$$

ROUND: The speed is given to three significant figures, so the result should be $\theta = 32.0°$.
DOUBLE-CHECK: A banking angle of $32°$ is reasonable for the SR-71.

9.75. **THINK:** From the linear speed and the radius, the centripetal acceleration can be determined. With the pilots' mass, the centripetal force can also be determined. The pilot's apparent weight is the combined effect of gravitational and centripetal accelerations.
SKETCH:



RESEARCH:

(a) $a_c = \dfrac{v^2}{r},\ F_c = \dfrac{mv^2}{r}$

(b) $F_c = \dfrac{mv^2}{r},\ F_g = mg,\ w = \dfrac{mv^2}{r} + mg$

SIMPLIFY: Simplification is not necessary.
CALCULATE:

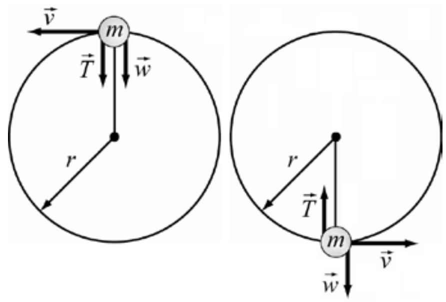
(a) $a_c = \dfrac{\left(500.\text{ m/s}\right)^2}{4000.\text{ m}} = 62.50 \text{ m/s}^2,\quad F_c = ma_c = \left(80.0 \text{ kg}\right)\left(62.50 \text{ m/s}^2\right) = 5.00\cdot 10^3 \text{ N}$

(b) $w = 5.00\cdot 10^3 \text{ N} + \left(80.0 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right) = 5784.8 \text{ N}$

ROUND: Round the results to three significant figures:
(a) $a_c = 62.5 \text{ m/s}^2$ and $F_c = 5.00\cdot 10^3 \text{ N}$
(b) $w = 5780 \text{ N}$
DOUBLE-CHECK: These are all reasonable values.

9.76. **THINK:** The net force on the ball is the centripetal force. Gravity and tension sum to produce this force. m = 1.00 kg, r = 1.00 m and v = 10.0 m/s. At the top of the circle, gravity and tension both point down. At the bottom of the circle, gravity still points down, but the tension points up.

SKETCH:



RESEARCH: $F_{\text{net}} = \dfrac{mv^2}{r}$

(a) $F_{\text{net}} = T + w$

(b) $F_{\text{net}} = T - w$

SIMPLIFY:

(a) $T = F_{\text{net}} - w = \dfrac{mv^2}{r} - mg$

(b) $T = F_{\text{net}} + w = \dfrac{mv^2}{r} + mg$

CALCULATE: $\dfrac{mv^2}{r} = \dfrac{(1.00\text{ kg})(10.0\text{ m/s})^2}{1.00\text{ m}} = 100.\text{ N},\;\; mg = (1.00\text{ kg})\left(9.81\text{ m/s}^2\right) = 9.81\text{ N}$

(a) $T = 100.\text{ N} - 9.81\text{ N} = 90.19\text{ N}$

(b) $T = 100.\text{ N} + 9.81\text{ N} = 109.8\text{ N}$

(c) The tension in the string is greatest at the bottom of the circle. As the ball moves away from the bottom, the tension decreases to its minimum value at the top of the circle. It then increases until the ball again reaches the bottom.

ROUND: Round the results to three significant figures.

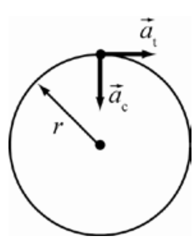
(a) $T = 90.2$ N

(b) $T = 110.$ N

DOUBLE-CHECK: If you are swinging the ball with a high speed like in this problem, the weight becomes almost negligible, and thus we should expect that the tensions at the bottom and top become almost identical. The tension is still highest at the bottom, as would be reasonably expected.

9.77. **THINK:** The car start slipping at the point where the magnitude of the total acceleration exceeds the maximum acceleration that can be provided by the friction force. The total acceleration of the car is composed of contributions from the centripetal and the tangential acceleration, which have to be added as vectors. Given here are R = 36.0 m, $a_{\text{t}} = 3.30\text{ m/s}^2$, $v_{\text{i}} = 0$ and $\mu = 0.950$.

SKETCH:



RESEARCH: The magnitude of the total acceleration is given by the tangential and radial acceleration, $a = \sqrt{a_{\text{t}}^2 + a_{\text{c}}^2}$. The centripetal acceleration is $a_{\text{c}} = v^2 / R$. Since the car accelerates at constant linear acceleration starting from rest, the speed as a function of time is $v = a_{\text{t}}t$. The maximum force of friction is given by $f = \mu mg$. So the maximum acceleration due to friction is $a_f = \mu g$. The distance traveled by then is $d = \frac{1}{2}a_t t^2$.

SIMPLIFY: Slippage occurs when $a_f = a$; so $a_f = \mu g = a = \sqrt{a_t^2 + a_c^2}$

$$\Rightarrow \mu^2 g^2 = a_t^2 + a_c^2 = a_t^2 + (v^2 / R)^2 = a_t^2 + (a_t^2 t^2 / R)^2 \quad \Rightarrow t^2 = R\sqrt{\mu^2 g^2 - a_t^2} \,/\, a_t^2$$

$$\Rightarrow d = \frac{1}{2}a_t t^2 = \frac{\frac{1}{2}R\sqrt{\mu^2 g^2 - a_t^2}}{a_t}$$

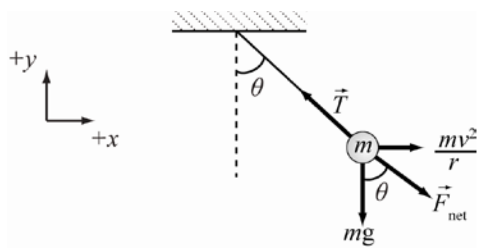
CALCULATE: $d = \dfrac{\frac{1}{2}(36.0 \text{ m})\sqrt{0.950^2(9.81 \text{ m/s}^2)^2 - (3.30 \text{ m/s}^2)^2}}{(3.30 \text{ m/s}^2)} = 47.5401 \text{ m}$

ROUND: Rounding to three significant figures gives $d = 47.5$ m.

DOUBLE-CHECK: d is proportional to R. This makes sense because a larger R implies less curvature and thus less centripetal force. d is also inversely proportional to a_t, which also makes sense since a smaller tangential acceleration implies a greater distance traveled before the maximum speed is attained.

9.78. **THINK:** The pendulum experiences a vertical force due to gravity and a horizontal centripetal force. These forces are balanced by the tension in the pendulum string. $r = 6.0$ m and $\omega = 0.020$ rev/s.

SKETCH:



RESEARCH: $F_{\text{net}} \sin\theta = \dfrac{mv^2}{r}$, $F_{\text{net}} \cos\theta = mg$, $v = \omega r$, $1 \text{ rev/s} = 2\pi \text{ rad/s}$

SIMPLIFY: $\dfrac{F_{\text{net}}}{m} = \dfrac{v^2}{r\sin\theta}$, $\dfrac{F_{\text{net}}}{m} = \dfrac{g}{\cos\theta}$

Equating the equations, $\dfrac{v^2}{r\sin\theta} = \dfrac{g}{\cos\theta} \Rightarrow \tan\theta = \dfrac{v^2}{rg} = \dfrac{\omega^2 r^2}{rg} = g\dfrac{\omega^2 r}{g} \Rightarrow \theta = \tan^{-1}\left(\dfrac{\omega^2 r}{g}\right)$.

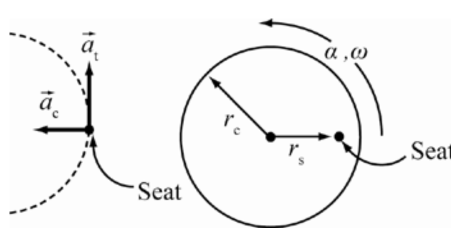
CALCULATE: $\theta = \tan^{-1}\left(\dfrac{\left(0.0200(2\pi)\text{s}^{-1}\right)^2(6.00 \text{ m})}{9.81 \text{ m/s}^2}\right) = 0.5533°$

ROUND: Rounding to three significant figures, $\theta = 0.553°$.

DOUBLE-CHECK: Such a small deviation is reasonable, given that the rotation is so slow.

9.79. **THINK:** Use the relationship between angular and centripetal acceleration. The given values are $r_s = 2.75$ m, $r_c = 6.00$ m, $\omega_i = 0$, $\omega_f = 0.600$ rev/s and $\Delta t = 8.00$ s.

SKETCH:



RESEARCH:

(a) $\alpha = \dfrac{\Delta\omega}{\Delta t}$

(b) $a_{\text{c}} = \dfrac{v_{\text{s}}^2}{r_{\text{s}}}, \quad v_{\text{s}} = \omega_{\text{s}} r_{\text{s}}$

(c) $\vec{a} = \vec{a}_{\text{c}} + \vec{a}_{\text{t}}, \quad a_{\text{t}} = \alpha r_{\text{s}}$

SIMPLIFY:

(a) Simplification is not necessary.

(b) $a_{\text{c}} = \omega_{\text{s}}^2 r_{\text{s}}$

(c) $a = \sqrt{a_{\text{c}}^2 + a_{\text{t}}^2}, \quad \tan\theta = \left(\dfrac{a_{\text{t}}}{a_{\text{c}}}\right)$

CALCULATE:

(a) $\alpha = \dfrac{0.600(2\pi)\text{ rad/s}}{8.00\text{ s}} = 0.4712\text{ rad/s}^2$

(b) At 8.00 s, $\omega_{\text{s}} = 0.600\text{ rev/s}$, so $a_{\text{c}} = \left(0.600(2\pi)\text{ s}^{-1}\right)^2(2.75\text{ m}) = 39.08\text{ m/s}^2$ and $\alpha = 0.4712\text{ rad/s}^2$.

(c) $a = \sqrt{\left(39.08\text{ m/s}^2\right)^2 + \left(0.4712\text{ s}^{-2}\right)^2(2.75\text{ m})^2} = 39.10\text{ m/s}^2 \quad \theta = \tan^{-1}\dfrac{\left(0.4712\text{ s}^{-2}\right)(2.75\text{ m})}{39.08\text{ m/s}^2} = 1.899°$

If the centripetal acceleration is along the positive x axis, then the direction of the total acceleration is $1.90°$ along the horizontal (rounded to three significant figures).

ROUND: Values are given to three significant figures, so the results should be rounded accordingly.

(a) $\alpha = 0.471\text{ rad/s}^2$

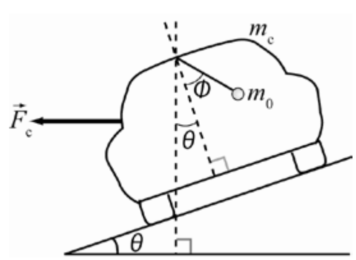
(b) $a_{\text{c}} = 39.1\text{ m/s}^2$ and $\alpha = 0.471\text{ rad/s}^2$.

(c) $a = 39.1\text{ m/s}^2$ at $\theta = 1.90°$.

DOUBLE-CHECK: The total acceleration is quite close to the centripetal acceleration, since the tangential acceleration and the angular acceleration are both quite small.

9.80. **THINK:** The forces acting on the ornament are the tension on the string and the force of gravity. The net force is the centripetal force acting towards the center of the track. The centripetal force is close to the car's friction with the ground. $m_{\text{c}}g = 10.0\text{ kN}$, $\theta = 20.0°$ and $\phi = 30.0°$. F_{f} is the frictional force acting on the car.

SKETCH:



RESEARCH: $T\cos(\theta+\phi) = m_0 g, \quad T\sin(\theta+\phi) = m_0\dfrac{v^2}{r}, \quad F_{\text{c}} = F_{\text{f}} = m_{\text{c}}\dfrac{v^2}{r}$

SIMPLIFY: $\dfrac{T}{m_0} = \dfrac{g}{\cos(\theta+\phi)}$

$\dfrac{T}{m_0} = \dfrac{v^2}{r\sin(\theta+\phi)}$

Equating the equations above gives $\dfrac{g}{\cos(\theta+\phi)}=\dfrac{v^2}{r\sin(\theta+\phi)} \Rightarrow \tan(\theta+\phi)=\dfrac{v^2}{rg} \Rightarrow \dfrac{v^2}{r}=g\tan(\theta+\phi).$

The force of friction then becomes $F_\text{f}=m_\text{c}\dfrac{v^2}{r}=m_\text{c}g\tan(\theta+\phi).$

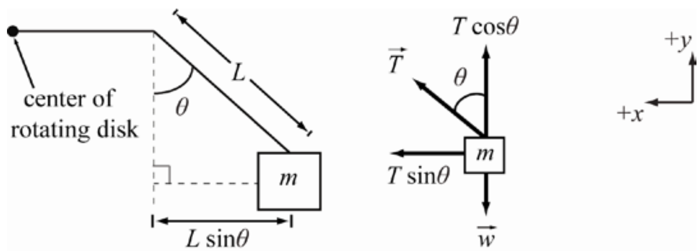
CALCULATE: $F_\text{f}=(10.0\cdot 10^3\text{ N})\tan(20.0°+30.0°)=1.192\cdot 10^4\text{ N}$

ROUND: Rounding to three significant figures gives $F_\text{f}=1.19\cdot 10^4$ N.

DOUBLE-CHECK: This is a reasonable value for a car of this weight.

9.81. **THINK:** Both gravity and tension act on the passenger. The net force is the centripetal force acting towards the center. The given values are as follows: $\theta=30.0°$, m = 65.0 kg, L = 3.20 m and $R_0=3.00$ m.

SKETCH:



RESEARCH: $T\cos\theta=w,\ w=mg,\ T\sin\theta=\dfrac{mv^2}{r},\quad r=R_0+L\sin\theta$

SIMPLIFY: $v^2=\dfrac{rT\sin\theta}{m},\quad T=\dfrac{w}{\cos\theta}=\dfrac{mg}{\cos\theta}$

(a) $v^2=\dfrac{r\sin\theta}{m}\left(\dfrac{mg}{\cos\theta}\right)=rg\tan\theta \Rightarrow v=\sqrt{rg\tan\theta}$

(b) $T=\dfrac{mg}{\cos\theta}$ or $T=\dfrac{mv^2}{r\sin\theta}$.

CALCULATE:

(a) $v=\sqrt{(3.00\text{ m}+3.20\sin 30.0°\text{ m})(9.81\text{ m/s}^2)(\tan 30.0°)}=5.104$ m/s

(b) $T=\dfrac{(65.0\text{ kg})(9.81\text{ m/s}^2)}{\cos 30.0°}=736.3$ N

ROUND: All values are given to three significant figures, so the results should be rounded accordingly.

(a) v = 5.10 m/s

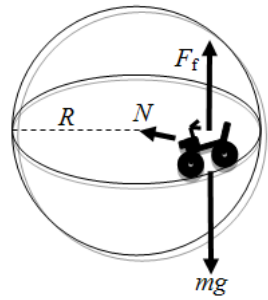
(b) T = 736 N

DOUBLE-CHECK: Note that the speed increases if the main disk, R_0, increases, or the length of the cable, L, increases, as it should.

Multi-Version Exercises

9.82. **THINK:** The only values given in this problem are the radius of the sphere and the coefficient of static friction between the motorcycle and the sphere. The motorcycle will stay on the surface as long as the vertical force exerted by the force of friction is at least as much as the weight of the motorcycle. The friction force is proportional to the normal force exerted by the wall of the dome, which is given by the centripetal force. Combine these to solve for the minimum velocity.

SKETCH:



RESEARCH: The centripetal force required to keep the motorcycle moving in a circle is $F_c = \frac{mv^2}{R}$. The friction force is $F_f = \mu_s N$, and it must support the weight of the motorcycle, s o $F_f \geq mg$.

SIMPLIFY: Since the normal force equals the centripetal force in this case, substitute F_c for N in the equation $F_f = \mu_s N$ to get $F_f = \mu_s F_c = \mu_s \frac{mv^2}{R}$. Combine this with the fact that the frictional force must be enough to support the weight of the motorcycle, so $mg \leq F_f = \mu_s \frac{mv^2}{R}$. Finally, solve the inequality for the velocity (keep in mind that the letters represent positive values):

$$\mu_s \frac{mv^2}{R} \geq mg \Rightarrow$$
$$\frac{R}{\mu_s m} \cdot \mu_s \frac{mv^2}{R} \geq \frac{R}{\mu_s m} \cdot mg \Rightarrow$$
$$v^2 \geq \frac{Rg}{\mu_s}$$
$$v \geq \sqrt{\frac{Rg}{\mu_s}}$$

CALCULATE: The radius of the sphere is 12.61 m, and the coefficient of static friction is 0.4601. The gravitational acceleration near the surface of the earth is about 9.81 m/s^2, so the speed must be:

$$v \geq \sqrt{\frac{Rg}{\mu_s}}$$
$$v \geq \sqrt{\frac{12.61 \text{ m} \cdot 9.81 \text{ m/s}^2}{0.4601}}$$
$$v \geq 16.3970579 \text{ m/s}$$

ROUND: Since the measured values are all given to four significant figures, the final answer will also have four figures. The minimum velocity is 16.40 m/s.

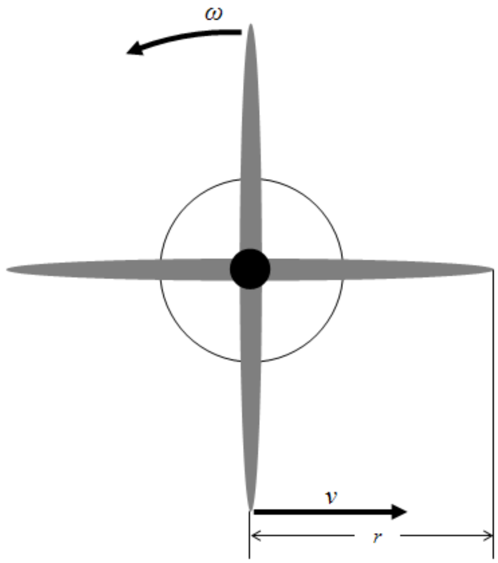
DOUBLE-CHECK: In this case, the motorcycle is traveling at 16.40 m/s, or about 59 kilometers per hour, which is a reasonable speed based on how fast motorcycles *can* go. It needs to travel 12.61(2π) = 79.23 meters to go all the way around the sphere, so it makes one revolution every 4.83 seconds, or between 12 and 13 revolutions per minute. These values all seem reasonable based on past experience with motorcycles.

9.83. $\mu_s = \frac{Rg}{v^2} = \frac{(13.75 \text{ m})(9.81 \text{ m/s}^2)}{(17.01 \text{ m/s})^2} = 0.4662$

9.84. $R = \frac{\mu_s v^2}{g} = \frac{(0.4741)(15.11)^2}{9.81 \text{ m/s}^2} = 11.03 \text{ m}$

9.85. **THINK:** The speed of a point on the tip of the propeller can be calculated from the angular speed and the length of the propeller blade. The angular speed of the propeller can be calculated from the frequency. Find the maximum length of the propeller blade such that the angular speed at the tip of the propeller blade is less than the indicated speed of sound.

SKETCH: A view, looking towards the airplane from the front, is shown.



RESEARCH: The linear velocity should be less than the speed of sound $v \le v_{\text{sound}}$. The magnitude of the linear velocity v is equal to the product of the radius of rotation r and the angular speed ω: $v = r\omega$. The angular speed is related to the rotation frequency by $\omega = 2\pi f$. The length of the propeller blade is twice the radius of the propeller ($d = 2r$). Finally, note that the rotation frequency is given in revolutions per minute and the speed of sound is given in meters per second, so a conversion factor of 60 seconds / minute will be needed.

SIMPLIFY: Use the equation for the linear speed ($v = r\omega$) and the equation for the rotation frequency to get $v = 2\pi f \cdot r$. Use this in the inequality $v \le v_{\text{sound}}$ to find that $2\pi f \cdot r \le v_{\text{sound}}$. Solve this for the length of the propeller blade r (note that ω is a positiv e number of revolutions per minute) to get $r \le \frac{v_{\text{sound}}}{2\pi f}$. The maximum length of the propeller blade is two times the largest possible value of $d = 2r = \frac{v_{\text{sound}}}{\pi f}$.

CALCULATE: The angular frequency f is given in the problem as 2403 rpm and the speed of sound is 343.0 m/s. The maximum length of the propeller blade is thus $d = \frac{343.0 \text{ m/s} \cdot 60 \text{ s/min}}{\pi \cdot 2403 \text{ rev/min}} = 2.726099649 \text{ m}$.

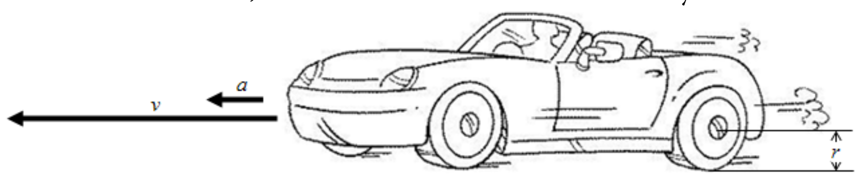
ROUND: The measured values from the problem (the angular frequency and speed of sound) are given to four significant figures, so the final answer should also have four significant figures. The maximum length of a propeller blade is 2.726 m.

DOUBLE-CHECK: For those familiar with propeller-driven aircraft, a total propeller length of about 2.7 m seems reasonable. Working backwards, if the propeller blade is 2.726 m and the linear speed at the tip of the propeller is 343.0 m/s, then the angular speed is $\omega = \frac{v}{r} = \frac{343.0 \text{ m/s}}{1.363 \text{ m}}$. The angular frequency is then $f = \frac{\omega}{2\pi} = \frac{343.0 \text{ m/s}}{2\pi \cdot 1.363 \text{ m}} = 40.05 \text{ rev/sec}$. Since there are 60 seconds in a minute, this agrees with the value of 2403 rev/min given in the problem, and the calculations were correct.

9.86. $f = \dfrac{v}{\pi d} = \dfrac{343.0 \text{ m/s}}{\pi(2.601 \text{ m})} = \left(41.98\dfrac{1}{\text{s}}\right)\left(\dfrac{60 \text{ s}}{\text{min}}\right) = 2519 \text{ rpm}$

9.87. **THINK:** The linear acceleration can be computed from the change in the speed of the car and the time required to accelerate, both of which are given in the problem. The angular acceleration can be calculated from the linear acceleration and the radius of the tires. Since the car's acceleration is constant and it starts at rest, the motion of the car occurs in only one direction, which can be taken to be the +x direction, and the time that the car starts moving can be taken as time zero.

SKETCH: The car starts at rest, so the constant acceleration and velocity are in the same direction.



RESEARCH: The constant linear acceleration is the change in speed per unit time $a = \dfrac{\Delta v}{\Delta t}$. The relationship between linear acceleration a and angular acceleration α is given by $a = r\alpha$, where r is the radius of the rotating object.

SIMPLIFY: Since there are two expressions for the linear acceleration, $a = \dfrac{\Delta v}{\Delta t}$ and $a = r\alpha$, they must be equal to one another: $r\alpha = \dfrac{\Delta v}{\Delta t}$. Solve for the angular acceleration α to get $\alpha = \dfrac{\Delta v}{r\Delta t}$. The car starts at rest at time zero, the final velocity is equal to Δv and the total time is equal to Δt, giving $\alpha = \dfrac{v}{rt}$.

CALCULATE: After 3.945 seconds, the car's final speed is 29.13 m/s. The rear wheels have a radius of 46.65 cm, or $46.65 \cdot 10^{-2}$ m. The angular acceleration is then

$$\alpha = \frac{29.13 \text{ m/s}}{46.65 \cdot 10^{-2} \text{ m} \cdot 3.945 \text{ s}} = 15.82857539 \text{ s}^{-2}$$

ROUND: The time in seconds, radius of the tires, and speed of the car are all given to four significant figures, so the final answer should also have four figures. The angular acceleration of the car is 15.83 s^{-2}.

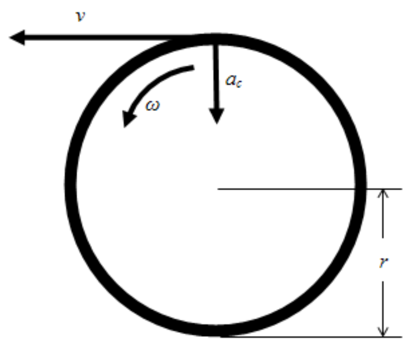
DOUBLE-CHECK: First note that the units (per second per second) are correct for angular acceleration. Working backwards, if the sports car accelerates with an angular acceleration of 15.83 s^{-2} for 3.945 seconds, it will have a final angular speed of $(15.83 \cdot 3.945)$ s^{-1}. With a tire radius of 46.65 cm, this means that the car's final speed will be $(46.65 \cdot 15.83 \cdot 3.945)$ cm/s, or 29.13 m/s (when rounded to four significant figures), which agrees with the problem statement. This confirms that the first set of calculations was correct.

9.88. $v = r\alpha t = (0.4895 \text{ m})(14.99 \text{ s}^{-2})(3.997 \text{ s}) = 29.33 \text{ m/s}$

9.89. $r = \dfrac{v}{\alpha t} = \dfrac{29.53 \text{ m/s}}{(17.71 \text{ s}^{-2})(4.047 \text{ s})} = 0.4120 \text{ m} = 41.20 \text{ cm}$

9.90. **THINK:** The frequency and radius of the flywheel can be used to calculate the speed at the edge of the flywheel. The centripetal acceleration can be calculated from the linear speed and the radius of the flywheel.

SKETCH:



RESEARCH: The centripetal acceleration at the edge of the flywheel is $a_c = \frac{v^2}{r}$, where v is the linear speed at the edge of the flywheel and r is the flywheel's radius. The linear speed v is equal to the angular speed times the radius of the flywheel ($v = r\omega$), and the angular speed ω is related to the frequency f by the equation $\omega = 2\pi f$. The numbers are given in centimeters and revolutions per minute, so conversion factors of $\frac{1 \text{ m}}{100 \text{ cm}}$ and $\frac{1 \text{ min}}{60 \text{ sec}}$ may be needed.

SIMPLIFY: First, find the equation for the velocity in terms of the angular frequency to get $v = r\omega = r(2\pi f)$. Use this in the equation for centripetal acceleration to find

$$a_c = \frac{v^2}{r} = \frac{(2\pi r f)^2}{r} = 4r(\pi f)^2 .$$

CALCULATE: The radius is 27.01 cm, or 0.2701 m and the frequency of the flywheel is 4949 rpm. So the angular acceleration is $4 \cdot 27.01 \text{ cm}\left(\pi \cdot 4949 \frac{\text{rev}}{\text{min}}\right)^2 = 2.611675581 \cdot 10^{10} \frac{\text{cm}}{\text{min}^2}$. Converting to more familiar units, this becomes

$$2.611675581 \cdot 10^{10} \frac{\text{cm}}{\text{min}^2} \cdot \frac{1 \text{ m}}{100 \text{ cm}} \cdot \left(\frac{1 \text{min}}{60 \text{sec}}\right)^2 = 7.254654393 \cdot 10^4 \text{ m/s}^2 .$$

ROUND: The radius and frequency of the flywheel both have four significant figures, so the final answer should also have four figures. The centripetal acceleration at a point on the edge of the flywheel is $7.255 \cdot 10^4$ m/s^2.

DOUBLE-CHECK: Work backwards to find the frequency from the centripetal acceleration and the radius of the flywheel. The linear velocity is $v = \sqrt{a_c r}$, the angular speed is $\omega = v / r = \frac{\sqrt{a_c r}}{r}$, and the frequency $f = \frac{\omega}{2\pi} = \frac{\sqrt{a_c r}}{2\pi r}$. The radius of the flywheel is 0.2701 m and the centripetal acceleration is $7.255 \cdot 10^4$ m/s^2, so the frequency is

$$\begin{aligned} f &= \frac{\sqrt{a_c r}}{2\pi r} \\ &= \frac{\sqrt{7.255 \cdot 10^4 \text{ m/s}^2 \cdot 0.2701 \text{ m}}}{2\pi \cdot 0.2701 \text{ m}} \\ &= 82.4853 \text{ s}^{-1} \cdot \frac{60 \text{sec}}{1 \text{min}} \\ &= 4949.117882 \text{ min}^{-1} \end{aligned}$$

After rounding to four significant figures, this agrees with the frequency given in the problem of 4949 rpm (revolutions per minute).

9.91. $a_c = r(2\pi f)^2$

$$f = \frac{1}{2\pi}\sqrt{\frac{a_c}{r}} = \frac{1}{2\pi}\sqrt{\frac{8.629 \cdot 10^4 \text{ m/s}^2}{0.3159 \text{ m}}} = \left(83.18 \ \frac{1}{\text{s}}\right)\left(\frac{60 \text{ s}}{\text{min}}\right) = 4991 \text{ rpm}$$

Chapter 10: Rotation

Concept Checks

10.1. c **10.2.** c **10.3.** a **10.4.** f **10.5.** b **10.6.** c **10.7.** c **10.8.** b **10.9.** b **10.10** b

Multiple-Choice Questions

10.1. b **10.2.** c **10.3.** b **10.4.** d **10.5.** c **10.6.** c **10.7.** c **10.8.** d **10.9.** b **10.10.** e **10.11.** b **10.12.** a **10.13.** c **10.14.** b **10.15.** c **10.16.** b **10.17.** a **10.18.** b **10.19.** c **10.20.** b

Conceptual Questions

10.21. Rotational kinetic energy is given by $K_{\text{rot}} = \frac{1}{2}cMv^2$

The total kinetic energy for an object rolling without slipping is given by:

$$K_{\text{total}} = K_{\text{trans}} + K_{\text{rot}} = \tfrac{1}{2}Mv^2(1+c) \text{ with } c = 2/5 \text{ for a sphere} \Rightarrow$$

$$\frac{K_{\text{rot}}}{K_{\text{total}}} = \frac{c}{1+c} = \frac{2/5}{1+2/5} = \frac{2}{7}$$

10.22. Assume negligible drag and no slipping. The object that reaches the bottom of the incline first will be the one with the lowest moment of inertia (that is, with the least resistance to rotation). The moments of inertia for the given objects are as follows: Thin ring: $I_{\text{r}} = MR^2$; Solid sphere: $I_{\text{ss}} = \frac{2}{5}MR^2$; Hollow sphere: $I_{\text{hs}} = \frac{2}{3}MR^2$; Homogeneous disk: $I_{\text{d}} = \frac{1}{2}MR^2$. Therefore, the order of the moments of inertia from smallest to greatest (assuming equal mass and radius) is I_{ss}, I_{d}, I_{hs} and I_{r}. Therefore, the order of finish of the objects in the race is: First: solid sphere; Second: homogeneous disk; Third: hollow sphere; Last: thin ring.

10.23. The net translational and rotational forces on both the solid sphere and the thin ring are, respectively, $F_{\text{net}} = ma = mg\sin\theta - f_{\text{static}}$ and $\tau = f_{\text{static}}r = I\alpha$, where the angular acceleration is $\alpha = a/r$. Since the moment of inertia for the solid sphere is $I = \left(2mr^2\right)/5$, the force of static friction is given by $f_{\text{static}} = 2ma/5$. Substitute this expression into the net force equation to solve for the acceleration of the solid sphere: $a_{\text{ss}} = 5g\left(\sin\theta\right)/7$. The moment of inertia for the thin ring is $I = mr^2$. Therefore, the force of static friction in this case is given by $f_{\text{static}} = ma$. Substitute this expression into the net force equation to solve for the acceleration of the thin ring: $a_{\text{r}} = g\left(\sin\theta\right)/2$. Therefore, the ratio of the acceleration is:

$$\frac{a_{\text{r}}}{a_{\text{ss}}} = \frac{\left(g\sin\theta\right)/2}{\left(5g\sin\theta\right)/7} = \frac{7}{10}.$$

10.24. The net translational and rotational forces on the solid sphere on the incline are, respectively, $F_{\text{net}} = ma = mg\sin\theta - f_{\text{static}}$ and $\tau = f_{\text{static}}r = I\alpha$, where the angular acceleration is $\alpha = a/r$. Since the moment of inertia for the solid sphere is $I = \left(2mr^2\right)/5$, the force of static friction is given by $f_{\text{static}} = 2ma/5$. Substituting this expression into the net force equation to solve for the acceleration gives $a = 5g\left(\sin\theta\right)/7$. Thus, $f_{\text{static}} = 2ma/5 = 2mg\left(\sin\theta\right)/7$. The limiting friction corresponding to a coefficient of static friction, μ_{s}, is $\max\left\{f_{\text{static}}\right\} = \mu_{\text{s}}F_{\text{normal}} = \mu_{\text{s}}mg\cos\theta$. For rolling without slipping to take place, it is required that

$$f_{\text{static}} \le \max\{f_{\text{static}}\} \Rightarrow \frac{2mg\sin\theta}{7} \le \mu_s mg\cos\theta. \text{ Therefore, } \tan\theta \le \frac{7\mu_s}{2} \Rightarrow \theta \le \tan^{-1}\left(\frac{7\mu_s}{2}\right).$$

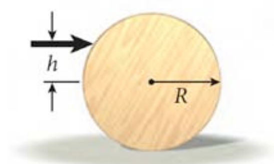
Thus, the maximum angle for which the sphere will roll without slipping is $\theta = \tan^{-1}\left(\frac{7\mu_s}{2}\right).$

10.25. The "sharp horizontal blow" means that a force of magnitude F acts horizontally on the object along the arrow in the figure. With this force, we have to apply Newton's Second Law for linear motion ($F = Ma$) and Newton's Second Law for rotation ($\tau = I\alpha$). According to the problem text, the round object rolls without slipping. In Section 10.3 we have learned that this condition implies $v = R\omega$ and $a = R\alpha$. The force exerts a torque of magnitude $\tau = Fh$ (= magnitude of force times perpendicular distance) around the center of mass of the round object. Using all these relationships we can write

$$\tau = Fh = (Ma)h = M(R\alpha)h = MRh\alpha = I\alpha.$$

Simplifying and rearranging, we get

$$I = MRh \quad\Rightarrow\quad \frac{I}{MR^2} = \frac{h}{R}.$$



As a double-check, let's calculate h assuming a uniform solid sphere

$$h = \frac{RI}{MR^2} = \frac{R\left(\frac{2}{5}MR^2\right)}{MR^2} = \frac{2}{5}R.$$

So, for example, you might strike a cue ball with a horizontal cue a distance of $2R/5$ above the center of the cue ball to start it rolling without sliding.

10.26. (a) Since the path of the projectile is not a straight line about the origin (which would give an angular momentum of zero), the angular momentum can be determined by considering that the velocity of the projectile changes continuously along its path because of the change in the vertical component of velocity under the gravitational pull. If θ is the angle of projection, the horizontal component of velocity, $v_0\cos\theta_0$, remains unchanged throughout the path and at the maximum height, the vertical component of velocity is zero and it has only the horizontal $v_0\cos\theta_0$. The 'lever arm' for angular momentum at the maximum height is the maximum height itself, $\left(v_0^2\sin^2\theta_0\right)/2g$ so that the angular momentum is

$$L = \frac{\left(mv_0\cos\theta_0\right)\left(v_0^2\sin^2\theta_0\right)}{2g}.$$

Since the angular momentum is conserved in this case, the angular momentum above is the same throughout the path.

(b) Since the angular momentum does not change throughout the path, the rate of change is zero.

(c) The rate of change of this angular momentum is the net torque about the origin, which also equals zero, that is:

$$\tau = \frac{dL}{dt} = \frac{d(0)}{dt} = 0.$$

10.27. For each object we convert the initial potential energy into kinetic energy at the bottom of the ramp.

Sphere: $Mgh_0 = \tfrac{1}{2}Mv^2(1+c_{\text{sphere}}) \Rightarrow v^2 = 2gh_0/(1+c_{\text{sphere}})$

Cylinder: $Mgh = \tfrac{1}{2}Mv^2(1+c_{\text{cylinder}}) \Rightarrow v^2 = 2gh/(1+c_{\text{cylinder}})$

If the speed is to be the same in both cases, this means:

$$2gh/(1+c_{\text{cylinder}}) = 2gh_0/(1+c_{\text{sphere}}) \Rightarrow$$

$$h = h_0\frac{1+c_{\text{cylinder}}}{1+c_{\text{sphere}}} = h_0\frac{1+1/2}{1+2/5} = h_0\frac{3/2}{7/5} = h_0\frac{15}{14}$$

10.28. To open a door (that is, to rotate a door about the hinges), a force must be applied so as to produce a torque about the hinges. Recall that torque is defined as $\vec{\tau} = \vec{r}\times\vec{F}$. The magnitude of this torque is then given by $\tau = rF\sin\theta$, where θ is the angle between the force applied at a point *p*, and the vector connecting the point *p* to the axis of rotation (to the axis of the hinges in this case). Therefore, torque is maximal when the applied force is perpendicular to the vector $\vec{r}$. That is, when the force is perpendicular to the plane of the door. Similarly, torque is minimal (i.e. zero) when the applied force is parallel to $\vec{r}$ (i.e. along the plane of the door).

10.29. Angular momentum is conserved, however, energy is not conserved; her muscles must provide an additional centripetal acceleration to her hands to pull them inwards. That force times the displacement is equal to the work that she does in pulling them in. Since she is doing work on the system, energy is not conserved.

10.30. Consider a particle of constant mass, *m*, which starts at position, r_0, moving with velocity, *v*, and having no forces acting on it. By Newton's first law of motion, the absence of forces acting on it means that it must continue to move in a straight line at the same speed, so its equation of motion is given by $r = r_0 + vt$. Its linear momentum is *mv*, so its angular momentum relative to the origin is given by $\vec{L} = \vec{r}\times m\vec{v} = (\vec{r}_0 + \vec{v}t)\times m\vec{v}$. The cross product is distributive over addition, so this can be rewritten as $\vec{L} = (\vec{r}_0\times m\vec{v}) + (\vec{v}t\times m\vec{v})$. Clearly the vectors $\vec{v}t$ and $m\vec{v}$ are parallel, since they are both in the direction of $\vec{v}$, and the cross product of two parallel vectors is zero. So, the last term in the sum above comes to zero, and the expression can be rewritten as $\vec{L} = \vec{r}_0\times m\vec{v}$. Now $\vec{r}_0$, *m* and $\vec{v}$ are all constants in this system, so it follows that $\vec{L}$ is also constant, as required by the law of conservation of angular momentum. Therefore, whether or not the particle has angular momentum is dependent on the r_0 vector, given non-zero velocity. If the path of the particle crosses the origin, $r_0 = 0$ and the particle has no angular momentum relative to the origin. In every other case, the particle will have constant, non-zero angular momentum relative to the origin.

10.31. Work is given by $W = Fd\cos\theta$ for linear motion, and by $W\tau = \theta$ for angular motion, where the torque, τ, is applied through a revolution of θ.

(a) Gravity points downward, therefore, $W_{\text{gravity}} = mg(s)\sin\theta$.

(b) The normal force acts perpendicular to the displacement. Therefore, $\cos 90 = 0 \Rightarrow W_{\text{normal}} = 0$.

(c) The frictional force considered in this problem is that of static friction since the cylinder is rolling without slipping. The direction of the static friction is opposite to that of the motion. Work done by the frictional force consists of two parts; one is the contribution by translational motion and the other is the contribution by rotational motion.

(i) Translational motion: $W_{\text{translation}} = (f_s)(s)\cos(180°) = -f_s s$

(ii) Rotational motion: $W\tau_{\text{rotational}}\ Rf\ \theta = (f\ s_s)\theta = \ _s$

Therefore, the total contribution to work done by the friction is zero.

10.32. Mechanical energy is conserved for the rolling motion without slipping. Setting the top as the vertical origin, the mechanical energy at the top is $E_t = K + U = 0 + 0.$ The mechanical energy at the bottom is $E_b = K(\text{translational} + \text{rotational}) + U = \left[(1+c)mv^2/2\right] + (-mgh),$ where h is the vertical height. By conservation of energy, $E_t = E_b$ implies $v = \sqrt{2gh/(1+c)} = \sqrt{\frac{4gh}{3}}, \sin\theta = \frac{h}{s} \Rightarrow v = \sqrt{\frac{4}{3}sg\sin\theta},$ where c is 1/2 for the cylindrical object. Using $v_f^2 - v_i^2 = 2as,$ the acceleration is

$$a = \frac{2gh/(1+c)}{2s} = \frac{2gs\sin\theta/(1+c)}{2S} = g\frac{\sin\theta}{(1+c)}.$$

For a cylinder, c = 1/2. Therefore, $a = \frac{2}{3}g\sin\theta.$

10.33. Prove that the pivot point about which the torque is calculated is arbitrary. First, consider the definition of torque, $\vec{\tau} = \vec{r} \times \vec{F}.$ Therefore, for each of the applied forces, $\vec{F}_1$ and $\vec{F}_2 = -\vec{F}_1,$ their contributions to the torque are given by $\vec{\tau}_1 = \vec{r}_1 \times \vec{F}_1$ and $\vec{\tau}_2 = \vec{r}_2 \times \vec{F}_2,$ where $\vec{r}_1$ and $\vec{r}_2$ are the respective distances to the pivot point. The net torque is calculated from the algebraic sum of the torque contributions, that is, $\vec{\tau}_{\text{net}} = \vec{\tau}_1 + \vec{\tau}_2 = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 = \vec{r}_1 \times \vec{F}_1 - \vec{r}_2 \times \vec{F}_1.$ Since the cross product is distributive, $\vec{\tau}_{\text{net}} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_1 = \vec{d} \times \vec{F}_1.$ Therefore, the net torque produced by a couple depends only on the distance between the forces, and is independent of the actual pivot point about which the contributing torques are calculated or the actual points where the two forces are applied.

10.34. It is actually the act of pulling in her arms that makes the figure skater increase her angular velocity. Since angular momentum is conserved $(I\omega_1 = I_2\omega_2 = \text{constant})$, by reducing her rotational inertia (by means of reducing the distance between her arms and hands to the axis of rotation), the figure skater increases her angular velocity.

10.35. By momentarily turning the handlebars to the left, the contact point of the motorcycle with the ground moves to the left of the center of gravity of the motorcycle so that the motorcycle leans to the right. Now the motorcycle can be turned to the right and the rider can lean to the right. The initial left turn creates a torque that is directed upwards, which deflects the angular momentum of the front tire upward and causes the motorcycle to lean to the right. In addition, as the motorcycle leans to the right, a forward pointing torque is induced that tends to straighten out the front wheel, preventing over-steering and oscillations. At low speeds, these effects are not noticeable, but at high speeds they must be considered.

10.36. The Earth-Moon system, to a good approximation, conserves its angular momentum (though the Sun also causes tides on the Earth). Thus, if the Earth loses angular momentum, the moon must gain it. If there were 400 days in a year in the Devonian period, the day was about 10% shorter, meaning the angular velocity of the Earth was about 10% greater. Since the rotational inertia of the Earth is virtually unchanged, this means that the rotational angular momentum of the Earth was then about 10% greater, and correspondingly the orbital angular momentum of the Moon was about 10% less.

10.37. In this problem, the key is that the monkey is trying to reach the bananas by climbing the rope. Since the monkey has the same mass as the bananas, if he didn't try to climb the rope, both the net torque and total angular momentum on the pulley would be zero. Take counterclockwise to be positive just for aesthetics.

(a) Consider the extra tension provided by the monkey on the rope by climbing (i.e. by pulling on the rope). Average out the force caused by the monkeys pulling with a constant force downwards, $\vec{T},$ on the monkey side. Therefore, the net torque on the pulley axis is provided by this extra force, $\vec{T},$ as

$$\vec{\tau}_{\text{net}} = \left[\left(\vec{F}_{\text{monkey}} + \vec{T}\right) - \vec{F}_{\text{banana}}\right]R = \vec{T}R.$$

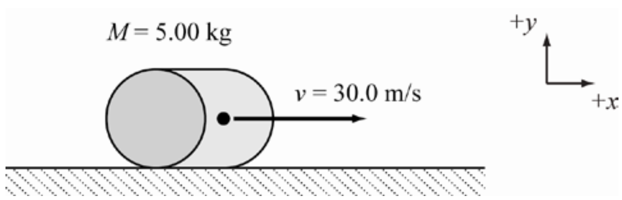
(b) Since there is now a non-zero net torque on the pulley, there is a non-zero total angular momentum given by

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \Rightarrow \vec{L} = \int \tau dt.$$

Using the results of part (a), the above expression can be rewritten as $\vec{L} = \int \vec{T}Rdt = \vec{T}Rt.$ Recall that the extra "climbing" force was taken to be constant. In reality, the monkey's pulling will be time dependent and this will affect the final form of the time dependent angular momentum.

Exercises

10.38. **THINK:** Determine the energy of a solid cylinder as it rolls on a horizontal surface. The mass of the cylinder is $M = 5.00$ kg and the translational velocity of the cylinder's center of mass is $v = 30.0$ m/s.
SKETCH:



RESEARCH: Since the motion occurs on a horizontal surface, consider only the total kinetic energy of the cylinder, $K_{\text{T}} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$. $I = cMR^2$ and, for a solid cylinder, $c = 1/2$ as in Table 10.1. For rolling without slipping, $v = \omega R$.

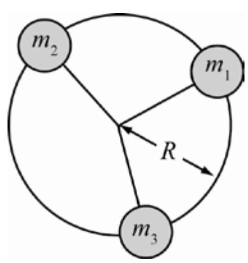
SIMPLIFY: $K_{\text{T}} = \frac{1}{2}Mv^2 + \frac{1}{2}\left(cMR^2\right)\left(\frac{v^2}{R^2}\right) = \frac{1}{2}Mv^2\left(1+c\right)$

CALCULATE: $K_{\text{T}} = \frac{1}{2}\left(5.00 \text{ kg}\right)\left(30.0 \text{ m/s}\right)^2\left(1+1/2\right) = 3375 \text{ J}$

ROUND: Both given values have three significant figures, so the result is rounded to $K_{\text{T}} = 3.38 \cdot 10^3$ J.

DOUBLE-CHECK: The calculated result has Joules as units, which are units of energy. This means that the calculated result is plausible.

10.39. **THINK:** The children can be treated as point particles on the edge of a circle and placed so they are all the same distance, R, from the center. Using the conversion, 1 kg = 2.205 lbs, the three masses are $m_1 = 27.2$ kg, $m_2 = 20.4$ kg and $m_3 = 36.3$ kg. Using the conversion 1 m = 3.281 ft, the distance is $R = 3.657$ m.
SKETCH:



RESEARCH: The moment of inertia for point particles is given by $I = \sum_i m_i r_i^2$.

SIMPLIFY: $I = \left(m_1 + m_2 + m_3\right)R^2$

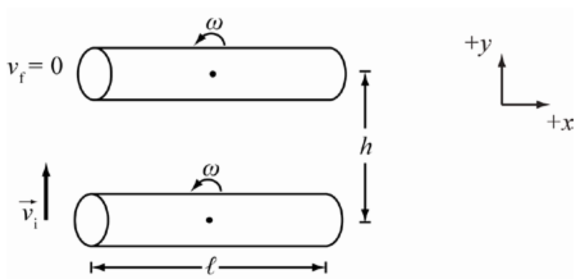
CALCULATE: $I = \left(27.2 \text{ kg} + 20.4 \text{ kg} + 36.3 \text{ kg}\right)\left(3.66 \text{ m}\right)^2 = 1123.9 \text{ kg m}^2$

ROUND: $I = 1.12 \cdot 10^3$ kg m^2

DOUBLE-CHECK: Since the children are located on the edge of the merry-go-round, a large value for I is expected.

10.40. **THINK:** Since the pen with a length of l = 24 cm rotates at a constant rate, rotational kinetic energy remains constant so only the translational energy is converted to potential energy at a height of h = 1.2 m from release. Use kinematics to determine the time it takes the pen to reach the top and make 1.8 revolutions, in order to determine ω.

SKETCH:



RESEARCH: The pen has a translational kinetic energy of $K_T = mv^2/2$, where v_i is the velocity at release. The potential energy at the top is given by $U = mgh$ and $mgh = mv_i^2/2$. The rotational kinetic energy is given by $K_R = I\omega^2/2$, where $I = ml^2/12$. The initial velocity of the pen is determined from $v_f^2 = v_i^2 - 2gh$ and the time of flight is given by $t = -(v_f - v_i)/g$. Angular velocity is given by $\omega = 2\pi(1.8 \text{ rev})/t$.

SIMPLIFY: The final velocity is zero, so the expression reduces to $0 = v_i^2 - 2gh \Rightarrow v_i = \sqrt{2gh}$. The expression for the time of flight also reduces to

$$t = -\frac{(0 - v_i)}{g} = \sqrt{\frac{2h}{g}}.$$

The angular velocity is given by $\omega = \dfrac{3.6\pi}{\sqrt{2h/g}} = 3.6\pi\sqrt{\dfrac{g}{2h}}$ rad/s. The ratio is then:

$$\frac{K_R}{K_T} = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}mv^2} = \frac{\frac{1}{2}\left(\frac{1}{12}ml^2\right)(3.6\pi)^2\frac{g}{2h}}{mgh} = \frac{12.96}{48}\pi^2\frac{l^2}{h^2}.$$

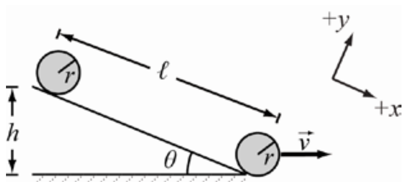
CALCULATE: $\dfrac{K_R}{K_T} = \dfrac{12.96}{48}\pi^2\left(\dfrac{0.24 \text{ m}}{1.2 \text{ m}}\right)^2 = 0.1066$

ROUND: To three significant figures: $\dfrac{K_R}{K_T} = 0.107.$

DOUBLE-CHECK: Remember that this is not a case of rolling without slipping. Here the translational motion is independent of the rotational motion, and so the ratio between translational and rotational kinetic energies could have almost any value. A simple double-check is thus not easily possible.

10.41. **THINK:** With no friction and no slipping, each object with mass, m = 1.00 kg, conserves energy. Since each object starts at the same height, they all have the same potential energy and hence kinetic energy after they travel a distance l = 3.00 m at an incline of $\theta = 35.0°$. Each ball has a radius of r = 0.100 m. Whichever object has the highest velocity at the bottom should reach the bottom first, and vice versa.

SKETCH:



RESEARCH: The constant c is related to the geometry of a figure. The values of c for different objects can be found in Table 10.1. The solid sphere has $c_1 = 2/5$, the hollow sphere has $c_2 = 2/3$ and the ice cube has $c_3 = 0$. Since energy is conserved, the velocity of each object at the bottom is $v = \sqrt{2gh/(1+c)}$. The incline shows that $h = l\sin\theta$.

SIMPLIFY: $v_1 = \sqrt{\dfrac{2gl\sin\theta}{1+c_1}}$, $v_2 = \sqrt{\dfrac{2gl\sin\theta}{1+c_2}}$, $v_3 = \sqrt{\dfrac{2gl\sin\theta}{1+c_3}}$

CALCULATE: $v_1 = \sqrt{\dfrac{10}{7}}\sqrt{gl\sin\theta}$, $v_2 = \sqrt{\dfrac{6}{5}}\sqrt{gl\sin\theta}$, $v_3 = \sqrt{2}\sqrt{gl\sin\theta}$

(a) Since the velocity is inversely proportional to c, the object with a smaller c will have a larger velocity than that of one with a greater c, and will reach the end first. Since $c_1 < c_2$, the solid sphere reaches the bottom first.

(b) Since $c_3 < c_1$, and the velocity is inversely proportional to c, the ice cube travels faster than the solid ball at the base of the incline.

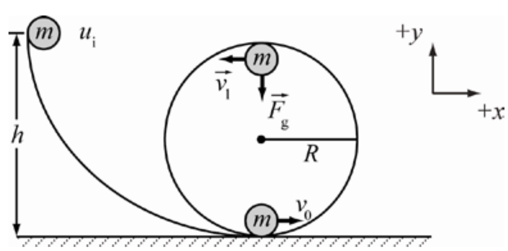
(c) $v_1 = \sqrt{\dfrac{10}{7}}\sqrt{\left(9.81 \text{ m/s}^2\right)\left(3.00 \text{ m}\right)\sin\left(35.0^\circ\right)} = 4.911 \text{ m/s}$

ROUND: Parts (a) and (b) do not need to be rounded. (c) $v_1 = 4.91$ m/s

DOUBLE-CHECK: It is reasonable that the ice cube reaches the bottom first since it does not have to contribute any energy to rotational motion. As expected, the velocity of the sphere is less than it would be if it were in freefall $\left(v = \sqrt{2gl} = 8 \text{ m/s}\right)$.

10.42. **THINK:** With no friction and no slipping, the object of mass, m, and radius, r, will conserve energy. Therefore, the potential energy of the ball at height, h, should equal the potential energy at the top of the loop of radius, R, plus translational and rotational kinetic energy. For the ball to complete the loop, the minimum velocity required is the one where the normal force of the loop on the ball is 0 N, so that the centripetal force is solely the force of gravity on the ball.

SKETCH:



RESEARCH: The only force on the ball at the top of the loop is $F_g = mg = mv_1^2/R$. The initial potential energy is given by $U_i = mgh$ and the final potential energy is given by $U_f = mg(2R)$. The kinetic energy at the top of the loop is $K = (1+c)mv_1^2/2$, where the c value for a solid sphere is 2/5.

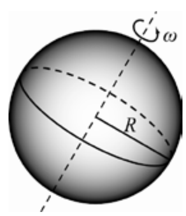
SIMPLIFY: The conservation of energy is given by

$$U_{\mathrm{i}} = U_{\mathrm{f}} + K \;\Rightarrow\; mgh = 2mgR + \frac{1}{2}(1+c)mv_1^{\,2}.$$

From the forces, $mg = \dfrac{mv_1^{\,2}}{R} \;\Rightarrow\; v_1^{\,2} = gR.$ Therefore, $mgh = 2mgR + \dfrac{1}{2}\left(\dfrac{7}{5}\right)mgR \Rightarrow h = R\left(2 + \dfrac{7}{10}\right) = \dfrac{27}{10}R.$

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: The height is greater than $2R$, which neglecting rotational energy would be the minimum energy needed, so the result is reasonable.

10.43. **THINK:** The change in energy should be solely that of the change in rotational kinetic energy. Assume the pulsar is a uniform solid sphere with $m \approx 2\cdot 10^{30}$ kg and $R = 12$ km. Initially, the pulsar rotates at $\omega = 60\pi$ rad/s and has a period, T which is increased by 10^{-5} s after 1 y. We calculate the power emitted by the pulsar by taking the time derivative of the rotational kinetic energy. The power output of the Sun is $P_{\mathrm{Sun}} = 4\cdot 10^{26}$ W.
SKETCH:



RESEARCH: The kinetic energy is given by $K = I\omega^2/2,$ so

$$P_{\mathrm{Crab}} = -\frac{dK}{dt} = -I\omega\frac{d\omega}{dt}.$$

The angular velocity is given by $\omega = 2\pi/T,$ so

$$\frac{d\omega}{dt} = -\frac{2\pi}{T^2}\frac{dT}{dt} = -\frac{2\pi}{\left(\dfrac{2\pi}{\omega}\right)^2}\frac{dT}{dt} = -\frac{\omega^2}{2\pi}\frac{dT}{dt}.$$

The moment of inertia of a sphere is

$$I = \frac{2}{5}mR^2.$$

SIMPLIFY: Combining our equations gives us

$$P_{\mathrm{Crab}} = \frac{2}{5}mR^2\omega\frac{\omega^2}{2\pi}\frac{dT}{dt} = \frac{mR^2\omega^3}{5\pi}\frac{dT}{dt}.$$

CALCULATE: First we calculate the change in the period over one year,

$$\frac{dT}{dt} = \frac{10^{-5}\text{ s}}{(365\text{ days})(24\text{ hour/day})(3600\text{ s/hour})} = 3.17\cdot 10^{-13}.$$

The power emitted by the pulsar is

$$P_{\mathrm{Crab}} = \frac{dK}{dt} = \frac{\left(2\cdot 10^{30}\text{ kg}\right)\left(12\cdot 10^{3}\text{ m}\right)^2\left(60\pi\,\text{rad/s}\right)^3}{5\pi}\left(3.17\cdot 10^{-13}\right) = 3.89\cdot 10^{31}\text{ W}.$$

So the ratio of the power emitted by the pulsar to the power emitted by the Sun is

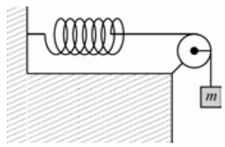
$$\frac{P_{\mathrm{Crab}}}{P_{\mathrm{Sun}}} = \frac{3.89\cdot 10^{31}\text{ W}}{4\cdot 10^{26}\text{ W}} = 9.73\cdot 10^{4}.$$

ROUND: $\dfrac{P_{\mathrm{Crab}}}{P_{\mathrm{Sun}}} = 1\cdot 10^{5}.$

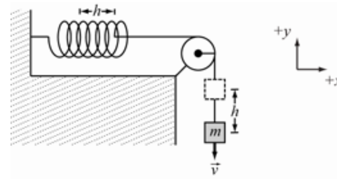
DOUBLE-CHECK: Our result for the ratio of the loss in rotational energy of the Crab Pulsar is close to the expected value of 100,000.

10.44. **THINK:** With no friction and no slipping, mechanical energy is conserved. This means that the potential energy of the block of mass m = 4.00 kg will be converted into the potential energy of the spring with a constant of k = 32.0 N/m, kinetic energy of the block and rotational energy of the pulley with a radius of R = 5.00 cm and mass M = 8.00 kg. If the block falls a distance h, then the spring is extended by a distance h as well. Consider the lower position of the block in parts (a) and (b) to be at zero potential. In part (a), the block falls a distance h = 1.00 m. In part (b), when the block comes to rest, the kinetic energy of the system is zero so that the block's potential energy is converted entirely into spring potential.

SKETCH: (a)



(b)



RESEARCH: The initial energy of the system is $E_\text{i} = U_\text{i} = mgh.$

(a) The final energy is $E_\text{f} = \frac{1}{2}k\left(x_0 - h\right)^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2.$

(b) The final energy is $E_\text{f} = k\left(x-h\right)^2/2.$ The moment of inertia of the wheel is $MR^2/2.$ With no slipping, $R\omega = v.$ Let $x_0 = 0$ for the spring equilibrium.

SIMPLIFY:

(a) $E_\text{i} = E_\text{f} \;\Rightarrow\; mgh = \frac{1}{2}kh^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v^2}{R^2}\right) + \frac{1}{2}mv^2.$ Therefore,

$$mgh - \frac{1}{2}kh^2 = \left(\frac{1}{4}M + \frac{1}{2}m\right)v^2 \;\Rightarrow\; v = \sqrt{\frac{mgh - \frac{1}{2}kh^2}{\frac{1}{4}M + \frac{1}{2}m}}.$$

(b) $E_\text{i} = E_\text{f} \;\Rightarrow\; mgh = \frac{1}{2}kh^2 \;\Rightarrow\; h = \frac{2mg}{k}$

CALCULATE:

(a) $$v = \sqrt{\frac{(4.00\text{ kg})(9.81\text{ m/s}^2)(1.00\text{ m}) - \frac{1}{2}(32.0\text{ N/m})(1.00\text{ m})^2}{\frac{1}{4}(8.00\text{ kg}) + \frac{1}{2}(4.00\text{ kg})}} = 2.410\text{ m/s}$$

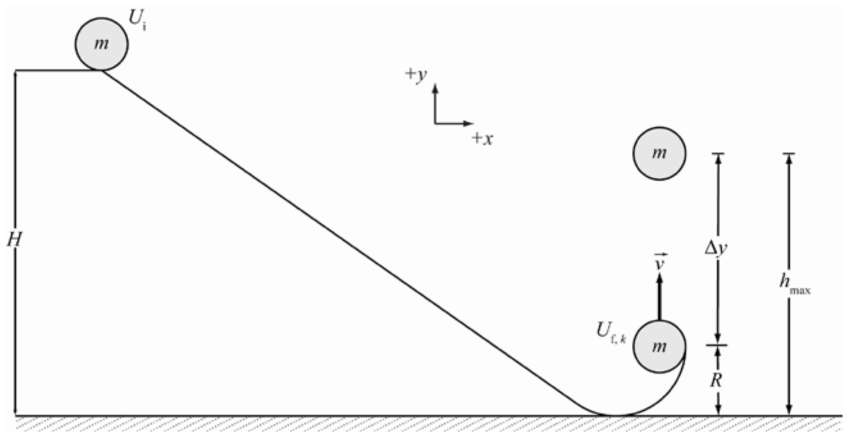
(b) $$h = \frac{2(4.00\text{ kg})(9.81\text{ m/s}^2)}{32.0\text{ N/m}} = 2.45\text{ m}$$

ROUND: Three significant figures:
(a) The block has a speed of v = 2.41 m/s after it has fallen 1.00 m.
(b) The maximum extension of the spring is h = 2.45 m.

DOUBLE-CHECK: For part (a), the speed should be less than it is for free fall $\left(v = \sqrt{2gh} = 4.4\text{ m/s}\right)$, which it is. For part (b), the distance is reasonable.

10.45. **THINK:** With no friction and no slipping, the energy of the object of mass, m, and radius, r, is conserved. This means that the initial potential energy at height H = 6.00 m is equal to the potential energy at height R = 2.50 m, plus its rotational and translational energy. The object has a c value of 0.400. Using conservation of energy, the velocity of the object can be determined. Then, using kinematics, the maximum height the object achieves can be determined. Let the subscript i indicate the ball is at the top of the ramp, and the subscript f indicate the ball is at the end of the ramp, at the launch point.

SKETCH:



RESEARCH: The initial energy of the ball is $E_i = U_i = mgH$. The final energy of the ball is $E_f = U_f + K \Rightarrow E_f = mgR + \left[(1+c)mv^2/2\right]$, where c = 0.400. The kinematics equation for the velocity is $v_f^2 = v_i^2 + 2g\Delta y$. Since the ball is at rest at the top, the equation becomes $v^2 = 2g\Delta y \Rightarrow \Delta y = v^2/2g$. The maximum height achieved is $\Delta y + R = h_{max}$.

SIMPLIFY: $E_i = E_f \Rightarrow mgH = mgR + \frac{1}{2}(1+c)mv^2 \Rightarrow mg(H-R) = \frac{1}{2}(1+c)mv^2 \Rightarrow v^2 = \frac{2g(H-R)}{1+c}$

$$h_{max} = \Delta y + R = \frac{v^2}{2g} + R = \frac{(H-R)}{1+c} + R$$

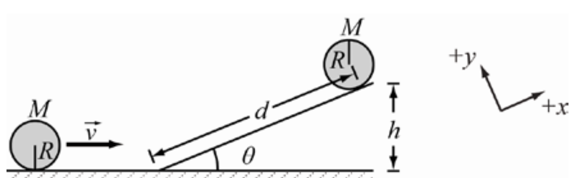
CALCULATE: $h_{max} = \frac{(6.00 \text{ m} - 2.50 \text{ m})}{1+0.400} + 2.50 \text{ m} = 5.000 \text{ m}$

ROUND: $h_{max} = 5.00 \text{ m}$

DOUBLE-CHECK: If the object did not rotate, the mass is expected to reach its original height of 6 m. Since the object does rotate, the height it reaches should be less than the original height.

10.46. **THINK:** In both cases, energy should be conserved. In part (a), if the ball of mass, *M*, and radius, *R*, continues to spin at the same rate, then there is no change in rotational kinetic energy and only the translational energy is converted to potential energy. In part (b), there is slipping so both rotational and translational kinetic energy are converted to potential energy. The ball has an initial velocity of 3.00 m/s and travels a distance, *d*, up an incline with an angle of $\theta = 23.0°$.

SKETCH:



RESEARCH:

(a) The initial translational kinetic energy is given by $K_T = mv^2/2$, the initial rotational energy is given by $K_R = \frac{1}{2}I\omega^2$, and the final potential energy is given by $U_f = mgh$.

(b) The initial kinetic energy is $K = (1+c)mv^2/2$ and the final potential energy is $U_f = mgh$. The height of the ball is given by the expression $h = d\sin\theta$, and c = 2/5 for a solid sphere.

SIMPLIFY:

(a) $E_\text{i} = E_\text{f} \Rightarrow \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgd\sin\theta + \frac{1}{2}I\omega^2 \Rightarrow d = \frac{1}{2}\frac{v^2}{g\sin\theta}$

(b) $E_\text{i} = E_\text{f} \Rightarrow \frac{1}{2}(1+c)mv^2 = mgd\sin\theta \Rightarrow d = \frac{v^2(1+c)}{2g\sin\theta} = \frac{7}{10}\frac{v^2}{g\sin\theta}$

CALCULATE:

(a) $d = \dfrac{(3.00\text{ m/s})^2}{2(9.81\text{ m/s}^2)\sin(23.0°)} = 1.174\text{ m}$

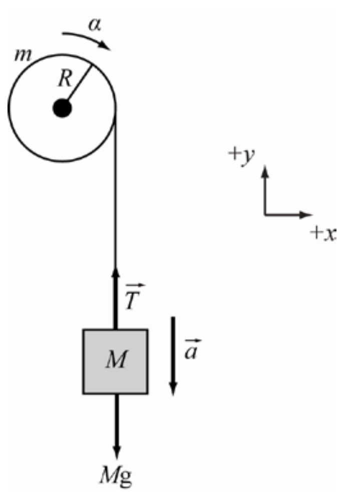
(b) $d = \dfrac{7(3.00\text{ m/s})^2}{10(9.81\text{ m/s}^2)\sin(23.0°)} = 1.644\text{ m}$

ROUND:
Rounding to three significant figures:
(a) d = 1.17 m
(b) d = 1.65 m

DOUBLE-CHECK: Since the ball in part (b) does contribute rotational energy to the potential, it is expected to go higher up the ramp and hence have a larger value for d.

10.47. **THINK:** The hanging block with a mass of M = 70.0 kg, will cause a tension, T, in the string that will in turn produce a torque, τ, in the wheel with a mass, m = 30.0 kg, and a radius, R = 40.0 cm. This torque will give the wheel an angular acceleration, α. If there is no slipping, then the angular acceleration of the wheel is directly related to the acceleration of the block.

SKETCH:



RESEARCH: The balance of forces is given by $T - Mg = -Ma$. The torque produced by the tension, T, is given by $\tau = TR = I\alpha$, where I of the wheel is $mR^2/2$. With no slipping, $R\alpha = a$.

SIMPLIFY: First, determine the tension, $T \Rightarrow T = M(g-a)$. This expression can be substituted into the torque equation to solve for a:

$$M(g-a)R = \frac{1}{2}mR^2\left(\frac{a}{R}\right) \Rightarrow MgR - MaR = \frac{1}{2}mRa \Rightarrow Mg = \left(\frac{1}{2}m + M\right)a \Rightarrow a = \frac{Mg}{\frac{1}{2}m + M}.$$

CALCULATE: $a = \dfrac{70.0\text{ kg}(9.81\text{ m/s}^2)}{\frac{1}{2}(30.0\text{ kg}) + 70.0\text{ kg}} = 8.079\text{ m/s}^2$

ROUND: $a = 8.08 \text{ m/s}^2$

DOUBLE-CHECK: Since there is tension acting opposite gravity, the overall acceleration of the hanging mass should be less than *g*.

10.48. **THINK:** The torque is simply the cross product of the vectors, $\vec{r} = (4\hat{x} + 4\hat{y} + 4\hat{z})$ m and $\vec{F} = (2\hat{x} + 3\hat{y})$ N.

SKETCH: Not applicable.

RESEARCH: $\vec{\tau} = \vec{r} \times \vec{F}$

SIMPLIFY: $\vec{\tau} = (4\hat{x} + 4\hat{y} + 4\hat{z}) \times (2\hat{x} + 3\hat{y})$ N m

$= \left[8(\hat{x} \times \hat{x}) + 12(\hat{x} \times \hat{y}) + 8(\hat{y} \times \hat{x}) + 12(\hat{y} \times \hat{y}) + 8(\hat{z} \times \hat{x}) + 12(\hat{z} \times \hat{y})\right]$ N m

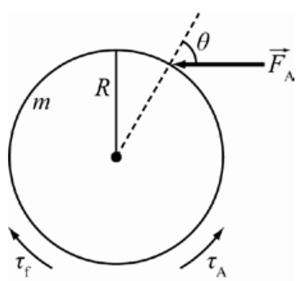
CALCULATE: $\vec{\tau} = \left[8(0) + 12(\hat{z}) + 8(-\hat{z}) + 12(0) + 8(\hat{y}) + 12(-\hat{x})\right]$ N m $= (-12\hat{x} + 8\hat{y} + 4\hat{z})$ N m

ROUND: Not applicable.

DOUBLE-CHECK: The magnitude of the calculated torque is about 15. As required, this number is smaller than (or at most equal to) the product of the magnitudes of the force and the position vectors, which is about 25 in this case.

10.49. **THINK:** There are two forces, $F_1 = 70.0$ N (applied from 0 to 2.00 seconds) and $F_2 = 24.0$ N (applied after 2.00 seconds). These forces are applied at an angle of $\theta = 37.0°$ on the surface of a disk of mass, *m* = 14.0 kg, and diameter of *d* = 30.0 cm (radius, *R* = 15.0 cm). After 2.00 seconds, the disk moves at a constant angular speed, ω. This means that the sum of the torques is zero, so the torque produced by friction is equal and opposite the torque produced by the applied force. Assuming the frictional torque is constant, the angular acceleration, α, of the disk from 0 to 2.00 seconds can be calculated and ω can be determined.

SKETCH:



RESEARCH: The torque that F_A produces is $\tau_A = RF_A \sin\theta$. After *t* = 2.00 s, when ω is constant, $\sum \tau = 0 = \tau_A - \tau_f$, where τ_f is the frictional torque. For *t* = 0 to *t* = 2 s, $\sum \tau = \tau_A - \tau_f = \tau_{net}$ and $\tau_{net} = I\alpha$. Starting from rest, $\omega = \alpha t$. The rotational kinetic energy of the wheel after *t* = 2.00 s is then $K = I\omega^2/2$, where $I = mR^2/2$.

SIMPLIFY:

(a) $\sum \tau = \tau_A - \tau_f = 0 \Rightarrow \tau_A = \tau_f = RF_2 \sin\theta$

(b) $\tau_{net} = \tau_A - \tau_f = RF_1 \sin\theta - RF_2 \sin\theta = I\alpha \Rightarrow \alpha = \dfrac{R\sin\theta(F_1 - F_2)}{\frac{1}{2}mR^2} = \dfrac{2\sin\theta(F_1 - F_2)}{mR}$

$\omega = \alpha t = \dfrac{2\sin\theta(F_1 - F_2)}{mR}t$

(c) $K = \frac{1}{2}I\omega^2 = \frac{1}{4}mR^2\omega^2$

CALCULATE:

(a) $\tau_{\text{f}} = (0.150\text{ m})(24.0\text{ N})\sin(37.0°) = 2.167\text{ Nm}$

(b) $\omega = \dfrac{2\sin(37.0°)(70.0\text{ N} - 24.0\text{ N})(2.00\text{ s})}{(14.0\text{ kg})(0.150\text{ m})} = 52.73\text{ rad/s}$

(c) $K = \dfrac{1}{4}(14.0\text{ kg})(0.150\text{ m})^2(52.73\text{ rad/s})^2 = 218.96\text{ J}$

ROUND:

(a) $\tau_{\text{f}} = 2.17\text{ Nm}$

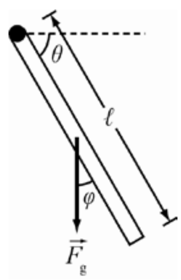
(b) $\omega = 52.7\text{ rad/s}$

(c) $K = 219\text{ J}$

DOUBLE-CHECK: Given the initial variables, these results are reasonable.

10.50. **THINK:** When the rod is at an angle of $\theta = 60.0°$ below the horizontal, the force of gravity acting at the center of mass of the rod, with mass, m = 2.00 kg, and length, l = 1.00 m, will produce a torque, τ, and hence an angular acceleration, α. If the rod has a uniform density, then the center of mass is at the geometric center of the rod.

SKETCH:



RESEARCH: From geometry it can be shown that $\theta + \varphi = 90°$. Therefore, $\varphi = 90° - \theta = 90° - 60° = 30°$. The torque that the force of gravity produces is $\tau = mg(l/2)\sin\varphi = I\alpha$, where $I = ml^2/3$.

SIMPLIFY: $mg\left(\dfrac{l}{2}\right)\sin\varphi = \dfrac{1}{3}ml^2\alpha \Rightarrow \alpha = \dfrac{3g\sin\varphi}{2l}$

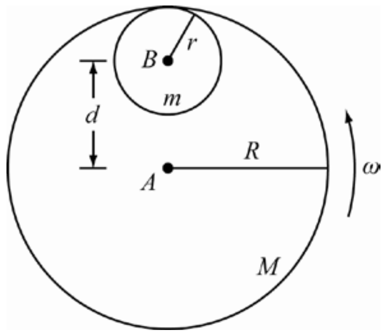
CALCULATE: $\alpha = \dfrac{3(9.81\text{ m/s}^2)\sin(30.0°)}{2(1.00\text{ m})} = 7.35\text{ rad/s}^2$

ROUND: To three significant figures: $\alpha = 7.35\text{ rad/s}^2$

DOUBLE-CHECK: The vertical component of the tangential acceleration at the end is given by $a_{\text{v}} = a_{\text{T}}\sin\theta = l\alpha\sin\theta \approx 6\text{ m/s}^2$, which is less than g. This is expected since the pivot is causing the rod to swing and the vertical displacement of the end is slowing down and a smaller acceleration is expected.

10.51. **THINK:** Each object has its own moment of inertia, I_{A} and I_{B}. Disk *A* with a mass, M = 2.00 kg, and a radius, R = 25.0 cm, rotates about its center of mass while disk *B* with a mass, m = 0.200 kg and a radius, r = 2.50 cm, rotates a distance, $d = R - r$, away from the axis. This means the parallel axis theorem must be used to determine the overall moment of inertia of disk *B*, I'_{B}. The total moment of inertia is the sum of the two. If a torque, $\tau = 0.200\text{ Nm}$, is applied then it will cause an angular acceleration, α. If the disk initially rotates at $\omega = -2\pi\text{ rad/s}$, then kinematics can be used to determine how long it takes to slow down.

SKETCH:



RESEARCH: The moment of inertia of disk A is $I_A = MR^2/2$. The moment of inertia of disk B is $I_B = mr^2/2$. Since disk B is displaced by $d = R - r$ from the axis of rotation, $I_B' = I_B + md^2$, by the parallel axis theorem. Therefore, the total moment of inertia is $I_{tot} = I_A + I_B'$. The torque that is applied produces $\tau = I_{tot}\alpha$, where $\alpha = (\omega_f - \omega_i)/\Delta t$.

SIMPLIFY:

(a) $I_{tot} = I_A + I_B' = I_A + I_B + m(R-r)^2$

$$= \frac{1}{2}MR^2 + \frac{1}{2}mr^2 + mR^2 - 2mRr + mr^2 = \left(\frac{1}{2}M + m\right)R^2 + \frac{3}{2}mr^2 - 2mRr$$

(b) $\tau = I_{tot}\alpha = \dfrac{I_{tot}(0-\omega_i)}{t} \Rightarrow t = -\dfrac{I_{tot}\omega_i}{\tau}$

CALCULATE:

(a) $I_{tot} = \left(\frac{1}{2}(2.00) + 0.200\right)(0.250)^2 + \frac{3}{2}(0.200)(0.0250)^2 - 2(0.200)(0.0250)(0.250) = 0.0726875 \text{ kg m}^2$

(b) $t = -\dfrac{(0.0726875 \text{ kg m}^2)(-2\pi \text{ rad/s})}{0.200 \text{ N m}} = 2.284 \text{ s}$

ROUND:

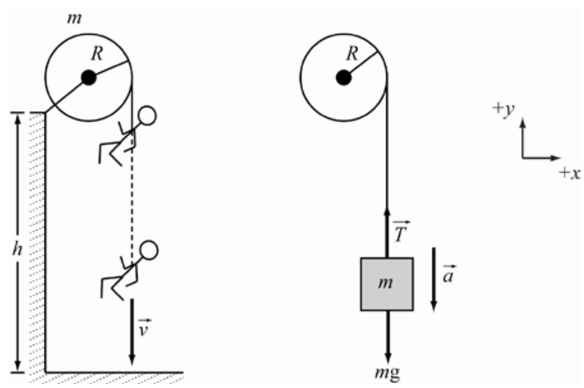
(a) $I_{tot} = 7.27 \cdot 10^{-2} \text{ kg m}^2$

(b) $t = 2.28$ s

DOUBLE-CHECK: Given the small masses and disk sizes, the moment of inertia should be small. Also, given the small torque and angular velocity, two seconds to come to a stop is reasonable.

10.52. **THINK:** The stuntman with a mass, m = 50.0 kg, will cause a tension, T, in the rope which produces a torque, τ, on the drum of mass, M = 100. kg and radius, R = 0.500 m. This torque will cause the drum to have an angular acceleration, α, and if the rope does not slip, then it will be directly related to the stuntman's translational acceleration, a. If the stuntman starts from rest and needs to accelerate to v = 4.00 m/s after dropping a height, $h = 20.0$ m, then kinematics can be used to determine the acceleration.

SKETCH:



RESEARCH: The sum of the forces yields $T - mg = -ma$. The torque produced by the tension is given by $\tau = TR = I\alpha$. With no slipping, $R\alpha = a$. The velocity of the stuntman after falling a height, h, at an acceleration of a is given by $v_{\text{f}}^2 = v_{\text{i}}^2 + 2ah$, where $v_{\text{i}} = 0$. Also, if there is no slipping, $v = \omega R$. The angle the barrel makes is given by $\omega_{\text{f}}^2 = \omega_{\text{i}}^2 + 2\alpha\Delta\theta$.

SIMPLIFY:

(a) The tension is given by $T = m(g-a)$. Therefore, the torque is given by $\tau = TR = m(g-a)R = I_0\alpha$. This implies:

$$mgR - maR = I_0\frac{a}{R} \Rightarrow mgR^2 = maR^2 + I_0 a \Rightarrow a = \frac{mgR^2}{mR^2 + I_0}.$$

(b) $v^2 = 0 + 2ah \Rightarrow a = \dfrac{v^2}{2h}$; From part (a), $I_0 = \dfrac{mgR^2 - maR^2}{a} = mR^2\left(\dfrac{g}{a} - 1\right)$.

(c) $\alpha = \dfrac{a}{R}$

(d) $\Delta\theta = \dfrac{\omega_{\text{f}}^2}{2\alpha} = \dfrac{v^2}{2\alpha R^2}$, $\#\text{revolutions} = \dfrac{\Delta\theta}{2\pi} = \dfrac{v^2}{4\pi\alpha R^2}$

CALCULATE:

(a) No calculation is necessary.

(b) $a = \dfrac{(4.00 \text{ m/s})^2}{2(20.0 \text{ m})} = 0.400 \text{ m/s}^2$, $I = (50.0 \text{ kg})(0.500 \text{ m})^2\left(\dfrac{(9.81 \text{ m/s}^2)}{0.400 \text{ m/s}^2} - 1\right) = 294.0625 \text{ kg m}^2$

(c) $\alpha = \dfrac{0.400 \text{ m/s}^2}{0.500 \text{ m}} = 0.800 \text{ rad/s}^2$

(d) $\#\text{revolutions} = \dfrac{(4.00 \text{ m/s})^2}{4\pi(0.800 \text{ rad/s}^2)(0.500 \text{ m})^2} = 6.366$

ROUND:

Rounding to thre significant figures:

(a) Not applicable.

(b) $a = 0.400 \text{ m/s}^2$ and $I = 294 \text{ kg m}^2$.

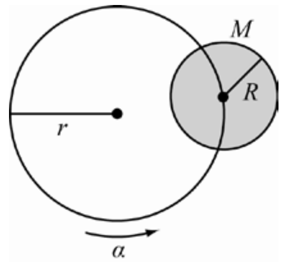
(c) $\alpha = 0.800 \text{ rad/s}^2$

(d) $\#\text{revolutions} = 6.37$

DOUBLE-CHECK: Given the large height and the small final velocity, the small accelerations and the few rotations of the drum are reasonable.

10.53. **THINK:** Since the center of mass of the tire with mass, $M = 23.5$ kg, is at a distance, $r = 1.10$ m, from the axis of rotation, the parallel axis theorem is used to determine the overall moment of inertia of the tire. Consider both cases where the tire is a thin hollow cylinder of radius, $R = 0.350$ m, and a thick hollow cylinder with radii, $R_1 = 0.300$ m and $R_2 = 0.400$ m. The torque, $\tau = 20.0$ N m, the athlete applies will cause an angular acceleration, α. Kinematics can be then be used to determine the linear speed after three rotations.

SKETCH:



RESEARCH: From the parallel axis theorem, the moment of inertia of a tire is $I = I_{\text{cm}} + Mr^2$. For a thin hollow cylinder, $I_{\text{cm}} = MR^2$ and for a thick hollow cylinder, $I_{\text{cm}} = M\left(R_1^{\,2} + R_2^{\,2}\right)/2$. The torque is given by $\tau = I\alpha$. Then time can be determined from $\Delta\theta = \alpha t^2/2$, where $\Delta\theta$ is three rotations or 6π radians. Since the tire starts from rest, its final angular velocity is $\omega = \alpha t$ and its tangential velocity is $v = \omega r$.

SIMPLIFY:

(a) $\tau = I\alpha = \left(MR^2 + Mr^2\right)\alpha \Rightarrow \alpha = \dfrac{\tau}{M\left(R^2 + r^2\right)}$, $\Delta\theta = \dfrac{1}{2}\alpha t_{\text{throw}}^{\;\;2} \Rightarrow t_{\text{throw}} = \sqrt{\dfrac{2\Delta\theta}{\alpha}} = \sqrt{\dfrac{12\pi\left(M\left(R^2 + r^2\right)\right)}{\tau}}$

(b) $v = \omega r = \alpha t r = \sqrt{2\alpha\Delta\theta}\, r = \sqrt{\dfrac{12\pi\tau}{M\left(R^2 + r^2\right)}}$

(c) $t_{\text{throw}} = \sqrt{\dfrac{12\pi\left(M\left(r^2 + \dfrac{R_1^{\,2} + R_2^{\,2}}{2}\right)\right)}{\tau}}$, $v = r\sqrt{\dfrac{12\pi\tau}{M\left(r^2 + \dfrac{R_1^{\,2} + R_2^{\,2}}{2}\right)}}$

CALCULATE:

(a) $t_{\text{throw}} = \sqrt{\dfrac{12\pi\left((23.5\text{ kg})\left((0.350\text{ m})^2 + (1.10\text{ m})^2\right)\right)}{20.0\text{ Nm}}} = 7.683\text{ s}$

(b) $v = (1.10\text{ m})\sqrt{\dfrac{12\pi(20.0\text{ Nm})}{(23.5\text{ kg})\left((0.350\text{ m})^2 + (1.10\text{ m})^2\right)}} = 5.39766\text{ m/s}$

(c) $t_{\text{throw}} = \sqrt{\dfrac{12\pi\left((23.5\text{ kg})\left((1.10\text{ m})^2 + \dfrac{(0.300\text{ m})^2 + (0.400\text{ m})^2}{2}\right)\right)}{20.0\text{ Nm}}} = 7.690\text{ s}$

$$v = (1.1\text{ m})\sqrt{\frac{12\pi(20.0\text{ Nm})}{(23.5\text{ kg})\left((1.10\text{ m})^2 + \frac{(0.300\text{ m})^2+(0.400\text{ m})^2}{2}\right)}} = 5.393\text{ m/s}$$

ROUND:

(a) $t_{\text{throw}} = 7.68$ s

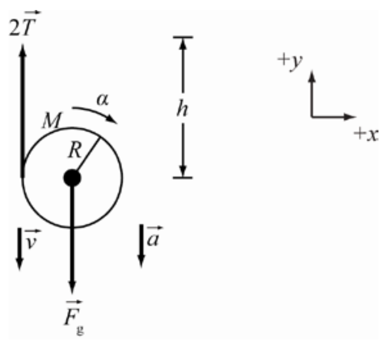
(b) $v = 5.40$ m/s

(c) $t_{\text{throw}} = 7.69$ s and $v = 5.39$ m/s

DOUBLE-CHECK: Given the small change in the two moments of inertia between the thin and thick cylinder, virtually identical values in time and velocity are reasonable.

10.54. **THINK:** Due to the symmetry of the barrel, assume the tension, T, in each rope is equal. The barrel with mass, M = 100. kg, and radius, R = 50.0 cm, will cause a tension, T, in the ropes that in turn produces a torque, τ, on the barrel and hence an angular acceleration, α. If the ropes do not slip, the angular acceleration will be directly related to the linear acceleration of the barrel. Once the linear acceleration is determined, kinematics can be used to determine the velocity of the barrel after it has fallen a distance, h = 10.0 m, assuming it starts from rest.

SKETCH:



RESEARCH: From the kinematic equations, $v_{\text{f}}^2 = v_{\text{i}}^2 + 2ah$, where the initial velocity is zero. The sum of the forces acting on the barrel is given by $2T - Mg = -Ma$. The tension in the ropes also cause a torque, $\tau = 2TR = I\alpha$, where $I = MR^2/2$.

SIMPLIFY: Summing the tensions in the ropes gives $2T = M(g-a)$. The torque this tension produces is

$$\tau = 2TR = MR(g-a) = \frac{1}{2}MR^2\left(\frac{a}{R}\right) \Rightarrow MgR = \frac{3}{2}MRa \Rightarrow a = \frac{2}{3}g.$$

The velocity is given by $v^2 = 2ah \Rightarrow v = \sqrt{\frac{4}{3}gh}$. The tension in one rope is $T = M(g-a)/2 = Mg/6$.

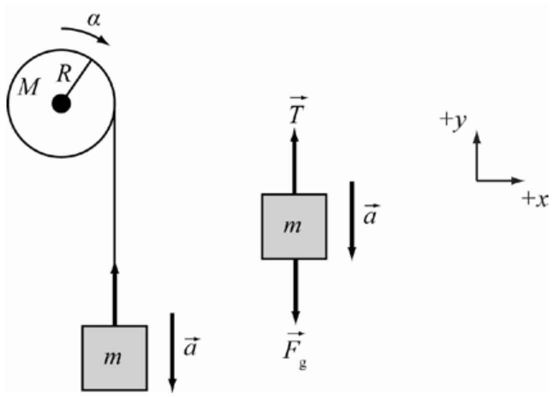
CALCULATE: $v = \sqrt{\frac{4}{3}(9.81\text{ m/s}^2)(10.0\text{ m})} = 11.437$ m/s, $T = \frac{1}{6}(100.\text{ kg})(9.81\text{ m/s}^2) = 163.5$ N

ROUND: Rounding to three significant figures, $v = 11.4$ m/s and T = 164 N.

DOUBLE-CHECK: If the barrel is in free fall, it would have a velocity of 14 m/s after falling 10 m, so a smaller velocity for this result is reasonable.

10.55. **THINK:** The hanging mass, m = 2.00 kg, will cause a tension, T, in the rope. This tension will then produce a torque, τ, on the wheel with a mass, M = 40.0 kg, a radius, R = 30.0 cm and a c value of 4/9. This torque will then give the wheel an angular acceleration, α. Assuming the rope does not slip, the angular acceleration of the wheel will be directly related to the linear acceleration of the hanging mass.

SKETCH:



RESEARCH: With no slipping, the linear acceleration is given by $a = \alpha R$. The tension can be determined by $T = m(g - a)$, which in turn produces a torque $\tau = TR = I\alpha$, where the moment of inertia of the wheel is $4MR^2 / 9$.

SIMPLIFY: To determine the angular acceleration:

$$TR = m(g - \alpha R)R = \frac{4}{9}MR^2\alpha \Rightarrow mgR = m\alpha R^2 + \frac{4}{9}MR^2\alpha \Rightarrow \alpha = \frac{mg}{\left(m + \frac{4}{9}M\right)R}.$$

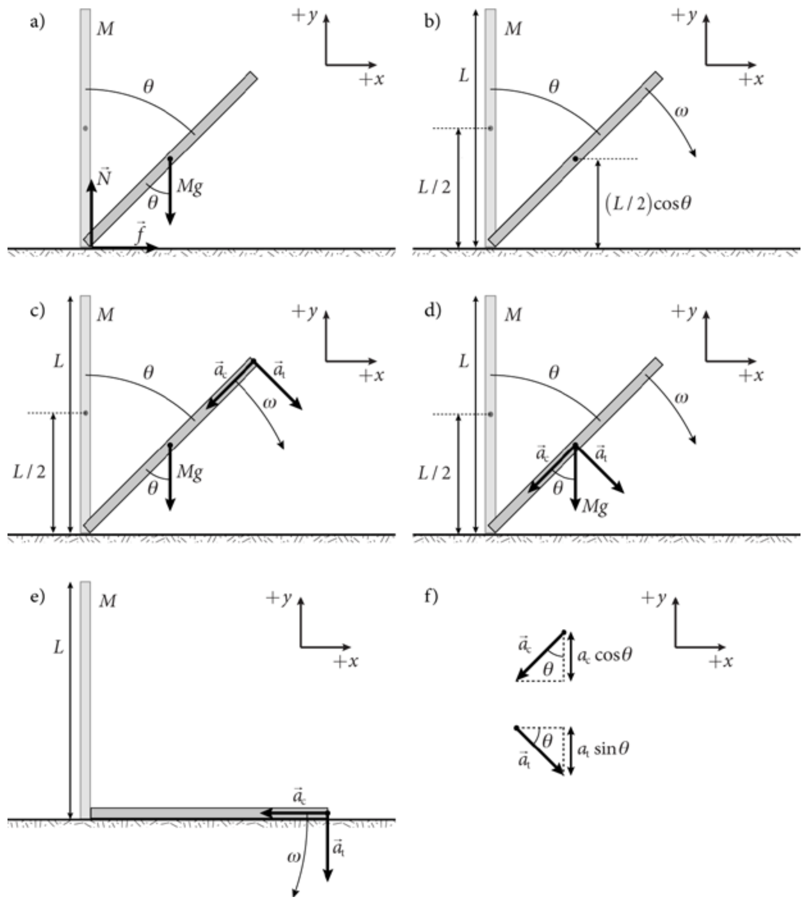
CALCULATE: $\alpha = \dfrac{2.00\text{ kg}\left(9.81\text{ m/s}^2\right)}{\left(2.00\text{ kg} + \frac{4}{9}(40.0\text{ kg})\right)(0.300\text{ m})} = 3.3067\text{ rad/s}^2$

ROUND: $\alpha = 3.31\text{ rad/s}^2$

DOUBLE-CHECK: Given the small hanging mass and the large mass of the wheel, this acceleration is reasonable.

10.56. **THINK:** As the rod with mass, M = 250.0 g and length, L = 50.0 cm, tips over, the torque, τ, caused by the force of gravity on the center of mass will change, which means the angular acceleration, α, of the rod will change with the angle it makes with the vertical. We can use energy conservation to calculate the angular velocity, ω, of the rod for any angle. The linear acceleration of any point on the rod is equal to the sum of the tangential acceleration plus the centripetal acceleration.

SKETCH:



RESEARCH: (a) In part a) of the sketch, we can see that the three forces acting on the rod are the normal force exerted by the table, the force of friction between the rod and the surface of the table, and the force of gravity.

(b1) To calculate the speed of the rod at $\theta = 45.0^\circ$, we can use energy conservation. Conservation of mechanical energy gives us $K + U = K_0 + U_0$. The kinetic energy before the rod begins to fall is zero and at angle θ the kinetic energy is given by the kinetic energy of rotation $K = (1/2)I\omega^2$ where ω is the angular velocity and $I = (1/3)ML^2$. The potential energy before is $U_0 = mg(L/2)$ and the potential energy at angle θ is $U = mg(L/2)\cos\theta$ as illustrated in part b) of the sketch.

(b2) To calculate the vertical acceleration of the moving end of the rod, we need to calculate the tangential acceleration and the centripetal acceleration. The tangential acceleration can be calculated by realizing that that the force of gravity exerts a torque on the rod given by $\tau = mg(L/2)\sin\theta$ assuming the pivot on the table at the end of the rod. The angular acceleration is given by $\tau = I\alpha$ where $I = (1/3)ML^2$. The tangential acceleration can then be calculated from $a_\text{t} = L\alpha$. The centripetal acceleration is given by $a_\text{c} = L\omega^2$ where ω was obtained in part b1). As shown in part f) of the sketch, the vertical component of the tangential acceleration is $a_\text{t}\sin\theta$ and the vertical component of the centripetal acceleration is $a_\text{c}\cos\theta$.

(b3) To calculate the normal force exerted by the table on the rod, we need to calculate the vertical component of the tangential and centripetal acceleration of the center of mass of the rod. The angular acceleration is the same as calculated in part b2) so the tangential acceleration is $a_\text{t} = (L/2)\alpha$. The centripetal acceleration is $a_\text{c} = (L/2)\omega^2$. The vertical components of the tangential and centripetal acceleration are then $a_t \sin\theta$ and $a_c \cos\theta$ respectively. The normal force is then given by $N - Mg = Ma_v$, where a_v is the vertical acceleration of the center of mass of the rod.

(c) When the rod falls on the table, $\theta = 90.0°$ and we have

$$a_t = \frac{3}{2} g \sin 90.0° = \frac{3}{2} g.$$

The centripetal acceleration is given by

$a_c = L\omega^2$ where

$$\omega = \sqrt{\frac{3g}{L}\left(1 - \cos 90.0°\right)} = \sqrt{\frac{3g}{L}} \Rightarrow a_c = L\frac{3g}{L} = 3g.$$

SIMPLIFY:

(b1) We can combine the equations in b1) to obtain

$$\frac{1}{2}I\omega^2 + mg\frac{L}{2}\cos\theta = mg\frac{L}{2}.$$

We can rewrite the previous equation as

$$\frac{1}{2}\left(\frac{1}{3}mL^2\right)\omega^2 = mg\frac{L}{2}\left(1-\cos\theta\right) \Rightarrow \omega = \sqrt{\frac{3g}{L}\left(1-\cos\theta\right)}.$$

(b2) We can combine the equations in (b2) to get the tangential acceleration a_t

$$mg\frac{L}{2}\sin\theta = \frac{1}{3}mL^2\frac{a_t}{L} \rightarrow g\frac{1}{2}\sin\theta = \frac{1}{3}a_t \Rightarrow a_t = \frac{3}{2}g\sin\theta.$$

We can then write the vertical component of the acceleration as

$$a_v = a_t \sin\theta + a_c \cos\theta = -\frac{3}{2}g\sin^2\theta - L\omega^2\cos\theta.$$

(b3) We can combine the equations in (b3) to get

$$mg\frac{L}{2}\sin\theta = \frac{1}{3}mL^2\frac{a_t}{(L/2)} \rightarrow g\frac{1}{2}\sin\theta = \frac{2a_t}{3} \Rightarrow a_t = \frac{3}{4}g\sin\theta.$$

We can now write the vertical component of the acceleration as

$$a_v = -\frac{3}{4}g\sin^2\theta - \frac{L}{2}\omega^2\cos\theta.$$

The normal force is

$$N = m\left(g + a_v\right) = m\left(g - \frac{3}{4}g\sin^2\theta - \frac{L}{2}\omega^2\cos\theta\right).$$

(c) The linear acceleration at $\theta = 90.0°$ is

$$a = \sqrt{a_c^2 + a_t^2} = \sqrt{(3g)^2 + \left(\frac{3}{2}g\right)^2} = 3g\sqrt{1 - \frac{1}{4}} = 3g\sqrt{\frac{3}{4}} = \frac{3\sqrt{3}}{2}g.$$

CALCULATE:

(a) Not necessary.

(b1) $\omega = \sqrt{\frac{3\left(9.81 \text{ m/s}^2\right)}{0.500 \text{ m}}\left(1 - \cos 45.0°\right)} = 4.1521 \text{ rad/s}.$

(b2) $a_v = -\frac{3}{2}\left(9.81 \text{ m/s}^2\right)\sin^2 45.0° - \left(0.500 \text{ m}\right)\left(4.1521 \text{ rad/s}\right)^2 \cos 45.0° = -13.453 \text{ m/s}^2.$

(b3) $N = \left(0.2500 \text{ kg}\right)\left(9.81 \text{ m/s}^2 - \frac{3}{4}\left(9.81 \text{ m/s}^2\right)\sin^2 45.0° - \frac{0.500 \text{ m}}{2}\left(4.1521 \text{ s}^{-1}\right)^2 \cos 45.0°\right) = 0.77091 \text{ N}.$

(c) $a = \frac{3\sqrt{3}}{2}\left(9.81 \text{ m/s}^2\right) = 25.487 \text{ m/s}^2.$

ROUND: Three significant figures:

(a) Not necessary

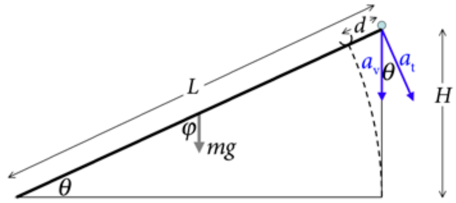
(b) $\omega = 4.15$ rad/s, $a_\text{v} = -13.5 \text{ m/s}^2$ and $N = 0.771$ N.

(c) $a = 25.5 \text{ m/s}^2$.

DOUBLE-CHECK: When $\theta \to 0$, $\omega = 0$, i.e. the rod is perfectly upright and not rotating, looking at the equation for the normal force it can be seen that the normal force is equal to the force of gravity. The values for the accelerations may seem surprising because they are larger than g. However, we have to remember that the force of friction and the normal force must provide the centripetal force necessary to keep the rod rotating around one end. Note that the assumption that the friction force can provide the required centripetal force all the way to $\theta = 90.0°$ is unrealistic.

10.57. THINK: If we place the ball (blue dot) at the end of the board, it can only be caught by the cup (half circle), if the end of the board falls with a vertical component of the acceleration a_v, which is greater (or at least equal) to g. The cup is to be placed at a distance d away from the end so that it can be vertically under the ball and catch it when the board lands on the ground.

SKETCH:



RESEARCH: The board rotates about its lower end and has the same moment of inertia as a rod, $I = \frac{1}{3}mL^2$. The torque equation is $\tau = I\alpha$, where the torque is given by $\tau = Fr\sin\varphi = mg \cdot \frac{1}{2}L \cdot \sin\varphi$. The angular and tangential acceleration are related to each other via $a_\text{t} = \alpha L$. The vertical component of the tangential acceleration is then (see sketch) $a_\text{v} = a_\text{t}\cos\theta$.

Geometrical relations: Since $\theta = 90° - \varphi$, (see sketch), we find that $\sin\varphi = \cos\theta$. Also from the sketch, we see that the height of the vertical support stick is $H = L\sin\theta$. In addition (dashed circular segment in the sketch), we see that $d = L - L\cos\theta = L(1-\cos\theta)$.

SIMPLIFY:

$\tau = \frac{1}{2}mgL\sin\varphi = \frac{1}{2}mgL\cos\theta = \frac{1}{3}mL^2\alpha = \frac{1}{3}mLa_\text{t} \Rightarrow a_\text{t} = \frac{3}{2}g\cos\theta$

Inserting this result into $a_\text{v} = a_\text{t}\cos\theta$ from above, we find $a_\text{v} = \frac{3}{2}g\cos^2\theta$. If, as required, $a_\text{v} \geq g$, this means $\frac{3}{2}g\cos^2\theta \geq g$ or $\cos\theta \geq \sqrt{\frac{2}{3}}$. Since $\sin^2\theta + \cos^2\theta = 1$, this implies $\sin\theta \leq \sqrt{\frac{1}{3}}$. So, finally, from $H = L\sin\theta$ we see $H \leq \sqrt{\frac{1}{3}}L$.

CALCULATE:

(a) $H_\text{max} = \sqrt{\frac{1}{3}}L = \sqrt{\frac{1}{3}}(1.00 \text{ m}) = 0.57735$ m.

(b) $d = L(1-\cos\theta) = (1.00 \text{ m})(1-\sqrt{\frac{2}{3}}) = 0.1835$ m.

ROUND: Rounding to 3 digits leaves us with

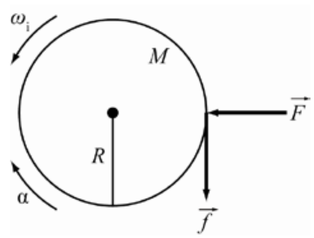
(a) $H_\text{max} = 0.577$ m.

(b) $d = 0.184$ m.

DOUBLE-CHECK: Clearly, this is a somewhat surprising result. However, it is also a standard lecture demonstration. When you see it you can convince yourself that these calculations for $\sin\theta$ are correct.

10.58. THINK: If the brakes applies an inward radial force, F = 100. N, and the contact has a coefficient of friction, $\mu_\text{k} = 0.200$, then this frictional force, *f*, will be perpendicular to *F* and cause a torque, τ, on the flywheel of mass, M = 120. kg, and radius, R = 80.0 cm. The torque can be used to determine the angular acceleration, α, of the wheel. Kinematics can then be used to determine the number of revolutions, *n*, the wheel will make and the time it will take for it to come to rest. The work done by torque should be the change in rotational energy, by conservation of energy. The flywheel has an initial angular speed of 500. rpm or $50\pi/3$ rad/s.

SKETCH:



RESEARCH: Similarly to the relation between the normal force and friction, $f = \mu_k F$, the friction causes a torque, $\tau = fR = I\alpha$, where the moment of inertia of the wheel is $I = MR^2/2$. Kinematics is used to determine the number of revolutions and the time it takes to come to an end, $\omega_f^2 = \omega_i^2 - 2\alpha\Delta\theta$ and $\omega_f - \omega_i = -\alpha t$. The work done by the friction is $W = \Delta K = -I\omega_i^2/2$.

SIMPLIFY: The angular acceleration is given by $\alpha = \dfrac{fR}{I} = \dfrac{\mu_k FR}{\frac{1}{2}MR^2} = \dfrac{2\mu_k F}{MR}$. Therefore, the total angular displacement is given by $\omega_f^2 = \omega_0^2 + 2\alpha\Delta\theta \Rightarrow |\Delta\theta| = \dfrac{\omega_0^2}{2\alpha}$. The number of revolutions, n, is given by $\dfrac{\Delta\theta}{2\pi} = \dfrac{\omega_0^2}{4\pi\alpha}$. The time to come to rest is $t = \omega_i/\alpha$. The work done is then $-MR^2\omega_i^2/4$.

CALCULATE:

$$\left(\frac{500 \text{ revolutions}}{1 \text{ min}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ revolution}}\right)\left(\frac{1 \text{ min}}{60 \text{ sec}}\right) = \frac{50\pi}{3} \text{ rad/sec}$$

$$\alpha = \frac{2(0.200)(100. \text{ N})}{(120. \text{ kg})(0.800 \text{ m})} = 0.4167 \text{ rad/s}^2, \quad n = \frac{(50\pi/3 \text{ rad/s})^2}{4\pi 0.4167 \text{ rad/s}^2} = 523.60 \text{ revolutions}$$

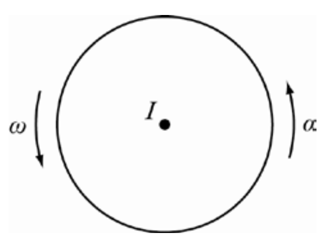
$$t = \frac{50\pi/3 \text{ rad/s}}{0.4167 \text{ rad/s}^2} = 125.66 \text{ s}, \quad W = -\frac{1}{4}(120. \text{ kg})(0.800 \text{ m})^2\left(\frac{50\pi}{3} \text{ rad/s}\right)^2 = -52638 \text{ J}$$

ROUND: To three significant figures: $n = 524$ revolutions, $t = 126$ s and $W = -5.26 \cdot 10^4$ J.

DOUBLE-CHECK: Given the small friction, and hence the small torque, and the fast speed of the wheel, it would take a long time to stop, so the results are reasonable.

10.59. **THINK:** Assuming a constant angular acceleration, α, and $\Delta t = 25$ s, regular kinematics can be used to determine α and $\Delta\theta$. The total work done by torque, τ, should be converted entirely into rotational energy. $I = 25.0 \text{ kg m}^2$ and $\omega_f = 150.$ rad/s.

SKETCH:



RESEARCH: From kinematics, $\omega_{\text{f}} - \omega_{\text{i}} = \alpha \Delta t$ and $\Delta\theta = \alpha(\Delta t)^2 / 2.$ The torque on the wheel is $\tau = I\alpha.$ Since the torque is constant, the work done by it is $W = \tau\Delta\theta.$ The kinetic energy of the turbine is $K = I\omega^2 / 2.$

SIMPLIFY:

(a) $\omega_{\text{f}} = \alpha \Delta t \Rightarrow \alpha = \omega_{\text{f}} / \Delta t$

(b) $\tau = I\alpha$

(c) $\Delta\theta = \frac{1}{2}\alpha(\Delta t)^2$

(d) $W = \tau\Delta\theta$

(e) $K = \frac{1}{2}I\omega_{\text{f}}^2$

CALCULATE:

(a) $\alpha = \frac{150. \text{ rad/s}}{25.0 \text{ s}} = 6.00 \text{ rad/s}^2$

(b) $\tau = (25.0 \text{ kg m}^2)(6.00 \text{ rad/s}^2) = 150. \text{ N m}$

(c) $\Delta\theta = \frac{1}{2}(6.00 \text{ rad/s}^2)(25.0 \text{ s})^2 = 1875 \text{ rad}$

(d) $W = (150. \text{ N m})(1875 \text{ rad}) = 281{,}250 \text{ J}$

(e) $K = \frac{1}{2}(25.0 \text{ kg m}^2)(150. \text{ rad/s})^2 = 281{,}250 \text{ J}$

ROUND:

(a) $\alpha = 6.00 \text{ rad/s}^2$

(b) $\tau = 150. \text{ N m}$

(c) $\Delta\theta = 1880 \text{ rad}$

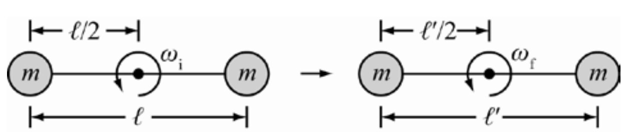
(d) $W = 281 \text{ kJ}$

(e) $K = 281 \text{ kJ}$

DOUBLE-CHECK: It is expected that the work and kinetic energy are equal. Since they were each determined independently and they are the same value, the procedure must have been correct.

10.60. **THINK:** Since the two masses have an equal mass of m = 6.00 kg, their center of mass will be at the geometric center, l/2, which is also the location of the axis of rotation. Initially, l = 1.00 m and then it extends to 1.40 m. When the length increases, the moment of inertia also increases. Since there are no external torques, conservation of angular momentum can be applied. The masses initially rotate at $\omega_{\text{i}} = 5.00 \text{ rad/s}.$

SKETCH:



RESEARCH: The angular momentum before and after are $L_{\text{i}} = I_{\text{i}}\omega_{\text{i}}$ and $L_{\text{f}} = I_{\text{f}}\omega_{\text{f}}$. The moments of inertia for before and after are $I_{\text{i}} = 2m(l/2)^2$ and $I_{\text{f}} = 2m(l'/2)^2$. The conservation of angular momentum is represented by $L_{\text{i}} = L_{\text{f}}$.

SIMPLIFY: $L_{\text{i}} = L_{\text{f}} \Rightarrow 2m\left(\frac{l^2}{4}\right)\omega_{\text{i}} = 2m\left(\frac{l'^2}{4}\right)\omega_{\text{f}} \Rightarrow l^2\omega_{\text{i}} = l'^2\omega_{\text{f}} \Rightarrow \omega_{\text{f}} = \left(\frac{l}{l'}\right)^2\omega_{\text{i}}$

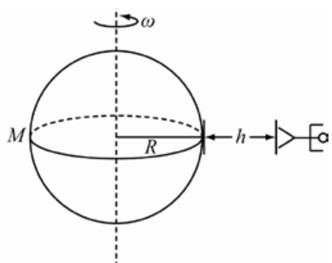
CALCULATE: $\omega_{\text{f}} = \left(\frac{1.00\text{ m}}{1.40\text{ m}}\right)^2 5.00\text{ rad/s} = 2.551\text{ rad/s}$

ROUND: To three significant figures: $\omega_{\text{f}} = 2.55\text{ rad/s}$

DOUBLE-CHECK: Since the string length and hence the moment of inertia increases, a smaller rotational speed is expected, since angular momentum is conserved.

10.61. **THINK:** For the moment of inertia of the Earth, treat it as a solid sphere with mass, $M = 5.977 \cdot 10^{24}$ kg, and radius, R = 6371 km. The Chinese ($n = 1.30 \cdot 10^9$ people and m = 70.0 kg each) can be treated as a point mass of total mass, nm, standing on the surface of the Earth and then also at h = 1.00 m above the surface. Conservation of angular momentum relates the change in moment of inertia to the change in angular frequency, and hence period, of the Earth.

SKETCH:



RESEARCH: The Earth's (solid sphere) moment of inertia is $I_{\text{E}} = 2MR^2/5$, while the Chinese have a moment of inertia of $I_{\text{C}} = nmR^2$ on the surface of the Earth and $I_C' = nm(R+h)^2$ when standing on the chair. The angular momentum is $L = I\omega$. The period of the Earth's rotation is 1 day or 86,400 s, and is related to the angular velocity by $\omega = 2\pi/T$.

SIMPLIFY:

(a) The moment of inertia of Earth is $I_{\text{E}} = 2MR^2/5.$

(b) The moment of inertia of the Chinese people on Earth is $I_{\text{C}} = nmR^2.$

(c) The moment of inertia of the Chinese people on chairs is $I_{\text{C}}' = nm(R+1)^2$. The change in the moment of inertia for the Chinese people is $\Delta I_{\text{C}} = I_{\text{C}}' - I_{\text{C}} = nm(R^2 + 2R + 1 - R^2) = nm(2R+1).$

(d) The conservation of angular momentum states, $(I_{\text{E}} + I_{\text{C}})\omega_{\text{i}} = (\Delta I_{\text{C}})\Delta\omega \Rightarrow (I_{\text{E}} + I_{\text{C}})\frac{2\pi}{T_{\text{i}}} = (\Delta I_{\text{C}})\frac{2\pi}{\Delta T}.$

Therefore, $\frac{\Delta T}{T} = \frac{\Delta I_{\text{C}}}{I_{\text{E}} + I_{\text{C}}}$ (fractional change) and $\Delta T = \frac{\Delta I_{\text{C}}}{I_{\text{E}} + I_{\text{C}}}T$ (total change).

CALCULATE:

(a) $I_{\text{E}} = \frac{2}{5}(5.977 \cdot 10^{24}\text{ kg})(6{,}371{,}000\text{ m})^2 = 9.704 \cdot 10^{37}\text{ kg m}^2$

(b) $I_{\text{C}} = (1.30 \cdot 10^9)(70.0\text{ kg})(6{,}371{,}000\text{ m})^2 = 3.694 \cdot 10^{24}\text{ kg m}^2$

(c) $\Delta I_{\text{C}} = (1.30 \cdot 10^9)(70.0\text{ kg})(2(6{,}371{,}000\text{ m}) + 1.00\text{ m}) = 1.1595 \cdot 10^{18}\text{ kg m}^2$

(d) $\frac{\Delta T}{T} = \frac{1.1595 \cdot 10^{18}\text{ kg m}^2}{9.704 \cdot 10^{37}\text{ kg m}^2 + 3.694 \cdot 10^{24}\text{ kg m}^2} = 1.1949 \cdot 10^{-20}$

ROUND:

(a) $I_{\text{E}} = 9.704 \cdot 10^{37}$ kg m^2

(b) $I_C = 3.69 \cdot 10^{24}$ kg m^2

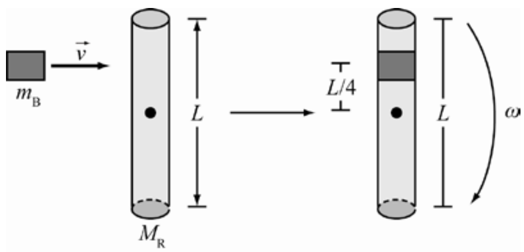
(c) $\Delta I_C = 1.16 \cdot 10^{18}$ kg m^2

(d) $\dfrac{\Delta T}{T} = 1.19 \cdot 10^{-20}$

DOUBLE-CHECK: Despite the large number of Chinese people, the Earth is so massive that the rotation is hardly affected by their jump onto the surface.

10.62. **THINK:** The bullet with a mass, $m_{\text{B}} = 1.00 \cdot 10^{-2}$ kg, has a linear momentum. When it strikes the rod with length, $L = 1.00$ m, and mass, $m_{\text{R}} = 5.00$ kg, the rod begins to rotate about its center and thus has an angular momentum. Conservation of momentum means the bullet's linear momentum is equal to the rod and bullet's angular momentum. Likewise, the bullet has a linear kinetic energy and the rod and bullet have a rotational kinetic energy so the change in kinetic energy is the difference between these two. The bullet can be treated as a point particle that is a distance $L/4$ from the axis of rotation.

SKETCH:



RESEARCH: The momentum of the bullet is given by $p = m_{\text{B}}v$. When it hits the rod at $L/4$ from the center, its linear momentum can be converted to angular momentum by $pL/4$. The moment of inertia of the rod is $m_{\text{R}}L^2/12$ and that of the bullet when in the rod is $m\left(L/4\right)^2$. The kinetic energy of the bullet is all translational, $K_{\text{T}} = m_{\text{B}}v^2/2$, while the kinetic energy of the bullet and rod together is all rotational, $K_{\text{R}} = I\omega^2/2$.

SIMPLIFY:

(a) The initial angular momentum is $L_{\text{i}} = m_{\text{B}}vL/4$. The final angular momentum is $L_{\text{f}} = I\omega = \left(m_{\text{R}}/12 + m_{\text{B}}/16\right)L^2\omega$. The angular velocity is then

$$L_{\text{i}} = L_{\text{f}} \Rightarrow \frac{m_{\text{B}}vL}{4} = \left(\frac{1}{12}m_{\text{R}} + \frac{1}{16}m_{\text{B}}\right)L^2\omega \Rightarrow \omega = \frac{m_{\text{B}}v}{\left(\frac{1}{3}m_{\text{R}} + \frac{1}{4}m_{\text{B}}\right)L}.$$

(b) $K_{\text{T}} = \dfrac{1}{2}m_{\text{B}}v^2$ and $K_{\text{R}} = \dfrac{1}{2}I\omega^2 = \left(\dfrac{1}{24}m_{\text{R}} + \dfrac{1}{32}m_{\text{B}}\right)L^2\omega^2$. Therefore, $\Delta K = K_{\text{R}} - K_{\text{T}}$.

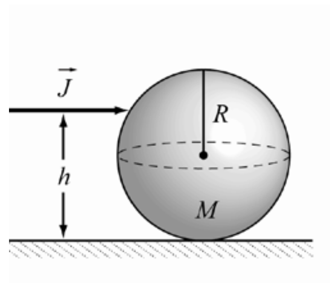
CALCULATE:

(a) $$\omega = \frac{\left(1.00 \cdot 10^{-2} \text{ kg}\right)\left(100. \text{ m/s}\right)}{\left(\dfrac{5.00 \text{ kg}}{3} + \dfrac{0.0100 \text{ kg}}{4}\right)\left(1.00 \text{ m}\right)} = 0.5991 \text{ rad/s}$$

(b) $$\Delta K = \left(\frac{5.00 \text{ kg}}{24} + \frac{1.00 \cdot 10^{-2} \text{ kg}}{32}\right)\left(1.00 \text{ m}\right)^2\left(0.599 \text{ rad/s}\right)^2 - \frac{1}{2}\left(1.00 \cdot 10^{-2} \text{ kg}\right)\left(100. \text{ m/s}\right)^2 = -49.925 \text{ J}$$

ROUND:
To three significant figures,
(a) $\omega = 0.599$ rad/s
(b) $\Delta K = -49.9$ J
DOUBLE-CHECK: Given that the rod is five hundred times heavier than the bullet and that the bullet will lose energy from imbedding itself in the rod, a small ω and a negative value for ΔK is reasonable.

10.63. **THINK:** The sphere of mass, M, spins clockwise when a horizontal impulse J is exerted at a height h above the tabletop when $R < h < 2R$.
SKETCH:



RESEARCH: To calculate the linear speed after the impulse is applied, we use the fact that the impulse J can be written as $J = \Delta p = M\Delta v$. To get the angular velocity, we write the change in the angular momentum of the sphere as $\Delta L = \Delta p(h-R)$. To calculate the height where the impulse must be applied, we have to apply Newton's Second Law for linear motion, $F = Ma$, and Newton's Second Law for rotation, $\tau = I\alpha$. The torque is given by $\tau = F(h-R)$. The object rolls without slipping so from Section 10.3 we know that $v = R\omega$ and $a = R\alpha$. In addition, we can write the impulse as $J = F\Delta t$.
SIMPLIFY: a) Combining these relationships to get the linear velocity gives us

$$J = \Delta p = M\Delta v = Mv \quad \Rightarrow \quad v = \frac{J}{M}.$$

Combining these relationships to get the angular velocity gives us

$$\Delta L = \Delta p(h-R) = J(h-R)$$
$$\Delta L = I\Delta\omega = I\omega = \frac{2}{5}MR^2\omega$$
$$J(h-R) = \frac{2}{5}MR^2\omega$$
$$\omega = \frac{5J(h-R)}{2MR^2}.$$

b) Combining these relationships to get the height h_0 at which the impulse must be applied for the sphere to roll without slipping we get

$$F = Ma \quad \Rightarrow \quad \frac{J}{\Delta t} = MR\alpha$$
$$F(h_0 - R) = I\alpha \quad \Rightarrow \quad \frac{J}{\Delta t}(h_0 - R) = \frac{2}{5}MR^2\alpha.$$

Dividing these two equations gives us

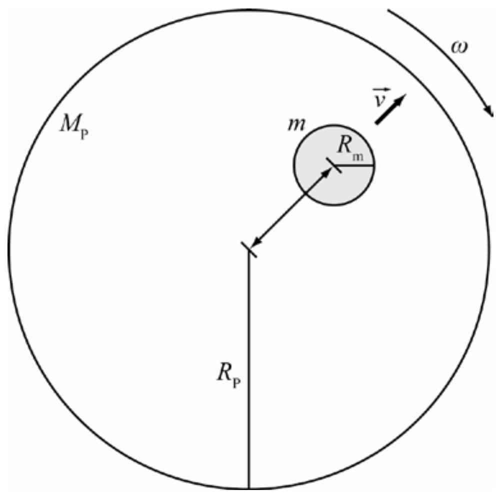
$$h_0 - R = \frac{2}{5}R \quad \Rightarrow \quad h_0 = \frac{7}{5}R.$$

CALCULATE: Not applicable.
ROUND: Not applicable.

DOUBLE-CHECK: The linear velocity will always be positive. However, the angular velocity can be positive or negative, depending on whether $h > R$ or $h < R$. The fact that $h_0 > R$ is consistent with the ball rolling to the right after the impulse is applied.

10.64. **THINK:** If the man, approximated by a cylinder of mass, m = 80.0 kg, and radius, $R_{\text{m}} = 0.200$ m, walks at a constant velocity, v = 0.500 m/s, then his distance, d, from the center of the platform of mass, $M_p = 400.$ kg, and radius, $R_{\text{p}} = 4.00$ m, will change linearly with time. The platform initially rotates at 6.00 rpm or 0.200π rad/s. Initially, the man and the platform have their center of mass on the axis of rotation, so their moments of inertia are summed. When the man is a distance, d, from the center, the parallel axis theorem is needed to determine his overall moment of inertia.

SKETCH:



RESEARCH: The distance, d, the man is from the center is $d = vt$. The moment of inertia of the platform is $I_{\text{p}} = M_{\text{p}}R_{\text{p}}^2/2$. The man has a moment of inertia of $I_{\text{m}} = mR_{\text{m}}^2/2$ and by the parallel axis theorem has a final moment of inertia of $I'_{\text{m}} = \left(mR_{\text{m}}^2/2\right) + md^2$. Conservation of angular momentum states $L_{\text{i}} = L_{\text{f}}$, where $L = \omega I$.

SIMPLIFY: The man's moment of inertia as a function of time is $I'_{\text{m}} = \left(mR_{\text{m}}^2/2\right) + mv^2t^2 = I_{\text{m}} + mv^2t^2$.

The initial angular momentum of the system is $L_{\text{i}} = \left(I_{\text{p}} + I_{\text{m}}\right)\omega_{\text{i}}$. The angular momentum at time t is $L_{\text{f}} = \left(I_{\text{p}} + I'_{\text{m}}\right)\omega_{\text{f}}$. Therefore, $\left(I_{\text{p}} + I_{\text{m}}\right)\omega_{\text{i}} = \left(I_{\text{p}} + I_{\text{m}} + mv^2t^2\right)\omega_{\text{f}}$

$$\Rightarrow \omega_{\text{f}} = \frac{\left(I_{\text{p}} + I_{\text{m}}\right)\omega_{\text{i}}}{\left(I_{\text{p}} + I_{\text{m}} + mv^2t^2\right)} \Rightarrow \omega_f(t) = \omega_{\text{i}}\left(1 + \frac{2mv^2t^2}{M_{\text{p}}R_{\text{p}}^2 + mR_{\text{m}}^2}\right)^{-1}.$$

When the man reaches the end, $d = R_{\text{p}} \Rightarrow t = R_{\text{p}}/v$. Therefore, $\omega_{\text{f}} = \omega_{\text{i}}\left(1 + \dfrac{2mR_{\text{p}}^2}{M_{\text{p}}R_{\text{p}}^2 + mR_{\text{m}}^2}\right)^{-1}$.

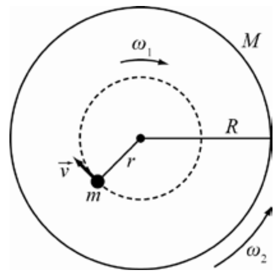
CALCULATE: $\omega_{\text{f}} = \left(0.200\pi \text{ rad/s}\right)\left(1 + \dfrac{2\left(80.0 \text{ kg}\right)\left(4.00 \text{ m}\right)^2}{\left(400. \text{ kg}\right)\left(4.00 \text{ m}\right)^2 + \left(80.0 \text{ kg}\right)\left(0.200 \text{ m}\right)^2}\right)^{-1} = 0.4489 \text{ rad/s}$

ROUND: To three significant figures: $\omega_{\text{f}} = 0.449$ rad/s.

DOUBLE-CHECK: As time increases (i.e. the man walks from the center), the overall moment of inertia increases, so a smaller angular velocity is expected.

10.65. **THINK:** Initially, the system has zero angular momentum. The boy with mass, m = 25.0 kg, can be treated as a point particle a distance, r = 2.00 m, from the center of the merry-go-round, which has a moment of inertia, $I_0 = 200.\text{ kg m}^2$. When the boy starts running with a velocity, v = 0.600 m/s, the merry-go-round will begin to rotate in the opposite direction in order to conserve angular momentum.

SKETCH:



RESEARCH: The initial angular momentum is $L_i = 0$. The angular velocity of the boy is $\omega_1 = v/r$. The moment of inertia of the boy is mr^2. The angular momentum is given by $L = I\omega$. The tangential velocity of the merry-go-round at r is $v_2 = \omega_2 r$. The boy's velocity relative to the merry-go-round (which is rotating in the opposite direction) is $v' = v + v_2$.

SIMPLIFY:

(a) $L_f = I_0\omega_2 - mr^2\omega_1 = 0 \Rightarrow \omega_2 = \dfrac{mr^2\omega_1}{I_0} \Rightarrow \omega_2 = \dfrac{mrv}{I_0}$

(b) $v' = v + r\omega_2$

CALCULATE:

(a) $\omega_2 = \dfrac{(25.0\text{ kg})(2.00\text{ m})(0.600\text{ m/s})}{200.\text{ kg m}^2} = 0.150\text{ rad/s}$

(b) $v' = 0.600\text{ m/s} + (2.00\text{ m})(0.150\text{ rad/s}) = 0.900\text{ m/s}$

ROUND:

(a) $\omega_2 = 0.150\text{ rad/s}$

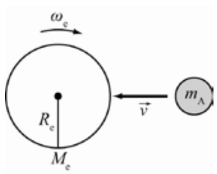
(b) $v' = 0.900\text{ m/s}$

DOUBLE-CHECK: Since the merry-go-round must move opposite to the boy, a relative velocity greater than the velocity compared to the ground makes sense. Also, since the boy and the merry-go-round have comparable moments of inertia, the comparable velocities are reasonable.

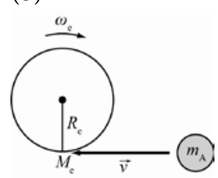
10.66. **THINK:** In every case, the momentum (angular and linear) must be conserved. If the asteroid with mass, $m_A = 1.00\cdot 10^{22}$ kg, and velocity, $v = 1.40\cdot 10^3$ m/s, hits the Earth, which has an angular speed of $\omega_E = 7.272\cdot 10^{-5}$ rad/s, dead on (radially inward), then it will not contribute any of its linear momentum to the angular momentum of the planet, meaning the change in the Earth's rotation is solely a result of it gaining mass. If the asteroid hits the planet tangentially, then the full amount of the asteroid's linear momentum is contributed to the angular momentum. If the asteroid hits in the direction of Earth's rotation, it will add its momentum and the Earth will spin faster and vice versa for the opposite direction. The mass of the Earth is $m_E = 5.977\cdot 10^{24}$ kg and the radius is $R_E = 6371$ km. The Earth can be treated as a solid sphere.

SKETCH:

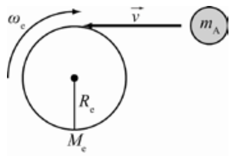
(a)



(b)



(c)



RESEARCH: The moment of inertia of Earth is $I_E = 2M_ER_E^2/5$. After the asteroid has collided, the moment of inertia of the system is then given by $I_T = I_E + m_AR_E^2$. The angular momentum is $L = I\omega$. Conservation of angular momentum applies in each case. The momentum the asteroid contributes is $p = m_Av$ and its linear momentum will be $\pm pR_E$, depending on which way it hits.

SIMPLIFY:

(a) $I_E\omega_E = I_T\omega_F \Rightarrow \omega_F = \dfrac{\frac{2}{5}M_ER_E^2}{\frac{2}{5}M_ER_E^2 + M_AR_E^2}\omega_E \Rightarrow \omega_F = \dfrac{2M_E}{2M_E + 5M_A}\omega_E$

(b) $I_E\omega_E + M_AvR_E = I_T\omega_F \Rightarrow \omega_F = \dfrac{\frac{2}{5}M_ER_E^2\omega_E + M_AvR_E}{\frac{2}{5}M_ER_E^2 + M_AR_E^2} \Rightarrow \omega_F = \dfrac{\frac{2}{5}M_ER_E\omega_E + M_Av_A}{\frac{2}{5}M_ER_E + M_AR_E}$

(c) $I_E\omega_E - M_AvR_E = I_T\omega_F \Rightarrow \omega_F = \dfrac{\frac{2}{5}M_ER_E\omega_E - M_Av_A}{\frac{2}{5}M_ER_E + M_AR_E}$

CALCULATE:

(a) $\omega_F = \dfrac{2\left(5.977\cdot 10^{24}\text{ kg}\right)\left(7.272\cdot 10^{-5}\text{ rad/s}\right)}{2\left(5.977\cdot 10^{24}\text{ kg}\right) + 5\left(1.00\cdot 10^{22}\text{ kg}\right)} = 7.2417\cdot 10^{-5}\text{ rad/s}$

(b)

$$\omega_F = \frac{\frac{2}{5}\left(5.977\cdot 10^{24}\text{ kg}\right)\left(6371000\text{ m}\right)\left(7.272\cdot 10^{-5}\text{ rad/s}\right) + \left(1.00\cdot 10^{22}\text{ kg}\right)\left(1.40\cdot 10^{3}\text{ m/s}\right)}{\frac{2}{5}\left(5.977\cdot 10^{24}\text{ kg}\right)\left(6371000\text{ m}\right) + \left(1.00\cdot 10^{22}\text{ kg}\right)\left(6371000\text{ m}\right)} = 7.333\cdot 10^{-5}\text{ rad/s}$$

(c)

$$\omega_F = \frac{\frac{2}{5}\left(5.977\cdot 10^{24}\text{ kg}\right)\left(6371000\text{ m}\right)\left(7.272\cdot 10^{-5}\text{ rad/s}\right) - \left(1.00\cdot 10^{22}\text{ kg}\right)\left(1.40\cdot 10^{3}\text{ m/s}\right)}{\left(\frac{2}{5}\left(5.977\cdot 10^{24}\text{ kg}\right) + \left(1.00\cdot 10^{22}\text{ kg}\right)\right)\left(6371000\text{ m}\right)} = 7.1502\cdot 10^{-5}\text{ rad/s}$$

ROUND:

To three significant figures:

(a) $\omega_F = 7.24\cdot 10^{-5}\text{ rad/s}$

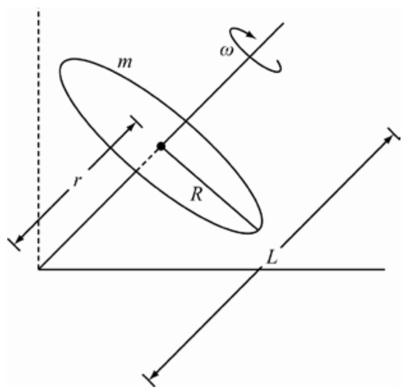
(b) $\omega_F = 7.33\cdot 10^{-5}$ rad/s

(c) $\omega_F = 7.15\cdot 10^{-5}$ rad/s

DOUBLE-CHECK: In part (a), it is expected that ω would be reduced very little, since the Earth gains a 0.4% mass on the surface and the moment of inertia is changed only slightly. In part (b), the asteroid would make the Earth spin faster, provided the velocity was great enough. In part (c), the asteroid would definitely make the Earth slow down its rotation.

10.67. **THINK:** If the disk with radius, R = 40.0 cm, is rotating at 30.0 rev/s, then the angular speed, ω, is 60.0π rad/s. The length of the gyroscope is L = 60.0 cm, so that the disk is located at $r = L/2$ from the pivot.

SKETCH:



RESEARCH: The precessional angular speed is given by $\omega_p = rmg / I\omega$. The moment of inertia of the disk is $I = mR^2/2$.

SIMPLIFY: $\omega_p = \dfrac{\frac{L}{2}mg}{\frac{1}{2}mR^2\omega} = \dfrac{Lg}{R^2\omega}$

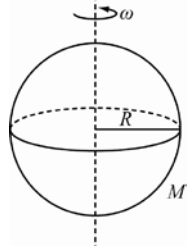
CALCULATE: $\omega_p = \dfrac{0.600\text{ m}\left(9.81\text{ m/s}^2\right)}{\left(0.400\text{ m}\right)^2 60.0\pi\text{ rad/s}} = 0.19516$ rad/s

ROUND: $\omega_p = 0.195$ rad/s

DOUBLE-CHECK: The precession frequency is supposed to be much less than the frequency of the rotating disk. In this example, the disk frequency is about one thousand times the precession frequency, so it makes sense.

10.68. **THINK:** Assume the star with a mass, $M = 5.00\cdot 10^{30}$ kg, is a solid sphere. After the star collapses, the total mass remains the same, only the radius of the star has changed. Initially, the star has radius, $R_i = 9.50\cdot 10^8$ m, and period, $T_i = 30.0\text{ days} = 2592000$ s, while after the collapse it has a radius, $R_f = 10.0$ km, and a period, T_f. To determine the final period, consider the conservation of angular momentum.

SKETCH:



RESEARCH: The moment of inertia of the star is $I = 2MR^2/5$, so initially it is $I_\text{i} = 2MR_\text{i}^2/5$ and afterwards it is $I_\text{f} = 2MR_\text{f}^2/5$. Angular momentum is conserved, so $L_\text{i} = L_\text{f}$, where $L = I\omega$. The period is related to the angular frequency by $T = 2\pi/\omega$ or $\omega = 2\pi/T$.

SIMPLIFY: $I_\text{i}\omega_\text{i} = I_\text{f}\omega_\text{f} \Rightarrow \dfrac{\frac{2}{5}MR_\text{i}^2(2\pi)}{T_\text{i}} = \dfrac{\frac{2}{5}MR_\text{f}^2(2\pi)}{T_\text{f}} \Rightarrow T_\text{f} = \dfrac{R_\text{f}^2}{R_\text{i}^2}T_\text{i}$

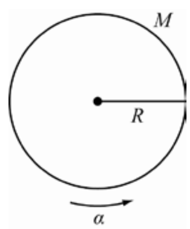
CALCULATE: $T_\text{f} = \dfrac{\left(10.0\cdot 10^3 \text{ m}\right)^2\left(2{,}592{,}000 \text{ s}\right)}{\left(9.50\cdot 10^8 \text{ m}\right)^2} = 2.872\cdot 10^{-4} \text{ s}$

ROUND: $T_\text{f} = 2.87\cdot 10^{-4}$ s

DOUBLE-CHECK: Given the huge reduction in size, a large reduction in period, or increase in angular velocity is expected.

10.69. **THINK:** The flywheel with radius, R = 3.00 m, and $M = 1.18\cdot 10^6$ kg, rotates from rest to $\omega_\text{f} = 1.95$ rad/s in $\Delta t = 10.0 \text{ min} = 600.$ s. The wheel can be treated as a solid cylinder. The angular acceleration α, can be determined using kinematics. The angular acceleration is then used to determine the average torque, τ.

SKETCH:



RESEARCH: The energy is all rotational kinetic energy, so $E = I\omega^2/2$. The moment of inertia of the wheel is $I = MR^2/2$. From kinematics, $\omega_\text{f} - \omega_\text{i} = \alpha\Delta t$. The torque is then given by $\tau = I\alpha$.

SIMPLIFY: The total energy is given by $E = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2 = \frac{1}{4}MR^2\omega^2$. The angular acceleration is given by $\omega_\text{f} - 0 = \alpha\Delta t \Rightarrow \alpha = \omega_\text{f}/\Delta t$. The torque needed is

$$\tau = I\alpha = \left(\frac{1}{2}MR^2\right)\left(\frac{\omega_\text{f}}{\Delta t}\right) = \frac{MR^2\omega_\text{f}}{2\Delta t}.$$

CALCULATE: $E = \frac{1}{4}\left(1.18\cdot 10^6 \text{ kg}\right)\left(3.00 \text{ m}\right)^2\left(1.95 \text{ rad/s}\right)^2 = 1.0096\cdot 10^7$ J

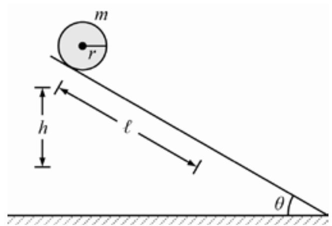
$$\tau = \frac{\left(1.18\cdot 10^6 \text{ kg}\right)\left(3.00 \text{ m}\right)^2\left(1.95 \text{ rad/s}\right)}{2\left(600. \text{ s}\right)} = 17257.5 \text{ N m}$$

ROUND: $E = 1.01\cdot 10^7$ J, $\tau = 17{,}300$ N m

DOUBLE-CHECK: The problem mentions that a huge amount of energy is needed for the experiment and the resulting energy is huge. It is reasonable that a huge torque would also be required.

10.70. **THINK:** With no friction and no slipping, energy is conserved. The potential energy of the hoop of mass, m = 2.00 kg, and radius, r = 50.0 cm, will be converted entirely into translational and rotational kinetic energy at l = 10.0 m down the incline with an angle of $\theta = 30.0°$. For a hoop, c = 1.

SKETCH:



RESEARCH: The change in height of the hoop is $h = l\sin\theta$. The initial potential energy of the hoop is *mgh*. The kinetic energy of the hoop is $K_{\text{f}} = (1+c)mv^2/2$. The *c* value for the hoop is 1.

SIMPLIFY: $U_{\text{i}} = K_{\text{f}} \Rightarrow mgl\sin\theta = (1+c)mv^2/2 \Rightarrow v = \sqrt{\frac{2gl\sin\theta}{1+c}}$

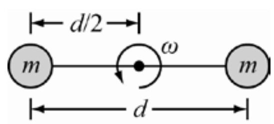
CALCULATE: $v = \sqrt{\frac{2(9.81\text{ m/s}^2)(10.0\text{ m})\sin(30.0°)}{1+1}} = 7.004\text{ m/s}$

ROUND: To three significant figures: $v = 7.00$ m/s

DOUBLE-CHECK: This value is less than the velocity the hoop would have going a distance, *h*, in free fall (v = 9.9 m/s), so it seems reasonable.

10.71. **THINK:** The oxygen atoms, $m = 2.66 \cdot 10^{-26}$ kg, can be treated as point particles a distance, *d*/2 (where $d = 1.21 \cdot 10^{-10}$ m) from the axis of rotation. The angular speed of the atoms is $\omega = 4.60 \cdot 10^{12}$ rad/s.

SKETCH:



RESEARCH: Since the masses are equal point particles, the moment of inertia of the two is $I = 2m(d/2)^2$. The rotational kinetic energy is $K = I\omega^2/2$.

SIMPLIFY:

(a) $I = 2m\left(\frac{d^2}{4}\right) = \frac{1}{2}md^2$

(b) $K = \frac{1}{2}I\omega^2 = \frac{1}{4}md^2\omega^2$

CALCULATE:

(a) $I = \frac{1}{2}(2.66 \cdot 10^{-26}\text{ kg})(1.21 \cdot 10^{-10}\text{ m})^2 = 1.9473 \cdot 10^{-46}\text{ kg m}^2$

(b) $K = \frac{1}{4}(2.66 \cdot 10^{-26}\text{ kg})(1.21 \cdot 10^{-10}\text{ m})^2(4.60 \cdot 10^{12}\text{ rad/s})^2 = 2.06 \cdot 10^{-21}\text{ J}$

ROUND:

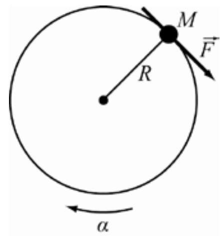
(a) $I = 1.95 \cdot 10^{-46}\text{ kg m}^2$

(b) $K = 2.06 \cdot 10^{-21}$ J

DOUBLE-CHECK: Since an oxygen molecule is so small, a very small moment of inertia and energy are expected.

10.72. **THINK:** If the force, *F*, is tangent to the circle's radius, then the angle between it and the radius, *R* = 0.40 m, is 90°. The bead with mass, *M* = 0.050 kg, can be treated as a point particle. The required angular acceleration, $\alpha = 6.0\text{ rad/s}^2$, is then found using the torque, τ.

SKETCH:



RESEARCH: The force produces a torque, $\tau = FR = I\alpha$. The moment of inertia of the bead is $I = MR^2$.

SIMPLIFY: $FR = I\alpha = MR^2\alpha \Rightarrow F = MR\alpha$.

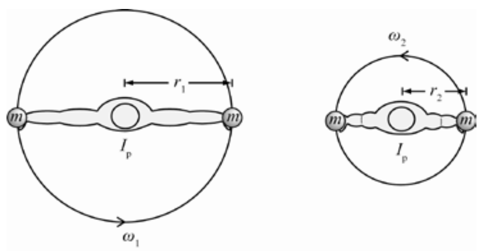
CALCULATE: $F = (0.0500 \text{ kg})(0.400 \text{ m})\left(6.00 \text{ rad/s}^2\right) = 0.120 \text{ N}$

ROUND: To three significant figures: $F = 0.120$ N

DOUBLE-CHECK: For a small mass, a small force is reasonable.

10.73. **THINK:** Angular momentum will be conserved when the professor brings his arms and the two masses, because there is no external torque.

SKETCH:



RESEARCH: Conservation of angular momentum states $I_i\omega_i = I_f\omega_f$, and the moment of inertia at any point is $I = I_{\text{body}} + 2r^2m$. We assume that $I_{\text{body,i}} = I_{\text{body,f}}$.

SIMPLIFY: Substituting the moments of inertia into the conservation equation gives

$$\omega_f = \frac{I_i}{I_f}\omega_i = \frac{I_{\text{body,i}} + 2r_i^2 m}{I_{\text{body,f}} + 2r_f^2 m}\omega_i.$$

CALCULATE:

The initial angular speed is $\omega_i = 2\pi f = 2\pi(1.00 \text{ rev/min}) = 0.1047$ rad/s.

So the final angular speed is

$$\omega_f = \frac{(2.80 \text{ kg m}^2) + 2(1.20 \text{ m})^2(5.00 \text{ kg})}{(2.80 \text{ kg m}^2) + 2(0.300 \text{ m})^2(5.00 \text{ kg})}(0.1047 \text{ rad/s}) = 0.4867135 \text{ rad/s}^{-1}.$$

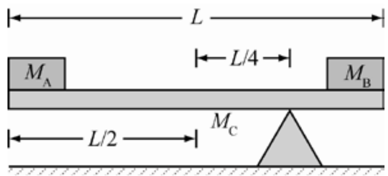
ROUND:

$\omega_f = 0.487$ rad/s.

DOUBLE-CHECK: We find that the angular velocity increases from 0.105 rad/s to 0.487 rad/s. Does it make sense that the professor speeds up by pulling in the arms? If you have ever watched a figure skating competition, you know that the answer is yes, and that speeding up the rate of rotation by a factor of ~3 is very reasonable.

10.74. **THINK:** Determine the angular acceleration, which can be obtained by first determining the total torque. Make sure that the moments of inertia are calculated with respect to the pivot point. $M_A = 1.00$ kg, $M_B = 10.0$ kg, $M_C = 20.0$ kg and $L = 5.00$ m.

SKETCH:



RESEARCH: $\sum \tau = I\alpha$

For M_A: $I_A = M_A(3L/4)^2$. For M_B: $I_B = M_B(L/4)^2$. For the rod: $I_C = (1/12)M_C L^2 + M_C(L/4)^2$.
$I = I_A + I_B + I_C$, $\sum \tau = \tau_A + \tau_C - \tau_B$, $\tau_A = M_A g(3L/4)$, $\tau_B = M_B g(L/4)$ and $\tau_C = M_C g(L/4)$.

SIMPLIFY: $\sum \tau = gL\left(\frac{3}{4}M_A - \frac{1}{4}M_B + \frac{1}{4}M_C\right) = \frac{gL}{4}(3M_A - M_B + M_C)$

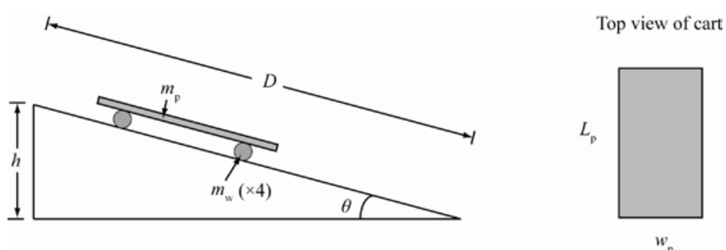
$$I = L^2\left(\frac{9}{16}M_A + \frac{1}{16}M_B + \left(\frac{1}{12}+\frac{1}{16}\right)M_C\right) = \frac{L^2}{16}\left(9M_A + M_B + \frac{7}{3}M_C\right),\quad \sum \tau = I\alpha \Rightarrow \alpha = \frac{\sum \tau}{I}$$

$$\alpha = \frac{gL}{4}\left(\frac{16}{L^2}\right)\left(\frac{3M_A - M_B + M_C}{9M_A + M_B + \frac{7}{3}M_C}\right) = \frac{4g}{L}\left(\frac{3M_A - M_B + M_C}{9M_A + M_B + \frac{7}{3}M_C}\right)$$

CALCULATE: $\alpha = \frac{4(9.81\text{ m/s}^2)}{5.00\text{ m}}\left(\frac{3(1.00\text{ kg}) - 10.0\text{ kg} + 20.0\text{ kg}}{9(1.00\text{ kg}) + 10.0\text{ kg} + \frac{7}{3}20.0\text{ kg}}\right) = 1.55\text{ rad/s}^2$

ROUND: Using three significant figures, $\alpha = 1.55\text{ rad/s}^2$. The positive sign indicates that the angular acceleration is counter clockwise.
DOUBLE-CHECK: Note that α decreases as *L* increases. This makes sense because *I* increases faster with *L* (*L* is squared) than does τ.

10.75. **THINK:** To determine the cart's final speed, use the conservation of energy. The initial gravitational potential energy is converted to kinetic energy. The total kinetic energy at the bottom is the sum of the translational and rotational kinetic energies. Use $m_p = 8.00$ kg, $m_w = 2.00$ kg, $L_p = 1.20$ m, $w_p = 60.0$ cm, r = 10.0 cm, D = 30.0 m and $\theta = 15.0°$.
SKETCH:



RESEARCH: The initial energy is $E_{tot} = U$ (potential energy). The final energy is $E_{tot} = K$ (kinetic energy). $U = M_{tot}gh$, $h = D\sin\theta$, $M_{tot} = m_p + 4m_w$, $K = M_{tot}v^2/2 + I\omega^2/2$, $\omega = v/r$ and $I = 4(m_w r^2/2)$.
SIMPLIFY:

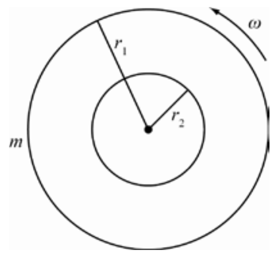
$$U = K \Rightarrow M_{tot}gh = \frac{1}{2}M_{tot}v^2 + \frac{1}{2}I\omega^2 \Rightarrow (m_p + 4m_w)gD\sin\theta = \frac{1}{2}(m_p + 4m_w)v^2 + \frac{1}{2}(4)\left(\frac{1}{2}m_w r^2\right)\left(\frac{v^2}{r^2}\right)$$

$$\Rightarrow\ \left(m_{\text{p}}+4m_{\text{w}}\right)gD\sin\theta=\left(\frac{1}{2}m_{\text{p}}+2m_{\text{w}}+m_{\text{w}}\right)v^2\ \Rightarrow\ v=\sqrt{\frac{\left(m_{\text{p}}+4m_{\text{w}}\right)gD\sin\theta}{\frac{1}{2}m_{\text{p}}+3m_{\text{w}}}}$$

CALCULATE: $v=\sqrt{\dfrac{\left(8.00\text{ kg}+4\left(2.00\text{ kg}\right)\right)\left(9.81\text{ m/s}^2\right)\left(30.0\text{ m}\right)\sin15.0^\circ}{\frac{1}{2}\left(8.00\text{ kg}\right)+3\left(2.00\text{ kg}\right)}}=11.04\text{ m/s}$

ROUND: The length of the incline is given to three significant figures, so the result should be rounded to $v = 11.0$ m/s.
DOUBLE-CHECK: This velocity is rather fast. In reality, the friction would slow the cart down. Note also that the radii of the wheels play no role.

10.76. **THINK:** Determining the moment of inertia is straightforward. To determine the torque, first determine the angular acceleration, α, and both $\Delta\omega$ and $\Delta\theta$ are known. Knowing α and I, the torque can be determined. $m = 15.0$ g, $r_1 = 1.5$ cm/2, $r_2 = 11.9$ cm/2, $\omega_{\text{i}} = 0$, $\omega_{\text{f}} = 4.3$ rev/s and $\Delta\theta = 0.25$ revs.
SKETCH:



RESEARCH: $\tau = I\alpha$, $I=\frac{1}{2}m\left(r_1^2+r_2^2\right)$, $\omega_{\text{f}}^2=\omega_{\text{i}}^2+2\alpha\Delta\theta$

SIMPLIFY:

(a) $I=\frac{1}{2}m\left(r_1^2+r_2^2\right)$

(b) $\alpha=\dfrac{\left(\omega_{\text{f}}^2-\omega_{\text{i}}^2\right)}{2\Delta\theta}$, $\tau=I\alpha=\dfrac{m}{4\Delta\theta}\left(r_1^2+r_2^2\right)\omega_{\text{f}}^2\ \left(\omega_{\text{i}}=0\right)$

CALCULATE:

(a) $I=\frac{1}{2}\left(15.0\cdot10^{-3}\text{ kg}\right)\left(\left(\frac{1.50\cdot10^{-2}}{2}\text{ m}\right)^2+\left(\frac{11.9\cdot10^{-2}\text{ m}}{2}\right)^2\right)=2.697\cdot10^{-5}\text{ kg m}^2$

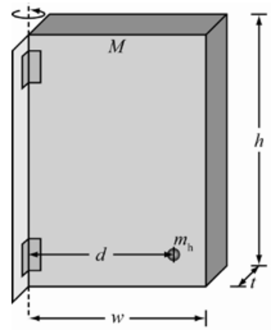
(b) $\Delta\theta=0.250\text{ revs}\left(\frac{2\pi\text{ rad}}{\text{rev}}\right)=1.571\text{ rad},\quad \omega_{\text{f}}=4.30\text{ rev/s}\left(\frac{2\pi\text{ rad}}{\text{rev}}\right)=27.02\text{ rad/s}$

$\alpha=\dfrac{\left(27.02\text{ rad/s}\right)^2}{2\left(1.571\text{ rad}\right)}=232.4\text{ rad/s}^2,\quad \tau=2.697\cdot10^{-5}\text{ kg m}^2\left(232.4\text{ rad/s}^2\right)=6.267\cdot10^{-3}\text{ N m}$

ROUND: Rounding to three significant figures, (a) $I=2.70\cdot10^{-5}\text{ kg m}^2$ and (b) $\tau=6.27\cdot10^{-3}\text{ N m}$.
DOUBLE-CHECK: These results are reasonable for the given values.

10.77. **THINK:** Begin with the moment of inertia of the door about an axis passing through its center of mass, then use the parallel axis theorem to shift the axis to the edge of the door, and then add the contribution of the handle, which can be treated as a point particle. $\rho = 550.\text{ kg/m}^3$, w = 0.550 m, h = 0.790 m, t = 0.0130 m, d = 0.450 m and $m_{\text{h}} = 0.150$ kg.

SKETCH:



RESEARCH: $M = \rho V = \rho wht$, $I_{\text{center}} = \frac{1}{12}M\left(w^2+t^2\right)$, $I_{\text{edge}} = I_{\text{center}} + M\left(\frac{w}{2}\right)^2$, $I_{\text{handle}} = m_{\text{h}}d^2$

$I = I_{\text{edge}} + I_{\text{handle}}$

SIMPLIFY: $I = \frac{M}{12}\left(w^2+t^2\right)+\frac{M}{4}w^2+m_{\text{h}}d^2 = \frac{M}{12}\left(4w^2+t^2\right)+m_{\text{h}}d^2$

CALCULATE: Substituting $M = \rho wht$ into the above equation yields:

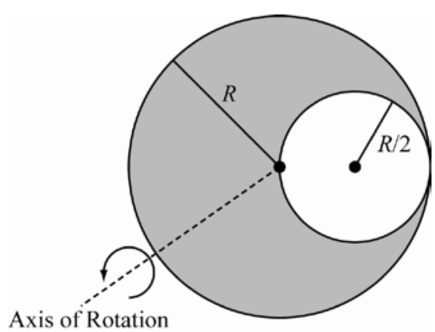
$$I = \frac{1}{12}\left(550.\text{ kg/m}^3\right)\left(0.550\text{ m}\right)\left(0.790\text{ m}\right)\left(0.0130\text{ m}\right)\left(4\left(0.550\text{ m}\right)^2+\left(0.0130\text{ m}\right)^2\right)+\left(0.150\text{ kg}\right)\left(0.450\text{ m}\right)^2$$
$$= 0.3437\text{ kg m}^2.$$

ROUND: Rounding to three significant figures gives $I = 0.344\text{ kg m}^2$.

DOUBLE-CHECK: This is a reasonable result for a door of this size. Note that the height of the door enters into only the calculation of the door's mass.

10.78. **THINK:** The moment of inertia of the machine part is the moment of inertia of the initial solid disk about its center, minus the moment of inertia of a solid disk of the amount of mass removed about its outside edge (which is at the center of the disk). M = mass of the disk without the hole cut out, and m = mass of the material cut out to make a hole.

SKETCH:



RESEARCH: $I_{\text{center}} = MR^2/2$ (disk spinning about its center). $I_{\text{edge}} = \frac{1}{2}m\left(R/2\right)^2 + m\left(R/2\right)^2$ (disk spinning about its edge). The area of the hole is $\pi R^2/4$. The area of the disk without the hole is πR^2. The area of the disk with the hole is $\pi R^2 - \pi R^2/4 = 3\pi R^2/4$. The area of the hole is 1/4 the area of the disk without the hole; therefore, because the disk has uniform density, $m = M/4$. The moment of inertia is

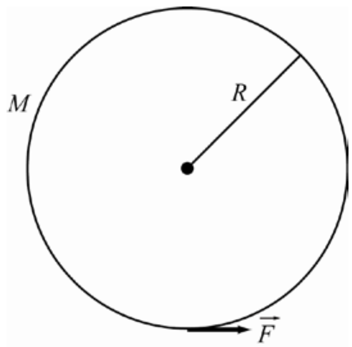
$$I = \frac{1}{2}MR^2 - \left(\frac{1}{2}m\left(\frac{R}{2}\right)^2 + m\left(\frac{R}{2}\right)^2\right).$$

SIMPLIFY: Substitute $m = M/4$ into the above equation to get

$$I = \frac{1}{2}MR^2 - \left[\frac{M}{8}\left(\frac{R^2}{4}\right) + \frac{M}{4}\left(\frac{R^2}{4}\right)\right] = \frac{16}{32}MR^2 - \frac{1}{32}MR^2 - \frac{2}{32}MR^2 = \frac{13}{32}MR^2.$$

CALCULATE: This step is not necessary.
ROUND: This step is not necessary.
DOUBLE-CHECK: As expected, the moment of inertia decreases when the hole is cut out.

10.79. **THINK:** If the angular momentum and the torque are determined, the time can be determined by recalling that torque is the time rate of change of angular momentum. To determine the angular momentum, first determine the angular speed required to produce a centripetal acceleration equal to Earth's gravitational acceleration. From this, the angular momentum, L, of the space station can be determined. Finally, the torque can be determined from the given force and the radius of the space station. $R = 50.0$ m, $M = 2.40\cdot 10^5$ kg and $F = 1.40\cdot 10^2$ N.
SKETCH:



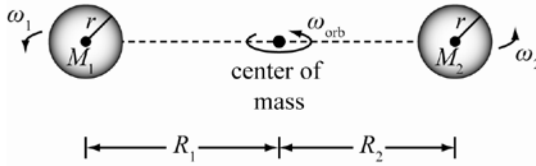
RESEARCH: $I = MR^2,\ L = \omega I,\ v = \omega R,\ \dfrac{v^2}{R} = g,\ \tau = FR,\ \tau = \dfrac{\Delta L}{\Delta t}$

SIMPLIFY: $\Delta t = \dfrac{\Delta L}{\tau} = \dfrac{\omega I - 0}{FR} = \dfrac{MR^2\omega}{FR} = \dfrac{MR\omega}{F},\quad \omega = \dfrac{v}{R} = \dfrac{1}{R}\sqrt{Rg} = \sqrt{\dfrac{g}{R}},\quad \Delta t = \dfrac{MR}{F}\sqrt{\dfrac{g}{R}} = \dfrac{M\sqrt{Rg}}{F}$

CALCULATE: $\Delta t = \dfrac{2.40\cdot 10^5\text{ kg}\sqrt{(50.0\text{ m})(9.81\text{ m/s}^2)}}{1.40\cdot 10^2\text{ N}} = 3.797\cdot 10^4\text{ s}$

ROUND: The radius of the space station is given to three significant figures, so the result should be rounded to $\Delta t = 3.80\cdot 10^4$ s.
DOUBLE-CHECK: The result is equal to about 10 hours. For such a relatively small thrust, this result is reasonable. As expected, this time interval increases if either the thrust decreases or the mass increases.

10.80. **THINK:** There is enough information given to determine the stars' rotational and translational kinetic energies directly and subsequently determine their ratio. Note that the orbital period is given as 2.4 hours. Use the values: $M_1 = 1.250M_{\text{Sun}}$, $M_2 = 1.337M_{\text{Sun}}$, $\omega_1 = 2\pi$ rad/2.8 s, $\omega_2 = 2\pi$ rad/0.023 s, r = 20.0 km, $R_1 = 4.54\cdot 10^8$ m, $R_2 = 4.23\cdot 10^8$ m and $\omega_{\text{orb}} = 2\pi$ rad/2.4 h.
SKETCH:



RESEARCH: $K_{\text{rot}} = \dfrac{1}{2}I\omega^2,\ I = \dfrac{2}{5}MR^2,\ K_{\text{orb}} = \dfrac{1}{2}Mv^2 = \dfrac{1}{2}MR^2\omega^2$

SIMPLIFY:

(a) $\frac{K_{1,\text{rot}}}{K_{2,\text{rot}}}=\frac{I_1\omega_1^2}{I_2\omega_2^2}=\frac{M_1r_1^2\omega_1^2}{M_2r_2^2\omega_2^2}=\frac{M_1\omega_1^2}{M_2\omega_2^2}$

(b) $\frac{K_{1,\text{rot}}}{K_{1,\text{orb}}}=\frac{\frac{1}{5}M_1r_1^2\omega_1^2}{\frac{1}{2}M_1R_1^2\omega_{\text{orb}}^2}=\frac{2r_1^2\omega_1^2}{5R_1^2\omega_{\text{orb}}^2},\quad \frac{K_{2,\text{rot}}}{K_{2,\text{orb}}}=\frac{2r_2^2\omega_2^2}{5R_2^2\omega_{\text{orb}}^2}$

CALCULATE:

(a) $\frac{K_{1,\text{rot}}}{K_{2,\text{rot}}}=\frac{1.25/(2.8)^2}{1.337/(0.023)^2}=6.308\cdot 10^{-5}$

(b) $\frac{K_{1,\text{orb}}}{K_{1,\text{rot}}}=\frac{2\left(20\cdot 10^3\right)^2/(2.8\text{ s})^2}{5\left(4.54\cdot 10^8\right)^2/(2.4\text{ h}\cdot 3600\text{ s/h})^2}=0.00739128$

$\frac{K_{2,\text{rot}}}{K_{2,\text{orb}}}=\frac{2\left(20\cdot 10^3\text{ m}\right)^2/(0.023\text{ s})^2}{5\left(4.23\cdot 10^8\right)^2/(2.4\text{ h}\cdot 3600\text{ s/h})^2}=126.186$

ROUND: The rotation periods are given to at least three significant figures, so the results should be rounded to:

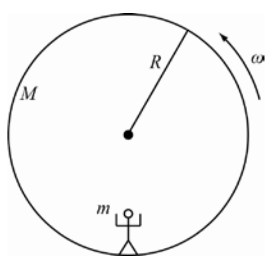
(a) $K_{1,\text{rot}}/K_{2,\text{rot}}=6.31\cdot 10^{-5}$

(b) $K_{1,\text{orb}}/K_{1,\text{rot}}=7.39\cdot 10^{-3},\quad K_{2,\text{orb}}/K_{2,\text{rot}}=126$

DOUBLE-CHECK: M_2 has a much faster rotational speed than M_1. The kinetic energy for M_1 is dominated by the orbit, while for M_2 it is dominated by rotational motion.

10.81. **THINK:** Conservation of angular momentum can be considered to determine the angular momentum of the merry-go-round. From this, the mass, M, of the merry-go-round can be determined. For parts (b) and (c), use the uniform acceleration equations to answer the problem. $R = 1.50$ m, $\omega = 1.30$ rad/s, $m = 52.0$ kg and $v = 6.80$ m/s (speed of the student just prior to jumping on).

SKETCH:



RESEARCH: $L_{\text{student}}=Rmv,\quad L=I\omega$ (merry-go-round), $I=\frac{1}{2}MR^2+mR^2,\quad \Delta\theta=\frac{1}{2}\alpha t^2+\omega_{\text{i}}t,$

$\omega_{\text{f}}=\alpha t+\omega_{\text{i}}$, and $\tau = I\alpha$.

SIMPLIFY:

(a) $L_{\text{student}}=L \Rightarrow Rmv=I\omega=\left(\frac{1}{2}MR^2+mR^2\right)\omega \Rightarrow \frac{1}{2}MR^2\omega=Rmv-mR^2\omega$

$\Rightarrow M=\frac{2}{\omega R^2}\left(Rmv-mR^2\omega\right) \Rightarrow \frac{2m}{R\omega}\left(v-R\omega\right)=2m\left(\frac{v}{\omega R}-1\right)$

(b) $\tau=I\alpha=\left(\frac{1}{2}MR^2+mR^2\right)\alpha,\quad \alpha=\frac{\Delta\omega}{\Delta t}=-\frac{\omega}{t}\quad\left(\omega_{\text{i}}=\omega,\ \omega_{\text{f}}=0,\ t_{\text{f}}=t,\ t_{\text{i}}=0\right),\quad \tau=-\frac{\omega R^2}{t}\left(\frac{M}{2}+m\right)$

(c) $\Delta\theta = \frac{1}{2}\alpha t^2 + \omega_{\text{i}} t = \frac{1}{2}\left(-\frac{\omega}{t}\right)t^2 + \omega t = -\frac{1}{2}\omega t + \omega t = \frac{1}{2}\omega t$

CALCULATE:

(a) $M = 2(52.0 \text{ kg})\left(\frac{6.80 \text{ m/s}}{1.30 \text{ rad s}^{-1}(1.50 \text{ m})} - 1\right) = 258.7 \text{ kg}$

(b) $\tau = -\frac{(1.30 \text{ rad s}^{-1})(1.50 \text{ m})^2}{35.0 \text{ s}}\left(\frac{258.7 \text{ kg}}{2} + 52.0 \text{ kg}\right) = -15.15 \text{ N m}$

(c) $\Delta\theta = \frac{1}{2}(1.30 \text{ rad/s})(35.0 \text{ s}) = 22.75 \text{ rad} = 22.75 \text{ rad}\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right) = 3.621 \text{ rev}$

ROUND: Round the results to three significant figures.

(a) $M = 259 \text{ kg}$

(b) $\tau = -15.2 \text{ N m}$

(c) $\Delta\theta = 3.62$ revolutions

DOUBLE-CHECK: The results are all consistent with the given information.

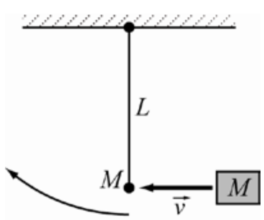
10.82. **THINK:**

(a) The speed of the pendulum just after the collision can be determined by considering the conservation of linear momentum. From the conservation of energy, the maximum height of the pendulum can be determined, since at this point, all of the initial kinetic energy will be stored as gravitational potential energy.

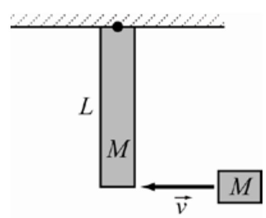
(b) From the conservation of angular momentum, the rotation speed of the pendulum just after collision can be determined. From the conservation of energy, the maximum height of the pendulum can be determined, since at this point, all of the initial rotational kinetic energy will be stored as gravitational potential energy. $L = 0.48$ m and $v = 3.6$ m/s.

SKETCH:

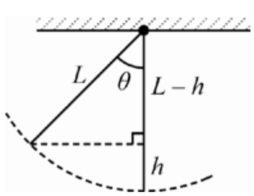
(a)



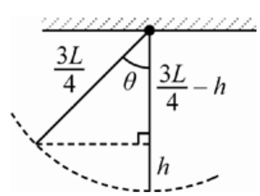
(b)



(c)



(d)



RESEARCH:

(a) $E = \text{constant} = K + U, \quad K = \frac{1}{2}mv^2, \quad U = mgh$

(b) $L = \text{constant} = I\omega, \quad I_{\text{rod}} = \frac{1}{3}ML^2, \quad I_{\text{proj}} = ML^2$

SIMPLIFY: v_0 is the speed of the projectile just prior to collision. v_{p} is the speed of the pendulum at the lower edge just after collision.

(a) $P_i = P_f \Rightarrow Mv_0 = (M+M)v_p \Rightarrow v_p = \frac{1}{2}v_0$; At the pendulum's maximum height,

$$K_f = (M+M)gh = \frac{1}{2}(M+M){v_p}^2.$$

$$h = \frac{{v_p}^2}{2g} = \frac{{v_0}^2}{8g} \Rightarrow \cos\theta = 1 - \frac{h}{L} = 1 - \frac{{v_0}^2}{8gL} \Rightarrow \theta = \cos^{-1}\left(1 - \frac{{v_0}^2}{8gL}\right)$$

(b) $L = \omega I$, $\omega = v/L$, $L_i = I_i \frac{v_0}{L} = MLv_0$

$$L_f = (I_{rod} + I_i)\omega_f = \frac{4}{3}ML^2\omega_f = MLv_0 \Rightarrow \omega_f = \frac{3}{4L^2}Lv_0 = \frac{3v_0}{4L} \Rightarrow \frac{3v_0}{4L} = \frac{v_p}{L} \Rightarrow v_p = \frac{3}{4}v_0$$

At maximum height: $\frac{1}{2}I{\omega_f}^2 = mgh \Rightarrow \frac{1}{2}\left(\frac{4}{3}ML^2\right)\left(\frac{3v_0}{4L}\right)^2 = 2Mgh \Rightarrow \frac{3}{8}{v_0}^2 = 2gh \Rightarrow h = \frac{3{v_0}^2}{16g}$.

This is the height attained by the center of mass of the pendulum and projectile system. By symmetry, the center of mass of the system is located $3L/4$ from the top. So,

$$\cos\theta = 1 - \frac{h}{3L/4} = 1 - \frac{4h}{3L} = 1 - \frac{4}{3L}\cdot\frac{3{v_0}^2}{16g} = 1 - \frac{{v_0}^2}{4gL} \Rightarrow \theta = \cos^{-1}\left(1 - \frac{{v_0}^2}{4gL}\right).$$

CALCULATE:

(a) $\theta = \cos^{-1}\left(1 - \frac{(3.6 \text{ m/s})^2}{8(9.81 \text{ m/s}^2)(0.48 \text{ m})}\right) = 49.01°$

(b) $\theta = \cos^{-1}\left(1 - \frac{(3.6 \text{ m/s})^2}{4(9.81 \text{ m/s}^2)(0.48 \text{ m})}\right) = 71.82°$

ROUND: Round the results to three significant figures.

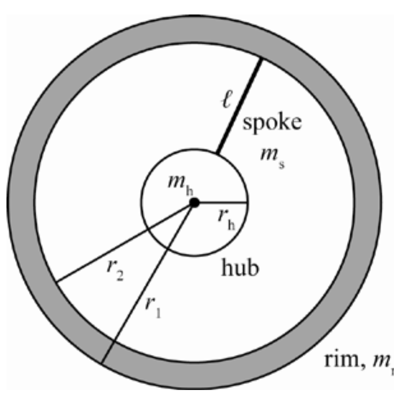
(a) $\theta = 49.0°$

(b) $\theta = 71.8°$

DOUBLE-CHECK: The rod swings higher. This is expected since the center of mass is higher than for the pendulum. The projectile exerts a greater torque on the rod.

10.83. **THINK:** The quantity of interest can be calculated directly from the given information. $m_r = 5.20$ kg, $m_h = 3.40$ kg, $m_s = 1.10$ kg, $r_1 = 0.900$ m, $r_2 = 0.860$ m, $r_h = 0.120$ m and $l = r_2 - r_h$.

SKETCH:



RESEARCH: $I_{\text{rim}}=\frac{1}{2}m_{\text{r}}\left(r_1^2+r_2^2\right)$, $I_{\text{spoke}}=\frac{1}{12}m_{\text{s}}l^2+m_{\text{s}}d^2$, (with $d=\frac{1}{2}l+r_{\text{h}}$), $I_{\text{hub}}=\frac{1}{2}m_{\text{h}}r_{\text{h}}^2$

SIMPLIFY: $I=I_{\text{rim}}+I_{\text{hub}}+12I_{\text{spoke}}$, $M=m_{\text{r}}+m_{\text{h}}+12m_{\text{s}}$, $R=r_1$

CALCULATE: $I_{\text{rim}}=\frac{1}{2}5.20\text{ kg}\left((0.900)^2+(0.860)^2\right)\text{ m}^2=4.029\text{ kg m}^2$

$I_{\text{hub}}=\frac{1}{2}(3.40\text{ kg})(0.120\text{ m})^2=2.448\cdot 10^{-2}\text{ kg m}^2$

$I_{\text{spoke}}=\frac{1}{12}(1.10\text{ kg})(0.860\text{ m}-0.120\text{ m})^2+(1.10\text{ kg})(0.490\text{ m})^2=3.143\cdot 10^{-1}\text{ kg m}^2$

$I=I_{\text{rim}}+I_{\text{hub}}+12I_{\text{spoke}}=7.825\text{ kg m}^2$, $M=\left[5.20+3.40+12(1.10)\right]\text{ kg}=21.8\text{ kg}$

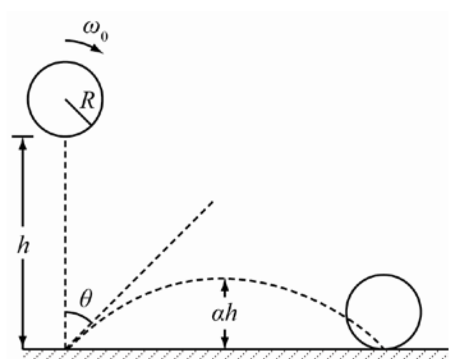
$c=\frac{I}{MR^2}=\frac{7.825\text{ kgm}^2}{(21.8\text{ kg})(0.900\text{ m})^2}=0.4431$

ROUND: Rounding to three significant figures, $c=0.443$.

DOUBLE-CHECK: It is reasonable that the moment of inertia is dominated by the rim and the spokes, and the hub is negligible.

10.84. **THINK:** To determine the angles in parts (a), the vertical and horizontal components of the velocity just after impact must be determined. To determine the vertical velocity, consider the conservation of energy. To determine the horizontal velocity, consider the linear and angular impulses experienced in either of the following two situations. Situation I: The ball slips on the floor during the entire impact time. Kinetic friction must be considered the entire time. Situation II: The ball stops slipping on the floor at some point during the impact. From this point for the duration of the impact, rolling motion is attained, and the usual equations relating angular and rotational speeds are applicable.

SKETCH:



RESEARCH: Energy conservation is given by $mgh=mv_0^2/2$. v_0 is the speed of the ball just prior to the impact for the first time. Also, from energy conservation: $mg(\alpha h)=\frac{1}{2}mv_{2y}^2$, where v_{2y} is the vertical velocity just after the impact of the ball with the ground. Linear impulse is given by

$$\int_{t_1}^{t_2}F(t)dt=p(t_2)-p(t_1).$$

Angular impulse is given by $\int_{t_1}^{t_2}\tau(t)dt=L(t_2)-L(t_1)=I(\omega_0-\omega_2)$.

SIMPLIFY: Just prior to impact: $mgh=mv_0^2/2\Rightarrow v_0=\sqrt{2gh}$. Just after impact:

$$\alpha mgh=\frac{1}{2}mv_{2y}^2\Rightarrow v_{2y}=+\sqrt{2\alpha gh}.$$

Situation I:

n is the normal force and μn is the frictional force. The impulses are as follows:

$$I_y = \int_{t_1}^{t_2} n(t)\,dt = m\sqrt{2\alpha gh} - \left(-m\sqrt{2gh}\right) = m\left(1+\sqrt{\alpha}\right)v_0, \quad I_x = \int_{t_1}^{t_2} \mu n(t)\,dt = \mu I_y = \mu m\left(1+\sqrt{\alpha}\right)v_0 = mv_{2x}$$

$$I_\theta = \int_{t_1}^{t_2} \mu n R\,dt = R\mu I_y = R\mu m\left(1+\sqrt{\alpha}\right)v_0 = I\left(\omega_0 - \omega_2\right) \Rightarrow v_{2x} = \mu\left(1+\sqrt{\alpha}\right)v_0 = \mu\left(1+\sqrt{\alpha}\right)\sqrt{2gh}$$

$$\omega_2 = \omega_0 - \frac{1}{I}R\mu m\left(1+\sqrt{\alpha}\right)v_0 = \omega_0 - \frac{R}{I}\mu m\left(1+\sqrt{\alpha}\right)\sqrt{2gh} = \omega_0 - \frac{Rmv_{2x}}{I}$$

(a) $\tan\theta = \left(\dfrac{v_{2x}}{v_{2y}}\right)^{-1} = \left(\dfrac{\mu\left(1+\sqrt{\alpha}\right)\sqrt{2gh}}{\sqrt{2\alpha gh}}\right)^{-1} = \left(\dfrac{\mu\left(1+\sqrt{\alpha}\right)}{\sqrt{\alpha}}\right)^{-1}$

(b) The time, t, it takes for the ball to fall is $t = \dfrac{2v_{2y}}{g} = \dfrac{2}{g}\sqrt{2\alpha gh} = \sqrt{\dfrac{8\alpha h}{g}}$.

The distance traveled during this time is $d = v_{2x}t = \mu\left(1+\sqrt{\alpha}\right)\left(\sqrt{2gh}\right)\left(\sqrt{\dfrac{8\alpha h}{g}}\right) = 4\mu\left(1+\sqrt{\alpha}\right)\left(\sqrt{\alpha}\right)h$.

(c) $\omega_2 = \omega_0 - \dfrac{Rmv_{2x}}{I}$. The minimum ω_0 occurs when $R\omega_2 = v_{2x}$, where $R\omega_2$ is the velocity of the contact point. $R\omega_{0,\min} - \dfrac{R^2mv_{2x}}{I} = v_{2x}$, $\omega_{0,\min} = v_{2x}\left(\dfrac{1}{R} + \dfrac{Rm}{I}\right) = \dfrac{v_{2x}}{R}\left(1+\dfrac{mR^2}{I}\right) = \dfrac{\mu\left(1+\sqrt{\alpha}\right)\sqrt{2gh}}{R}\left(1+\dfrac{mR^2}{I}\right)$

Situation II:

After the ball stops slipping, there is a rolling motion and $\omega_2 R = v_{2x}$. The impulses are as follows.

$$I_x = \int_{t_1}^{t_2} \mu n\,dt = mv_{2x}, \quad I_\theta = \int_{t_1}^{t_2} \mu n R\,dt = I\left(\omega_0 - \omega_2\right) = RI_x$$

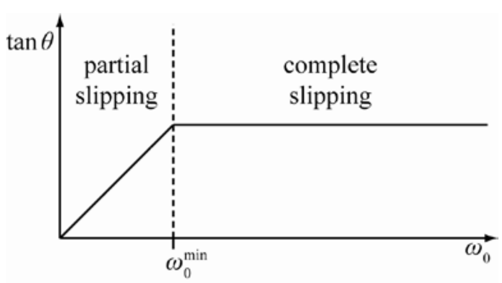
$$\Rightarrow I\left(\omega_0 - \omega_2\right) = Rmv_{2x} \Rightarrow I\left(\omega_0 - \frac{v_{2x}}{R}\right) = Rmv_{2x}.$$

Solve for v_{2x} by substituting $I = \left(2mR^2/5\right)$ into the above equation to get:

$$\frac{2}{5}mR^2\left(\omega_0 - \frac{v_{2x}}{R}\right) = Rmv_{2x} \Rightarrow \frac{2}{5}\left(\omega_0 R - v_{2x}\right) = v_{2x} \Rightarrow v_{2x}\left(1+\frac{2}{5}\right) = \frac{2}{5}\omega_0 R \Rightarrow v_{2x} = \frac{5}{7}\left(\frac{2}{5}\omega_0 R\right) = \frac{2}{7}\omega_0 R.$$

(d) $\tan\theta = \dfrac{v_{2x}}{v_{2y}} = \dfrac{\frac{2}{7}\omega_0 R}{\sqrt{2\alpha gh}}$

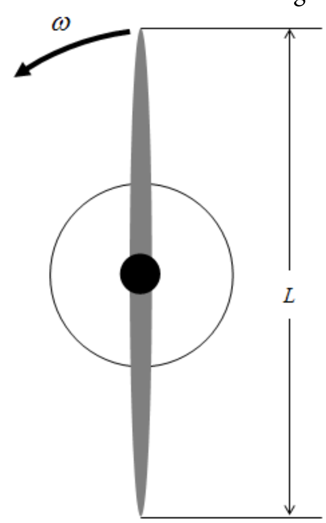
(e) $d = v_{2x}t = \dfrac{2}{7}\omega_0 R\sqrt{\dfrac{8\alpha h}{g}}$



CALCULATE: This step is not necessary.
ROUND: This step is not necessary.
DOUBLE-CHECK: When only partial slipping occurs, the horizontal distance traveled should and does depend on ω_0.

Multi-Version Exercises

10.85. **THINK:** The length and mass of the propeller, as well as the frequency with which it is rotating, are given. To find the kinetic energy of rotation, it is necessary to find the moment of inertia, which can be calculated from the mass and radius by approximating the propeller as a rod with constant mass density.
SKETCH: The propeller is shown as it would be seen looking directly at it from in front of the plane.



RESEARCH: The kinetic energy of rotation is related to the moment of inertia and angular speed by the equation $K_{\text{rot}} = \frac{1}{2}I\omega^2$. The angular speed $\omega = 2\pi f$ can be computed from the frequency of the propeller's rotation. Approximating the propeller as a rod with constant mass density means that the formula $I = \dfrac{1}{12}mL^2$ for a long, thin rod rotating about its center of mass can be used.
SIMPLIFY: Combine the equations for the moment of inertia and angular speed to get a single equation for the kinetic energy $K_{\text{rot}} = \frac{1}{2}I\omega^2 = \dfrac{1}{2}\left(\dfrac{1}{12}mL^2\right)\cdot(2\pi f)^2$. Using algebra, this can be simplified to $K_{\text{rot}} = \dfrac{m}{6}(\pi L f)^2$. Since the angular speed is given in revolutions per minute, the conversion 1 minute = 60 seconds will also be needed.
CALCULATE: The propeller weighs m = 17.36 kg, it is L = 2.012 m long, and it rotates at a frequency of f = 3280. rpm. The rotational kinetic energy is

$$\begin{aligned}K_{\text{rot}} &= \frac{m}{6}(\pi L f)^2 \\ &= \frac{17.36\text{ kg}}{6}\left(\pi \cdot 2.012\text{ m} \cdot 3280.\text{ rpm} \cdot \frac{1\text{min}}{60\text{sec}}\right)^2 \\ &= 345,461.2621\text{ J}\end{aligned}$$

ROUND: The values in the problem are all given to four significant figures, so the final answer should have four figures. The propeller has a rotational kinetic energy of $3.455\cdot 10^5$ J or 345.5 kJ.
DOUBLE-CHECK: Given the large amount of force needed to lift a plane, it seems reasonable that the energy in the propeller would be in the order of hundreds of kilojoules. Working backwards, if a propeller weighing 17.36 kg and having length 2.012 m has rotational kinetic energy 345.5 kJ, then it is turning at

$f=\frac{1}{\pi L}\sqrt{\frac{6K_{\text{rot}}}{m}}=\frac{1}{\pi\cdot 2.012\text{ m}}\sqrt{\frac{6\cdot 3.455\cdot 10^5\text{ J}}{17.36\text{ kg}}}$. This is 54.667 revolutions per second, which agrees with the given value of 54.667·60 = 3280 rpm. This confirms that the calculations were correct.

10.86. $K_{\text{rot}}=\frac{m}{6}(\pi Lf)^2$

$$f=\frac{1}{\pi L}\sqrt{\frac{6K_{\text{rot}}}{m}}$$

$$=\frac{1}{\pi(2.092\text{ m})}\sqrt{\frac{6\left(422.8\cdot 10^3\text{ J}\right)}{17.56\text{ kg}}}=57.8\text{ rev/s}=3470.\text{ rpm}$$

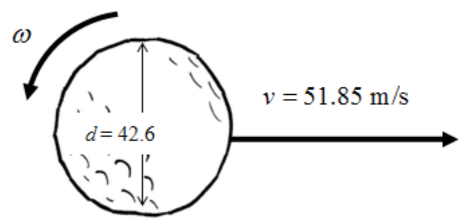
10.87. $K_{\text{rot}}=\frac{m}{6}(\pi Lf)^2$

$$m=\frac{6K_{\text{rot}}}{(\pi Lf)^2}$$

$$=\frac{6\left(124.3\cdot 10^3\text{ J}\right)}{\left[\pi(1.812\text{ m})\left(2160.\text{ rpm}\cdot\frac{1}{60}\text{ s/min}\right)\right]^2}=17.76\text{ kg}$$

10.88. **THINK:** The total kinetic energy of the golf ball is the sum of the rotational kinetic energy and the translational kinetic energy. The translational kinetic energy can be calculated from the mass of the ball and the speed of the center of mass of the golf ball, both of which are given in the question. To find the rotational kinetic energy, it is necessary to find the moment of inertia of the golf ball. Though the golf ball is not a perfect sphere, it is close enough that the moment of inertia can be computed from the mass and diameter of the golf ball using the approximation for a sphere.

SKETCH: The golf ball has both rotational and translational motion.



RESEARCH: The total kinetic energy is equal to the translational kinetic energy plus the rotational kinetic energy $K=K_{\text{trans}}+K_{\text{rot}}$. The translational kinetic energy is computed from the speed and the mass of the golf ball using the equation $K_{\text{trans}}=\frac{1}{2}mv^2$. The rotational kinetic energy is computed from the moment of inertia and the angular speed by $K_{\text{rot}}=\frac{1}{2}I\omega^2$. It is necessary to compute the moment of inertia and the angular speed. The angular speed $\omega=2\pi f$ depends only on the frequency. To find the moment of inertia, first note that golf balls are roughly spherical. The moment of inertia of a sphere is given by $I=\frac{2}{5}mr^2$. The question gives the diameter d which is twice the radius (d / 2 = r). Since the frequency is given in revolutions per minute and the speed is given in meters per second, the conversion factor $\frac{1\text{ min}}{60\text{ sec}}$ will be necessary.

SIMPLIFY: First, find the moment of inertia of the golf ball in terms of the mass and diameter to get $I=\frac{1}{10}md^2$. Substituted for the angular speed and moment of inertia in the equation for rotational kinetic energy to get $K_{\text{rot}} = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{10}md^2\right)(2\pi f)^2$. Finally, use the equations $K_{\text{trans}} = \frac{1}{2}mv^2$ and $K_{\text{rot}} = \frac{1}{2}\left(\frac{1}{10}md^2\right)(2\pi f)^2$ to find the total kinetic energy and simplify using algebra:

$$\begin{aligned} K &= K_{\text{trans}} + K_{\text{rot}} \\ &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}\left(\tfrac{1}{10}md^2\right)(2\pi f)^2 \\ &= \tfrac{1}{2}mv^2 + \tfrac{1}{5}m(\pi d f)^2 \end{aligned}$$

CALCULATE: The mass of the golf ball is 45.90 g = 0.04590 kg, its diameter is 42.60 mm = 0.04260 m, and its speed is 51.85 m/s. The golf ball rotates at a frequency of 2857 revolutions per minute. The total kinetic energy is

$$\begin{aligned} K &= \tfrac{1}{2}mv^2 + \tfrac{1}{5}m(\pi d f)^2 \\ &= \tfrac{1}{2}(0.04590 \text{ kg})(51.85 \text{ m/s})^2 + \tfrac{1}{5}(0.04590 \text{ kg})\left(\pi \cdot 0.04260 \text{ m} \cdot 2857 \text{ rpm} \cdot \frac{1 \text{ min}}{60 \text{ sec}}\right)^2 \\ &= 62.07209955 \text{ J} \end{aligned}$$

ROUND: The mass, speed, frequency, and diameter of the golf ball are all given to four significant figures, so the translational and rotational kinetic energies should both have four significant figures, as should their sum. The total energy of the golf ball is 62.07 J.

DOUBLE-CHECK: The golf ball's translational kinetic energy alone is equal to $\frac{1}{2}(0.04590 \text{ kg})(51.85 \text{ m/s})^2 = 61.7 \text{ J}$, and it makes sense that a well-driven golf ball would have much more energy of translation than energy of rotation.

10.89. $K = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2}mv^2 + \frac{1}{5}m(\pi d f)^2$

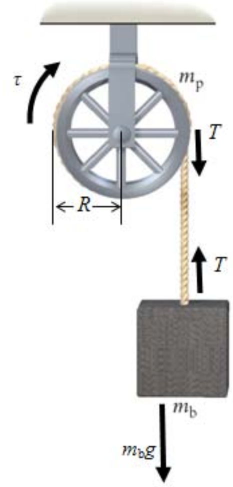
$$\begin{aligned} f &= \frac{1}{\pi d}\sqrt{5\left(\frac{K}{m} - \frac{1}{2}v^2\right)} \\ &= \frac{1}{\pi(0.04260 \text{ m})}\sqrt{5\left(\frac{67.67 \text{ J}}{0.04590 \text{ kg}} - \frac{1}{2}(54.15 \text{ m/s})^2\right)} = 47.79 \text{ rev/s} = 2867 \text{ rpm} \end{aligned}$$

10.90. $K = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2}mv^2 + \frac{1}{5}m(\pi d f)^2$

$$\begin{aligned} v &= \sqrt{2\left(\frac{K}{m} - \frac{1}{5}(\pi d f)^2\right)} \\ &= \sqrt{2\left(\frac{73.51 \text{ J}}{0.04590 \text{ kg}} - \frac{1}{5}\left(\pi \cdot 0.04260 \text{ m} \cdot 2875 \text{ rpm} \cdot \frac{1}{60} \text{ min/s}\right)^2\right)} = 56.45 \text{ m/s} \end{aligned}$$

10.91. **THINK:** The gravitational force on the block is transmitted through the rope, causing a torque on the pulley. The torque causes an angular acceleration, and the linear acceleration is calculated from the angular acceleration.

SKETCH: Use the figure from the text:



RESEARCH: The torque on the pulley is given by the tension on the rope times the radius of the pulley $\tau = TR$. This torque will cause an angular acceleration $\tau = I\alpha$, where the moment of inertia of the pulley is given by $I = m_p R^2$. The tension on the rope is given by $T - m_b g = -m_b a$ (the minus indicates that the block is accelerating downward). The linear acceleration of the block a is related to the angular acceleration of the pulley α by the equation $a = R\alpha$.

SIMPLIFY: First, substitute for the tension on the pulley $T = m_b g - m_b a$ in the equation for the torque τ to get $\tau = (m_b g - m_b a)R$. Then, substitute for the moment of inertia ($I = m_p R^2$) and angular acceleration ($\alpha = a/R$) in the equation $\tau = I\alpha$ to get $\tau = (m_p R^2)\left(\frac{a}{R}\right) = m_p Ra$. Combine these two expressions for the torque to get $(m_b g - m_b a)R = m_p Ra$. Finally, solve this expression for the linear acceleration a of the block:

$$(m_b g - m_b a)R = m_p aR$$
$$m_b gR - m_b aR + m_b aR = m_p aR + m_b aR$$
$$m_b gR = (m_p R + m_b R)a$$
$$\frac{m_b gR}{m_p R + m_b R} = a$$
$$a = \frac{m_b g}{m_p + m_b}$$

CALCULATE: The mass of the block is m_b = 4.243 kg and the mass of the pulley is m_p = 5.907 kg. The acceleration due to gravity is –9.81 m/s^2. So, the total (linear) acceleration of the block is

$$a = \frac{m_b g}{m_p + m_b} = \frac{-9.81 \text{ m/s}^2 \cdot 4.243 \text{ kg}}{5.907 \text{ kg} + 4.243 \text{ kg}} = -4.100869951 \text{ m/s}^2 .$$

ROUND: The masses of the pulley and block are given to four significant figures, and the sum of their masses has five figures. On the other hand, the gravitational constant g is given only to three significant figures. So, the final answer should have three significant figures. The block accelerates downward at a rate of 4.10 m/s^2.

DOUBLE-CHECK: A block falling freely would accelerate (due to gravity near the surface of the Earth) at a rate of 9.81 m/s^2 towards the ground. The block attached to the pulley will still accelerate downward, but the rate of acceleration will be less (the potential energy lost when the block falls 1 meter will equal the kinetic energy of a block in free fall, but it will equal the kinetic energy of the block falling plus the rotational kinetic energy of the pulley in the problem). The mass of the pulley is close to, but a bit larger than, the mass of the block, so the acceleration of the block attached to the pulley should be a bit less than half of the acceleration of the block in free fall. This agrees with the final acceleration of 4.10 m/s^2, which is a bit less than half of the acceleration due to gravity.

10.92. $$a = \frac{m_\text{b} g}{m_\text{p} + m_\text{b}}$$

$$m_\text{p} = \frac{m_\text{b} g}{a} - m_\text{b} = m_\text{b}\left(\frac{g}{a} - 1\right) = (4.701 \text{ kg})\left(\frac{9.81 \text{ m/s}^2}{4.330 \text{ m/s}^2} - 1\right) = 5.95 \text{ kg}$$

10.93. $$a = \frac{m_\text{b} g}{m_\text{p} + m_\text{b}}$$

$$m_\text{p} = \frac{m_\text{b} g}{a} - m_\text{b} = m_\text{b}\left(\frac{g}{a} - 1\right)$$

$$\Rightarrow m_\text{b} = \frac{m_\text{p}}{\frac{g}{a} - 1} = \frac{5.991 \text{ m}}{\frac{9.81 \text{ m/s}^2}{4.539 \text{ m/s}^2} - 1} = 5.16 \text{ kg}$$

Chapter 11: Static Equilibrium

Concept Checks

11.1. d **11.2.** b **11.3.** e **11.4.** c **11.5.** a **11.6.** c **11.7.** a

Multiple-Choice Questions

11.1. c **11.2.** c **11.3.** b **11.4.** e **11.5.** c **11.6** b **11.7.** a **11.8** d **11.9.** c **11.10.** a **11.11.** c **11.12.** d

Conceptual Questions

11.13. The upward force exerted on the wing gears is four times that exerted on the nose gear. By symmetry, the center of mass must lie along the centerline of the plane. In order for the torque about the center of mass to be zero, the center of mass must be one fifth of the distance from the midpoint of the wing gears and the nose gear.

11.14. By symmetry, a vertical force equal to half the weight of the arch, $f_{\text{v}} = W/2$ must be supplied at the base of each leg. The torque about the base of each leg must be zero. The weight of each leg exerts a torque about the base equal to the weight of the leg, $W/2$, times the horizontal distance between the center of mass of the leg and the base. The center of mass of each leg is not at its midpoint. Rather, if the legs are of uniform cross section and density, the horizontal distance from the center line of the arch to the center of mass of the leg is

$$x = \frac{\int_0^{\pi/2} a^2 \cos\theta \, d\theta}{\int_0^{\pi/2} a \, d\theta} = \frac{a^2 \sin\theta \Big|_0^{\pi/2}}{a\dfrac{\pi}{2}} = \frac{2a}{\pi}.$$

The lever arm of the weight of the leg about its base is therefore $a(\pi - 2)/\pi$. The horizontal force exerted by each leg on the other has lever arm, a, about the base. The condition of zero torque requires that this force be $(\pi - 2)/\pi$ times the weight of the leg. Hence, a horizontal force of $f_{\text{h}} = \dfrac{\pi - 2}{\pi}\dfrac{W}{2} = 0.18169W$ must be supplied at the base of each leg. Of course, it is possible to stand an arch on a frictionless surface, with no horizontal force at the base. In that case, the torque required to keep the legs standing must be supplied by the stresses at the top of the arch in the form of a *couple* – compression and tension in equal measure. But stone, concrete, and brick are not strong under tension.

11.15. The previous two problems show that there is a limit to the number of unknowns which can be determined.

(a) In a two-dimensional problem, the equilibrium of forces can provide only two equations. All torques must be perpendicular to the plane of the problem, so torque equilibrium provides only a single equation. Hence, at most three unknown force components can be determined in the general case.

(b) In three dimensions, the equilibrium conditions can involve three components of force and three components of torque. Hence, six unknown force components can be determined in the general case.

(c) This is more subtle, because the cross product used to define torque is not a vector quantity except in three dimensions. The generalization of the cross product is the antisymmetric tensor product, a two-index object with components of the form $x_i F_0 - x_j F_1$, where x_i and x_j are vector quantities with indices running from 1 to n. The number of independent components of this form is the number of pairs of *distinct* values $\{i, j\}$, which is

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

These plus n independent force components provide equations to determine at most $n(n+1)/2$ unknown force components in the general case. Note that this coincides with the preceding results for $n=2$ and $n=3$.

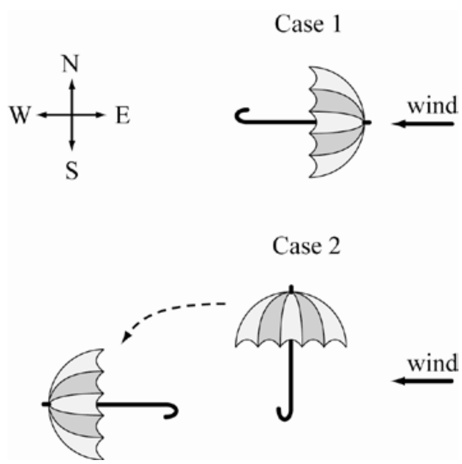
11.16. No. A homogenous meter stick would balance at the 50 cm mark, the geometric center.

11.17. Yes. An example of an inhomogeneous meter stick that balances at the 50 cm mark is one whose density at position x is the same as its density at position 100 – x. The ruler must balance at the 50 cm mark by symmetry.

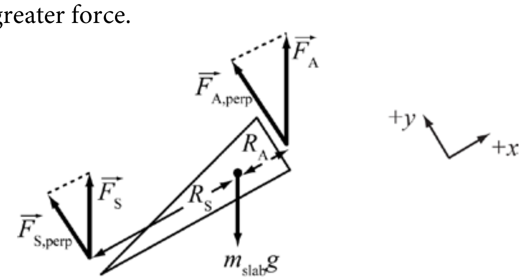
11.18. Without the small rotor on its tail, the helicopter and the main rotor would rotate in opposite directions. The small rotor provides a torque to offset the rotational motion that the helicopter would otherwise have.

11.19. The system is at equilibrium in the symmetrical position. Indeed, the two normal forces exerted by the cylinders on the board will be equal, and therefore the opposite dynamic friction forces will be equal. A small displacement of the system from equilibrium will increase one normal force at the cost of reducing the other. The friction force on one side becomes larger than that on the other side, and the net force will tend to bring the system back to equilibrium. Thus, this is a stable equilibrium position. If the board is given a small nudge, the system will perform an oscillatory motion about the equilibrium.

11.20. An umbrella that is pointing east into a strong wind (as shown in the diagram Case 1) is in equilibrium, however it is an unstable one. If the umbrella is held perfectly into the wind, it will stay in this orientation. However, even a small change in orientation will cause the umbrella to move away from this equilibrium. An umbrella being held perpendicular to the wind (as shown in the diagram Case 2) is not in equilibrium. The person holding the umbrella will have to overcome the torque caused by the wind.



11.21. As can be seen in the figure, the forces cancel if $F_A + F_S = m_{slab}g$, but since the center of mass of the slab is closer to the assistant, $R_A < R_S$, and the torques ($R_A F_{A,perp}$ and $R_S F_{S,perp}$) cancel only if $F_A > F_S$. So the assistant has to hold with a greater force.



11.22. (a) First, I use the fact that the sum of forces on the rod, and the sum of torques around any pivot point must be zero in order for the rod to be hanging in equilibrium: $\sum F = 0 = T_L + T_R + W = T_L + T_R - Mg = 0$.

It is useful to consider torques about a point in which at least one of the applied forces acts, so that it can be excluded from the calculation. I choose the point at which the right wire is attached to the rod, so

$$\sum \tau = 0 = \tau_{\mathrm{L}} + \tau_{\mathrm{R}} = -\left(T_{\mathrm{L}}\frac{L}{2}\right) + T_{\mathrm{R}}\left(\frac{2L}{3} - \frac{L}{2}\right) = 0 \;\Rightarrow\; \left(T_{\mathrm{L}}\frac{L}{2}\right) = T_{\mathrm{R}}\left(\frac{L}{6}\right) \;\Rightarrow\; T_{\mathrm{R}} = 3T_{\mathrm{L}}.$$

Substitute $T_{\mathrm{R}} = 3T_{\mathrm{L}}$ into $T_{\mathrm{L}} + T_{\mathrm{R}} - Mg = 0$ to get $4T_{\mathrm{L}} = Mg \;\Rightarrow\; T_{\mathrm{L}} = \frac{\mathrm{M}g}{4}$ and $T_{\mathrm{R}} = \frac{3Mg}{4}$.

(b) Now, I need to add another mass, *m,* on the right end so the tension in the left end wire vanishes. If the tension in the left wire is to be zero, then what must happen is that the mass of the rod must be balanced by the mass hung at the right edge of the rod. The force of gravity is applied at a distance, as calculated above, of *L*/6 to the left of the right wire. The string and hanging mass apply a force downward a distance of *L*/3 to the right of the right wire. The moment arm for the hung mass is twice that for the center of mass of the rod. Therefore, if no tension is to be in the left wire, the torques about the right wire's attachment point must balance. This means the torque from the hanging mass is equal and opposite to the torque due to the force of gravity. Thus the force applied by the hanging mass must be half that of the force of gravity on the rod. Thus $m = M/2$.

11.23. (a) By symmetry, the center of gravity must lie on the horizontal line passing through the center of the disk. Take the origin of the *x*-axis to be the left edge of the disk. Consider a disk of mass M_1 and radius R_1 without a hole in it, whose center of gravity is then at $x = R_1$. Also consider a smaller disk of mass M_2 and radius $R_2 = R_1/2$ whose center of gravity is at $x = R_1 + R_2$. The equation for *x*-coordinate of the center of gravity for the disk is then

$$X = \frac{R_1M_1 - (R_1 + R_2)M_2}{M_1 - M_2} = \frac{(R_1)\pi R_1^2 - (R_1 + R_2)\pi R_2^2}{\pi R_1^2 - \pi R_2^2} = \frac{(R_1)\pi R_1^2 - \left(\frac{3}{2}R_1\right)\pi\left(\frac{R_1}{2}\right)^2}{\pi R_1^2 - \pi\left(\frac{R_1}{2}\right)^2},$$

which can be simplified to $X = \dfrac{R_1 - \frac{3}{8}R_1}{\frac{3}{4}} = \frac{5}{8}R_1\left(\frac{4}{3}\right) = \frac{5}{6}R_1.$

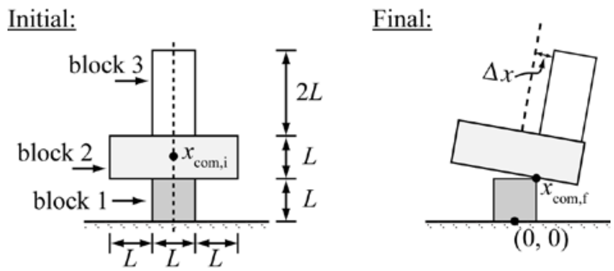
(b) There will be two equilibrium positions for the disk resting on its side; one will be a stable equilibrium and one unstable. The unstable equilibrium will have the hole oriented directly downwards, and the stable equilibrium will have the hole oriented directly upwards.

11.24. In order for the system not to undergo rotation, the pivot point should be placed to the right of the center. Suppose the pivot point is placed a distance, *s*, to the right of the center of the rod. In order for the system not to rotate the sum of the torques about the pivot point must be equal to zero.

$$Mg\left(\frac{L}{2} + s\right) + 10Mgs - 2Mg\left(\frac{L}{2} - s\right) = 0 \;\Rightarrow\; \left(\frac{L}{2} + s\right) + 10s - 2\left(\frac{L}{2} - s\right) = 0 \;\Rightarrow\; -\frac{L}{2} + 13s = 0 \;\Rightarrow\; s = \frac{L}{26}.$$

The pivot point must be moved $L/26$ to the right.

11.25. The first block is a cube with sides of length, *L* and mass M_1. The second block is three cubes together lengthwise, so its longest edge is 3*L* in length, and its mass is $M_2 = 3M_1$. The third block is two cubes stuck together, so its longest edge is 2*L* in length, and its mass is $M_3 = 2M_1$. Initially, the centers of each block are in a vertical line. I want to calculate the furthest distance the top block can be slid along the middle block before the middle block tips. Take the center of mass of block 1 to have an *x*-coordinate of zero.



I can treat the bottom block as a pivot. The middle block will start to tip when the composite center of mass, $x_{\text{com,f}}$, of the two top blocks is at $L/2$. The general equation for the center of mass is:

$$x_{\text{com,f}} = \frac{M_1x_1 + M_2x_2 + M_3x_3}{M_1 + M_2 + M_3}.$$

The mass of the system consisting of block 2 and block 3 is $M_{2,3} = 3M_1 + 2M_1 = 5M_1$. I can treat block 2 as a point mass whose center of mass is at $(0, L/2)$, and block 3 (after it has moved Δx) as a point mass whose center of mass is at $(\Delta x, 5L/2)$. I can simplify the equation for $x_{\text{com,f}}$ in terms of Δx. The x-coordinate of the center of mass of the block 2 and block 3 system is:

$$x_{\text{com,f}} = \frac{3M_1(0) + 2M_1\Delta x}{5M_1} = \frac{2\Delta x}{5}.$$

When the x-coordinate of the center of mass is $L/2$, block 3 will tip. This is when $\frac{2\Delta x}{5} = \frac{L}{2}$, which is when the top block has moved $\Delta x = \frac{5L}{4}$.

11.26. A broader base enables more freedom in placing successive stones with respect to balancing each level of stones. There is no pivot point for a successive layer provided its ends do not extend beyond the ends of previous layers.

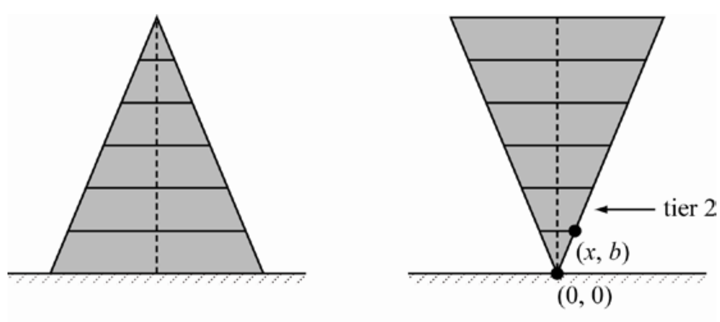


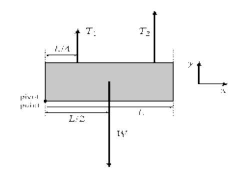
Figure (a) Figure (b)

Consider the situation in figure (b). The stones that make up tier 2 would have to be placed so that the center of mass of tier 2 was at the origin. If tier 2's center of mass was at the position (x, b), then there would be a net torque about the pivot causing the pyramid to lean over. This would be true for any successive layer shown in figure (b). While both situations are equilibria, the first situation is stable and the second is unstable. This is because the slightest change in position in the second situation will cause the pyramid to topple. This is not true in the first situation.

Exercises

11.27. **THINK:** We know that the weight of the crate is W = 1000. N. The length of the crate is L. Two vertical ropes are pulling the crate upwards. We know that the tension in the left rope is T_1 = 400. N, and it is attached a distance $L/4$ from the left edge of the crate. The crate does not move and does not rotate when the platform is lowered. Assume that the crate is of uniform density, so that its center of mass is its geometric center.

SKETCH:



RESEARCH: The crate does not move so the combined tensions, $T_1 + T_2$, must equal the weight of the crate, W = 1000. N. Thus we can write $T_1 + T_2 = W$. The crate does not rotate, so there is no net torque acting on the system. The tension T_1 acts at a distance $x_1 = L/4$ from the assumed pivot point at the lower left corner. The force from the weight of the crate acts at a distance of $L/2$ from the pivot point. The force T_2 acts a distance x_2 from the pivot point. The sum of the torques about the pivot point is given by $\tau_{\text{net}} = -W\frac{L}{2} + T_2x_2 + T_1x_1 = 0$.

SIMPLIFY: Solving for the maximum value of T_2 from the force equations we get

$$T_2 = W - T_1 = 1000.\text{ N} - 400.\text{ N} = 600.\text{ N}.$$

Solving for x_2 from the torque equations gives us

$$0 = -W\frac{L}{2} + T_2x_2 + T_1x_1$$

$$T_2x_2 = W\frac{L}{2} - T_1x_1$$

$$x_2 = \frac{W\frac{L}{2} - T_1x_1}{T_2} = \frac{WL - 2T_1x_1}{2T_2} = \frac{WL - 2T_1(L/4)}{2T_2} = \frac{WL - T_1L/2}{2T_2}.$$

CALCULATE: The tension in the right rope is $T_2 = 600.$ N. The rope on the right is attached at

$$x_2 = \frac{WL - T_1L/2}{2T_2} = \frac{(1000.\text{ N})L - (400.\text{N})L/2}{2(600.\text{N})} = \frac{(800.)L}{1200.} = \frac{2}{3}L.$$

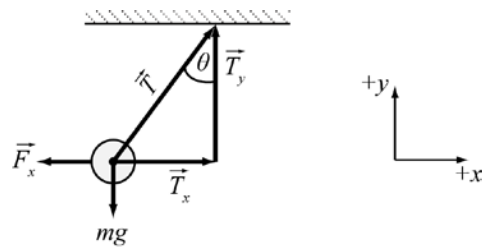
ROUND: Not applicable.

DOUBLE-CHECK: To double-check, we show that the counterclockwise torque has the same magnitude as the clockwise torque when we assume that the pivot point is located at the center of mass of the crate:

$$T_2\left(\frac{2L}{3}-\frac{L}{2}\right)=T_1\left(\frac{L}{2}-\frac{L}{4}\right)$$
$$(600.\text{N})\left(\frac{L}{6}\right)=(400.\text{ N})\left(\frac{L}{4}\right)$$
$$(100.\text{N})L=(100.\text{N})L.$$

11.28. **THINK:** The mass of the bowling ball is $m = 5.00$ kg. The ball is attached to a rope of length, $L = 4.00$ m. The professor pulls the ball an angular displacement of $\theta = 20.0°$ with respect to the vertical position. Assume the professor's force, F_x, on the ball is entirely in the horizontal direction. Calculate F_x and the tension in the rope, T.

SKETCH:



RESEARCH: In order to determine the tension in the rope, T, and the force, F_x, the forces in the x and y directions must be summed separately. The system is in a state of equilibrium, so it is known that there is no net force in the x or y directions and the equations are $\vec{F}_{\text{net},y} = 0 = T_y - mg$ and $\vec{F}_{\text{net},x} = 0 = -F_x + T_x$.

SIMPLIFY: $T_y = mg$

From the sketch it can be seen that $T_y = T\cos\theta$. Therefore, the equation can be written as $T\cos\theta = mg$. Solving for T gives $T = mg/\cos\theta$. Similarly, the equation for the net force in the x direction can be simplified to $F_x = T\sin\theta$. Substituting $T = mg/\cos\theta$ into this equation gives

$$F_x = \frac{mg}{\cos\theta}\sin\theta = mg\tan\theta.$$

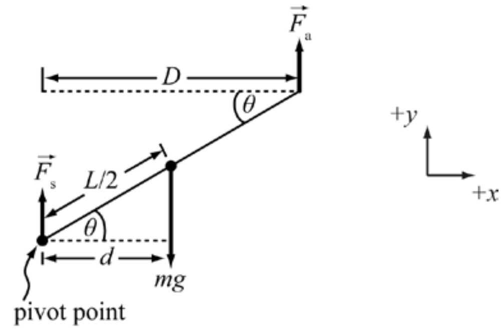
CALCULATE: $F_x = (5.00\text{ kg})\left(9.81\text{ m/s}^2\right)\tan(20.0°) = 17.853\text{ N}$, $T = \dfrac{(5.00\text{ kg})\left(9.81\text{ m/s}^2\right)}{\cos(20.0°)} = 52.198\text{ N}$

ROUND: $F_x = 17.9$ N and $T = 52.2$ N.

DOUBLE-CHECK: Dimensional analysis confirms the answers have the correct units. It is necessary that the horizontal force is less than the tension in the rope. The calculated values are reasonable given the mass and displacement of the ball.

11.29. **THINK:** The marble slab has length, $L = 2.00$ m, and mass, $m = 75.0$ kg. The mass of the slab is uniformly distributed along its length. The sculptor and his assistant are pulling directly up, with force $\vec{F}_\text{s}$ and $\vec{F}_\text{a}$, respectively. The slab is at an angle, $\theta = 30.0°$, with respect to the horizontal. I want to calculate the magnitude of $\vec{F}_\text{s}$ and $\vec{F}_\text{a}$ that will ensure the slab remains stationary during the break. Take the sculptor's end of the slab as the pivot point so only two forces have to be considered. Let torques in the counter-clockwise direction be positive.

SKETCH:



RESEARCH: In order for the slab to remain stationary, its net torque, τ_{net}, about the pivot point must equal zero. The net torque is given by the formula: $\tau_{\text{net}} = \sum_{j=1}^{n} \tau_j = 0;\ \tau_j = \vec{F}_j \times \vec{d}_j = \left|F_j\right|\left|d_j\right|\sin\theta_j.$

The horizontal distance, *D*, from the assistant to the pivot point is $D = L\cos\theta.$ The horizontal distance, *d*, from the slab's center of mass to the pivot point is $d = L(\cos\theta)/2.$

SIMPLIFY: $\tau_{\text{net}} = F_s(0) + F_aD - mgd = 0$, which is equivalent to $F_aD = mgd.$ Substituting the equations for *D* and *d* into this equation gives: $F_a\left(L\cos\theta\right) = mg\left(\frac{L}{2}\cos\theta\right) \Rightarrow F_a = \frac{mg}{2}.$ The force the sculptor exerts can be determined by adding the forces in the *y* direction:

$$F_{\text{net},y} = 0 = F_a + F_s - mg \Rightarrow F_s = mg - \frac{mg}{2} = \frac{mg}{2}.$$

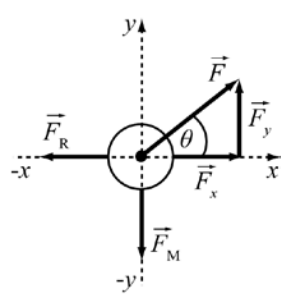
CALCULATE: $F_a = \frac{(75.0 \text{ kg})\left(9.81 \text{ m/s}^2\right)}{2} = 367.875 \text{ N}, F_s = F_a = 367.875 \text{ N}$

ROUND: $F_s = F_a = 368 \text{ N}.$

DOUBLE-CHECK: Half the force must be supplied by each person. This makes sense since each person is the same distance and same angle from the center of mass of the slab.

11.30. **THINK:** During the 3-way tug of war, Roberta pulls west with a force $F_R = 420.$ N, and Michael pulls south with a force $F_M = 610.$ N. I want to determine the magnitude and direction of the force with which I need to pull to keep the knot stationary. Let the force I supply be $\vec{F}$, with component forces: F_x in the *x*-direction and F_y in the *y*-direction.

SKETCH:



RESEARCH: To keep the knot from moving I must apply a force that balances the forces applied by Roberta and Michael. This means that the sum of the net forces in the *x* and *y* directions must be zero:

$$\sum_{i=1}^{n} F_{y,i} = F_{\text{net},y} = 0,\ \sum_{i=1}^{n} F_{x,i} = F_{\text{net},x} = 0.$$

SIMPLIFY: The magnitude of the force is $\left|\vec{F}\right| = \sqrt{(F_x)^2 + (F_y)^2} = \sqrt{(F_R)^2 + (F_M)^2}$. The values of the net forces in each direction are: $F_{\text{net},y} = 0 = -F_M + F_y$, and $F_{\text{net},x} = 0 = -F_R + F_x$. These imply that $F_y = F_M$, and $F_x = F_R$. The direction of the force is $\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{F_M}{F_R}\right)$.

CALCULATE: $\left|\vec{F}\right| = \sqrt{(420.\text{ N})^2 + (610.\text{ N})^2} = 740.6079\text{ N}$

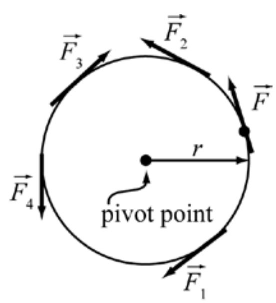
$\theta = \tan^{-1}\left(\frac{610.\text{ N}}{420.\text{ N}}\right) = 55.4516°$ with respect to the positive x-axis. This direction is in the first quadrant.

ROUND: Values with three significant figures were provided in the question. Therefore, the answers must be written $F = 741$ N and $\theta = 55.5°$.

DOUBLE-CHECK: The magnitude of the force calculated is on the same order of magnitude as the forces that were given, and the calculated direction is in the first quadrant, which is appropriate to balance the forces in the south and west directions.

11.31. **THINK:** The magnitudes of the four other forces acting on the merry-go-round are $F_1 = 104.9$ N, $F_2 = 89.1$ N, $F_3 = 62.8$ N, and $F_4 = 120.7$ N. I will use the convention that a torque that acts in the clockwise direction is negative and a torque that acts in the counterclockwise direction is positive. Consider the torque about the center as a pivot point.

SKETCH:



RESEARCH: The merry-go-round has radius, r. The forces are all applied in a tangential direction, so the torque due to each force is given by $\tau_i = F_i r$. In order for the merry-go-round to remain stationary, the sum of the torques about the pivot point must equal zero:

$$\tau_{\text{net}} = \sum_{i=1}^{n} \tau_i = r \sum_{i=1}^{n} F_i.$$

SIMPLIFY: $\tau_{\text{net}} = r\left(-F_1 + F_2 - F_3 + F_4 + F\right)$. I want to prevent the merry-go-round from moving, so the net torque is zero. Since the radius is not zero, $F = F_1 - F_2 + F_3 - F_4$.

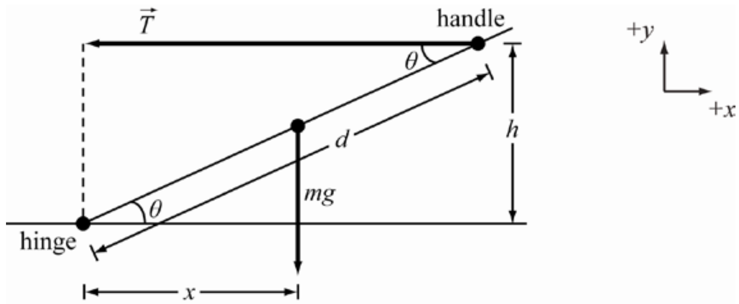
CALCULATE: $F = 104.9\text{ N} - 89.1\text{ N} + 62.8\text{ N} - 120.7\text{ N} = -42.1\text{ N}$. The negative sign indicates that the force should be applied in the clockwise direction.

ROUND: The least number of significant figures provided in the question is three, so no rounding is necessary. Write the answer as, 42.1 N clockwise.

DOUBLE-CHECK: The calculated force is of the same order of magnitude as the forces which were given. The sum of the magnitudes of the forces in the counter-clockwise direction is more than the sum of the magnitudes of the forces in the clockwise direction. In order to counter-balance this, the fifth child should push in the clockwise direction. This is consistent with the calculated answer.

11.32. **THINK:** A trap door has uniform thickness and density. Its mass is m = 19.2 kg, and its width is w = 1.50 m. The door has a handle a distance, d = 1.41 m, away from the hinge. If the rope that is tied to the handle is horizontal and the door is open so that the handle has a height of h =1.13 m, what is the tension, T, in the rope? Compute the torque using the hinge as the pivot point. Use the convention that a counter-clockwise torque is positive. It will be vital to know the perpendicular distances from the forces to the pivot point. The perpendicular distance from $\vec{T}$ to the pivot point is *h*. Let the perpendicular distance from $\vec{F}_{\text{g}} = mg$ to the pivot point be *x*. It will be important to compute *x*.

SKETCH:



RESEARCH: In order to compute the value of *x*, find the value of θ using the top triangle and use this value in the bottom small triangle. In the top triangle, the hypotenuse is *d*, and the leg opposite θ is *h*. Then, $\theta = \arcsin(h/d)$. The value of θ can be used in the lower (small) triangle to find the value of *x*, using a cosine. The hypotenuse is *w*/2, so $x = (w/2)\cos\theta$.

With respect to the pivot point, the torque due to the rope is positive and the torque due to gravity is negative. The door has uniform thickness and density, so its center of mass is located at $w/2$. Thus, the torque due to the force of gravity is $\tau_{\text{gravity}} = -mgx$. The torque due to the rope is $\tau_{\text{rope}} = +Th$. The system is in static equilibrium, so the sum of the torques about the pivot point equals zero: $\tau_{\text{net}} = \tau_{\text{rope}} + \tau_{\text{gravity}} = 0$.

SIMPLIFY: $0 = \tau_{\text{rope}} + \tau_{\text{gravity}} = Th - mgx = Th - mg\left(\frac{w}{2}\cos\theta\right) = Th - \frac{mgw\cos(\arcsin(h/d))}{2}$. Solving for *T*:

$$T = \frac{mgw\cos(\arcsin(h/d))}{2h}.$$

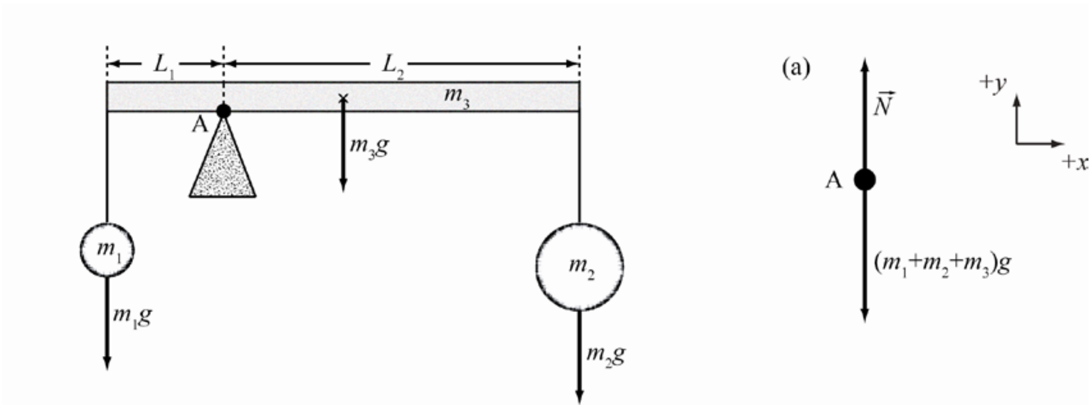
CALCULATE: $T = \frac{(19.2 \text{ kg})(9.81 \text{ m/s}^2)(1.50 \text{ m})\cos(\arcsin(1.13 \text{ m}/1.41\text{m}))}{2(1.13 \text{ m})} = 74.77042 \text{ N}$

ROUND: The values given in the question have three significant figures, so the answer should be rounded to $T = 74.8$ N.

DOUBLE-CHECK: Dimensional analysis confirms the tension is in the correct units of force. The magnitude of the force is reasonable considering the quantities given.

11.33. **THINK:** The mass of the rigid rod is $m_3 = 3m_1$, and the masses of the two hanging weights are m_1 and $m_2 = 2m_1$. The pivot point for the rod is at point *A*. I want to calculate the normal force acting on the pivot, and the ratio of L_1 to L_2 which are the distances from the pivot to m_1 and m_2, respectively. The rod is in static equilibrium, so the sum of the forces acting on the pivot is zero, and also the sum of the torques about the pivot point is zero. Use the convention that a torque that would cause a counterclockwise rotation is positive, and a torque that would cause a clockwise rotation is negative.

SKETCH:



RESEARCH:
(a) The rod is in static equilibrium, so the sum of the forces acting on the pivot is zero. All of the forces are acting in the y-direction, so the equation is

$$F_{\text{net},y} = \sum_{i=1}^{n} F_{i,y} = 0 = \vec{N} - m_1 g - m_2 g - m_3 g.$$

(b) The rod is in static equilibrium, so the sum of the torques about the pivot point equals zero:

$$\tau_{\text{net}} = \sum_{i=1}^{n} \tau_i = 0.$$

The torque due to m_1 is positive, the torque due to m_2 is negative, and the torque due to m_3 is negative. There is no torque due to the normal force at the pivot point because its distance is zero. The perpendicular distances from the forces to the pivot point are: L_1 for m_1, L_2 for m_2, and since the length of the whole rod is $L_1 + L_2$, the distance from the midpoint to the right end is $\frac{L_1 + L_2}{2}$, and the distance from the pivot to the midpoint then is $L_2 - \frac{L_1 + L_2}{2} = \frac{L_2 - L_1}{2}$, which is the distance to the pivot point from m_3 by treating the rod as a point-mass.

SIMPLIFY:
(a) From considering the balanced forces, $N = (m_1 + m_2 + m_3)g = (m_1 + 2m_1 + 3m_1)g = 6m_1 g$.
(b) By setting the sum of the torques about the pivot point equal to zero:

$$\tau_{\text{net}} = 0 = m_1 g L_1 - m_2 g L_2 - m_3 g\left(\frac{L_2 - L_1}{2}\right) = m_1 g L_1 - (2m_1) g L_2 - (3m_1) g\left(\frac{L_2 - L_1}{2}\right) = \frac{5}{2} m_1 g L_1 - \frac{7}{2} m_1 g L_2.$$

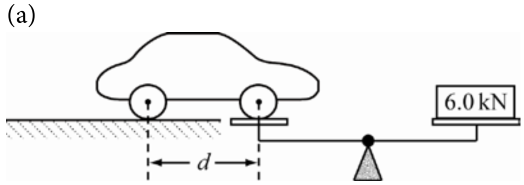
Since the mass and acceleration due to gravity are non-zero, divide them out, and solve for $\frac{L_1}{L_2}$. The ratio is $\frac{L_1}{L_2} = \frac{7}{5}$.

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: The value calculated for the normal forces seems reasonable considering the given values. The ratio of L_1 to L_2 also seems reasonable considering that m_2 is more massive than m_1.

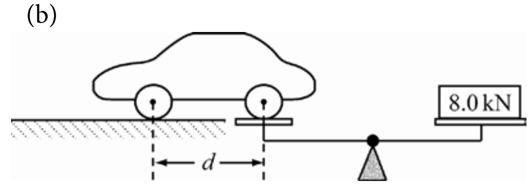
11.34. **THINK:** The distance between the axles of the vehicle is d = 2.80 m. The balance weight when the front wheels are on the scale is $W_{\text{f}} = 8.00$ kN. The balance weight when the rear wheels are on the scale is $W_{\text{r}} = 6.00$ kN. I want to calculate the weight of the vehicle, and determine how far its center of gravity, x_{com}, is behind the front axle.

SKETCH:

(a)



(b)



RESEARCH: The total weight of the vehicle is $W = W_f + W_r$. The position of the center of gravity is

$$x_{com} = \frac{1}{W}\sum_{i=1}^{n} W_i x_i.$$

SIMPLIFY: I can take the position of the vehicle's front wheels to be at $x_f = 0$, so the distance from the front wheels to the rear wheels will be $x_r = d$. Substituting these values into the equation for x_{cog} gives

$$x_{com} = \frac{W_f(0) + W_r d}{W} = \frac{W_r d}{W_r + W_f}.$$

CALCULATE: $W = 8.00 \text{ kN} + 6.00 \text{ kN} = 14.0 \text{ kN}$, $x_{com} = \dfrac{(6.00 \text{ kN})(2.80 \text{ m})}{14.0 \text{ kN}} = 1.20 \text{ m}$

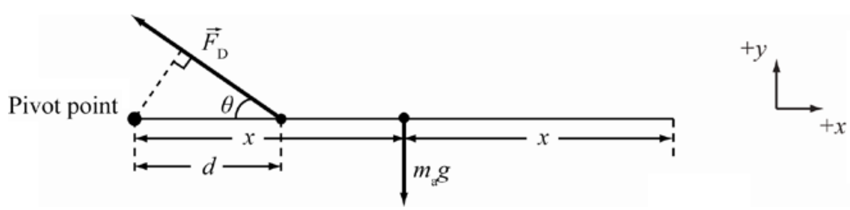
ROUND: The weights were provided to three significant figures, so the answers must be written as $W = 14.0$ kN and $x_{com} = 1.20$ m.

DOUBLE-CHECK: It is reasonable that the total weight of the vehicle is the sum of its components. It is also reasonable that the center of gravity is closer to the front of the vehicle because the front is heavier than the rear. The ratio of the masses 8:6 is the inverse of the ratio of the distances 1.2:1.6. This suggests the result is correct.

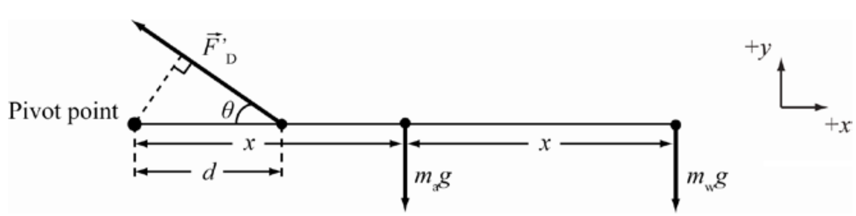
11.35. **THINK:** I must estimate the distance, *d*, from the shoulder pivot to the point where the deltoid muscle connects to the upper arm. I must also estimate the distance from the shoulder pivot point to the center of mass of the arm, *x*. The deltoid muscle will have to pull up the arm at an angle θ. The mass of the arms is m_a. The mass of the weight is $m_w = (10.0 \text{ lb})\left(\dfrac{1 \text{ kg}}{2.20 \text{ lb}}\right) = 4.55$ kg. I want to calculate the force, F_D, the deltoid muscle would have to exert to hold my arm at shoulder lever, and also the force, F_D', required to hold the 4.55 kg weight at shoulder level.

SKETCH:

(a)



(b)



RESEARCH: To hold the arm at shoulder level in either situation, the sum of the torques about the pivot point must equal zero:

$$\sum_{i=1}^{n} \tau_i = 0.$$

In order to calculate values, estimates must be made for d, x, θ, and m_a. I will assume that the center of mass of my arm is in the middle of the arm. I estimate d = 0.120 m, x = 0.300 m, θ = 20.0°, and m_a = 4.00 kg.

SIMPLIFY: For the first situation where the arm is extended at shoulder level, the equation is

$$\sum_{i=1}^{n} \tau_i = 0 = -m_\text{a} gx + F_\text{D} d \sin\theta \Rightarrow F_\text{D} = \frac{m_\text{a} gx}{d\sin\theta}.$$

For the second situation, the torque about the pivot due to the force of the weight must be considered:

$$\sum_{i=1}^{n} \tau_i = 0 = -m_\text{a} gx - m_\text{W} g2x + F_\text{D}' d \sin\theta \Rightarrow F_D' = \frac{m_\text{a} gx + m_\text{W} g(2x)}{d\sin\theta}.$$

CALCULATE: $F_\text{D} = \dfrac{(4.00 \text{ kg})(9.81 \text{ m/s}^2)(0.300 \text{ m})}{(0.120 \text{ m})\sin(20.0°)} = 286.83 \text{ N}$

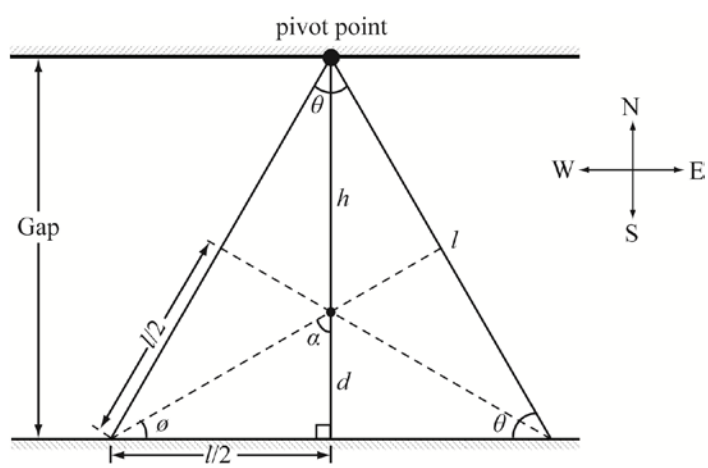
$$F_\text{D}' = \frac{(4.00 \text{ kg})(9.81 \text{ m/s}^2)(0.300 \text{ m}) + (4.55 \text{ kg})(9.81 \text{ m/s}^2)(0.600 \text{ m})}{(0.120 \text{ m})\sin(20.0°)} = 939.35 \text{ N}$$

ROUND: $F_\text{D} = 287$ N and $F_\text{D}' = 939$ N.

DOUBLE-CHECK: Dimensional analysis confirms the answers are given in the correct units of force. It makes sense that the deltoid muscle must exert a much larger force to hold the 10.0 lb weight compared to the force required to hold up just the arm.

11.36. **THINK:** The uniform, equilateral triangle has sides of length, $l = 2.00$ m, and weight, $W = 4.00 \cdot 10^3$ N. The triangle is placed across a gap. The point of the triangle is contacting the north end, and the base of the triangle is contacting the south end. I want to calculate the force, F_n, on the north side and the force, F_s, on the south side. By the z-direction, I refer to down (not to be confused with south). Take the apex of the triangle as the pivot point.

SKETCH:



RESEARCH: The triangle is in static equilibrium, so the sum of the torques about the pivot point must equal zero:

$$\sum_{i=1}^{n} \tau_i = 0.$$

Also, the sum of the forces in the *z* direction must equal zero:

$$\sum_{i=1}^{n} F_{z,i} = 0.$$

Since the triangle is equilateral, $\theta = 60°$, $\phi = 30°$, and $\alpha = 60°$.

SIMPLIFY: From the sketch, it can be seen that R_{com} is located at $l/2$ in the west to east direction, and *d* in the south to north direction. Choose the point of the triangle as the pivot point. Then the torque about that point is $0 = -Wh + F_{\text{s}}(h+d) \Rightarrow F_{\text{s}} = Wh/(h+d)$. From the geometry shown in the sketch, $(h+d) = l\sin\theta$ and $d = (l\tan\phi)/2$. Substitute the equation for *d* into the equation for (*h*+*d*) and solve for *h*: $h = l\sin\theta - (l\tan\phi)/2$. Substituting the expressions for *h* and (*h*+*d*) into the equation for F_{s} gives

$$F_{\text{s}} = \frac{W\left(l\sin\theta - \dfrac{l}{2}\tan\phi\right)}{l\sin\theta} = W - \frac{W\tan\phi}{2\sin\theta}.$$

The sum of the forces in the *z* direction can be written as $0 = F_{\text{s}} + F_{\text{n}} - W \Rightarrow F_{\text{n}} = W - F_{\text{s}}$. Substituting the equation for F_{s} into the expression for F_{n} gives:

$$F_{\text{n}} = W - \left(W - \frac{W\tan\phi}{2\sin\theta}\right) = \frac{W\tan\phi}{2\sin\theta}.$$

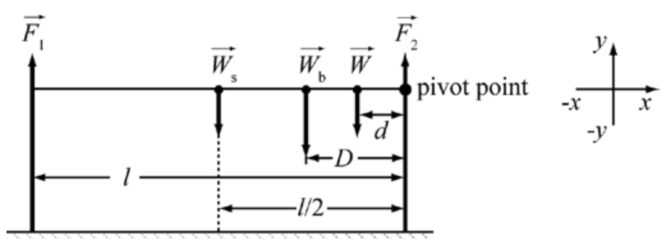
CALCULATE: $F_{\text{s}} = 4.00\cdot 10^{3}\text{ N} - \dfrac{\left(4.00\cdot 10^{3}\text{ N}\right)\tan 30°}{2\sin 60°} = 2666.67\text{ N}$, $F_{\text{n}} = \left(2000\text{ N}\right)\dfrac{\tan 30°}{\sin 60°} = 1333.33\text{ N}$

ROUND: The least precise value given in the question has three significant figures, so the answers must be written as $F_{\text{s}} = 2670$ N and $F_{\text{n}} = 1330$ N.

DOUBLE-CHECK: The center of mass of the triangle is closer to the south end so it makes sense that the force of the south end is greater than the force on the north end.

11.37. **THINK:** The bricklayer has a weight *W* = 600.0 N and is a distance, *d* = 1.50 m, from one end of a uniform scaffold. The scaffold has a length, *l* = 7.00 m, and a weight, W_{s} = 800.0 N. There is a pile of bricks of weight, W_{b} = 500.0 N, at a distance, *D* = 3.00 m, from the end of the scaffold. Determine the net force on each end of the scaffold, which can be denoted as F_1 and F_2. Choose the pivot point to be located at the end of the scaffold where the force, F_2, is being applied. Use the convention that counterclockwise torque is positive and clockwise torque is negative.

SKETCH:



RESEARCH: The unknowns in the problem are F_1 and F_2. Use two equations to solve for the two unknowns. The scaffold is in static equilibrium, so the sum of the torques about the pivot point equals zero:

$$\tau_{\text{net}} = \sum_{i=1}^{n} \tau_i = 0 \quad (1).$$

Also, the sum of the forces in the *y*-direction equal zero because the scaffold is in static equilibrium:

$$F_{\text{net}} = F_{\text{net},y} = \sum_{i=1}^{n} F_{y,i} = 0 \quad (2).$$

SIMPLIFY: From equation (1): $\sum_{i=1}^{n}\tau_i = 0 = W_s\dfrac{l}{2} + W_bD + Wd - F_1l \Rightarrow F_1 = \dfrac{W_s\dfrac{l}{2} + W_bD + Wd}{l}.$

From equation (2): $\sum_{i=1}^{n}F_{y,i} = 0 = F_1 + F_2 - W_s - W_b - W \Rightarrow F_2 = W_s + W_b + W - F_1.$

CALCULATE: $F_1 = \dfrac{(800.0\text{ N})(3.50\text{ m}) + (500.0\text{ N})(3.00\text{ m}) + (600.0\text{ N})(1.50\text{ m})}{7.00\text{ m}} = 742.86\text{ N}$

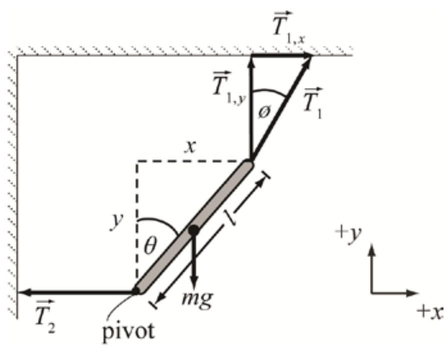
$F_2 = 800.0\text{ N} + 500.0\text{ N} + 600.0\text{ N} - 742.86\text{ N} = 1157.14\text{ N}$

ROUND: The least precise value provided in the question has three significant figures. Therefore, the answers must be written as $F_1 = 743$ N and $F_2 = 1160$ N.

DOUBLE-CHECK: Most of the weight is positioned closer to the scaffold end where the force, F_2, is applied. It is therefore expected that the force applied by this support is greater than the force applied at the other end of the scaffold.

11.38. **THINK:** The uniform rod is supported at its ends by two strings. The rod is tilted at an angle, θ, with respect to the vertical. The string attached to the ceiling is at an angle, $\phi = 30.0°$, with respect to the vertical. The string attached to the wall is horizontal. Determine the value of θ. Use the convention that counter-clockwise torques are positive.

SKETCH:



RESEARCH: A convenient choice for a pivot is the point where the string attached to the wall connects to the rod. The uniform rod is in static equilibrium, so the sum of the torques about its pivot point is equal to zero. Also, the sum of the forces acting on it is zero:

$$\tau_{\text{net}} = \sum_{i=1}^{n}\tau_i = 0; \quad F_{\text{net},y} = \sum_{i=1}^{n}F_{y,i} = 0; \quad F_{\text{net},x} = \sum_{i=1}^{n}F_{x,i} = 0.$$

The force *mg* acts at a distance, $l/2$, along the rod.

SIMPLIFY: The sum of the vertical forces is: $\sum_{i=1}^{n}F_{y,i} = 0 = -mg + T_{1,y} \Rightarrow mg = T_{1,y} = T_1\cos\phi.$ From the sketch, it can be seen that $x = l\sin\theta,\ x/2 = (l\sin\theta)/2,\ y = l\cos\theta.$ The sum of the torques is:

$$\sum_{i=1}^{n}\tau_i = 0 = T_{1,y}x - (mg)\frac{x}{2} - T_{1,x}y = (T_1\cos\phi)x - mg\frac{x}{2} - (T_1\sin\phi)y$$

$$\Rightarrow (T_1\cos\phi)(l\sin\theta) - mg\left(\frac{l}{2}\sin\theta\right) - T_1\sin\phi(l\cos\theta) = 0.$$

Use the equation from the balanced forces: Substitute $mg = T_1\cos\phi$ into the equation and solve for θ:

$$T_1\cos\phi(l\sin\theta) - T_1\cos\phi\frac{l\sin\theta}{2} = T_1\sin\phi(l\cos\theta) \Rightarrow \frac{T_1\cos\phi(l\sin\theta)}{2} = T_1\sin\phi(l\cos\theta).$$

Dividing both sides by $T_1 l\cos\theta$ gives

$$\frac{\cos\phi}{2}\tan\theta=\sin\phi \Rightarrow \frac{\tan\theta}{2}=\tan\phi \Rightarrow \theta=\tan^{-1}(2\tan\phi).$$

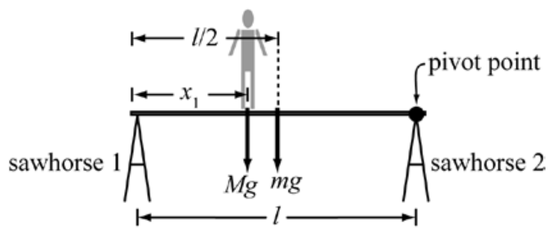
CALCULATE: $\theta=\tan^{-1}(2\tan 30.0°)=49.1066°$

ROUND: $\theta=49.1°$.

DOUBLE-CHECK: The calculated angle seems reasonable considering the geometry of the problem and the value of the angle that was provided.

11.39. **THINK:** The construction supervisor has mass, $M=92.1$ kg, and stands a distance, $x_1=1.07$ m, away from sawhorse 1. He is standing on a board that is supported by sawhorse 1 and sawhorse 2 on either end. The board has a mass, $m=27.5$ kg. The two sawhorses are separated by a distance, $l=3.70$ m. The question asks for the force, F, that the board exerts on sawhorse 1. Use the convention that counter-clockwise torques are positive.

SKETCH:



RESEARCH: Assuming the board is of uniform density and thickness, its center of mass should be at $l/2$. The board is in static equilibrium, so the sum of the torques about the pivot point is zero:

$$\tau_{\text{net}}=\sum_{i=1}^{n}\tau_i=0.$$

The force that the board exerts on sawhorse 1 should be equal in magnitude and opposite in direction to the normal force sawhorse 1 exerts on the board.

SIMPLIFY: $\sum_{i=1}^{n}\tau_i=0=-N_1 l+mg\frac{l}{2}+Mg(l-x_1) \Rightarrow N_1=\frac{mg\frac{l}{2}+Mg(l-x_1)}{l}$ and $F=-N_1$, so $F=\frac{-mg\frac{l}{2}-Mg(l-x_1)}{l}$.

CALCULATE:

$$F=\frac{-(27.5\text{ kg})(9.81\text{ m/s}^2)\left(\frac{3.70\text{ m}}{2}\right)-92.1\text{ kg}(9.81\text{ m/s}^2)(3.70\text{ m}-1.07\text{ m})}{3.70\text{ m}}=-777.106\text{ N}$$

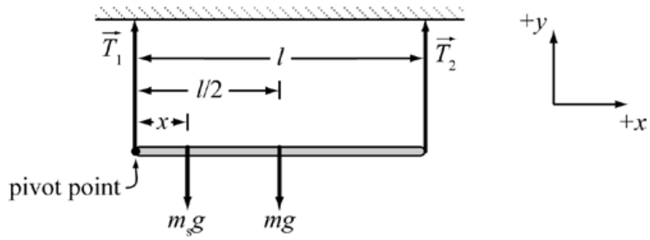
The negative sign indicates that the force of the board is directed down on sawhorse 1.

ROUND: The question provided three significant figures, so the answer is $F=777$ N downward.

DOUBLE-CHECK: Dimensional analysis confirms that the answer is in the correct units of force, Newtons. The weight of the board is about 300 N, and the weight of the man is about 900 N. The magnitude and direction of the calculated force seem reasonable for the values given in the question.

11.40. **THINK:** A horizontal bar of mass, $m=4.00$ kg, and length, $l=1.20$ m, is suspended by two vertical wires at its ends. A sausage of mass $m_s=2.40$ kg is hung a distance $x=0.20$ m from the left end of the bar. Determine the tension in the two wires, denoted T_1 and T_2, respectively. A convenient choice for the pivot point is the left end of the bar where the wire is attached. Use the convention that a counter-clockwise torque is positive.

SKETCH:



RESEARCH: The system is in static equilibrium, so the sum of the torques about the pivot point is zero:

$$\tau_{\text{net}} = \sum_{i=1}^{n} \tau_i = 0.$$

Assuming the rod is uniform, its center of mass will be at $l/2$. The sum of the forces in the y-direction is also zero due to the system being in static equilibrium:

$$F_{\text{net},y} = \sum_{i=1}^{n} F_{y,i} = 0.$$

SIMPLIFY: $\sum_{i=1}^{n} \tau_i = 0 = -m_s g x - \dfrac{mgl}{2} + T_2 l \;\Rightarrow\; T_2 = \dfrac{m_s g x + \dfrac{mgl}{2}}{l} = \dfrac{g\left(m_s x + \dfrac{ml}{2}\right)}{l}$

$\sum_{i=1}^{n} F_{y,i} = 0 = T_1 + T_2 - m_s g - mg \;\Rightarrow\; T_1 = (m_s + m)g - T_2$

Substitute the value found for T_2 in the above equation to solve for T_1.

CALCULATE: $T_2 = \dfrac{\left(9.81 \text{ m/s}^2\right)\left[2.40 \text{ kg}\left(0.20 \text{ m}\right) + 4.00 \text{ kg}\left(0.60 \text{ m}\right)\right]}{1.20 \text{ m}} = 23.544 \text{ N}$

$T_1 = \left[(2.40 \text{ kg} + 4.00 \text{ kg})\left(9.81 \text{ m/s}^2\right)\right] - 23.544 \text{ N} = 39.240 \text{ N}$

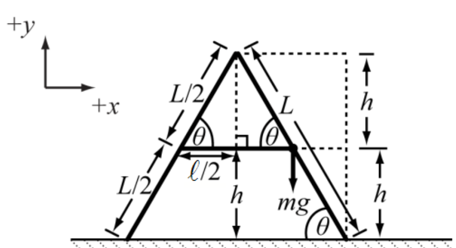
ROUND: The distance 0.20 m given in the question has two significant figures, so the answers must be written $T_1 = 39$ N and $T_2 = 24$ N.

DOUBLE-CHECK: It is reasonable that $T_1 > T_2$ because the sausage was hung closer to the end supported by wire 1. Furthermore, the total upward force is about 39 N + 24 N = 63 N, while the total downward force is about 10 m/s^2(6.4 kg) = 64 N. The two tensions agree within rounding errors.

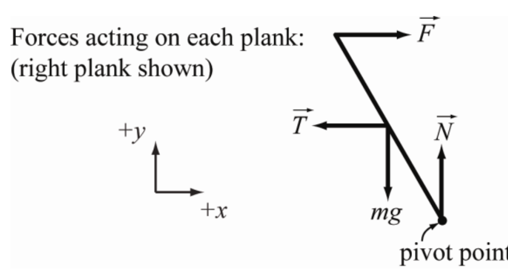
11.41. **THINK:** The two uniform planks each have mass, m, and length, L. They are connected at the top by a hinge and held together at their centers by a chain of length, ℓ, and negligible mass. I want to find the tension, T, in the chain, the force, F, of the hinge on each plank, and the force, N, of the ground on each plank as a function of the chain length. Note that the system sits on a frictionless surface. Use the convention that a counter-clockwise torque is positive.

SKETCH:

(a)



(b)



RESEARCH: The two planks are in static equilibrium, so the sums of the forces in the x and y directions are zero. Also, the sum of the torques about the pivot point equals zero:

$$F_{\text{net},x} = \sum_{i=1}^{n} F_{x,i} = 0;\ F_{\text{net},y} = \sum_{i=1}^{n} F_{y,i} = 0;\ \tau_{\text{net}} = \sum_{i=1}^{n} \tau_i = 0$$

SIMPLIFY: $\sum_{i=1}^{n} F_{x,i} = 0 = -T + F \Rightarrow F = T$, $\sum_{i=1}^{n} F_{y,i} = 0 = -mg + N \Rightarrow N = mg$, and $\sum_{i=1}^{n} \tau_i = 0 = mg\dfrac{L\cos\theta}{2} + Th - 2Fh.$ From sketch (a), it can be seen that $\cos\theta = \dfrac{\ell/2}{L/2} = \dfrac{\ell}{L}.$ Using the Pythagorean Theorem, $h^2 + \left(\dfrac{\ell}{2}\right)^2 = \left(\dfrac{L}{2}\right)^2 \Rightarrow h = \dfrac{\sqrt{L^2 - \ell^2}}{2}$. Substituting these values into the torque equation gives $0 = \left(\dfrac{mgL}{2}\right)\dfrac{\ell}{L} + T\dfrac{\sqrt{L^2-\ell^2}}{2} - 2F\dfrac{\sqrt{L^2-\ell^2}}{2}.$ Substitute $F = T$ into this equation to further simplify it to $\dfrac{mg\ell}{2} - T\dfrac{\sqrt{L^2-\ell^2}}{2} = 0.$

(a) $T = \dfrac{mg\ell}{\sqrt{L^2-\ell^2}}$

(b) $F = T = \dfrac{mg\ell}{\sqrt{L^2-\ell^2}}$

(c) $N = mg.$

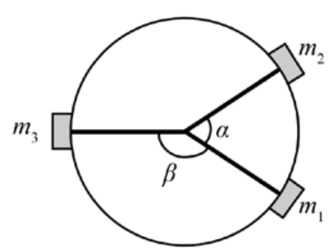
CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: It makes sense that the force the ground exerts on the planks is equal to the weight of the planks. It also makes sense that the tension in the chain equals the force of the hinge on the planks because the planks lean against each other and do not fall.

11.42. **THINK:** The knot is at the center of the table, so each of the masses is an equal distance r from the center. The known masses are $m_1 = 4.30$ kg and $m_2 = 5.40$ kg. The third mass, m_3, is not known. The measure of the angle between strings 1 and 2 is $\alpha = 74°$. Determine the angle, β, between strings 1 and 3. The strings and masses are not moving and thus are in force equilibrium.

SKETCH: Choose the x-axis to lie along the string connecting m_1 to the center.



RESEARCH: Set up two equations for the force components in x- and y-direction.

x-direction: $gm_1 + gm_2\cos\alpha + gm_3\cos\beta = 0 \Rightarrow m_1 + m_2\cos\alpha = -m_3\cos\beta$

y-direction: $0 + gm_2\sin\alpha - gm_3\sin\beta = 0 \Rightarrow m_2\sin\alpha = m_3\sin\beta$

These are two equations for two unknowns, m_3 and β. First, square both and add them:

$$\left.\begin{aligned}(m_1 + m_2\cos\alpha)^2 &= m_3^2\cos^2\beta \\ m_2^2\sin^2\alpha &= m_3^2\sin^2\beta\end{aligned}\right\}\ (m_1 + m_2\cos\alpha)^2 + m_2^2\sin^2\alpha = m_3^2$$

In other words, $m_3 = \sqrt{m_1^2 + 2m_1m_2\cos\alpha + m_2^2}$. Insert this back into the x- equation:

$$\beta = \cos^{-1}\left(-\frac{m_1 + m_2\cos\alpha}{\sqrt{m_1^2 + 2m_1m_2\cos\alpha + m_2^2}}\right)$$

CALCULATE: $\beta = \cos^{-1}\left(-\frac{4.30 + 5.40\cos 74°}{\sqrt{4.30^2 + 2(4.30)(5.40)\cos 74° + 5.40^2}}\right) = 138.116°$

ROUND: The angle, α, was given to two significant figures. Therefore, the answer is $\beta = 140°$.

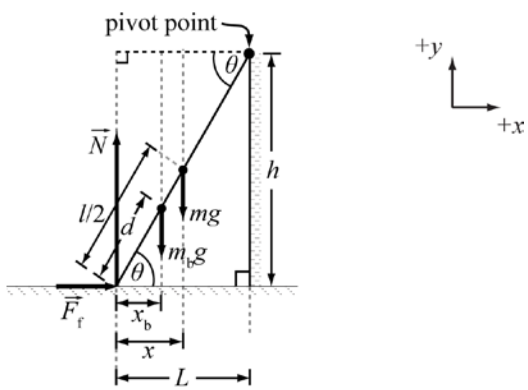
DOUBLE-CHECK: We can double-check our result in the limits $\alpha \to 0$ and $\alpha \to 180°$. For $\alpha \to 0$, the result is that $\beta \to 180°$ (and $m_3 = m_1 + m_2$). For $\alpha \to 180°$ the result is that $\beta \to 0°$ (and $m_3 = m_2 - m_1$).

11.43. **THINK:** The uniform ladder has length, $l = 10.0$ m, and its mass is $m = (20.0 \text{ lb})\left(\frac{1 \text{ kg}}{2.2046 \text{ lb}}\right) = 9.072$ kg. The ladder is leaning against a frictionless wall at an angle $\theta = 60.0°$ with respect to the horizontal. The mass of the boy is

$$m_\text{b} = (61.0 \text{ lb})\left(\frac{1 \text{ kg}}{2.2046 \text{ lb}}\right) = 27.67 \text{ kg}.$$

The boy climbs a distance $d = 4.00$ m up the ladder. I want to calculate the magnitude of the frictional force exerted on the ladder by the floor. I will use the convention that counterclockwise torque is positive and clockwise torque is negative. Choose the top of the ladder as the pivot point.

SKETCH:



RESEARCH: The system is in static equilibrium, so the sum of the torques about the pivot point is zero:

$$\tau_\text{net} = \sum_{i=1}^{n} \tau_i = 0.$$

SIMPLIFY: $\sum_{i=1}^{n} \tau_i = F_\text{f}h - NL + m_\text{b}g(L - x_\text{b}) + mg(L - x) = 0,\; F_\text{f} = \frac{NL - g\left(m_\text{b}(L - x_\text{b}) + m(L - x)\right)}{h}$

From the sketch I can derive expressions for $L, h, x,$ and x_b:

$$L = l\cos\theta,\; h = l\sin\theta,\; x_\text{b} = d\cos\theta \;\Rightarrow\; L - x_\text{b} = \cos\theta(l - d),\; x = \frac{l\cos\theta}{2} \;\Rightarrow\; L - x = \frac{l\cos\theta}{2}.$$

Substituting these values into the expression for F_f gives:

$$F_\text{f} = \frac{Nl\cos\theta - g\left[m_\text{b}(l - d)\cos\theta + m\frac{l\cos\theta}{2}\right]}{l\sin\theta}.$$

N is unknown, but because the system is in static equilibrium, the sum of the forces in the y direction is zero. $\sum_{i=1}^{n} F_{y,i} = N - m_b g - mg = 0 \Rightarrow N = (m_b + m)g$. Substituting this expression for N into the equation for F_f gives:

$$F_f = \frac{m_b gl\cos\theta + mgl\cos\theta - m_b gl\cos\theta + m_b gd\cos\theta - mg\dfrac{l\cos\theta}{2}}{l\sin\theta} = \frac{mg\dfrac{l\cos\theta}{2} + m_b gd\cos\theta}{l\sin\theta}.$$

CALCULATE:

$$F_f = \frac{9.072\text{ kg}\left(9.81\text{ m/s}^2\right)\dfrac{(10.0\text{ m})\cos 60.0^\circ}{2} + (27.67\text{ kg})\left(9.81\text{ m/s}^2\right)(4.00\text{ m})\cos 60.0^\circ}{(10.0\text{ m})\sin 60.0^\circ} = 88.378\text{ N}$$

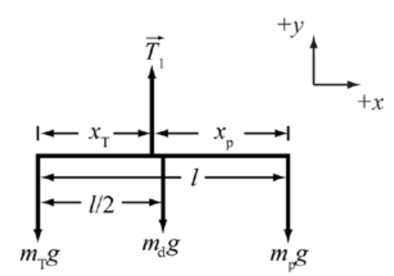
ROUND: All of the values given in the question have three significant figures. Therefore the answer should also be rounded to this precision. The final answer is is $F_f = 88.4$ N.

DOUBLE-CHECK: Given that the force of static friction is $f_s = \mu_s N$, the calculated values indicate $\mu_s \approx 88.4\text{ N}/360\text{ N} \approx 0.25$, which is a reasonable value for a coefficient of static friction between the ladder and the floor.

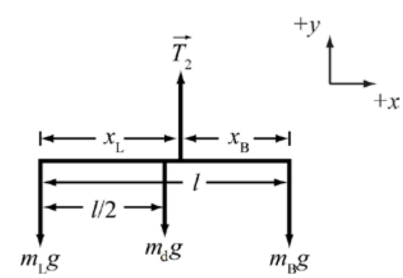
11.44. **THINK:** The masses of the four stuffed animals are: $m_T = 0.0160$ kg, $m_L = 0.0180$ kg, $m_P = 0.0220$ kg and $m_B = 0.0150$ kg. The three wooden dowels each have the same mass, $m_d = 0.00500$ kg, and same length, $l = 0.150$ m. Robin wants to hang m_T and m_P from the ends of dowel 1, and m_L and m_B from the ends of dowel 2. She wants to suspend dowels 1 and 2 from the ends of dowel 3 and hang the entire system from the ceiling. Assume that the thread used to attach the stuffed animals and the dowels has negligible mass. Determine where each thread must be positioned in order for the entire assembly to hang level. Use the convention that a torque in the counter-clockwise direction is positive.

SKETCH:

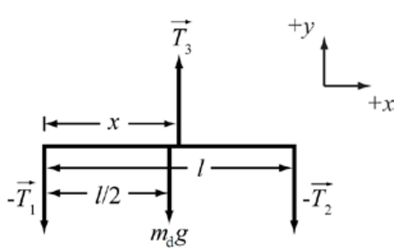
(a)



(b)



(c)



RESEARCH: The system is in static equilibrium, so the sum of the torques around the pivot point equals zero and the sum of all forces acting on the system must also equal zero:

$$\tau_{net} = \sum_{i=1}^{n} \tau_i = 0\ ;\ F_{net,y} = \sum_{i=1}^{n} F_{y,i} = 0.$$

Note that in this system all the forces are acting in the y-direction.

SIMPLIFY: For the parts of the system shown in sketch (a), I will assume that m_{T} and m_{P} are attached at the ends of the dowel. I will choose the point where m_{T} is attached as the pivot point. The distance from this end to the point where the string is attached is x_{T}.

$$\sum_{i=1}^{n}\tau_i = 0 = T_1 x_{\text{T}} - m_{\text{P}} g l - m_{\text{d}} g\frac{l}{2} \Rightarrow x_{\text{T}} = \frac{m_{\text{P}} g l + m_{\text{d}} g\left(l/2\right)}{T_1}$$

In this section, the net forces in the y direction are $T_1 - m_{\text{T}} g - m_{\text{P}} g - m_{\text{d}} g = 0$, so $T_1 = g(m_{\text{T}} + m_{\text{P}} + m_{\text{d}})$. Substituting this expression into the equation for x_T gives

$$x_{\text{T}} = \frac{m_{\text{P}} l + m_{\text{d}}\left(l/2\right)}{m_{\text{T}} + m_{\text{P}} + m_{\text{d}}}.$$

For the system shown in sketch (b), I will assume m_{L} and m_{B} are hung from the ends of the dowels. The pivot point will be where m_{L} is attached. The distance x_{L} will be the distance from the pivot end to the point where the string is attached.

$$\sum_{i=1}^{n}\tau_i = 0 = T_2 x_{\text{L}} - m_{\text{B}} g l - m_{\text{d}} g\frac{l}{2} \Rightarrow x_{\text{L}} = \frac{m_{\text{B}} g l + m_{\text{d}} g\frac{l}{2}}{T_2}$$

But $T_2 = g(m_{\text{L}} + m_{\text{B}} + m_{\text{d}})$, so, $x_{\text{L}} = \dfrac{m_{\text{B}} l + m_{\text{d}}\frac{l}{2}}{m_{\text{L}} + m_{\text{B}} + m_{\text{d}}}$. For the system shown in sketch (c), I will assume that dowels 1 and 2 will be hung from the ends of dowel 3. I choose the system's pivot point to be the end with the string that has tension, T_1. The distance x will be the distance from this end to the point where the string is attached.

$\sum_{i=1}^{n}\tau_i = 0 = T_3 x - T_2 l - m_{\text{d}} g\frac{l}{2}$. But, $T_2 = (m_{\text{L}} + m_{\text{B}} + m_{\text{d}})g$ and $T_3 = (m_{\text{T}} + m_{\text{P}} + m_{\text{L}} + m_{\text{B}} + 3m_{\text{d}})g$.

Substituting these expressions into the torque equation,

$$x = \frac{(m_{\text{L}} + m_{\text{B}} + m_{\text{d}})gl + m_{\text{d}} g\frac{l}{2}}{(m_{\text{T}} + m_{\text{B}} + m_{\text{P}} + m_{\text{L}} + 3m_{\text{d}})g} = \frac{(m_{\text{L}} + m_{\text{B}} + \left(3m_{\text{d}}\right)/2)l}{(m_{\text{T}} + m_{\text{B}} + m_{\text{P}} + m_{\text{L}} + 3m_{\text{d}})}.$$

The values of x_{P} and x_{B} are given by $x_{\text{P}} = 0.150 - x_{\text{T}}$, $x_{\text{B}} = 0.150 - x_L$.

CALCULATE: $$x_{\text{T}} = \frac{(0.0220\text{ kg})(0.150\text{ m}) + (0.00500\text{ kg})\left(\frac{0.150\text{ m}}{2}\right)}{(0.0160\text{ kg} + 0.0220\text{ kg} + 0.00500\text{ kg})} = 0.0855\text{ m}$$

$$x_{\text{P}} = 0.150\text{ m} - 0.0855\text{ m} = 0.0645\text{ m}$$

$$x_{\text{L}} = \frac{(0.0150\text{ kg})(0.150\text{ m}) + (0.00500\text{ kg})\left(\frac{0.150\text{ m}}{2}\right)}{(0.0150\text{ kg} + 0.0180\text{ kg} + 0.00500\text{ kg})} = 0.06908\text{ m}$$

$$x_{\text{B}} = 0.150\text{ m} - 0.06907\text{ m} = 0.08092\text{ m}$$

$$x = \frac{(0.0180\text{ kg} + 0.0150\text{ kg} + 3\left(0.00500\text{ kg}\right)/2)(0.150\text{ m})}{[0.0180\text{ kg} + 0.0150\text{ kg} + 0.0220\text{ kg} + 0.0160\text{ kg} + 3(0.00500\text{ kg})]} = 0.07064\text{ m}$$

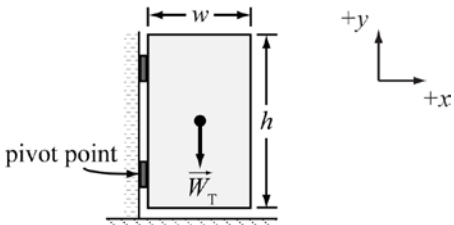
ROUND: $x_{\text{T}} = 0.0855\text{ m} = 8.55\text{ cm}$, $x_{\text{P}} = 0.0645\text{ m} = 6.45\text{ cm}$, $x_{\text{L}} = 0.0691\text{ m} = 6.91\text{ cm}$, $x_{\text{B}} = 0.0809\text{ m} = 8.09\text{ cm}$, and $x = 0.0706\text{ m} = 7.06\text{ cm}$.

DOUBLE-CHECK: The value for x_{T} is appropriate because the mass of the pony is greater than the mass of the teddy bear, so the string would need to be attached closer to the pony. Similarly, the mass of the lamb is greater than the mass of the bird, so the string should be attached closer to the lamb.

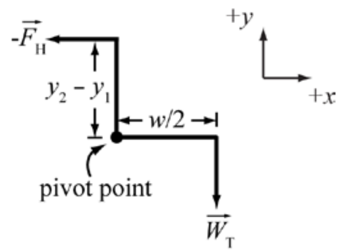
11.45. **THINK:** The door is uniform and has height $h = 2.00$ m, width $w = 0.800$ m, and weight $W = 100.0$ N. The door is supported at one end by two hinges at positions $y_1 = 0.300$ m and $y_2 = 1.70$ m with respect to the bottom of the door. I want to calculate the horizontal components of the forces F_{H} on the hinges.

SKETCH:

(a)



(b)



RESEARCH: The system is in static equilibrium, so the sum of the torques about the pivot point is equal to zero: $\sum_{i=1}^{n} \tau_i = 0$. The force on the upper hinge will be equal in magnitude but opposite in direction to the force on the lower hinge. The weight of the door will pull out on the upper hinge and push in on the lower hinge, so the force on the upper hinge will be positive while the force on the lower hinge will be negative. In sketch (b) the direction of F_{H} indicates the force that the hinge exerts on the door to keep it from pivoting about the lower hinge.

SIMPLIFY: $\sum_{i=1}^{n} \tau_i = 0 = F_{\text{H}}(y_2 - y_1) - (W)\left(\frac{w}{2}\right) \Rightarrow F_{\text{H}} = \frac{W\left(\frac{w}{2}\right)}{y_2 - y_1}$

CALCULATE: $F_{\text{H}} = \frac{(100.0 \text{ N})\left(\frac{0.800 \text{ m}}{2}\right)}{1.70 \text{ m} - 0.300 \text{ m}} = 28.57 \text{ N}$

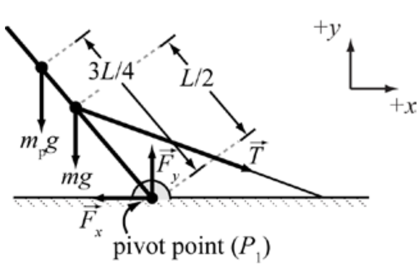
ROUND: The width provided in the question has three significant figures, so $F_{\text{H}} = 28.6$ N away from the wall. The horizontal force on the lower hinge is 28.6 N towards the wall.

DOUBLE-CHECK: Newtons are appropriate units for force. The calculated value for the horizontal force is reasonable considering the weight and dimensions of the door.

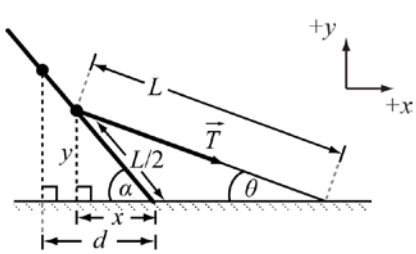
11.46. **THINK:** The mass of the ladder is $m = 20.0$ kg, and its length is L. The ladder is hinged on a horizontal platform at a point P_1, and anchored by a steel cable that is attached a length $L/2$ up the ladder. A person of mass $m_{\text{p}} = 80.0$ kg stands on the ladder at $3L/4$. I want to calculate the tension T in the cable and the reaction forces F_x and F_y. I will use the convention that a counter-clockwise torque is positive.

SKETCH:

(a)



(b)



RESEARCH: The system is in static equilibrium, so the sum of the torques about the pivot point is zero, and the sums of the forces in the *x*- and *y*-directions are also zero:

$$\tau_{\text{net}} = \sum_{i=1}^{n} \tau_i = 0; \quad F_{\text{net},y} = \sum_{i=1}^{n} F_{y,i} = 0; \quad F_{\text{net},x} = \sum_{i=1}^{n} F_{x,i} = 0.$$

SIMPLIFY: For the torque, $\sum_{i=1}^{n} \tau_i = 0 = mgx + m_{\text{p}}gd - T_x y + T_y x.$ From sketch (b):

$$x = \frac{L}{2}\cos\alpha;\ y = \frac{L}{2}\sin\alpha;\ d = \frac{3L}{4}\cos\alpha;\ T_x = T\cos\theta;\ T_y = -T\sin\theta.$$

Substituting these expressions into the torque equation gives

$$0 = mg\frac{L}{2}\cos\alpha + m_p g\frac{3L}{4}\cos\alpha - T\cos\theta\frac{L}{2}\sin\alpha - T\sin\theta\frac{L}{2}\cos\alpha$$

$$\Rightarrow T\frac{L}{2}(\sin\alpha\cos\theta + \sin\theta\cos\alpha) = \frac{gL}{4}\left(2m\cos\alpha + 3m_{\text{p}}\cos\alpha\right)$$

$$\Rightarrow T\left(\sin\alpha\cos\theta + \sin\theta\cos\alpha\right) = \frac{g}{2}\left(\cos\alpha\right)(2m + 3m_{\text{p}}).$$

Using the trigonometric identity $\sin\alpha\cos\theta + \sin\theta\cos\alpha = \sin(\alpha+\theta)$:

$$T = \frac{g\cos\alpha(2\text{ m} + 3m_{\text{p}})}{2\sin(\alpha+\theta)}.$$

For the reaction forces:

$$\sum_{i=1}^{n} F_{x,i} = 0 = -F_x + T\cos\theta \ \Rightarrow\ F_x = T\cos\theta,\ \ \sum_{i=1}^{n} F_{y,i} = 0 = F_y - mg - m_{\text{p}}g - T\sin\theta \Rightarrow F_y = T\sin\theta + mg + m_{\text{p}}g.$$

CALCULATE: $T = \dfrac{\left(9.81\text{ m/s}^2\right)\left(\cos 50.0^\circ\right)\left(2\cdot\left(20.0\text{ kg}\right) + 3\cdot\left(80.0\text{ kg}\right)\right)}{2\sin(50.0^\circ + 30.0^\circ)} = 896.42\text{ N}$

$F_x = \left(896.42\text{ N}\right)\cos 30.0^\circ = 776.33\text{ N},$

$F_y = \left(896.42\text{ N}\right)\sin 30.0^\circ + \left(20.0\text{ kg}\right)\left(9.81\text{ m/s}^2\right) + \left(80.0\text{ kg}\right)\left(9.81\text{ m/s}^2\right) = 1429.21\text{ N}$

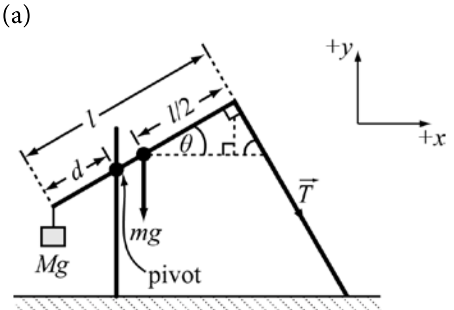
ROUND: $T = 896$ N, $F_x = 776$ N, and $F_y = 1430$ N.

DOUBLE-CHECK: The tension should depend on the mass of the person on the ladder, the mass of the ladder, and the angles α and θ. The calculated tension is a function of these values. It makes sense that the horizontal reaction force cancels the *x* component of the tension. It is also reasonable that the *y* component of the reaction force depends on the masses and the *y* component of the tension in the cable.

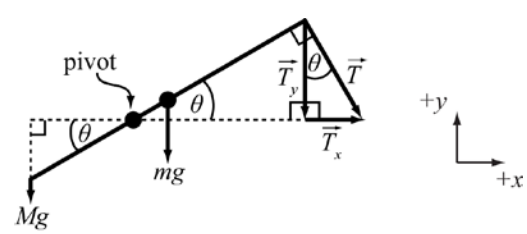
11.47. **THINK:** The beam has length $l = 8.00$ m and mass $m = 100.$ kg. The beam is attached by a bolt to a support a distance $d = 3.00$ m from one end. The beam makes an angle $\theta = 30.0^\circ$ with respect to the horizontal. A mass of $M = 500.$ kg is attached to one end of the beam by a rope. A rope attaches the other end of the beam to the ground. I want to calculate the tension T in the rope, and the force $\vec{F} = F_{\text{bx}}\hat{x} + F_{\text{by}}\hat{y}$ exerted on the beam by the bolt.

SKETCH:

(a)



(b)



RESEARCH: The sum of the torques about the pivot point is zero because the system is in static equilibrium; $\sum_{i=1}^{n}\tau_i = 0$. The sums of the forces in the x and y directions are also zero:

$$\sum_{i=1}^{n}F_{y,i} = 0;\ \sum_{i=1}^{n}F_{x,i} = 0.$$

Counterclockwise torque is considered positive and clockwise torque is considered negative.

SIMPLIFY: $\sum_{i=1}^{n}\tau_i = 0 = Mgd\cos\theta - mg\left(\frac{l}{2}-d\right)\cos\theta - T(l-d) \Rightarrow T = \dfrac{Mgd\cos\theta - mg\left(\frac{l}{2}-d\right)\cos\theta}{l-d}$

$$\sum_{i=1}^{n}F_{yi} = 0 = -Mg - mg - T_y + F_{by}$$

From sketch (b), $T_y = T\cos\theta$, so, $F_{by} = Mg + mg + T\cos\theta$.

$$\sum_{i=1}^{n}F_{x,i} = 0 = -F_{bx} + T\sin\theta \Rightarrow F_{bx} = T\sin\theta$$

CALCULATE:

$$T = \frac{(500.\text{ kg})\left(9.81\text{ m/s}^2\right)(3.00\text{ m})\cos 30.0° - (100.\text{ kg})\left(9.81\text{ m/s}^2\right)(4.00\text{ m} - 3.00\text{ m})\cos 30.0°}{8.00\text{ m} - 3.00\text{ m}} = 2378.799\text{ N}$$

$F_{by} = (500.\text{ kg})\left(9.81\text{ m/s}^2\right) + (100.\text{ kg})\left(9.81\text{ m/s}^2\right) + (2378.799\text{ N})\cos 30.0° = 7946.100\text{ N}$

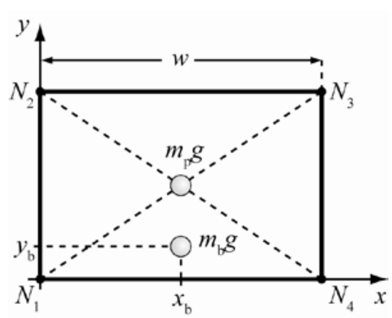
$F_{bx} = (2378.799\text{ N})\sin 30.0° = 1189.399\text{ N}$

ROUND: To three significant figures: $T = 2380\text{ N}$, $F_{by} = 7950\text{ N}$, $F_{bx} = 1190\text{ N}$.

DOUBLE-CHECK: The calculated values are reasonable given the masses and their configuration in the system.

11.48. **THINK:** The ball's mass is $m_b = 15.49$ kg. The glass plate's mass is $m_p = 12.13$ kg. The table's dimensions are $h = 0.720$ m, $w = 1.380$ m, and $d = 0.638$ m. The ball is located at (x_b, y_b) = (0.690 m, 0.166 m) with respect to corner 1. What is the force that the plate exerts on each leg, N_1', N_2', N_3', N_4'?

SKETCH:



RESEARCH: In static equilibrium, $\sum F_x = 0$, $\sum F_y = 0$, $\sum F_z = 0$, and $\tau_{net} = \sum\tau_{ccw} - \sum\tau_{cw} = 0$. Use the sides of the table as pivot points. Note the forces that the glass plate exerts on each leg, N_1', N_2', N_3', N_4', have the same magnitude as the force of each leg on the table, N_1, N_2, N_3, N_4, by Newton's third law. In addition, because the ball is placed in the middle with respect to the x-axis $(x_b = w/2)$, symmetry exists such that $N_1 = N_4$ and $N_2 = N_3$.

SIMPLIFY: Choose the bottom edge as a pivot point:

$$\tau_{net} = m_b g y_b + m_p g\frac{d}{2} - N_2 d - N_3 d = 0 \Rightarrow m_b g y_b + m_p g\frac{d}{2} - 2N_2 d = 0 \Rightarrow N_2 = \frac{m_b g y_b}{2d} + \frac{m_p g}{4}.$$

From $\sum F_z = 0,$

$$N_1 + N_2 + N_3 + N_4 - m_p g - m_b g = 0 \Rightarrow 2N_1 + 2N_2 - m_p g - m_b g = 0 \Rightarrow N_1 = \frac{m_p g}{2} + \frac{m_b g}{2} - N_2.$$

CALCULATE: $N_3 = N_2 = \frac{(15.49 \text{ kg})(9.81 \text{ m/s}^2)(0.166 \text{ m})}{2(0.638 \text{ m})} + \frac{(12.13 \text{ kg})(9.81 \text{ m/s}^2)}{4} = 49.52 \text{ N}$

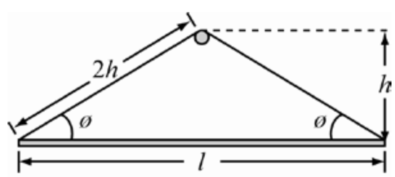
$N_4 = N_1 = \frac{(12.13 \text{ kg})(9.81 \text{ m/s}^2)}{2} + \frac{(15.49 \text{ kg})(9.81 \text{ m/s}^2)}{2} - 49.52 \text{ N} = 85.96 \text{ N}$

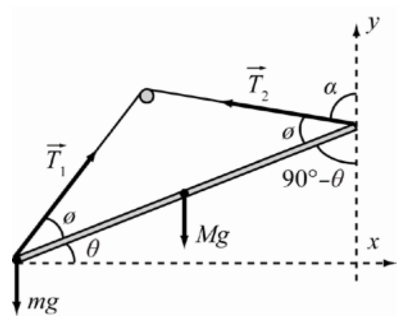
ROUND: Since the table dimensions have three significant figures, $N_1' = N_4' = 86.0 \text{ N}$ and $N_2' = N_2' = 49.5 \text{ N}.$

DOUBLE-CHECK: These forces seem reasonable given the masses of the ball and the glass plate.

11.49. **THINK:** The plank's length density is $\lambda = 2.00$ kg/m. The height is $h = 10.0$ m. Each rope is $L = 2h$ in length. Their maximum tension is $T_{\text{max}} = 2000.$ N . A hiker on the left side of the bridge causes it to tip $\theta = 25.0°$. Find the hiker's mass, m. Since the mass density of the plank is constant, we know that the plank's weight acts at its geometric center. For all of these problems of static equilibrium, we need to find the equilibrium equations for the forces and for the torques. For the latter we need to pick pivot point. Usually, the choice of pivot point does not matter, and one can find the result for any choice. However, sometimes it really pays to put a little thought into picking the correct pivot point, and the equations become much simpler. For this problem, let's make a somewhat unfortunate choice and pick the left edge of the plank is the pivot point first. Then, in the double-checking process we will pick another pivot point.

SKETCH:





RESEARCH: Note $l = 2\sqrt{3}h$ from the Pythagorean Theorem.

$$\phi = \sin^{-1}\left(\frac{h}{2h}\right) = \sin^{-1}\left(\frac{1}{2}\right) \text{ and } M\text{, the plank's mass is } M = \lambda l = 2\sqrt{3}\lambda h.$$

For static equilibrium, $\sum F_x = 0,$ $\sum F_y = 0,$ and $\sum \tau_{\text{net}} = \sum \tau_{\text{ccw}} - \sum \tau_{\text{cw}} = 0.$

SIMPLIFY: Choose the left edge as a pivot point. Then

$$\tau_{\text{net}} = -Mg\left(\frac{l}{2}\right)\sin(90° - \theta) + T_2 l \sin\phi = 0 \Rightarrow T_2 = \frac{Mg\cos\theta}{2\sin\phi}.$$

Note $T_{2x} = T_2 \sin\alpha$ and $T_{2y} = T_2 \cos\alpha$ where $\alpha = 180° - \phi - (90° - \theta) = 90° + \theta - \phi$. Also, $T_{1,x} = T_1 \cos(\theta + \phi)$ and $T_{1,y} = T_1 \sin(\theta + \phi)$. From $\sum F_x = 0$,

$$T_{1,x} = T_{2,x} \Rightarrow T_1 \cos(\theta + \phi) = T_2 \sin\alpha \Rightarrow T_1 = \frac{T_2 \sin\alpha}{\cos(\theta + \phi)}.$$

From $\sum F_y = 0,$

$$T_{1,y} + T_{2,y} - mg - Mg = 0 \Rightarrow m = \frac{(T_{1,y} + T_{2,y})}{g} - M = \frac{T_1 \sin(\theta + \phi) + T_2 \cos\alpha}{g} - M$$

$$= \frac{(T_2 \sin\alpha \tan(\theta+\phi) + T_2 \cos\alpha)}{g} - M = \frac{M\cos\theta}{2\sin\phi}(\sin\alpha\tan(\theta+\phi)+\cos\alpha) - M$$

$$= M\left[\frac{\cos\theta}{2\sin\phi}(\sin\alpha\tan(\theta+\phi)+\cos\alpha)-1\right].$$

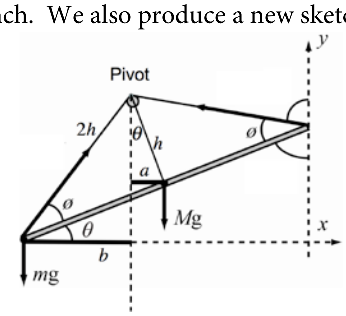
CALCULATE: $M = \lambda l = (2.00 \text{ kg/m})\left(2\sqrt{3}\right)(10.0 \text{ m}) = 69.282 \text{ kg}$, $\phi = \sin^{-1}\left(\frac{1}{2}\right) = 30.0°$

$\alpha = 90.0° + 25.0° - 30.0° = 85.0°$, $\phi + \theta = 30° + 25° = 55°$

$$m = 69.282 \text{ kg}\left[\left(\frac{\cos 25.0°}{2\sin 30.0°}\right)\left(\left(\sin 85.0°\right)\left(\tan 55.0°\right)+\left(\cos 85.0°\right)\right)-1\right] = 25.52 \text{ kg}$$

ROUND: We round to three significant figures, $m = 25.5$ kg.

DOUBLE-CHECK: As previously advertised, let's not pick a different pivot point. The "natural" pivot point is the location of the branch. We also produce a new sketch.



We draw in a vertical line through the new pivot point, and from geometry we can calculate the perpendicular distances of *M* and *m* to this line, *a* and *b*, respectively. They are $a = h\sin\theta$ and $b = 2h\cos(\theta+\phi)$. With this choice of pivot point the only torques are due to *M* and *m*, and they act in opposite directions. This leads to

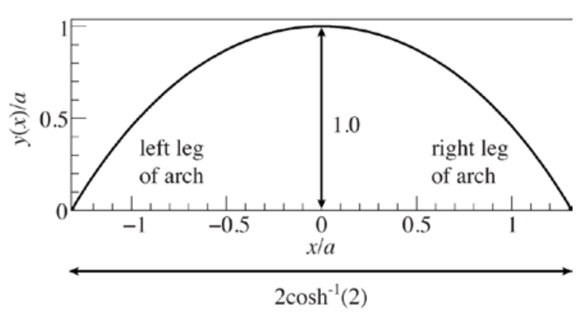
$$mgb = Mga \Rightarrow$$

$$m = M\frac{a}{b} = M\frac{\sin\theta}{2\cos(\theta+\phi)}$$

Inserting the number, we again find $m = (69.282 \text{ kg})\dfrac{\sin 25°}{2\cos(25°+30°)} = 25.52 \text{ kg}$. This double-check step shows that we can reach the same final result in very different ways, but that one often can save a lot of work by thinking about the problem beforehand.

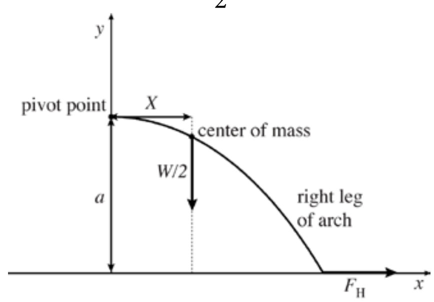
11.50. **THINK:** The arch has a functional form that is described by $y(x) = 2a - a\cosh(x/a)$, where a is the maximum height and *x* varies from $-a\cosh^{-1}(2)$ to $+a\cosh^{-1}(2)$. We can find the force on the legs of the arch using the fact that the forces must sum to zero and that the sum of the torques must be zero. The arch has a uniform density and cross section, so we can treat the mass in one dimension, x.

SKETCH:



RESEARCH: For static equilibrium, and $\tau_{\text{net}} = \sum \tau_{\text{cw}} - \sum \tau_{\text{ccw}}$. To find the vertical forces on the legs, balance all the forces in that direction, $\sum F_y = 0$. The horizontal forces on the legs can be found by considering the torque of the weight of one leg and the horizontal force on the bottom of the leg. The x-coordinate of the center of mass of the left leg is given by $X = \dfrac{\int_0^k xy(x)dx}{\int_0^k y(x)dx}$, where $k = a\cosh^{-1} 2$. To find θ, consider $\theta = \tan^{-1}\left(\dfrac{F_{\text{V}}}{F_{\text{H}}}\right)$.

SIMPLIFY: From $\sum F_y = 0,\ F_{\text{V}} + F_{\text{V}} - W = 0$. $F_{\text{V}} = \dfrac{1}{2}W$ (for each leg). For one leg,



Then $\tau_{\text{net}} = -\dfrac{1}{2}WX + F_{\text{H}}a = 0.$ For uniform density, and considering the bridge as a two dimensional object,

$$X = \frac{\int_0^k x\left(2a - a\cosh(x/a)\right)dx}{\int_0^k \left(2a - a\cosh(x/a)\right)dx} = \frac{\int_0^k 2x\,dx - \int_0^k x\left(\cosh(x/a)\right)dx}{\int_0^k 2\,dx - \int_0^k \left(\cosh(x/a)\right)dx}$$

$$X = \frac{k^2 - \left[a\left(x\sinh(x/a) - a\cosh(x/a)\right)\right]_0^k}{2k - \left[a\sinh(x/a)\right]_0^k}$$

Note that $\cosh\left(\left(a\cosh^{-1} 2\right)/a\right) = 2$ and $\sinh\left(\left(a\cosh^{-1} 2\right)/a\right) = \sqrt{3}$

$$X = \frac{k^2 - a\left[\sqrt{3}k - 2a + a\right]}{2k - \sqrt{3}a} = \frac{\left(a\cosh^{-1} 2\right)^2 - \sqrt{3}a^2\cosh^{-1} 2 + a^2}{2a\cosh^{-1} 2 - \sqrt{3}a} = \frac{a\left(\cosh^{-1} 2\right)^2 - \sqrt{3}a\cosh^{-1} 2 + a}{2\cosh^{-1} 2 - \sqrt{3}}$$

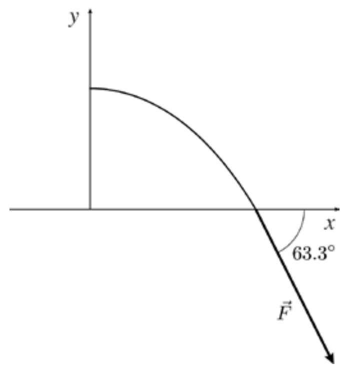
With X known, $F_{\text{H}} = \dfrac{WX}{2a}$. Then, $\theta = \tan^{-1}\left(\dfrac{F_{\text{V}}}{F_{\text{H}}}\right)$.

CALCULATE: $X = a\dfrac{\left(\cosh^{-1} 2\right)^2 - \sqrt{3}\cosh^{-1} 2 + 1}{2\cosh^{-1} 2 - \sqrt{3}} = 0.50267a$

$k = a\cosh^{-1} 2 = 1.31696a$

Then, $F_{\text{H}} = \dfrac{W(0.50267a)}{2a} = 0.251335W$ and $F_{\text{V}} = \dfrac{1}{2}W$, so $\theta = \tan^{-1}\dfrac{\frac{1}{2}W}{0.251335W} = 63.3127°$.

ROUND: We report our answer to three significant digits, $\theta = 63.3°$. This force points down and to the right as shown below.

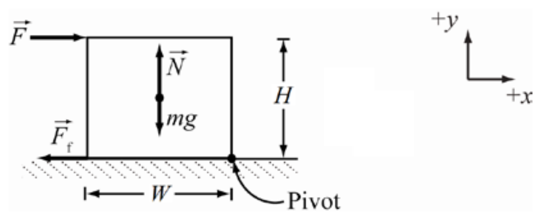


So the bottom face of each leg should make an angle of $63.3°$ with respect to the ground.

DOUBLE-CHECK: An angle of $\theta = 63.3°$ for an inverse catenary curve arch seems reasonable according to our sketch. Note that the Gateway Arch in Saint Louis is a flattened inverse catenary and does not have a uniform cross section or density.

11.51. **THINK:** The bookcase has height H, mass m, and width $W = H/2$. The bookcase is to be pushed with a constant velocity v across a level floor. The bookcase is pushed with a force F horizontally at its top edge a distance H above the floor. I want to calculate the maximum coefficient of kinetic friction, μ_{k}, between the bookcase and the floor so that the bookcase does not tip over while being pushed.

SKETCH:



RESEARCH: The condition for the bookcase to not tip over is that the sum of the torques about the pivot point must equal zero, $\sum_{i=1}^{n} \tau_i = 0$. The bookcase stays in contact with the ground, so the sum of forces in the y-direction is zero, $\sum F_y = 0$. The bookcase is being pushed in the x-direction at a constant velocity, which implies its acceleration is zero, therefore the sum of the forces in the x-direction is zero, $\sum F_x = 0$.

SIMPLIFY: $\sum F_y = 0 = -mg + N \Rightarrow N = mg$, $\sum F_x = 0 = F - F_{\text{f}} \Rightarrow F = F_{\text{f}}$. The definition of the frictional force is $F_{\text{f}} = \mu_{\text{k}} N$. Substituting this into the equation gives $F = \mu_{\text{k}} N$.

$$\sum_{i=1}^{n} \tau_i = 0 = mg\frac{W}{2} - FH$$

CALCULATE: Substituting $W = H/2$, $F = \mu_{\text{k}} N$, and $N = mg$ into this equation, and solving for μ_{k}:

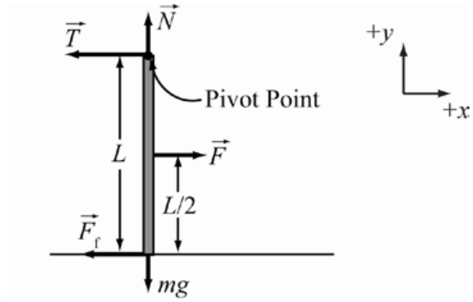
$$mg\frac{H}{4} = \mu_{\text{k}} mgH \Rightarrow \mu_{\text{k}} = 0.25.$$

ROUND: Not applicable.

DOUBLE-CHECK: The coefficient of kinetic friction calculated is reasonable.

11.52. **THINK:** The rod has length L and mass M. It is held in an upright position. The top of the rod is tied to a fixed surface with a string. A force, F is applied at $L/2$. The coefficient of static friction between the rod and the horizontal surface is μ_{s}. I want to calculate the maximum force, $F_{\max}$ that can be applied such that the rod maintains a condition of static equilibrium.

SKETCH:



RESEARCH: The system is in static equilibrium, so the sum of the torques about the pivot point is zero: $\sum_{i=1}^{n}\tau_i = 0.$ Also, the sum of the forces in the x and y directions are zero: $\sum_{i=1}^{n}F_{y,i} = 0$ and $\sum_{i=1}^{n}F_{x,i} = 0$

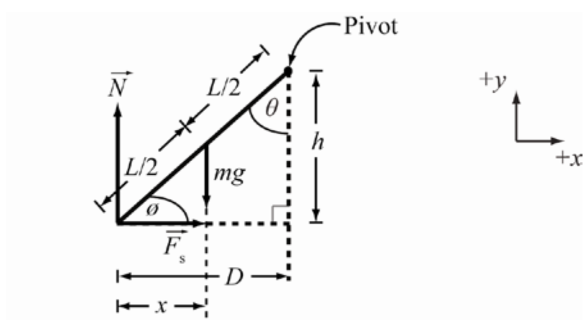
SIMPLIFY: $\sum_{i=1}^{n}F_{y,i} = 0 = -Mg + N \Rightarrow N = Mg,\ \sum_{i=1}^{n}\tau_i = 0 = -F_f L + F\frac{L}{2}$

By definition $F_f = \mu_s N$ and $N = Mg$. Substituting these expressions into the above equation gives

$$F\frac{L}{2} = \mu_s MgL \Rightarrow F = 2\mu_s Mg$$

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: This seems like a reasonable force as it has to balance the tension in the string and the frictional force between the rod and the horizontal surface.

11.53. **THINK:** The ladder has mass, $m = 37.7$ kg, length, $L = 3.07$ m, and leans against a wall at an angle, θ. The coefficient of static friction between the ladder and the floor is $\mu_s = 0.313$. Assume that the wall is frictionless. I want to calculate the maximum angle, θ_{max}, before the ladder starts to slip.
SKETCH:



RESEARCH: The system is in static equilibrium, so the sum of the torques about the pivot point is zero. This also implies that the sum of the forces in the x and y directions equals zero. Mathematically this can be stated as $\sum_{i=1}^{n}\tau_i = 0$, $\sum_{i=1}^{n}F_{y,i} = 0$, and $\sum_{i=1}^{n}F_{x,i} = 0$.

SIMPLIFY: $\sum_{i=1}^{n}\tau_i = 0 = F_s h + mg(D-x) - ND$. From the sketch it can be seen that $D = L\sin\theta$, $x = L\cos\phi/2$, and $\phi = 90° - \theta$. Also, $h = L\cos\theta$. Substituting these values gives:

$$F_s L\cos\theta + mg\left(L\sin\theta - \frac{L}{2}\cos(90° - \theta)\right) = NL\sin\theta.$$

Simplify further using the identity $\cos(90°-\theta)=\sin\theta$ and the fact that $F_s=\mu_s N$. Substituting these values into the equation gives:

$$\mu_s NL\cos\theta+mg\frac{L}{2}\sin\theta=NL\sin\theta \Rightarrow \mu_s N\cos\theta+mg\left(\frac{1}{2}\right)\sin\theta=N\sin\theta.$$

By summing the forces in the y direction, $N=mg$, which can be substituted into the torque balance equation:

$$\mu_s mg\cos\theta+\frac{mg}{2}\sin\theta=mg\sin\theta \Rightarrow \mu_s\cos\theta=\frac{1}{2}\sin\theta \Rightarrow \tan\theta=2\mu_s \Rightarrow \theta=\tan^{-1}(2\mu_s).$$

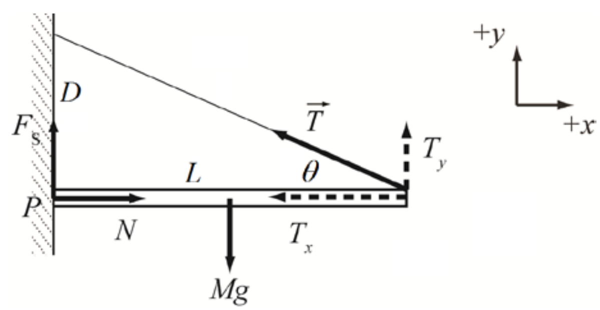
CALCULATE: $\theta=\tan^{-1}[2(0.313)]=32.047°$

ROUND: There are three significant figures provided in the question, so the result is $32.0°$.

DOUBLE-CHECK: This seems like quite a large angle with which to lean a ladder against a wall, so it is reasonable that exceeding this angle would cause the ladder to slip.

11.54. **THINK:** The uniform, rigid pole has length, L, and mass, M. The vertical force on the pole, where it meets the wall, is provided by the frictional force F_s. The coefficient of friction is μ_s. The other end of the pole is supported by a rope of negligible mass that is attached to the wall at height, D, above the rod. Determine the minimum value for μ_s, as a function of L and D, so the pole does not slide down the wall. The rope has tension T.

SKETCH:



RESEARCH: The system is in static equilibrium so the sum of the torques about the pivot point is zero: $\sum_{i=n}^{n}\tau_i=0$. This also implies that the sum of the forces in the x and y directions are zero.

SIMPLIFY: $\sum_{i=1}^{n}\tau_i=0=T_yL-Mg\frac{L}{2}$. From the sketch it is seen that $T_y=T\sin\theta$, so the equation can be written as $T=\frac{Mg}{2\sin\theta}$. The balance of the forces in the y-direction is given by $\sum_{i=1}^{n}F_{y,i}=0=-Mg+F_s+T_y$. But $T_y=T\sin\theta$ and $F_s=\mu_s N$, so substitute these values into the equation for the forces in the y-direction to get $\mu_s N=Mg-T\sin\theta$. The balance of the forces in the x-direction is given by $\sum_{i=1}^{n}F_{x,i}=0=-T_x+N$, which becomes $N=T\cos\theta$. Substitute this expression into the equation for the forces in the y direction to get $\mu_s T\cos\theta=Mg-T\sin\theta$. Substituting $T=\frac{Mg}{2\sin\theta}$ into this equation gives:

$$\mu_s\frac{Mg}{2\sin\theta}\cos\theta=Mg-\frac{Mg}{2\sin\theta}\sin\theta \Rightarrow \frac{\mu_s}{2}\frac{\cos\theta}{\sin\theta}=\frac{1}{2} \Rightarrow \mu_s=\frac{\sin\theta}{\cos\theta}=\tan\theta.$$

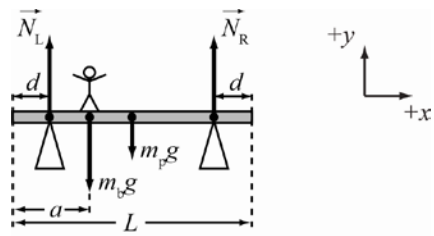
From the sketch it can be seen that $\tan\theta=D/L$, therefore,

$$\mu_s=\frac{D}{L}.$$

CALCULATE: Not applicable.

ROUND: Not applicable.
DOUBLE-CHECK: This answer is reasonable. The coefficient of friction should depend on the geometry of the system, as is shown here.

11.55. **THINK:** The boy's weight is 60.0 lb, so his mass is $m_{\text{b}} = 27.2$ kg. The plank weighs 30.0 lb, so its mass is $m_{\text{p}} = 13.6$ kg. Its length is $L = 8.00 \text{ ft} = 2.44 \text{ m},$ and lies on two supports, each $d = 2.00 \text{ ft} = 0.6096 \text{ m}$ from each end of the plank. Determine (a) the force exerted by each support, N_{L}, N_{R}, when the boy is $a = 3.00 \text{ ft} = 0.9144 \text{ m}$ from the left end, and (b) the distance the boy can go to the right before the plank tips, X_{b}.
SKETCH:



RESEARCH:
(a) Use $\sum F_y = 0$ and $\tau_{\text{net}} = \sum \tau_{\text{ccw}} - \sum \tau_{\text{cw}} = 0$ for the plank in static equilibrium.
(b) The plank will tip when the boy-plank system's center of mass, X is to the right of the right support. Solve for X_b from $X = \frac{1}{M}\sum x_i m_i$. Assume the plank's weight acts at its center, $L/2$.
SIMPLIFY:
(a) Choose the left support as a pivot point:

$$\tau_{\text{net}} = -m_{\text{b}}g(a-d) - m_{\text{p}}g\left(\frac{L}{2}-d\right) + N_{\text{R}}(L-2d) = 0 \Rightarrow N_{\text{R}} = \frac{g\left[m_{\text{b}}(a-d) + m_{\text{p}}\left(\frac{L}{2}-d\right)\right]}{L-2d}.$$

From $\sum F_y = 0,\ N_{\text{R}} + N_{\text{L}} - m_{\text{p}}g - m_{\text{b}}g = 0 \ \Rightarrow\ N_{\text{L}} = g(m_{\text{p}} + m_{\text{b}}) - N_{\text{R}}$.
(b) Choose the left edge of the plank as the origin of the coordinate system. Then

$$X = \frac{1}{M}\left(m_{\text{b}}x_{\text{b}} + m_{\text{p}}\frac{L}{2}\right) \Rightarrow x_{\text{b}} = \frac{XM - m_{\text{p}}\frac{L}{2}}{m_{\text{b}}} = \frac{(L-d)(m_{\text{p}}+m_{\text{b}}) - \frac{1}{2}m_{\text{p}}L}{m_{\text{b}}} = \frac{L\left(\frac{1}{2}m_{\text{p}} + m_{\text{b}}\right) - d(m_{\text{p}}+m_{\text{b}})}{m_{\text{b}}}.$$

CALCULATE:

(a) $$N_{\text{R}} = \frac{\left(9.81 \text{ m/s}^2\right)\left[27.2 \text{ kg}(0.9144 \text{ m} - 0.6096 \text{ m}) + 13.6 \text{ kg}\left(\frac{2.44 \text{ m}}{2} - 0.6096 \text{ m}\right)\right]}{2.44 \text{ m} - 2(0.6096 \text{ m})} = 133.33 \text{ N}$$

$$N_{\text{L}} = \left(9.81 \text{ m/s}^2\right)(13.6 \text{ kg} + 27.2 \text{ kg}) - 133.33 \text{ N} = 266.918 \text{ N}$$

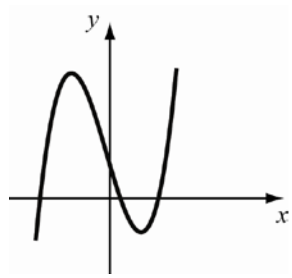
(b) $$x_{\text{b}} = \frac{2.44 \text{ m}\left(\frac{13.6 \text{ kg}}{2} + 27.2 \text{ kg}\right) - 0.6096 \text{ m}(13.6 \text{ kg} + 27.2 \text{ kg})}{27.2 \text{ kg}} = 2.1356 \text{ m}$$

ROUND: Since all of the given values have three significant figures,
(a) The right support applies an upward force of $N_{\text{R}} = 133 \text{ N (29.9 lb)}$. The left support applies an upward force of $N_{\text{L}} = 267 \text{ N (60.0 lb)}$.
(b) $x_{\text{b}} = 2.14 \text{ m (7.02 ft)}$ from the left edge of the board.

DOUBLE-CHECK: Since the boy is closer to the left edge, $N_L > N_R$. It is expected that the board will tip when the boy is just past the right support, $x_b > L - d = 2.44 \text{ m} - 0.61 \text{ m} = 1.83 \text{ m}$.

11.56. **THINK:** The track's height varies with x as $h(x) = x^3 + 3x^2 - 24x + 16$. Where are the stable locations? What kind of equilibrium do they have?

SKETCH:



RESEARCH: Consider the height as a potential energy landscape (e.g. $u = mgh$ is directly proportional to h). The marble will be stable for all positions x_o where $dh/dx = 0$. These equilibrium points can be classified based on the sign of the second derivative $\left.\frac{d^2h}{dx^2}\right|_{x_0}$, where x_0 is an equilibrium point. If the sign is negative, the equilibrium is unstable. If the sign is positive, it is a stable equilibrium.

SIMPLIFY: $\frac{dh}{dx} = \frac{d}{dx}(x^3 + 3x^2 - 24x + 16) = 3x^2 + 6x - 24$ and $\frac{d^2}{dx^2}h = \frac{d}{dx}(3x^2 + 6x - 24) = 6x + 6.$

CALCULATE: Solving the derivative equal to zero gives the values: $3x_0^2 + 6x_0 - 24 = 0$ $\Rightarrow x_0^2 + 2x_0 - 8 = 0 \Rightarrow (x_0 + 4)(x_0 - 2) = 0.$ So, $x_0 = -4$ and $x_0 = 2$ are the two equilibrium points. Now,

$$\text{At } x_0 = -4,\ \left.\frac{d^2h}{dx^2}\right|_{x_0=-4} = -18 < 0, \text{ so this is an unstable equilibrium point.}$$

$$\text{At } x_0 = 2,\ \left.\frac{d^2h}{dx^2}\right|_{x_0=2} = 18 > 0, \text{ so this is a stable equilibrium point.}$$

ROUND: Not applicable.

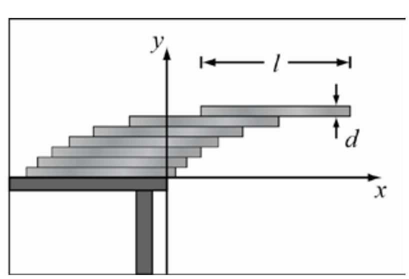
DOUBLE-CHECK: The first derivative of $h(x)$ is a second order polynomial, and should have at most two roots. This means that there will be at most two equilibrium points. This is consistent with the computed values.

11.57. **THINK:** Each block has length $l = 0.159$ m and thickness $d = 0.0220$ m.

(a) For seven blocks, find the maximum distance, x, between the right edge of the table and the right edge of the right-most block.

(b) Find the minimum height, h, of blocks for which the left edge of the top block is located off of the table.

SKETCH:



RESEARCH:

(a) From the text, use $x_1 = x_{n+1} + \frac{1}{2}l\left(\sum_{i=1}^{n}\frac{1}{i}\right)$, where x_1 is the location of the right edge of the top block with respect to taking x_{n+1} to be zero. The minimum height will be $h = dn$, where n is the minimum number of blocks. n can be determined from when $x_1 - l \geq 0$, as this is the condition that the top block's left edge is right of the table's edge.

SIMPLIFY:

(a) $x_1 = \frac{1}{2}l\sum_{i=1}^{7}\frac{1}{i} = \frac{1}{2}l\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right) = \frac{363l}{280}$

(b) $n = 4$ is the minimum number of blocks required. Then, $h = 4d$.

CALCULATE:

(a) $x_1 = \frac{363(0.159 \text{ m})}{280} = 0.20613 \text{ m}$

(b) $h = 4(0.0220 \text{ m}) = 0.0880 \text{ m}$

ROUND:

(a) l has three significant figures, so $x_1 = 0.206$ m.

(b) d has three significant figures, so $h = 0.0880$ m.

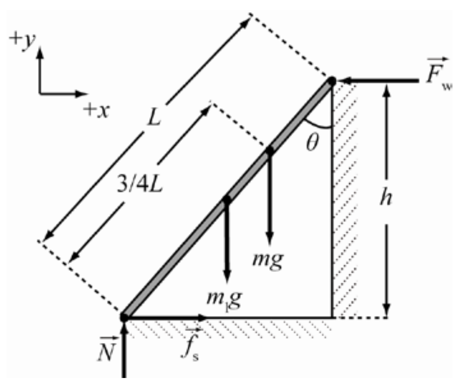
DOUBLE-CHECK: x_1 should be greater than l for seven blocks. It is expected for part (b) that n should be less than seven blocks.

11.58. **THINK:** The ladder is $L = 5.00 \text{ m}$ long, and it touches the house at a height $h = 4.00 \text{ m}$ above the ground. Its mass is $m_1 = 20.0 \text{ kg}$. Your mass is $m = 60.0 \text{ kg}$, and you are 3/4 of the way up the ladder. Treat "3/4 of the way up the ladder" as precise. Determine:

(a) The forces exerted by the side wall, F_w, and the ground, N, on the ladder, and

(b) The coefficient of static friction, μ_s, between the ground and the ladder.

SKETCH:



RESEARCH: Use $\tau_\text{net} = \sum\tau_\text{ccw} - \sum\tau_\text{cw} = 0$, and $F_{\text{net},x} = \sum F_x = 0$, and $F_{\text{net},y} = \sum F_y = 0$. Assume the ladder's center of mass (com) is at its center $(L/2)$.

SIMPLIFY:

(a) To determine F_w, use $\tau_\text{net} = 0$ and choose the point where the ladder touches the ground as a pivot point. It is useful to note that $\theta = \cos^{-1}\left(\frac{h}{L}\right)$.

$$\tau_\text{net} = \sum\tau_\text{ccw} - \sum\tau_\text{cw} = 0 \Rightarrow 0 = (F_w)(L)\sin(90^\circ - \theta) - (m_1 g)\left(\frac{L}{2}\right)\sin\theta - (mg)\left(\frac{3L}{4}\right)\sin\theta$$

$$\Rightarrow\ 0 = F_{\text{w}}L\cos\theta - \frac{1}{2}m_{\text{l}}gL\sin\theta - \frac{3}{4}mgL\sin\theta \ \Rightarrow\ F_{\text{w}} = g\tan\theta\left(\frac{1}{2}m_{\text{l}} + \frac{3}{4}m\right)$$

To determine N, use $\sum F_y = 0$: $0 = N - m_l g - mg \ \Rightarrow\ N = g(m_l + m).$

(b) To determine μ_{s} use $\sum F_x = 0$: $0 = f_{\text{s}} - F_{\text{w}} \ \Rightarrow\ 0 = \mu_{\text{s}}N - F_{\text{w}} \ \Rightarrow\ \mu_{\text{s}} = \frac{F_{\text{w}}}{N}.$

CALCULATE:

(a) $F_{\text{w}} = \left(9.81\text{ m/s}^2\right)\tan(36.87°)\left(\frac{20.0\text{ kg}}{2} + \frac{3(60.0\text{ kg})}{4}\right) = 405\text{ N},\ \ N = \left(9.81\text{ m/s}^2\right)(20\text{ kg} + 60\text{ kg}) = 785\text{ N}$

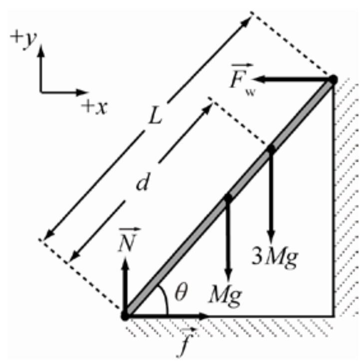
(b) $\mu_{\text{s}} = \frac{405\text{ N}}{785\text{ N}} = 0.516$

ROUND: Each value given has three significant figures, so $F_{\text{w}} = 405$ N, $N = 785$ N, and $\mu_{\text{s}} = 0.516.$

DOUBLE-CHECK: It is reasonable for $N > F_{\text{w}}$. The coefficient μ_{s} is reasonable as well.

11.59. **THINK:** The ladder's mass is M and its length is $L = 4.00$ m. The coefficient of static friction between the floor and the ladder is $\mu_{\text{s}} = 0.600$. The angle is $\theta = 50.0°$ between the ladder and the floor. The man's mass is $3M$. Determine the distance up the ladder, d, the man can climb before the ladder starts to slip.

SKETCH:



RESEARCH: Use $\tau_{\text{net}} = \sum\tau_{\text{ccw}} - \sum\tau_{\text{cw}} = 0$, $F_{\text{net},x} = \sum F_x = 0$, and $F_{\text{net},y} = \sum F_y = 0.$ Assume the ladder's center of mass is at its center, $L/2$. Since the force of the wall on the ladder, F_w, is not known, choose the point where the ladder touches the wall as a pivot point when evaluating τ_{net}.

SIMPLIFY: Determine N first: $F_{\text{net},y} = 0 \ \Rightarrow\ N - Mg - 3Mg = 0 \ \Rightarrow\ N = 4Mg.$

Consider $\tau_{\text{net}} = 0$:

$$Mg\frac{L}{2}\cos\theta + 3Mg(L-d)\cos\theta + fL\sin\theta - NL\cos\theta = 0.$$

$$Mg\frac{L}{2}\cos\theta + 3MgL\cos\theta - 3Mgd\cos\theta + 4\mu_{\text{s}}MgL\sin\theta - 4MgL\cos\theta = 0$$

$$\Rightarrow d = \frac{4\mu_{\text{s}}L\sin\theta - \frac{1}{2}L\cos\theta}{3\cos\theta} = \frac{4}{3}\mu_{\text{s}}L\tan\theta - \frac{1}{6}L.$$

CALCULATE: $d = \frac{4}{3}(0.600)(4.00\text{ m})\tan(50.0°) - \frac{1}{6}(4.00\text{ m}) = 3.15\text{ m}$

ROUND: Since all values have three significant figures, $d = 3.15$ m. The man can go 3.15 m up the ladder before it starts to slip.

DOUBLE-CHECK: It was determined that $d < L$, which it must be.

11.60. **THINK:** The short cylinder is brass, with $\rho_B = 8.60 \text{ g/cm}^3$, with dimensions $r_2 = 4.00$ cm and $d_2 = 4.00$ cm. The longer cylinder is aluminum, with $\rho_A = 2.70 \text{ g/cm}^3$ and dimensions $r_1 = 2.00$ cm and $d_1 = 20.0$ cm. Determine (a) its center of mass (com) and (b) if it is in equilibrium, and if it is a stable equilibrium.

SKETCH: Not necessary.

RESEARCH: Since these objects have uniform densities, their individual center of mass will be at their geometric centers, $(d/2, r)$. Then determine the composite center of mass from

$$x = \frac{1}{M}\sum x_i m_i \text{ and } y = \frac{1}{M}\sum y_i m_i .$$

Their individual masses can be found from $m = \rho V$ where $V = \pi r^2 d$ for a cylinder. The object will be in equilibrium if its *X* com is located within its support base, in this case the shorter brass cylinder.

SIMPLIFY: $m_B = \rho_B V_B = \rho_B(\pi r_2^2 d_2) = \pi \rho_B r_2^2 d_2$, $m_A = \pi \rho_A r_1^2 d_1$, $M = m_B + m_A$

Taking the bottom left corner as the origin of the coordinate system,

$$x = \frac{1}{M}(x_B m_B + x_A m_A) = \frac{1}{M}\left(\frac{1}{2}d_2 m_B + (d_2 + \frac{1}{2}d_1)m_A\right), \quad y = \frac{1}{M}(y_B m_B + y_A m_A) = \frac{1}{M}(r_2 m_B + r_2 m_A) = r_2$$

(m_A is centered at r_2 in the *y*-direction).

CALCULATE:

(a) $m_B = \pi\left(8.60 \text{ g/cm}^3\right)\left(4.00 \text{ cm}\right)^2\left(4.00 \text{ cm}\right) = 1729 \text{ g}$,

$m_A = \pi\left(2.70 \text{ g/cm}^3\right)\left(2.00 \text{ cm}\right)^2\left(20.0 \text{ cm}\right) = 678.6 \text{ g}$,

$M = 1729 \text{ g} + 678.6 \text{ g} = 2408 \text{ g}$,

$$x = \frac{1}{2408 \text{ g}}\left(\left(\frac{1}{2}4.00 \text{ cm}\left(1729 \text{ g}\right)\right) + \left(4.00 \text{ cm} + \frac{1}{2}20.0 \text{ cm}\right)\left(678.6 \text{ g}\right)\right) = 5.38 \text{ cm, and}$$

$$y = \frac{1}{2408 \text{ g}}\left(\left(4.00 \text{ cm}\right)\left(1729 \text{ g}\right) + \left(4.00 \text{ cm}\right)\left(678.6 \text{ g}\right)\right) = 3.9998 \text{ cm}.$$

(b) Since the *x* center of mass is to the right, outside of the brass cylinder, the object is not in equilibrium. It will tip over.

ROUND: To three significant figures, the center of mass is located at (5.38 cm, 4.00 cm).

DOUBLE-CHECK: Given how the object is assembled, we expect *y* com to be at r_2. The value for *x* center of mass should be closer to the more massive object's *x* center of mass, as it is.

11.61. **THINK:** An object's potential energy is $U(x) = a\left(x^4 - 2b^2x^2\right)$, where a and b are both positive. Determine the locations of any equilibrium points, and their classification (stable, unstable, indifferent).

SKETCH: Not necessary.

RESEARCH: Equilibrium points exist where $dU(x)/dx = 0$. Their classification can be determined from

$$\left.\frac{d^2U(x)}{d\left(x^2\right)}\right|_{x_0}$$

, where x_0 is the equilibrium point.

SIMPLIFY: First, find the derivative of *U*:

$$\frac{d}{dx}U(x) = \frac{d}{dx}\left(a\left(x^4 - 2b^2x^2\right)\right).$$

Setting the derivative of $U(x)$ to zero yields $4ax_0\left(x_0 - b\right)\left(x_0 + b\right) = 0$. Then, $x_0 = 0$ and $x_0 = \pm b$ are the three equilibrium points. Now,

$$\frac{d^2U(x)}{dx^2} = \frac{d}{dx}\left(4ax^3 - 4ab^2x\right) = 12ax^2 - 4ab^2.$$

When $x_0 = 0$,

$$\left.\frac{d^2U(x)}{d(x^2)}\right|_{x_0=0} = -4ab^2 < 0,$$ since a > 0. Therefore $x = 0$ is an unstable equilibrium point.

When $x = \pm b$,

$$\left.\frac{d^2U(x)}{d(x^2)}\right|_{x_0=\pm b} = 12ab^2 - 4ab^2 = 8ab^2 > 0,$$ since a > 0. Therefore $x = \pm b$ are stable equilibrium points.

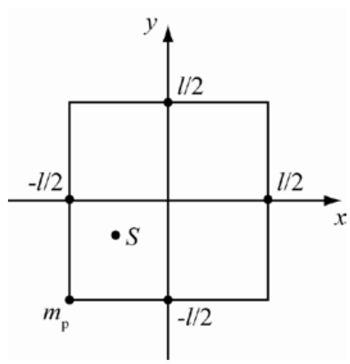
CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: The first derivative of $U(x)$ is a third order polynomial, and should have a maximum of three distinct roots, corresponding to at most three equilibrium points.

11.62. **THINK:** The two-dimensional square is centered at the origin of the *x-y* plane, and has mass, $m_s = 2.00$ kg, and sides of length, $l = 20.0$ cm. A point mass, $m_p = 2.00 \cdot 10^2$ g $= 0.200$ kg is placed at one corner of the square. Where must the support, *S*, be placed to keep the system in equilibrium? *S* has coordinates (x, y). The square has a uniform mass density.

SKETCH:



RESEARCH: The support must be placed at the system's center of mass. Since the square has a uniform mass density, its center of mass is at the origin. The point mass's center of mass is at $(x_p, y_p) = (-l/2, -l/2)$, since it is a point mass. Use the formulas for finding the center of mass in terms of the *x*- and *y*-coordinates:

$$x = \frac{1}{M}\sum x_i m_i \text{ and } y = \frac{1}{M}\sum y_i m_i.$$

SIMPLIFY: The combined mass is $M = m_s + m_p$. The *x*-coordinate of the point mass is:

$$x = \frac{1}{M}\left(x_s m_s + x_p m_p\right) = \frac{1}{M}x_p m_p = \frac{1}{M}\left(\frac{-l}{2}\right)m_p = -\frac{1}{M}\frac{l}{2}m_p,$$

and the *y*-coordinate is:

$$y = \frac{1}{M}\left(y_s m_s + y_p m_p\right) = \frac{1}{M}y_p m_p = \frac{1}{M}\left(\frac{-l}{2}\right)m_p = -\frac{1}{M}\frac{l}{2}m_p = x.$$

CALCULATE: $M = 2.00 \text{ kg} + 0.200 \text{ kg} = 2.200 \text{ kg}$,

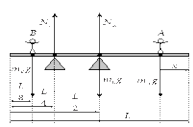
$$x = y = -\frac{1}{2.200 \text{ kg}}\left(\frac{20.0 \text{ cm}}{2}\right)(0.200 \text{ kg}) = -0.90909 \text{ cm}$$

ROUND: With three significant figures in each given value, the center of mass, and therefore the support's location, is $(-0.909 \text{ cm}, -0.909 \text{ cm})$.

DOUBLE-CHECK: The center of mass should be in the same quadrant as the point mass, but closer to the more massive object's center of mass. The fact that $x=y$ is consistent with the fact that the center of mass of two objects is on the line joining their center of masses. The ratio of the distances from m_p to S and S to the origin is 10:1, which is the inverse of the ratio of the masses, 0.200:2.00. This indicates that the correct result was obtained.

11.63. **THINK:** Person B has a mass twice that of person A, that is, $m_B = 2m_A$. The board's mass is $m_\text{b} = m_A/2$. Assume the board's weight acts at its center, $L/2$, where L is the length of the board. To determine the distance, x, from the edge of the board that person A can stand without tipping the board, balance the torques around a pivot point. The two natural choices for the pivot point are the two support points of the board. In this case, we note that if the board tips in a counterclockwise direction, the right support will not contribute to the torque. If the board tips in a clockwise direction, the left support will not contribute to the torque.

SKETCH:



RESEARCH: First, consider the case of the board tipping in the clockwise direction. The pivot point must be the right support and the left support will not contribute. The torque due to the weight of the board is zero because the moment arm is zero. The counterclockwise torque is $m_B g(3L/8) = (2m_A)g(3L/8) = (3/4)m_A gL.$ The maximum clockwise torque with $x=0$ is $m_A g(L/2) = (1/2)m_A gL.$ Thus, the board cannot tip in the clockwise direction.

We then consider the case of the board tipping in the counterclockwise direction. The pivot point is the left support. The counterclockwise torque due to person B is $\tau_B = m_B g(L/8)$. The clockwise torque due to person A is $\tau_A = m_A g((3L/4)-x),$ and the clockwise torque due to the board is $\tau_\text{b} = m_\text{b} g(L/4).$

SIMPLIFY: In static equilibrium, we have $\tau_\text{net} = 0$, so we can write

$$\sum_i \tau_{\text{counterclockwise},i} = \sum_j \tau_{\text{clockwise},j}$$

$$m_\text{B} g(L/8) = m_A g((3L/4)-x) + m_\text{b} g(L/4).$$

Now we express all terms as multiples of $m = m_A$

$$(2m)g(L/8) = mg((3L/4)-x) + (m/2)g(L/4).$$

This gives us

$$(2)(L/8) = ((3L/4)-x) + (1/2)(L/4)$$

$$L/4 = 3L/4 - x + L/8$$

$$x = (5/8)L.$$

CALCULATE: Not applicable.

ROUND: Not Applicable

DOUBLE-CHECK: Our result for x is somewhat surprising since this means that person A is standing to the left of the right support. However, it seems reasonable because person B is standing so close to the pivot point at the left support and the center of mass of the board is to the right of the left support. Thus person A must stand closer to the left support to tip the board in a counterclockwise direction.

11.64. **THINK:** The SUV has height, h, wheelbase, b, and its center of mass is marked on the diagram, αh above the ground and midway between the wheels. It enters a turn of radius $R >> b$ with speed, v, and the coefficient of static friction is μ_s. Determine the following.

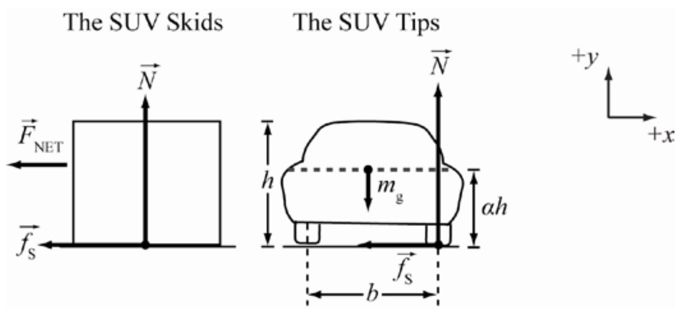
(a) the speed required to skid, v_{skid}, when $f_s = \mu N$

(b) the speed required to tip, v_{tip}

(c) the maximum value for α (in terms of b, h, and μ) for $v_{\text{skid}} < v_{\text{tip}}$

Use the convention that a counter-clockwise torque is positive. Choose the center of mass as the pivot point.

SKETCH:



RESEARCH: To determine v_{skid}, use $F_{\text{net},x} = \sum F_x$. To determine v_{tip}, balance the torque on the SUV so that $\tau_{\text{net}} = 0$. $f_s = \mu_s N$, and in this case, $N = mg$. $a_c = v^2 / R$ is the centripetal acceleration.

SIMPLIFY:

(a) $v = v_{\text{skid}}$; $F_{\text{net},x} = f_s \Rightarrow ma_c = \mu_s N = \mu_s mg \Rightarrow v^2/R = \mu_s g \Rightarrow v_{\text{skid}} = \sqrt{\mu_s gR}$

(b) The moment arm from N to mg is $b/2$. The moment arm from f_s to mg is αh. Balancing these torques yields $-f_s(\alpha h) + N(b/2) = 0$. From $\sum F_x = F_{\text{net}}$, it is known that $f_s = ma_c = mv_{\text{tip}}{}^2/R$. Then,

$$\left(\frac{mv^2{}_{\text{tip}}}{R}\right)(\alpha h) = \frac{1}{2}Nb = \frac{1}{2}(mg)b \Rightarrow v_{\text{tip}} = \sqrt{\frac{gbR}{2\alpha h}}.$$

(c) From $v_{\text{skid}} < v_{\text{tip}}$, $\sqrt{\mu_s gR} < \sqrt{\dfrac{gbR}{2\alpha h}} \Rightarrow \mu_s gR < \dfrac{gbR}{2\alpha h} \Rightarrow \alpha < \dfrac{b}{2\mu_s h}.$

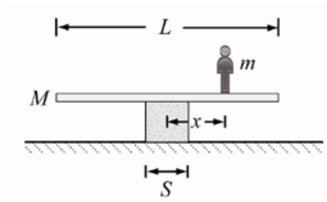
CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: v_{skid} should depend on μ_s and R, while v_{tip} should depend on the dimensions of the SUV and R.

11.65. **THINK:** The plank's length is L = 8.00 m and its mass is M = 100. kg. The cube's width is S = 2.00 m. The person's mass is m = 65.0 kg. Determine the distance from the center of the plank, x, the person can reach before tipping the plank.

SKETCH:



RESEARCH: The plank will tip when the plank and person system's center of mass is on the edge of the cube (1.00 m from the center). The person's location can be determined by using

$$X = \frac{1}{M_{\text{tot}}}\sum x_i m_i,$$

and taking the plank's center as the origin. Assume the plank's center of mass is at its center (at the origin). Then the position of the center of mass of the plank $x_M = 0.$

SIMPLIFY: $X = \frac{1}{M_{\text{tot}}}\left(Mx_M + mx_m\right) = \frac{1}{M_{\text{tot}}}mx_m \Rightarrow x_m = \frac{1}{m}M_{\text{tot}}X$

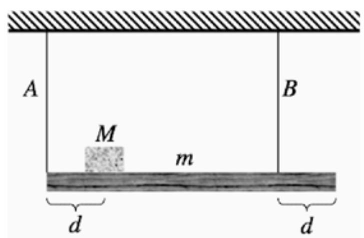
CALCULATE: $x_m = \left(\frac{1}{65.0 \text{ kg}}\right)\left(100. \text{ kg} + 65.0 \text{ kg}\right)\left(1.00 \text{ m}\right) = 2.54 \text{ m}$

ROUND: The answer should have three significant figures, so $x_m = 2.54$ m. The man can walk 2.54 m from the center before tipping the plank.

DOUBLE-CHECK: x_m must be less than half the length of the board in order for the answer to be valid.

11.66. **THINK:** The board's weight is *mg* = 120.0 N, its length is *L* = 5.00 m, the distance is *d* = 1.00 m, and the box's weight is *Mg* = 20.0 N. Determine the tension in each rope, T_{A} and T_{B}.

SKETCH:



RESEARCH: Two equations are required because there are two unknowns. Use the static equilibrium equations, $F_{y,\text{net}} = \sum F_y = 0$ and $\tau_{\text{net}} = \sum \tau_{\text{ccw}} - \sum \tau_{\text{cw}} = 0.$ Assume the board's weight acts at its center, *L*/2.

SIMPLIFY: From $F_{y,\text{net}} = 0,$ $T_{\text{A}} + T_{\text{B}} - Mg - mg = 0 \Rightarrow T_{\text{A}} = Mg + mg - T_{\text{B}}.$ Choose the left edge as a pivot point. Then from $\tau_{\text{net}} = 0,$

$$T_{\text{B}}\left(L-d\right) - Mgd - mg\frac{L}{2} = 0,\ \ T_{\text{B}} = \frac{Mgd + \frac{1}{2}mgL}{L-d}.$$

CALCULATE:

$$T_{\text{B}} = \frac{\left(20.0 \text{ N}\right)\left(1.00 \text{ m}\right) + \frac{1}{2}\left(120.0 \text{ N}\right)\left(5.00 \text{ m}\right)}{5.00 \text{ m} - 1.00 \text{ m}} = 80.0 \text{ N},$$

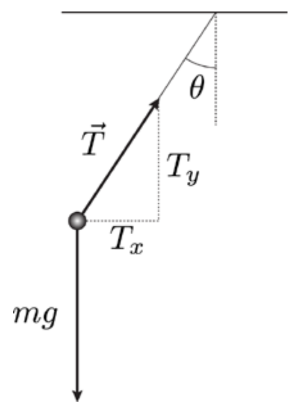
$T_{\text{A}} = 20.0 \text{ N} + 120.0 \text{ N} - 80.0 \text{ N} = 60.0 \text{ N}$

ROUND: Since *L*, *d* and *Mg* have three significant figures, the results remain $T_{\text{A}} = 60.0$ N and $T_{\text{B}} = 80.0$ N.

DOUBLE-CHECK: The total tension must equal the total weight of the system, and due to the configuration, it is not expected that $T_{\text{A}} = T_{\text{B}}$.

11.67. **THINK:** The only forces acting on the air freshener are the tension in the string and the force of gravity. The vertical component of the tension must balance the force of gravity while the horizontal component of the tension will cause the air freshener to accelerate in the positive x-direction.

SKETCH:



RESEARCH: The sum of the forces in the x-direction is $\sum F_x = T\sin\theta = ma.$ The sum of the forces in the y-direction is $\sum F_y = T\cos\theta - mg = 0 \Rightarrow T\cos\theta = mg.$

SIMPLIFY: Divide the two equations to get: $\dfrac{T\sin\theta}{T\cos\theta} = \dfrac{ma}{mg} \Rightarrow \dfrac{\sin\theta}{\cos\theta} = \dfrac{a}{g} = \tan\theta,$ which implies $\theta = \tan^{-1}\left(\dfrac{a}{g}\right).$

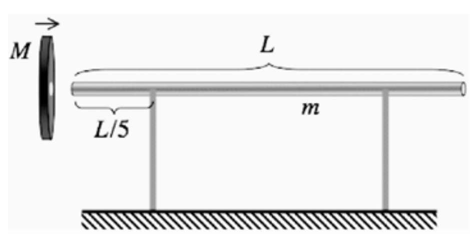
CALCULATE: $\theta = \tan^{-1}\left(\dfrac{5.00 \text{ m/s}^2}{9.81 \text{ m/s}^2}\right) = 27.007°$

ROUND: Since a has three significant figures, $\theta = 27.0°.$

DOUBLE-CHECK: θ is expected to be $0 < \theta < 90°.$ The size of the angle, $\theta = 27.0°$ is consistent with an angle someone might realistically observe in their own car.

11.68. **THINK:** The barbell has length, L = 2.20 m, and is supported at L /5 from each end. The weight has a mass of M = 22.0 kg. Determine the barbell's mass, m. The barbell is a uniform rod, so its center of mass is located at $x_{\text{m}} = L/2.$

SKETCH:



RESEARCH: The barbell will not tip if the weight-barbell system's center of mass is not less than $X = L/5$ from the end that the weight is being attached to. The mass, m, can be determined from

$$X = \frac{1}{M_{\text{tot}}}\sum x_i m_i.$$

Take the end at which the weight is attached as the origin of the coordinate system.

SIMPLIFY: $X = \dfrac{1}{M_{\text{tot}}}\left(Mx_M + mx_{\text{m}}\right)$

$x_M = 0,$ so, $m = \dfrac{1}{x_{\text{m}}}M_{\text{tot}}X = \dfrac{2}{L}\left(M+m\right)X = \dfrac{2}{L}MX + \dfrac{2}{L}mX.$ Then,

$$m=\frac{\frac{2}{L}MX}{\left(1-\frac{2}{L}X\right)}=\frac{\frac{2}{L}M\left(\frac{L}{5}\right)}{1-\frac{2}{L}\left(\frac{L}{5}\right)}=\frac{\frac{2}{5}M}{1-\frac{2}{5}}=\frac{2}{3}M.$$

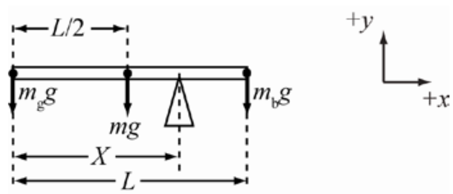
CALCULATE: $m=\frac{2}{3}(22.0\text{ kg})=14.67\text{ kg}$

ROUND: Since M has three significant figures, m = 14.7 kg.

DOUBLE-CHECK: Given the barbell's length and support locations, $m < M$.

11.69. **THINK:** The board's length is L = 5.00 m and its mass is M = 50.0 kg. A girl of mass, $\text{m}_\text{g}=45.0\text{ kg}$, sits at the left end and a boy of mass, $\text{m}_\text{b}=60.0\text{ kg}$, sits at the right end. For static equilibrium, determine the pivot's position, X.

SKETCH:



RESEARCH: For static equilibrium, the pivot must be placed at the boy-girl-board system's center of mass. Take the girl's position to be the origin of the coordinate system and assume the board's weight acts at its center, $L/2$. X can be determined from

$$X=\frac{1}{M_{\text{tot}}}\sum x_i m_i.$$

SIMPLIFY: $M_{\text{tot}}=m+m_\text{b}+m_\text{g}$, $X=\frac{1}{M_{\text{tot}}}\left(x_\text{M}m+x_\text{b}m_\text{b}+x_\text{g}m_\text{g}\right)\Rightarrow X=\frac{1}{M_{\text{tot}}}\left(\frac{1}{2}mL+m_\text{b}L\right)\ \left(x_\text{g}=0\right)$

CALCULATE: $M_{\text{tot}}=50.0\text{ kg}+60.0\text{ kg}+45.0\text{ kg}=155.0\text{ kg}$

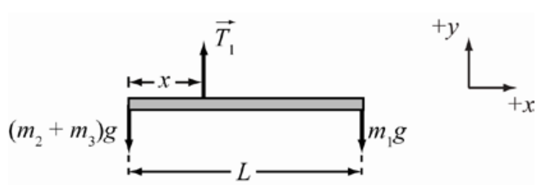
$$X=\frac{1}{155.0\text{ kg}}\left(\frac{1}{2}(50.0\text{ kg})(5.00\text{ m})+(60.0\text{ kg})(5.00\text{ m})\right)=\frac{1}{155.0\text{ kg}}(125\text{ kg}\cdot\text{m}+300\text{ kg}\cdot\text{m})=2.7419\text{ m}$$

ROUND: To three significant figures, X = 2.74 m.

DOUBLE-CHECK: It must be that $X < L$, and it is expected that X is closer to the boy's end than the girl's end since he has more mass.

11.70. **THINK:** The masses of objects 1 and 3 are $m_1=6.40\text{ kg}$ and $m_3=3.20\text{ kg}$. The bar's length is L = 0.400 m and the distance is x = 0.160 m. Determine object 2's mass, m_2.

SKETCH:



RESEARCH: The total mass of m_2 and m_3 must first be determined. Use $\tau_{\text{net}}=\sum\tau_{\text{ccw}}-\sum\tau_{\text{cw}}=0$ to determine m_2+m_3, as the system is in static equilibrium.

SIMPLIFY: Since T_1 is unknown, choose its location as a pivot point. Then,

$$\tau_{\text{net}}=(m_2+m_3)gx-m_1g(L-x)=0\Rightarrow\ m_2+m_3=\frac{m_1(L-x)}{x}\text{, and }m_2=\frac{m_1(L-x)}{x}-m_3.$$

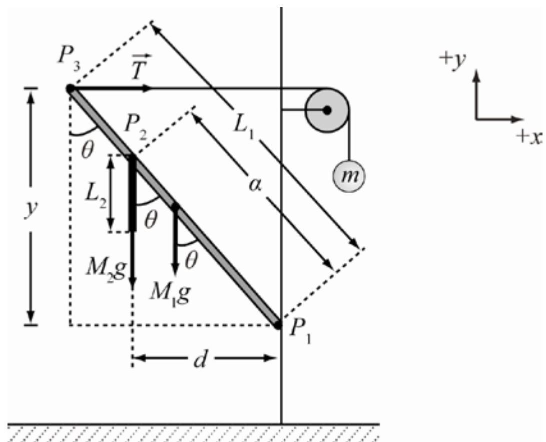
CALCULATE: $m_2 + m_3 = \dfrac{(6.40\text{ kg})(0.400\text{ m} - 0.160\text{ m})}{0.160\text{ m}} = 9.60\text{ kg}, \quad m_2 = 9.60\text{ kg} - 3.20\text{ kg} = 6.40\text{ kg}$

ROUND: Since m_2, L and x have three significant figures, the result remains $m_2 = 6.40$ kg.

DOUBLE-CHECK: Since the pivot point, T_1, is closer to m_2, it is expected that $m_2 > m_3$, as it is.

11.71. **THINK:** The given values are $L_1 = 1.00$ m, $M_2 = 0.200$ kg, $L_2 = 0.200$ m, d = 0.550 m, m = 0.500 kg and y = 0.707 m. Determine M_1.

SKETCH:



RESEARCH: The beam, B_1, is in static equilibrium, so use $\tau_{\text{net}} = \sum \tau_{\text{ccw}} - \sum \tau_{\text{cw}} = 0$ to determine M_1. Assume M_1 acts at the center of B_1 (a distance $L_1 / 2$ along the beam).

SIMPLIFY: The force of the support on B_1 at point, P_1, is not known, so choose this as a pivot point. Note $\theta = \cos^{-1}\left(y / L_1\right)$ and $T = mg$. Now,

$$\tau_{\text{net}} = M_1 g \frac{L_1}{2} \sin\theta + M_2 g d - T L_1 \cos\theta = 0 \;\Rightarrow\; 0 = \frac{1}{2} M_1 g L_1 \sin\theta + M_2 g d - m g y \;\Rightarrow\; M_1 = \frac{2\left(my - M_2 d\right)}{L_1 \sin\theta}.$$

CALCULATE: $\theta = \cos^{-1}\left(\dfrac{0.707\text{ m}}{1.00\text{ m}}\right) = 45.0^\circ$

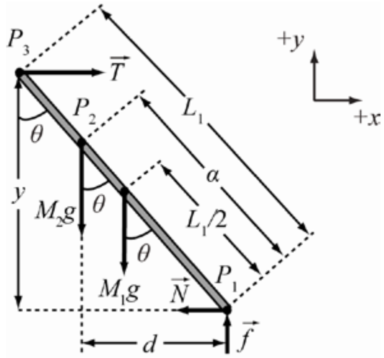
$$M_1 = \frac{2\big((0.500\text{ kg})(0.707\text{ m}) - (0.200\text{ kg})(0.550\text{ m})\big)}{(1.00\text{ m})\sin(45.0^\circ)} = 2\left(\frac{0.3535\text{ kg m} - 0.110\text{ kg m}}{0.707\text{ m}}\right) = 0.6887\text{ kg}$$

ROUND: To three significant figures, $M_1 = 0.689$ kg.

DOUBLE-CHECK: Compared to the other masses given in the problem, this result is reasonable.

11.72. **THINK:** The length and mass of beam, B_1, are $L_1 = 1.00$ m and $M_1 = 0.6887$ kg (from the result of the previous problem). In addition, $L_2 = 0.200$ m, $M_2 = 0.200$ kg, d = 0.550 m, m = 0.500 kg and y = 0.707 m. Determine the net torque, τ_{net}, at P_1, P_2 and P_3.

SKETCH: Consider B_1 only.



RESEARCH: To determine τ_{net}, use $\tau_{\text{net}} = \sum \tau_{\text{ccw}} - \sum \tau_{\text{cw}}$. Note $\theta = \cos^{-1}(y/L_1)$ and since B_1 is assumed to have no acceleration in any direction, it must be that $\sum F_x = 0$ and $\sum F_y = 0.$

SIMPLIFY: From $\sum F_x = 0,$ the normal force, N, at P_1 from the stand (not drawn) is $N = T$, where $T = mg$. From $\sum F_y = 0,$ the frictional force, f, at P_1 from the stand is $f = g(M_1 + M_2)$. Also, note in the sketch that $\alpha = d/\sin\theta > L_1/2.$ The balance of forces about the pivot points are as follows.

For pivot point P_1: $\tau_{\text{net}} = M_2 g\alpha \sin\theta + M_1 g \frac{L_1}{2} \sin\theta - TL_1 \sin(90° - \theta)$

$$= M_2 g\alpha\sin\theta + \frac{1}{2}M_1 gL_1 \sin\theta - mgL_1\cos\theta = g\left(M_2 d + \frac{1}{2}M_1 L_1 \sin\theta - my\right).$$

For pivot point P_2:

$$\tau_{\text{net}} = f\alpha\sin\theta - T(L_1 - \alpha)\sin(90° - \theta) - M_1 g\left(\alpha - \frac{L_1}{2}\right)\sin(180° - \theta) - N\alpha\sin(90° - \theta)$$

$$= g(M_1 + M_2)\alpha\sin\theta - T(L_1 - \alpha)\cos\theta - M_1 g\left(\alpha - \frac{L_1}{2}\right)\sin\theta - T\alpha\cos\theta$$

$$= M_1 g\alpha\sin\theta + M_2 g\alpha\sin\theta - TL_1\cos\theta + T\alpha\cos\theta - M_1 g\alpha\sin\theta + \frac{1}{2}M_1 gL_1\sin\theta - T\alpha\cos\theta$$

$$= g\left(M_2 d - mL_1\cos\theta + \frac{1}{2}M_1 L_1 \sin\theta\right).$$

For pivot point P_3:

$$\tau_{\text{net}} = fL_1\sin\theta - M_2 g(L_1 - \alpha)\sin(180° - \theta) - M_1 g\frac{L_1}{2}\sin(180° - \theta) - NL_1\sin(90° - \theta)$$

$$= g(M_1 + M_2)L_1\sin\theta - M_2 g(L_1 - \alpha)\sin\theta - \frac{1}{2}M_1 gL_1\sin\theta - TL_1\cos\theta$$

$$= M_1 gL_1\sin\theta + M_2 gL_1\sin\theta - M_2 gL_1\sin\theta + M_2 g\alpha\sin\theta - \frac{1}{2}M_1 gL_1\sin\theta - mgL_1\cos\theta$$

$$= g\left(\frac{1}{2}M_1 L_1\sin\theta + M_2 d - mL_1\cos\theta\right).$$

CALCULATE: $\theta = \cos^{-1}\left(\frac{0.707 \text{ m}}{1.00 \text{ m}}\right) = 45.0°$

For P_1: $\tau_{\text{net}} = (9.81 \text{ m/s}^2)\left((0.200 \text{ kg})(0.550 \text{ m}) + \frac{1}{2}(0.6887 \text{ kg})(1.00 \text{ m})\sin(45°) - (0.500 \text{ kg})(0.707 \text{ m})\right)$

$$= (9.81 \text{ m/s}^2)(0.110 \text{ kg m} + 0.2435 \text{ kg m} - 0.3535 \text{ kg m}) = 0.$$

For P_2:

$$\tau_{\text{net}} = \left(9.81 \text{ m/s}^2\right)\left(\left(0.200 \text{ kg}\right)\left(0.550 \text{ m}\right) - \left(0.500 \text{ kg}\right)\left(1.00 \text{ m}\right)\cos\left(45°\right) + \frac{1}{2}\left(0.6887 \text{ kg}\right)\left(1.00 \text{ m}\right)\sin\left(45°\right)\right)$$

$$= \left(9.81 \text{ m/s}^2\right)\left(0.110 \text{ kg m} - 0.3535 \text{ kg m} + 0.2435 \text{ kg m}\right) = 0.$$

For P_3:

$$\tau_{\text{net}} = \left(9.81 \text{ m/s}^2\right)\left(\frac{1}{2}\left(0.6887 \text{ kg}\right)\left(1.00 \text{ m}\right)\sin 45° + \left(0.200 \text{ kg}\right)\left(0.550 \text{ m}\right) - \left(0.500 \text{ kg}\right)\left(1.00 \text{ m}\right)\cos 45°\right)$$

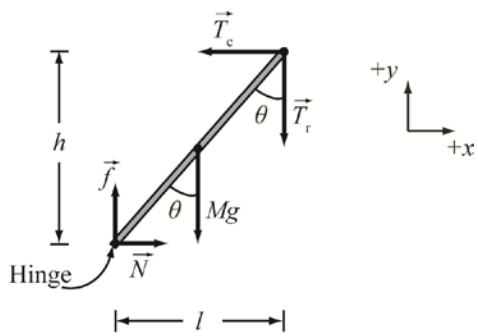
$$= \left(9.81 \text{ m/s}^2\right)\left(0.2435 \text{ kg m} + 0.110 \text{ kg m} - 0.3535 \text{ kg m}\right) = 0.$$

ROUND: Not applicable.

DOUBLE-CHECK: In static equilibrium, the torque about any pivot point must be zero.

11.73. **THINK:** The beam's mass is M = 50.0 kg, the hanging mass is m = 20.0 kg, the cable length is l = 3.00 m and its attachment height above the hinge is h = 4.00 m. Determine (a) the tension in the cable and the rope, T_c and T_r, respectively, and (b) the forces that the hinge exerts on the beam, *f* and *N*.

SKETCH:



RESEARCH: The beam is in static equilibrium, so use $\sum F_x = 0$, $\sum F_y = 0$ and $\tau_{\text{net}} = \sum \tau_{\text{ccw}} - \sum \tau_{\text{cw}} = 0.$

SIMPLIFY: Note $\theta = \tan^{-1}\left(l/h\right)$ and $L = \sqrt{l^2 + h^2}$ is the beam length.

(a) From $\sum F_y = 0$ on the hanging mass, $T_r = mg$. To determine T_c, use the hinge as a pivot point. Also, assume the beam's weight, Mg, acts at its center, $L/2$. Then,

$$\tau_{\text{net}} = T_c L \sin\left(90° - \theta\right) - Mg\left(\frac{L}{2}\right)\sin\theta - T_r L \sin\theta = 0$$

$$\Rightarrow\ 0 = T_c L\cos\theta - \frac{1}{2}MgL\sin\theta - T_r L\sin\theta \ \Rightarrow\ T_c = \frac{1}{2}Mg\tan\theta + T_r\tan\theta = \frac{l}{h}\left(\frac{1}{2}Mg + T_r\right).$$

(b) From $\sum F_x = 0$ on the beam, it can be seen that $N = T_c$. From $\sum F_y = 0$ on the beam, it can be seen that $f = Mg + T_r$.

CALCULATE:

(a) $T_r = \left(20.0 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right) = 196.2 \text{ N},\ T_c = \left(\dfrac{3.0 \text{ m}}{4.0 \text{ m}}\right)\left(\dfrac{1}{2}\left(50.0 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right) + 196.2 \text{ N}\right) = 331.1 \text{ N}$

(b) $f = \left(50.0 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right) + 196.2 \text{ N} = 686.7 \text{ N}$

ROUND: The least precise values given in the question have three significant figures. The results should be rounded to:

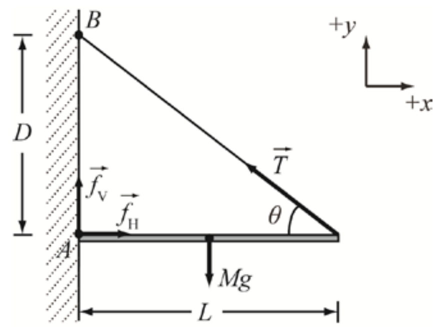
(a) $T_r = 196 \text{ N},\ T_c = 331 \text{ N}$

(b) $F_x = N = T_c = 331 \text{ N},\ F_y = f = 687 \text{ N}$

DOUBLE-CHECK: All of the results have units of force, as they should. Since the steel cable is supporting both the beam and the hanging mass, it is expected to have a greater tension than the rope, which only supports the hanging mass. This is consistent with the calculated results.

11.74. **THINK:** The bar's mass and length are M = 100. kg and L = 5.00 m. The cable is attached to the wall a distance D = 2.00 m above the bar. Determine (a) the tension, T, on the cable, and (b) the horizontal and vertical components of the force, f_{H} and f_{V}, on the bar at point A.

SKETCH:



RESEARCH: Use $\tau_{\text{net}} = \sum \tau_{\text{ccw}} - \sum \tau_{\text{cw}} = 0$, since the bar is in static equilibrium. Also, $\sum F_x = 0$ and $\sum F_y = 0$. Assume the bar's weight acts at its center, L/2.

SIMPLIFY: Note that $\theta = \tan^{-1}(D/L)$.

(a) Choose point A as a pivot point: $\tau_{\text{net}} = TL\sin\theta - Mg\left(\frac{L}{2}\right)\sin(90°) = 0 \Rightarrow T = \frac{Mg}{2\sin\theta}$.

(b) From $\sum F_x = 0$, $f_{\text{H}} = T\cos\theta$ and from $\sum F_y = 0$, $f_{\text{V}} = Mg - T\sin\theta$.

CALCULATE: $\theta = \tan^{-1}\left(\frac{2.00 \text{ m}}{5.00 \text{ m}}\right) = 21.8°$

(a) $T = \frac{(100. \text{ kg})(9.81 \text{ m/s}^2)}{2\sin(21.8°)} = 1321 \text{ N}$

(b) $f_{\text{H}} = (1321 \text{ N})\cos(21.8°) = 1227 \text{ N}$,

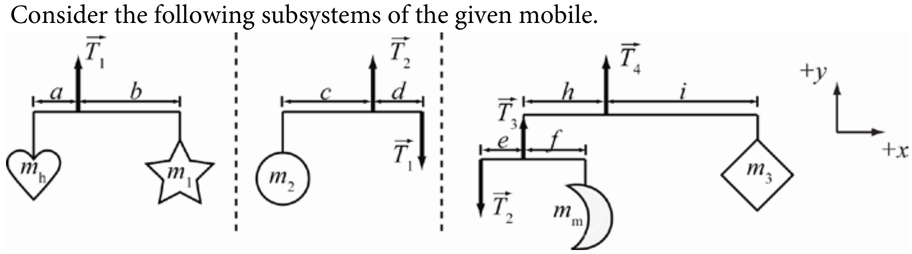
$f_{\text{V}} = (100 \text{ kg})(9.81 \text{ m/s}^2) - (1321 \text{ N})\sin(21.8°) = 490 \text{ N}$

ROUND: Since each given value has three significant figures, the results should be rounded accordingly.
(a) T = 1320 N
(b) $f_{\text{H}} = 1230$ N and $f_{\text{V}} = 490.$ N.

DOUBLE-CHECK: Since T point up and towards the wall, it is expected that $f_{\text{H}} > f_{\text{V}}$.

11.75. **THINK:** Given the mobile sketched in the problem, determine m_1, m_2 and m_3 using the equations of static equilibrium.

SKETCH: Consider the following subsystems of the given mobile.



$a = 1.50"$, $b = 3.00"$, $c = 3.00"$, $d = 1.00"$, $e = 1.50"$, $f = 7.50"$, $h = 6.00"$, $i = 9.00"$, $m_{\text{h}} = 0.0600$ kg and $m_{\text{m}} = 0.0240$ kg.

RESEARCH: The mobile is in static equilibrium, so use $\sum F_x = 0,\ \sum F_y = 0,\ \tau_{\text{net}} = \sum \tau_{\text{ccw}} - \sum \tau_{\text{cw}} = 0.$ Note all angles are $90°$, so $Fd\sin\theta$ becomes Fd.

SIMPLIFY: m_1 : choose T_1 as a pivot point on the *a*-*b* bar. Then,

$$\tau_{\text{net}} = -bm_1 g + am_{\text{h}} g = 0 \ \Rightarrow\ m_1 = \frac{a}{b} m_{\text{h}}.$$

m_2 : From $\sum F_y = 0$ on the *a*-*b* bar, it is seen that $T_1 = g\left(m_1 + m_h\right)$. Choose T_2 as a pivot point on the *c*-*d* bar. Then, $\tau_{\text{net}} = -T_1 d + m_2 gc = 0 \ \Rightarrow\ m_2 = \left(\dfrac{T_1 d}{gc}\right)$.

m_3 : From $\sum F_y = 0$ on the *c*-*d* bar, it can be seen that $T_2 = T_1 + m_2 g$. From $\sum F_y = 0$ on the *e*-*f* bar, it can be seen that $T_3 = T_2 + m_{\text{m}} g = T_1 + g\left(m_{\text{m}} + m_2\right)$. Choose T_4 as pivot point on the *h*-*i* bar. Then, $\tau_{\text{net}} = -m_3 gi + T_3 h = 0 \ \Rightarrow\ m_3 = \dfrac{T_3 h}{gi}$.

CALCULATE: $m_1 = \dfrac{1.50\text{ "}}{3.00\text{ "}}\left(0.0600\text{ kg}\right) = 0.0300\text{ kg},\ T_1 = \left(9.81\text{ m/s}^2\right)\left(0.0300\text{ kg} + 0.0600\text{ kg}\right) = 0.8829\text{ N}$

$m_2 = \dfrac{\left(0.8829\text{ N}\right)\left(1.00\text{ "}\right)}{\left(9.81\text{ m/s}^2\right)\left(3.00\text{ "}\right)} = 0.0300\text{ kg},\ T_3 = 0.8829\text{ N} + \left(9.81\text{ m/s}^2\right)\left(0.0240\text{ kg} + 0.0300\text{ kg}\right) = 1.41264\text{ N}$

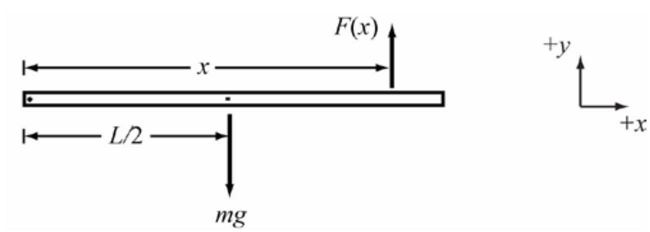
$m_3 = \dfrac{\left(1.41264\text{ N}\right)\left(6.00\text{ "}\right)}{\left(9.81\text{ m/s}^2\right)\left(9.00\text{ "}\right)} = 0.0960\text{ kg}$

ROUND: Three significant figures: $m_1 = 0.0300\text{ kg},\ m_2 = 0.0300\text{ kg}$ and $m_3 = 0.0960\text{ kg}.$

DOUBLE-CHECK: Given the mobile arrangement, it is expected that $m_1 < m_{\text{h}}$ and $m_3 > m_1, m_2$.

11.76. **THINK:** The rod of length, *L*, has a mass of *m* = 2.00 kg. The variable force is given by $F\left(x\right) = \left(15.0\text{ N}\right)\left(x/L\right)^4$ from the left end (where *x* = 0, at the pivot point). Determine where the point *x* where the force should be applied for static equilibrium. Use the convention that counter-clockwise torques are positive.

SKETCH:



RESEARCH: For static equilibrium, $\tau_{\text{net}} = \sum_j \tau_j = 0.$ Assume the rod's weight acts at its center, L/2.

SIMPLIFY: $\tau_{\text{net}} = -mg\left(\dfrac{L}{2}\right) + F\left(x\right)x = 0 \quad \Rightarrow\ -\dfrac{1}{2}mgL + \left(15.0\text{ N}\right)\left(\dfrac{x^4}{L^4}\right)x \Rightarrow x = L\left(\dfrac{mg}{2\left(15.0\text{ N}\right)}\right)^{1/5}$

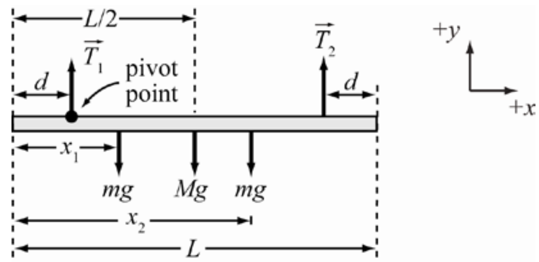
CALCULATE: $x = L\left(\dfrac{\left(2.00\text{ kg}\right)\left(9.81\text{ m/s}^2\right)}{2\left(15.0\text{ N}\right)}\right)^{1/5} = 0.9186L$

ROUND: Since each given value has three significant figures, so round the final answer to *x* = 0.919*L*.

DOUBLE-CHECK: It is expected that $L/2 < x < L.$ The force supplied at *x* = 0.919*L* is $F(0.9186L) = \left(15.0\text{ N}\right)\left(\dfrac{0.9186L}{L}\right)^4 = 10.7\text{ N}.$ Since the weight is about 20 N, this is a reasonable force.

11.77. **THINK:** The rod is L = 2.20 m long and has a mass, M = 8.13 kg. The two chains are d = 0.20 m from each end. The lights of mass, m = 7.89 kg, are $x_1 = 0.65$ m and $x_2 = 1.14$ m from the left end of the rod. Find the tension, T_1 and T_2, in each chain. Assume the rod's weight acts at its center, $L/2$, and that it is in static equilibrium. Use the convention that a counter-clockwise torque is positive. Choose the point where the left chain is attached as the pivot point.

SKETCH:



RESEARCH: For static equilibrium, $\tau_{\text{net}} = \sum_j \tau_j = 0$ and $\sum F_y = 0$.

SIMPLIFY: Choose T_1 as a pivot point and solve for T_2:

$$\tau_{\text{net}} = -mg(x_1 - d) - Mg\left(\frac{L}{2} - d\right) - mg(x_2 - d) + T_2(L - 2d) = 0 \Rightarrow T_2 = \frac{mg(x_1 + x_2 - 2d) + Mg\left(\frac{L}{2} - d\right)}{L - 2d}.$$

From $\sum F_y = 0$, $T_1 + T_2 - Mg - 2mg = 0 \Rightarrow T_1 = g(2m + M) - T_2$.

CALCULATE:

$$T_2 = \frac{(7.89 \text{ kg})(9.81 \text{ m/s}^2)(0.65 \text{ m} + 1.14 \text{ m} - 2(0.20 \text{ m})) + (8.13 \text{ kg})(9.81 \text{ m/s}^2)\left(\frac{2.20 \text{ m}}{2} - 0.20 \text{ m}\right)}{2.20 \text{ m} - 2(0.20 \text{ m})}$$

$$= 99.648 \text{ N}$$

$$T_1 = (9.81 \text{ m/s}^2)(2(7.89 \text{ kg}) + 8.13 \text{ kg}) - 99.648 \text{ N} = 134.91 \text{ N}$$

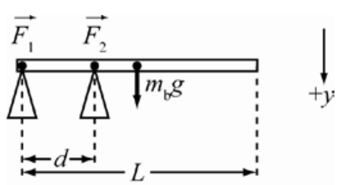
ROUND: In the calculation, each factor in the quotient has three significant figures. Therefore, the answers should be rounded to $T_2 = 99.6$ N, and $T_1 = 135$ N.

DOUBLE-CHECK: The tensions both have units of newtons, which is appropriate. From the pipe's configuration, T_1 and T_2 should be different, which is the case.

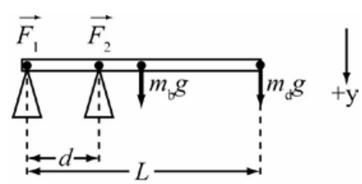
11.78. **THINK:** The diving board's length and mass are L = 2.00 m and m_b = 12.0 kg and it is h = 3.00 m above the water. Two attachments, one at the back edge and the other at d = 0.250 m from the edge, hold the board in place. Assume the board has uniform density. Determine (a) the forces on each attachment (take downward as positive), and (b) the forces on each attachment if a diver with mass, m_d = 65.0 kg, stands on the front end. Use the convention that a counter-clockwise torque is positive. Let F_1 be the force at the first attachment, and let F_2 be the force at the second attachment. Start with the assumption that F_1 and F_2 are both upward. If the signs of their forces are found to be negative, then they are downward forces.

SKETCH:

(a)



(b)



RESEARCH: Assume static equilibrium. $\tau_{\text{net}} = \sum_j \tau_j = 0$ and $\sum F_y = 0.$ Note $\theta = 90°,$ so $Fd\sin\theta = Fd.$
Since the board has uniform density, the board's weight acts at its center, $L/2$ from the end.

SIMPLIFY:

a) No diver: Choose F_2 as a pivot point. The net torque is $\tau_{\text{net}} = -F_1 d - m_{\text{b}} g\left(\frac{L}{2} - d\right) = 0$

$\Rightarrow F_1 = -m_{\text{b}} g\left(\frac{L}{2d} - 1\right)$**.** This value will be negative when $L > 2d$**.** Next, choose F_1 as a pivot point. The net torque is given by $\tau_{\text{net}} = F_2 d - m_{\text{b}} g\left(\frac{L}{2}\right) = 0 \Rightarrow F_2 = \frac{m_{\text{b}} gL}{2d}$. Note that $F_2 > 0,$ implying the force is upward.

b) With the diver: Choose F_2' as a pivot. The net torque is given by

$$\tau_{\text{net}} = -F_1' d - m_{\text{b}} g\left(\frac{L}{2} - d\right) - m_{\text{d}} g\left(L - d\right) = 0 \Rightarrow F_1' = -g\left(m_{\text{b}}\left(\frac{L}{2d} - 1\right) + m_{\text{d}}\left(\frac{L}{d} - 1\right)\right).$$

As before, F_1' acts downward. Next, choose F_1 as a pivot point. The net torque is given by

$$\tau_{\text{net}} = F_2' d - m_{\text{b}} g\left(\frac{L}{2}\right) - m_{\text{d}} gL = 0 \Rightarrow F_2' = \frac{gL}{d}\left(\frac{1}{2}m_{\text{b}} + m_{\text{d}}\right).$$

As before, F_2' acts upward.

CALCULATE:

(a) $F_1 = -\left(12.0 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right)\left(\frac{2.00 \text{ m}}{2\left(0.250 \text{ m}\right)} - 1\right) = -353.2 \text{ N}$

$F_2 = \left(12.0 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right)\left(\frac{2.00 \text{ m}}{2\left(0.250 \text{ m}\right)}\right) = 470.9 \text{ N}$

(b) $F_1' = -9.81 \text{ m/s}^2\left(\left(12.0 \text{ kg}\right)\left(\frac{2.00 \text{ m}}{2\left(0.250 \text{ m}\right)} - 1\right) + 65.0 \text{ kg}\left(\frac{2.00 \text{ m}}{0.250 \text{ m}} - 1\right)\right) = -4817 \text{ N}$

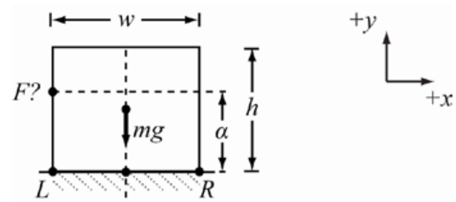
$F_2' = \frac{\left(9.81 \text{ m/s}^2\right)\left(2.00 \text{ m}\right)}{0.250 \text{ m}}\left(\frac{1}{2}\left(12.0 \text{ kg}\right) + 65.0 \text{ kg}\right) = 5572 \text{ N}$

ROUND: Each given value has three significant figures, so the results should be rounded to
(a) $F_1 = -353 \text{ N},$ and $F_2 = 471 \text{ N}$
(b) $F_1' = -4820 \text{ N},$ and $F_2' = 5570 \text{ N}$

DOUBLE-CHECK: For static equilibrium, F_1 must be downward on the board while F_2 is upward on the board. With the diver, $\left|F_1\right|$ and $\left|F_2\right|$ must increase significantly. This is what the computed values confirm.

11.79. **THINK:** Depending on whether the box is pushed or pulled, it will pivot about point *R* or *L*, respectively. This will not affect the solution; so choose *L* as the pivot point. If we pivot about *L*, then the force we apply at the handle should be perpendicular to its moment arm in order to generate maximum torque for a given force, i.e. point in horizontal direction. If we want to find the minimum force needed for tipping, then it would be a good starting point to try this first. However, we need to be able to generate a force of static friction at least as big as this horizontal force; otherwise we cannot prevent slipping. As we will see below, the numbers indeed work out in such a way that the force needed is bigger than the friction force that the weight of the box alone can provide. In order to increase the friction force, we need to increase the normal force between the box and the ground beyond the weight of the box. We can accomplish this by applying some downward force component at the handle. (BTW, this downward force component does not contribute to the torque, because it is parallel to the moment arm.) Let's call the horizontal force component F_x and the vertical component F_y.

SKETCH:



RESEARCH: The equations for equilibrium are for the

- x-component of the forces: $-F_x + f = 0$, where f is the friction force between box and ground
- y-component of the forces: $F_y + N - mg = 0$
- torques: $\tau_{\text{net}} = F_x\alpha - mg\frac{1}{2}w = 0$

Here we assume that the force is directed to the right and upward. We do not know if the force has a positive or negative y-component. We assumed a positive value, but if it turns out to be negative, then our assumption was incorrect, and the y-component is negative.

For the case of the maximum friction force without slipping, we have $f = \mu_s N$.

SIMPLIFY: From the equation for zero net torque we obtain

$$\tau_{\text{net}} = F_x\alpha - mg\tfrac{1}{2}w = 0 \;\Rightarrow\; F_x = \frac{mgw}{2\alpha}.$$

From $F_y + N - mg = 0$ we can solve for the y-component of the force:

$$F_y = mg - N = mg - f/\mu = mg - F_x/\mu.$$

Then the magnitude of the force is $F = \sqrt{F_x^2 + F_y^2}$.

With F_x and F_y known, the direction of F is given by $\theta = \tan^{-1}\left(F_y / F_x\right)$.

CALCULATE:

(a) $F_x = \dfrac{(20.0\text{ kg})(9.81\text{ m/s}^2)(0.300\text{ m})}{2(0.500\text{ m})} = 58.86\text{ N},$

and $F_y = (20.0\text{ kg})(9.81\text{ m/s}^2) - \dfrac{58.86\text{ N}}{0.280} = -14.01\text{ N}.$

Thus the y-component of the force is in the negative y-direction.

Then, $F = \sqrt{(58.86\text{ N})^2 + (14.01\text{ N})^2} = 60.50\text{ N}.$

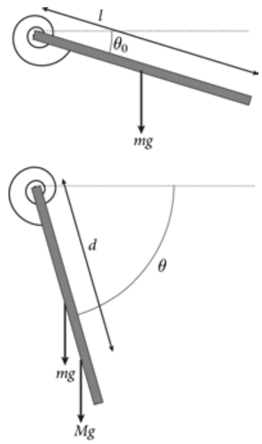
(b) $\theta = \tan^{-1}\left(\dfrac{-14.01\text{ N}}{58.86\text{ N}}\right) = -13.39°$, so below the horizontal.

ROUND: The least precise value given in the question has two significant figures. The answers should be rounded so they also have two significant figures. Therefore, the minimum force is 60.5 N and is directed at an angle of $13.4°$ below the horizontal.

DOUBLE-CHECK: Since the y-component of the force turned out to have a negative value, this indeed implies that we had to apply some downward force to prevent the box from slipping. Just to make sure that our solution is consistent, we can calculate the product of the box's weight and the coefficient of friction and make sure that this product is really smaller than our result for the horizontal component of the force, $\mu_s mg = (0.280)(20.0\text{ kg})(9.81\text{ m/s}^2) = 54.94\text{ N}$. This is indeed smaller than our result for F_x, which shows that some downward force component was indeed needed.

11.80. **THINK:** The torque exerted by a torsional spring is proportional to the angle over which it is displaced. The initial angular displacement gives the spring constant. The additional torque added by the hanging mass will create further angular displacement.The arm's mass is $m = 0.0450$ kg and is $l = 0.120$ m long. The equilibrium angle is $\theta_0 = 17.0°$. A mass, M = 0.420 kg, hung d = 0.0900 m from the axle creates a new angular displacement, θ, where $\tau = \kappa\theta$ (κ is a constant). It is useful to use radians, when dealing with angular displacement, so $17.0° = 0.2967$ rad. Use the convention that counter-clockwise torques are positive.

SKETCH:



RESEARCH: κ can be determined from the initial condition with no hanging mass, and then θ can be determined when the mass hangs from the arm. Assume the arm's weight acts at its center, $l/2$.

SIMPLIFY: Initially, $\tau = \kappa\theta_0 \Rightarrow \kappa\theta_0 = -mg\left(\frac{l}{2}\right)\sin\left(90° + \theta_0\right) = \Rightarrow \kappa = -\frac{1}{2}\frac{mgl\cos\theta_0}{\theta_0}$. With the added mass, $\tau = \kappa\theta \Rightarrow -mg\left(\frac{l}{2}\right)\sin\left(90° + \theta\right) + -Mgd\sin\left(90° + \theta\right) = \kappa\theta \Rightarrow \frac{\cos\theta}{\theta} = \frac{-\kappa}{\frac{1}{2}mgl + Mgd}$.

CALCULATE: $\kappa = -\frac{1}{2}\frac{(0.045\text{ kg})(9.81\text{ m/s}^2)(0.12\text{ m})\cos(0.2967\text{ rad})}{0.2967\text{ rad}} = -0.085371\text{ J/rad}$

$$\frac{\cos\theta}{\theta} = \frac{0.085371\text{ J/rad}}{\frac{1}{2}(0.045\text{ kg})(9.81\text{ m/s}^2)(0.12\text{ m}) + (0.42\text{ kg})(9.81\text{ m/s}^2)(0.090\text{ m})} = 0.21488$$

Solve for θ numerically. Here is a table of θ in degrees and $\cos(\theta)/\theta$ with θ in radians:

For $\theta = 73.9°$, $\cos(\theta)/\theta = 0.21501$, which is close to our value of 0.21488.

ROUND: The answer should be rounded to three significant figures, so $\theta = 73.9°$.

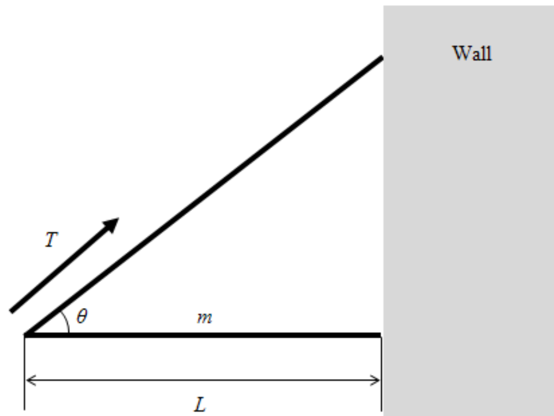
DOUBLE-CHECK: For an object hanging off the arm, θ must be $\theta_0 < \theta < 180°$.

θ	cos(θ)/θ
73.0	0.22947
73.1	0.22785
73.2	0.22623
73.3	0.22462
73.4	0.22301
73.5	0.22140
73.6	0.21980
73.7	0.21820
73.8	0.21660
73.9	0.21501
74.0	0.21342

Multi-Version Exercises

11.81. **THINK:** If the hinge where the bar is attached to the wall is the rotation point, then the sum of the clockwise torque and the counterclockwise torque is zero. The counterclockwise torque can be found from the mass and the length of the bar, and the clockwise torque can be used to find the tension in the cable.

SKETCH:



RESEARCH: The net torque is zero, to the magnitude of the counterclockwise torque is equal to the magnitude of the clockwise torque, $\tau_{\text{CCW}} = \tau_{\text{CW}}$. The gravitational force pulling down on the bar ($F_g = mg$) causes a counterclockwise torque that is given by the equation $\tau_{\text{CCW}} = mg(L/2)$. The clockwise torque, due to the tension of the cable, is given by $\tau_{\text{CW}} = TL\sin\theta$.

SIMPLIFY: Since the counterclockwise and clockwise torques are equal, $TL\sin\theta = mg\left(L/2\right)$. Solve for the tension on the cable: $T = \dfrac{mg}{2\sin\theta}$.

CALCULATE: The mass of the bar is 81.95 kg and the angle between the bar and the wall is θ = 38.89°. Near the surface of the earth, gravitational acceleration is g = 9.81 m/s^2. The tension on the cable is thus

$$T = \frac{81.95 \text{ kg} \cdot 9.81 \text{ m/s}^2}{2\sin 38.89°} = 640.2474085 \text{ N}.$$

ROUND: The mass and angle are given to four significant figures, so the final answer should also have four figures. The tension in the cable has a magnitude of 640.2 N.

DOUBLE-CHECK: The bar has a weight of 81.95 kg · 9.81 m/s^2 = 803.9 N. Since much of the weight of the bar is supported by the wall, it is reasonable to expect the tension on the cable to be less than the weight of the bar, confirming that the calculated value is a reasonable one.

11.82.

$$T = \frac{mg}{2\sin\theta}$$

$$\theta = \sin^{-1}\left(\frac{mg}{2T}\right)$$

$$= \sin^{-1}\left(\frac{\left(82.45 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right)}{2\left(618.8 \text{ N}\right)}\right) = 40.81°$$

11.83.

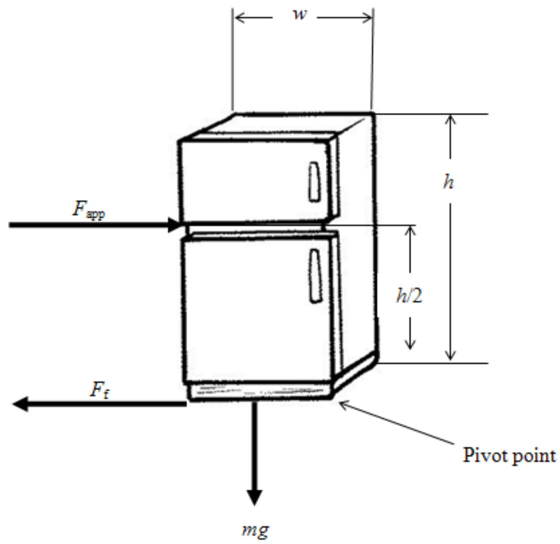
$$T = \frac{mg}{2\sin\theta}$$

$$m = \frac{2T\sin\theta}{g}$$

$$= \frac{2\left(599.3 \text{ N}\right)\sin\left(42.75°\right)}{9.81 \text{ m/s}^2} = 82.94 \text{ kg}$$

11.84. **THINK:** Since the velocity of the refrigerator is constant, the magnitudes of the friction force and the applied force must be equal (the total force on the refrigerator is zero so $F_{\text{app}} = F_{\text{f}}$). The refrigerator is on the verge of tipping but does not tip over, so the weight will cause a counterclockwise torque that exactly balances the clockwise torque from the force pushing the refrigerator. Find equations for the torques, set them equal to one another, and solve for the maximum coefficient of kinetic friction.

SKETCH: The refrigerator is being pushed from left to right by an applied force, F_{app}.



RESEARCH: The counterclockwise torque due to weight has the same magnitude as the clockwise torque due to the force pushing the refrigerator, $\tau_{\text{CCW}} = \tau_{\text{CW}}$. The clockwise torque is due to the applied force on the refrigerator, and is given by $\tau_{\text{CW}} = F_{\text{app}}(h/2)$. The counterclockwise torque is given by $\tau_{\text{CCW}} = mg(w/2)$. The friction force $F_{\text{f}} = \mu_k mg$ is computed from the coefficient of kinetic friction and the normal force (the normal force is equal to the gravitational force on the refrigerator, but opposite in direction).

SIMPLIFY: Since the applied force and frictional force have the same magnitude, the clockwise torque can be expressed in terms of the frictional force as $\tau_{\text{CW}} = F_{\text{f}}(h/2)$. The clockwise and counterclockwise torques must be equal ($\tau_{\text{CCW}} = \tau_{\text{CW}}$), so $mg(w/2) = F_{\text{f}}(h/2)$. Replace the frictional force in this equation with $F_{\text{f}} = \mu_k mg$, and solve for the coefficient of kinetic friction to get:

$$mg(w/2) = (\mu_k mg)(h/2)$$
$$w = \mu_k h$$
$$w/h = \mu_k$$

CALCULATE: The width and height of the refrigerator are given in the problem as 1.247 m and 2.177 m, respectively. The maximum coefficient of kinetic friction is then $\mu_k = w/h = 1.247\text{ m}/2.177\text{ m} = 0.5728066146$.

ROUND: The dimensions of the refrigerator are given to four significant figures, so the final answer should also have four figures. The maximum coefficient of kinetic friction is 0.5728.

DOUBLE-CHECK: For a refrigerator sliding across the floor, the coefficient of friction must be between 0 and 1. The calculated value is close to the vale for steel sliding on steel, and between the value of rubber sliding or wet and dry concrete. Since the bottom of most refrigerators is made of metal or smooth plastic, and the floor might be linoleum or carpet, a number in this range makes sense. Based on an understanding of how things move in the real world, the calculated value is reasonable.

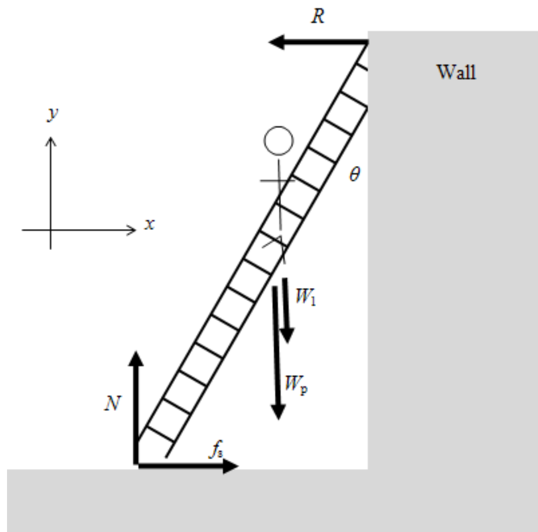
11.85. $w = \mu_k h = (0.4696)(2.187\text{ m}) = 1.027\text{ m}$

11.86. $w = \mu_k h$

$$h = \frac{w}{\mu_k} = \frac{1.059 \text{ m}}{0.4820} = 2.197 \text{ m}$$

11.87. **THINK:** In this problem, the maximum angle will occur when the torque due to gravity and the torque due to friction exactly cancel one another. In this case, both the sum of the torques and the sum of the forces will be zero. For this problem, it will be easier to break down the forces into their horizontal and vertical components.

SKETCH: The weight of the ladder is labeled W_l and the weight of the person climbing the ladder is labeled W_p.



RESEARCH: Both the net force and the net torque are zero in this situation. The only horizontal forces in the *x*-direction are the force of static friction and the horizontal force of the wall. These must cancel one another, so $f_s - R = 0 \Rightarrow f_s = R$. In the *y*-direction, the normal force exerted by the floor, the weight of the person, and the weight of the ladder must also cancel out, giving that $N - W_l - W_p = 0 \Rightarrow N = W_l + W_p$. The weight is easily calculated from the mass and the gravitational acceleration: $W_l = m_l g$ and $W_p = m_p g$. The maximum force of static friction is calculated from the normal force using the equation $f_s = \mu_s N$. If l is the length of the ladder, and the pivot point is the place where the ladder touches the floor, then the clockwise torque is given by $\tau_{CW} = W_l \frac{l}{2}\sin\theta + W_p \frac{l}{2}\sin\theta$. The counterclockwise torque is given by $\tau_{CCW} = Rl\cos\theta$. Since the net torque is zero, $\tau_{CCW} - \tau_{CW} = 0 \Rightarrow \tau_{CCW} = \tau_{CW}$.

SIMPLIFY: Since the net torque is zero ($\tau_{CCW} = \tau_{CW}$), $Rl\cos\theta = W_l \frac{l}{2}\sin\theta + W_p \frac{l}{2}\sin\theta$. Substitute in for the weights to get $Rl\cos\theta = m_l g \frac{l}{2}\sin\theta + m_p g \frac{l}{2}\sin\theta$. Solve for theta to get:

$$Rl\cos\theta = \left(m_l g + m_p g\right)\frac{l}{2}\sin\theta$$

$$\frac{\cos\theta}{\sin\theta} = \frac{\left(m_l g + m_p g\right)l}{2Rl}$$

$$\cot\theta = \frac{\left(m_l g + m_p g\right)}{2R}$$

$$\theta = \cot^{-1}\left(\frac{m_l g + m_p g}{2R}\right)$$

Use the fact that $R = f_s = \mu_s N$, and the normal force $N = W_l + W_p = m_l g + m_p g$ to get an expression for *R*: $R = \mu_s(m_l g + m_p g)$. Finally, substitute this into the equation for theta to get a final expression for the angle $\theta = \cot^{-1}\left(\frac{m_l g + m_p g}{2\mu_s(m_l g + m_p g)}\right) = \cot^{-1}\left(\frac{1}{2\mu_s}\right) = \tan^{-1}(2\mu_s)$.

CALCULATE: The coefficient of friction is 0.2881, so the maximum angle will be:

$$\begin{aligned}\theta &= \tan^{-1}(2\cdot 0.2881)\\ &= \tan^{-1}(0.5762)\\ &= 29.9505462^\circ\end{aligned}$$

ROUND: The coefficient of friction is given to four significant figures, so the final answer should also have four figures. The maximum angle between the ladder and the wall is 29.95°.

DOUBLE-CHECK: Based on real-world experience, the maximum angle between a ladder of this type and the wall against which it leans is about 30 degrees. Furthermore, it makes intuitive sense that the maximum angle would depend only on the coefficient of friction between the ladder and the floor. If the ladder is at a fixed angle, then it will either slide or not slide regardless of the weight of the person standing at the ladder's midpoint.

11.88. $\theta = \tan^{-1}(2\mu_s)$

$$\Rightarrow \mu_s = \frac{1}{2}\tan\theta = \frac{1}{2}\tan(27.30^\circ) = 0.2581$$

Chapter 12: Gravitation

Concept Checks

12.1. d **12.2.** a **12.3.** d **12.4.** a **12.5.** d **12.6.** b **12.7.** c

Multiple-Choice Questions

12.1. a **12.2.** b **12.3.** c **12.4.** c **12.5.** c **12.6.** b **12.7.** e **12.8.** a **12.9.** a **12.10.** e **12.11.** c **12.12.** c **12.13.** a **12.14.** b **12.15.** a

Conceptual Questions

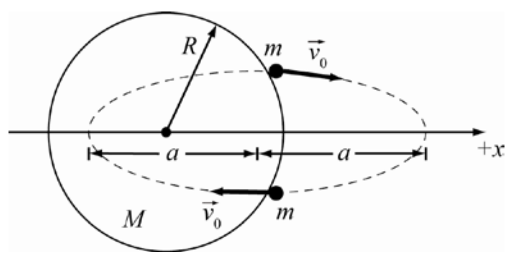
12.16. No, one cannot. The expression for gravitational potential energy $U_{\text{g}}(y) = mgy$ is only accurate when the height y is much less than the radius of the Earth, R_{E}. The correct expression for the potential energy is:

$U = -\frac{GM_{\text{E}}m}{(R_{\text{E}}+y)} = -\frac{GM_{\text{E}}m}{R_{\text{E}}}\left(1+\frac{y}{R_{\text{E}}}\right)^{-1}$. Expanding this in powers of Y/R_{E} gives:

$$U = -\frac{GM_{\text{E}}m}{R_{\text{E}}}\left[1-\frac{y}{R_{\text{E}}}+\frac{1}{2}\frac{y^2}{R_{\text{E}^2}}+\ldots\right].$$

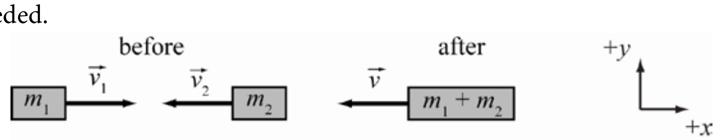
The potential energy is approximately linear, in the same manner as $U = mgy$, only when $(y/R_{\text{E}})^2$ is small. So, $U_{\text{g}}(y) = mgy$ is only valid when $y \ll R_{\text{E}}$.

12.17.



A parabolic trajectory only occurs when the gravitational acceleration due to gravity is constant. Instead of a uniform gravitational force downwards, consider two bodies in orbit as their gravitation attraction constantly changes. It is known from Kepler's laws that these orbits are elliptical. Now consider a projectile fired into the air on the Moon where air friction is not a factor. Once this projectile is in the air it is simply an object in an elliptical orbit with the center of mass of the Moon at one focus. If the initial velocity is not great enough, the projectile will not be able to complete the orbit since it will collide with the surface of the Moon. Therefore, the actual shape of the projectile's path will be that of an ellipse. A parabolic trajectory for a projectile is an approximation that is only valid when the gravitational acceleration is nearly constant. This is a good approximation for short and shallow trajectories. However, if the path is high and long then a parabolic trajectory is no longer valid.

12.18. In order to determine whether the two satellites would crash into the Earth, the final velocity after the collision is needed.



The velocities before the collision are determined from the motion of a satellite in circular orbit where the centripetal acceleration is equal to the gravitational acceleration, that is:

$$F = ma \Rightarrow \frac{GM_E m}{(R_E + h)^2} = m\frac{v_s^{\,2}}{(R_E + h)} \Rightarrow v_s = \sqrt{\frac{GM_E}{(R_E + h)}}.$$

Since v_s does not depend on the mass m of the satellite and both satellites are orbiting at the same height $h = 1000.$ km, $v_1 = v_2 = v_s$. The final velocity after collision is obtained using conservation of momentum, $p_i = p_f$: $m_1 v_s - m_2 v_s = v(m_1 + m_2)$. The final velocity is: $v = \frac{(m_1 - m_2)}{(m_1 + m_2)} v_s = \frac{(m_1 - m_2)}{(m_1 + m_2)}\sqrt{\frac{GM_E}{(R_E + h)}}.$

Since the final velocity is less than v_s, the satellites move in an elliptical orbit after the collision. The closest distance of this orbit to the center of the Earth is determined from the net energy of the elliptical orbit:

$$E = -\frac{GM_E m}{2a} = K + U = \frac{1}{2}mv^2 - \frac{GM_E m}{(R_E + h)},$$

where a is the semi-major axis. Therefore, the semi-major axis of the elliptical orbit is: $a = \frac{GM_E}{\frac{2GM_E}{(R_E + h)} - v^2}.$

Substituting $v^2 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 \frac{GM_E}{(R_E + h)}$ yields: $a = \frac{GM_E}{\frac{2GM_E}{(R_E + h)} - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 \frac{GM_E}{(R_E + h)}} = \frac{R_E + h}{2 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2}.$

Since the maximum distance for this new orbit must be where the collision took place, $(R_E + h)$, the distance of closest approach (to the center of the Earth) is:

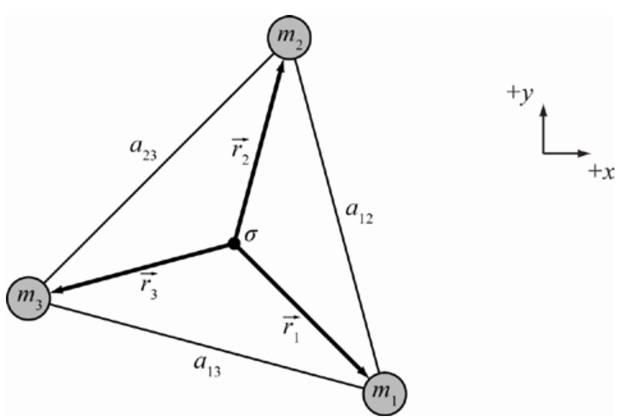
$$R = 2a - (R_E + h) = \frac{2(R_E + h)}{2 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2} - (R_E + h).$$

Therefore, the closest distance is:

$$R = \frac{2(6370 \text{ km} + 1000. \text{ km})}{2 - \left(\frac{250. \text{ kg} - 600. \text{ kg}}{250. \text{ kg} + 600. \text{ kg}}\right)^2} - (6370 \text{ km} + 1000. \text{ km}) = 683 \text{ km}.$$

Since R is less than R_E, the satellite would crash into the Earth.

12.19.



Since a_{12}, a_{23} and a_{13} are constant in time, the net force on the system must be zero and the centripetal force must be zero as well. Let the origin of an inertial frame located at the centre of mass of the system. The positions of m_1, m_2 and m_3 are given by vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$. The net force on m_1 is $\vec{F}_{\text{net},1} = G\frac{m_1 m_2}{a_{12}^3}(\vec{r}_2 - \vec{r}_1) + G\frac{m_1 m_3}{a_{13}^3}(\vec{r}_3 - \vec{r}_1)$. Similarly, for m_2 and m_3, the net forces are:

$$\vec{F}_{\text{net},2} = G\frac{m_1 m_2}{a_{12}^3}(\vec{r}_1 - \vec{r}_2) + G\frac{m_2 m_3}{a_{23}^3}(\vec{r}_3 - \vec{r}_2) \text{ and } \vec{F}_{\text{net},3} = G\frac{m_1 m_3}{a_{13}^3}(\vec{r}_1 - \vec{r}_3) + G\frac{m_2 m_3}{a_{23}^3}(\vec{r}_2 - \vec{r}_3).$$

The net forces on m_1, m_2 and m_3 must be equal to the centripetal forces on m_1, m_2 and m_3, which are: $\vec{F}_{\text{c},1} = -m_1\omega^2\vec{r}_1$, $\vec{F}_{\text{c},2} = -m_2\omega^2\vec{r}_2$, and $\vec{F}_{\text{c},3} = -m_3\omega^2\vec{r}_3$ Therefore, three equations are obtained:

$$-m_1\omega^2\vec{r}_1 = G\frac{m_1 m_2}{a_{12}^3}(\vec{r}_2 - \vec{r}_1) + G\frac{m_1 m_3}{a_{13}^3}(\vec{r}_3 - \vec{r}_1) \quad (1)$$

$$-m_2\omega^2\vec{r}_2 = G\frac{m_1 m_2}{a_{12}^3}(\vec{r}_1 - \vec{r}_2) + G\frac{m_2 m_3}{a_{23}^3}(\vec{r}_3 - \vec{r}_2) \quad (2)$$

$$-m_3\omega^2\vec{r}_3 = G\frac{m_3 m_1}{a_{13}^3}(\vec{r}_1 - \vec{r}_3) + G\frac{m_3 m_2}{a_{23}^3}(\vec{r}_2 - \vec{r}_3) \quad (3)$$

Eliminate m_1 in (1) and using $\vec{F}_{\text{c,net}} = m_1\omega^2\vec{r}_1 + m_2\omega^2\vec{r}_2 + m_3\omega^2\vec{r}_3 = 0$ to eliminate $m_3\vec{r}_3$ in (1) gives:

$$-\omega^2\vec{r}_1 = \frac{Gm_2}{a_{12}^3}(\vec{r}_2 - \vec{r}_1) + \frac{G}{a_{13}^3}\left(-(m_1\vec{r}_1 + m_2\vec{r}_2) - m_3\vec{r}_1\right)$$

$$\left(\frac{Gm_2}{a_{12}^3} + \frac{G(m_1 + m_3)}{a_{13}^3} - \omega^2\right)\vec{r}_1 = Gm_2\left(\frac{1}{a_{12}^3} - \frac{1}{a_{13}^3}\right)\vec{r}_2.$$

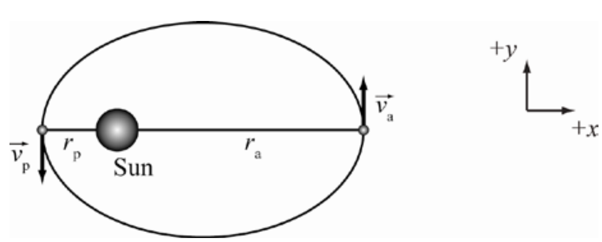
Since $\vec{r}_1$ and $\vec{r}_2$ are not collinear, the coefficients in front of $\vec{r}_1$ and $\vec{r}_2$ must be zero. Thus:

$$\frac{1}{a_{12}^3} - \frac{1}{a_{13}^3} = 0 \Rightarrow a_{12} = a_{13} = a \text{ and } \frac{Gm_2}{a_{12}^3} + \frac{G(m_1 + m_3)}{a_{13}^3} - \omega^2 = 0.$$

Using $a_{12} = a_{13} = a$, this becomes $G(m_1 + m_2 + m_3)/a^3 = \omega^2$ or $GM = \omega^2 a^3$, where $M = m_1 + m_2 + m_3$. In similar manner, for equation (2), the conditions $a_{12} = a_{23} = a$ and $GM = \omega^2 a^3$ are obtained. For equation (3), the conditions $a_{13} = a_{23} = a$ and $GM = \omega^2 a^3$ are also obtained. Therefore, the conditions for the system to rotate around the axis σ as a rigid body are $a_{12} = a_{13} = a_{23} = a$ and $GM = \omega^2 a^3$.

12.20. The escape velocity of an object from a planet's surface is given by $v_{\text{E}} = \sqrt{(2GM/R)}$, where M is the mass of the planet and R is the radius of the planet. Also, the gravitational acceleration on a planet's surface is given by $g = GM/R^2$, which can be rewritten as $GM/R = gR$. Substituting this expression into the equation for escape velocity yields $v_{\text{E}} = \sqrt{2gR}$. Therefore, v_{E} is directly proportional to g and R. This means that even though g is smaller on Uranus, the escape velocity is larger on Uranus because its radius is much larger than Earth's radius.

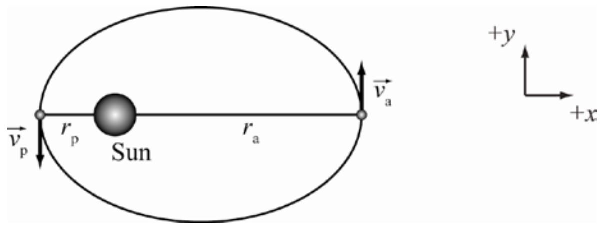
12.21.



As the Earth moves in an elliptical orbit around the Sun, the angular momentum of the Earth is conserved. This means that the angular momentum when the Earth is closest to the Sun (at perihelion, r_{p}) and when the Earth is farthest from the Sun (at aphelion, r_{a}) must be the same; that is, $mv_{\text{p}}r_{\text{p}} = mv_{\text{a}}r_{\text{a}}$. Therefore, $v_{\text{p}} / v_{\text{a}} = r_{\text{a}} / r_{\text{p}}$. Since $r_{\text{a}} > r_{\text{p}}$, $v_{\text{p}} > v_{\text{a}}$. The orbital speed at the perihelion is larger than the orbital speed at the aphelion.

12.22. The only flaw in the statement is *"This explains the main cause of seasons (Summer – Winter) on Earth."* The seasons on Earth are not caused by the change in the Earth-Sun distance as Earth follows its elliptical orbit. The seasons are caused by the tilt of the axis of Earth's rotation relative to the elliptical plane. As the Earth travels along its elliptical orbit the orientation of Earth's axis relative to the Sun changes.

12.23.



As the comet moves in an elliptical orbit around the Sun, the angular momentum of the comet is conserved. This means that the angular momentum when the comet is closest to the Sun (at perihelion, r_{p}) and when the comet is farthest from the Sun (at aphelion, r_{a}) must be the same; that is, $mv_{\text{p}}r_{\text{p}} = mv_{\text{a}}r_{\text{a}}$. Therefore, $v_{\text{p}} / v_{\text{a}} = r_{\text{a}} / r_{\text{p}}$. Since $r_{\text{a}} > r_{\text{p}}$, $v_{\text{p}} > v_{\text{a}}$. Since the orbital speed is largest at perihelion, the kinetic energy of the comet is a maximum at perihelion. Since the total energy of the comet is conserved, the maximum gravitational potential energy must be a maximum at aphelion. This is confirmed by inspecting the equation for gravitational potential energy: $U \propto -1/r$. As the distance r increases the gravitational potential energy increases (by becoming less negative).

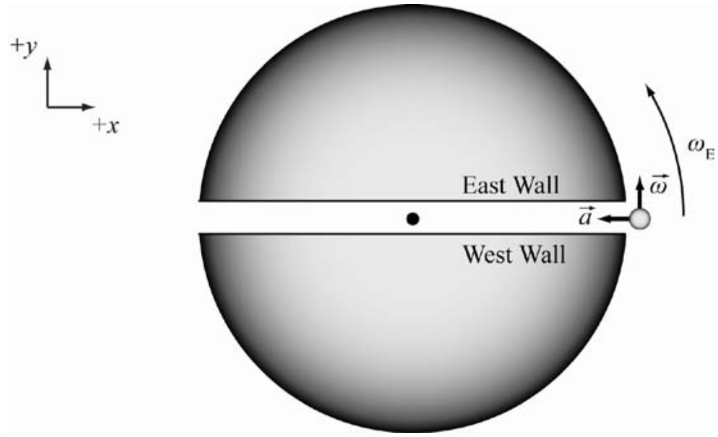
12.24. If an astronaut measures his/her weight on a scale, the measured weight is $W = \text{Normal force} = N = mg - ma_{\text{c}}$. Since the gravitational acceleration is equal to the centripetal acceleration, the weight is zero. This means also that the normal force exerted by a wall or the floor of the space station on the astronaut is zero. Therefore, the astronaut floats in the space station because both are moving in an orbit where their gravitational forces are equal to their centripetal forces.

12.25. In this case, there are two factors that cause a change in the kinetic energy of the satellite. First, the kinetic energy is reduced by the collision with the tenuous atmosphere. Second, the kinetic energy is increased by the decrease in potential energy of the satellite (it speeds up as it falls inward). This increase is much greater than the decrease due to the collision with the atmosphere. Therefore, as the satellite falls, its kinetic energy increases until it reaches a terminal velocity, whereby the gravitational force is equal to the drag force.

12.26. Neither, the magnitude of the two forces must be equal. This is Newton's third law, $F_{\text{E}\to\text{M}} = F_{\text{M}\to\text{E}} = \dfrac{GM_{\text{E}}M_{\text{M}}}{r^2}$.

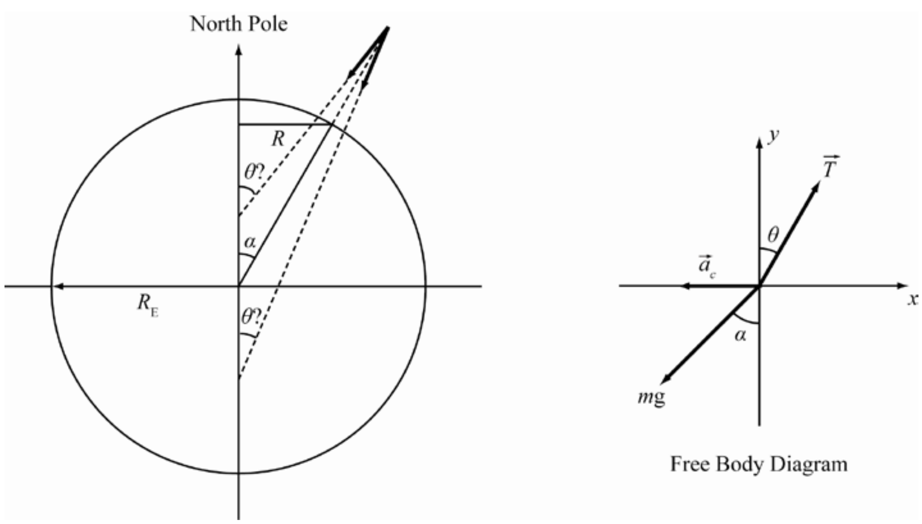
12.27. The ball released in Tunnel 1 reaches the center of the Earth first. This is because the ball in Tunnel 1 moves in free fall with acceleration given by its gravitational acceleration a_{g}, while the ball in Tunnel 2 moves with an acceleration of $a_{\text{g}} - a_{\text{c}}$, where a_{c} is the centripetal acceleration due to the rotation of the Earth. Therefore, the acceleration of the ball in Tunnel 1 is larger than the acceleration of the ball in Tunnel 2. This means that the ball in Tunnel 1 reaches the center of the Earth first.

12.28.



Initially, the ball has angular speed $\omega_i = \omega_E$. When the ball is released it experiences a force only in the radial direction. This means the angular momentum is conserved. When the ball reaches a distance r from the center of the Earth, the angular speed of the ball is: $L_i = L_f \Rightarrow m\omega_E R_E{}^2 = m\omega_f r^2 \Rightarrow \omega_f = \omega_E \left(\frac{R_E}{r}\right)^2$. Since r is less than R_E, its angular speed increases as it falls. After its angular speed is large enough, it catches up with the east wall and hits the wall. After the ball hits the wall, the angular momentum of the ball is no longer conserved. However, the angular momentum of the Earth- ball system is conserved.

12.29.



Using Newton's second law and $a_c = \frac{v^2}{R}$ for centripetal acceleration, it is found that: $\sum F_x = F_c \Rightarrow T\sin\theta - mg\sin\alpha = -m\frac{v^2}{R}$. Since $v = \omega R$ and $R = R_E \sin\alpha$:

$$T\sin\theta = mg\sin\alpha - m\omega^2 R_E \sin\alpha. \quad 1)$$

Similarly,

$$\sum F_y = 0 \Rightarrow T\cos\theta - mg\cos\alpha = 0 \Rightarrow T\cos\theta = mg\cos\alpha. \quad (2)$$

Dividing (1) by (2) gives:

$$\frac{T\sin\theta}{T\cos\theta} = \tan\theta = \frac{mg\sin\alpha - m\omega^2 R_E \sin\alpha}{mg\cos\alpha} \Rightarrow \tan\theta = \left(\frac{g-\omega^2 R_E}{g}\right)\tan\alpha = \left(1-\frac{\omega^2 R_E}{g}\right)\tan\alpha$$

Using $\omega = 7.27\cdot 10^{-5}$ rad/s, $\alpha = 90.0° - 55.0° = 35.0°$ and $R_E = 6.37\cdot 10^6$ m, the angle θ is:

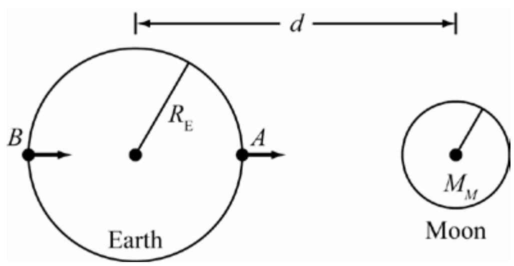
$$\tan\theta = \left[1 - \frac{\left(7.27\cdot 10^{-5} \text{ rad/s}\right)^2\left(6.37\cdot 10^6 \text{ m}\right)}{9.81 \text{ m/s}^2}\right]\tan\left(35.0°\right) = 0.698 \Rightarrow \theta = 34.9°.$$

Since the angle θ is less than α, the line intersects the Earth's axis south of the Earth's center.

Exercises

12.30. **THINK:** To solve this problem, the mass of the Moon, the Earth's radius, and the distance between the Earth and the Moon are needed. These values can be found from a table in the textbook. The mass of the Moon is $M_m = 7.36\cdot 10^{22}$ kg, the radius of the Earth is $R_E = 6.37\cdot 10^6$ m and the distance between the Earth and the Moon is $d = 3.82\cdot 10^8$ m.

SKETCH:



RESEARCH: Note in the above diagram that the position A is the nearest to the Moon and the position B is the farthest from the Moon. The gravitational accelerations due to the Moon at the positions A and B are $a_A = GM_m / \left(d - R_E\right)^2$ and $a_B = GM_m / \left(d + R_E\right)^2$.

SIMPLIFY: The difference in the accelerations is: $\Delta a = a_A - a_B = GM_m\left(\frac{1}{\left(d-R_E\right)^2} - \frac{1}{\left(d+R_E\right)^2}\right)$

CALCULATE:

$$\Delta a = \left(6.67\cdot 10^{-11} \text{ N m}^2/\text{kg}^2\right)\left(7.36\cdot 10^{22} \text{ kg}\right)\left(\frac{1}{\left(3.82\cdot 10^8 \text{ m} - 6.37\cdot 10^6 \text{ m}\right)^2} - \frac{1}{\left(3.82\cdot 10^8 \text{ m} + 6.37\cdot 10^6 \text{ m}\right)^2}\right)$$

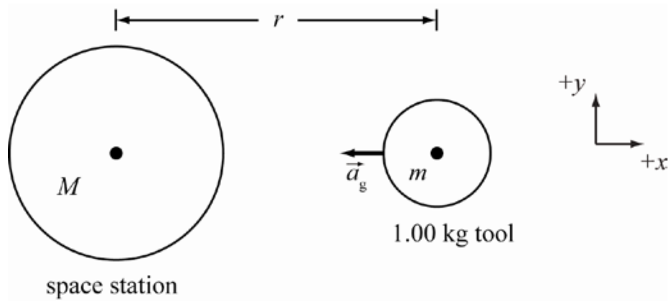
$$= 2.2452\cdot 10^{-6} \text{ m/s}^2$$

ROUND: Three significant figures: $\Delta a = 2.25\cdot 10^{-6}$ m/s^2

DOUBLE-CHECK: Since $d >> R_E$ it is reasonable that the difference in the gravitational acceleration between two points separated by $2R_E$ is very small.

12.31. **THINK:** Since the objects are floating in outer space, it can be assumed that the only force on each object is their mutual force of gravity. The space station and the tool exert forces with the same magnitude on each other (Newton's Third Law). In order to find the distance that the objects drift towards each other, we have to calculate the acceleration of the objects, which is the ratio of the gravitational force divided by the mass. Since the space station is MUCH more massive than the tool, it will experience negligible acceleration, and it is sufficient to just calculate the distance the tool moves after an hour needs to be calculated. Even though this force will increase slightly since they will be closer as they accelerate towards each other, it will be assumed that the force of gravity is constant. This assumption can only be made, however, if the distance the tool moves is small compared to the initial separation, and we will have to check this after we are done with our calculation.

SKETCH:



RESEARCH: The two objects attract each other by their mutual gravitational force: $F_g = GMm/r^2$. Using Newton's Second Law, the acceleration of the tool a_{tool} towards the space station due to gravity is $a_{\text{tool}} = GM/r^2$. Assuming that the objects were initially at rest and constant acceleration over a time interval *t*, the distance traveled by the tool is given by: $x = \frac{1}{2}a_{\text{tool}}t^2$.

SIMPLIFY: $x = \frac{1}{2}a_{\text{tool}}t^2 = \frac{1}{2}\frac{GM}{r^2}t^2$

CALCULATE: Substituting the given values:

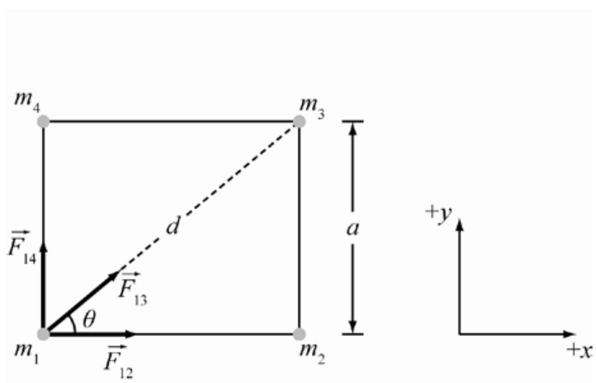
$$x = \frac{1}{2}\frac{(6.67\cdot 10^{-11}\text{ N m}^2/\text{kg}^2)(2.00\cdot 10^4\text{ kg})}{(50.0\text{ m})^2}(3600.\text{ s})^2 = 3.45773\cdot 10^{-3}\text{ m}$$

ROUND: $x = 3.46$ mm

DOUBLE-CHECK: Since the displacement *x* is much smaller than the initial separation $r = 50$ m, the assumption of constant acceleration is justified.

12.32. **THINK:** There are four masses $(m_1 = m_2 = m_3 = m_4 = m)$ as shown in the sketch. Calculate the forces on mass m_1.

SKETCH:



RESEARCH: Use Newton's Law of Gravity: $F = \frac{Gm_1m_2}{r^2}$.

SIMPLIFY: The forces on m_1 are as follows.

(1) $\vec{F}_{12} = G\frac{m_1m_2}{a^2}\hat{x} = G\frac{m^2}{a^2}\hat{x}$

(2) $\vec{F}_{13} = G\frac{m_1 m_3}{d^2}\left(\cos\theta\hat{x} + \sin\theta\hat{y}\right), \quad \theta = 45°$

Since d is the diagonal of the square, $d = \sqrt{a^2 + a^2} = a\sqrt{2}$.

$$\vec{F}_{13} = G\frac{m^2}{2a^2}\left(\cos\left(45°\right)\hat{x} + \sin\left(45°\right)\hat{y}\right) = G\frac{m^2}{2a^2}\left(\frac{1}{\sqrt{2}}\left(\hat{x} + \hat{y}\right)\right)$$

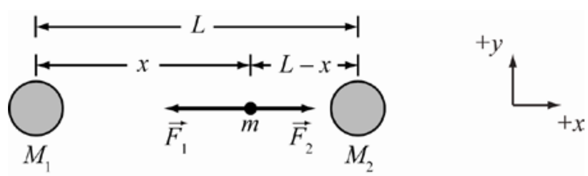
$$(3) \quad \vec{F}_{14} = G\frac{m_1 m_4}{a^2}\hat{y} = G\frac{m^2}{a^2}\hat{y}$$

The net force is: $\vec{F}_{\text{net}} = G\frac{m^2}{a^2}\left(\hat{x} + \frac{1}{2\sqrt{2}}\left(\hat{x} + \hat{y}\right) + \hat{y}\right) = G\frac{m^2}{a^2}\left(1 + \frac{1}{2\sqrt{2}}\right)\left(\hat{x} + \hat{y}\right)$. Using $\left|\hat{x} + \hat{y}\right| = \sqrt{2}$, the magnitude of net force is: $\left|\vec{F}_{\text{net}}\right| = G\frac{m^2}{a^2}\left(1 + \frac{1}{2\sqrt{2}}\right)\sqrt{2} = G\frac{m^2}{a^2}\left(\sqrt{2} + \frac{1}{2}\right)$.

CALCULATE: No numerical values were given. The solution is algebraic.
ROUND: No rounding is necessary since the answer is algebraic.

DOUBLE-CHECK: In the above equation the $\sqrt{2}$ term is due to the m_2 and m_4 combination at the adjacent corners, while the (1/2) term is due to m_3 at the opposite corner.

12.33. **THINK:** The location of a spacecraft located between two planets is desired, such that the net force on the spacecraft is zero. The two planets have masses M_1 and M_2. The distance between the two planets is L.
SKETCH:



RESEARCH: The variable x needs to be determined in the above diagram. Use Newton's Law of Gravity, $F = G\left(\frac{m_1 m_2}{r^2}\right)$. The forces on the spacecraft are $F_1 = G\left(\frac{mM_1}{x^2}\right)$ and $F_2 = G\left(\frac{mM_2}{\left(L-x\right)^2}\right)$.

SIMPLIFY: Since the net force is zero, $\vec{F}_1 = -\vec{F}_2$. This equation means that the magnitudes of the two forces are equal, but the forces are pointing in the opposite directions (hence, the negative sign). Therefore, $G\left(mM_1 / x^2\right) = G\left(mM_2 / \left(L-x\right)^2\right) \Rightarrow M_1\left(L-x\right)^2 = M_2 x^2$. Expanding $\left(L-x\right)^2$ gives:

$$L^2 - 2Lx + x^2 = \frac{M_2}{M_1}x^2 \Rightarrow \left(1 - \frac{M_2}{M_1}\right)x^2 - 2Lx + L^2 = 0$$

Solving for x using the quadratic formula:

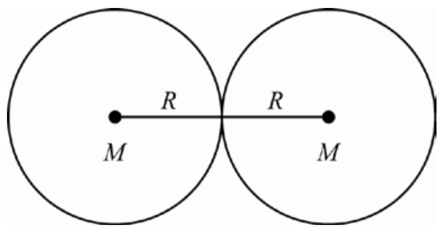
$$x = \frac{2L \pm \sqrt{4L^2 - 4\left(1 - \frac{M_2}{M_1}\right)L^2}}{2\left(1 - \frac{M_2}{M_1}\right)} = \frac{L\left(1 \pm \sqrt{\frac{M_2}{M_1}}\right)}{1 - \frac{M_2}{M_1}}$$

The spacecraft is only between the planets when the negative sign is used in front of the square root:

$$x=\frac{L\left(1-\sqrt{\dfrac{M_2}{M_1}}\right)}{1-\dfrac{M_2}{M_1}}=\frac{L}{1+\sqrt{\dfrac{M_2}{M_1}}}$$

CALCULATE: No numerical values were given. The solution is algebraic.
ROUND: No rounding is necessary since the answer is algebraic.
DOUBLE-CHECK: Using the above formula for $M_1=M_2$, $x=L/2$. Since the force would be zero midway between two equal masses, the derived formula is correct.

12.34. **THINK:** Using the given density of iron $\left(7860 \text{ kg/m}^3\right)$ the gravitational force between two 1.00 kg iron spheres that are touching can be calculated using Newton's Law of Gravity.
SKETCH:



RESEARCH: Using the mass and volume of the sphere, $V=4/3\pi R^3$, the radius of the sphere can be calculated. Newton's Law of Gravity: $F=G\left(MM/\left(2R\right)^2\right)=GM^2/4R^2$.

SIMPLIFY: Since $\rho=\dfrac{M}{V}=\dfrac{M}{\frac{4}{3}\pi R^3}$, the radius of the spheres is $R=\sqrt[3]{\dfrac{3M}{4\pi\rho}}$. Therefore, the force is:

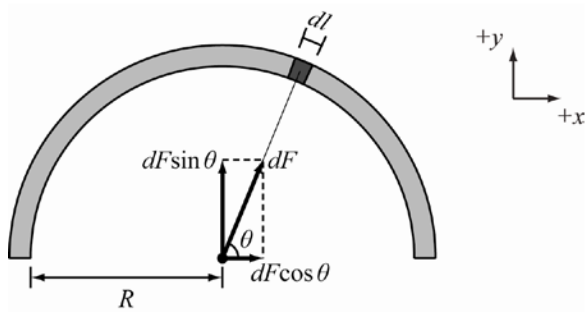
$$F=\frac{GM^2}{4\left(\dfrac{3M}{4\pi\rho}\right)^{\frac{2}{3}}}.$$

CALCULATE: $F=\dfrac{\left(6.67\cdot 10^{-11}\text{ N m}^2/\text{kg}^2\right)\left(1.00\text{ kg}\right)^2}{4\left(\dfrac{3\left(1.00\text{ kg}\right)}{4\pi\left(7860\text{ kg/m}^3\right)}\right)^{\frac{2}{3}}}=1.713\cdot 10^{-8}\text{ N}$

ROUND: To three significant figures, the gravitational force between the two spheres is $1.71\cdot 10^{-8}$ N.
DOUBLE-CHECK: A dimensional analysis gives units of Newton's which is expected for the unit of force. Since the masses of the two spheres are small it is reasonable that the force is small as well.

12.35. **THINK:** Consider a uniform rod of mass $M=333$ kg in the shape of a semi-circle of radius $R=5.00$ m. To calculate the magnitude of the force on a 77.0 kg point mass *m* at the centre of this semi-circle it can be assumed that the density of the rod is uniform such that $\rho=M/L$, where *L* is the length of the rod. If the rod is divided into small elements each of mass $dM=\rho dl$, integration can be used to find the total force on mass *m*.

SKETCH:



RESEARCH: Use Newton's law of Gravity, $F = G\left(m_1 m_2 / r^2\right)\hat{r}$. The gravitational force on mass m at the center of the semi-circle caused by a small element dM is $d\vec{F} = G\left(\frac{mdM}{R^2}\right)\hat{r}$, or in component form, $dF_x = G\left(mdM/R^2\right)\cos\theta$, $dF_y = G\left(mdM/R^2\right)\sin\theta$.

SIMPLIFY: The components of the force on mass m is then,

$$F_x = \int_0^M \frac{Gm\cos\theta}{R^2}dM \text{ and } F_y = \int_0^M \frac{Gm\sin\theta}{R^2}dM.$$

Using $dM = \rho dl = \rho R d\theta$,

$$F_x = \int_0^\pi \frac{Gm\rho R\cos\theta}{R^2}d\theta \text{ and } F_y = \int_0^\pi \frac{Gm\rho R\sin\theta}{R^2}d\theta.$$

Since G, m, ρ and R are constant, these can be simplified further to

$$F_x = \frac{Gm\rho}{R}\int_0^\pi \cos\theta d\theta \text{ and } F_y = \frac{Gm\rho}{R}\int_0^\pi \sin\theta d\theta.$$

From a mathematical table of integrals,

$$\int_0^\pi \cos\theta d\theta = \left[\sin\theta\right]_0^\pi = \sin\pi - \sin 0 = 0 \text{ and } \int_0^\pi \sin\theta d\theta = \left[-\cos\theta\right]_0^\pi = -\cos\pi + \cos 0 = 2.$$

Therefore, $F_x = 0$ and $F_y = \frac{2Gm\rho}{R}$. Using $\rho = \frac{M}{L}$ and $L = \pi R$ gives $F_y = \frac{2GmM}{\pi R^2}$.

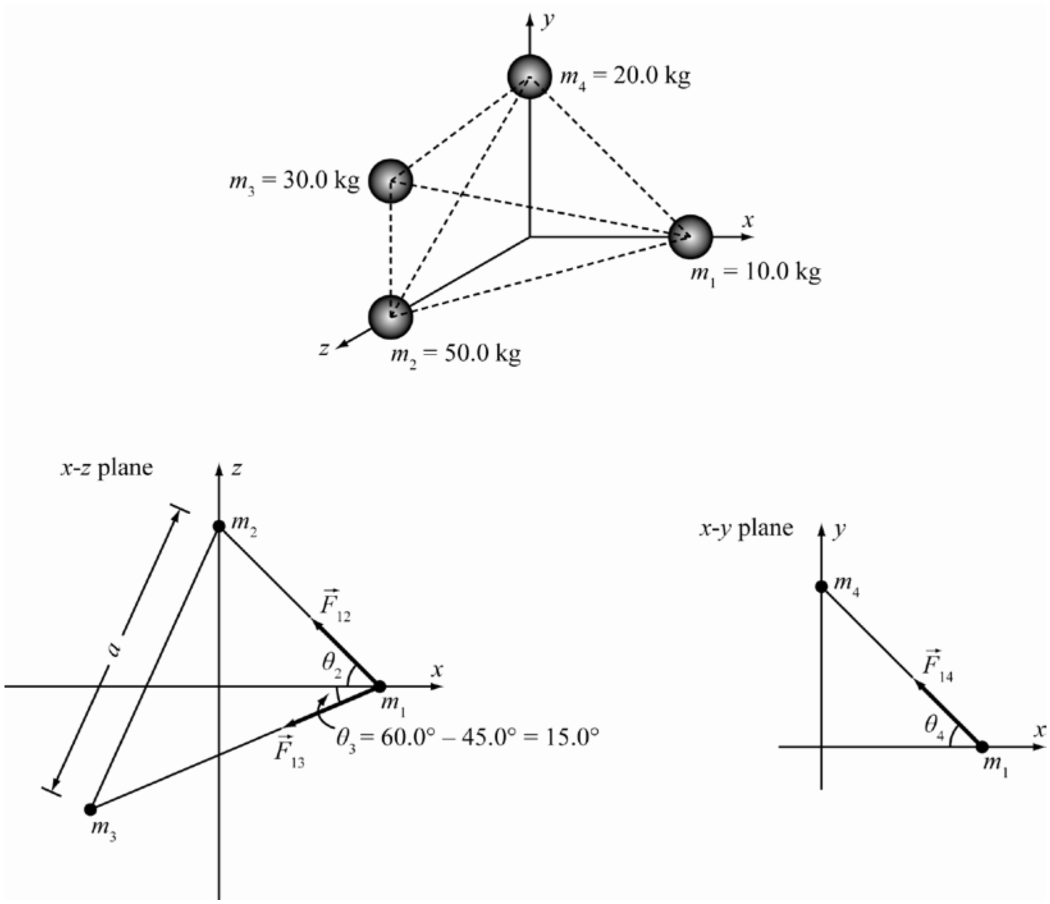
CALCULATE: $F_y = \frac{2\left(6.67 \cdot 10^{-11} \text{ N m}^2/\text{kg}^2\right)(77.0 \text{ kg})(333 \text{ kg})}{\pi(5.00 \text{ m})^2} = 4.355 \cdot 10^{-8}$ N

ROUND: To three significant figures, $F_y = 4.36 \cdot 10^{-8}$ N. The total net force is in the positive y-direction.

DOUBLE-CHECK: From the symmetry of the shape of the semi-circle, it is clear that $F_x = 0$, since a force due to mass dm on one side will have the same magnitude and opposite direction of a force due to mass dm on the other side.

12.36. **THINK:** There are four masses as shown in the figure in the sketch step. I need to determine the magnitude and the direction of the gravitational force on the 10.0 kg mass. We choose the axis so that the distance of m_1 along the x-axis, the distance of m_2 along the z-axis and the distance of m_4 along the y-axis are all the same.

SKETCH:



RESEARCH: The net force on m_1 is $\vec{F}_{\text{net}} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14}$ as shown in the figures above. The forces are calculated using Newton's law of Gravity: $F_{\text{ij}} = G\left(m_{\text{i}} m_{\text{j}} / r_{\text{ij}}\right)\hat{r}_{\text{ij}}$, where $\hat{r}_{\text{ij}}$ is the directional unit vector from *i* to *j*. Note that $r_{12} = r_{13} = r_{14} = a = 10.0$ cm, $\theta_2 = \theta_4 = 45.0°$ and $\theta_3 = 15.0°$.

SIMPLIFY:
$$\vec{F}_{12} = \frac{Gm_1 m_2}{a^2}\left(-\cos(\theta_2)\hat{x} + \sin(\theta_2)\hat{z}\right)$$
$$\vec{F}_{13} = \frac{Gm_1 m_3}{a^2}\left(-\cos(\theta_3)\hat{x} - \sin(\theta_3)\hat{z}\right)$$
$$\vec{F}_{14} = \frac{Gm_1 m_4}{a^2}\left(-\cos(\theta_4)\hat{x} + \sin(\theta_4)\hat{y}\right)$$

The net force is:

$$\vec{F}_{\text{net}} = \frac{Gm_1}{a^2}\Big(m_2\left(-\cos(\theta_2)\hat{x} + \sin(\theta_2)\hat{z}\right) + m_3\left(-\cos(\theta_3)\hat{x} - \sin(\theta_3)\hat{z}\right) + m_4\left(-\cos(\theta_4)\hat{x} + \sin(\theta_4)\hat{y}\right)\Big)$$
$$= \frac{Gm_1}{a^2}\Big[\left(-m_2\cos(\theta_2) - m_3\cos(\theta_3) - m_4\cos(\theta_4)\right)\hat{x} + m_4\sin(\theta_4)\hat{y} + \left(m_2\sin(\theta_2) - m_3\sin(\theta_3)\right)\hat{z}\Big]$$

CALCULATE: $\vec{F}_{\text{net}} = \dfrac{\left(6.67\cdot 10^{-11}\text{ N m}^2/\text{kg}^2\right)(10.0\text{ kg})}{(0.100\text{ m})^2}\left[z_1\hat{x} + z_2\hat{y} + z_3\hat{z}\right]$, where

$z_1 = (-50.0\text{ kg})\cos(45.0°) - (30.0\text{ kg})\cos(15.0°) - (20.0\text{ kg})\cos(45.0°) = -78.5$ kg,

$z_2 = (20.0\text{ kg})\sin(45.0°) = 14.1$ kg, and $z_3 = (50.0\text{ kg})\sin(45.0°) - (30.0\text{ kg})\sin(15.0°) = 27.6$ kg.

So $\vec{F}_{\text{net}} = 6.67\cdot 10^{-8}\text{ N}\left(-78.5\hat{x}+14.1\hat{y}+27.6\hat{z}\right)$. The magnitude of the net force is

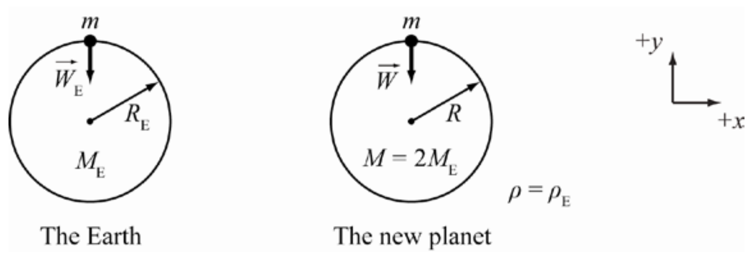
$\left|\vec{F}_{\text{net}}\right| = 6.67\cdot 10^{-8}\text{ N}\sqrt{\left(-78.5\right)^2+\left(14.1\right)^2+\left(27.6\right)^2} = 5.629\cdot 10^{-6}\text{ N}.$

ROUND: Three significant figures: $5.63\cdot 10^{-6}$ N. The direction of the force is given by the vector $-78.5\hat{x}+14.1\hat{y}+27.6\hat{z}$.

DOUBLE-CHECK: The magnitude of F_{net} is reasonable. From the above diagram it is noted that the direction of $\vec{F}_{\text{net}}$ is mainly in negative x-direction. This is in agreement with the above vector where the x-component is dominant and negative.

12.37. **THINK:** The new planet has mass $2M_{\text{E}}$ and density $\rho = \rho_{\text{E}}$. Once the radius of the new planet us found, Newton's law of Gravity can be used to determine the weight of the object on the new planet.

SKETCH:



RESEARCH: Density of an object: $\rho = M/V$, volume of a sphere $V = \left(4/3\right)\pi R^3$, and Newton's law of Gravity: $F = G\left(m_1 m_2 / R^2\right)$.

SIMPLIFY: The density is $\rho = \rho_{\text{E}} = \dfrac{M}{\left(4/3\right)\pi R^3} = \dfrac{M_{\text{E}}}{\left(4/3\right)\pi R_{\text{E}}^3}$. Since $M = 2M_{\text{E}}, \dfrac{2M_{\text{E}}}{R^3} = \dfrac{M_{\text{E}}}{R_{\text{E}}^3} \Rightarrow R = 2^{\frac{1}{3}} R_{\text{E}}$.

The weight of the object on the surface of the new planet is

$$W = \frac{GMm}{R^2} = 2\frac{GM_{\text{E}}m}{2^{\frac{2}{3}} R_{\text{E}}^2} = \sqrt[3]{2}W_{\text{E}}.$$

This means that the weight of the object on the new planet is $\sqrt[3]{2}$ times the weight of the object on the surface of the Earth.

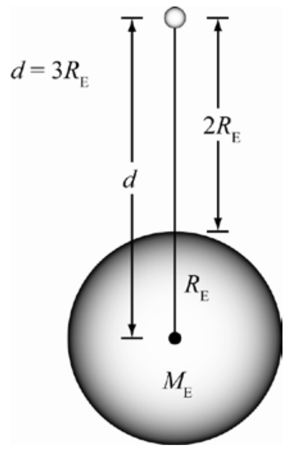
CALCULATE: No numerical values were given. The solution is algebraic.

ROUND: No rounding is necessary since the answer is algebraic.

DOUBLE-CHECK: Since the new planet is larger in size (twice the mass with the same density) than the Earth, it is expected that the weight of an object is larger on the new planet since there will be a stronger gravitational force.

12.38. **THINK:** The free fall acceleration is just the gravitational acceleration, which comes from Newton's Law of Gravity. The mass of the Earth can be placed at the center of the Earth.

SKETCH:



RESEARCH: The free fall acceleration is given by $g = GM_{\text{E}} / d^2$.

SIMPLIFY: If the ball is at an altitude of $2R_{\text{E}}$ then the distance from the center of the Earth is $d = 3R_{\text{E}}$. Therefore, the free fall acceleration of the ball is $g_{\text{b}} = \dfrac{GM_{\text{E}}}{\left(3R_{\text{E}}\right)^2} = \dfrac{1}{9}\dfrac{GM_{\text{E}}}{R_{\text{E}}^2} = \dfrac{g}{9}$.

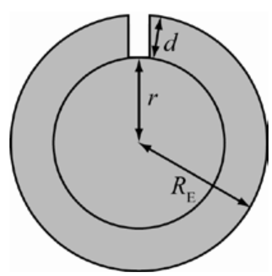
CALCULATE: $g_{\text{b}} = \dfrac{9.81 \text{ m/s}^2}{9} = 1.09 \text{ m/s}^2$

ROUND: Since the value of the gravitational acceleration on Earth is given to three significant figures, the free fall acceleration is 1.09 m/s^2.

DOUBLE-CHECK: It was expected that the free fall acceleration at this altitude is smaller than that at the surface of the Earth since it is inversely proportional to distance.

12.39. **THINK:** The gravitational acceleration decreases towards the center of the Earth since the exterior shell of mass no longer contributes a gravitational force. The equation for gravitational acceleration can be used and it can be assumed that the Earth has a uniform density, $\rho_{\text{E}} = M_{\text{E}} / V_{\text{E}}$.

SKETCH:



RESEARCH: The gravitational acceleration at the bottom of the mine shaft is given by $a = GM_{\text{int}} / r^2$, where M_{int} is the interior portion of Earth's mass.

SIMPLIFY: $M_{\text{int}} = \dfrac{4}{3}\pi r^3 \rho_{\text{E}} = \dfrac{4}{3}\pi r^3 \dfrac{M_{\text{E}}}{V_{\text{E}}} = \dfrac{(4/3)\pi r^3 M_{\text{E}}}{(4/3)\pi R_{\text{E}}^3} = M_{\text{E}}\dfrac{r^3}{R_{\text{E}}^3}$. Therefore, the gravitational acceleration is:

$$a = \frac{GM_{\text{E}}}{r^2}\frac{r^3}{R_{\text{E}}^3} = \frac{GM_{\text{E}}}{R_{\text{E}}^2}\left(\frac{r}{R_{\text{E}}}\right) = g\left(\frac{r}{R_{\text{E}}}\right).$$

CALCULATE:

(a) Substituting $a = g/2$ gives $r = R_E/2 = (6370\text{ km})/2 = 3185\text{ km}$. Therefore, the mine depth required for the gravitational acceleration to be reduced by a factor of 2 is $d = R_E - r = 3185\text{ km}$.

(b) The percentage difference of the gravitational acceleration at the bottom of 3.5 km deep mine relative to that at the Earth's surface is:

$$\frac{a_{surf} - a_{3.5km}}{a_{surf}} = \frac{g - \frac{R_E - d}{R_E}g}{g} = 1 - \left(1 - \frac{d}{R_E}\right) = \frac{d}{R_E} = \frac{3.5\text{ km}}{6370\text{ km}} = 5.495 \cdot 10^{-4}$$

ROUND:

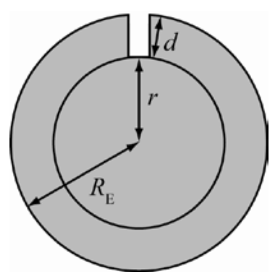
(a) To three significant figures, the depth required is 3190 km.

(b) To two significant figures the percentage difference is 0.055%.

DOUBLE-CHECK: Since 3.5 km is a relatively small distance into the surface of the Earth it is expected that the percentage change in the gravitational acceleration is very small.

12.40. **THINK:** The depth of a vertical shaft needs to be calculated using the equation for gravitational acceleration and projectile motion concepts developed in previous chapters. To measure the gravitational acceleration at the bottom of the shaft, a ball is tossed vertically with an initial speed $v_0 = 10.0\text{ m/s}$. The ball reaches a maximum height of 5.113 m. The projectile motion equations which assume constant acceleration can be used here since the maximum height is small (gravitational acceleration does not change much over such distances).

SKETCH:



RESEARCH: The gravitational acceleration at the bottom of the shaft is determined by $v^2 = v_0^2 - 2ax$, where $v = 0$ when the ball is at maximum height. Therefore, $a = v_0^2/2x$. This acceleration must be equal to the gravitational acceleration given by $a = GM_{int}/r^2$, where M_{int} is the interior portion of Earth's mass.

SIMPLIFY: $M_{int} = \frac{4}{3}\pi r^3 \rho_E = \frac{4}{3}\pi r^3 \frac{M_E}{V_E} = \frac{(4/3)\pi r^3 M_E}{(4/3)\pi R_E^3} = M_E \frac{r^3}{R_E^3}$.

Therefore, the gravitational acceleration is:

$$a = \frac{GM_E}{r^2}\frac{r^3}{R_E^3} = \frac{GM_E}{R_E^2}\left(\frac{r}{R_E}\right) = g\left(\frac{r}{R_E}\right) = g\left(\frac{R_E - d}{R_E}\right) = g\left(1 - \frac{d}{R_E}\right).$$

Since $a = v_0^2/2x$, $g\left(\frac{d}{R_E}\right) = g - \left(\frac{v_0^2}{2x}\right) \Rightarrow d = R_E\left(1 - \frac{v_0^2}{2gx}\right)$.

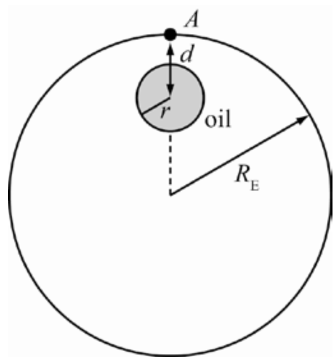
CALCULATE: $d = (6370\text{ km})\left(1 - \frac{(10.0\text{ m/s})^2}{2(9.81\text{ m/s}^2)(5.113\text{ m})}\right) = 20.133\text{ km}$

ROUND: The least precise value given in the question had three significant figures, so the final answer should also be rounded to three significant figures. The depth of the shaft is 20.1 km.

DOUBLE-CHECK: The unit of the depth is distance, as it should be, and the calculated value is plausible.

12.41. THINK: To determine the gravitational acceleration at a location on the surface of the Earth near a deposit of oil Newton's Law of Gravity needs to be used. Let $r = 1.00$ km, and $d = 2.00$ km.

SKETCH:



RESEARCH: From Newton's Law of Gravity, the gravitational acceleration is given by $a = GM/R^2$, where mass M can be expressed as $M = \rho V$. The density of the Earth is $\rho_E = 5500.\text{ kg/m}^3$ and the density of the oil is $\rho_{oil} = 900.\text{ kg/m}^3$. Instead of considering the Earth and the oil as one object, consider it as a superposition of two objects: the Earth with a uniform density and the Earth with a density given by $\rho = \rho_{oil} - \rho_E$, which is a negative density.

SIMPLIFY: The gravitational acceleration at the location above the oil deposit is:

$a = \dfrac{G\rho_E V_E}{R_E^2} + \dfrac{G(\rho_{oil} - \rho_E)V_{oil}}{d^2}$. Using $V_E = \dfrac{4}{3}\pi R_E^3$ and $V_{oil} = \dfrac{4}{3}\pi r^3$ gives

$$(1) \quad a = \frac{4G\rho_E \pi R_E^3}{3R_E^2} + \frac{4G(\rho_{oil} - \rho_E)\pi r^3}{3d^2} = \frac{4\pi G}{3}\left[\rho_E\left(R_E - \left(\frac{r}{d}\right)^2 r\right) + \rho_{oil}\left(\frac{r}{d}\right)^2 r\right].$$

The ratio of this acceleration relative to the acceleration due to a uniform Earth (the first term in equation (1) is given by:

$$\text{Ratio} = \frac{a}{a_E} = \frac{\frac{4\pi G}{3}\left[\rho_E\left(R_E - \left(\frac{r}{d}\right)^2 r\right) + \rho_{oil}\left(\frac{r}{d}\right)^2 r\right]}{\frac{4}{3}\pi G \rho_E R_E}$$

$$= 1 - \left(\frac{r}{d}\right)^2\left(\frac{r}{R_E}\right) + \frac{\rho_{oil}}{\rho_E}\left(\frac{r}{d}\right)^2\left(\frac{r}{R_E}\right)$$

$$= 1 - \left(1 - \frac{\rho_{oil}}{\rho_E}\right)\left(\frac{r}{d}\right)^2\left(\frac{r}{R_E}\right)$$

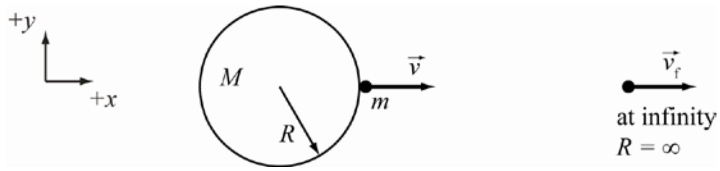
CALCULATE: $\text{Ratio} = 1 - \left(1 - \dfrac{900.\text{ kg/m}^3}{5500.\text{ kg/m}^3}\right)\left(\dfrac{1.00\text{ km}}{2.00\text{ km}}\right)^2\left(\dfrac{1.00\text{ km}}{6370\text{ km}}\right) = 1 - 3.2824\cdot 10^{-5}$

ROUND: To three significant figures the ratio is $0.99997 \approx 1.00$. The fractional deviation, rounded to three significant digits, is $(g_{\text{no oil}} - g_{\text{oil}})/g_{\text{no oil}} = 3.28\cdot 10^{-5}$.

DOUBLE-CHECK: It is expected that the ratio of the gravitational acceleration with and without oil would be very close to 1 since the volume of oil is much smaller than the volume of the Earth. It becomes clear from this exercise that finding even very large oil deposits as the one in this problem would require extremely precise measurements of the gravitational acceleration in order to have any chance of success.

12.42. **THINK:** The speed of a spaceship is needed as its distance from the Earth approaches infinity. The initial speed of the spaceship on the surface of the Earth is v. The principle of conservation of energy and the equation for gravitational potential energy can be used to find the final speed. The gravitational potential energy at infinity can be set to zero.

SKETCH:



RESEARCH: From the principle of conservation of energy, $E_i = E_f$,

$$K_i + U_i = K_f + U_f$$

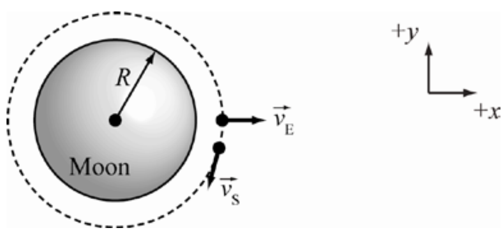
$$\frac{1}{2}mv^2 - \frac{GMm}{R} = \frac{1}{2}mv_f^2 + 0.$$

SIMPLIFY: Solving for v_f, it is found that $v_f^2 = v^2 - \frac{2GM}{R}$ or $v_f = \sqrt{v^2 - \frac{2GM}{R}}$.

CALCULATE: No numerical values were given. The solution is algebraic.
ROUND: No rounding is necessary since the answer is algebraic.
DOUBLE-CHECK: From the equation it is seen that the final speed is less than the initial speed. This is expected since the spaceship is gaining gravitational potential energy as it moves away from the Earth.

12.43. **THINK:** The ratio of the escape speed to the satellite speed at the surface of the Moon is needed. The principle of conservation of energy can be used to solve this problem.

SKETCH:



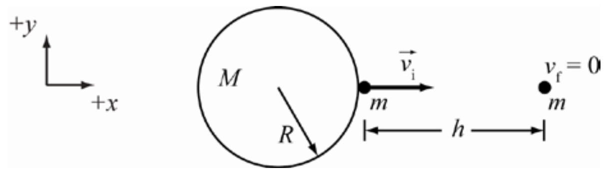
RESEARCH: The escape speed v_E is found by using conservation of energy, $E_i = E_f$, at the surface of the Moon and at infinity. Because the speed at infinity is zero and the gravitational potential energy is taken to be zero at infinity, the final energy is also zero. Therefore, $(1/2)mv_E^2 - (GMm/R) = 0$. This gives $v_E = \sqrt{2GM/R}$. The satellite speed v_s is obtained from the condition that the gravitational acceleration is equal to the centripetal acceleration; that is, $GM/R^2 = v_s^2/R$. This yields $v_s = \sqrt{GM/R}$.

SIMPLIFY: Therefore, the ratio of the escape speed to the satellite speed is $\frac{v_E}{v_s} = \frac{\sqrt{2GM/R}}{\sqrt{GM/R}} = \sqrt{2}$.

CALCULATE: No numerical values were given. The solution is algebraic.
ROUND: No rounding is necessary since the answer is algebraic.
DOUBLE-CHECK: In our algebraic simplification, notice that the mass and radius of the Moon dropped out of the equation, and the escape speed was simply a factor of $\sqrt{2}$ larger than the orbital speed. This result confirms the universal result obtained in equation (12.22) in the textbook.

12.44. **THINK:** An astronaut throws a rock with a mass $m = 2.00$ kg and a speed $v_\text{i} = 40.0$ m/s. The astronaut is on a small spherical moon whose radius is $6.30 \cdot 10^4$ m and mass is $8.00 \cdot 10^{18}$ kg. The principle of conservation of energy and the equation of gravitational potential energy can be used to find the maximum height.

SKETCH:



RESEARCH: At the maximum height, the kinetic energy of the rock is zero. Using the principle of conservation of energy, the total energy of the rock on the surface of the moon must be equal to its total energy at the maximum height, that is:

$$K_\text{i} + U_\text{i} = K_\text{f} + U_\text{f}$$

$$\frac{1}{2}mv_\text{i}^2 - \frac{GMm}{R} = 0 - \frac{GMm}{R+h}.$$

SIMPLIFY: Canceling the factor m, and then rearranging the above equation gives $R + h = \dfrac{GM}{\dfrac{GM}{R} - \dfrac{1}{2}v_\text{i}^2}$.

Therefore, the maximum height is $h = \dfrac{GM}{\dfrac{GM}{R} - \dfrac{1}{2}v_\text{i}^2} - R.$

CALCULATE: The maximum height is

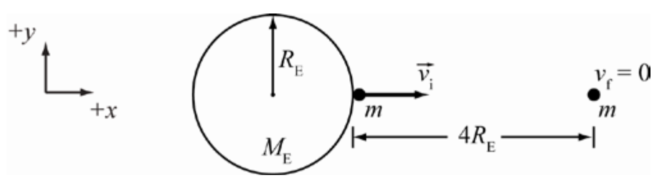
$$h = \frac{\left(6.67 \cdot 10^{-11} \text{ Nm}^2/\text{kg}^2\right)\left(8.00 \cdot 10^{18} \text{ kg}\right)}{\dfrac{\left(6.67 \cdot 10^{-11} \text{ Nm}^2/\text{kg}^2\right)\left(8.00 \cdot 10^{18} \text{ kg}\right)}{6.30 \cdot 10^4 \text{ m}} - \dfrac{1}{2}\left(40.0 \text{ m/s}\right)^2} - 6.30 \cdot 10^4 \text{ m} = 6.5712 \cdot 10^3 \text{ m}.$$

ROUND: To three significant figures, the maximum height of the rock is $h = 6.57 \cdot 10^3$ m.

DOUBLE-CHECK: In Solved Problem 12.1 the inverse problem is solved for the same small moon. In that case the final height of 2.2 km was specified, and the initial speed was calculated as 23.9 m/s. Here we find a final height of 6.57 km for an initial velocity of 40 m/s. These two pairs of numbers are consistent with each other.

12.45. **THINK:** In order to find the minimum speed required for an object to reach a height of $4R_\text{E}$, the principle of conservation of energy and the equation for gravitational potential energy can be used.

SKETCH:



RESEARCH: Using the principle of conservation energy, $\text{E}_\text{i} = \text{E}_\text{f}$, it is found that

$$K_\text{i} + U_\text{i} = K_\text{f} + U_\text{f}$$

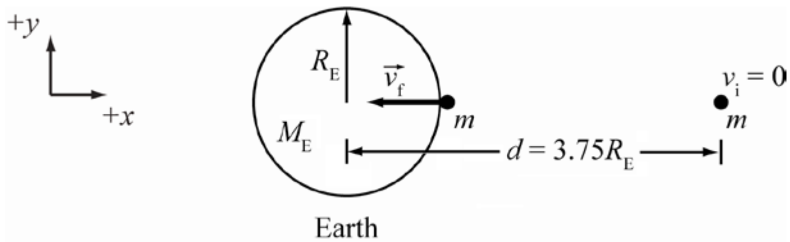
$$\frac{1}{2}mv_\text{i}^2 - \frac{GM_\text{E}m}{R_\text{E}} = \frac{1}{2}mv_\text{f}^2 - \frac{GM_\text{E}m}{5R_\text{E}}.$$

SIMPLIFY: The minimum speed v_{i} is when $v_{\text{f}}=0.$ This leads to

$$v_{\text{min}}^2=2GM_{\text{E}}\left(\frac{1}{R_{\text{E}}}-\frac{1}{5R_{\text{E}}}\right)=2GM_{\text{E}}\left(\frac{4}{5R_{\text{E}}}\right)\Rightarrow v_{\text{min}}=\sqrt{\frac{8GM_{\text{E}}}{5R_{\text{E}}}}.$$

CALCULATE: The requested equation has been derived.
ROUND: Not required.
DOUBLE-CHECK: Since the object only reaches a height of $4R_{\text{E}}$, it has not escaped the gravitational pull of the Earth. This would only be achieved at the escape speed. Therefore, it was expected that v_{min} is less than the escape speed from Earth, given by $v_{\text{E}}=\sqrt{2GM_{\text{E}}/R_{\text{E}}}$.

12.46. **THINK:** A satellite is orbiting the Earth at a distance of $3.75R_{\text{E}}$, with a speed of 4.08 km/s. The speed of the satellite when it impacts the Earth is required. It is assumed that the initial speed of the satellite is $v_{\text{i}}=0$ (since it was suddenly stopped).
SKETCH:



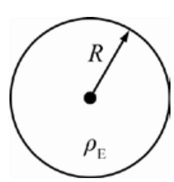
RESEARCH: To determine the final speed of the satellite when it impacts the Earth, the principle of conservation energy is used, $E_{\text{i}}=E_{\text{f}}$. Thus $(1/2)mv_{\text{i}}^2-(GM_{\text{E}}m/3.75R_{\text{E}})=(1/2)mv_{\text{f}}^2-(GM_{\text{E}}m/R_{\text{E}})$.

SIMPLIFY: Since $v_{\text{i}}=0$, this simplifies to $v_{\text{f}}=\sqrt{(2GM_{\text{E}}/R_{\text{E}})(1-1/3.75)}$.
CALCULATE: The final speed is:

$$v_{\text{f}}=\sqrt{\frac{2\left(6.67\cdot 10^{-11}\text{ N m}^2/\text{kg}^2\right)\left(5.98\cdot 10^{24}\text{ kg}\right)}{6.37\cdot 10^{6}\text{ m}}\left(1-\frac{1}{3.75}\right)}=9.5832\cdot 10^{3}\text{ m/s}$$

ROUND: Three significant figures: $v_{\text{f}}=9.58\cdot 10^{3}$ m/s
DOUBLE-CHECK: It is expected that an object falling to Earth from such a great distance would achieve a very large speed, particularly since the effects of air resistance are being ignored.

12.47. **THINK:** To solve this problem, a reasonable value of the initial speed of the jump is needed. It is assumed that the density ρ_{E} of the asteroid is uniform.
SKETCH:



RESEARCH: Take the maximum height of a jump on Earth to be $x=1.0$ m. Using this height, the initial speed can be found by using $v_{\text{f}}^2=v^2-2gx$ (g can be assumed constant over a distance of 1.0 m). At maximum height on Earth, $v_{\text{f}}=0$ so: $v=\sqrt{2gx}$. The escape speed from the asteroid is given by $v=\sqrt{2GM/R}$.

SIMPLIFY: Setting the equations equal gives: $2gx=\dfrac{2GM}{R}\Rightarrow R=\dfrac{GM}{gx}$. The mass of the asteroid is $M=\dfrac{4}{3}\pi R^3\rho_{\text{E}}=M_{\text{E}}\left(\dfrac{R}{R_{\text{E}}}\right)^3$. Therefore, $R=\dfrac{R^3}{R_{\text{E}}gx}\left(\dfrac{GM_{\text{E}}}{R_{\text{E}}^2}\right)=\dfrac{R^3}{xR_{\text{E}}}\Rightarrow R=\sqrt{xR_{\text{E}}}$.

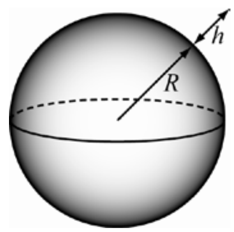
CALCULATE: $R=\sqrt{(1.0\text{ m})(6.37\cdot 10^6\text{ m})}=2.524$ km

ROUND: To two significant figures the largest asteroid that one could escape from by jumping is about $R=2.5$ km.

DOUBLE-CHECK: The initial velocity of a 1.0 m jump on Earth is $v=\sqrt{2(9.81\text{ m/s}^2)(1.0\text{ m})}=4.4$ m/s. Since this is a small velocity, it is reasonable that this would be the escape speed from a very small asteroid.

12.48. **THINK:** The radius and surface gravity of Eris are known, $R=1200$ km and $g=0.77\text{ m/s}^2$. The escape velocity can be found by using the equation for gravitational acceleration. By using the principle of conservation of energy, the maximum height can be found for an initial speed of half of the escape speed.

SKETCH:



RESEARCH:

(a) The escape speed is given by $v=\sqrt{2GM/R}$ while the surface gravity is $a_g=GM/R^2$.

(b) In order to find the height, the conservation of energy is useful:

$$K_{\text{i}}+U_{\text{i}}=K_{\text{f}}+U_{\text{f}}$$

$$\frac{1}{2}mv_{\text{i}}^2-\frac{GMm}{R}=\frac{1}{2}mv_{\text{f}}^2-\frac{GMm}{R+h}.$$

SIMPLIFY:

(a) The escape speed is $v_{\text{e}}=\sqrt{\dfrac{2GM}{R}}=\sqrt{2R\dfrac{GM}{R^2}}=\sqrt{2gR}$.

(b) At the maximum height the velocity is zero, $v_{\text{f}}=0$. The conservation of energy gives $\dfrac{1}{2}v_{\text{i}}^2-\dfrac{GM}{R}=-\dfrac{GM}{R+h}$. Substituting $v_{\text{i}}=\dfrac{1}{2}v_{\text{e}}=\dfrac{1}{2}\sqrt{\dfrac{2GM}{R}}=\sqrt{\dfrac{GM}{2R}}$ and solving for h yields:

$$\frac{1}{2}\frac{GM}{2R}-\frac{GM}{R}=-\frac{GM}{R+h}\Rightarrow\frac{1}{4R}-\frac{1}{R}=-\frac{1}{R+h}\Rightarrow-\frac{3}{4R}=-\frac{1}{R+h}\Rightarrow h=\frac{4}{3}R-R$$

Therefore, the maximum height above the surface is $h=\dfrac{1}{3}R.$

CALCULATE:

(a) $v_{\text{e}}=\sqrt{2(0.77\text{ m/s}^2)(1200\cdot 10^3\text{ m})}=1359.41$ m/s

(b) $h=\dfrac{1}{3}(1200\text{ km})=400.$ km

ROUND: The answers are limited to two significant figures since both values have this accuracy.

(a) The escape speed of Eris is 1400 m/s.

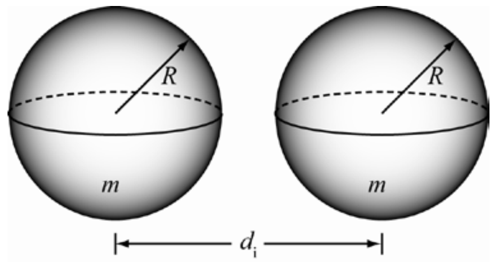
(b) The maximum height attained for half the escape velocity is $4.0\cdot 10^2$ km above the surface of Eris.

DOUBLE-CHECK: Earth has an escape velocity around 11000 m/s which is greater than Eris, as expected since Earth is much more massive.

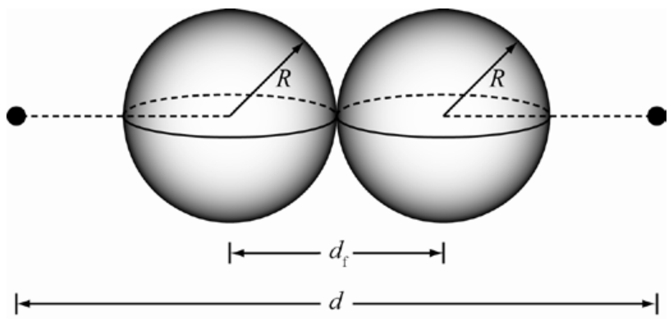
12.49. **THINK:** There are two objectives for this question. The first is to find the speed at which the two spheres collide. The second is to find the energy that is required to separate the two spheres by 1.00 m once they are in contact. The spheres each have a mass of $m = 20.0$ kg with a radius of $R = 10.0$ cm and are separated by $d_{\text{i}} = 30.0$ cm. The spheres are initially at rest.

SKETCH:

(a)



(b)



RESEARCH: To find the speed of the spheres, use the principle of conservation of energy, $\text{E}_{\text{i}} = \text{E}_{\text{f}}$. Note that the energy is the sum of the kinetic and potential energies. Conservation of energy can also be used to find the energy required to separate the spheres.

SIMPLIFY:

(a) Since there are two spheres that share the decrease in gravitational potential energy as they fall toward each other, the principle of conservation of energy gives:

$$K_{\text{i}} + U_{\text{i}} = K_{\text{f}} + U_{\text{f}}$$

$$0 - \frac{Gm^2}{d_{\text{i}}} = 2\left(\frac{1}{2}mv_{\text{f}}^2\right) - \frac{Gm^2}{d_{\text{f}}}.$$

Solving for the final velocity:

$$v_{\text{f}}^2 = Gm\left(\frac{1}{d_{\text{f}}} - \frac{1}{d_{\text{i}}}\right) \Rightarrow v_{\text{f}} = \sqrt{Gm\left(\frac{1}{d_{\text{f}}} - \frac{1}{d_{\text{i}}}\right)}.$$

(b) The energy to separate the two spheres is equal to the change in potential energy:

$$E = \Delta U = Gm^2\left(\frac{1}{d_{\text{f}}} - \frac{1}{d}\right).$$

CALCULATE:

(a) The final velocity is of each sphere is:

$$v_{\text{f}} = \sqrt{\left(6.674 \cdot 10^{-11} \text{ N m}^2/\text{kg}^2\right)\left(20.0 \text{ kg}\right)\left(\frac{1}{0.200 \text{ m}} - \frac{1}{0.300 \text{ m}}\right)} = 4.7166 \cdot 10^{-5} \text{ m/s}.$$

(b) The energy required to separate the spheres is:

$$E = \Delta U = \left(6.674 \cdot 10^{-11}\ \text{N m}^2/\text{kg}^2\right)\left(20.0\ \text{kg}\right)^2\left(\frac{1}{0.200\ \text{m}} - \frac{1}{1.00\ \text{m}}\right) = 1.0678 \cdot 10^{-7}\ \text{J}.$$

ROUND: Rounding will be done to three significant figures.

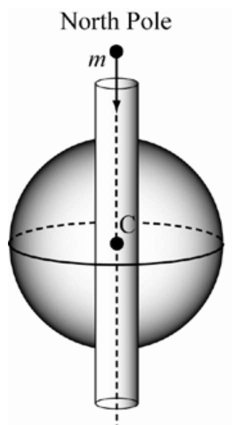
(a) The velocity of a ball when it touches the other is $v_\text{f} = 4.72 \cdot 10^{-5}$ m/s.

(b) The energy needed to separate the balls by 1.00 m is $E = 1.07 \cdot 10^{-7}$ J.

DOUBLE-CHECK: In our day-to-day life we don't see 20.0 kg balls moving towards each other. This means the calculated values should be very small which they are.

12.50. **THINK:** The object of this question is to find the potential energy of a ball with mass 5.00 kg, and the speed at which it passes through the center of the Earth. The speed of the ball at the center is found using the principle of the conservation of energy.

SKETCH:



RESEARCH: As seen in this chapter, the force on the ball inside the Earth is $\text{F} = (-4/3)\pi G\rho_\text{E} mr$. This is related to the potential energy by $U = -\int \vec{F} \cdot d\vec{r}$.

SIMPLIFY: The potential energy is $U(r) - U(0) = -\int_0^r \left(-\frac{4}{3}\right)\pi G\rho_\text{E} mr\,dr = \frac{2}{3}\pi G\rho_\text{E} mr^2\Big|_0^r = \frac{2}{3}\pi G\rho_\text{E} mr^2.$

The potential at the center is defined as zero so $U(r) = \frac{2}{3}\pi G\rho_\text{E} mr^2$. The speed at the center of the Earth is found by solving the following for v:

$$K_\text{i} + U_\text{i} = K_\text{f} + U_\text{f}$$

$$0 + \frac{2}{3}\pi G\rho_\text{E} mR_\text{E}^2 = \frac{1}{2}mv^2 + 0 \Rightarrow v^2 = \frac{4}{3}\pi G\rho_\text{E} R_\text{E}^2 \Rightarrow v = \sqrt{\frac{4}{3}\pi G\rho_\text{E} R_\text{E}^2}$$

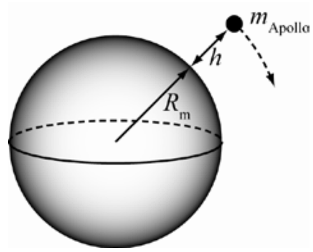
CALCULATE: $v = \sqrt{\frac{4}{3}\pi\left(6.674 \cdot 10^{-11}\ \text{N m}^2/\text{kg}^2\right)\left(5.5 \cdot 10^3\ \text{kg/m}^3\right)\left(6370 \cdot 10^3\ \text{m}\right)^2} = 7898.75\ \text{m/s}$

ROUND: The equation of the potential as a function of the distance from the center of the Earth is $U(r) = (2/3)\pi G\rho_\text{E} mr^2$. The speed of the ball has the same accuracy as the density of the Earth. The ball reaches a speed of 7900 m/s at the center of the Earth.

DOUBLE-CHECK: The ball is traveling extremely fast; about 23 times faster than the speed of sound. This is a reasonable value because of the large distance travelled to the center of the Earth.

12.51. **THINK:** The orbit is 111 km above the surface of the Moon. The Moon has a mass and radius of $M_{\text{m}} = 7.35 \cdot 10^{22}$ kg and $R_{\text{m}} = 1.74 \cdot 10^{6}$ m, respectively.

SKETCH:



RESEARCH: Using Kepler's third law, the period is $T = 2\pi\sqrt{\dfrac{r^3}{GM}}$.

SIMPLIFY: $T = 2\pi\sqrt{\dfrac{(R_{\text{m}}+h)^3}{GM_{\text{m}}}}$

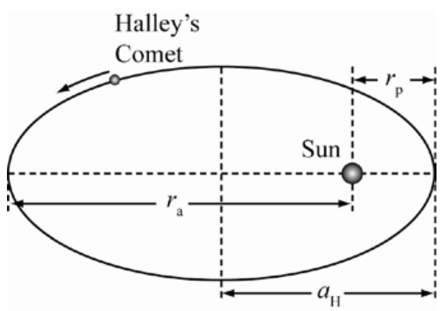
CALCULATE: $T = 2\pi\sqrt{\dfrac{\left(1.74\cdot 10^{6}\text{ m} + 0.111\cdot 10^{6}\text{ m}\right)^3}{\left(6.674\cdot 10^{-11}\text{ N m}^2/\text{kg}^2\right)\left(7.35\cdot 10^{22}\text{ kg}\right)}} = 7144.19\text{ s}$

ROUND: To three significant figures, the period of this orbit is 7140 s.

DOUBLE-CHECK: This trip of about 2 hours around the Moon is fast, but reasonable since the orbit is so close to the surface.

12.52. **THINK:** The goal is to find the semi-major axis, the maximum distance from the Sun, and the eccentricity of the orbit of Halley's comet. The comet has a period of $T_{\text{H}} = 75.3$ years and the perihelion of its orbit is $r_{\text{p}} = 0.586$ AU.

SKETCH:



RESEARCH: The period and semi-major axis of the orbits of the Earth and the comet are related by Kepler's third law: $T_{\text{E}}^2/a_{\text{E}}^3 = T_{\text{H}}^2/a_{\text{H}}^3$. The sum of the distances at perihelion and aphelion is equal to twice the semi-major axis $r_{\text{a}} + r_{\text{p}} = 2a_{\text{H}}$. The eccentricity can be calculated with the equation $r_{\text{p}} = (1-e)a_{\text{H}}$.

SIMPLIFY:

(a) The semi-major axis of the orbit is $a_{\text{H}}^3 = \left(\dfrac{T_{\text{H}}^2}{T_{\text{E}}^2}\right)a_{\text{E}}^3 \Rightarrow a_{\text{H}} = a_{\text{E}}\sqrt[3]{\dfrac{T_{\text{H}}^2}{T_{\text{E}}^2}}$.

(b) The maximum distance from the Sun occurs at aphelion, $r_a = 2a_H - r_p = 2a_E \sqrt[3]{\frac{T_H^2}{T_E^2}} - r_p$, and the eccentricity of the orbit is given by $1 - e = \frac{r_p}{a_H} \Rightarrow e = 1 - \frac{r_p}{a_H} = 1 - \frac{r_p}{a_E}\sqrt[3]{\frac{T_E^2}{T_H^2}}$.

CALCULATE:

(a) Semi-major axis: $a_H = (1\text{ AU})\left(\frac{(75.3\text{ yr})^2}{(1\text{ yr})^2}\right)^{\frac{1}{3}} = 17.8319\text{ AU}$,

(b) Maximum distance at aphelion: $r_a = 2(1\text{ AU})\left(\frac{(75.3\text{ yr})^2}{(1\text{ yr})^2}\right)^{\frac{1}{3}} - (0.586\text{ AU}) = 35.0777\text{ AU}$, and the eccentricity: $e = 1 - \left(\frac{0.586\text{AU}}{1\text{ AU}}\right)\left(\frac{(1\text{ yr})^2}{(75.3\text{ yr})^2}\right)^{\frac{1}{3}} = 0.967137$.

ROUND:

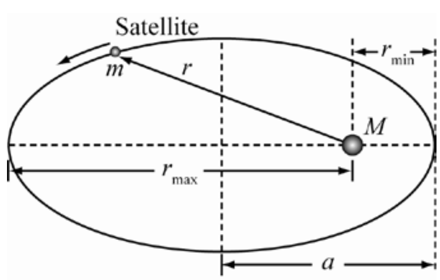
(a) To three significant figures the semi-major axis of Halley's comet is $a_H = 17.8$ AU.

(b) To three significant figures, the distance at aphelion is $r_a = 35.1$ AU and the eccentricity of the orbit is $e = 0.967$.

DOUBLE-CHECK: The measured values for Halley's comet are: $a_H = 17.8$ AU, and $r_a = 35.1$ AU, and $e = 0.967$. So our results seem reasonable.

12.53. **THINK:** The goal of this question is to derive expressions for the semi-major axis and the eccentricity in terms of the energy and angular momentum of the satellite. The principle of the conservation of energy and angular momentum can be used here. The object has a mass of *m* and orbits another body of mass *M*, such that $M >> m$.

SKETCH:



RESEARCH:

(a) $E = K + U$ with $K = (1/2)mv^2$ and $U = -GMm/r$.

(b) At the extremes of the orbit, the angular momentum is given by $L = mrv$, so $v = L/(mr)$.

(c) Multiply the equation for energy that will be obtained in part (b) by the square of the radius. This will result in a quadratic equation in terms of the radius. Its two roots will be the maximum and minimum radii.

(d) The minimum and maximum radii are related to the semi-major axis by the equations $r_{max} = a(1+e)$ and $r_{min} = a(1-e)$. The semi-major axis is found by using the equation $a(1+e) + a(1-e) = r_{max} + r_{min}$. The eccentricity is found by $e = (r_{max} - r_{min})/(2a)$.

SIMPLIFY:

(a) The energy of the satellite is $E = \frac{1}{2}mv^2 - \frac{GMm}{r}.$

(b) Eliminating v allows the energy at the extreme distances to be written in terms of the angular momentum:

$$E = \frac{1}{2}m\left(\frac{L}{mr}\right)^2 - \frac{GMm}{r} = \frac{L^2}{2mr^2} - \frac{GMm}{r}.$$

(c) $Er^2 = \frac{L^2}{2m} - GMmr,$ which is equivalent to $Er^2 + GMmr - \frac{L^2}{2m} = 0.$ Solving for the radius:

$$r = \frac{-GMm \pm \sqrt{G^2M^2m^2 - 4E\left(-\frac{L^2}{2m}\right)}}{2E} = \frac{-GMm \pm \sqrt{G^2M^2m^2 + \frac{2L^2E}{m}}}{2E}.$$

Since $E < 0,$

$$r_{\min} = \frac{-GMm + \sqrt{G^2M^2m^2 + \frac{2L^2E}{m}}}{2E} = -\frac{GMm}{2E} + \sqrt{\frac{G^2M^2m^2}{4E^2} + \frac{2L^2E}{4mE^2}}$$

$$r_{\min} = -\frac{GM}{2(E/m)} + \sqrt{\frac{G^2M^2}{\left(2(E/m)\right)^2} + \frac{(L/m)^2}{2(E/m)}}$$

and

$$r_{\max} = \frac{-GMm - \sqrt{G^2M^2m^2 + \frac{2L^2E}{m}}}{2E} = -\frac{GMm}{2E} - \sqrt{\frac{G^2M^2m^2}{4E^2} + \frac{2L^2E}{4mE^2}}$$

$$r_{\max} = -\frac{GM}{2(E/m)} - \sqrt{\frac{G^2M^2}{\left(2(E/m)\right)^2} + \frac{(L/m)^2}{2(E/m)}}.$$

(d) $2a = r_{\max} + r_{\min}$ and $e = (r_{\max} - r_{\min})/(r_{\max} + r_{\min})$. Substituting the equations from part (c) yields:

$$a = \frac{1}{2}\left[\frac{\left(-GMm - \sqrt{G^2M^2m^2 + \frac{2L^2E}{m}}\right)}{2E} + \frac{\left(-GMm + \sqrt{G^2M^2m^2 + \frac{2L^2E}{m}}\right)}{2E}\right] = \frac{1}{2}\left[-\frac{2GMm}{2E}\right] = -\frac{GMm}{2E}$$

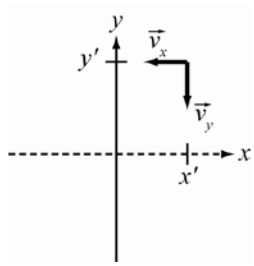
$$e = \frac{\left[\frac{\left(-GMm - \sqrt{G^2M^2m^2 + \frac{2L^2\mathrm{E}}{m}}\right)}{2\mathrm{E}} - \frac{\left(-GMm + \sqrt{G^2M^2m^2 + \frac{2L^2\mathrm{E}}{m}}\right)}{2\mathrm{E}}\right]}{\left(\frac{-GMm}{E}\right)} = \frac{\sqrt{\left(G^2M^2m^2 + \frac{2L^2\mathrm{E}}{m}\right)}}{GMm}$$

Thus, the semi-major axis and the eccentricity in terms of the angular momentum per unit mass and energy per unit mass are $a = -\frac{GMm}{2E} = -\frac{GM}{2(E/m)}$ and

$$e=\sqrt{1+\frac{2L^2E}{G^2M^2m^3}}=\sqrt{1+\frac{(L/m)^2\left(2(E/m)\right)}{G^2M^2}}.$$

CALCULATE: Since the result is entirely algebraic, there are no calculations to perform.
ROUND: Since the result is entirely algebraic, no rounding is necessary.
DOUBLE-CHECK: The formula for the eccentricity of the orbit will always produce a result less than 1, since $E<0$. The formula for the semi-major axis will always produce a positive number as well, and the formula matches that derived in the text.

12.54. **THINK:** The goal is to find the speed and distance of an asteroid at closest approach to the Sun. At a given time, the velocity and position of the asteroid are $\left(-9.00\cdot 10^3\text{ m/s},-7.00\cdot 10^3\text{ m/s}\right)$ and $\left(2.00\cdot 10^{11}\text{ m},3.00\cdot 10^{11}\text{ m}\right)$, respectively.
SKETCH:



RESEARCH: Two equations are required to find the velocity and the distance at closest approach, since there are two unknowns. Both energy and angular momentum are conserved in the orbit. The total energy is $E=\frac{1}{2}mv^2-\frac{GMm}{r}$ and the angular momentum is $\vec{L}=m\vec{r}\times\vec{v}$.
SIMPLIFY: The conservation of energy gives the first equation:

$$\frac{1}{2}m\left(v_{0x}^2+v_{0y}^2\right)-\left(\frac{GMm}{\sqrt{r_{0x}^2+r_{0y}^2}}\right)=\frac{1}{2}mv_{\text{max}}^2-\left(\frac{GMm}{r_{\text{min}}}\right)\;\Rightarrow\;\frac{1}{2}\left(v_{0x}^2+v_{0y}^2\right)-\frac{GM}{\sqrt{r_{0x}^2+r_{0y}^2}}=\frac{1}{2}v_{\text{max}}^2-\frac{GM}{r_{\text{min}}}.$$

Conservation of angular momentum gives the second equation:

$$\left|\vec{L}\right|=m\left|\vec{r}\times\vec{v}\right|=m\left(r_{0x}v_{0y}-r_{0y}v_{0x}\right)=mr_{\text{min}}v_{\text{max}}\;\Rightarrow\;r_{\text{min}}v_{\text{max}}=\left(r_{0x}v_{0y}-r_{0y}v_{0x}\right).$$

With two equations, it is now possible to solve for r_{min} and v_{max}. To simplify, let

$$\varepsilon=\frac{1}{2}\left(v_{0x}^2+v_{0y}^2\right)-\frac{GM}{\sqrt{r_{0x}^2+r_{0y}^2}}\text{ and }L\equiv\left(r_{0x}v_{0y}-r_{0y}v_{0x}\right).$$

So the equations are $\varepsilon=(1/2)v_{\text{max}}^2-GM/r_{\text{min}}$ and $L=r_{\text{min}}v_{\text{max}}$. Plug $r_{\text{min}}=L/v_{\text{max}}$ into the energy equation $\frac{1}{2}v_{\text{max}}^2-\frac{GM}{L}v_{\text{max}}-\varepsilon=0$. This is a quadratic equation with solution:

$$v_{\text{max}}=\frac{\frac{GM}{L}+\sqrt{\frac{G^2M^2}{L^2}-4\left(\frac{1}{2}\right)(-\varepsilon)}}{2\left(\frac{1}{2}\right)}=\frac{GM}{L}+\sqrt{\frac{G^2M^2}{L^2}+2\varepsilon},$$

where the positive sign was chosen because this gives the maximum value, and

$$r_{\text{min}}=\frac{L}{v_{\text{max}}}=\frac{L^2}{GM+\sqrt{G^2M^2+2\varepsilon L^2}}.$$

CALCULATE:

$$\varepsilon = \frac{1}{2}\left[\left(-9.00\cdot10^{3}\text{ m/s}\right)^{2} + \left(-7.00\cdot10^{3}\text{ m/s}\right)^{2}\right] - \frac{\left(6.674\cdot10^{-11}\text{ N m}^{2}/\text{kg}^{2}\right)\left(1.99\cdot10^{30}\text{ kg}\right)}{\sqrt{\left(2.00\cdot10^{11}\text{ m}\right)^{2} + \left(3.00\cdot10^{11}\text{ m}\right)^{2}}} = -3.03356\cdot10^{8}\text{ J/kg}$$

$$L = \left[\left(2.00\cdot10^{11}\text{ m}\right)\left(-7.00\cdot10^{3}\text{ m/s}\right)\right] - \left[\left(3.00\cdot10^{11}\text{ m}\right)\left(-9.00\cdot10^{3}\text{ m/s}\right)\right] = 1.30\cdot10^{15}\text{ m}^{2}/\text{s}$$

$$v_{\text{max}} = \frac{\left(6.674\cdot10^{-11}\text{N m}^{2}/\text{kg}^{2}\right)\left(1.99\cdot10^{30}\text{ kg}\right)}{1.30\cdot10^{15}\text{ m}^{2}/\text{s}} + \sqrt{\frac{\left(6.674\cdot10^{-11}\text{N m}^{2}/\text{kg}^{2}\right)^{2}\left(1.99\cdot10^{30}\text{ kg}\right)^{2}}{\left(1.30\cdot10^{15}\text{ m}^{2}/\text{s}\right)^{2}} + 2\left(-3.03356\cdot10^{8}\text{ J/kg}\right)}$$

$$= 2.0131\cdot10^{5}\text{ m/s}$$

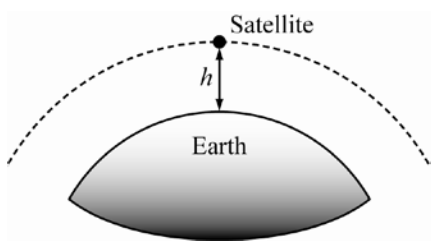
$$r_{\text{min}} = \frac{1.30\cdot10^{15}\text{ m}^{2}/\text{s}}{2.013112\cdot10^{5}\text{ m/s}} = 6.4577\cdot10^{9}\text{ m}$$

ROUND: To three significant figures, the speed and position of closest approach are $v = 2.01\cdot10^{5}$ m/s and $r = 6.46\cdot10^{9}$ m.

DOUBLE-CHECK: The calculated distance of closest approach is less than the given distance $\left(3.6\cdot10^{11}\text{ m}\right)$. Due to conservation of angular momentum, at the distance of closest approach the speed will be a maximum. Since the calculated speed is greater than the magnitude of the given velocity, the results are reasonable.

12.55. **THINK:** The goal is to find the orbital speed and period of a satellite $h = 700.$ km above the Earth. The speed of a circular orbit is found by setting the force of gravity equal to the centripetal force.

SKETCH:



RESEARCH: The orbital speed is equal to $v = \sqrt{GM/r}$, where the radius r is the sum of the radius of the Earth and the height above the surface $r = R_{\text{e}} + h$. The period is just the distance travelled divided by the speed: $T = \frac{2\pi r}{v}$.

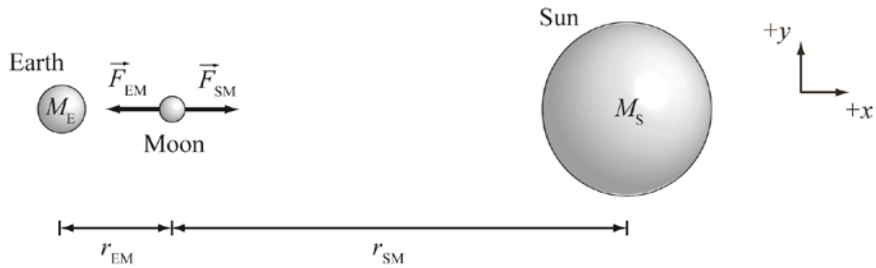
SIMPLIFY: $v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{R_{\text{e}} + h}}$ and $T = \frac{2\pi r}{v}$.

CALCULATE: $v = \sqrt{\frac{\left(6.67\cdot10^{-11}\text{ N m}^{2}/\text{kg}^{2}\right)\left(5.97\cdot10^{24}\text{ kg}\right)}{\left(6370 + 700.\right)\cdot10^{3}\text{ m}}} = 7504.82\text{ m/s}$ and

$$T = 2\pi\frac{\left(6370 + 700.\right)\cdot10^{3}\text{ m}}{7504.82\text{ m/s}} = 5919.15\text{ s}$$

ROUND: To three significant figures, the orbital speed and period are $v = 7.50$ km/s and $T = 5920$ s.
DOUBLE-CHECK: A speed of 7.50 km/s is a reasonable value for a satellite in orbit. Notice that the period $T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{GM}} = 2\pi\sqrt{\frac{r^3}{GM}}$, which is Kepler's third law.

12.56. **THINK:** The goal is to find the ratio of the gravitational force on the Moon due to the Earth to the gravitational force on the Moon due to the Sun and to determine why the Sun doesn't pull the Moon away from the Earth.
SKETCH:

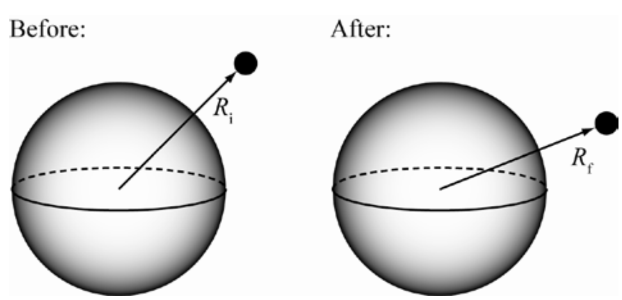


RESEARCH: The force of gravity is given by $F = GMm/r^2$.
SIMPLIFY: The ratio then is $\frac{F_{EM}}{F_{SM}} = \left(\frac{GM_E m_M}{r_{EM}^2}\right) / \left(\frac{GM_S m_M}{r_{SM}^2}\right) = \frac{M_E r_{SM}^2}{M_S r_{EM}^2}$.
CALCULATE: From Table 12.1, you can see that the ratio of the masses of Earth and Sun is $M_E/M_E = 3\cdot 10^{-6}$, and the ratio of the distances is $r_{SM}/r_{EM} = 4\cdot 10^2$. Therefore

$$\frac{F_{EM}}{F_{SM}} = (3\cdot 10^{-6})\cdot(4\cdot 10^2)^2 \approx 0.5$$

ROUND: There is no need to round.
DOUBLE-CHECK: This answer is somewhat surprising: The Sun's gravitational force on the Moon is approximately twice as big as that from the Earth on the Moon. Again, the question: Why does the Sun not pull the Moon away from Earth? The answer is that the Suns gravity does not only act on the Moon, but also on the Earth. Because the Moon orbits around the Earth, the Sun's gravitational pull acts on the center of mass of the Earth-Moon system, and both Earth and Moon orbit the Sun.

12.57. **THINK:** The goal is to find the speed of the shuttle before and after the retrorocket is fired. The radius of the orbit is $r = 6.60\cdot 10^6$ m before the shuttle loses 10% of its total energy. The speed of a circular orbit is found by setting the force of gravity equal to the centripetal force.
SKETCH:



RESEARCH: The orbital speed is given by $v = \sqrt{GM/r}$. The energy of the orbit, $E = -GMm/(2r)$, can be used to relate the radius of the orbit before and after the retrorocket is fired.

SIMPLIFY: The speed before the retrorocket is fired is: $v_{\text{b}} = \sqrt{GM/R_{\text{i}}}$. The initial energy of the space shuttle is $E_{\text{i}} = -\dfrac{GMm}{2R_{\text{i}}}$. Since the energy of the orbit is reduced by 10% the final energy of the orbit is $E_{\text{f}} = -\dfrac{(1.1)GMm}{2R_{\text{i}}} = -\dfrac{GMm}{2R_{\text{f}}} \Rightarrow R_{\text{f}} = \dfrac{R_{\text{i}}}{1.1}$. This means the speed after the retrorocket fires is $v_{\text{a}} = \sqrt{\dfrac{GM}{R_{\text{f}}}} = \sqrt{\dfrac{(1.1)GM}{R_{\text{i}}}}$.

CALCULATE: The speed before the retrorocket fires is:

$$v_{\text{b}} = \sqrt{\left(6.674\cdot 10^{-11}\,\text{Nm}^2/\text{kg}^2\right)\left(5.97\cdot 10^{24}\,\text{kg}\right)/\left(6.60\cdot 10^{6}\,\text{m}\right)} = 7769.77 \text{ m/s},$$

and the after the retrorocket fires the speed is:

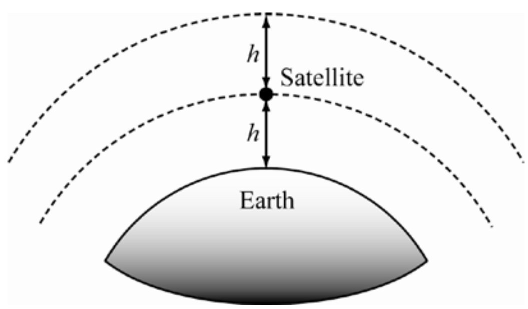
$$v_{\text{a}} = \sqrt{1.1\left(6.674\cdot 10^{-11}\,\text{Nm}^2/\text{kg}^2\right)\left(5.97\cdot 10^{24}\,\text{kg}\right)/\left(6.60\cdot 10^{6}\,\text{m}\right)} = 8149.01 \text{ m/s}.$$

ROUND: To three significant figures, the speeds before and after the retrorocket fires are $v_{\text{b}} = 7770$ m/s and $v_a = 8150$ m/s, respectively.

DOUBLE-CHECK: Since the space shuttle goes into a lower orbit after the retrorocket is fired, gravitational potential energy must be converted into kinetic energy. The calculation shows that the speed increases after firing so the result is reasonable.

12.58. **THINK:** The work required to double the height of a satellite of mass $m = 200.$ kg, orbiting the Earth at a speed of $v = 5.00\cdot 10^3$ m/s is just the change in kinetic energy. The speed of a circular orbit is found by setting the force of gravity equal to the centripetal force.

SKETCH:



RESEARCH: The speed is related to the radius of the satellite's orbit by $v = \sqrt{GM/r}$. This relation can be used to find the final speed. The work done on the satellite is equal to the change in kinetic energy: $W = -\Delta K = -\dfrac{1}{2}m\left(v_{\text{f}}^2 - v_{\text{i}}^2\right)$.

SIMPLIFY: The initial and final speeds are given by: $v_{\text{i}}^2 = \dfrac{GM}{\left(R_{\text{E}} + h\right)}$ and $v_{\text{f}}^2 = \dfrac{GM}{\left(R_{\text{E}} + 2h\right)}$. The initial height h can be found from the first expression: $h = \dfrac{GM}{v_{\text{i}}^2} - R_{\text{E}}$. The final speed can be given in terms of the initial speed:

$$v_{\text{f}}^2 = \frac{GM}{R_{\text{E}} + 2\left(\dfrac{GM}{v_{\text{i}}^2} - R_{\text{E}}\right)} = \frac{GM}{\dfrac{2GM}{v_{\text{i}}^2} - R_{\text{E}}} = \frac{v_{\text{i}}^2}{2 - \dfrac{v_{\text{i}}^2 R_{\text{E}}}{GM}}.$$

The work needed then is:

$$W = -\frac{1}{2}m\left(v_\text{f}^2 - v_\text{i}^2\right) = -\frac{1}{2}mv_\text{i}^2\left(\frac{1}{2-\dfrac{v_\text{i}^2 R_\text{E}}{GM}} - 1\right).$$

CALCULATE:

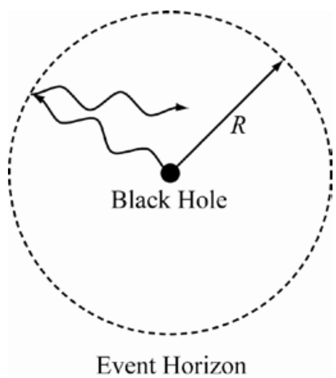
$$W = -\frac{1}{2}(200.\text{ kg})\left(5.00\cdot 10^3\text{ m/s}\right)^2\left(\frac{1}{2-\dfrac{\left(5.00\cdot 10^3\text{m/s}\right)^2\left(6370\cdot 10^3\text{m}\right)}{\left(6.674\cdot 10^{-11}\text{Nm}^2/\text{kg}^2\right)\left(5.97\cdot 10^{24}\text{kg}\right)}} - 1\right) = 9.3781\cdot 10^8\text{ J}$$

ROUND: The work done that must be done on the satellite is $9.38\cdot 10^8$ J.

DOUBLE-CHECK: This is a large amount of energy, but reasonable since the objects and distances involved are very large. The sign of the work is expected to be positive since work must be done on the satellite in order to put it into a higher orbit.

12.59. **THINK:** The goal of this question is to find the Schwarzschild radius of a black hole with twice the mass of the Sun, the radius at which the orbital speed is the same as the speed of light, and the radius of a black hole with Earth's mass. It can be assume that the orbit is circular so that the speed can be found by setting the force of gravity equal to the centripetal force.

SKETCH:



RESEARCH: The escape speed is given by $v = \sqrt{2GM/R}$. The radius of the orbital velocity can be found with $v = \sqrt{GM/R}$. The mass of the Sun is $1.9891\cdot 10^{30}$ kg. The mass of the Earth is $5.9742\cdot 10^{24}$ kg.

SIMPLIFY: (a) The Schwarzschild radius can be found by setting the escape speed to the speed of light and solving for R: $v = c = \sqrt{2GM/R} \Rightarrow c^2 = 2GM/R \Rightarrow R_\text{s} = 2GM/c^2$.

(b) The radius when the orbital speed is the speed of light is given by: $v = c = \sqrt{GM/R} \Rightarrow R = GM/c^2$.

(c) $R_\text{s} = 2GM/c^2$

CALCULATE:

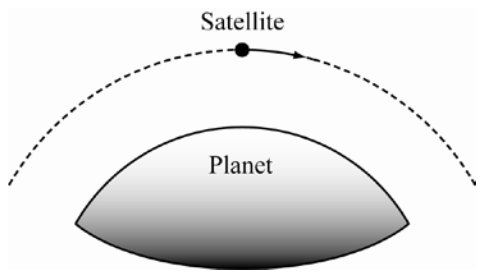
(a) $$R_\text{s} = \frac{2GM}{c^2} = \frac{2\left(6.674\cdot 10^{-11}\text{ N m}^2/\text{kg}^2\right)\left((2)1.9891\cdot 10^{30}\text{ kg}\right)}{\left(2.998\cdot 10^8\text{ m/s}\right)^2} = 5907.99\text{ m}$$

(b) $R = \frac{GM}{c^2} = \frac{\left(6.674\cdot 10^{-11}\text{ N m}^2/\text{kg}^2\right)\left((2)1.9891\cdot 10^{30}\text{ kg}\right)}{\left(2.998\cdot 10^{8}\text{ m/s}\right)^2} = 2953.99\text{ m}$

(c) $R_s = \frac{2GM}{c^2} = \frac{2\left(6.674\cdot 10^{-11}\text{ N m}^2/\text{kg}^2\right)\left(5.9742\cdot 10^{24}\text{ kg}\right)}{\left(2.998\cdot 10^{8}\text{ m/s}\right)^2} = 0.0088722\text{ m}$

ROUND: To four significant figures:
(a) The radius of the black hole with a mass twice that of the sun is $R_s = 5.908$ km.
(b) The orbital radius when the orbital speed is equal to the speed of light is $R = 2.954$ km.
(c) The radius of a black hole with a mass equal to the mass of the Earth is $R_s = 8.872$ mm.
DOUBLE-CHECK: Such extreme values are typical when dealing with black holes since they are so dense and massive.

12.60. **THINK:** The object is to find the ratio of the kinetic energy to the potential energy of a satellite in a circular orbit. For a circular orbit, the force of gravity and the centripetal force are equal.
SKETCH:



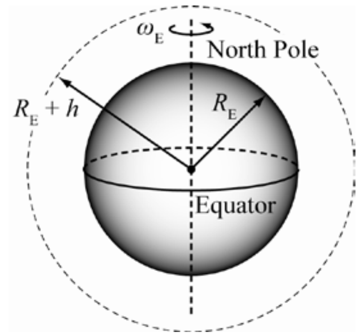
RESEARCH: To show that the ratio is constant, the expression for both types of energy are needed: $K = \frac{1}{2}mv^2$ and $U_g = -\frac{GMm}{r}$, $U_g = 0$ at $r = \infty$. Centripetal force is $F_c = \frac{mv^2}{r}$, and Newton's Law of Gravity is $F = \frac{GMm}{r^2}$.

SIMPLIFY: Setting the forces equal gives or $v^2 = \frac{GM}{r}$. Using this result in the expression for the kinetic energy gives $K = \frac{1}{2}mv^2 = \frac{GMm}{2r} = -\frac{U_g}{2}$. Thus, the ratio of the kinetic and potential energy of an object in a circular orbit is $\frac{K}{U_g} = -\frac{1}{2}$.

CALCULATE: Since the result is entirely algebraic, there are no calculations to perform.
ROUND: Since the result is entirely algebraic, no rounding is necessary.
DOUBLE-CHECK: Since gravitational potential energy is negative, it is expected that the ratio is negative. The ratio is independent of masses and the size and speed of the orbit, as expected.

12.61. **THINK:** The goal is to find the energy needed to launch a projectile into circular orbit from the North Pole and the equator. The launch site that requires the least amount of energy is better since less fuel is used and the cost is lower. The projectile has a mass $m = 100.0$ kg and the orbit is $h = 10.00$ km above the surface of the Earth.

SKETCH:



RESEARCH: Before being launched from the surface, the projectile has a total energy of $E_i = \frac{1}{2}mv^2 - \frac{GMm}{R_E}$. While in orbit, the projectile has an energy of $E_{orbit} = \frac{1}{2}U = -\frac{GMm}{2(R_E + h)}$.

SIMPLIFY:

(a) At the North Pole, the projectile starts from rest so its energy is $E_{NP,\,i} = 0 - \frac{GMm}{R_E}$. The energy required to launch the projectile at the North Pole is:

$$E_{NP} = E_{orbit} - E_{NP,\,i} = GMm\left[\frac{1}{R_E} - \frac{1}{2(R_E + h)}\right].$$

(b) At the equator the rotation of the Earth gives the projectile an initial speed of $v = \omega_E R_E$, so the initial energy of the projectile launched at the equator is $E_{eq,\,i} = \frac{1}{2}mv^2 - \frac{GMm}{R_E} = \frac{1}{2}m(\omega_E R_E)^2 - \frac{GMm}{R_E}$. The energy required to launch the projectile at the equator is:

$$E_{eq} = E_{orbit} - E_{eq,\,i} = GMm\left[\frac{1}{R_E} - \frac{1}{2(R_E + h)}\right] - \frac{1}{2}m\omega_E^2 R_E^2 = E_{NP} - \frac{1}{2}m\omega_E^2 R_E^2.$$

CALCULATE:

(a) The energy required to launch at the North Pole is:

$$E_{NP} = \left(6.674 \cdot 10^{-11} \text{ N m}^2/\text{kg}^2\right)\left(5.974 \cdot 10^{24} \text{ kg}\right)\left(100.0 \text{ kg}\right)\left[\frac{1}{6.357 \cdot 10^6 \text{ m}} - \frac{1}{2\left(6.357 \cdot 10^6 \text{ m} + 10.00 \cdot 10^3 \text{ m}\right)}\right]$$

$$= 3.14088 \cdot 10^9 \text{ J}.$$

(b) The energy required to launch at the equator is:

$$E_{eq} = E_{NP} - \frac{1}{2}m\omega_E^2 R_E^2 = \left(3.13052 \cdot 10^9 \text{ J}\right) - \frac{1}{2}(100.0 \text{ kg})\left(\frac{2\pi}{86{,}400 \text{ s}}\right)^2 \left(6.378 \cdot 10^6 \text{ m}\right)^2$$

$$= 3.130123 \cdot 10^9 \text{ J}.$$

ROUND: To four significant figures, the energy required to launch the projectile from each site is:

(a) $E_{NP} = 3.141 \cdot 10^9$ J

(b) $E_{eq} = 3.130 \cdot 10^9$ J

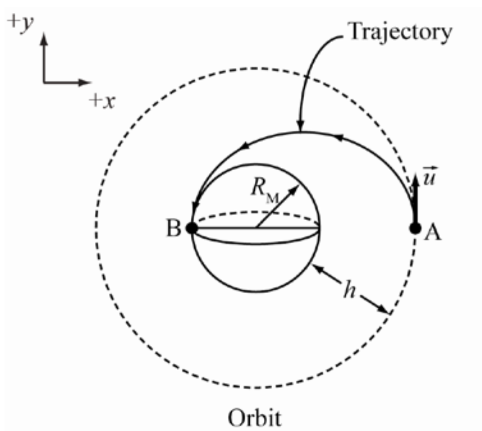
The difference in energy is $1.1 \cdot 10^7$ J which is large, but it is only about 0.4% of the energy required to make the launch. Thus, the difference is not very significant.

DOUBLE-CHECK: This result might have been expected since the rotation of the Earth is relatively slow. Therefore, the initial speed at the surface is small and there is only a small energy advantage in launching projectiles from the equator.

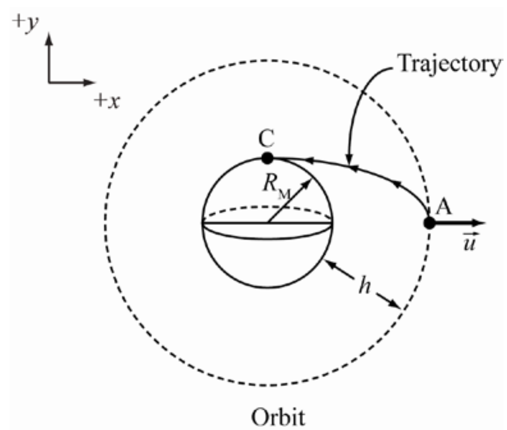
12.62. **THINK:** The rocket has a mass of $M = 12.0 \text{ metric tons} = 1.20 \cdot 10^4 \text{ kg}$ and orbits the Moon at a height $h = 100. \text{ km}$ above the surface. The Moon has a radius $R_{\text{M}} = 1.74 \cdot 10^3 \text{ km}$ and an acceleration of gravity of $g_{\text{M}} = 1.62 \text{ m/s}^2$ near the surface. In the first case, the rocket fuel, ejected at a velocity of $\vec{u} = 1.00 \cdot 10^4 \text{ m/s}$ forward, is used to land the rocket at point B. In the second case, the rocket fuel is ejected at the same speed away from the Moon, in order to land that rocket at point C. It can be assumed that the gravitational acceleration is constant throughout the descent in each case. The equation for the speed of a circular orbit is found by setting the force of gravity equal to the centripetal force. The conservation of energy and momentum can be used to find the amount of fuel needed in each case.

SKETCH:

(a)



(b)



RESEARCH: The first step is to find the speed of the rocket in orbit $v = \sqrt{GM/r}$. In order to accomplish this, rewrite G using $g_{\text{M}} = GM_{\text{M}} / R_{\text{M}}^2$. Using the conservation of energy and angular momentum, the change in speed can be determined. The conservation of linear momentum is then used to find the amount of fuel needed for each trip.

SIMPLIFY: The speed of the orbit is $v = \sqrt{\dfrac{GM_{\text{M}}}{R}} = \sqrt{\dfrac{g_{\text{M}} R_{\text{M}}^2}{R}} = \sqrt{\dfrac{g_{\text{M}} R_{\text{M}}^2}{R_{\text{M}} + h}}$.

(a) If v_{A} is the speed of the rocket at A after the braking, and m_{B} is the mass of fuel required to send the rocket to point B, so that the rocket's remaining mass is $M' = M - m_{\text{B}}$, then conservation of energy gives $\dfrac{M' v_{\text{A}}^2}{2} - \dfrac{GM' M_{\text{M}}}{R_{\text{M}} + h} = \dfrac{M' v_{\text{B}}^2}{2} - \dfrac{GM' M_{\text{M}}}{R_{\text{M}}}$, and conservation of angular momentum gives $M' v_{\text{A}} \left(R_{\text{M}} + h \right) = M' v_{\text{B}} R_{\text{M}}$. After M' cancels, there are two unknowns in two equations, so the velocity v_{A} at point A can be found:

$$v_{\text{B}} = v_{\text{A}} \frac{R_{\text{M}} + h}{R_{\text{M}}}$$

$$\frac{v_A^2}{2}-\frac{v_B^2}{2}=GM_M\left(\frac{1}{R_M+h}-\frac{1}{R_M}\right)$$

$$\frac{v_A^2}{2}-\frac{v_A^2}{2}\left(\frac{R_M+h}{R_M}\right)^2=GM_M\left(-\frac{h}{R_M\left(R_M+h\right)}\right)$$

$$\frac{v_A^2\left(R_M^2-\left(R_M+h\right)^2\right)}{R_M^2}=2GM_M\left(-\frac{h}{R_M\left(R_M+h\right)}\right)$$

$$v_A=\sqrt{2GM_M\frac{R_M^2}{R_M^2-\left(R_M+h\right)^2}\cdot\frac{-h}{R_M\left(R_M+h\right)}}=\sqrt{\frac{2GM_MR_M^2h}{\left(2R_Mh+h^2\right)R_M\left(R_M+h\right)}}$$

$$=\sqrt{\frac{2GM_MR_M}{\left(2R_M+h\right)\left(R_M+h\right)}}=\sqrt{\frac{2g_MR_M^3}{\left(2R_M+h\right)\left(R_M+h\right)}}=v\sqrt{\frac{2R_M}{2R_M+h}}$$

The change in velocity is $\Delta v=v-v_A=v-v\sqrt{\frac{2R_M}{2R_M+h}}=v\left(1-\sqrt{\frac{2R_M}{2R_M+h}}\right)$. Conservation of linear momentum gives $\left(M-m_B\right)\Delta v=m_Bu$. Therefore, $M\Delta v=m_B\left(u+\Delta v\right)$ or $m_B=\frac{M\Delta v}{u+\Delta v}$.

(b) In the second case, the change in direction is perpendicular to the motion so the velocity at point A is $v_A=\sqrt{v^2+\left(\Delta v\right)^2}$. Conservation of energy gives: $\frac{M'\left(v^2+\left(\Delta v\right)^2\right)}{2}-\frac{GM'M_M}{R_M+h}=\frac{M'v_C^2}{2}-\frac{GM'M_M}{R_M}$, and conservation of angular momentum gives: $M'v\left(R_M+h\right)=M'v_CR_M$, where m_C is the mass of fuel required to send the rocket to point C and now $M'=M-m_C$. Solving these equations for Δv gives $\Delta v=\sqrt{\frac{g_Mh^2}{R_M+h}}$. Conservation of linear momentum in the direction perpendicular to the original direction gives $\left(M-m_C\right)\Delta v=m_Cu$ or $m_C=M\Delta v/\left(u+\Delta v\right)$.

CALCULATE:

(a) In the first case, the change in velocity is

$$\Delta v=\sqrt{\frac{\left(1.62\text{ m/s}^2\right)\left(1.74\cdot10^6\text{ m}\right)^2}{\left(1.74\cdot10^6\text{ m}+100.\cdot10^3\text{ m}\right)}}\left(1-\sqrt{\frac{2\left(1.74\cdot10^6\text{ m}\right)}{2\left(1.74\cdot10^6\text{ m}\right)+100.\cdot10^3\text{ m}}}\right)=22.964\text{ m/s}.$$

Therefore, the mass of spent fuel is $m_B=\frac{\left(1.20\cdot10^4\text{ kg}\right)\left(22.964\text{ m/s}\right)}{\left(1.00\cdot10^4\text{ m/s}+22.964\text{ m/s}\right)}=27.494\text{ kg}.$

(b) For the second case the change in velocity is

$$\Delta v=\sqrt{\frac{\left(1.62\text{ m/s}^2\right)\left(100.\cdot10^3\text{ m}\right)^2}{\left(1.74\cdot10^6\text{ m}+100.\cdot10^3\text{ m}\right)}}=93.831\text{ m/s}.$$

Therefore, the mass of spent fuel is $m_C=\frac{\left(1.20\cdot10^4\text{ kg}\right)\left(93.831\text{ m/s}\right)}{\left(1.00\cdot10^4\text{ m/s}+93.831\text{ m/s}\right)}=111.55\text{ kg}.$

ROUND:

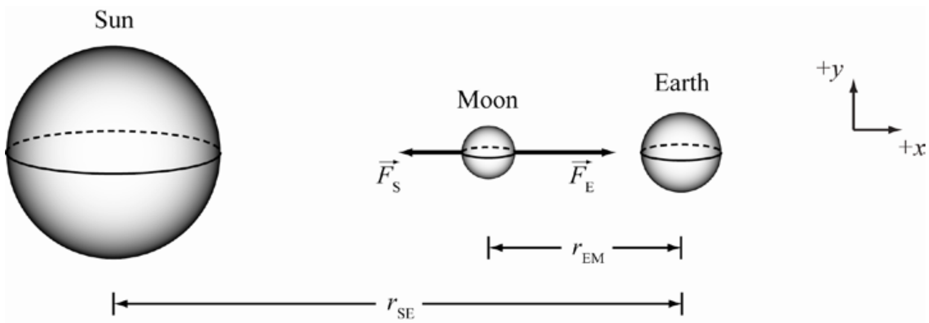
(a) The fuel required to send the rocket to point B is 27.5 kg.

(b) The fuel required to send the rocket to point C is 112 kg.

DOUBLE-CHECK: The rocket requires less fuel to reach point B than point C. The path to C is shorter so it is expected that a greater deceleration and, therefore, more fuel would be required.

12.63. **THINK:** The force of gravity acting on the Moon due to the Sun and the Earth are calculated separately. The net force due to gravity from the Earth and the Sun can then be found by summing up the two forces.

SKETCH:



RESEARCH: Use Newton's Law of Gravity, $F = Gm_1m_2/r^2$, to calculate both forces.

SIMPLIFY: $\vec{F}_S = -\frac{Gm_S m_M}{(r_{SE} - r_{EM})^2}\hat{x}$, $\vec{F}_E = \frac{Gm_E m_M}{(r_{EM})^2}\hat{x}$. The net force acting on the Moon is

$$\vec{F}_M = \sum \vec{F} = \vec{F}_S + \vec{F}_E = Gm_M\left[-\frac{m_S}{(r_{SE} - r_{EM})^2} + \frac{m_E}{(r_{EM})^2}\right]\hat{x}.$$

CALCULATE:

$$\vec{F}_S = -\frac{(6.674\cdot 10^{-11}\text{ N m}^2/\text{kg}^2)(1.99\cdot 10^{30}\text{ kg})(7.344\cdot 10^{22}\text{ kg})}{(1.496\cdot 10^{11}\text{ m} - 3.844\cdot 10^{8}\text{ m})^2}\hat{x} = -4.381\cdot 10^{20}\text{ N }\hat{x}$$

$$\vec{F}_E = \frac{(6.674\cdot 10^{-11}\text{ N m}^2/\text{kg}^2)(5.97\cdot 10^{24}\text{ kg})(7.344\cdot 10^{22}\text{ kg})}{(3.844\cdot 10^{8}\text{ m})^2}\hat{x} = 1.980\cdot 10^{20}\text{ N }\hat{x}$$

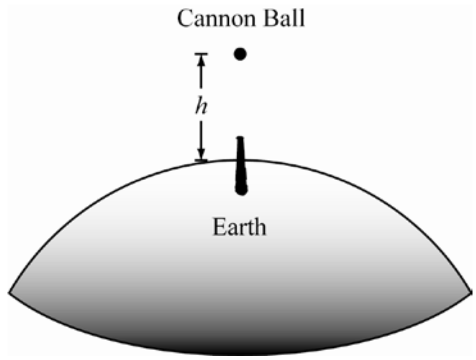
$$\vec{F}_M = (-4.381\cdot 10^{20}\text{ N} + 1.980\cdot 10^{20}\text{ N})\hat{x} = -2.401\cdot 10^{20}\text{ N }\hat{x}$$

ROUND: To three significant figures, the force on the Moon due to the Sun is $F_S = 4.38\cdot 10^{20}$ N towards the Sun. The force on the Moon due to the Earth is $F_E = 1.98\cdot 10^{20}$ N towards the Moon. The total force on the Moon is $F_M = 2.40\cdot 10^{20}$ N towards the Sun. The Moon's orbit never curves away from the Sun toward the Earth.

DOUBLE-CHECK: Use the answer of problem 12.56 as a double-check, the ratio of the magnitude of the two forces was calculated to be approximately 0.5, in accordance with the present result.

12.64. **THINK:** Conservation of energy can be used to find the initial speed of the projectile.

SKETCH:



RESEARCH: At the surface the initial energy is $E_{\text{i}} = \frac{1}{2}mv^2 - \frac{GMm}{R}$, where v is the initial speed, M is the mass of the Earth, R is the radius of the Earth, and m is the mass of the projectile. At its highest point the projectile's speed is zero so its final energy is $E_{\text{f}} = -\frac{GMm}{R}$. Conservation of energy demands that $E_{\text{i}} = E_{\text{f}}$.

SIMPLIFY: $\frac{1}{2}mv^2 - \frac{GMm}{R} = -\frac{GMm}{R+h} \Rightarrow v^2 = 2GM\left(\frac{1}{R} - \frac{1}{R+h}\right) \Rightarrow v = \sqrt{2GM\left(\frac{1}{R} - \frac{1}{R+h}\right)}$

CALCULATE: $v = \sqrt{2\left(6.674\cdot 10^{-11}\text{ N m}^2/\text{kg}^2\right)\left(5.97\cdot 10^{24}\text{ kg}\right)\left[\frac{1}{6.37\cdot 10^6\text{ m}} - \frac{1}{6.37\cdot 10^6\text{ m} + 55.0\cdot 10^3\text{ m}}\right]}$

$= 1034.83\text{ m/s}$

ROUND: To three significant figures, the initial speed of the projectile is 1.03 km/s.

DOUBLE-CHECK: Such a large speed is reasonable because the projectile travels to such a high altitude.

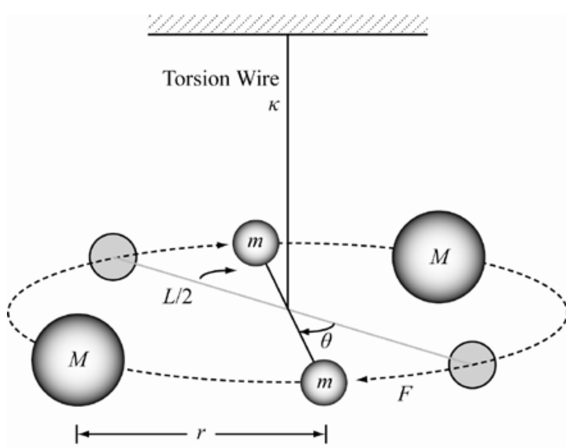
12.65. **THINK:** The force due to gravity between the large and small balls can be found by using Newton's Law of Gravity. Since there are two small balls and two large balls, the force measured by the apparatus is twice the force due to one set of balls. Converting the data to SI units gives:

$$m = (1.61\text{ lb})(0.4536\text{ kg/lb}) = 0.7303\text{ kg},$$

$$M = (348\text{ lb})(0.4536\text{ kg/lb}) = 157.85\text{ kg},$$

$$r = 9.00\text{ in.}(0.0254\text{ m/in.}) = 0.2286\text{ m}.$$

SKETCH:



RESEARCH: The force that is measured between the balls is $F_b = 2F = \dfrac{2GmM}{r^2}$. The weight of the small balls is $F_g = 2mg$.

SIMPLIFY: The ratio is $F_b / F_g = \left(2GmM/r^2\right)/\left(2mg\right) = GM/gr^2$.

CALCULATE: $F_b = \dfrac{2\left(6.674\cdot 10^{-11}\text{ N m}^2/\text{kg}^2\right)\left(0.7303\text{ kg}\right)\left(157.85\text{ kg}\right)}{\left(0.2286\text{ m}\right)^2} = 2.944\cdot 10^{-7}\text{ N}$

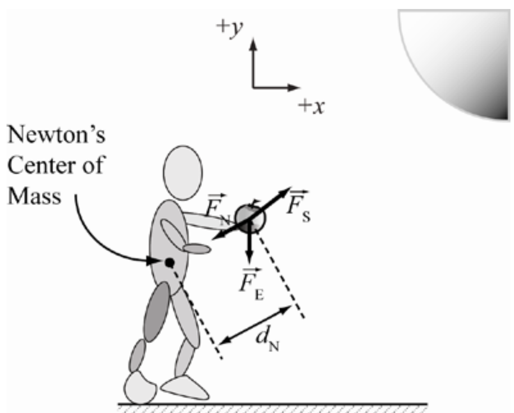
$$F_b/F_g = \frac{\left(6.674\cdot 10^{-11}\text{ N m}^2/\text{kg}^2\right)\left(157.85\text{ kg}\right)}{\left(9.81\text{ m/s}^2\right)\left(0.2286\text{ m}\right)^2} = 2.056\cdot 10^{-8}$$

ROUND: To three significant figures, the force measured by the apparatus to the gravitational attraction of the large and small balls is $F_b = 2.94\cdot 10^{-7}$ N. The force of gravity of the Earth on the small balls is much larger than the force due to the balls. The ratio of the ball forces to the weight of the small balls is $2.06\cdot 10^{-8}$.

DOUBLE-CHECK: Since the gravitational force is proportional to the mass of the objects involved, it is expected that the Earth would exert a much greater force on the balls than the balls exert on each other.

12.66. **THINK:** The point of this question is to show the major gravitational force on an apple. The apple has a mass of 100. g. Newton's mass is 80.0 kg and is 50.0 cm away from the apple.

SKETCH:



RESEARCH: The gravitational force is given by $F = GMm/R^2$.

SIMPLIFY: Not required.

CALCULATE: The force due to Newton is

$F_N = \left(6.674\cdot 10^{-11}\text{ N m}^2/\text{kg}^2\right)\left(80.0\text{ kg}\right)\left(0.100\text{ kg}\right)/\left(0.500\text{ m}\right)^2 = 2.13568\cdot 10^{-9}$ N. The force due to the Earth is $F_E = mg = \left(0.100\text{ kg}\right)\left(9.81\text{ m/s}^2\right) = 0.981$ N. The force due to the Sun is

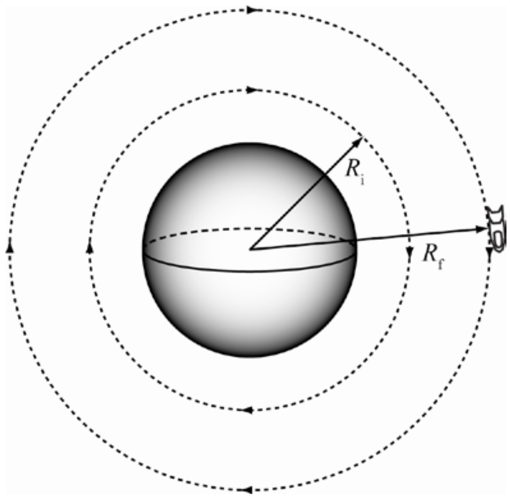
$$F_s = \frac{\left(6.674\cdot 10^{-11}\text{ N m}^2/\text{kg}^2\right)\left(1.99\cdot 10^{30}\text{ kg}\right)\left(0.100\text{ kg}\right)}{\left(149.6\cdot 10^9\text{ m}\right)^2} = 5.9344\cdot 10^{-4}\text{ N}.$$

ROUND: The answers should have three significant figures. The force of gravity due to Newton is $F_N = 2.14\cdot 10^{-9}$ N, the force of gravity due to the Earth is $F_E = 0.981$ N and the force of gravity due to the Sun is $F_S = 5.93\cdot 10^{-4}$ N.

DOUBLE-CHECK: This makes sense; the force due to the Earth at close range is much bigger than that due to the Sun, and greater than that exerted by the much less massive Newton. This is consistent with everyday experience – when the apple is released it falls to the Earth.

12.67. **THINK:** The principle of the conservation of energy can be used to determine the energy supplied by the rockets in order to increase the altitude of the orbit. The energy supplied by the rockets is the difference in orbital energies of the two orbits.

SKETCH:



RESEARCH: The total energy of the orbit is $E=-\frac{GMm}{2R}$.

SIMPLIFY: $E=E_f-E_i=-\frac{GMm}{2R_f}+\frac{GMm}{2R_i}=\frac{GMm}{2}\left(\frac{1}{R_i}-\frac{1}{R_f}\right)$

CALCULATE: $E=\frac{\left(6.674\cdot 10^{-11}\text{ N m}^2/\text{kg}^2\right)\left(5.97\cdot 10^{24}\text{ kg}\right)\left(1000.\text{ kg}\right)}{2}\left[\frac{1}{7.00\cdot 10^{6}\text{ m}}-\frac{1}{5.00\cdot 10^{7}\text{ m}}\right]$

$=2.448\cdot 10^{10}\text{ J}$

ROUND: To three significant figures, the energy supplied by the rockets to move the satellite into a higher orbit is $2.45\cdot 10^{10}$ J.

DOUBLE-CHECK: This is the same amount of energy required to lift a 2 million kilogram object 1 kilometer into the air (assuming gravitational acceleration of Earth is constant). This large value was expected since the force of gravity between the Earth and the satellite is rather large.

12.68. **THINK:** By Newton's third law, the force that the apple exerts on the Earth is the same as the force that the Earth exerts on the apple. Whether the apple is tied to a tree or is in free fall does not matter. The force of gravity only depends on the mass of the two objects and their separation. Since the height of an apple in a tree is much less than the radius of the Earth, the distance can be approximated to be the radius of the Earth.

SKETCH:

RESEARCH: This gravitational force is $F=\frac{GM_{\text{E}}m}{\left(R_E+h\right)^2}$, where M_{E} is the mass of the Earth, m is the mass of the apple, R_{E} is the radius of the Earth, and h is the height of the apple above the surface of the Earth.

SIMPLIFY: Because $R_{\text{E}} >> h$, the equation reduces to the familiar $F=m\left(\frac{GM_{\text{E}}}{{R_E}^2}\right)=mg$.

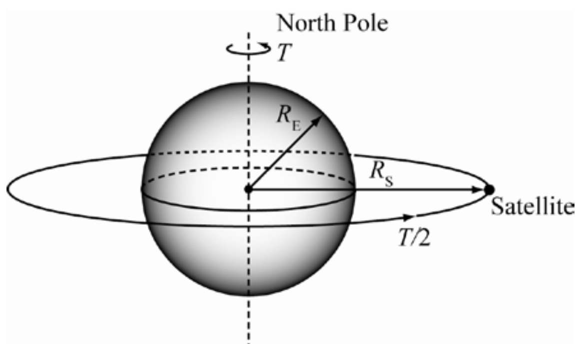
CALCULATE: $F=\left(0.300\text{ kg}\right)\left(9.81\text{ m/s}^2\right)=2.943\text{ N}.$

ROUND: To three significant figure, the force that the apple exerts on the Earth is $F=2.94\text{ N},$ towards the apple. Because h is very small compared to R_{E}, this force remains constant whether the apple is on the tree of falling toward the ground.

DOUBLE-CHECK: This small force is reasonable since the mass of the apple is very small. Newton's third law states that each force has an equal magnitude, but opposite direction. Since the Earth is very massive, it's acceleration towards the apple is minuscule. Since the apple as a small mass it accelerates towards the Earth.

12.69. **THINK:** Kepler's third law can be used to determine the height required to have half the period of the Earth about its axis.

SKETCH:



RESEARCH: Kepler's third law relates the period of an orbit to its radius: $\frac{T^2}{r^3}=\frac{4\pi^2}{GM}$.

SIMPLIFY: Solving for the radius gives: $R_{\text{s}}=\left(\frac{GMT^2}{4\pi^2}\right)^{\frac{1}{3}}$. The height above the surface of the Earth is given by the equation $R_{\text{s}}=R_{\text{E}}+h$ or $h=\left(\frac{GMT^2}{4\pi^2}\right)^{\frac{1}{3}}-R_{\text{E}}$.

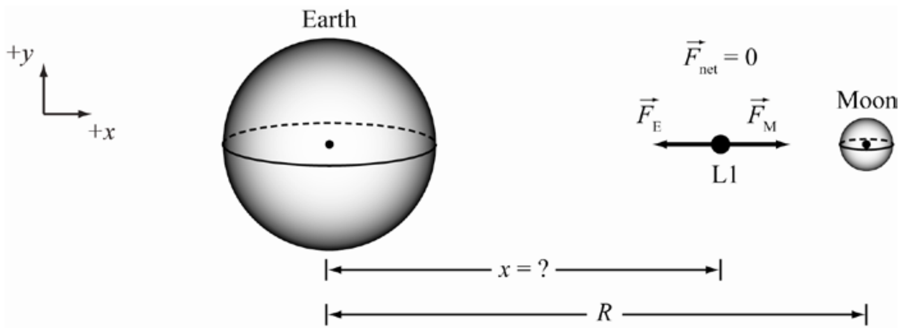
CALCULATE:
$$h=\left[\frac{\left(6.674\cdot 10^{-11}\text{ Nm}^2/\text{kg}^2\right)\left(5.97\cdot 10^{24}\text{ kg}\right)\left(\frac{86400\text{ s}}{2}\right)^2}{4\pi^2}\right]^{\frac{1}{3}}-6370\cdot 10^3\text{ m}=2.024\cdot 10^7\text{ m}$$

ROUND: To three significant figures, the satellite must be at a height of $2.02\cdot 10^7$ m or $2.02\cdot 10^4$ km.

DOUBLE-CHECK: This answer is of a reasonable order of magnitude for the height of a satellite in orbit.

12.70. **THINK:** The neutral point for the Earth/Moon system is the point between the Earth and the Moon where the gravitational force of the Moon exactly equals the gravitational force of the Earth (or the net force of gravity is zero). Using Newton's Law of Gravity the ratio of the forces between the Sun and the Earth, and the Sun and the Moon can be calculated. Let the distance from the center of the Earth to the neutral point be *x*, and let *R* denote the distance between the Earth and the Moon. Assume the mass of the Moon is 1/81 the mass of the Earth.

SKETCH:



RESEARCH: Newton's Law of Gravity gives the gravitational force: $F = GMm/r^2$. The distance between the Moon and the neutral point is $R - x$. The gravitational force of the Earth on an object of mass *m* at the neutral point is $F_E = \dfrac{GM_E m}{x^2}$. The gravitational force of the Moon on an object at the neutral point is

$F_M = \dfrac{G\left(\frac{1}{81}M_E\right)m}{(R-x)^2}$.

SIMPLIFY:

(a) By definition, $F_E = F_M$ at L1, so

$$\frac{GM_E m}{x^2} = \frac{G\left(\frac{1}{81}M_E\right)m}{(R-x)^2} \Rightarrow \frac{1}{x^2} = \frac{1}{81(R-x)^2} \Rightarrow 80x^2 - 162Rx + 81R^2 = 0$$

This factors to $(10x - 9R)(8x - 9R) = 0$. Therefore the possible values of *x* are $x = 9R/10$ or $x = 9R/8$. Since the neutral point is between the Earth and the Moon, the value of *x* has to be between 0 and *R*. Therefore, $x = 9R/10$ is the proper solution.

(c) The ratio of the gravitational force due to the Sun to that due to the Earth or the Moon is

$$\frac{F_S}{F_E} = \frac{F_S}{F_M} = \frac{\left(GM_S m / R_S^2\right)}{\left(GM_E m / x^2\right)} = \frac{M_S x^2}{M_E R_S^2}.$$

Recall that $F_E = F_M$ at the neutral point. Since the Moon revolves around the Earth, the exact distance between the Sun and the neutral point varies. But it is reasonable to use the Earth-Sun distance R_S as an approximation.

CALCULATE:

(a) The distance of the neutral point from the Earth is:

$$x = \frac{9\left(3.844 \cdot 10^5 \text{ km}\right)}{10} = 3.4596 \cdot 10^5 \text{ km}$$

(b) If an object at the neutral point is pushed towards the Earth or the Moon the object will have a net gravitational force in the direction of the push. Therefore, since the object does not settle back into the neutral point after a perturbation, the neutral point is an unstable equilibrium.

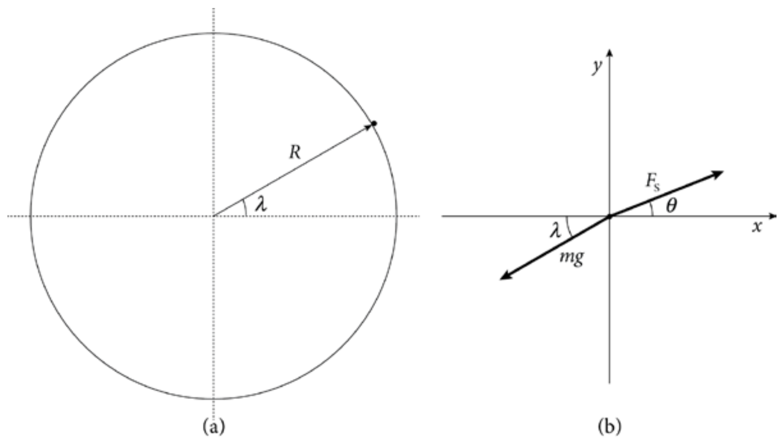
(c) $\frac{F_{\text{S}}}{F_{\text{E}}}=\frac{F_{\text{S}}}{F_{\text{M}}}=\frac{\left(1.99\cdot 10^{30}\text{ kg}\right)\left(3.4596\cdot 10^{5}\text{ km}\right)^{2}}{\left(5.97\cdot 10^{24}\text{ kg}\right)\left(1.496\cdot 10^{8}\text{ km}\right)^{2}}=1.7827$.

ROUND: To three significant figures, the neutral point is $3.46\cdot 10^{5}$ km from the Earth in the direction of the Moon. The ratio of the forces of gravity is 1.78 at the neutral point.

DOUBLE-CHECK: Since the Earth is much more massive than the Moon, the neutral point should be substantially closer to the Moon than it is to the Earth. Since the Sun is much more massive than the Earth or the Moon, it is reasonable that the force of gravity due to the Sun is greater than the force of gravity due to the Earth or the Moon at the neutral point.

12.71. **THINK:** The effective gravitational force acting on a particle on the Earth's surface at $\lambda = 30.0°$ north of the equator can be found by summing the forces acting on the particle. The only two forces acting on the particle are the force of gravity, pointing toward the center of the Earth, and the force exerted by the surface of the Earth. Because the Earth is rotating and the particle is traveling in a circle, the difference between the x-component of the force of gravity and the x-component of the force exerted by the surface of the Earth must be equal to the centripetal force required to keep the particle traveling in a circle.

SKETCH:



RESEARCH: The y-components of the forces must sum to zero

$$F_{\text{S},y}-mg\sin\lambda=0 \quad\Rightarrow\quad mg\sin\lambda=F_{\text{S},y}.$$

The x-components of the forces must equal the centripetal force required to keep the particle traveling in a circle with radius $r=R\cos\lambda$

$$F_{\text{S},x}-mg\cos\lambda=-mr\omega^2=-mR\cos\lambda\omega^2$$

$$F_{\text{S},x}=mg\cos\lambda-mR\cos\lambda\omega^2.$$

SIMPLIFY:

We can write the magnitude of the force exerted by the surface as

$$F_{\text{S}}=\sqrt{\left(mg\cos\lambda-mR\cos\lambda\omega^2\right)^2+\left(mg\sin\lambda\right)^2}$$

$$F_{\text{S}}=\sqrt{\left(mg\cos\lambda-mR\cos\lambda\omega^2\right)^2+\left(mg\sin\lambda\right)^2}=m\sqrt{\left(g\cos\lambda-R\cos\lambda\omega^2\right)^2+\left(g\sin\lambda\right)^2}$$

$$F_{\text{S}}=m\sqrt{\left(g-R\omega^2\right)^2\cos^2\lambda+g^2\sin^2\lambda}.$$

The angle of the force exerted by the surface is

$$\theta=\tan^{-1}\left(\frac{mg\sin\lambda}{mg\cos\lambda-mR\cos\lambda\omega^2}\right)=\tan^{-1}\left(\frac{g\sin\lambda}{g\cos\lambda-R\cos\lambda\omega^2}\right)=\tan^{-1}\left(\frac{g\tan\lambda}{g-R\omega^2}\right).$$

The deviation can be expressed as

$$\Delta = \lambda - \tan^{-1}\left(\frac{g\tan\lambda}{g - R\omega^2}\right).$$

CALCULATE:

(a) The magnitude of the force exerted by the surface for $\lambda = 30.0°$ is

$$F_S = m\sqrt{\left(g - R\omega^2\right)^2\cos^2\lambda + g^2\sin^2\lambda}$$

$$F_S = m\sqrt{\left(9.81\text{ m/s}^2 - \left(6.37\cdot 10^6\text{ m}\right)\left(7.27\cdot 10^{-5}\text{ rad/s}\right)^2\right)^2\cos^2 30.0° + \left(9.81\text{ m/s}^2\right)^2\sin^2 30.0°}$$

$$F_s = m\left(9.7848\text{ m/s}^2\right).$$

The angular deviation at $\lambda = 30.0°$ is

$$\Delta = \lambda - \tan^{-1}\left(\frac{g\tan\lambda}{g - R\omega^2}\right) = 30.00° - \tan^{-1}\left(\frac{\left(9.81\text{ m/s}^2\right)\tan 30.0°}{9.81\text{ m/s}^2 - \left(6.37\cdot 10^6\text{ m}\right)\left(7.27\cdot 10^{-5}\text{ rad/s}\right)^2}\right)$$

$$\Delta = -0.085365°.$$

We can get the maximum by taking the derivative, setting it equal to zero, and solving for λ,

Let $a = \dfrac{g}{g - R\omega^2} = 1.00344$

$$\frac{d\Delta}{d\lambda} = \frac{d}{d\lambda}\left[\lambda - \tan^{-1}\left(a\tan\lambda\right)\right] = 1 - \frac{a\sec^2\lambda}{1 + a^2\tan^2\lambda} = 0$$

$$1 + a^2\tan^2\lambda = a\sec^2\lambda$$

$$\cos^2\lambda + a^2\sin^2\lambda = a$$

Solving numerically (in this case, using Mathematica)

$$\lambda = 2\tan^{-1}\left(\sqrt{1 + 2a - 2\sqrt{a(1+a)}}\right) = 0.784539\text{ rad} = 44.95°.$$

ROUND:

(a) The effective force of gravitation is $F_S = \left(9.78\text{ m/s}^2\right)m$.

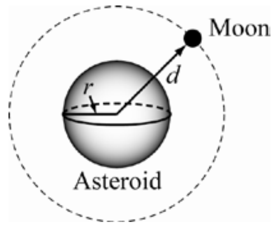
(b) The direction of the effective gravitational force is $29.9°$ above the horizontal.

(c) The angle of λ that maximizes the deviation is $45.0°$ or $\pi/4$ radians.

DOUBLE-CHECK: It is expected that the effective gravitational acceleration is less than *g*. If the Earth were to rotate fast enough, objects would not stay on the surface. The direction points toward the equator from the radial direction. This is the same effect that makes the Earth's oceans bulge at the equator. For $\lambda = 0$, we see that $F_S = m\sqrt{\left(g - R\omega^2\right)^2\cos^2 0 + g^2\sin^2 0} = mg - mR\omega^2$, which is what we would expect at the equator.

12.72. **THINK:** The goal of the exercise is to find the acceleration of gravity and the escape speed of the asteroid. The asteroid has a radius of $r = 20.0$ km. There is a tiny moon in a circular orbit about the asteroid at a distance of $d = 100.$ km and a period of $T = 40.0$ hr or 144000 s. The equation for the acceleration of gravity and Kepler's third law can be used here.

SKETCH:



RESEARCH: The acceleration of gravity is defined as $g = GM/r^2$. Using Kepler's third law $T^2/d^3 = 4\pi^2/GM$, the acceleration of gravity of the asteroid can be found. The escape speed is given by $v = \sqrt{2GM/r}$.

SIMPLIFY:

(a) From Kepler's third law: $GM = \dfrac{4\pi^2 d^3}{T^2}$. Therefore, the acceleration due to gravity is $g = \dfrac{GM}{r^2} = \dfrac{4\pi^2 d^3}{r^2 T^2}$.

(b) The escape speed is $v_E = \sqrt{\dfrac{2GM}{r}} = \sqrt{2gr}$.

CALCULATE:

(a) The acceleration of gravity at the surface of the asteroid is:

$$g = \frac{4\pi^2 \left(1.00 \cdot 10^5 \text{ m}\right)^3}{\left(1.44 \cdot 10^5 \text{ s}\right)^2 \left(2.00 \cdot 10^4 \text{ m}\right)^2} = 4.7596 \cdot 10^{-3} \text{ m/s}^2$$

(b) The escape speed from the asteroid is: $v_E = \sqrt{2\left(4.7596 \cdot 10^{-3} \text{ m/s}^2\right)\left(2.00 \cdot 10^4 \text{ m}\right)} = 13.798 \text{ m/s}$.

ROUND:

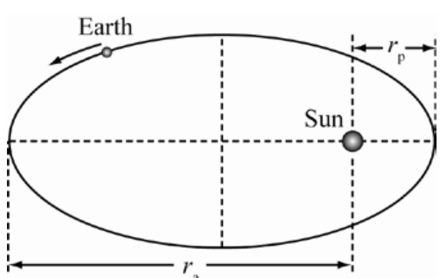
(a) The acceleration of gravity at the surface of the asteroid is $g = 4.76 \cdot 10^{-3} \text{ m/s}^2$.

(b) The escape speed is $v_E = 13.8 \text{ m/s}$.

DOUBLE-CHECK: These values are much smaller than those of Earth, as they should be since the asteroid is very small.

12.73. **THINK:** The values for the distances at perihelion and aphelion will need to be used to find a value for the percentage change in the gravitational potential energy. By using conservation of energy, the percentage change in kinetic energy can be found.

SKETCH:



RESEARCH: The potential energy is given by $U = -GM_E M_S / r$, where M_S is the mass of the Sun and M_E is the mass of the Earth. The percentage change is given by $\left(\dfrac{U_f - U_i}{U_i}\right) \cdot 100\%$.

SIMPLIFY:
(a) The percentage change for the potential energy is:

$$\left(\frac{U_{\text{a}}-U_{\text{p}}}{U_{\text{p}}}\right)(100\%)=\frac{\left(-GM_{\text{E}}M_{\text{S}}/r_{\text{a}}\right)-\left(-GM_{\text{E}}M_{\text{S}}/r_{\text{p}}\right)}{\left(-GM_{\text{E}}M_{\text{S}}/r_{\text{p}}\right)}(100\%)=\left(\frac{1/r_{\text{a}}-1/r_{\text{p}}}{1/r_{\text{p}}}\right)(100\%)$$

$$\left(\frac{U_{\text{a}}-U_{\text{p}}}{U_{\text{p}}}\right)(100\%)=\left(\frac{\left(r_{\text{p}}-r_{\text{a}}\right)/r_{\text{p}}r_{\text{a}}}{1/r_{\text{p}}}\right)(100\%)=\left(\frac{r_{\text{p}}-r_{\text{a}}}{r_{\text{a}}}\right)(100\%).$$

(b) The kinetic energy is related to the potential energy through $E=U+K$, so we an express the kinetic energy as $K=E-U$. The percentage change for kinetic energy is

$$\Delta K=\frac{K_{\text{a}}-K_{\text{p}}}{K_{\text{p}}}(100\%)=\frac{E-U_{\text{a}}-\left(E-U_{\text{p}}\right)}{E-U_{p}}(100\%)=\frac{U_{\text{p}}-U_{\text{a}}}{E-U_{p}}(100\%).$$

Remembering that $E=-\frac{1}{2}G\frac{M_{\text{E}}m}{a}$ where a is the semimajor axis , we can write

$$\Delta K=\frac{U_{\text{p}}-U_{\text{a}}}{E-U_{p}}(100\%)=\frac{\frac{-GM_{\text{E}}M_{\text{S}}}{r_{\text{p}}}+\frac{GM_{\text{E}}M_{\text{S}}}{r_{\text{a}}}}{-\frac{1}{2}G\frac{M_{\text{E}}M_{\text{S}}}{a}+\frac{GM_{\text{E}}M_{\text{S}}}{r_{\text{p}}}}(100\%)=$$

$$\Delta K=\frac{\frac{1}{r_{\text{a}}}-\frac{1}{r_{\text{p}}}}{\frac{1}{r_{\text{p}}}-\frac{1}{2a}}=\frac{\frac{r_{\text{p}}-r_{\text{a}}}{r_{\text{a}}r_{\text{p}}}}{\frac{2a-r_{\text{p}}}{2ar_{\text{p}}}}(100\%)=\frac{2ar_{\text{p}}\left(r_{\text{p}}-r_{\text{a}}\right)}{r_{\text{a}}r_{\text{p}}\left(2a-r_{\text{p}}\right)}(100\%).$$

CALCULATE:

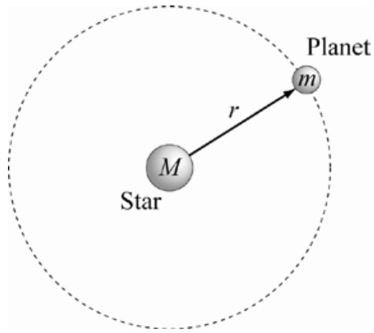
(a) $\Delta U=\frac{\left(147.1\cdot 10^{6}\text{ km}\right)-\left(152.1\cdot 10^{6}\text{ km}\right)}{\left(152.1\cdot 10^{6}\text{ km}\right)}(100\%)=-3.28731\%.$

(b) $\Delta K=\frac{2\left(149.6\cdot 10^{6}\text{ km}\right)\left(147.1\cdot 10^{6}\text{ km}\right)\left(147.09\cdot 10^{6}\text{ km}-152.1\cdot 10^{6}\text{ km}\right)}{\left(152.1\cdot 10^{6}\text{ km}\right)\left(147.1\cdot 10^{6}\text{ km}\right)\left(2\left(149.6\cdot 10^{6}\text{ km}\right)-147.1\cdot 10^{6}\text{ km}\right)}(100\%)=-6.46656\%.$

ROUND: The answer will be rounded to four significant figures.
(a) The potential energy increases (becomes less negative) by 3.287% from going from the perihelion to the aphelion.
(b) To conserve energy, the kinetic energy must decrease by 6.467%.
DOUBLE-CHECK: Such a modest change in gravitational potential energy and kinetic energy is reasonable, because the orbit of the Earth around the Sun is nearly circular.

12.74. **THINK:** The linear velocity, period, and the total mechanical energy are desired for a planet of mass $m=7.00\cdot 10^{21}$ kg in circular orbit around a star of mass $M=2.00\cdot 10^{30}$ kg. The radius of the orbit is $r=3.00\cdot 10^{10}$ m.

SKETCH:



RESEARCH:
(a) For a circular orbit the linear velocity (or orbital speed) is found by setting the centripetal force equal to the force of gravity. The linear velocity is equal to $v = \sqrt{\frac{GM}{r}}$ for a body in a circular orbit.

(b) The period of a circular orbit is $T = \frac{2\pi r}{v}$.

(c) The total energy for an orbiting object is simply $E = \frac{U}{2} = -\frac{GMm}{2r}$.

SIMPLIFY: The equations are already in simplified form.

CALCULATE:

(a) The linear velocity is: $v = \sqrt{\frac{\left(6.674 \cdot 10^{-11} \text{ N m}^2/\text{kg}^2\right)\left(2.00 \cdot 10^{30} \text{ kg}\right)}{\left(3.00 \cdot 10^{10} \text{ m}\right)}} = 66.703 \text{ km/s}.$

(b) The period is $T = \frac{2\pi\left(3.00 \cdot 10^{10} \text{ m}\right)}{6.6703 \cdot 10^4 \text{ m/s}} = 2825880 \text{ s}.$

(c) The total mechanical energy of the planet is:

$$E = -\frac{\left(6.674 \cdot 10^{-11} \text{ N m}^2/\text{kg}^2\right)\left(2.00 \cdot 10^{30} \text{ kg}\right)\left(7.00 \cdot 10^{21} \text{ kg}\right)}{2\left(3.00 \cdot 10^{10} \text{ m}\right)} = -1.5573 \cdot 10^{31} \text{ J}.$$

ROUND: The answers should be rounded to three significant figures.

(a) The linear velocity is $v = 6.67 \cdot 10^4$ m/s.

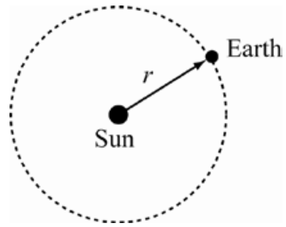
(b) The period of the circular orbit is $T = 32.7$ days.

(c) The total mechanical energy of the orbit is $E = -1.56 \cdot 10^{31}$ J.

DOUBLE-CHECK: These are typical values for an orbit. Recall that the value for the mechanical energy of an orbit is always negative because the kinetic energy of a circular orbit is half the gravitational potential energy.

12.75. **THINK:** Given the distance $r = 1.4960 \cdot 10^{11}$ m and period $T = 3.1557 \cdot 10^7$ s of the Earth's orbit about the Sun, along with the gravitational constant $G = 6.6738 \cdot 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$, the mass M of the Sun can be calculated by using Kepler's third law. The radius and period for the orbit are $1.4960 \cdot 10^{11}$ m and $3.1557 \cdot 10^7$ s, respectively.

SKETCH:



RESEARCH: The mass of the Sun can be found by using Kepler's third law: $\frac{T^2}{r^3}=\frac{4\pi^2}{GM}$.

SIMPLIFY: Solving for M gives: $M=\frac{4\pi^2 r^3}{GT^2}$.

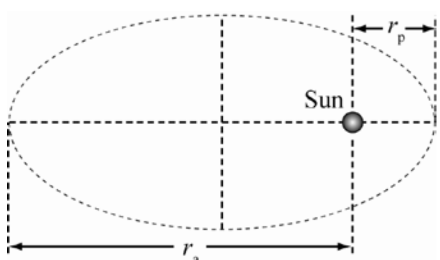
CALCULATE: $M=\frac{4\pi^2\left(1.4960\cdot 10^{11}\text{ m}\right)^3}{\left(6.6738\cdot 10^{-11}\text{ m}^3\text{kg}^{-1}\text{s}^{-2}\right)\left(3.1557\cdot 10^7\text{ s}\right)^2}=1.98879\cdot 10^{30}\text{ kg}$

ROUND: The values are given to five significant figures, thus, the answer also has this accuracy. The mass of the Sun is $1.9888\cdot 10^{30}$ kg.

DOUBLE-CHECK: This agrees with the known value of $1.99\cdot 10^{30}$ kg.

12.76. **THINK:** The question asks to find the ratio of the speed of Pluto at perihelion versus aphelion. The perihelion and aphelion distances are $r_{\text{p}}=4410\cdot 10^6$ km and $r_{\text{a}}=7360\cdot 10^6$ km.

SKETCH:



RESEARCH: The conservation of angular momentum is all that is needed to answer this question: $L=mrv$.

SIMPLIFY: $\frac{v_{\text{p}}}{v_{\text{a}}}=\frac{L/mr_{\text{p}}}{L/mr_{\text{a}}}=\frac{1/r_{\text{p}}}{1/r_{\text{a}}}=\frac{r_{\text{a}}}{r_{\text{p}}}$

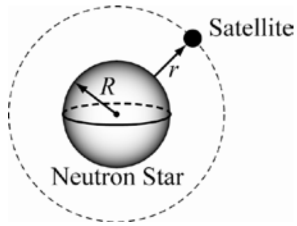
CALCULATE: $\frac{v_{\text{p}}}{v_{\text{a}}}=\frac{7360\cdot 10^6\text{ km}}{4410\cdot 10^6\text{ km}}=1.66893$

ROUND: This value is known to three significant figures. The ratio of Pluto's speed at perihelion relative to aphelion is 1.67.

DOUBLE-CHECK: This indicates that the speed at the perihelion is greater than at the aphelion, which is known to be true.

12.77. **THINK:** This question explores the properties of neutron stars. Consider a neutron star of radius $R=10.0$ km and mass $M_{\text{N}}=2M_{\text{S}}$. The question asks to calculate several things: the speed at the equator of the neutron star, the g value of the star, the weight of a 1.00 kg mass on its surface, the number of revolutions a satellite makes in a minute while orbiting at a distance $r=10.0$ km from the surface of a neutron star and the radius of the geostationary orbit. The period of rotation of the star about its axis is $T_N=1$ s. The satellite is orbiting a distance of 10.0 km above the star.

SKETCH:



RESEARCH:

(a) The speed of a point on the equator is given by $T = \dfrac{2\pi r}{v}$.

(b) The acceleration of gravity at the surface is $g_{\text{N}} = \dfrac{GM_{\text{N}}}{r^2}$.

(c) The weight on the surface is given by $F_{\text{N}} = mg_{\text{N}}$ for the Neutron star and $F_{\text{E}} = mg$ on Earth.

(d) and (e) The revolutions per minute of the satellite and the radius of the geostationary orbit are found by using Kepler's third law, $\dfrac{T^2}{r^3} = \dfrac{4\pi^2}{GM_{\text{N}}}$.

SIMPLIFY:

(a) The velocity at the equator is $v = \dfrac{2\pi r}{\text{T}_{\text{N}}}$.

(d) The revolutions per minute of the satellite is $S_{\text{rpm}} = \dfrac{1}{T} = \sqrt{\dfrac{GM_{\text{N}}}{4\pi^2 (r+R)^3}} \left(\dfrac{60 \text{ s}}{\text{min}} \right)$.

(e) The radius of the geostationary orbit is $r_{\text{geo}} = \left(\dfrac{GM_{\text{N}} T_{\text{N}}^2}{4\pi^2} \right)^{\frac{1}{3}}$.

CALCULATE:

(a) $v = \dfrac{2\pi \left(10.0 \cdot 10^3 \text{ m}\right)}{(1.00 \text{ s})} = 6.28319 \cdot 10^4 \text{ m/s}$

(b) $g_{\text{N}} = \dfrac{\left(6.674 \cdot 10^{-11} \text{ N m}^2/\text{kg}^2\right)\left(2 \cdot 1.99 \cdot 10^{30} \text{ kg}\right)}{\left(10.0 \cdot 10^3 \text{ m}\right)^2} = 2.65625 \cdot 10^{12} \text{ m/s}^2$

(c) $\dfrac{F_{\text{N}}}{F_{\text{E}}} = \dfrac{g_{\text{N}}}{g} = \dfrac{2.65625 \cdot 10^{12} \text{ m/s}^2}{9.81 \text{ m/s}^2} = 2.7077 \cdot 10^{11}$

(d) $S_{\text{rpm}} = \sqrt{\dfrac{\left(6.674 \cdot 10^{-11} \text{ N m}^2/\text{kg}^2\right)\left(2 \cdot 1.99 \cdot 10^{30} \text{ kg}\right)}{4\pi^2 \left(20.0 \cdot 10^3 \text{ m}\right)^3}} \left(\dfrac{60 \text{ s}}{\text{min}} \right) = 5.5025 \cdot 10^4 \text{ rpm}$

(e) $r_{\text{geo}} = \left(\dfrac{\left(6.674 \cdot 10^{-11} \text{ N m}^2/\text{kg}^2\right)\left(2 \cdot 1.99 \cdot 10^{30} \text{ kg}\right)(1.00 \text{ s})^2}{4\pi^2} \right)^{\frac{1}{3}} = 1.8879 \cdot 10^6 \text{ m}$

ROUND: The values should be rounded to three significant figures.

(a) The speed of a point on the equator is $6.28 \cdot 10^4$ m/s for this neutron star. This is very fast, the speed about the equator of the Earth is about 464 m/s or 135 times smaller.

(b) The acceleration of gravity at the surface of the neutron star is $2.66 \cdot 10^{12} \text{ m/s}^2$.

(c) The weight of a 1.00 kg object is $2.71 \cdot 10^{11}$ times bigger than on the Earth.

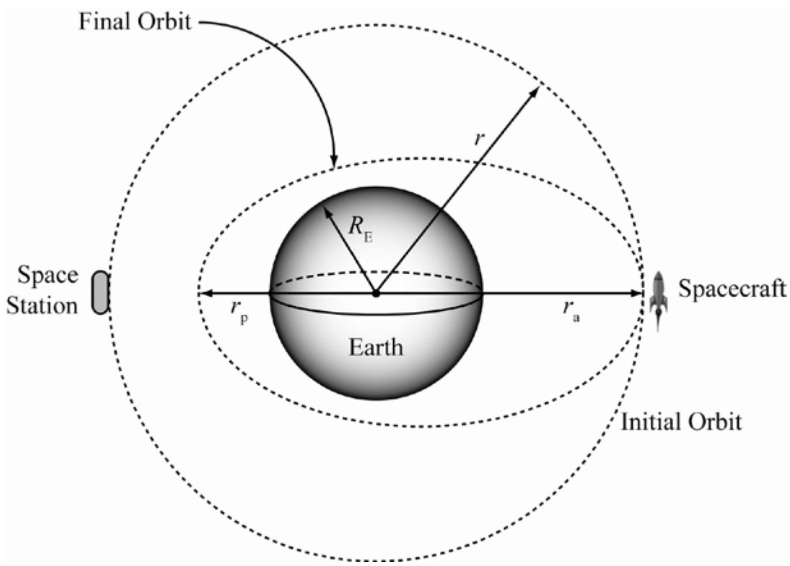
(d) The satellite completes $5.50\cdot 10^4$ revolutions per minute.

(e) The distance for a geostationary orbit is $1.89\cdot 10^6$ m.

DOUBLE-CHECK: These extreme values are reasonable since neutrons stars are extremely dense. They rotate very fast because as the star compresses and becomes smaller the star must rotate faster in order to conserve angular momentum.

12.78. **THINK:** The minimum distance at perihelion for the new elliptical orbit must be greater than the radius of the Earth or I will crash into the surface of the Earth. The new period is half the old period and the radius is $2.5000\cdot 10^4$ km.

SKETCH:



RESEARCH: Use the subscript 1 to refer to the circular orbit, and the subscript 2 to denote the new elliptical orbit. This means that $r = a_1 = r_a$. To find the new semi-major axis a_2 use Kepler's third law, $T_1^2/a_1^3 = T_2^2/a_2^3$. The perihelion distance r_p can be found using $r_a + r_p = 2a_2$.

SIMPLIFY: $a_2 = \left(\frac{T_2^2}{T_1^2}\right)^{\frac{1}{3}} a_1 = \left(\frac{T_2}{T_1}\right)^{\frac{2}{3}} r = \left(\frac{1}{2}\right)^{\frac{2}{3}} r$. Using this I can find the distance at perihelion:

$$r_a + r_p = r + r_p = 2\left(\frac{1}{2}\right)^{\frac{2}{3}} r \Rightarrow r_p = \left(2\left(\frac{1}{2}\right)^{\frac{2}{3}} - 1\right) r.$$

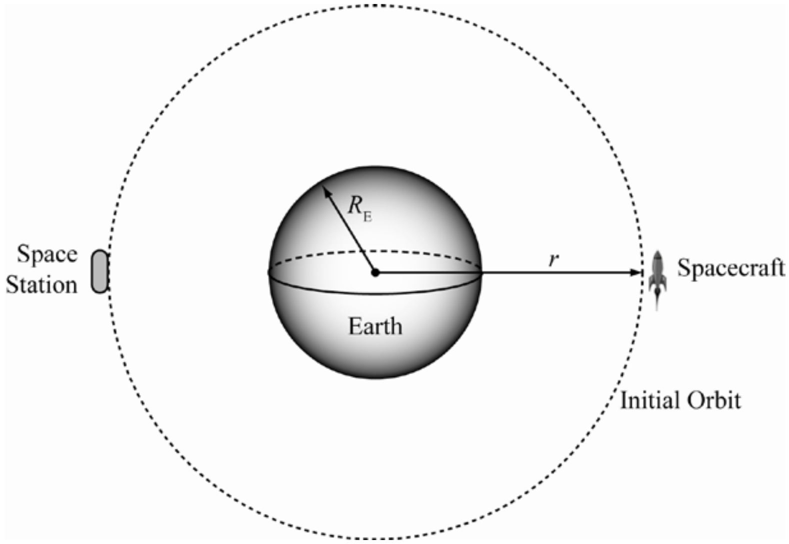
CALCULATE: $r_p = \left(2\left(\frac{1}{2}\right)^{\frac{2}{3}} - 1\right)\left(2.5000\cdot 10^4 \text{ km}\right) = 6498.026 \text{ km}$

ROUND: To five significant figures, the minimum distance at perihelion is 6498.0 km which is greater than the radius of the Earth, but not by much.

DOUBLE-CHECK: The space ship with come close to the Earth but not crash. The minimum radius is just less than the original radius, so the result is reasonable.

12.79. **THINK:** The space ship starts at 6720 km from the center of the Earth. Instead of slowing the spacecraft, the forward boosters will be applied to put the craft in an elliptical orbit. The current position is the distance at perihelion, r_{p}. The new period should be 1.5 times greater than the present one so that the spaceship meets the station.

SKETCH:



RESEARCH: Use the subscript 1 to refer to the circular orbit, and the subscript 2 to denote the new elliptical orbit. This means that $r = a_1 = r_{\text{p}}$. To find the new semi-major axis we use Kepler's third law, $T_1^2/a_1^3 = T_2^2/a_2^3$. The distance at aphelion, r_{a}, can be found by using the equation $r_{\text{a}} + r_{\text{p}} = 2a$. The period of the new orbit can be found by using $T^2/a^3 = 4\pi^2/GM$.

SIMPLIFY: The distance at perihelion is r, the radius of the original orbit. $a_2 = \left(\frac{T_2^2}{T_1^2}\right)^{\frac{1}{3}} a_1 = \left(\frac{T_2}{T_1}\right)^{\frac{2}{3}} r = \left(\frac{3}{2}\right)^{\frac{2}{3}} r.$ Using this, find the distance at aphelion: $r_{\text{a}} + r_{\text{p}} = r_{\text{a}} + r = 2\left(\frac{3}{2}\right)^{\frac{2}{3}} r \Rightarrow r_{\text{a}} = \left(2\left(\frac{3}{2}\right)^{\frac{2}{3}} - 1\right) r.$ The new period is $\text{T}_2^2 = \left(\frac{4\pi^2}{GM}\right) a_2^3 \Rightarrow T_2 = \sqrt{\left(\frac{9\pi^2}{GM}\right) r^3}.$

CALCULATE: $r_{\text{a}} = \left(2\left(\frac{3}{2}\right)^{\frac{2}{3}} - 1\right)(6720 \text{ km}) = 10891 \text{ km}$

$$T_2 = \sqrt{\frac{9\pi^2}{\left(6.674 \cdot 10^{-11} \text{ Nm}^2/\text{kg}^2\right)\left(5.97 \cdot 10^{24} \text{ kg}\right)}\left(6720 \cdot 10^3 \text{ m}\right)^3} = 8225.2 \text{ s}$$

ROUND: The answers should be rounded to three significant figures. For the new orbit, the distance at perihelion is $6.72 \cdot 10^3$ km, the distance at aphelion is $1.09 \cdot 10^4$ km, and the period is $8.23 \cdot 10^3 \text{ s} = 2.28$ hours.

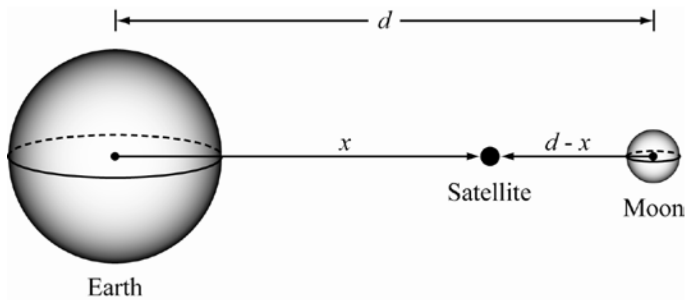
DOUBLE-CHECK: The period of the original orbit was

$$T_1 = \sqrt{\left(\frac{4\pi^2}{GM}\right)r^3} = \sqrt{\frac{4\pi^2}{\left(6.674\cdot 10^{-11}\text{ Nm}^2/\text{kg}^2\right)\left(5.97\cdot 10^{24}\text{ kg}\right)}\left(6720\cdot 10^3\text{ m}\right)^3} = 5480\text{ s} = 1.52\text{ hours}.$$

Since $\frac{T_2}{T_1} = \frac{2.28\text{ hours}}{1.52\text{ hours}} = \frac{3}{2}$, the spacecraft will rendezvous with the station, as was required.

12.80. **THINK:** For the satellite to stay in orbit between the Earth and the Moon, the period of the orbit of the satellite must be the same as the period of the orbit of the Moon. The satellite is to stay between the Earth and the Moon with a period the same as that of the Moon: $T = 27.3$ days. Newton's second law and Newton's Law of Gravity can be used to calculate the correct placement of the satellite.

SKETCH:



RESEARCH: The gravitational forces on the satellite must equal its centripetal force, $F_c = F_E - F_M$. The period is related to the velocity by $T = \frac{2\pi r}{v}$.

SIMPLIFY: Newton's second law: $F_c = F_E - F_M = \frac{mv^2}{x} = \frac{GM_E m}{x^2} - \frac{GM_M m}{(d-x)^2}$.

Using $v = \frac{2\pi x}{T}$, $\frac{4\pi^2 x}{T^2} = \frac{GM_E}{x^2} - \frac{GM_M}{(d-x)^2}$. This equation can be solved using math software such as a computer algebra system.

CALCULATE: Using $G = 6.674\cdot 10^{-11}\text{ N m}^2/\text{kg}^2$, $M_E = 5.97\cdot 10^{24}$ kg, $M_M = 7.36\cdot 10^{22}$ kg, $T = 27.3\text{ days} = 2358720$ s, and $d = 3.84\cdot 10^8$ m a computer program will verify that there is one real root, with a value of approximately $x = 3.25695821\cdot 10^8$ m.

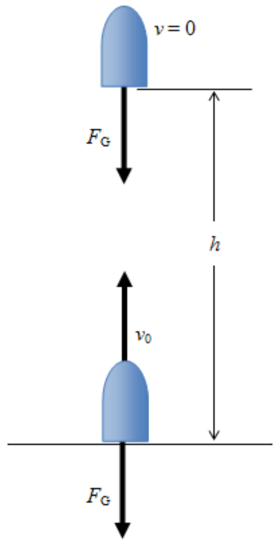
ROUND: To three significant figures, the proper radius of the orbit is $3.26\cdot 10^8$ m from the Earth's center.

DOUBLE-CHECK: The distance the satellite is from Earth is reasonable since it is 85% of the distance to the Moon. It is expected that the satellite should be closer to the Moon since the Moon is less massive than the Earth. Note that this distance not the same as the distance where the gravitational force from the Earth balances the gravitational force from the Moon, which is 90% of the distance from the Earth to the Moon.

Multi-Version Exercises

12.81. **THINK:** When it is launched, the projectile has kinetic energy. At its highest point, it is not moving at all; the energy is now in the form of gravitational potential energy. So, find the kinetic energy of the projectile when it was launched, and that should equal the gravitational potential energy of the projectile at its maximum height.

SKETCH: The projectile when it is launched and at maximum height are shown, with arrows indicating the gravitational force and velocity vectors. Take the surface of the moon to be height $y = 0$ m.



RESEARCH: The acceleration due to gravity on the surface of the moon can be found from Newton's Law of Gravity to be $g_{\text{M}} = \frac{Gm_{\text{M}}}{R_{\text{M}}^2}$. The kinetic energy of the projectile when it is launched is given by $K_0 = \frac{1}{2}mv_0^2$. The change in potential energy of the projectile is given by $\Delta U = -\frac{Gmm_{\text{M}}}{R_{\text{M}} + h} + \frac{Gmm_{\text{M}}}{R_{\text{M}}}$. Conservation of energy gives us $K_0 = \Delta U$. Since the radius of the moon is given in kilometers and the initial velocity of the projectile is given in meters per second, the conversion factor of 1 km = 1000 m will also be necessary.

SIMPLIFY: Since $K_0 = \Delta U$, substitute in on both sides for the kinetic energy and change in potential energy to get $\frac{1}{2}mv_0^2 = -\frac{Gmm_{\text{M}}}{R_{\text{M}} + h} + \frac{Gmm_{\text{M}}}{R_{\text{M}}}$. Use algebra to solve for the maximum height h:

$$m\left(\tfrac{1}{2}v_0^2\right) = m\left(\frac{Gm_{\text{M}}}{R_{\text{M}}} - \frac{Gm_{\text{M}}}{R_{\text{M}} + h}\right)$$

$$\tfrac{1}{2}v_0^2 = \frac{Gm_{\text{M}}}{R_{\text{M}}} - \frac{Gm_{\text{M}}}{R_{\text{M}} + h}$$

$$\frac{Gm_{\text{M}}}{R_{\text{M}} + h} = \frac{Gm_{\text{M}}}{R_{\text{M}}} - \frac{v_0^2}{2}$$

$$R_{\text{M}} + h = \frac{Gm_{\text{M}}}{\left[\frac{Gm_{\text{M}}}{R_{\text{M}}} - \frac{v_0^2}{2}\right]}$$

$$h = \frac{Gm_{\text{M}}}{\left[\frac{Gm_{\text{M}}}{R_{\text{M}}} - \frac{v_0^2}{2}\right]} - R_{\text{M}}$$

CALCULATE: According to the question, the mass of the moon is $m_{\text{M}} = 7.348 \cdot 10^{22}$ kg, the radius of the moon is R_{M} = 1737 km = 1,737,000 m, and the initial speed of the projectile is $v_0 = 114.5$ m/s. The gravitational constant is $G = 6.674 \cdot 10^{-11}$ N·m² / kg². Using these values, the maximum height of the projectile is

$$h=\frac{Gm_{\text{M}}}{\left[\dfrac{Gm_{\text{M}}}{R_{\text{M}}}-\dfrac{v_0^2}{2}\right]}-R_{\text{M}}$$

$$=\frac{6.674\cdot 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2\cdot 7.348\cdot 10^{22}\text{ kg}}{\left[\dfrac{6.674\cdot 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2\cdot 7.348\cdot 10^{22}\text{ kg}}{1.737\cdot 10^{6}\text{ m}}-\dfrac{(114.5\text{ m/s})^2}{2}\right]}-1.737\cdot 10^{6}\text{ m}$$

$$=4042.358098\text{ m}$$

ROUND: The numbers used in the calculation all have four significant figures, and the final answer should also have four figures. The projectile reaches a height of 4042 m or 4.042 km.

DOUBLE-CHECK: On Earth, a projectile shot up with an initial velocity v_0 = 114.5 m/s would reach a maximum height of about 670 m. The mass of the Moon is about one sixth that of the Earth, so the object will go much higher.

12.82. As determined in the preceding solution, $\frac{1}{2}v_0^2=\dfrac{Gm_{\text{M}}}{R_{\text{M}}}-\dfrac{Gm_{\text{M}}}{R_{\text{M}}+h}$. Then

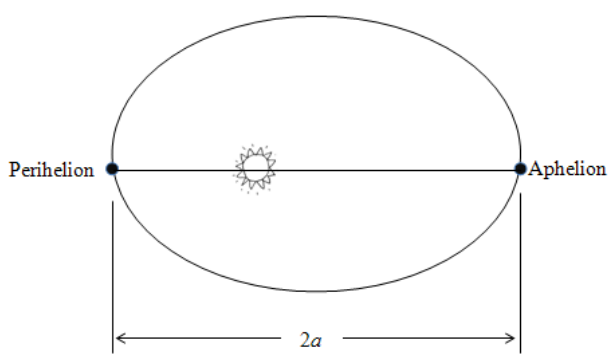
$$v_0=\sqrt{2Gm_{\text{M}}\left(\frac{1}{R_{\text{M}}}-\frac{1}{R_{\text{M}}+h}\right)}$$

$$=\sqrt{2\left(6.674\cdot 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2\right)\left(7.348\cdot 10^{22}\text{ kg}\right)\left(\frac{1}{1.737\cdot 10^{6}\text{ m}}-\frac{1}{\left(1.737\cdot 10^{6}\text{ m}\right)+\left(4.905\cdot 10^{3}\text{ m}\right)}\right)}$$

$$=126.1\text{ m/s}$$

12.83. **THINK:** Kepler's law can be used to compute the semimajor axis from the period. Then, use the geometric fact that the location of the comet at aphelion is equal to twice the semimajor axis minus the location at perihelion to find the aphelion.

SKETCH: The semimajor axis (length 2*a*), as well as the location of the comet at perihelion and aphelion are labeled.



RESEARCH: According to Kepler's third law, for all of the objects orbiting a given star $\dfrac{T^2}{a^3}=\text{constant}$.

Since both the comet and the Earth orbit the Sun, $\dfrac{T_{\text{Earth}}^2}{a_{\text{Earth}}^3}=\dfrac{T_{\text{Comet}}^2}{a_{\text{Comet}}^3}$. To find the location of the comet at perihelion, use the fact that the perihelion + aphelion = 2*a*.

SIMPLIFY: Since $\dfrac{T_{\text{Earth}}^2}{a_{\text{Earth}}^3}=\dfrac{T_{\text{Comet}}^2}{a_{\text{Comet}}^3}$, the semimajor axis of the comet's orbit is given by $a_{\text{Comet}}=\sqrt[3]{\dfrac{T_{\text{Comet}}^2\cdot a_{\text{Earth}}^3}{T_{\text{Earth}}^2}}$. Then the location of the comet at aphelion is given by

$$\text{aphelion} = 2a_{\text{Comet}} - \text{perihelion}$$
$$= 2\sqrt[3]{\frac{T_{\text{Comet}}^2 \cdot a_{\text{Earth}}^3}{T_{\text{Earth}}^2}} - \text{perihelion.}$$

CALCULATE: The comet orbits the sun with a period of 89.17 years, and it is 1.331 AU from the sun at perihelion. The Earth orbits the sun with a period of 1 year and has a semimajor axis of 1 AU. At aphelion, the distance between the comet and the sun is

$$\text{aphelion} = 2\sqrt[3]{\frac{T_{\text{Comet}}^2 \cdot a_{\text{Earth}}^3}{T_{\text{Earth}}^2}} - \text{perihelion}$$
$$= 2\sqrt[3]{\frac{(89.17\text{ yr})^2 \cdot (1\text{ AU})^3}{(1\text{ yr})^2}} - 1.331\text{ AU}$$
$$= 38.5876495\text{ AU.}$$

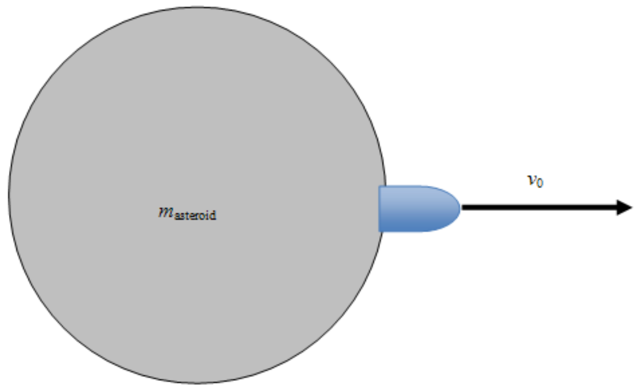
ROUND: Since the numbers used in the calculations all have four figures, the final answer should also have four figures. The distance between the comet and the sun at aphelion is 38.59 AU.
DOUBLE-CHECK: Generally, comets have very eccentric orbits, so it is reasonable that the comet is more than ten times further from the sun at aphelion than it is at perihelion. The comet travels a much greater distance than the Earth, which makes sense as it takes almost 90 times longer to orbit the Sun.

12.84. $$\text{aphelion} = 2\sqrt[3]{\frac{T_{\text{Comet}}^2 \cdot a_{\text{Earth}}^3}{T_{\text{Earth}}^2}} - \text{perihelion}$$
$$\text{perihelion} = 2\sqrt[3]{\frac{T_{\text{Comet}}^2 \cdot a_{\text{Earth}}^3}{T_{\text{Earth}}^2}} - \text{aphelion}$$
$$= 2\sqrt[3]{\frac{(98.11\text{ yr})^2 (1\text{ AU})^3}{(1\text{ yr})^2}} - 41.19\text{ AU}$$
$$= 1.35\text{ AU}$$

12.85. $$\text{aphelion} = 2\sqrt[3]{\frac{T_{\text{Comet}}^2 \cdot a_{\text{Earth}}^3}{T_{\text{Earth}}^2}} - \text{perihelion}$$
$$T_{\text{Comet}} = \sqrt{\frac{T_{\text{Earth}}^2}{a_{\text{Earth}}^3}\left(\frac{\text{aphelion} + \text{perihelion}}{2}\right)^3}$$
$$= \sqrt{\frac{(1\text{ yr})^2}{(1\text{ AU})^3}\left(\frac{31.95\text{ AU} + 1.373\text{ AU}}{2}\right)^3}$$
$$= 68.01\text{ yr}$$

12.86. **THINK:** To escape from the asteroid's gravitational influence, the total kinetic energy of the object must be greater than or equal to the gravitational potential energy.

SKETCH: The escape speed is the minimum speed that a projectile will need to escape the pull of the asteroid's gravity.



RESEARCH: The kinetic energy of a projectile when it is shot with a speed v from the surface of the asteroid is $K = \frac{1}{2}m_0v_0^2$. The (absolute value) gravitational potential energy is $U = \frac{Gm_0m_{\text{asteroid}}}{R_{\text{asteroid}}}$. To escape from the asteroid, the kinetic energy must be greater than or equal to the potential energy, $K \geq U$. The minimum escape speed occurs when they are equal.

SIMPLIFY: First, set the kinetic and potential energies equal to one another $\frac{1}{2}m_0v_0^2 = K = U = \frac{Gm_0m_{\text{asteroid}}}{R_{\text{asteroid}}}$.

Solve this for the escape speed:

$$\frac{1}{2}m_0v_0^2 = \frac{Gm_0m_{\text{asteroid}}}{R_{\text{asteroid}}}$$
$$v_0^2 = \frac{2Gm_{\text{asteroid}}}{R_{\text{asteroid}}}$$
$$v_0 = \sqrt{\frac{2Gm_{\text{asteroid}}}{R_{\text{asteroid}}}}$$

CALCULATE: The gravitational constant is $G = 6.674 \cdot 10^{-11}$. The mass of the asteroid is $m_{\text{asteroid}} = 1.869 \cdot 10^{20}$ kg, and the asteroid has a radius of $R_{\text{asteroid}} = 358.9$ km = 358,900 m. The escape speed is

$$v_0 = \sqrt{\frac{2Gm_{\text{asteroid}}}{R_{\text{asteroid}}}} = \sqrt{\frac{2 \cdot 6.674 \cdot 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2 \cdot 1.869 \cdot 10^{20}\ \text{kg}}{358900\ \text{m}}} = 263.6489345\ \text{m/s}.$$

ROUND: Since all of the numbers in this problem have four significant figures, the final answer should also have four significant figures. The escape speed from this asteroid is 263.6 m/s.

DOUBLE-CHECK: 263.6 m/s is faster than most objects on Earth, but it is less than the escape speed from Earth, which makes sense as the asteroid is less massive than Earth. Although the energy that is needed to escape from the asteroid will depend on the mass of the object, the minimum speed that it needs to achieve depends only on the mass and radius of the asteroid (and the gravitational constant). Any object that leaves the asteroid, heading away from the asteroid at the escape speed or greater, will have enough energy to escape the gravitational pull of the asteroid.

12.87. $v_0 = \sqrt{\frac{2Gm_{\text{asteroid}}}{R_{\text{asteroid}}}}$

$$R_{\text{asteroid}} = \frac{2Gm_{\text{asteroid}}}{v_0^2}$$
$$= \frac{2\left(6.674 \cdot 10^{-11} \text{ N} \cdot \text{m}^2 / \text{kg}^2\right)\left(1.769 \cdot 10^{20} \text{ kg}\right)}{\left(273.7 \text{ m/s}\right)^2}$$
$$= 315.2 \text{ km}$$

12.88. $v_0 = \sqrt{\frac{2Gm_{\text{asteroid}}}{R_{\text{asteroid}}}}$

$$m_{\text{asteroid}} = \frac{v_0^2 R_{\text{asteroid}}}{2G}$$
$$= \frac{\left(319.2 \text{ m/s}\right)^2\left(365.1 \cdot 10^3 \text{ m}\right)}{2\left(6.674 \cdot 10^{-11} \text{ N} \cdot \text{m}^2 / \text{kg}^2\right)}$$
$$= 2.787 \cdot 10^{20} \text{ kg}$$

Chapter 13: Solids and Fluids

Concept Checks

13.1. a **13.2.** d **13.3.** d **13.4.** a **13.5.** c **13.6.** i) b ii) a **13.7.** c **13.8.** e

Multiple-Choice Questions

13.1. a **13.2.** c **13.3.** d **13.4.** b **13.5.** $F_2 = F_3 > F_1$ **13.6.** c **13.7.** d **13.8.** d **13.9.** e **13.10.** c **13.11.** b **13.12.** c **13.13.** a **13.14.** a **13.15.** a

Conceptual Questions

13.16. When the car decelerates, loose heavy objects within the car tend to continue forward because of inertia. Air, which is denser than helium, will have more inertia than helium and thus moves forward, displacing the lighter helium balloon, which then moves to the rear. One can also view the accelerated frame of reference inside the car having a fictitious "gravity", pointing to the right during braking (right being toward the front on the car). This gives rise to a "buoyant force" directed to the left that "buoys" the less dense helium balloon.

13.17. The paper will move down, narrowing the gap between the paper and the table. The higher velocity air between the paper and the table results in a lower pressure on the bottom of the paper. The net force on the paper is then down and the gap narrows.

13.18. The air inside the shower is being pushed by the water so the speed inside is greater than the air speed outside. From Bernoulli's equation, that means that the pressure inside is lower so that the curtain will be pushed inward, toward the shower.

13.19. The flaw in this statement concerns the amount of work done by the input and output forces in a hydraulic lift device. While the output force is much larger than the input force, the corresponding displacement is much smaller, and the work done by both forces is the same.

13.20. The steel has the higher spring constant. As the motion of the spring tends to cause a shear deformation and not volume deformation of individual segments, the system depends heavily on the shear modulus. Since steel resists this more, it has the higher spring constant.

13.21. No, a higher density is not necessarily due to a material's heavier molecules. The first material could have a higher density because its molecules are more closely packed together than the molecules of the second, although the molecules of the second are heavier. Gold and lead are typical examples of this case. Gold has an atomic number of 79 and a density of 19.3 g/cm^3. Lead has atomic number 82, but its density, 11.35 g/cm^3, which is less than that of gold.

13.22. The force exerted on the balance by an object of mass, m, and density, ρ, is the weight of the object minus the buoyant force exerted on the object by the air:

$$F = mg - \rho_\text{a}\frac{m}{\rho}g = mg\left(1-\rho_\text{a}/\rho\right),$$

where m/ρ is the volume of the object. The balance is calibrated to compensate for this for steel test masses. Hence, the mass displayed by the balance is

$$m_\text{bal} = \frac{F}{g\left(1-\rho_\text{a}/\rho_\text{s}\right)} = m\left(\frac{1-\rho_\text{a}/\rho}{1-\rho_\text{a}/\rho_\text{s}}\right).$$

To obtain the true mass, one must correct the balance reading with the factor:

$$m = m_{\text{bal}}\left(\frac{1-\rho_{\text{a}}/\rho_{\text{s}}}{1-\rho_{\text{a}}/\rho}\right).$$

Of course, the density of the sample must be known to make this correction. Yes, it can matter. For example, if the sample being weighed is aluminum, with a density of $\rho = 2694.1\text{ kg/m}^3$, then the correction is

$$m_{\text{Al}} = m_{\text{bal}}\left(\frac{1-\dfrac{\rho_{\text{a}}}{\rho_{\text{s}}}}{1-\dfrac{\rho_{\text{a}}}{\rho_{\text{Al}}}}\right) = m_{\text{bal}}\left(\frac{1-\dfrac{1.205\text{ kg/m}^3}{8000.00\text{ kg/m}^3}}{1-\dfrac{1.205\text{ kg/m}^3}{2694.1\text{ kg/m}^3}}\right) = 1.000297 m_{\text{bal}}.$$

The correction amounts to almost 300 parts per million. But a good analytical balance can weight a hundred-gram mass to tenths of a milligram, i.e. to parts per million. For measurements at the full sensitivity of the balance, then the buoyancy correction is quite significant.

13.23. The flow rate of an incompressible fluid like water is constant. This means that the flow rate through any cross-section of the system is constant. But the speed of the water is greater near the bottom of the sink compared to near the spigot since the water accelerates due to gravity. Since the flow rate is constant, the cross-sectional area of the stream must be smaller near the bottom of the sink compared to near the spigot. That is, the stream narrows as it leaves the spigot.

13.24. In fact, most fluid-flow problems become horribly complicated without viscosity, because only viscosity preserves laminar flow. Without viscous forces between elements of the fluid, the elements can move independently of their neighbors, thus producing the tremendous complexity of turbulence. The lower the viscosity, the higher the Reynolds number, and high Reynolds numbers lead to turbulent flow.

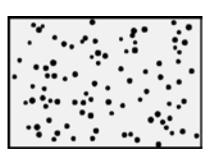
13.25. The buoyant force on the sphere in fluid A is less than the buoyant force on the sphere in fluid B. The weights of the spheres are the same; therefore, the buoyant force, which points upward, will determine if the object sinks or floats. If it sinks, F_{b} is less than the weight and if it floats, F_{b} is greater than the weight.

13.26. The water free-falls at a speed given by $v(y) = \left(v_0^2 + 2gy\right)^{1/2}$, where g is the acceleration of gravity. By continuity, the radius of the stream follows from $\pi r_0^2 v_0 = \pi r^2(y)v(y) = \pi r^2(y)\left(v_0^2 + 2gy\right)^{1/2}$. This yields $r(y) = \dfrac{r_0}{\left(1+\dfrac{2g}{v_0^2}y\right)^{1/4}}$ for the radius of the falling stream.

Exercises

13.27. **THINK:** The density at sea-level and the volume can used to determine the total mass. From the molar mass, the number of moles and thus the number of molecules can be determined. V = 0.50 L, $A = 28.95\text{ g/mol}$ and $\rho = 1.229\text{ kg/m}^3$.

SKETCH:



RESEARCH: $m = \rho V$

The number of moles, n, is given by $n = m/A$ (m in grams and A in grams per mole). The number of molecules, N, is given by $N = nN_{\text{A}}$, where $N_{\text{A}} = 6.02\cdot 10^{23}$ molecules/mol.

SIMPLIFY: $N = nN_\text{A} = \dfrac{m}{A}N_\text{A} = \dfrac{\rho V N_\text{A}}{A}$

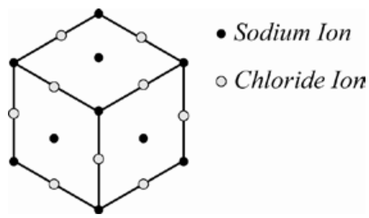
CALCULATE: $N = \dfrac{\left(1.229 \text{ kg/m}^3\right)0.50 \text{ L}\left(1 \text{ m}^3 / 10^3 \text{ L}\right)6.02\cdot 10^{23} \text{ molecules/mol}}{28.95\cdot 10^{-3} \text{ kg/mol}} = 1.278\cdot 10^{22} \text{ molecules}$

ROUND: Rounding to two significant figures, $N = 1.3\cdot 10^{22}$ molecules.

DOUBLE-CHECK: *N* is proportional to both the density and the volume. This is a reasonable result.

13.28. **THINK:** The eight sodium ions at the corners of each unit cell are each shared with adjacent cells; each constitutes one-eighth of an ion to a single cell. The six ions at the centers of the cell faces are each shared by two adjacent cells; each constitutes one half to a single cell. Thus, a single unit cell contains four sodium and four chloride ions. The total mass contained in a unit cell can be determined. Through the density, the volume of the unit cell can be determined. This is a cubic lattice and the volume gives the edge length as well. The distance between an adjacent chloride and sodium ion is half the edge length. $m_\text{Na} = 22.99u$, $m_\text{Cl} = 35.45u$ and $\rho = 2.165\cdot 10^3 \text{ kg/m}^3$.

SKETCH:



RESEARCH: $m = \rho V$. The cube edge length is given by $a = V^{1/3}$. The distance between the sodium and chloride ions is given by $d = a/2$.

SIMPLIFY: $m = 4\left(m_\text{Na} + m_\text{Cl}\right)$, $V = \dfrac{m}{\rho} = \dfrac{4}{\rho}\left(m_\text{Na} + m_\text{Cl}\right)$, $d = \dfrac{1}{2}a = \dfrac{1}{2}V^{\frac{1}{3}} = \dfrac{1}{2}\sqrt[3]{\dfrac{4\left(m_\text{Na} + m_\text{Cl}\right)}{\rho}}$

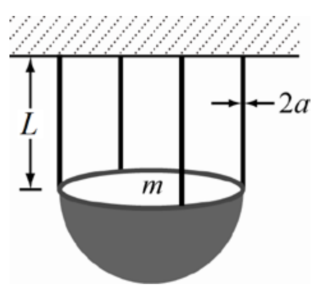
CALCULATE: $d = \dfrac{1}{2}\sqrt[3]{\dfrac{4\left(22.99 \text{ u} + 35.45 \text{ u}\right)\left(1.661\cdot 10^{-27} \text{ kg/u}\right)}{2.165\cdot 10^3 \text{ kg/m}^3}} = 2.820\cdot 10^{-10} \text{ m}$

ROUND: Keeping four significant figures, $d = 2.820$ Å.

DOUBLE-CHECK: This is a typical spacing for ions in a solid and can be measured, for example, by X-ray diffraction.

13.29. **THINK:** Each wire supports 1/4 of the weight of the chandelier. This weight can be determined, then using Young's modulus, determine the amount of stretching. Use the values L = 1 m, $2a$ = 2 mm, m = 20 kg and $Y = 200\cdot 10^9 \text{ N/m}^2$.

SKETCH:



RESEARCH: $F = AY\dfrac{\Delta L}{L}$ and $A = \pi a^2$.

SIMPLIFY: $\Delta L = \dfrac{FL}{AY} = \dfrac{FL}{\pi a^2 Y} = \dfrac{mgL}{4\pi a^2 Y}$

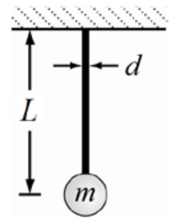
CALCULATE: $\Delta L = \dfrac{(20 \text{ kg})(9.81 \text{ m/s}^2)(1 \text{ m})}{4\pi(1\cdot 10^{-3} \text{ m})^2 200\cdot 10^9 \text{ N/m}^2} = 7.807\cdot 10^{-5} \text{ m}$

ROUND: Rounding to one significant figure, $\Delta L = 0.08$ mm.

DOUBLE-CHECK: This is a reasonable stretch for a steel wire.

13.30. **THINK:** From the total weight of the object, use Young's modulus to determine the amount of stretching. From this, the required diameter can be determined. Use the values: L = 50.0 m, m = 70.0 kg, $\Delta L = 1.00$ cm and $Y = 3.51\cdot 10^9 \text{ N/m}^2$.

SKETCH:



RESEARCH: $F = AY\dfrac{\Delta L}{L}$, $A = \pi(d/2)^2$ and $F = mg$.

SIMPLIFY: $A = \dfrac{FL}{Y\Delta L} \Rightarrow \dfrac{\pi d^2}{4} = \dfrac{mgL}{Y\Delta L} \Rightarrow d = \sqrt{\dfrac{4mgL}{Y\Delta L\pi}}$

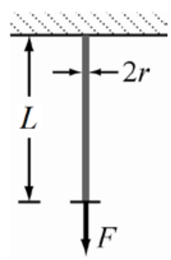
CALCULATE: $d = \sqrt{\dfrac{4(70.0 \text{ kg})(9.81 \text{ m/s}^2)(50.0 \text{ m})}{(3.51\cdot 10^9 \text{ N/m}^2)(0.0100 \text{ m})\pi}} = 3.529\cdot 10^{-2} \text{ m}$

ROUND: Rounding to three significant figures, $d = 3.53$ cm.

DOUBLE-CHECK: The string is rather thick since it is so long. Note that d increases if L increases. This is reasonable.

13.31. **THINK:** Young's modulus can be used to determine the change of length for the given tension. Use the values: F = 90.0 N, L = 2.00 m, r = 0.300 mm and $Y = 20.0\cdot 10^{10} \text{ N/m}^2$.

SKETCH:



RESEARCH: $F = AY\dfrac{\Delta L}{L}$, and $A = \pi r^2$.

SIMPLIFY: $\Delta L = \dfrac{FL}{YA} = \dfrac{FL}{Y\pi r^2}$

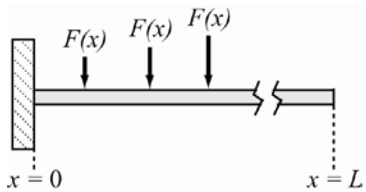
CALCULATE: $\Delta L = \dfrac{(90.0 \text{ N})(2.00 \text{ m})}{(20.0\cdot 10^{10} \text{ N/m}^2)\pi(0.300\cdot 10^{-3} \text{ m})^2} = 0.3183 \text{ cm}$

ROUND: Rounding to three significant figures, $\Delta L = 0.318$ cm.

DOUBLE-CHECK: This is a reasonable result for a wire of the given dimension.

13.32. **THINK:** The shear force can be determined at an arbitrary distance x from the wall. From this, the shear force at the right, center and left ends of the rod can be determined.

SKETCH:



RESEARCH: $dF(x)=\frac{Wx}{L}dx$. Shear force is given by $F(x)=\int_x^L dF(x)$.

SIMPLIFY: $F(x)=\int_x^L \frac{Wx'}{L}dx'=\left[\frac{Wx'^2}{2L}\right]_x^L=\frac{W}{2L}\left(L^2-x^2\right)=\frac{WL}{2}-\frac{Wx^2}{2L}$

CALCULATE:

(a) $F(L)=\frac{WL}{2}-\frac{WL^2}{2L}=0$

(b) $F\left(\frac{L}{2}\right)=\frac{WL}{2}-\frac{W}{2L}\left(\frac{L^2}{4}\right)=\frac{WL}{2}-\frac{WL}{8}=\frac{3}{8}WL$

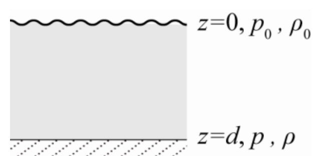
(c) $F(0)=\frac{WL}{2}-0=\frac{WL}{2}$

ROUND: This step is not necessary.

DOUBLE-CHECK: The shear force decreases as a quadratic as the distance along the rod increases. This is reasonable for an applied force which increases linearly.

13.33. **THINK:** The change in pressure can be determined first. Then the change in density can be determined by using the equation for the bulk modulus. For the following, d = 10.922 km is the depth, $p_0=101.3$ kPa, $\rho_0=1024\text{ kg/m}^3$, and $B(p)=B_0+6.67(p-p_0)$ with $B_0=2.19\cdot 10^9$ Pa. We simplify the calculation by assuming that that the density of the water is constant *until just above* depth d at the bottom.

SKETCH:



RESEARCH: $p=p_0+\rho_0 gz,\ \ p=B\frac{\Delta V}{V_0},\ \ V=\frac{m}{\rho}$

SIMPLIFY: The pressure at depth d is given by $p=p_0+\rho_0 gd$. For a given mass of water a decrease in volume results in an increase in pressure, and vice versa. Therefore, $\Delta p=-B\frac{\Delta V}{V_0}$. Taking m as the mass of a volume of seawater at depth d that is smaller than the same mass on the surface by ΔV,

$$\Delta V=\frac{m}{\rho}-\frac{m}{\rho_0}=-m\left(\frac{\rho-\rho_0}{\rho\rho_0}\right)$$

$$\Delta p=p-p_0=\frac{mB}{V_0}\left(\frac{\rho-\rho_0}{\rho\rho_0}\right)=\rho_0 B\left(\frac{\rho-\rho_0}{\rho\rho_0}\right)=B\left(\frac{\rho-\rho_0}{\rho}\right)$$

$$p-p_0=\left(B_0+6.67(p-p_0)\right)\left(\frac{\rho-\rho_0}{\rho}\right)$$

Solving for the density ρ:

$$\rho(p-p_0)=(B_0+6.67(p-p_0))\rho-(B_0+6.67(p-p_0))\rho_0$$
$$\rho\left[(B_0+6.67(p-p_0))-(p-p_0)\right]=(B_0+6.67(p-p_0))\rho_0$$
$$\rho=\frac{B_0+6.67(p-p_0)}{B_0+5.67(p-p_0)}\rho_0$$
$$\rho=\frac{B_0+6.67\rho_0 gd}{B_0+5.67\rho_0 gd}\rho_0$$

CALCULATE: $p=101.3\cdot 10^3\text{ Pa}+(1024\text{ kg/m}^3)(9.807\text{ m/s}^2)(10.922\cdot 10^3\text{ m})=1.098\cdot 10^8\text{ Pa}$

$$\rho=\frac{(2.19\cdot 10^9\text{ Pa})+6.67(1024\text{ kg/m}^3)(9.807\text{ m/s}^2)(10.922\cdot 10^3\text{ m})}{(2.19\cdot 10^9\text{ Pa})+5.67(1024\text{ kg/m}^3)(9.807\text{ m/s}^2)(10.922\cdot 10^3\text{ m})}(1024\text{ kg/m}^3)=1.06394\cdot 10^3\text{ kg/m}^3$$

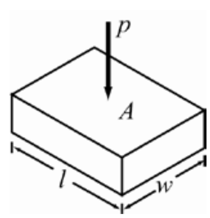
The percent change in density is $\frac{\rho-\rho_0}{\rho_0}=\frac{1.06394\cdot 10^3\text{ kg/m}^3-1024\text{ kg/m}^3}{1024\text{ kg/m}^3}=0.039=3.9\%.$ Therefore, it is a good approximation to treat the density of seawater as being constant.

ROUND: Rounding to three significant figures, the pressure at the bottom of Challenger Deep is $p=1.10\cdot 10^8\text{ Pa}$ and the density is $\rho=1.06\cdot 10^3\text{ kg/m}^3$.

DOUBLE-CHECK: The change in pressure is very large, but the change in density is quite small. This is consistent with the fact that water, like most liquids, is nearly incompressible.

13.34. **THINK:** Pressure is defined as the force per unit area. Determine the area of the textbook cover, and calculate the force exerted on the textbook by the atmosphere. Find the mass of an object whose weight would be equal to this force. w = 21.9 cm, l = 28.3 cm, p = 101.3 kPa.

SKETCH:



RESEARCH: $F_\text{p}=pA=plw,\ \ F_\text{g}=mg$

SIMPLIFY: $F_\text{g}=F_p\ \Rightarrow\ mg=plw\ \Rightarrow\ m=\frac{plw}{g}$

CALCULATE: $F_\text{p}=(101.3\cdot 10^3\text{ Pa})(0.219\text{ m})(0.283\text{ m})=6278.27\text{ N}$

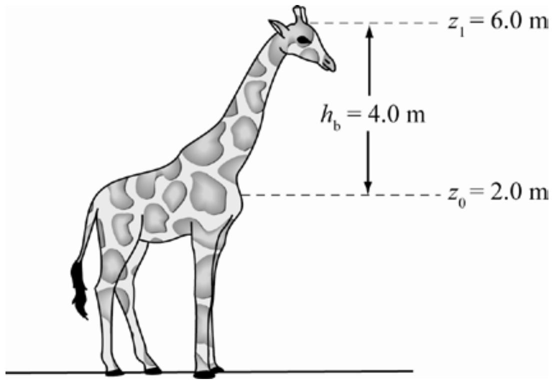
$$m=\frac{(101.3\cdot 10^3\text{ Pa})(0.219\text{ m})(0.283\text{ m})}{(9.81\text{ m/s}^2)}=639.99\text{ kg}$$

ROUND: Rounding to three significant figures gives $F=6280\text{ N}$ and m = 640. kg.

DOUBLE-CHECK: The equivalent mass is large but reasonable.

13.35. **THINK:** From the height and the density, the required pressure can be determined. $\rho_{\text{b}} = 1.00 \text{ g/cm}^3$ (blood) $\rho_{\text{Hg}} = 13.6 \text{ g/cm}^3$ (mercury).

SKETCH:



RESEARCH: $p = \rho g h$

SIMPLIFY: $p_{\text{b}} = \rho_{\text{b}} g h_{\text{b}}, \; p_{\text{Hg}} = \rho_{\text{Hg}} g h_{\text{Hg}}, \; p_{\text{b}} = p_{\text{Hg}} \;\Rightarrow\; h_{\text{Hg}} = \dfrac{\rho_{\text{b}} h_{\text{b}}}{\rho_{\text{Hg}}}$

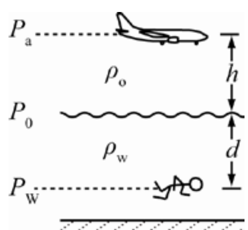
CALCULATE: $h_{\text{Hg}} = \dfrac{\left(1.00 \text{ g/cm}^3\right)\left(4.0 \cdot 10^2 \text{ cm}\right)}{13.6 \text{ g/cm}^3} = 2.941 \cdot 10^1 \text{ cm} = 294.1 \text{ mm}$

ROUND: Rounding to two significant figures gives $h_{\text{Hg}} = 290$ mm.

DOUBLE-CHECK: The systolic blood pressure of a giraffe must be two to three times that of a human. Since the relationship between pressure and height is linear, and a giraffe is about three times the height of a human, this seems like a reasonable result.

13.36. **THINK:** Pressure changes differently in water than it does in air. Determine the changes separately and add the results. $h = 5000.$ m, $d = 20.0$ m, $\rho_w = 1024 \text{ kg/m}^3$, $\rho_0 = 1.229 \text{ kg/m}^3$ and $p_0 = 101.3$ kPa.

SKETCH:



RESEARCH: $p_{\text{w}} - p_0 = \rho_{\text{w}} g d, \; p_0 - p_{\text{a}} = p_0 - p_0 e^{-h\rho_0 g/P_0}$

SIMPLIFY: $p_{\text{w}} - p_{\text{a}} = \rho_{\text{w}} g d + p_0\left(1 - e^{-h\rho_0 g/P_0}\right)$

CALCULATE: $p_{\text{w}} - p_{\text{a}} = \left(1024 \text{ kg/m}^3\right)\left(9.81 \text{ m/s}^2\right)\left(20.0 \text{ m}\right) +$

$$101.3 \text{ kPa}\left(\frac{1000 \text{ Pa}}{1 \text{ kPa}}\right)\left(1 - \exp\frac{-5000.\left(1.229 \text{ kg/m}^3\right)\left(9.81 \text{ m/s}^2\right)}{101.3 \cdot 10^3 \text{ Pa}}\right)$$

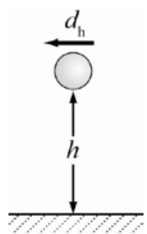
$= 246340$ Pa

ROUND: Rounding to three significant figures gives $p_{\text{w}} - p_{\text{a}} = 246$ kPa.

DOUBLE-CHECK: Note that the pressure of the atmosphere at sea level is 101 kPa. Our pressure difference found here is 246 kPa, 201 kPa of which are generated by the 20 m of water above the diver's head in the dive. The part about the unpressurized airplane ride only contributes 45 kPa. Even if the plane was replaced by a rocket, and our poor diver were to leave the Earth's atmosphere altogether, the pressure difference of this part of the journey relative to sea level could never be bigger than 101 kPa.

13.37. **THINK:** The pressure inside the balloon is equal to the pressure outside the balloon. Since the mass of the air inside the balloon is constant, the volume of the air inside the balloon changes as the pressure changes. $d_0 = 20.0$ cm, $\rho_0 = 1.229$ kg/m^3, $p_0 = 101.3$ kPa and $d_h = 2r_h$.

SKETCH:



RESEARCH: $\rho = m/V$, $\rho_h = \rho_0 e^{-h\rho_0 g/p_0}$ and $V_h = 4\pi r_h^3/3$.

SIMPLIFY: $r_h^3 = \frac{3V_h}{4\pi} = \frac{3}{4\pi}\frac{m}{\rho_h} = \frac{3m}{4\pi}\left(\frac{1}{\rho_0}\right)\left(e^{h\rho_0 g/p_0}\right) \Rightarrow r_h = \left(\frac{3m}{4\pi\rho_0}\right)^{\frac{1}{3}} e^{h\rho_0 g/(3p_0)}$

Take $r_0 = \left(\frac{3m}{4\pi\rho_0}\right)^{\frac{1}{3}}$ and the expression becomes $r_\text{h} = r_0 e^{h\rho_0 g/(3p_0)} \Rightarrow d_\text{h} = 2r_0 e^{h\rho_0 g/(3p_0)}$.

CALCULATE: $d_h = (0.200\text{ m})\exp\left(\frac{h(1.229\text{ kg/m}^3)(9.81\text{ m/s}^2)}{3(101.3\cdot 10^3\text{ Pa})}\right) = 0.2e^{(3.967\cdot 10^{-5} h)}$ m

(a) $d_{1\text{ km}} = 0.2e^{(3.967\cdot 10^{-5}\cdot 1000.)}$ m $= 0.2081$ m

(b) $d_{2\text{ km}} = 0.2e^{(3.967\cdot 10^{-5}\cdot 2000.)}$ m $= 0.2165$ m

(c) $d_{5\text{ km}} = 0.4e^{(3.967\cdot 10^{-5}\cdot 5000.)}$ m $= 0.2439$ m

ROUND: Round the results to three significant figures.

(a) $d_{1\text{ km}} = 20.8$ cm

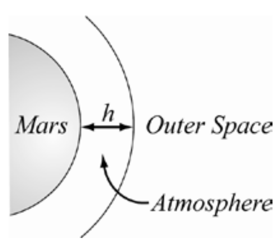
(b) $d_{2\text{ km}} = 21.7$ cm

(c) $d_{5\text{ km}} = 24.4$ cm

DOUBLE-CHECK: The diameter grows exponentially with increasing altitude, which is reasonable since the air is less dense at higher altitudes.

13.38. **THINK:** The question refers to the atmosphere of Mars. The values given are an atmospheric pressure at the surface, $p_0 = 600.$ Pa, and a density of $\rho_0 = 0.0200$ kg/m^3. Determine (a) the thickness of the atmosphere based on the pressure at the boundary with space being 0.0100% of p_0, (b) the atmospheric pressure, p_{-8}, at a depth of 8.18 km, (c) the atmospheric pressure, p_{21}, at an altitude of 21.3 km and (d) the relative change in pressure $\frac{\Delta p}{p}$ between the results of parts (b) and (c) and compare with the relative change in pressure on Earth between a depth of 400. m and an altitude of 8850 m.

SKETCH:



RESEARCH:

(a) The thickness, h, of the Martian atmosphere can be determined using the relationship $p(h)=p_0e^{-h\rho_0 g/p_0}$, and solving for h when the pressure is 0.01 % of the pressure at p_0. Note that g is the acceleration due to gravity at the surface of Mars (i.e. not 9.81 m/s^2 !).

(b) Make use of the barometric pressure formula, $p(h)=p_0e^{-h\rho_0 g/p_0}$, where h = -8.18 km.

(c) Make use of the barometric pressure formula, $p(h)=p_0e^{-h\rho_0 g/p_0}$, where h = 21.3 km.

(d) Determine the relative change in pressure by $\dfrac{\Delta p}{p}=\dfrac{p_{\text{low}}-p_{\text{high}}}{p_{\text{surface}}}$. Then compare with that of Earth for the given locations.

SIMPLIFY:

(a) Since the boundary between atmosphere and space is defined when the pressure is 0.0100% or 0.000100 of p_0,

$$p(h)=(0.000100)p_0=p_0e^{-h\rho_0 g/p_0} \Rightarrow \ln(0.000100)=-\frac{h\rho_0 g}{p_0} \Rightarrow h=-\frac{p_0}{\rho_0 g}\ln(0.000100).$$

(b, c and d) Simplification is not necessary.

CALCULATE:

(a) First, determine g at the surface of mars. From Newton's law of gravitation:

$$\text{g}=\text{G}\frac{M}{R^2}=\left(6.67300\cdot 10^{-11}\text{ m}^3\text{ kg}^{-1}\text{ s}^{-2}\right)\frac{\left(6.4186\cdot 10^{23}\text{ kg}\right)}{\left(3.3895\cdot 10^{6}\text{ m}\right)^2}=3.73\text{ m/s}^2.$$

Plugging in the values, the thickness is

$$h=-\frac{600.\text{ Pa}}{\left(0.0200\text{ kg/m}^3\right)\left(3.73\text{ m/s}^2\right)}\ln(0.000100)=74.1\text{ km}.$$

(b) $P(-8.18\text{ km})=(600.\text{ Pa})e^{\left(8.18\cdot 10^3\text{ m}\right)\left(0.0200\text{ kg/m}^3\right)\left(3.73\text{ m/s}^2\right)/(600.\text{ Pa})}=1659.01\text{ Pa}$

(c) $P(21.3\text{ km})=(600.\text{ Pa})e^{\left(-2.13\cdot 10^4\text{ m}\right)\left(0.0200\text{ kg/m}^3\right)\left(3.73\text{ m/s}^2\right)/(600.\text{ Pa})}=42.4629\text{ Pa}$

(d) First, calculate the change in pressure for Mars:

$$\frac{\Delta P}{P}=\frac{1659.01\text{ Pa}-42.4629\text{ Pa}}{600.\text{ Pa}}=2.69425$$

Determine the pressures at -400 m and 8850 m for Earth:

$$P(-400.)=\left(1.01\cdot 10^5\text{ Pa}\right)e^{(400.\text{ m})\left(1.229\text{ kg/m}^3\right)\left(\left(9.81\text{ m/s}^2\right)\right)/\left(1.01\cdot 10^5\text{ Pa}\right)}=106\text{ kPa and}$$

$$P(8850\text{ m})=\left(1.01\cdot 10^5\text{ Pa}\right)e^{-(8850\text{ m})\left(1.229\text{ kg/m}^3\right)\left(\left(9.81\text{ m/s}^2\right)\right)/\left(1.01\cdot 10^5\text{ Pa}\right)}=35.1\text{ kPa},$$

respectively. Now, the change in pressure for Earth is:

$$\frac{\Delta P}{P}=\frac{106\text{ kPa}-35.1\text{ kPa}}{1.01\cdot 10^5\text{ Pa}}=0.702=70.2\%.$$

ROUND: Round the results to three significant figures.

(a) The thickness of the atmosphere of Mars is h = 74.1 km.

(b) $P(-8.18\text{ km})=1660\text{ Pa}$

(c) $P(21.3\text{ km})=42.5\text{ Pa}$

(d) $(\Delta P/P)_{\text{Mars}}=2.69$. So, it can be seen that Mars' atmospheric pressure changes about 269% from the lowest to highest points, compared with only 70.2% for Earth's atmospheric pressure.

DOUBLE-CHECK: The units are correct. As expected, the pressure increases as the height decreases, and vice versa. The results are also reasonable for the given parameters.

13.39. **THINK:** The question asks for the height of Mount McKinley when it is known that the atmospheric pressure on top is 47.7 % of the pressure at sea level and that the air density on Mount Everest is 34.8 % of that at sea level. The height of Mount Everest is given as $h_{\text{Everest}} = 8850$ m.

SKETCH: Not needed.

RESEARCH: We make use of the equations for barometric pressure and for density in a compressible fluid, that is, $P(h) = P_0 e^{-h\rho_0 g/P_0}$, and $\rho(h) = \rho_0 e^{-h\rho_0 g/p_0}$. By solving for the constant $\rho_0 g / p_0$ from the density equation, we need only use the values given by the question without actual knowledge of the constants. We can then use this information in the barometric pressure equation to obtain the height of Mount McKinley.

SIMPLIFY: The air density on Mount Everest is given by $\rho_{\text{Everest}}(h) = 0.348\rho_0 = \rho_0 e^{-h_{\text{Everest}}\rho_0 g/P_0}$, which can be solved for the set of constants as $\dfrac{\rho_0 g}{p_0} = -\dfrac{\ln(0.348)}{h_{\text{Everest}}}$. Now, the pressure on Mount McKinley is given by $P_{\text{McKinley}}(h) = 0.477P_0 = P_0 e^{-h_{\text{McKinley}}\rho_0 g/P_0}$. If we solve for the height of Mount McKinley, h_{McKinley}, we get $h_{\text{McKinley}} = -\dfrac{p_0}{\rho_0 g}\ln(0.477)$. Using the result above, we can write

$$h_{\text{McKinley}} = -\frac{p_0}{\rho_0 g}\ln(0.477) = h_{\text{Everest}}\frac{\ln(0.477)}{\ln(0.348)}.$$

CALCULATE: Inserting the given value for the height of Mount Everest we find

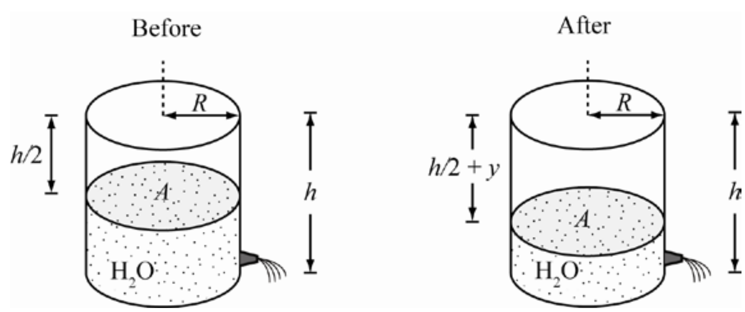
$$h_{\text{McKinley}} = 8850 \text{ m}\frac{\ln(0.477)}{\ln(0.348)} = 6206.33 \text{ m}.$$

ROUND: Since the input values have been given to at least three significant figures, our result for the height of Mount McKinley is $h_{\text{McKinley}} = 6210$ m.

DOUBLE-CHECK: You can look up the height of Mount McKinley and see if what we calculate is reasonably close. (The official height of the mountain is 6194 m, which make it the tallest mountain in North America). However, in an exam situation this check would not be an option. Instead, we can make simple checks of the right unit of our answer, which is meters, and the right order of magnitude. Since we are talking about a mountain, it should be at least a few thousand meters high, which is true for our answer. And of course our answer should come out less than the height of Mt Everest, which is the tallest peak on Earth.

13.40. **THINK:** The question asks for the change in height of a column of water in a cylinder after a small valve is opened at the bottom of the cylinder. The cylinder has a total height of h = 0.60 m and is initially half filled with water. The pressure of the air trapped inside the top half of the cylinder is initially at $p_0 = 1.01 \cdot 10^5$ Pa. Use $\rho = 1000.$ kg/m^3 for the density of water. By summing the forces acting on the column of water, the distance that the water drops by can be found.

SKETCH:



RESEARCH: In equilibrium, the downward force of gravity and the force due to the pressure of the air in the cylinder is equal to the upward force of atmospheric pressure acting on the bottom of the cylinder: $F_{\text{net}} = 0 = p_0 A - pA - mg$, where A is the cross-sectional area of the cylinder, and y is the distance that the water level drops below its initial depth of $h/2$. At constant temperature, the pressure of the air is inversely proportional to volume. Therefore, $p = p_0\left(V_0 / V\right)$, where the initial volume of air is $V_0 = Ah/2$ and the final volume of air is $V = A\left(h/2 + y\right)$. In terms of the density and volume, the mass of the water is given by $m = \rho_{\text{w}} V_{\text{w}}$.

SIMPLIFY: Using the force balance equation,

$$\left(p_0 - p\right)A - mg = 0$$

$$\left(p_0 - p_0\frac{V_0}{V}\right)A - \rho_{\text{w}} V_{\text{w}} g = 0$$

$$p_0\left(1 - \frac{V_0}{V}\right)A - \rho_{\text{w}} A\left(\frac{h}{2} - y\right)g = 0$$

$$p_0\left(1 - \frac{Ah/2}{A\left(h/2 + y\right)}\right)A - \rho_{\text{w}} A\left(\frac{h}{2} - y\right)g = 0$$

$$p_0\left(\left(h/2 + y\right) - h/2\right) - \rho_{\text{w}}\left(\frac{h}{2} - y\right)\left(\frac{h}{2} + y\right)g = 0$$

$$p_0 y - \rho_{\text{w}} g\frac{h^2}{4} + \rho_{\text{w}} g y^2 = 0$$

$$y^2 + \frac{p_0}{\rho_{\text{w}} g}y - \frac{h^2}{4} = 0$$

The positive solution of this quadratic equation is:

$$y = \frac{-\dfrac{p_0}{\rho_{\text{w}} g} + \sqrt{\left(\dfrac{p_0}{\rho_{\text{w}} g}\right)^2 + h^2}}{2}$$

CALCULATE: Plug in the values to get:

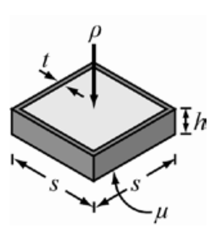
$$y = \frac{-\dfrac{\left(1.01\cdot 10^5 \text{ Pa}\right)}{\left(1000.\text{ kg/m}^3\right)\left(9.81 \text{ m/s}^2\right)} + \sqrt{\left(\dfrac{\left(1.01\cdot 10^5 \text{ Pa}\right)}{\left(1000.\text{ kg/m}^3\right)\left(9.81 \text{ m/s}^2\right)}\right)^2 + \left(0.60 \text{ m}\right)^2}}{2} = 8.734\cdot 10^{-3} \text{ m} = 8.734 \text{ mm}$$

ROUND: To two significant figures, the depth of the water is lowered by 8.7 mm.

DOUBLE-CHECK: Since the initial height of the water is quite small, it is reasonable that the amount that the water level drops is also small.

13.41. **THINK:** The question presents a pool embedded in a parking lot. $\mu_s = 0.450$ and $\rho = 2.50 \text{ g/cm}^3$.

SKETCH:



RESEARCH: For the pool to fail, the net force on the walls due to the water pressure must exceed the force necessary to move the blocks, that is, the force of static friction. The pressure as a function of depth h

is $P(h)=P_0+\rho_{\text{w}}gh.$ The force necessary to move the blocks is $F_{\text{failure}}=\mu_{\text{s}}F_{\text{N}}=\mu_{\text{s}}m_{\text{c}}g=\mu_{\text{s}}\rho_{\text{c}}Vg=\mu_{\text{s}}\rho_{\text{c}}sthg$, The total force due to the pressure is the integrated force for a small differential depth *dx*:

$$F_{\text{Pressure}}=\sum P(h)\Delta A=\int_0^x P(h)dA.$$

In this case, $dA = s\,dh$, where dh is the thickness of a layer at depth h. The total force on a wall for a pool of a particular depth x is:

$$F_{\text{Pressure}}=\int_0^x P(h)dA=\int_0^x\left(P_0+\rho_{\text{w}}gh\right)sdh=\left[s\left(P_0h+\frac{1}{2}\rho_{\text{w}}gh^2\right)\right]_0^x=s\left(P_0x+\frac{1}{2}\rho_{\text{w}}gx^2\right).$$

However, the force of the air pressure on the other side of the blocks (pushing against the water) must not be forgotten. Assuming constant pressure $F_{air}=P_0A=P_0sx.$ The depth necessary for the pool to fail is the depth when $F_{\text{Pressure}}-F_{\text{air}}>F_{\text{failure}}$, which is equivalent to when $F_{\text{Pressure}}-F_{\text{air}}-F_{\text{failure}}>0.$ Compute the value of x when $F_{\text{Pressure}}-F_{\text{air}}-F_{\text{failure}}=0$. This will be a quadratic equation in x that opens upwards. The corresponding inequality will be satisfied *outside* the two roots of the quadratic equation.

SIMPLIFY:

$$s\left(P_0x+\frac{1}{2}\rho_{\text{w}}gx^2\right)-P_0sx-\mu_{\text{s}}\rho_{\text{c}}sthg=0$$

$$\Rightarrow\ sP_0x+s\frac{1}{2}\rho_{\text{w}}gx^2-P_0sx-\mu_{\text{s}}\rho_{\text{c}}stgx=0$$

$$\Rightarrow\ x\left(\frac{s\rho_{\text{w}}gx}{2}-\mu_{\text{s}}\rho_{\text{c}}stg\right)=0$$

$$\Rightarrow\ x=0\text{ or }x=\frac{2\mu_{\text{s}}\rho_{\text{c}}t}{\rho_{\text{w}}}.$$

Therefore the depth of the pool will be sufficient to collapse the wall if $x<0$ (which does not make sense since the depth has to be non-negative) or if $x>\dfrac{2\mu_s\rho_c t}{\rho_w}$. Compute this value of *x*.

CALCULATE: $x=\dfrac{2(0.450)\left(2.50\cdot 10^3\text{ kg/m}^3\right)(0.500\text{ m})}{1000.\text{ kg/m}^3}=1.125\text{ m}.$

ROUND: Since the values have three significant figures, the result should be rounded to $x=1.13$ m.

DOUBLE-CHECK: The computed depth has units of meters, and the depth is a reasonable size in a parking lot.

13.42. (a) Consider a thin slice of the atmosphere at height *r* and with thickness *dr*. The upward force that this slice of the atmosphere experiences from the gas below it is $F_{\text{up}}=p(r)A$, where *A* is the area and *p* is the pressure. The downward force from the gas above is $F_{\text{down}}=p(r+dr)A$. The net force due to the pressure experienced by this slice of the atmosphere is then $F_{\text{net}}=F_{\text{up}}-F_{\text{down}}=(p(r)-p(r+dr))A$. This force has to be equal that of gravity, $mg(r)=V\rho(r)g(r)=Adr\rho(r)g(r)$, where we have used the fact that the volume of the thin slice of the atmosphere is the product of the area and the thickness. So we find

$$(p(r)-p(r+dr))A=Adr\rho(r)g(r)$$

or

$$\frac{dp(r)}{dr}=-\rho(r)g(r)$$

which is the equation of hydrostatic equilibrium.

(b) If, as stated in the problem, $p(r)=\gamma\rho(r)$ with a constant γ , then our differential equation becomes

$$\gamma\frac{d\rho(r)}{dr}=-\rho(r)g(r)$$

The gravitational acceleration for this spherical distribution is given by the mass interior to the radius *r*, see Chapter 12, $g(r)=Gm(r)/r^2=G\left(4\pi\int_0^r \rho(r')r'^2\,dr'\right)/r^2$. So our equation to solve becomes

$$\gamma\frac{d\rho(r)}{dr}=-\rho(r)G\left(4\pi\int_0^r \rho(r')r'^2\,dr'\right)/r^2 \qquad (*)$$

This looks very complicated, but we already have a proposed solution, and all we need to show is that it works. Our solution is $\rho(r)=\alpha/r^2$ with some constant α. With this *ansatz*, $g(r)$ simply becomes:

$$g(r)=G\left(4\pi\int_0^r \frac{\alpha}{r'^2}r'^2\,dr'\right)/r^2=G\left(4\pi\int_0^r \alpha dr'\right)/r^2=4\pi\alpha G/r$$

With this, our right-hand side of equation (*) is: $-\rho(r)(4\pi\alpha G/r)=-(\alpha/r^2)(4\pi\alpha G/r)=-4\pi\alpha^2G/r^3$. Our left-hand side of the same equation is $\gamma\frac{d\rho(r)}{dr}=\gamma\frac{d(\alpha/r^2)}{dr}=-\gamma 2\alpha/r^3$.

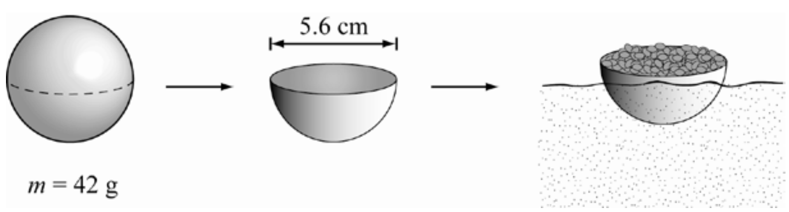
In other words, this *ansatz* implies that $-\gamma 2\alpha/r^3=-4\pi\alpha^2G/r^3$, which means that the constant has to have the value $\alpha=\frac{\gamma}{2\pi G}$. Our solution for the density is a function of radius is therefore:

$$\rho(r)=\frac{\gamma}{2\pi Gr^2}$$

The difficulty with this solution as a model of a star is that the density, ρ, does not go to zero at any infinite radius, *r*. If the model is cut off at some radius representing the surface of the star, the infinite pressure gradient at the surface would lift the surface layer off the star; the model could not be in equilibrium. But if the model is extended to an infinite radius, the total mass diverges to infinity. Real stars in equilibrium are not isothermal, of course; they produce energy in their cores and radiate it into space from their surfaces. A star which can no longer do this collapses under its own gravity, either to a configuration (white dwarf or neutron star) with a different equation of state than $p=\gamma\rho$, or to a black hole.

13.43. **THINK:** The question asks for the number of pennies that can be placed in half of a floating racquetball such that the racquetball does not sink. The values given are: the diameter of the racquetball, d = 5.6 cm, the mass of the racquetball, $m_{\text{ball}}=42$ g, the volume of a penny, $v_{\text{penny}}=0.36\text{ cm}^3$, and the mass of a penny, $m_{\text{penny}}=2.5$ g.

SKETCH:



RESEARCH: In static equilibrium, the net force is zero and the buoyant force equals the weight of the racquetball boat and the pennies. The maximum buoyant force occurs when the maximum amount of water is displaced. The maximum amount of displaced water occurs when the top of the boat is level with the surface of the water and $V_{\text{water}}=V_{\text{ball}}/2$. Therefore, $F_{\text{buoyant}}=F_{\text{boat}}$. Recalling that the buoyant force is equal to the weight of the fluid displaced gives $F_{\text{buoyant}}=\rho_{\text{water}}V_{\text{water}}g=\frac{1}{2}\rho_{\text{water}}V_{\text{ball}}g$. Since the weight of the half-racquetball plus the pennies is $F_{\text{boat}}=mg=\left(\frac{m_{\text{ball}}}{2}+m_{\text{pennies}}\right)g$: $\left(\frac{m_{\text{ball}}}{2}+m_{\text{pennies}}\right)g=\frac{1}{2}\rho_{\text{water}}V_{\text{ball}}g$.

SIMPLIFY: Solving for m_{pennies} gives $m_{\text{pennies}} = \frac{1}{2}\rho_{\text{water}}V_{\text{ball}} - \frac{m_{\text{ball}}}{2}$. Now, determine the volume of the racquetball:

$$V_{\text{ball}} = \frac{\pi d^3}{6} = \frac{\pi\left(5.6\text{ cm}\right)^3}{6} = 91.95\text{ cm}^3.$$

The maximum number of pennies the racquetball boat can hold without sinking is then given by:

$$N = \frac{m_{\text{pennies}}}{m_{\text{penny}}}.$$

CALCULATE: First, calculate the volume of the racquetball: $V_{\text{ball}} = \frac{\pi d^3}{6} = \frac{\pi\left(5.6\text{ cm}\right)^3}{6} = 91.95\text{ cm}^3.$

The maximum number of pennies the racquetball boat can hold without sinking is then given by:

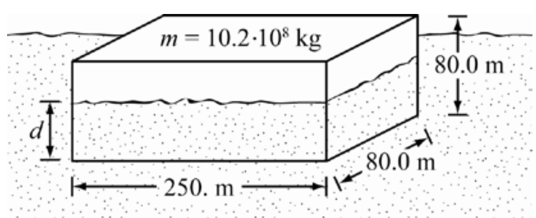
$$N = \frac{m_{\text{pennies}}}{m_{\text{penny}}} = \frac{\rho_{\text{water}}V_{\text{ball}} - m_{\text{ball}}}{2m_{\text{penny}}} = \frac{\left(1.0\text{ g/cm}^3\right)\left(91.95\text{ cm}^3\right) - 42\text{ g}}{2\left(2.5\text{ g}\right)} = 9.99.$$

ROUND: Since the number of pennies is an integer, we must round to either 9 or 10. And since rounding up would cause the racquetball boat to sink, the answer must be rounded down, to $N = 9$ pennies.

DOUBLE-CHECK: Based on the given weight of a penny and that of the racquetball (half) compared to the weight of the fluid displaced, the answer is reasonable. It should be noted that the copper and zinc content of an American penny was changed in 1982 and pennies made before 1982 have a different density.

13.44. **THINK:** The question asks for the depth, d, that a supertanker is submerged based on its dimensions. The length is L= 250. m, the width is w = 80.0 m and the height is h = 80.0 m. Assume the mass of its contents is $m = 10.2\cdot 10^8$ kg, and the density of sea water is $\rho_{\text{sea}} = 1024\text{ kg/m}^3$.

SKETCH:



RESEARCH: The buoyant force, F_{buoyant}, is equal to the weight of the water displaced by the tanker. Therefore, $F_{\text{buoyant}} = \rho_{\text{sea}}V_{\text{water}}g = \rho_{\text{sea}}\left(Lwd\right)g$, where d is the depth of the bottom of the tanker below sea level. Recall that an object will only float on a fluid if the buoyant force balances the weight of the object, that is, $F_{\text{gravity}} = F_{\text{buoyant}}$.

SIMPLIFY: $F_{\text{gravity}} = F_{\text{buoyant}} \Rightarrow mg = \rho_{\text{sea}}\left(Lwd\right)g \Rightarrow d = \frac{m}{\rho_{\text{sea}}Lw}$

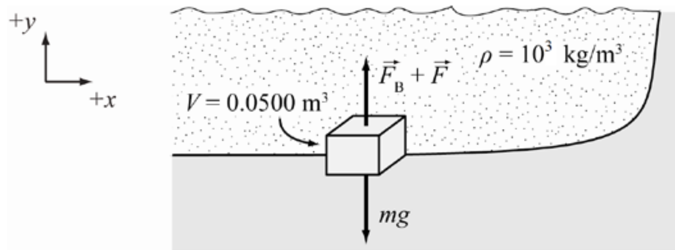
CALCULATE: Plugging in the values gives: $d = \frac{10.2\cdot 10^8\text{ kg}}{\left(250.\text{ m}\right)\left(80.0\text{ m}\right)\left(1024\text{ kg/m}^3\right)} = 49.8047\text{ m}.$

ROUND: Since the dimensions of the tanker are given to three significant figures, the depth the supertanker sinks into the water is x = 49.8 m.

DOUBLE-CHECK: Based on the given information, the density of the supertanker is about 60 % that of seawater. Therefore, it's reasonable that the tanker has approximately 60 % of its volume under water.

13.45. **THINK:** A box has volume, $V = 0.0500\text{ m}^3$, and the density of lake water is $\rho = 1.00\cdot 10^3\text{ kg/m}^3$. Determine the force necessary to lift a box which lies at the bottom of a lake when its mass is (a) 1000. kg, (b) 100. kg and (c) 55.0 kg.

SKETCH:



RESEARCH: The buoyant force will point along the same direction as the applied force, whereas the force of gravity will point downwards. If the box is to lifted with no acceleration, then the sum of the forces on the box are $F + F_B - mg = 0 \Rightarrow F = mg - F_B$.

SIMPLIFY: Now, recall that $F_B = \rho gV$, where V is the volume of the box, so the above equation becomes $F = mg - \rho gV$.

CALCULATE:

(a) For m = 1000. kg: $F = (1000.\text{ kg})(9.81\text{ m/s}^2) - (1.00 \cdot 10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.0500\text{ m}^3) = 9319.5\text{ N}$.

(b) For m = 100. kg: $F = (100.\text{ kg})(9.81\text{ m/s}^2) - (1.00 \cdot 10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.0500\text{ m}^3) = 490.5\text{ N}$.

(c) For m = 55.0 kg, $F = (55.0\text{ kg})(9.81\text{ m/s}^2) - (1.00 \cdot 10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.0500\text{ m}^3) = 49.05\text{ N}$.

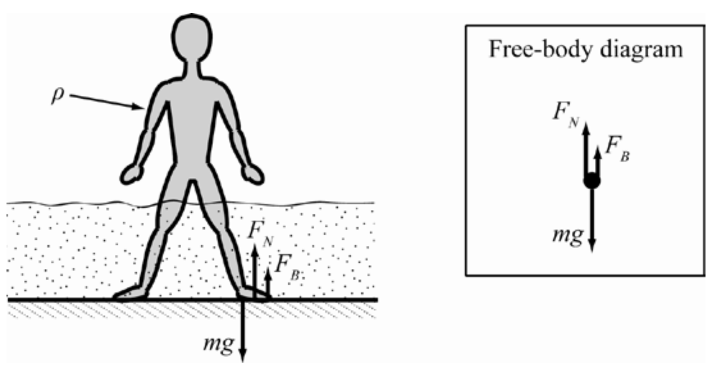
ROUND: Since all values are given to three significant figures, the answers should be:

(a) 9320 N
(b) 491 N
(c) 49.1 N

DOUBLE-CHECK: The box displaces the same volume of water no matter what its mass, so the buoyant force remains the same. It is expected that more force will be required to lift the box as it gets heavier.

13.46. **THINK:** The question asks to determine the normal force on a man standing in a pool where 32% of his body is submerged. The given values are the mass of the man, m = 64 kg, and the density of the man $\rho = 970\text{ kg/m}^3$.

SKETCH:



RESEARCH: To determine the normal force, look at the free-body diagram above. When the body is in equilibrium, an upward normal force, F_N, is required along with the buoyant force, F_B, to balance the weight of the man. That is, $F_{net} = 0 = F_N + F_B - mg \Rightarrow F_N = mg - F_B$. To determine F_B, recall that the magnitude of the buoyant force is equal to the weight of the fluid displaced. Therefore, $F_B = m_{fluid}g = \rho_{fluid}V_{displaced}g$. The volume displaced is equal to 32% of the volume of the man, therefore,

$$V_{displaced} = (0.32)V_{man} = (0.32)\frac{m}{\rho}.$$

SIMPLIFY: The above expressions can be combined and simplified as follows:

$$F_N = mg - F_B = mg - \rho_{\text{fluid}} V_{\text{displaced}} g = mg\left(1-(0.32)\frac{\rho_{\text{fluid}}}{\rho}\right).$$

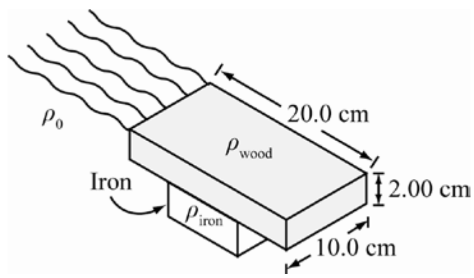
CALCULATE: $F_N = (64 \text{ kg})(9.81 \text{ m/s}^2)\left(1-(0.32)\frac{1000. \text{ kg/m}^3}{970 \text{ kg/m}^3}\right) = 420.72 \text{ N}$

ROUND: Since the given values have two significant figures, the result should be rounded to $F_N = 4.2\cdot 10^2$ N.

DOUBLE-CHECK: A quick check shows that the 68% of the man's volume that is not in the water would weigh 426 N, which is very close to the normal force calculated. This is consistent with the fact that the density of the man and the density of the water are fairly close.

13.47. **THINK:** The question asks for the volume of a piece of iron such that when glued to a piece of wood, this will submerge completely but not sink. The given values are: the length of the piece of wood, l = 20.0 cm, the width of the piece of wood, w = 10.0 cm, the thickness of the piece of wood, t = 2.00 cm, the density of the piece of wood, $\rho_{\text{wood}} = 800. \text{ kg/m}^3$, the density of iron, $\rho_{\text{iron}} = 7860 \text{ kg/m}^3$, and the density of water, $\rho_0 = 1000. \text{ kg/m}^3$.

SKETCH:



RESEARCH: If the object is to be at equilibrium, the weight of the wood piece plus the weight of the iron piece have to be equal to the buoyant force, that is, $F_B = W_{\text{wood}} + W_{\text{iron}}$. Recall that the buoyant force is equal to the weight of the fluid displaced: $W_{\text{water}} = W_{\text{wood}} + W_{\text{iron}} \Rightarrow \rho_0 V_{\text{displaced}} g = \rho_{\text{wood}} V_{\text{wood}} g + \rho_{\text{iron}} V_{\text{iron}} g$, where $V_{\text{displaced}} = V_{\text{wood}} + V_{\text{iron}}$.

SIMPLIFY: $\rho_0 (V_{\text{wood}} + V_{\text{iron}}) = \rho_{\text{wood}} V_{\text{wood}} + \rho_{\text{iron}} V_{\text{iron}} \Rightarrow V_{\text{iron}} = V_{\text{wood}}\left(\frac{\rho_0 - \rho_{\text{wood}}}{\rho_{\text{iron}} - \rho_0}\right)$

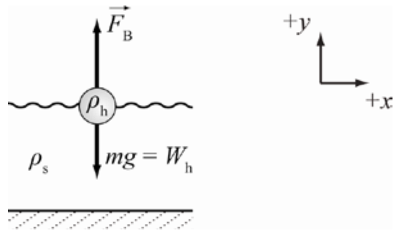
CALCULATE: $V_{\text{iron}} = (0.200 \text{ m})(0.100 \text{ m})(0.0200 \text{ m})\left(\frac{1000. \text{ kg/m}^3 - 800. \text{ kg/m}^3}{7860 \text{ kg/m}^3 - 1000. \text{ kg/m}^3}\right) = 1.1662\cdot 10^{-5} \text{ m}^3$

ROUND: The result should be rounded to three significant figures: $V_{\text{iron}} = 1.17\cdot 10^{-5} \text{ m}^3$.

DOUBLE-CHECK: This corresponds to a cube of iron with sides about 2.27 cm, and mass approximately 91.7 g. The result is reasonable based on the given values.

13.48. **THINK:** The question gives the average density of the human body as $\rho_h = 985 \text{ kg/m}^3$, and the density of sea water as $\rho_s = 1024 \text{ kg/m}^3$. (a) Draw a free-body diagram and determine the percentage of the human body submerged in sea water and (b) the percentage of the volume of the human body submerged when the density of the body is 985 kg/m^3 compared to when it is 945 kg/m^3. (c) If two thirds of the body is submerged, what is the density of the water?

SKETCH:



RESEARCH:

(a) Looking at the sketch above, the weight of the body must be balanced by the buoyant force. Since the magnitude of the buoyant force is the weight of the fluid displaced:

$$F_B = W_h \;\Rightarrow\; \rho_s V_{submerged} g = \rho_h V_h g \;\Rightarrow\; V_{submerged} = V_h \frac{\rho_h}{\rho_s}.$$

(b) The difference between the volumes submerged under the two densities is $\Delta V_{submerged} = V_{985} - V_{945}$.

(c) For the body to be in equilibrium, the weight of the body must be balanced by the weight of the fluid displaced, therefore,

$$\rho_h V_h g = \rho_s V_{sub} g \;\Rightarrow\; \rho_s = \rho_h \frac{V_h}{V_{sub}}.$$

SIMPLIFY:

(a) Therefore, the percentage submerged is given by $\%_{submerged} = (100\ \%)\dfrac{\rho_h}{\rho_s}$.

(b) From part (a), $V_{submerged} = V_h \rho_h / \rho_s$, therefore, $\Delta V_{submerged} = V_{985} - V_{945} = V_h\left(\dfrac{\rho_{985}}{\rho_s} - \dfrac{\rho_{945}}{\rho_s}\right)$. Therefore, the percentage bobbing up and down is given by:

$$\%_{bob} = (100\ \%)\frac{\Delta V_{submerged}}{V_h} = (100\ \%)\left(\frac{\rho_{985} - \rho_{945}}{\rho_s}\right).$$

(c) No simplification is necessary.

CALCULATE:

(a) $\%_{submerged} = (100\ \%)\dfrac{985}{1024} = 96.191\ \%$

(b) $\%_{bob} = (100\ \%)\left(\dfrac{985 - 945}{1024}\right) = 3.91\ \%$

(c) $\rho_s = \left(985\ \text{kg/m}^3\right)\dfrac{3}{2} = 1477.5\ \text{kg/m}^3$

ROUND: Round the results to three significant figures.

(a) $\%_{submerged} = 96.2\ \%$

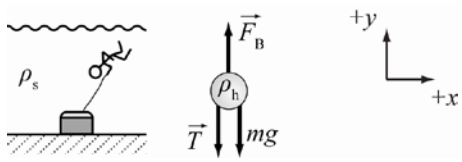
(b) $\%_{bob} = 3.91\ \%$

(c) $\rho_s = 1480\ \text{kg/m}^3$

DOUBLE-CHECK: All results are reasonable for the given values. It is expected that the density of the Dead Sea is greater than that of regular seawater.

13.49. **THINK:** The problem gives the following information about the diver: mass, m = 60.0 kg, density, $\rho_h = 945\ \text{kg/m}^3$, and the density of sea water is $\rho_s = 1024\ \text{kg/m}^3$. Determine (a) the tension on the chain, (b) the mass whose weight is equivalent to the tension of part (a), and (c) the upward acceleration of the diver due to the buoyant force.

SKETCH:



RESEARCH:

(a) To determine the tension on the chain, consider the free-body diagram above. The tension is given by $T = F_B - mg$, where F_B is the buoyant force, given by the weight of the water displaced by the diver. Since the diver is completely submerged, the weight of the water displaced is given by

$$F_B = W_w = \rho_s V_s g = \rho_s V_{diver} g.$$

(b) From Newton's second law, $T = mg \Rightarrow m = \frac{T}{g}$.

(c) Since the net force on the diver is given by $F_{net} = ma = F_B - mg = T$, as obtained in part (a). The corresponding acceleration is given by $a = T / m$.

SIMPLIFY:

(a) Since $V_{diver} = m / \rho_h$, the tension is given by $T = \rho_s V_{diver} g - mg = \rho_s \frac{m}{\rho_h} g - mg = mg\left(\frac{\rho_s}{\rho_h} - 1\right)$.

(b) Simplification is not necessary.

(c) Simplification is not necessary.

CALCULATE:

(a) $T = 60.0 \text{ kg}\left(9.81 \text{ m/s}^2\right)\left(\frac{1024}{945} - 1\right) = 49.2057 \text{ N}$

(b) $m = \frac{49.2057 \text{ N}}{9.81 \text{ m/s}^2} = 5.01587 \text{ kg}$

(c) $a = \frac{49.2057 \text{ N}}{60.0 \text{ kg}} = 0.820095 \text{ m/s}^2$

ROUND:

(a) $T = 49.2$ N.

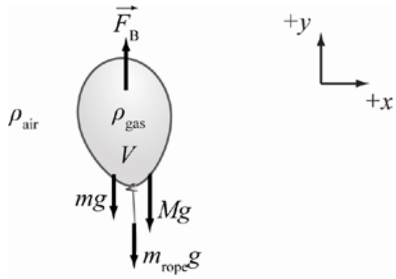
(b) $m = 5.02$ kg

(c) $a = 0.820 \text{ m/s}^2$

DOUBLE-CHECK: The units for all of our answers are the appropriate ones, which is comforting. The magnitude of the answer in part (b) may cause us to hesitate, because the equivalent mass seems rather small at only around 5 kg. But if you are a diver this sounds about right, because this is on the order of the extra ballast lead weights that you need to strap on to float under water. Could you really only lift a mass of 5 kg off the bottom of the sea? No, you can do quite a bit better by grabbing the chain and making swimming motions towards the surface, thus propelling you with an additional force beyond the buoyant force, which was the only one assumed to be in play for this problem.

13.50. **THINK:** Determine the maximum mass that can be added to a balloon such that the balloon hovers without sinking to the ground. The gas in the balloon has density $\rho_{gas} = 0.20 \text{ kg/m}^3$, the mass of the balloon is M = 10.0 kg, the volume of the balloon is $V = 20.0 \text{ m}^3$ and the density of air is given to be $\rho_{air} = 1.30 \text{ kg/m}^3$.

SKETCH:



RESEARCH: In order for the balloon not to fall to the ground, the sum of the forces acting on the balloon must equal zero. There are a total of five forces acting on the balloon. These forces are the buoyant force (pointing upwards), the weight of the balloon, the weight of the gas, the weight of the rope and the weight of the unknown mass. Therefore, $F_B = Mg + \rho_{gas}Vg + m_{rope}g + mg,$ where $F_B = \rho_{air}Vg.$

SIMPLIFY: Solving for the mass tied to the balloon, $m = V\left(\rho_{air} - \rho_{gas}\right) - \left(M + m_{rope}\right).$

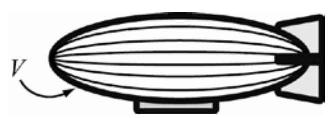
CALCULATE: $m = 20.0 \text{ m}^3\left(1.30 \text{ kg/m}^3 - 0.20 \text{ kg/m}^3\right) - \left(10.0 \text{ kg} + 2.00 \text{ kg}\right) = 10.0 \text{ kg}$

ROUND: m = 10.0 kg.

DOUBLE-CHECK: This result is reasonable for the given values.

13.51. THINK: The volume of the Hindenburg zeppelin is $V = 2.000 \cdot 10^5 \text{ m}^3$ and the useful lift is $W_{useful} = 1.099 \cdot 10^6 \text{ N}.$ The densities of air, hydrogen and helium are $\rho_{air} = 1.205 \text{ kg/m}^3,$ $\rho_H = 0.08988 \text{ kg/m}^3$ and $\rho_{He} = 0.1786 \text{ kg/m}^3,$ respectively. Determine (a) the weight of the structure of the zeppelin and (b) the useful lift capacity of helium compared with that of hydrogen. Since these values are given to four significant figures, the value of the acceleration due to gravity used will be treated as having four significant figures, $g = 9.810 \text{ m/s}^2.$

SKETCH:



RESEARCH:

(a) Recall that the magnitude of the buoyant force is given by the weight of the fluid displaced (in this case, air), therefore, $F_B = \rho_{air}Vg.$ Then the weight of the zeppelin can be determined from the difference in the total lift capacity and the useful lift capacity. The total lift capacity is given by $W_{tot} = F_B - W_H,$ where W_H is the weight of the hydrogen inside the zeppelin. Therefore, the weight of the zeppelin is given by $W_{zep} = W_{tot} - W_{useful}.$

(b) If the zeppelin had been filled with helium instead of hydrogen, the total weight that could be lifted would be $W_1 = F_B - W_{He}.$ Therefore, the useful lift would be $W_{He,useful} = W_1 - W_{zep}.$ By switching from helium to hydrogen, the useful lift is increased by only,

$$\frac{W_{useful} - W_{He,useful}}{W_{He,useful}}.$$

SIMPLIFY:

(a) $W_{zep} = \rho_{air}Vg - \rho_H Vg - W_{useful}.$

(b) $W_{He,useful} = F_B - W_{He} - W_{zep} = \rho_{air}Vg - \rho_{He}Vg - W_{zep}$

CALCULATE:

(a) $W_{zep} = \left(1.205 \text{ kg/m}^3 - 0.08988 \text{ kg/m}^3\right)\left(2.000 \cdot 10^5 \text{ m}^3\right)\left(9.810 \text{ m/s}^2\right) - \left(1.099 \cdot 10^6 \text{ N}\right) = 1.0888 \cdot 10^6 \text{ N}$

(b) $W_{\text{He,useful}} = \left(2.000\cdot 10^5 \text{ m}^3\right)\left(9.810 \text{ m/s}^2\right)\left(1.205 \text{ kg/m}^3 - 0.1786 \text{ kg/m}^3\right) - \left(1.0888\cdot 10^6 \text{ N}\right)$

$= 9.2493\cdot 10^5 \text{ N}$

The increase in lift is then $\dfrac{1.099\cdot 10^6 \text{ N} - 9.2498\cdot 10^5 \text{ N}}{9.2498\cdot 10^5 \text{ N}} = 0.188196$ or 18.8196%.

ROUND: Round the results to four significant figures.

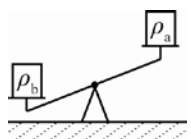
(a) $W_{\text{zep}} = 1.089\cdot 10^6 \text{ N}$

(b) $W_{\text{He,useful}} = 9.249\cdot 10^5 \text{ N},$ which is an 18.82% increase from using hydrogen rather than helium.

DOUBLE-CHECK: This result is somewhat counterintuitive, since the density of hydrogen is half the density of helium. The explanation is in the fact that it is the difference between the air density and the filling gas density that matters, which changes by a small fraction when the gas is changed from hydrogen to helium. By this account, filling the Hindenburg with hydrogen rather than helium was a risk not worth taking. The initial plans called for the Hindenburg to be filled with helium. It was the blockade imposed on Germany in the years preceding World War II that determined the use of hydrogen rather than helium.

13.52. **THINK:** The first question we need to answer is "what changes between the measurements in dry air and in humid air?" And the answer is that the brass and aluminum objects displace a different amount of air; so changing the humidity changes (ever so slightly!) the mass of the air that is displaced. The question asks about the mass of a sample in an analytical balance, given a sensitivity of $m_0 = 0.100$ mg.

SKETCH:



RESEARCH: Suppose in dry air the scale is exactly balanced between the aluminum object on one side and the brass weight on the other. This means that the net force, including the buoyant force on both sides, is zero. Since the weight is given by the product of the mass times the gravitational acceleration, and the mass is the product of the density and the volume, we get

$$V_A(\rho_A - \rho_{\text{dry}}) - V_B(\rho_B - \rho_{\text{dry}}) = 0$$

In the humid air, however, there will be a slight imbalance, *m*, given by

$$V_A(\rho_A - \rho_{\text{humid}}) - V_B(\rho_B - \rho_{\text{humid}}) = m$$

Let's set *m* to the size of the sensitivity of the balance scale given in the problem and solve for the two unknown volumes of the aluminum and brass objects.

SIMPLIFY: We solve the upper of the two equations for V_B and find

$$V_B = V_A(\rho_A - \rho_{\text{dry}})/(\rho_B - \rho_{\text{dry}})$$

Inserting this result into the lower of our two equations then gives us:

$$V_A(\rho_A - \rho_{\text{humid}}) - V_A(\rho_A - \rho_{\text{dry}})(\rho_B - \rho_{\text{humid}})/(\rho_B - \rho_{\text{dry}}) = m \Rightarrow$$

$$V_A = m\frac{\rho_B - \rho_{\text{dry}}}{(\rho_B - \rho_A)(\rho_{\text{dry}} - \rho_{\text{humid}})}$$

Multiplying both side of this equation with the density of aluminum gives us the minimum mass needed to show an observable effect of the change of the air density due to humidity changes on the balance scale:

$$m_A = \rho_A V_A = \rho_A m\frac{\rho_B - \rho_{\text{dry}}}{(\rho_B - \rho_A)(\rho_{\text{dry}} - \rho_{\text{humid}})}$$

CALCULATE: We insert the given numbers for the densities and use the scale sensitivity $m_0 = 0.100$ mg for *m*. We find

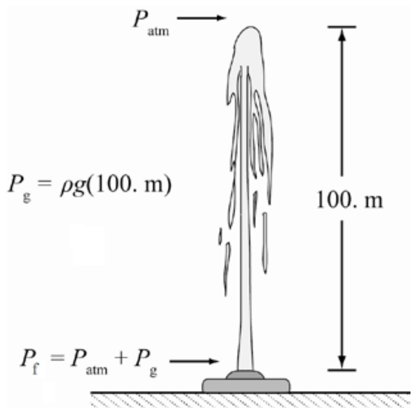
$$m_A = (0.1\cdot 10^{-6} \text{ kg})\frac{(2.70\cdot 10^3 \text{ kg/m}^3)(8.50\cdot 10^3 \text{ kg/m}^3 - 1.2285 \text{ kg/m}^3)}{(8.50\cdot 10^3 \text{ kg/m}^3 - 2.70\cdot 10^3 \text{ kg/m}^3)(1.2285 \text{ kg/m}^3 - 1.2273 \text{ kg/m}^3)} = 0.329694 \text{ kg}$$

ROUND: The input values are given to at least three significant figures, so the result should be rounded to $m_A \geq 0.330$ kg.

DOUBLE-CHECK: First, as in most cases, we check the units of our answer to make sure that it is dimensionally correct. (Obviously, in a Solutions Manual this double-check should not turn up any problems, though …). More importantly, we can check the limits of our derived answer for m_A. The effect we are looking for can only exists if the displaced air volume is large; therefore the smaller the difference between the dry and humid air densities is, the bigger we expect the volume and therefore the mass of the aluminum object to be. This is born out by our formula above.

13.53. **THINK:** Determine the difference in pressure between the fountain and a point 100. m above.

SKETCH:



RESEARCH: The total pressure at the fountain is the sum of the gauge pressure at a depth of 100. m, and the atmospheric pressure. That is, $P_f = P_{gauge} + P_{atm}$. I can then determine the pressure at 100. m above by subtracting the gauge pressure because the only relevant pressure is the atmospheric pressure. The total pressure at the top is then $P_{atm} = P_f - P_{gauge}$. Thus, the difference in pressure is given by $P_f - P_{atm} = P_{gauge}$.

SIMPLIFY: Since gauge pressure is defined as $\rho_{fluid}gh$, the difference in pressure is then given by:

$$P_f - P_{atm} = \rho_{water}gh.$$

CALCULATE: Inserting the values gives:

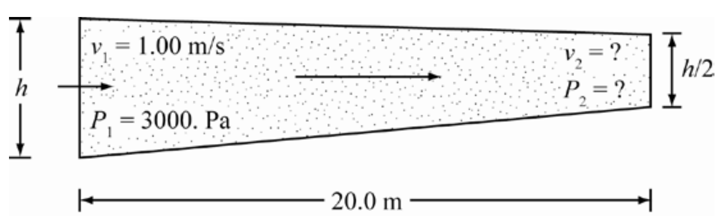
$$P_{gauge} = \rho_{water}gh = (1000.\text{ kg/m}^3)(9.81\text{ m/s}^2)(100.\text{ m}) = 9.81\cdot 10^5\text{ N/m}^2.$$

ROUND: The only value given is the height of the water column with three significant figures, so the difference in pressure is $9.81\cdot 10^5\text{ N/m}^2$.

DOUBLE-CHECK: For a height of 100. m, the result is reasonable.

13.54. **THINK:** Determine the pressure in a rectangular pipe where the width remains constant but the height decreases by half. The question gives the initial velocity as $v_1 = 1.00$ m/s, and the initial pressure is $P_1 = 3000.$ Pa.

SKETCH:



RESEARCH: Since the pipe is rectangular and the width remains the same but the height decreases by half, the cross-sectional area also decreases by half, that is,

$$A_1 = (\text{width})(\text{height}) \Rightarrow A_2 = (\text{width})\left(\frac{1}{2}\text{height}\right) = \frac{1}{2}A_1.$$

Assuming that the water is incompressible, then the continuity equation must hold: $A_1v_1 = A_2v_2$. Therefore, the final velocity must increase by a factor of 2. By making use of the Bernoulli equation:

$$P_1 + \frac{1}{2}\rho v_1^{\,2} + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^{\,2} + \rho g h_2.$$

Since the overall height of the water is essentially fixed, then $h_1 = h_2$.

SIMPLIFY: Solving for the final pressure:

$$P_2 = P_1 + \frac{1}{2}\rho\left(v_1^{\,2} - v_2^{\,2}\right).$$

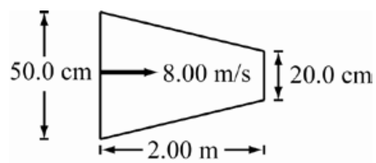
CALCULATE: $P_2 = P_1 + \frac{1}{2}\rho\left(v_1^{\,2} - 4v_1^{\,2}\right) = 3000.\text{ Pa} - \frac{3}{2}\left(1000.\text{ kg/m}^3\right)\left(1.00\text{ m/s}\right)^2 = 1500.\text{ Pa}$

ROUND: The values were given to three significant figures, so the answer should be rounded to $P_2 = 1.50 \cdot 10^3$ Pa.

DOUBLE-CHECK: The pressure is expected to decrease as the velocity increases (Bernoulli's effect!).

13.55. **THINK:** The question presents a nozzle with square sides of side length 50.0 cm and 20.0 cm, respectively. While this looks like an awfully big nozzle, it still may be possible to find such a device inside a hydroelectric power plant, for example. Since we are dealing with flowing water, our equations for fluid flow, in particular the continuity equation will come in handy. One word of caution: in part a) we are interested in the fluid speed at the exit end, while the fluid speed at the entrance is given; so it may be tempting to use some kind of kinematic equation to solve for the acceleration in part b). However, this would be wrong, because the condition of constant acceleration is not fulfilled in this case, and so the kinematic equations derived for point particles in Chapters 2 and 3 do not apply.

SKETCH:



RESEARCH:

(a) From the equation of continuity, $A_1v_1 = A_2v_2 = R_\text{V}$. Therefore, the flow rate at the exit will be the same as that of the entrance. Since the velocity of the fluid at the entrance is known, the flow rate can be readily obtained.

(b) If we call the coordinate along the direction of the fluid flow x, (measured in units of m) then the area is given as a function of this coordinate as

$$A(x) = (0.500 - 0.150x)^2 \text{ m}^2$$

From the continuity equation we then obtain for the fluid speed as a function of the x-coordinate:

$$v(x) = R_\text{V} / A(x) = R_\text{V} / (0.500 - 0.150x)^2 \text{m}^2$$

In order to take the derivative of the fluid speed with respect to time and thus obtain the local acceleration, we have to make use of a change of variables:

$$a \equiv \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

For the derivative dv/dx we find

$$dv(x)/dx = 0.300R_\text{V} / (0.500 - 0.150x)^3 \text{m}^3$$

(c) The increased flow rate increases the velocity, and we can obtain our result by simply inserting a new value of R_V at the exit.

SIMPLIFY: Simplification is not necessary for part a). In part b) we obtain:

$$a = \frac{dv}{dx}v = \frac{0.300R_\text{V}}{(0.500 - 0.150x)^3 \text{m}^3}\frac{R_\text{V}}{(0.500 - 0.150x)^2 \text{m}^2} = \frac{0.300R_\text{V}^{\,2}}{(0.500 - 0.150x)^5 \text{m}^5}$$

CALCULATE:

(a) The flow rate throughout is given by $R_V = (8.00 \text{ m/s})(0.500 \text{ m})(0.500 \text{ m}) = 2.00 \text{ m}^3/\text{s}$.

(b) The acceleration at the exit is then given by $a(x{=}2) = \dfrac{0.300(2.00 \text{ m}^3/\text{s})^2}{(0.500 - 0.150{\cdot}2)^5 \text{ m}^5} = 3750 \text{ m/s}^2$

(c) In part a) we found a flow rate of $2.00 \text{ m}^3/\text{s}$, and in part b) we found that the acceleration depends on the square of the flow rate. Increasing the flow rate to $6.00 \text{ m}^3/\text{s}$ then increases the acceleration by a factor of 9 to $33{,}750 \text{ m/s}^2$.

ROUND:

(a) The flow rate is $2.00 \text{ m}^3/\text{s}$.

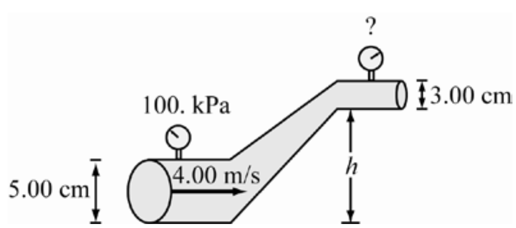
(b) $a(x=2) = 3{,}750 \text{ m/s}^2$

(c) $a(x=2) = 3.38 \cdot 10^4 \text{ m/s}^2$

DOUBLE-CHECK: Our calculations resulted in extremely high accelerations of hundreds of *g*. This might make us suspicious. However, a nozzle like the one described in the problem is not something we might encounter in our everyday experiences, and so we have no easy reference point to check how reasonable our answer is. It is comforting, though, that the units work out properly. We can also calculate the average acceleration and see how this compares to our answer for part b). The average acceleration is the velocity change between the beginning and end of the nozzle, divided by the average time the water took to cross the nozzle. To do this, we can use our continuity equation and find that the speed at the end of the nozzle is 50 m/s. This means that the velocity change is $\Delta v = 50 \text{ m/s} - 8 \text{ m/s} = 42 \text{ m/s}$. We can obtain the time interval from taking the ratio of the length of the nozzle, divided by the average velocity $\Delta t = \Delta x / \bar{v}$. With $\Delta x = 2 \text{ m}$ and $\bar{v} = \frac{1}{2}(v + v_i) = 29 \text{ m/s}$ we get $\Delta t = 2/29 \text{ s}$ and thus for the average acceleration $\bar{a} = 609 \text{ m/s}^2$. This is also a very large number, but not nearly as large as what we found in part b). It indicates that the acceleration rises sharply along the nozzle. For comparison, if we insert $x = 0$ into our formula for the acceleration, we find a value of 38.4 m/s^2 at the beginning of the nozzle.

13.56. **THINK:** The question asks for the pressure at the right-most side of the pipe. The velocity at the entrance of the pipe is $v_1 = 4.00 \text{ m/s}$, the pressure in the left section of the pipe is $P_1 = 100. \text{ kPa}$, and the diameter of the left section of the pipe is $d_1 = 5.00 \text{ cm}$. The diameter of the right section of the pipe is $d_2 = 3.00 \text{ cm}$, and the right section of the pipe is h = 1.50 m higher than the left section.

SKETCH:



RESEARCH: The equation of continuity gives the velocity of the fluid at the upper gauge:

$$v_1 A_1 = v_2 A_2 \;\Rightarrow\; v_2 = v_1 \frac{A_1}{A_2}.$$

Now, Bernoulli's equation states that $P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$. Setting $h_1 = 0$ gives:

$$P_2 + \frac{1}{2}\rho {v_2}^2 + \rho g h_2 = P_1 + \frac{1}{2}\rho {v_1}^2.$$

SIMPLIFY: The pressure at the right section of the pipe is given by:

$$P_2 = P_1 + \frac{1}{2}\rho {v_1}^2 - \frac{1}{2}\rho {v_2}^2 - \rho g h_2 = P_1 + \frac{1}{2}\rho {v_1}^2 - \frac{1}{2}\rho\left(v_1 \frac{A_1}{A_2}\right)^2 - \rho g h_2.$$

CALCULATE:

$$P_2 = 100.\text{ kPa} + \frac{1}{2}\left(1000.\text{ kg/m}^3\right)\left(4.00\text{ m/s}\right)^2 - \frac{1}{2}\left(1000.\text{ kg/m}^3\right)\left(\left(4.00\text{ m/s}\right)\frac{\left(0.00250\text{ m}^2\right)}{\left(0.000900\text{ m}^2\right)}\right)^2 -$$

$$\left(1000.\text{ kg/m}^3\right)\left(9.81\text{ m/s}^2\right)\left(1.50\text{ m}\right)$$

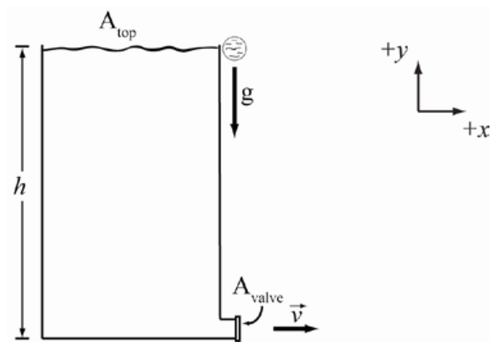
$$= 31556\text{ Pa}$$

ROUND: $P_2 = 31.6\text{ kPa}$.

DOUBLE-CHECK: It is expected that the pressure will decrease as the velocity increases.

13.57. **THINK:** The question asks for the speed of the water at the valve. It is given that the area of the valve is 1/10th the area at the top of the tank.

SKETCH:



RESEARCH: From Bernoulli's equation,

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2.$$

Consider that the atmospheric pressure at both ends is the same; $p_1 = p_2$. Now, by the continuity equation:

$$v_{\text{valve}} = v_{\text{top}}\frac{A_{\text{top}}}{A_{\text{valve}}} = 10 v_{\text{top}}.$$

Therefore, from Bernoulli's equation,

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2 \implies \frac{1}{2}\frac{v_{\text{valve}}^2}{10^2} + gh = \frac{1}{2}v_{\text{valve}}^2.$$

SIMPLIFY: Solving for v_{valve} gives $v_{\text{valve}} = \sqrt{\frac{2gh}{1-10^{-2}}}$. Consider a drop released from rest. The velocity after a height, *h*, will given by the kinematic equation, $v^2 = v_0^2 + 2gh = 2gh \Rightarrow v = \sqrt{2gh}$.

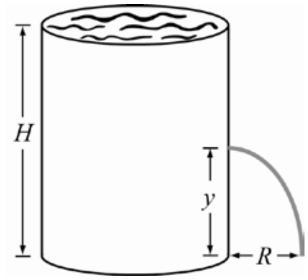
CALCULATE: $v_{\text{valve}} = \sqrt{\frac{2\left(9.81\text{ m/s}^2\right)\left(1.0\text{ m}\right)}{1-10^{-2}}} = 4.45\text{ m/s},\ v = \sqrt{2\left(9.81\text{ m/s}^2\right)\left(1.0\text{ m}\right)} = 4.43\text{ m/s}$

ROUND: Since the height is given to two significant figures, the velocity of the water at the valve is $v_{\text{valve}} = 4.5\text{ m/s}$ and the velocity of a drop of water from rest is $v = 4.4$ m/s.

DOUBLE-CHECK: When the area of the tank is much greater than that of the release valve, the speed of the water at the top relative to the speed at the valve is almost negligible, and Bernoulli's equation essentially gives the kinematic equation for free fall. The velocity at the valve is slightly larger than the velocity of a drop after free fall because the water at the tank surface has nonzero downward velocity.

13.58. **THINK:** The question asks for the height from the ground that would maximize the range of the outgoing water stream. It is given that the range is zero both at the top and at the bottom of the tank.

SKETCH:



RESEARCH: Apply Bernoulli's equation to determine the speed at which the water is exiting at the height, h:

$$\frac{1}{2}\rho_1 v_1^{\;2} + \rho g h_1 = \frac{1}{2}\rho_2 v_2^{\;2} + \rho g h_2.$$

Substituting $v_1 = 0$ at the height, H, the velocity, $v_2 = v$, at height, y, can be determined:

$$\rho g H = \frac{1}{2}\rho v^2 + \rho g y. \text{ Therefore, } v = \sqrt{\frac{2\rho g(H-y)}{\rho}}. \quad (12)$$

Next, determine how long it will take the water to hit the ground by using the following equation:

$$y_{\text{f}} - y_0 = v_{0y}t + \frac{1}{2}a_y t^2.$$

Taking $y_{\text{f}} = 0,$ $y_0 = y,$ $v_{0y} = 0$ and $a_y = -\text{g}$, the above equation becomes: $0 - y = 0 - \left(gt^2/2\right),$ which gives $t = \sqrt{2y/g}$. The velocity calculated in (12) is along the x direction. Knowing this velocity and the time, the range, x, can be determined from

$$v = \sqrt{\frac{2\rho g(H-y)}{\rho}} = \frac{x}{t}. \text{ Therefore, } x = t\sqrt{\frac{2\rho g(H-y)}{\rho}} \text{ or } x = \sqrt{4y(H-y)}.$$

SIMPLIFY: To determine the y-value for which the range, x, would be a maximum, determine when $dx/dy = 0$. From equation (3),

$$\frac{dx}{dy} = \sqrt{\frac{H-y}{y}} - \sqrt{\frac{y}{H-y}} = 0.$$

This gives $\sqrt{(H-y)/y} = \sqrt{y/(H-y)}$ or $y = H/2$. Therefore, the maximum range would be obtained when the hole in the tank would be half of its height.

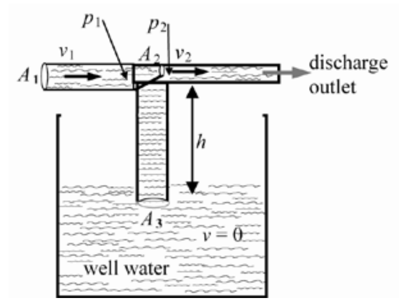
CALCULATE: This step is not necessary.

ROUND: This step is not necessary.

DOUBLE-CHECK: There is nothing to double check.

13.59. **THINK:** The question describes a water pump mechanism. By using tap water and drastically reducing the size of the pipe one can create much higher water streaming speed and thus create pressure below atmospheric pressure. (This is similar to putting your thumb on the garden hose, which enables you to spray farther.) The reduced pressure can then be used to suck water out of the sump well and keep the basement dry. In this problem we need to proceed in two steps. First we deal with the horizontal pipe and calculate the speed and pressure in the part of the pipe that has the reduced pipe size. Of course, the Bernoulli equation in the special case of equal height seems tailor-made to do this. Then, in the second step, we take the calculated pressure difference and see to what height we can lift water with it. Again we can use a limit of the Bernoulli equation. We can obtain information on the maximum height we can lift the water to, if we consider the case of very slowly flowing water (v=0), in which we can neglect the v^2 term in the Bernoulli equation).

SKETCH:



RESEARCH:

(a) To determine the speed at the discharge outlet, recall the continuity equation:

$$A_1v_1 = A_2v_2 \Rightarrow v_2 = v_1\frac{A_1}{A_2}.$$

(b) Recall Bernoulli's equation: $p_1 + \frac{1}{2}\rho v_1^2 + \rho gh_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho gh_2$. Since $h_1 = h_2$,

$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$.

(c) Making use of Bernoulli's equation:

$$p_3 + \frac{1}{2}\rho v_3^2 + \rho gh_3 = p_2 + \frac{1}{2}\rho v_2^2 + \rho gh_2,$$

where $h_3 = 0$ and $h_2 = h$. For negligible v^2 we then find with $p_3 = p_{\text{atm}}$,

$$p_3 + \rho gh_3 = p_2 + \rho gh_2 \Rightarrow p_3 - p_2 = \rho g(h_2 - h_3) = \rho gh$$

SIMPLIFY:

(a) Since $A_2 = A_1/10$, the speed is given by $v_2 = v_1(10)$.

(b) The pressure, p_2, is given by $p_2 = p_1 + \frac{1}{2}\rho\left(v_1^2 - v_2^2\right)$.

(c) $h = \dfrac{p_3 - p_2}{\rho g}$

CALCULATE:

(a) $v_2 = (2.05 \text{ m/s})(10) = 20.5 \text{ m/s}$

(b) $p_2 = 3.03 \cdot 10^5 \text{ Pa} + \frac{1}{2}\left(1000. \text{ kg/m}^3\right)\left((2.05 \text{ m/s})^2 - (20.5 \text{ m/s})^2\right) = 94.98 \text{ kPa}$

(c) $h = \dfrac{(101 \text{ kPa}) - (94.98 \text{ kPa})}{(1000. \text{ kg/m}^3)(9.81 \text{ m/s}^2)} = 0.614 \text{ m}$

ROUND: Round the results to three significant figures.

(a) $v_2 = 20.5$ m/s

(b) $p_2 = 95.0$ kPa

(c) $h = 0.614$ m

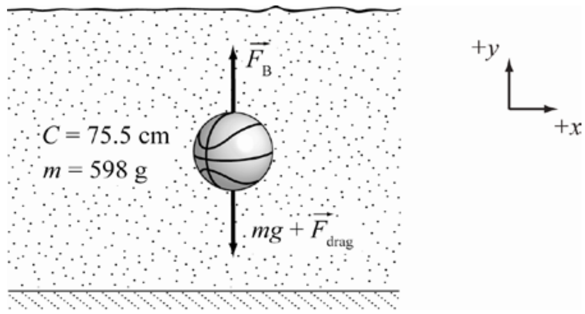
DOUBLE-CHECK: The last number is a bit worrisome, because lifting water 2 ft usually does not get it out of the basement. But if you examine the input numbers, you see that the final result depends strongly on the ratio by which the pipe area is reduced and on the initial streaming speed of the water. Adjusting these one can get pumps operating on this principle, which lift water by 10 ft, which is certainly enough for the intended purpose. Water-powered sump pump are a great safety device in case that electricity fails (which is not unusual in some parts of the country during strong thunderstorms …).

13.60. **THINK:** A basketball with circumference, C = 75.5 cm or 0.755 m and mass, m = 598 g or 0.598 kg, is forced to the bottom of a swimming pool then released.

(a) Determine the buoyant force on the basketball.

(b) Determine the drag force the basketball experiences while rising at a constant velocity.

SKETCH:



RESEARCH:

(a) Recall that the magnitude of the buoyant force is equal to the magnitude of the weight of the fluid displaced. Therefore, $F_B = \rho_{water} Vg$, where the volume is given by:

$$V = \frac{4}{3}\pi r^3.$$

(b) Since the velocity of the ball is constant, the net force acting on it must be zero. From the free-body diagram: $0 = F_B - mg - F_{drag} \Rightarrow F_{drag} = F_B - mg$.

SIMPLIFY:

(a) Nothing to simplify.

(b) Nothing to simplify.

CALCULATE:

(a) First, determine the volume:

$$V = \frac{4}{3}\pi\left(\frac{C}{2\pi}\right)^3 = \frac{4}{3}\pi\left(\frac{0.755 \text{ m}}{2\pi}\right)^3 = 7.27 \cdot 10^{-3} \text{ m}^3.$$

Then, the buoyant force is given by $F_B = \left(1000. \text{ kg/m}^3\right)\left(7.27 \cdot 10^{-3} \text{ m}^3\right)\left(9.81 \text{ m/s}^2\right) = 71.3 \text{ N}.$

(b) Plugging in the result from part (a) gives $F_{drag} = 71.3 \text{ N} - 0.598 \text{ kg}\left(9.81 \text{ m/s}^2\right) = 65.4 \text{ N}.$

ROUND: The given values had three significant figures, so the results remain as they are.

(a) $F_B = 71.3$ N

(b) $F_{drag} = 65.4$ N

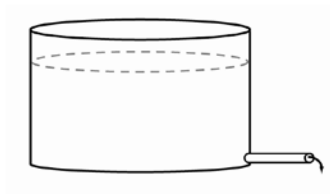
DOUBLE-CHECK:

(a) 71.3 N corresponds to about 7 kg or 15 lbs on Earth. Since the buoyancy force is simply the amount of water displaced by the object, it is easy to imagine a water filled basketball weighing this amount (ignoring the thickness of the basketball's material, the volume of the basketball is virtually the volume inside the basketball).

(b) The drag force should be smaller since the basketball moves upward.

13.61. **THINK:** The question asks for the amount of oil flowing out of a container in a period of 10.0 s. The oil has a viscosity of 0.300 Pa s and a density of $670. \text{ kg/m}^3$. The ejection tube is 20.0 cm long with a diameter of 0.200 cm.

SKETCH:



RESEARCH: Determine the volume of oil per second: $R_v = \dfrac{\pi r^4 \Delta p}{8\eta l}$.

Assume that the change in pressure does not vary much in 10 s. Then, the volume of oil ejected in time Δt is $V = R_v \Delta t$.

SIMPLIFY: $V = R_v \Delta t = \dfrac{\pi r^4 \Delta p}{8\eta l}\Delta t = \dfrac{\pi r^4 \rho g \Delta y}{8\eta l}\Delta t$

CALCULATE: $V = \dfrac{\pi(0.00100 \text{ m})^4 (670. \text{ kg/m}^3)(9.81 \text{ m/s}^2)(0.500 \text{ m})}{8(0.300 \text{ Pa}\cdot\text{s})(0.200 \text{ m})}(10.0 \text{ s}) = 2.1509\cdot 10^{-7} \text{ m}^3$

ROUND: Rounding to three significant figures, $V = 2.15\cdot 10^{-7} \text{ m}^3$.

DOUBLE-CHECK: This volume is equal to 0.215 mL, which is a reasonable amount of oil to drip out of such an opening.

13.62. **THINK:** The question asks for the pressure inside Hurricane Rita. Use $v_1 \approx 0$ as the outside wind speed. After converting the wind speed inside the hurricane to meters per second, use $v_2 = 290 \text{ km/h} = 80.6 \text{ m/s}$. From Example 13.5 from the text, use $\rho = 1.205 \text{ kg/m}^3$ as the density of air (a typical value for the density of air at a temperature of $20°$ C).

SKETCH: A sketch is not needed to solve the problem.

RESEARCH: Use $P_1 = P_{\text{atm}} = 101 \text{ kPa}$, and take $h_1 = h_2$ in Bernoulli's equation:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2.$$

SIMPLIFY: $P_{\text{atm}} = P_{\text{hur}} + \frac{1}{2}\rho v_2^2 \quad \Rightarrow \quad P_{\text{hur}} = P_{\text{atm}} - \frac{1}{2}\rho v_2^2$

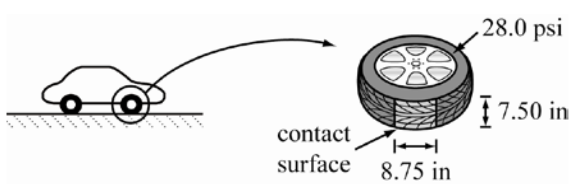
CALCULATE: $P_{\text{hur}} = 101\cdot 10^3 \text{ Pa} - \frac{1}{2}(1.205 \text{ kg/m}^3)(80.6 \text{ m/s})^2 = 97.1 \text{ kPa}.$

ROUND: The wind speed was given to two significant figures, so the answer should be rounded to the same precision. The atmospheric pressure inside the hurricane was 97 kPa.

DOUBLE-CHECK: NASA maintains data they obtain using Hurricane hunter aircraft. They report that Hurricane Rita had a minimum recorded central pressure of 89.7 kPa. The calculated pressure is reasonably close to, but greater than the minimum recorded central pressure, so the calculated value is reasonable.

13.63. **THINK:** The question asks for the weight of a car if it is known that the tire pressure is 28.0 psi, and the width and length of the contact surface of each tire is 7.50 in, and 8.75 in, respectively.

SKETCH:



RESEARCH: Recall that the definition of pressure is:

$$P=\frac{\text{Force}}{\text{Area}}=\frac{F}{A}.$$

Therefore, the weight of the car can be taken as the force due to the pressure on all four tires. That is, $F_{\text{car}}=4PA$.

SIMPLIFY: Simplification is not necessary.

CALCULATE: Recall also that 1 pound per square inch = 6894.75729 Pascals, and that 1 inch = 0.0254 m.

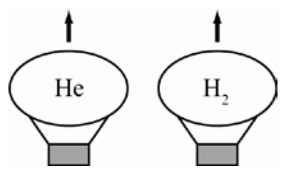
Therefore, $F_{\text{car}}=4(28.0)(6894.75729\text{ Pa})(0.0254\text{ m})^2(7.50)(8.75)=32694.4\text{ N}.$

ROUND: To three significant figures, the weight of the car is 32700 N.

DOUBLE-CHECK: Although the result is somewhat high for the weight of an average car, it is reasonable for the given values.

13.64. **THINK:** The question asks for the ratio of the lifting powers of helium to hydrogen gases. The molar mass of air is 28.95 g/mol and that of hydrogen gas is 2.016 g/mol. (Hydrogen is a diatomic molecule, and the mass number of the average hydrogen atom is 1.0079, which is slightly larger than 1 because of the small admixtures of the heavier hydrogen isotopes deuterium and tritium.) Helium is a noble gas and therefore monatomic. Its mass number is 4.003 and therefore the molar mass of helium gas is 4.003 g/mol.

SKETCH:



RESEARCH: Avogadro's Hypothesis states that equal volumes of any gases at the same temperature and pressure contain equal amount (number of particles or moles) of gas. Hence, the mass of a given volume of any gas is proportional to its molar mass. By Archimedes' Principle, then, the lifting power of any gas is proportional to the difference between the equivalent molar mass of air and the molar mass of the gas. Hence, the desired ratio of lifting powers is:

$$\frac{L\text{He}}{L\text{H}_2}=\frac{m_{\text{air}}-m_{\text{He}}}{m_{\text{air}}-m_{\text{H}_2}}.$$

SIMPLIFY: Simplification is not necessary.

CALCULATE: $\dfrac{L\text{He}}{L\text{H}_2}=\dfrac{28.95-4.003}{28.95-2.016}=0.926227$

ROUND: Since the equivalent molar mass of air is given to four significant figures, the answer is rounded to 0.9262.

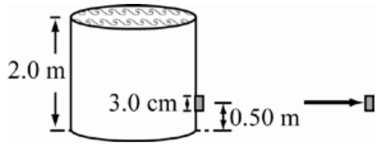
DOUBLE-CHECK: Our first guess might have been that perhaps the lifting power of hydrogen might be about twice as high as that of helium, because it weighs only half of helium. The lifting power comes from the weight of the displaced air minus the weight of the gas. Since helium and hydrogen are both very light compared to air, this means they should have approximately the same lifting power, with hydrogen having a slight advantage. Our answer shows this. Clearly the only slight extra lifting power of hydrogen is far outweighed by the difficulty and danger of handling it.

13.65. **THINK:** The problem describes a large barrel filled with water with a small cork plugging a hole 1.5 m below the top of the barrel.

(a) Determine the magnitude of the static friction, assuming the cork pops out as the water reaches the top of the barrel.

(b) Would the cork pop out if the water was replaced with sea water?

SKETCH:



RESEARCH:

(a) The cork will fly out of the barrel when the force on it from inside the barrel exceeds the combination of force due to air pressure from the outside and the maximal static friction force. The cork flies out when the barrel is completely full, so the pressure at depth, h, below the top of the barrel can be determined as follows. Inside the barrel, at depth, h = 2.0 m – 0.50 m = 1.5 m, the pressure is given by $P = P_0 + \rho gh$, where P_0 is the external air pressure. Then, the force on the cork due to the water inside the barrel can be determined as $F_{\text{int}} = PA = (P_0 + \rho gh)\left(\pi (d/2)^2\right)$ and that from the outside as $F_{\text{ext}} = P_0 A + F_{\text{s}}$. These forces will be equal to each immediately before the cork flies out.

(b) Seawater is slightly denser than fresh water $\left(1024 \text{ kg/m}^3 \text{ versus } 1000. \text{ kg/m}^3\right)$, so the pressure inside the barrel will be slightly increased compared to the fresh water case. Thus the cork would have flown out of the barrel somewhat before it became full.

SIMPLIFY:

(a) Thus, the static force can be determined from:

$$F_{\text{ext}} = F_{\text{int}} = P_0 A + F_{\text{s}} = (P_0 + \rho gh)\left(\pi\left(\frac{d}{2}\right)^2\right),$$

which implies $F_{\text{s}} = (P_0 + \rho gh)\left(\pi\left(\frac{d}{2}\right)^2\right) - P_0 A \Rightarrow (P_0 + \rho gh)\left(\pi\left(\frac{d}{2}\right)^2\right) - P_0\left(\pi\left(\frac{d}{2}\right)^2\right) = \rho gh\pi\left(\frac{d}{2}\right)^2.$

(b) No simplification is necessary.

CALCULATE:

(a) $F_{\text{s}} = \left(1000. \text{ kg/m}^3\right)\left(9.81 \text{ m/s}^2\right)(1.5 \text{ m})\pi\left(\frac{0.030 \text{ m}}{2}\right)^2 = 10.4 \text{ N}.$

(b) Nothing to calculate.

ROUND:

(a) Since the values are given to two significant figures, the result should be rounded to $F_{\text{s}} = 10.$ N.

(b) Rounding is not necessary.

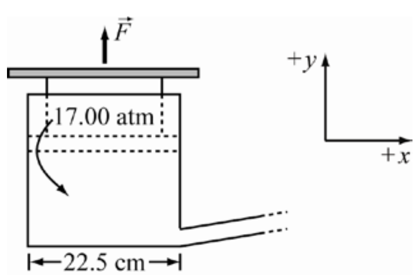
DOUBLE-CHECK:

(a) The value obtained for the static friction is reasonable based on the given values.

(b) It is expected that by increasing the density of the fluid, the gauge pressure will also increase, and thus the corresponding force. Thus, for a greater force, it is expected that the threshold of static friction will be reached sooner.

13.66. **THINK:** The problem asks for the maximum weight a hydraulic lift can support given that the maximum gauge pressure in the lift is 17.00 atm, and the diameter of the output line is d = 22.5 cm = 0.225 m.

SKETCH:



RESEARCH: Recall that the pressure can be written in terms of the force per unit area as $P = \frac{F}{A}$.

SIMPLIFY: Therefore, the maximum force the lift can sustain is:

$$P_{\text{max}} = \frac{F_{\text{max}}}{A} \Rightarrow F_{\text{max}} = P_{\text{max}} A, \text{ where } A = \pi\left(d/2\right)^2.$$

CALCULATE: Recall that 1 atm = 101325 Pa, thus:

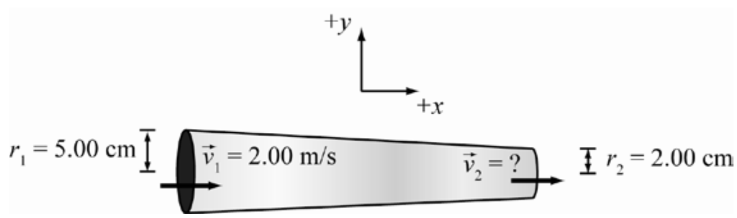
$$F_{\text{max}} = P_{\text{max}} A = (17)(101325 \text{ Pa})\pi\left(\frac{0.225 \text{ m}}{2}\right)^2 = 68489 \text{ N}.$$

ROUND: Since the diameter of the output line is given to three significant figures, the maximum weight the lift can sustain is 68.5 kN.

DOUBLE-CHECK: It is reasonable that a lift will have such great lift capacity since it should be well above the weight of a heavy vehicle.

13.67. **THINK:** The question asks about the velocity of water in a pipe when the radius of the pipe is $r_1 = 5.00$ cm and the water is flowing into the pipe at this end with a speed of 2.00 m/s. The pipe narrows down to a radius of $r_2 = 2.00$ cm and we are interested in finding the speed of the water at this end.

SKETCH:



RESEARCH: Using the continuity equation: $A_1 v_1 = A_2 v_2$. Therefore, $v_1 \pi r_1^2 = v_2 \pi r_2^2$.

SIMPLIFY: Solving for v_2 : $v_2 = v_1 \left(\frac{r_1}{r_2}\right)^2$.

CALCULATE: $v_2 = (2.00 \text{ m/s})\left(\frac{5.00 \text{ cm}}{2.00 \text{ cm}}\right)^2 = 12.5 \text{ m/s}$

ROUND: Since the values are given to three significant figures, the result should remain as $v_2 = 12.5$ m/s.

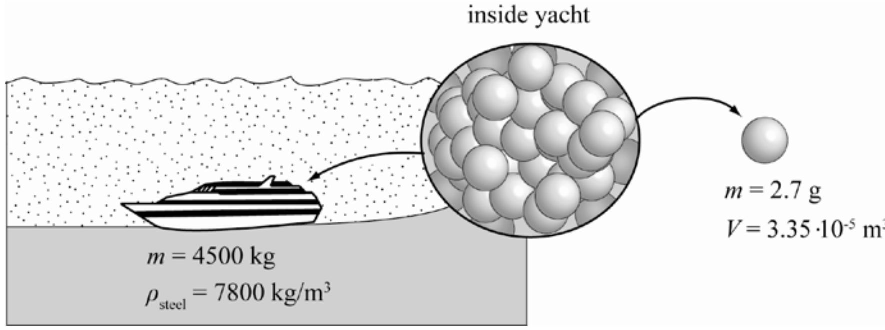
DOUBLE-CHECK: It is expected that the velocity of the fluid will increase as the cross-sectional area of the pipe decreases.

13.68. **THINK:** The problem describes Uncle Scrooge's boat being sunk. The mass of the boat is m = 4500 kg, and it is made of steel with density, $\rho_{\text{steel}} = 7800 \text{ kg/m}^3$. The mass of a ping pong ball is $m_{\text{ball}} = 2.7$ g, and the volume of a ping pong ball is $V_{\text{ball}} = 3.35 \cdot 10^{-5} \text{ m}^3$.

(a) Determine the buoyant force on one ball in water.

(b) Determine how many balls are necessary to float the ship.

SKETCH:



RESEARCH:
(a) Recall by Archimedes' principle that the magnitude of the buoyant force is equal to the magnitude of the weight of the water displaced. Therefore, the buoyant force on one ping-pong ball is $F_B = \rho_{\text{water}} V_{\text{ball}} g$.
(b) The total buoyant force when the boat is filled with N ping-pong balls is equal to the weight of water displaced by the N ping-pong balls and by the steel boat, $F_{\text{B,total}} = N\left(\rho_{\text{water}} V_{\text{ball}} g\right) + \rho_{\text{water}} V_{\text{steel}} g$. The minimum buoyant force necessary for the boat to float is equal to the weight of the boat plus the weight of the N ping-pong balls, $F_{\text{g,total}} = m_{\text{steel}} g + N m_{\text{ball}} g$. The number of balls can be determined solving for N.
SIMPLIFY:
(a) There is nothing to simplify.
(b) Using the fact that $F_{\text{B,total}} = F_{\text{g,total}}$, solving for N yields the equation:

$$N\left(\rho_{\text{water}} V_{\text{ball}} g\right) + \rho_{\text{water}} V_{\text{steel}} g = m_{\text{steel}} g + N m_{\text{ball}} g \Rightarrow N\left(\rho_{\text{water}} V_{\text{ball}} - m_{\text{ball}}\right) = m_{\text{steel}} - \rho_{\text{water}} V_{\text{steel}}$$

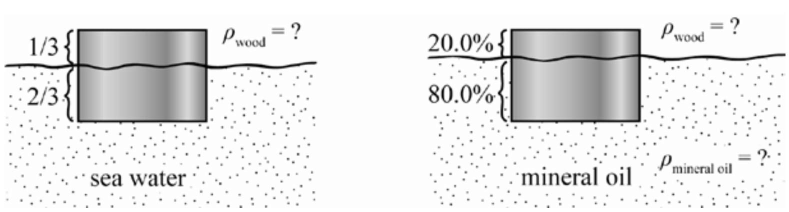
$$\Rightarrow N = \frac{m_{\text{steel}} - \rho_{\text{water}} V_{\text{steel}}}{\rho_{\text{water}} V_{\text{ball}} - m_{\text{ball}}} = \frac{m_{\text{steel}} - \rho_{\text{water}}\left(\dfrac{m_{\text{steel}}}{\rho_{\text{steel}}}\right)}{\rho_{\text{water}} V_{\text{ball}} - m_{\text{ball}}} = \frac{m_{\text{steel}}\left(1 - \dfrac{\rho_{\text{water}}}{\rho_{\text{steel}}}\right)}{\rho_{\text{water}} V_{\text{ball}} - m_{\text{ball}}}.$$

CALCULATE:
(a) $F_B = \left(1000.\text{ kg/m}^3\right)\left(3.35 \cdot 10^{-5}\text{ m}^3\right)\left(9.81\text{ m/s}^2\right) = 0.3286\text{ N}$

(b) $$N = \frac{\left(4500\text{ kg}\right)\left(1 - \dfrac{1000.\text{ kg/m}^3}{7800\text{ kg/m}^3}\right)}{\left(1000.\text{ kg/m}^3\right)\left(3.35 \cdot 10^{-5}\text{ m}^3\right) - \left(2.7 \cdot 10^{-3}\text{ kg}\right)} = 127373$$

ROUND:
(a) Keeping two significant figures, the buoyant force on each ball is $F_B = 0.33$ N.
(b) In order to lift the boat the minimum number of ping pong balls needed is $1.3 \cdot 10^5$.
DOUBLE-CHECK:
It is expected that a very large number of balls is needed to overcome the weight of the boat. Note that if the weight of the balls and volume of the steel is neglected (reasonable as the balls are light and steel is dense), $N = \dfrac{\text{weight of boat}}{\text{buoyant force on a single ball from (a)}} = \dfrac{\left(4500\text{ kg}\right)\left(9.81\text{ m/s}^2\right)}{0.33\text{ N}} = 1.338 \cdot 10^5$ balls, which is equal to the above answer to two significant figures.

13.69. **THINK:** The question asks about the density of (a) wooden block and (b) mineral oil, given that the block will be two thirds submerged in sea water, and 80.0% submerged in mineral oil.
SKETCH:



RESEARCH:
(a) Since the wooden block is floating, its weight must be equal to the displaced sea water.
(b) It is the same core idea for determining the density of the mineral oil.
SIMPLIFY:

(a) $\rho_{\text{block}} V_{\text{block}} g = \rho_{\text{sea water}} V_{\text{submerged}} g \Rightarrow \rho_{\text{block}} V_{\text{block}} g = \rho_{\text{sea water}}\left(\dfrac{2}{3}\right) V_{\text{block}} g \Rightarrow \rho_{\text{block}} = \rho_{\text{sea water}}\left(\dfrac{2}{3}\right).$

(b) $\rho_{\text{block}} V_{\text{block}} g = \rho_{\text{oil}} V_{\text{submerged}} g \;\Rightarrow\; \rho_{\text{block}} V_{\text{block}} g = \rho_{\text{oil}} (0.80) V_{\text{block}} g \;\Rightarrow\; \rho_{\text{oil}} = \dfrac{\rho_{\text{block}}}{0.80}.$

CALCULATE:

(a) $\rho_{\text{block}} = \dfrac{2}{3}\left(1.024 \text{ g/cm}^3\right) = 0.683 \text{ g/cm}^3.$

(b) $\rho_{\text{oil}} = \dfrac{0.683 \text{ g/cm}^3}{0.800} = 0.853 \text{ g/cm}^3.$

ROUND:

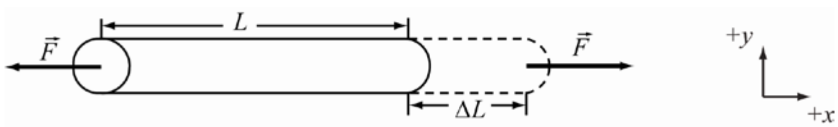
(a) Keeping three significant figures, the answer remains $\rho_{\text{block}} = 0.683 \text{ g/cm}^3$.

(b) Keeping three significant figures, the answer is $\rho_{\text{oil}} = 0.853 \text{ g/cm}^3$.

DOUBLE-CHECK: Both results are reasonable based on the given values. It is expected that since more of the wooden block is submerged in the mineral oil than in sea water, then the density of mineral oil must be lower than that of sea water.

13.70. **THINK:** Use the change in tendon length, $\Delta l = 0.37$ mm, to determine the strain. Use the force, F = 13.4 N, to determine the stress. Assume the tendon is a cylinder of length, L = 15 cm, and diameter, d = 3.5 mm.

SKETCH:



RESEARCH: The cross-sectional area of the tendon is $A = \pi (d/2)^2$. The strain on the tendon is $\Delta L / L$. The stress on the tendon is F/A. Young's modulus is Y = stress/strain.

SIMPLIFY: $Y = \dfrac{\text{stress}}{\text{strain}} = \dfrac{F/A}{\Delta L / L} = \dfrac{4FL}{\pi d^2 \Delta L}$

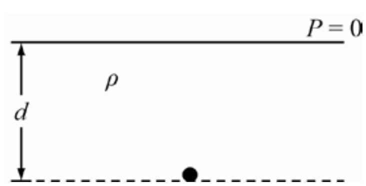
CALCULATE: $Y = \dfrac{4(13.4 \text{ N})(0.15 \text{ m})}{\pi (0.0035 \text{ m})^2 (0.00037 \text{ m})} = 5.64636 \cdot 10^8 \text{ N/m}^2$

ROUND: $Y = 0.56 \cdot 10^9 \text{ N/m}^2$

DOUBLE-CHECK: A check of the proper units works out fine. Table 13.1 shows that 10^9 N/m^2 is approximately the right order of magnitude for Young's modulus. Tendons should be much harder to stretch than rubber and easier than wood, and according to Table 13.1 this is true for our result.

13.71. **THINK:** On another planet, the pressure under water is determined in the same manner as it is on Earth. The only difference is the new value of gravity, $g' = 0.135g$. The depth is d = 1.00 km and the density of water is $\rho = 1000. \text{ kg/m}^3$. Assume the pressure in the atmosphere is zero.

SKETCH:



RESEARCH: The pressure at a certain depth is given by $P = P_0 + \rho g d$.

SIMPLIFY: $P = \rho g' d = 0.135 \rho g d$

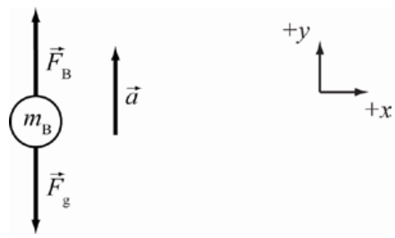
CALCULATE: $P = 0.135\left(1000. \text{ kg/m}^3\right)\left(9.81 \text{ m/s}^2\right)\left(1.00 \cdot 10^3 \text{ m}\right) = 1.324 \cdot 10^6 \text{ Pa}$

ROUND: P = 1.32 MPa

DOUBLE-CHECK: This pressure is still greater than the atmospheric pressure (P_0) of Earth, which is expected even on another planet.

13.72. **THINK:** Since the density of each ball, $\rho_A = 0.90 \text{ g/cm}^3$ and $\rho_B = 0.80 \text{ g/cm}^3$, is less than the density of water, $\rho_w = 1.00 \text{ g/cm}^3$, they should float to the top, i.e. the buoyant force, F_B, is greater than the gravitational force, F_g. The volumes are equal and irrelevant in calculating their accelerations. Whichever has the higher acceleration wins.

SKETCH:



RESEARCH: The buoyant force on each sphere is the same, $F_B = \rho_w Vg$. The mass of each sphere is $m_A = \rho_A V$ and $m_B = \rho_B V$. The net force on each sphere gives the sphere its acceleration such that, in general, $F_B - F_g = ma$.

SIMPLIFY:

(a) Using $F_g = m_A g$, and balancing the forces gives

$$\rho_w Vg - m_A g = m_A a_A \Rightarrow \rho_w Vg - \rho_A Vg = \rho_A V a_A \Rightarrow a_A = \left(\frac{\rho_w - \rho_A}{\rho_A}\right)g.$$

(b) Using $F_g = m_B g$, and balancing the forces gives

$$\rho_w Vg - m_B g = m_B a_B \Rightarrow \rho_w Vg - \rho_B Vg = \rho_B V a_B \Rightarrow a_B = \left(\frac{\rho_w - \rho_B}{\rho_B}\right)g.$$

CALCULATE:

(a) $a_A = \left(\frac{1.00 \text{ g/cm}^3 - 0.90 \text{ g/cm}^3}{0.90 \text{ g/cm}^3}\right)(9.81 \text{ m/s}^2) = 1.09 \text{ m/s}^2$

(b) $a_B = \left(\frac{1.00 \text{ g/cm}^3 - 0.80 \text{ g/cm}^3}{0.80 \text{ g/cm}^3}\right)(9.81 \text{ m/s}^2) = 2.45 \text{ m/s}^2$

ROUND:

(a) $a_A = 1.1 \text{ m/s}^2$

(b) $a_B = 2.5 \text{ m/s}^2$

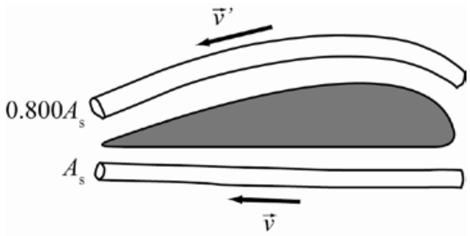
(c) Ball B reaches the top first.

DOUBLE-CHECK: Since the densities of the balls are close to that of the water, they should have a relatively small acceleration.

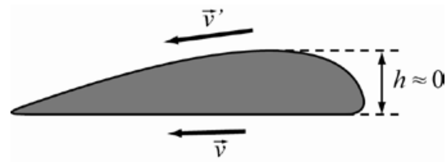
13.73. **THINK:** It is the cross-sectional area of the stream lines which are reduced by 80.0%. If A_s is the cross-sectional area of the streamline under the wing (where no distortion occurs), then it is $0.800A_s$ above the wings. The continuity equation can be used to determine the velocity above the wing. The Bernoulli equation can then be used to determine the pressure above the wing, assuming the change in height from the bottom to the top of the wing is negligible. The pressure difference multiplied by the cross-sectional area of the wing should provide the lift force. The velocity of air under the wing is v = 200. m/s, the area of the wing is $A_w = 40.0 \text{ m}^2$ and the density of air is $\rho_a = 1.30 \text{ kg/m}^3$.

SKETCH:

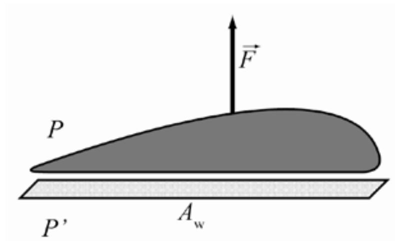
(a)



(b)



(c)



RESEARCH: The continuity equation is $A_1v_1 = A_2v_2 = \text{constant}$. The Bernoulli equation (for same height) is $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$. The force produced when a pressure difference is applied on an area is $F = \Delta PA$.

SIMPLIFY:

(a) $A_1v_1 = A_\text{s}v$ and $A_2v_2 = 0.800A_\text{s}v'$. The continuity equation gives $v' = \dfrac{A_\text{s}v}{0.800A_\text{s}} = \dfrac{v}{0.800} = \dfrac{5v}{4}$.

(b) $P_1 + \frac{1}{2}\rho v_1^2 = P' + \frac{1}{2}\rho_\text{a}v^2$ and $P_2 + \frac{1}{2}\rho v_2^2 = P + \frac{1}{2}\rho_\text{a}v'^2$. Equating these expressions gives

$$P' - P = \frac{1}{2}\rho_\text{a}\left(v'^2 - v^2\right).$$

(c) $F = \Delta PA \Rightarrow F = \left(P' - P\right)A$

CALCULATE:

(a) $v' = \dfrac{5\left(200.\text{ m/s}\right)}{4} = 250.\text{ m/s}$

(b) $P' - P = \frac{1}{2}\left(1.30\text{ kg/m}^3\right)\left(\left(250.\text{ m/s}\right)^2 - \left(200.\text{ m/s}\right)^2\right) = 14625\text{ Pa}$

(c) $F = \left(14625\text{ Pa}\right)\left(40.0\text{ m}^2\right) = 585{,}000\text{ N}$

ROUND:

(a) $v' = 250.\text{ m/s}$

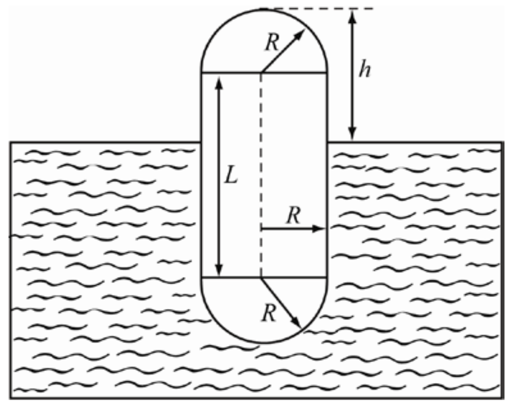
(b) $P' - P = 14.6\text{ kPa}$

(c) $F = 585\text{ kN}$

DOUBLE-CHECK: Overall, the force from the lift is equivalent to around 60 tons, which is a reasonable mass for an airplane.

13.74. **THINK:** Since the buoy is partially submerged and not moving, the buoyant force and gravitational force are equal. Since the forces are equal, the ratio of densities is equal to the ratio of volumes. The density of the buoy of mass, $M = 75.0$ kg, can be calculated by first determining the volume of the buoy. $L = 0.600$ m, $R = 0.200$ m and the density of water is $\rho_\text{w} = 1027\text{ kg/m}^3$.

SKETCH:



RESEARCH: Each cap is half a sphere, so the total volume is the volume of the cylinder and a sphere summed:

$$V_B = \pi R^2 L + \frac{4}{3}\pi R^3.$$

The density of the buoy is $\rho_B = M / V_B$. The volume of the displaced water is the same as the volume of the buoy that is submerged, which is part of a cylinder and half of a sphere:

$$V_w = \pi R^2 \left(L - (h - R)\right) + \frac{1}{2}\left(\frac{4}{3}\pi R^3\right).$$

Since the forces are equal: $\frac{\rho_B}{\rho_w} = \frac{V_w}{V_B}$.

SIMPLIFY: $\frac{\rho_B}{\rho_w} = \frac{V_w}{V_B} \Rightarrow \frac{M}{V_B \rho_w} = \frac{V_w}{V_B} \Rightarrow \frac{M}{\rho_w} = V_w \Rightarrow \frac{M}{\rho_w} = \pi R^2 \left(L - (h - R)\right) + \frac{4}{6}\pi R^3$

$\Rightarrow \frac{M}{\pi R^2 \rho_w} = \frac{10R}{6} + L - h \Rightarrow h = \frac{10R}{6} + L - \frac{M}{\pi R^2 \rho_w}$

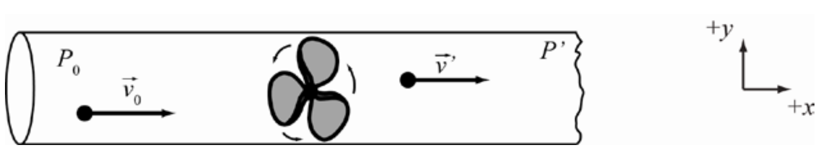
CALCULATE: $h = \frac{10}{6}(0.200 \text{ m}) + 0.600 \text{ m} - \frac{75.0 \text{ kg}}{\pi (0.200 \text{ m})^2 (1027 \text{ kg/m}^3)} = 0.3522 \text{ m}$

ROUND: $h = 0.352$ m

DOUBLE-CHECK: Since most of the mass is at the bottom, it is expected that most of it is submerged. Its total length is $L + 2R = 1.00$ m, so most of the buoy is submerged ((1.00 – 0.352) m = 0.648 m).

13.75. **THINK:** Assume the height of the water remains the same. The Bernoulli equation can then be used to determine the velocity at which it boils. The pressure when the water boils is $P' = 2.3388$ kPa. The water is at atmospheric pressure, $P_0 = 101.3$ kPa, when stationary $(v_0 \approx 0 \text{ m/s})$. The density of water is $\rho_w = 998.2 \text{ kg/m}^3$.

SKETCH:



RESEARCH: The Bernoulli equation for constant height is $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$.

SIMPLIFY: $P_0 = P' + \frac{1}{2}\rho_w v'^2 \Rightarrow v' = \sqrt{\frac{2(P - P')}{\rho_w}}$

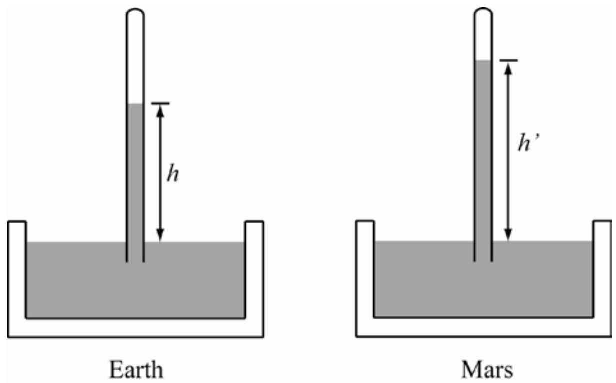
CALCULATE: $v' = \sqrt{\dfrac{2(101.3\text{ kPa} - 2.3388\text{ kPa})}{998.2\text{ kg/m}^3}} = 14.081\text{ m/s}$

ROUND: $v' = 14.08\text{ m/s}$

DOUBLE-CHECK: This velocity is about 32 mph, which is a fairly brisk speed for a boat to travel, but certainly seems in the right order of magnitude.

13.76. **THINK:** To determine the gravity on the surface of Mars, use the mass, $M = 6.42 \cdot 10^{23}$ kg, and the radius, $R = 3.39 \cdot 10^6$ m, of Mars. Since the scale is to still read 760 mm for $P_0 = 101.325$ kPa, the change in gravity will produce a change in height and this height must still read 760 mm, so the length of a millimeter will change. The density of mercury, ρ_m, is the same on either planet.

SKETCH:



RESEARCH: The force of gravity on the surface of Mars is $g_\text{M} = \text{G}M / R^2$. Since the pressure in the manometer is zero, the atmospheric pressure is the depth pressure, so $P_0 = \rho_\text{m} gh$ on Earth and $P_0 = \dfrac{\rho_\text{m}\text{G}Mh'}{R^2}$ on Mars. The scale needs to be stretched by a factor of h'/h.

SIMPLIFY: For the same pressure: $\rho_\text{m} gh = \rho_\text{m}\dfrac{\text{G}M}{R^2}h' \Rightarrow \dfrac{h'}{h} = \dfrac{gR^2}{\text{G}M}$.

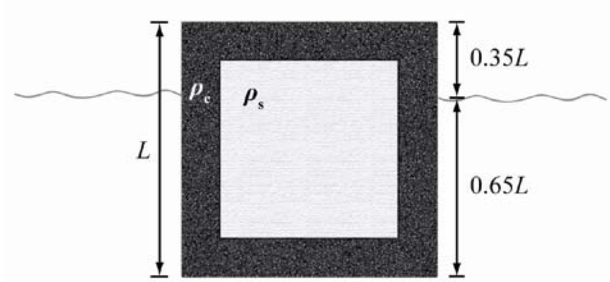
CALCULATE: $\dfrac{h'}{h} = \dfrac{(9.81\text{ m/s}^2)(3.39 \cdot 10^6\text{ m})^2}{(6.673 \cdot 10^{-11}\text{ N m}^2/\text{kg}^2)(6.42 \cdot 10^{23}\text{ kg})} = 2.63155$

ROUND: $h'/h = 2.63$, i.e. 1 mm on Earth's barometer needs to be 2.63 mm on Mars' barometer.

DOUBLE-CHECK: Since Mars' gravity is about (2/5)g, it is expected that the mercury rises about 2.5 times that on Earth.

13.77. **THINK:** If the mass is distributed uniformly and if 35.0 % of the volume is above, then 65.0 % is below, which is the same volume of the displaced water. If the block floats, the buoyant and gravitational forces are equal and the ratio of volumes is equal to the ratio of densities.

SKETCH:



RESEARCH: The overall density of the block is the total mass over the total volume: $\rho_0 = \dfrac{M_c + M_s}{V_c + V_s}$.

Since the forces are equal, $\dfrac{V_w}{V_0} = \dfrac{\rho_0}{\rho_w}$. The volume of displaced water is the same as the submerged volume of the block, so $V_w = 0.650V_0$.

SIMPLIFY: $\dfrac{\rho_0}{\rho_w} = \dfrac{V_w}{V_0} = 0.650 \Rightarrow \rho_0 = 0.650\rho_w$

The mass of concrete is $M_c = \rho_c V_c$. The mass of Styrofoam is $M_s = \rho_s V_s$. The overall density is then:

$$\rho_0 = \frac{M_c + M_s}{V_c + V_s} = \frac{\rho_c V_c + \rho_s V_s}{V_c + V_s} = \frac{\rho_c (V_c / V_s) + \rho_s}{(V_c / V_s) + 1} \Rightarrow \rho_0 + \rho_0 (V_c / V_s) = \rho_c (V_c / V_s) + \rho_s$$

$$\Rightarrow \frac{V_c}{V_s} = \frac{\rho_s - \rho_0}{\rho_0 - \rho_c} = \frac{\rho_s - 0.650\rho_w}{0.650\rho_w - \rho_c}.$$

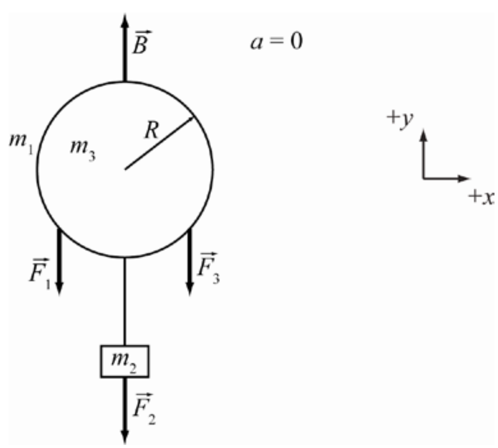
CALCULATE: $\dfrac{V_c}{V_s} = \dfrac{50.0 \text{ kg/m}^3 - 0.650(1000. \text{ kg/m}^3)}{0.650(1000. \text{ kg/m}^3) - 2200 \text{ kg/m}^3} = 0.3871$

ROUND: Two significant figures: $\dfrac{V_c}{V_s} = 0.39$.

DOUBLE-CHECK: For the block to float, there must be more Styrofoam than concrete in the block. As expected, the ratio is less than one.

13.78. THINK: Assume the balloon is thin, so the volume of the balloon is the same as the volume of helium. If it is not moving, the buoyant force of the balloon is balanced by the gravitational force of the balloon with mass, $m_1 = 1.0$ g, and the hanging mass, $m_2 = 4.0$ g. Consider the balloon to be a sphere. The density of helium is $\rho_H = 0.179 \text{ kg/m}^3$, and the density of air is $\rho_a = 1.3 \text{ kg/m}^3$.

SKETCH:



RESEARCH: The sum of the forces indicates that $F_B = F_1 + F_2 + F_3$, where $F_1 = m_1 g$ is the force of the balloon, $F_2 = m_2 g$ is the force of the hanging mass, and $F_3 = m_3 g$ is the force of the helium. The buoyant force is $F_B = \rho_a V g$. The volume of the balloon with the helium is $V = 4\pi R^3 / 3$. The mass of helium is $m_3 = \rho_H V$.

SIMPLIFY: $F_B = F_1 + F_2 + F_3 \Rightarrow \rho_a Vg = (m_1 + m_2 + m_3)g \Rightarrow \rho_a V = m_1 + m_2 + \rho_H V \Rightarrow V = \dfrac{m_1 + m_2}{\rho_a - \rho_H}$ and

$V = \dfrac{4}{3}\pi\left(\dfrac{d}{2}\right)^3 = \dfrac{4}{3}\pi\left(\dfrac{d^3}{8}\right) = \dfrac{\pi}{6}d^3.$ Equating the volume expressions gives $d = \left[\dfrac{6}{\pi}\left(\dfrac{m_1 + m_2}{\rho_a - \rho_H}\right)\right]^{\frac{1}{3}}.$

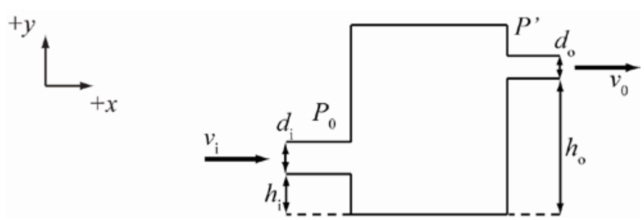
CALCULATE: $d = \left[\dfrac{6}{\pi}\left(\dfrac{0.0010 \text{ kg} + 0.0040 \text{ kg}}{1.3 \text{ kg/m}^3 - 0.179 \text{ kg/m}^3}\right)\right]^{\frac{1}{3}} = 0.2042 \text{ m}$

ROUND: $d = 0.20$ m

DOUBLE-CHECK: A diameter of 20. cm is reasonable for a balloon.

13.79. **THINK:** The continuity equation states that the volumes per second at the inlet and outlet pipe are the same. The Bernoulli equation can then be used to determine gauge pressure in the outlet pipe. Assume gauge pressure of the inlet pipe is $P_o = 101.3$ kPa. The diameters of the inlet and the outlet pipes are 2.00 cm and 5.00 cm, respectively. The heights of the pipes are 1.00 m and 6.00 m, respectively. The density of water is $\rho = 1000.$ kg/m^3. The volume through the pipes is $\Delta V = 0.300$ m^3 in $\Delta t = 60.0$ s.

SKETCH:



RESEARCH: The area of each pipe is $A_i = \pi(d_i/2)^2$ and $A_o = \pi(d_o/2)^2$. The continuity equation states $A_i v_i = A_o v_o = \Delta V/\Delta t$. The Bernoulli equation states $p_i + \rho g h_i + \frac{1}{2}\rho v_i^2 = p_o + \rho g h_o + \frac{1}{2}\rho v_o^2.$

SIMPLIFY:

(a) The velocity in the outlet pipe can be determined from $A_o v_o = \dfrac{\Delta V}{\Delta t} \Rightarrow v_o = \dfrac{\Delta V}{\Delta t}\left(\dfrac{4}{\pi d_o^2}\right).$

(b) First, determine the velocity in the inlet pipe: $A_i v_i = \dfrac{\Delta V}{\Delta t} \Rightarrow v_i = \dfrac{\Delta V}{\Delta t}\left(\dfrac{4}{\pi d_i^2}\right).$

Then, the gauge pressure at the outlet is $p_o = p_i + \rho g(h_i - h_o) + \frac{1}{2}\rho(v_i^2 - v_o^2)$.

CALCULATE:

(a) $v_o = \left(\dfrac{0.300 \text{ m}^3}{60.0 \text{ s}}\right)\left(\dfrac{4}{\pi(0.0500 \text{ m})^2}\right) = 2.546$ m/s

(b) $v_i = \left(\dfrac{0.300 \text{ m}^3}{60.0 \text{ s}}\right)\left(\dfrac{4}{\pi(0.0200 \text{ m})^2}\right) = 15.92$ m/s

$p_o = 101.3 \text{ kPa} + (1000. \text{ kg/m}^3)(9.81 \text{ m/s}^2)(1.00 \text{ m} - 6.00 \text{ m}) + \frac{1}{2}(1000. \text{ kg/m}^3)\left((15.92 \text{ m/s})^2 - (2.546 \text{ m/s})^2\right)$

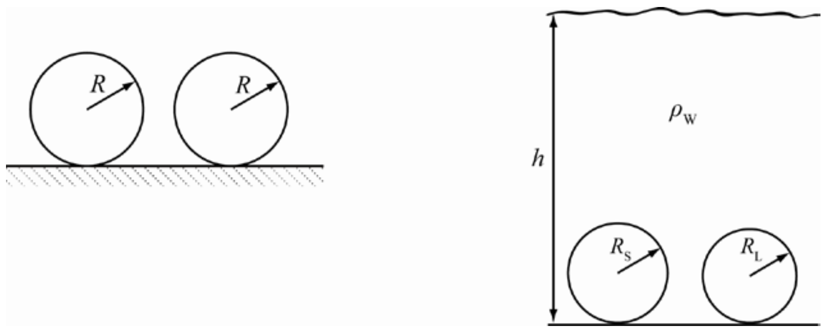
$= 175732.1$ Pa

ROUND: $p_o = 176$ kPa

DOUBLE-CHECK: In the limit that the inlet and outlet pipes have the same diameter, this whole problem only would involve pumping water up a height of 5 m, which would result in a gauge pressure drop of 50 kPa, reducing the outlet pressure to 50 kPa. Since our outlet pipe is larger than the inlet pipe, and since they have to carry the same water flow, the speed in the inlet pipe is greater. This, in turn, means that the pressure in the outlet pipe has to be greater than 50 kPa, which our solution fulfills.

13.80. **THINK:** As the two spheres of steel, $B_s = 160 \cdot 10^9$ Pa, and lead, B_L, go under water, they experience the same pressure at similar heights. However, the different bulk moduli cause them to compress differently. Even thought their initial volumes, V_0, are the same, when submerged a distance, h = 2000. m, under water with a density of $\rho_w = 1024 \text{ kg/m}^3$, they have different volumes, V_s and V_L, such that $V_s / V_L = 1.001206$.

SKETCH:



RESEARCH: The relative change in volume of a sphere is $-\frac{\Delta V}{V} B = \Delta P$. For each sphere, the change in pressure is $\Delta P = \rho_w gh$. The volume of a sphere after it is submerged is $V_0 + \Delta V$.

SIMPLIFY: $\Delta P = \rho_w gh$. The new volume of the steel ball is $V_s = V_0 + \Delta V_s = V_0 \left(1 - \frac{\Delta P}{B_s}\right)$. The new volume of the lead ball is $V_L = V_0 + \Delta V_L = V_0 \left(1 - \frac{\Delta P}{B_L}\right)$. Therefore,

$$\frac{V_s}{V_L} = \frac{1 - \frac{\Delta P}{B_s}}{1 - \frac{\Delta P}{B_L}} = \frac{\frac{B_s - \Delta P}{B_s}}{\frac{B_L - \Delta P}{B_L}} = \frac{B_L}{B_s}\left(\frac{B_s - \Delta P}{B_L - \Delta P}\right).$$

This implies that:

$$\frac{V_s}{V_L}B_s\left(B_L-\Delta P\right)=B_L\left(B_s-\Delta P\right)$$
$$\Rightarrow \left(\frac{V_s}{V_L}B_s-B_s+\Delta P\right)B_L=\frac{V_s}{V_L}\Delta PB_s$$
$$\Rightarrow B_L=\frac{\frac{V_s}{V_L}\Delta PB_s}{\left(\frac{V_s}{V_L}-1\right)B_s+\Delta P}.$$

CALCULATE: $\Delta P=\left(1024 \text{ kg/m}^3\right)\left(9.81 \text{ m/s}^2\right)\left(2000.\text{ m}\right)=2.009\cdot 10^7 \text{ Pa}$

$$B_L=\frac{\left(1.001206\right)\left(2.009\cdot 10^7 \text{ Pa}\right)\left(160\cdot 10^9 \text{ Pa}\right)}{\left(1.001206-1\right)\left(160\cdot 10^9 \text{ Pa}\right)+2.009\cdot 10^7 \text{ Pa}}=1.511\cdot 10^{10} \text{ Pa}$$

ROUND: $B_L=15 \text{ GPa}$

DOUBLE-CHECK: The published value for the bulk modulus of lead is 46 GPa. An alloy composed of lead and a more compressible metal would be expected to have a smaller bulk modulus.

Multi-Version Exercises

13.81. **THINK:** The diving bell is at a fixed depth, so there is no net force moving it up or down (the weight of the diving bell, buoyant force from the water, and the tension from the chain holding the diving bell cancel one another exactly). The net force on the viewing port will depend on the difference in pressure inside and outside of the diving bell. The pressure inside the diving bell is equal to atmospheric pressure, and the pressure outside the diving bell will depend on the depth of the viewing port.

SKETCH: The depth of the diving bell (h) and the diameter of the viewing port (d) are not shown to scale. The forces on the diving bell due to gravity, buoyancy, and the tension on the chain are equal (the diving bell is submerged at a fixed depth and it is not moving), so there is no net upward or downward force.

RESEARCH: The pressure at depth h is $P_{ext}=\rho gh+p_{atm}$, where ρ is the density of water and p_{atm} is the atmospheric pressure. The pressure inside the diving bell is $P_{int}=p_{atm}$. The total pressure on the viewing port is $P=P_{ext}-P_{int}$. Since pressure is defined as force per unit area, the total force on the viewing port, surface area of the viewing port, and pressure are related by the equation $F = PA$, where A is the surface area of the viewing port. The area of the round viewing port is given by $A = \pi r^2 = \pi \left(\frac{d}{2}\right)^2$.

SIMPLIFY: The total pressure at depth h is given by

$$\begin{aligned} P &= P_{\text{ext}} - P_{\text{int}} \\ &= \rho g h + p_{\text{atm}} - p_{\text{atm}} \\ &= \rho g h. \end{aligned}$$

The net force on the viewing port is then $F = PA = (\rho g h)\left(\frac{\pi}{4} d^2\right)$.

CALCULATE: The depth h = 129.1 m and diameter d = 22.89 cm = 0.2289 m are given in the question. The gravitational acceleration near the surface of the earth is g = 9.81 m/s^2, and the density of fresh water is ρ = 1000 kg/m^3. The net force on the viewing port is:

$$\begin{aligned} F &= (\rho g h)\left(\frac{\pi}{4} d^2\right) \\ &= 1000 \text{ kg/m}^3 \cdot 9.81 \text{ m/s}^2 \cdot 129.1 \text{ m} \cdot \frac{\pi}{4}(0.2289 \text{ m})^2 = 52116.67693 \text{ N} \end{aligned}$$

ROUND: All of the numbers in this calculation had four significant figures, so the final answer will also have four figures. The viewing port experiences a force of $5.212 \cdot 10^4$ N = 52.12 kN towards the interior of the diving bell.

DOUBLE-CHECK: As a rule of thumb, divers expect the pressure to increase by 1 atmosphere = $1.103 \cdot 10^5$ Pa for every 10 m of depth, so the expected pressure is about $1.424 \cdot 10^6$ Pa. The window experiences a force of magnitude $5.212 \cdot 10^4$ N, so the pressure is $P = F/A = 1.267 \cdot 10^6$ Pa. This is the correct order of magnitude, so this rough estimate confirms that the calculation is of the correct order of magnitude.

13.82. $F = \frac{1}{4} \rho g h \pi d^2$

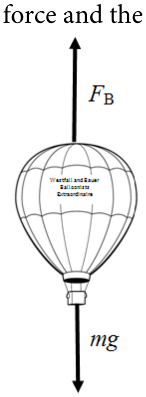
$$h = \frac{4F}{\rho g \pi d^2} = \frac{4\left(6.251 \cdot 10^4 \text{ N}\right)}{\left(1000 \text{ kg/m}^3\right)\left(9.81 \text{ m/s}^2\right)\pi\left(0.2311 \text{ m}\right)^2} = 151.9 \text{ m}$$

13.83. $F = \frac{1}{4} \rho g h \pi d^2$

$$d = \sqrt{\frac{4F}{\rho g \pi h}} = \sqrt{\frac{4\left(7.322 \cdot 10^4 \text{ N}\right)}{\left(1000 \text{ kg/m}^3\right)\left(9.81 \text{ m/s}^2\right)\pi\left(174.9 \text{ m}\right)}} = 0.2331 \text{ m} = 23.31 \text{ cm}$$

13.84. **THINK:** The balloon experience an upward force from buoyancy and a downward force from gravity. When the balloon is lifting the maximum weight, the upward force and downward force are equal.

SKETCH: The sketch shows the buoyant force and the gravitational force:



RESEARCH: The weight of the air the balloon displaces is $F_B = \rho_{outside} Vg$. The weight of the hot air filling the balloon is $W_{hot\ air} = \rho_{inside} Vg$. The weight that can be lifted, plus the weight of the hot air filling the balloon, is equal to the weight of the air the balloon displaces: $F_B = W_{hot\ air} + W$.

SIMPLIFY: The goal is to find the weight W of the load that the balloon can lift. Using algebra, the weight of the load is $W = F_B - W_{hot\ air}$. Substituting in for $F_B = \rho_{outside} Vg$ and $W_{hot\ air} = \rho_{inside} Vg$ gives $W = \rho_{outside} Vg - \rho_{inside} Vg = Vg(\rho_{outside} - \rho_{inside})$.

CALCULATE: The acceleration due to gravity is g = 9.81 m/s. The volume of the balloon is V = 2979 m^3. The density of the air outside the balloon is $\rho_{outside}$ = 1.205 kg/m^3 and the density of the air inside the balloon is ρ_{inside} = 0.9441 kg/m^3. The total weight that the balloon can lift is then

$$\begin{aligned} W &= Vg(\rho_{outside} - \rho_{inside}) \\ &= 2979 \text{ m}^3 \cdot 9.81 \text{ m/s}^2 \left(1.205 \text{ kg/m}^3 - 0.9441 \text{ kg/m}^3\right) \\ &= 7624.538991 \text{ N} \end{aligned}$$

ROUND: The volume of the balloon and the density of the air are all given to four significant figures. However, the difference in the density outside the balloon and the density inside the balloon $\rho_{outside} - \rho_{inside} = 0.2609$, has only three significant figures, so the final answer should also have only three figures. The balloon can lift a maximum of 7.62·10^3 N = 7.62 kN.

DOUBLE-CHECK: To check this, convert the weight from Newton to pounds. The balloon can lift $7.62 \cdot 10^3 \text{ N} \cdot \frac{1 \text{ lb}}{4.448 \text{ N}} = 1710 \text{ lb}$. For a hot air balloon with a wicker basket, nylon balloon, propane or other compressed gas heating mechanism, and a few human passengers, this seems like a realistic weight. (Keep in mind that, if the maximum weight is too low, the balloon will never get off the ground, and if the maximum weight is too high, it will take too long for the balloon to return to earth.)

13.85. $W = Vg(\rho_{outside} - \rho_{inside})$

$$V = \frac{W}{g(\rho_{outside} - \rho_{inside})} = \frac{5626 \text{ N}}{(9.81 \text{ m/s}^2)(1.205 \text{ kg/m}^3 - 0.9449 \text{ kg/m}^3)} = 2205 \text{ m}^3$$

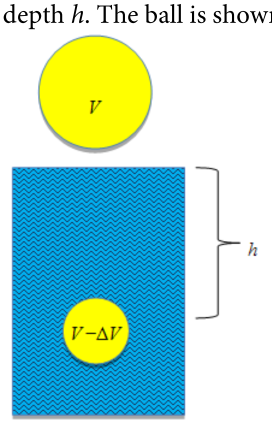
13.86. $W = Vg(\rho_{outside} - \rho_{inside})$

$$\frac{W}{Vg} = \rho_{outside} - \rho_{inside}$$

$$\rho_{inside} = \rho_{outside} - \frac{W}{Vg} = 1.205 \text{ kg/m}^3 - \frac{6194 \text{ N}}{(2435 \text{ m}^3)(9.81 \text{ m/s}^2)} = 0.946 \text{ kg/m}^3$$

13.87. **THINK:** The bulk modulus can be used to compute the fractional change in volume from the pressure. The pressure can be computed from the depth of the water, which is given in the question.

SKETCH: The ball is submerged to depth h. The ball is shown before and after it has been submerged.



RESEARCH: The pressure at depth h is given by $p=\rho gh$, where ρ is the density of the water in which the ball is submerged. The pressure is defined to be the force per unit area $p=\frac{F}{A}$, and the equation for volume compression is $\frac{F}{A}=B\frac{\Delta V}{V}$.

SIMPLIFY: The goal is to find the fractional change in the volume of the ball, $\frac{\Delta V}{V}$. Replace force per unit area in the equation for volume compression, substitute in for the pressure at depth h, and finally use algebra to solve for the fractional change in volume.

$$B\frac{\Delta V}{V}=\frac{F}{A}\Rightarrow$$
$$B\frac{\Delta V}{V}=p\Rightarrow$$
$$B\frac{\Delta V}{V}=\rho gh\Rightarrow$$
$$\frac{\Delta V}{V}=\frac{\rho gh}{B}$$

CALCULATE: The density of water is 1000 kg/m^3 and the acceleration due to gravity near the surface of the earth is 9.81 m/s^2. The question states that the ball is submerged to a depth of 55.93 m and the bulk modulus is $6.309\cdot 10^7$ N/m^2. The fractional change in volume is then

$$\frac{\Delta V}{V}=\frac{\rho gh}{B}$$
$$=\frac{1000\text{ kg/m}^2\cdot 9.81\text{ m/s}^2\cdot 55.93\text{ m}}{6.309\cdot 10^7\text{ N/m}^2}$$
$$=8.696676177\cdot 10^{-3}$$

ROUND: The height and bulk modulus are both given to four figures, so the final answer should also have four significant figures. The fractional change in volume is $8.697\cdot 10^{-3}$ or 0.8697%.

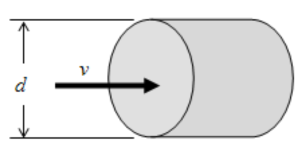
DOUBLE-CHECK: The pressure is increasing, so it is natural to expect the volume to decrease. From experience, when a ball is submerged at the bottom of a pool or pond, the decrease in radius is minimal. It is reasonable that, even when the ball is submerged to a depth of over 50 meters, the fractional change in volume is less than 1 percent.

13.88. $$\rho gh=B\frac{\Delta V}{V}$$
$$h=\frac{B}{\rho g}\frac{\Delta V}{V}=\frac{8.141\cdot 10^7\text{ N/m}^2}{\left(1000\text{ kg/m}^3\right)\left(9.81\text{ m/s}^2\right)}\left(6.925\cdot 10^{-3}\right)=57.47\text{ m}$$

13.89. $$\rho gh=B\frac{\Delta V}{V}$$
$$B=\frac{\rho gh}{\Delta V/V}=\frac{\left(1000\text{ kg/m}^3\right)\left(9.81\text{ m/s}^2\right)\left(59.01\text{ m}\right)}{2.937\cdot 10^{-2}}=1.971\cdot 10^7\text{ N/m}^2$$

13.90. **THINK:** The Betz limit calculated in Example 13.7 applies to seawater. Use the Betz limit, speed of the current, and geometry of the turbine to determine the maximum power that can be extracted.

SKETCH: The diameter of the turbine and velocity of the seawater are shown. The seawater is flowing into the turbine from left to right.



RESEARCH: The Betz limit states that a fraction of 16/27 of the total power can be extracted from the fluid. The total power of a fluid flowing at a velocity v with a density ρ flowing through an area A is $P_{\text{in}} = \frac{1}{2}Av^3\rho$. In this case, the area is the surface area of one end of the turbine and can be expressed in terms of the turbine's diameter as $A = \pi(d/2)^2$. The power extracted by the turbine is 16/27 of the total, so the maximum power extracted is $P = \frac{16}{27}P_{\text{in}}$.

SIMPLIFY: Use the Betz limit to express the maximum power $P = \frac{16}{27}\left(\frac{1}{2}Av^3\rho\right)$. Express the area in terms of the diameter of the turbine ($A = \pi(d/2)^2$) and simplify to get $P = \frac{2}{27}\pi d^2 v^3 \rho$.

CALCULATE: The density of seawater is given in the problem as ρ = 1024 kg/m^3, and it flows through the turbine with a speed of 1.35 m/s. The turbine's rotors have a diameter of 24.5 m. The maximum power that can be extracted is

$$\begin{aligned} P &= \tfrac{2}{27}\pi d^2 v^3 \rho \\ &= \frac{2\pi}{27}(24.5 \text{ m})^2 (1.35 \text{ m/s})^3 1024 \text{ kg/m}^3 \\ &= 3.519245266 \cdot 10^5 \text{ W} \\ &= 351.9245266 \text{ kW}. \end{aligned}$$

ROUND: Although the other values have four significant figures, the velocity of the seawater is given to only three figures, so the final answer should have only three significant figures. The maximum power that the turbine can extract under these conditions is 352 kW.

DOUBLE-CHECK: Flowing water, such as a current, river, or waterfall, has a tremendous amount of power. For seawater flowing fairly quickly (a speed greater than 1 m/s), the expected power output is on the order of 10^3 times the square of the diameter. For a turbine with diameter d = 24.5 m, the order of magnitude of the square of the diameter is about 10^2, so it is reasonable to expect that the answer should have an order of magnitude of 10^5 Watts. This agrees with the calculated value (352 kW = $3.52 \cdot 10^5$ W).

13.91. $$P = \frac{16}{27}\left(\frac{1}{2}Av^3\rho\right) = \frac{16}{27}\left[\frac{1}{2}\left(\pi\left(\frac{d}{2}\right)^2\right)v^3\rho\right] = \frac{2}{27}\pi d^2 v^3 \rho$$

$$d = \sqrt{\frac{27P}{2\pi v^3 \rho}} = \sqrt{\frac{27(571{,}800 \text{ W})}{2\pi(1.57 \text{ m/s})^3(1024 \text{ kg/m}^3)}} = 24.9 \text{ m}$$

13.92. $$P = \frac{2}{27}\pi d^2 v^3 \rho$$

$$v = \sqrt[3]{\frac{27P}{2\pi d^2 \rho}} = \sqrt[3]{\frac{27(918{,}800 \text{ W})}{2\pi(25.5 \text{ m})^2(1024 \text{ kg/m}^3)}} = 1.81 \text{ m/s}$$

Chapter 14: Oscillations

Concept Checks

14.1. e **14.2.** c **14.3.** a **14.4.** a **14.5.** b **14.6.** c **14.7.** c **14.8.** c

Multiple-Choice Questions

14.1. c **14.2.** c **14.3.** b **14.4.** d **14.5.** b **14.6.** c **14.7.** a **14.8.** c **14.9.** a **14.10.** b **14.11.** b **14.12.** c **14.13.** c **14.14.** e **14.15.** d

Conceptual Questions

14.16. The amplitude of the oscillations decreases to 5 % in 2 seconds. The equation of motion for a damped particle is $x(t) = Ae^{-\omega_\gamma t}\sin(\omega' t + \theta_0)$. The decrease in amplitude implies $A(t) = Ae^{-\omega_\gamma t}$, with period $T = \dfrac{2k}{\omega'}$. Using this decrease in amplitude equation at time $t = 0$, $A(0) = A = 1$. At time $t = 2s$ the amplitude is 5% or less of what it was at $t = 0$, $A(2) = (1)e^{-\omega_\gamma 2} = e^{-2\omega_\gamma} \leq 0.05$, so $-2\omega_\gamma \approx \ln 0.05$, and therefore $\omega_\gamma = \dfrac{\ln 0.05}{-2} = 1.5$. Using the fact that $\omega_\gamma = \dfrac{b}{2m}$, b can be computed using the formula $b = 2m\omega_\gamma = 2(2200)(1.5) = 6600$ kg/s. To complete three oscillations in 2 seconds the period has to satisfy the equation $T = \dfrac{2}{3} = \dfrac{2\pi}{\omega'} \Rightarrow \omega' = 3\pi$. It follows that because $\omega'^2 = \omega_0^2 - \omega_\gamma^2 = 9\pi^2 - 2.25 = \dfrac{k}{m} \Rightarrow \quad k = \left(9\pi^2 - 2.25\right)m = 91m = (91)(2200) = 2\cdot 10^5$ N/m. The vehicle has an effective spring constant of $2\cdot 10^5$ N/m, and a damping constant of $6\cdot 10^3$ kg/s.

14.17. The frequency of a pendulum is $\omega = \sqrt{g/l}$. That is, it is dependent on the acceleration of gravity. If the acceleration increases, so will the angular frequency. The digital watch keeps its time using the vibrations of a crystal, which do not change. During the acceleration of the shuttle, the g's felt by the pendulum increase and in turn the frequency of the clock increases. The clock will then report a time that is ahead of the digital watch.

14.18. The cylinder only has the force of the spring acting on it. Because it is rotating, the force on the spring must equal the centripetal force at equilibrium, $m\omega_0^2 r = kr$, where ω_0^2 is the angular frequency of the turntable. This means the equilibrium position of the cylinder is r = 0, unless $\omega_0^2 = k/m$. The period of the cylinder is the same as it is when the turntable is not spinning, $T = 2\pi\sqrt{m/k}$, and is independent of ω_0. If the condition $\omega_0^2 = k/m$ is met, then the equilibrium position has no bounds. If the cylinder is placed at r, there will be no oscillations.

14.19. Critical damping will close the door the fastest without having the door overshoot and slam into the frame (see Fig. 14.16). Overdamping will cause the door to close and open very slowly. Underdamping will cause the door to close quickly, but the door will then overshoot the closed position, hitting the frame. If there is no frame, and the door is underdamped, the door will swing back and forth about the closed position. As a result, critical damping, or damping slightly below critical is the best compromise, providing a relatively quick return with no overshoot.

14.20. The period a stretched string is $T = 2\pi/\omega = 2\pi\sqrt{m/k}$. Since the period does not depend on the amplitude, any change in the amplitude does not affect the period.

14.21. The elasticity of a spring may change after repeated use or due to aging. The resulting change in the spring constant changes the oscillation period. A pendulum does not have the same problem, although more care must be taken to keep the oscillation amplitude the same on each measurement (since the period does depend on the amplitude, particularly as the amplitude gets larger).

14.22. The oscillations of a mass and spring do not depend on gravity. The period, T, of the oscillations is related to the mass, m, and spring constant, k, via $T = 2\pi\sqrt{m/k}$. The unknown spring constant can be determined by measuring the oscillation period with the standard mass attached to the spring:

$$k = \frac{4\pi^2 m_{\text{std}}}{T_{\text{std}}^{\ 2}}.$$

An unknown mass can be determined by attaching the mass to the spring and measuring the period. The relation between the mass and period is:

$$m = k\frac{T^2}{4\pi^2} \text{ or } m = m_{\text{std}}\frac{T^2}{T_{\text{std}}^{\ 2}},$$

where all the variables on the right-hand side are known or measurable. An oscillator used to measure masses in this way is known as an inertial balance; it measures inertial mass directly, rather than (passive) gravitational mass. An apparatus of this sort is used to measure the body masses of astronauts on a space shuttle or the International Space Station. Loss of bone and muscle mass is a serious concern on prolonged space flights.

14.23. The frequency of a pendulum is given by $\omega = \sqrt{g/l}$. Since the frequency is not dependent on the mass, the frequencies of pendulum A and B are equal.

14.24. The bell can be considered a harmonic oscillator driven simultaneously at all possible frequencies with equal amplitude. The response at each frequency is proportional to the height of the bell's amplitude resonance curve at that frequency. A well-made bell has very low damping; the peak of the resonance curve is very high and narrow. The response of the bell is therefore very strongly dominated by the response at the resonant frequency, i.e. its natural frequency – that is, the frequency which is heard.

Exercises

14.25. The angular frequency is given by $\omega = \sqrt{k/m}$ or $k = m\omega^2$. The spring constant is then $k = m\omega^2 = (5.00 \text{ kg})(5.00 \text{ s}^{-1})^2 = 125 \text{ N/m}$.

14.26. The angular frequency is given by: $\omega = \sqrt{k/m}$.

(a) $\omega = \sqrt{k_1/m} = \sqrt{\dfrac{1000. \text{ N/m}}{0.200 \text{ kg}}} = 70.7 \text{ s}^{-1}$

(b) $\omega = \sqrt{k_2/m} = \sqrt{\dfrac{1500. \text{ N/m}}{0.200 \text{ kg}}} = 86.6 \text{ s}^{-1}$

(c) The spring constant is not straightforward for this problem. There are two springs in series, which creates an effective spring constant obtained as follows. If the force in the spring is F and the two springs lengthen by x_1 and x_2, respectively, then

$$F = k_1x_1 = k_2x_2 = k_{\text{eff}}(x_1 + x_2)$$

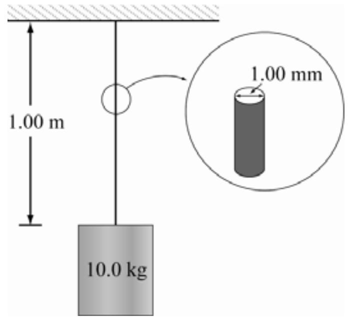
from which it follows that

$$k_{\text{eff}} = \frac{F}{x_1 + x_2} = \frac{F}{F/k_1 + F/k_2} = \frac{1}{1/k_1 + 1/k_2} = \frac{k_1k_2}{k_1 + k_2}$$

The angular frequency is: $\omega = \sqrt{\dfrac{k_{\text{eff}}}{m}} = \sqrt{\dfrac{k_1k_2}{(k_1 + k_2)m}} = \sqrt{\dfrac{(1000. \text{ N/m})(1500. \text{ N/m})}{(1000. \text{ N/m} + 1500. \text{ N/m})(0.200 \text{ kg})}} = 54.8 \text{ s}^{-1}.$

14.27. **THINK:** The spring constant, k, can be determined from the definition of Young's modulus. In turn, the frequency can be determined.

SKETCH:



RESEARCH: The definition of Young's modulus is: $\frac{F}{A} = -Y\frac{x}{l}$. Remember that A here is the cross-sectional area of the wire, not amplitude.
The spring force is given by $F = -kx$. The angular frequency is defined as $\omega = \sqrt{k/m}$, and $\omega = 2\pi f$.
SIMPLIFY: Equating the force from the Young's modulus to the force of a spring gives:

$$F = -Y\frac{A}{l}x = -kx.$$

The spring constant is: $k = Y\frac{A}{l} = Y\frac{\pi r^2}{l} = Y\frac{\pi d^2}{4l}$.

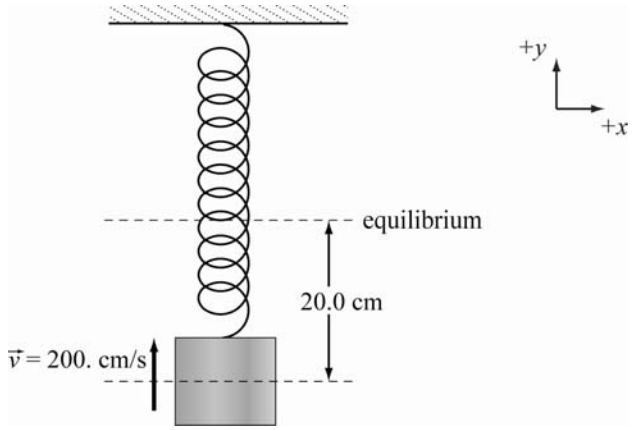
This means the frequency is: $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{Y\frac{\pi d^2}{4lm}}$.

CALCULATE: $f = \frac{1}{2\pi}\sqrt{\left(2.0\cdot 10^{11}\text{ N/m}^2\right)\frac{\pi\left(0.00100\text{ m}\right)^2}{4\left(1.00\text{ m}\right)\left(10.0\text{ kg}\right)}} = 19.947\text{ Hz}$

ROUND: Young's modulus for steel is given to two significant figures, so the frequency of the mass is reported as 20. Hz.
DOUBLE-CHECK: Dimensional analysis shows that the calculation yields the correct units of frequency: $\left(\left(\frac{\text{N}}{\text{m}^2}\right)\left(\frac{(\text{m})^2}{(\text{m})(\text{kg})}\right)\right)^{1/2} = \left(\frac{\text{kg m m}^2}{\text{s}^2\text{ m}^2\text{ m kg}}\right)^{1/2} = \text{s}^{-1} = \text{Hz}.$

14.28. **THINK:** The time taken for the displacement to reach $x_1 = 30.0\text{ cm}$ can be determined from the equation of oscillation. The time can then be used to determine the speed of the block. The mass is m = 0.100 kg and the spring constant is k = 5.00 N/m = 5.00 kg/s^2. At t = 0 s, the block is at $x_0 = -20.0\text{ cm}$ and moving with a speed of v = 200. cm/s.

SKETCH:



RESEARCH: The angular frequency is given by $\omega = \sqrt{k/m}$. The equations of the displacement and the speed of the block are $x(t) = A\sin(\omega t + \theta)$ and $v(t) = A\omega\cos(\omega t + \theta)$, respectively.

SIMPLIFY: The amplitude and phase shift are determined using the initial conditions:

$$x_0 = A\sin(\omega(0)+\theta) = A\sin\theta \text{ and } v_0 = A\omega\cos(\omega(0)+\theta) = A\omega\cos\theta.$$

The phase can be determined by dividing the equations:

$$\frac{A\sin\theta}{A\omega\cos\theta} = \frac{\tan\theta}{\omega} = \frac{x_0}{v_0} \Rightarrow \theta = \tan^{-1}\left(\omega\frac{x_0}{v_0}\right) = \tan^{-1}\left(\sqrt{\frac{k}{m}}\frac{x_0}{v_0}\right).$$

The amplitude is determined using the phase shift: $A = \dfrac{x_0}{\sin\theta}$. Next, determine the time at which the displacement is x_1:

$$x_1 = A\sin(\omega t + \theta) \Rightarrow \frac{x_1}{A} = \sin(\omega t+\theta) \Rightarrow \omega t + \theta = \sin^{-1}\left(\frac{x_1}{A}\right) \Rightarrow t = \frac{\sin^{-1}(x_1/A) - \theta}{\omega}.$$

The speed at time t is given by:

$$v = A\omega\cos(\omega t+\theta) = A\omega\cos\left[\omega\left(\frac{\sin^{-1}(x_1/A)-\theta}{\omega}\right)+\theta\right] = A\omega\cos\left[\sin^{-1}(x_1/A)\right].$$

CALCULATE: The phase is: $\theta = \tan^{-1}\left[\sqrt{\dfrac{5.00 \text{ kg/s}^2}{0.100 \text{ kg}}}\left(-\dfrac{20.0 \text{ cm}}{200. \text{ cm/s}}\right)\right] = -0.6155 \text{ rad}.$

The displacement is: $A = \dfrac{x_0}{\sin\theta} = -\dfrac{20.0 \text{ cm}}{\sin(-0.6155 \text{ rad})} = 34.640 \text{ cm}.$

The angular frequency is: $\omega = \sqrt{\dfrac{5.00 \text{ kg/s}^2}{0.100 \text{ kg}}} = 7.071 \text{ s}^{-1}.$

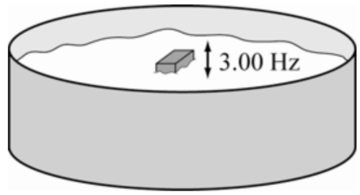
The speed is: $v = (34.640 \text{ cm})(7.071 \text{ s}^{-1})\cos\left[\sin^{-1}\left(\dfrac{30.0 \text{ cm}}{34.640 \text{ cm}}\right)\right] = 122.46 \text{ cm/s}.$

ROUND: To three significant figures, the result is: v = 122 cm/s.

DOUBLE-CHECK: The speed must be smaller further from the equilibrium position of the spring because more energy is stored as potential energy in the spring. Therefore, our result is reasonable.

14.29. **THINK:** The spring constant can be determined from the mass and the frequency. The mass is $m_1 = 55.0$ g and bobs with a frequency of $f_1 = 3.00$ Hz. The mass is then changed to $m_2 = 250.$ g.

SKETCH:



RESEARCH: The angular frequency is given by $\omega=\sqrt{k/m}$ and $f=\omega/2\pi$.

SIMPLIFY: The spring constant is $k=m_1\omega^2=m_1\left(2\pi f_1\right)^2=4\pi^2 m_1 f_1^2$. Assume this spring constant does not change with the new mass. The new frequency is:

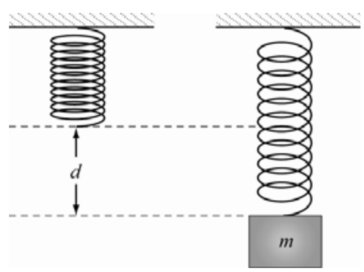
$$f_2=\frac{\omega}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{k}{m_2}}=\frac{1}{2\pi}\sqrt{\frac{4\pi^2 m_1 f_1^2}{m_2}}=\frac{2\pi f_1}{2\pi}\sqrt{\frac{m_1}{m_2}}=f_1\sqrt{\frac{m_1}{m_2}}.$$

CALCULATE: $k=4\pi^2\left(0.0550\text{ kg}\right)\left(3.00\text{ Hz}\right)^2=19.54\text{ kg/s}^2=19.54\text{ N/m}.$ The frequency is:

$$f=3\text{ Hz}\sqrt{\frac{55.0\text{ g}}{250.\text{ g}}}=1.407\text{ Hz}.$$

ROUND: The results should be rounded to three significant figures.
(a) The spring constant is $k=19.5$ N/m.
(b) The frequency is 1.41 Hz.
DOUBLE-CHECK: Since the frequency is inversely proportional to the mass, a larger mass should oscillate with a smaller frequency. The results are reasonable.

14.30. **THINK:** The oscillation period, *T*, can be determined from the given values. The balance of forces on the spring can then be used to determine the stretch distance, *d*.
SKETCH:



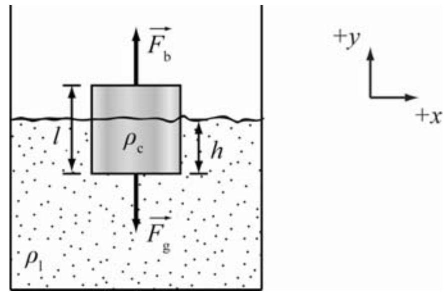
RESEARCH: The force of gravity is balanced by the spring force and this balance is given by $mg=kd$. The oscillation period, *T*, is related to *k* and *m* via $T=2\pi\sqrt{m/k}$.

SIMPLIFY: $T=2\pi\sqrt{m/k}\;\Rightarrow\; m/k=\left(\frac{T}{2\pi}\right)^2$. Rearranging the balance of forces equation for *d* yields $d=\frac{mg}{k}$. The expression for m/k can be inserted into the above equation to get $d=\left(\frac{T}{2\pi}\right)^2 g$.

CALCULATE: From the given values, the period is: $T=\frac{30\text{ s}}{50}=0.60\text{ s}.$ $d=\left(\frac{0.60\text{ s}}{2\pi}\right)^2\left(9.81\text{ m/s}^2\right)=0.08945\text{ m}=8.945\text{ cm}$

ROUND: Given value have three significant figures, so the result should be rounded to $d=8.95$ cm.
DOUBLE-CHECK: This is a reasonable distance for a spring to stretch.

14.31. **THINK:** When the cube is put into the liquid, it will feel a buoyant force acting upward. The cube will sink until the buoyant force equals the weight of the cube. If the cube is then given a small downward push, the buoyant force will act as a restoring force, proportional to the distance it is moved downward. Thus we can make an analogy with Hooke's Law. When the cube is released, it will undergo simple harmonic motion, in analogy with a mass on a spring.
SKETCH:



RESEARCH: To determine the effective spring constant, we apply an analogy to Hooke's Law. The weight of the cube will force it to sink at distance h until the buoyant force is equal to the weight of the cube, just as when we attach an object to a spring, it will stretch until the upward force of the spring equals the weight of the object. The analogy to Hooke's Law gives us

$$|F| = |kx| \quad \Rightarrow \quad m_c g = kh \quad \Rightarrow \quad \rho_c l^3 g = kh.$$

The frequency of an object in simple harmonic motion is $\omega = \sqrt{k/m}$. The mass of the cube is $m_c = \rho_c l^3$.

SIMPLIFY: We can write effective spring constant as

$$k = \frac{\rho_c l^3 g}{h}.$$

The frequency of the oscillatory motion of the cube is:

$$f = \frac{1}{2\pi}\omega = \frac{1}{2\pi}\sqrt{\frac{k}{m_c}} = \frac{1}{2\pi}\sqrt{\frac{\rho_c l^3 g/h}{\rho_c l^3}} = \frac{1}{2\pi}\sqrt{\frac{g}{h}}.$$

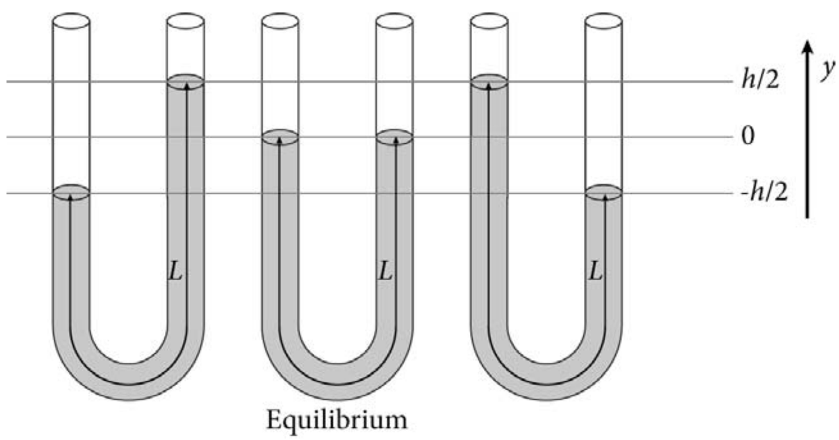
CALCULATE: There are no calculations necessary.

ROUND: Rounding is not necessary.

DOUBLE-CHECK: The units of the result are s^{-1}, which is correct for frequency. The frequency of oscillation does not depend on the density of the liquid as long as the density of the cube is less than the density of the liquid.

14.32. **THINK:** The water levels are different, so the system will push the liquid down that is at a higher level. This action is similar to a ball in a half pipe. The water will continue to oscillate up and down due to the conservation of energy.

SKETCH:



RESEARCH: The mass on which any force will act is the mass of the fluid in the system given by $M = \rho V = \rho AL$ where A is the cross sectional area of the tube and ρ is the density of the fluid. The restoring force is given by the force of gravity on the part of the fluid above the lower fluid level. We define the position of the fluid in terms of the position of the top of the fluid in the right half of the tube, y. We can write the restoring force as $F = -(2\rho Ay)g = -(2\rho Ag)y$, where $2\rho Ay$ is the mass of the fluid between the upper and lower level of the fluid. Now we can make the analogy with the spring force, which has the form $F = -kx$. The period of a mass m on a spring with spring constant k is $T = 2\pi/\omega = 2\pi\sqrt{m/k}$.

SIMPLIFY: The effective spring constant for this case is $k = 2\rho Ag$ The period is then:

$$T = 2\pi\sqrt{\frac{M}{k}} = 2\pi\sqrt{\frac{\rho AL}{2\rho Ag}} = 2\pi\sqrt{\frac{L}{2g}}.$$

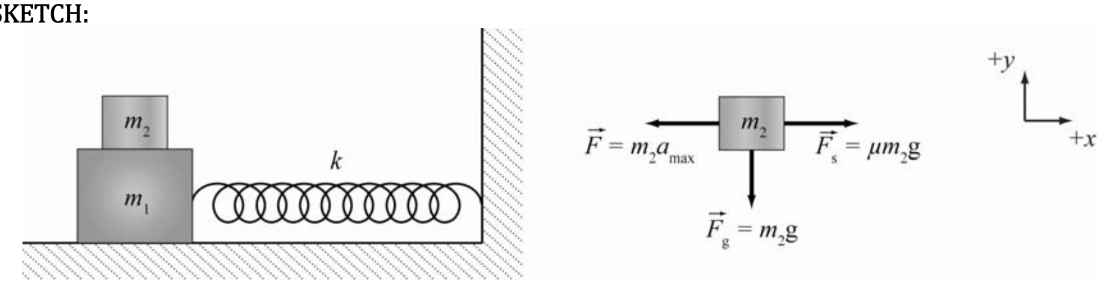
CALCULATE: It is not necessary to do any calculations.

ROUND: Rounding is not necessary.

DOUBLE-CHECK: Notice that the period is independent of the mass density of the liquid and the cross-sectional area of the tube (as long as the tube has a uniform cross-section). Also, if the tube is long compared to the curve at the bottom of the tube, then the length, *L*, can be considered to be simply the length of both arms of the tube and we obtain a result that is similar to the period of a pendulum.

14.33. **THINK:** Block m_2 will slide off of block m_1 if the maximum acceleration of oscillation produces a force greater than the force of static friction. The spring constant is $k = 10.0$ N/m and the masses are $m_1 = m_2 = 20.0$ g. The coefficient of static friction is $\mu = 0.600$.

SKETCH:



RESEARCH: The maximum acceleration of the oscillation is $a = \omega^2 A$. The angular frequency is $\omega = \sqrt{k/m}$.

SIMPLIFY: The maximum force on the second mass is given by:

$$F_{max} = m_2 a_{max} = m_2\omega^2 A = \frac{m_2 kA}{m_1 + m_2}.$$

The maximum force is equal to the force of static friction:

$$F_s = F_{max} \Rightarrow \mu m_2 g = \frac{m_2 kA}{m_1 + m_2} \Rightarrow A = \frac{\mu g(m_1 + m_2)}{k}.$$

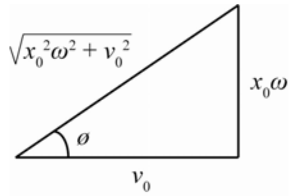
CALCULATE: $A = \dfrac{(0.600)(9.81\text{ m/s}^2)(0.0200\text{ kg} + 0.0200\text{ kg})}{(10.00\text{ N/m})} = 0.023544\text{ m}$

ROUND: The given values have three significant figures, so the maximum amplitude the system can have without having the second mass slip is $A = 0.0235$ m = 2.35 cm.

DOUBLE-CHECK: Dimensional analysis of the calculation shows that the answer is in the correct units of length: $\dfrac{[\text{m/s}^2][\text{kg}]}{[\text{N/m}]} \Rightarrow \dfrac{[\text{m}][\text{kg}][\text{m}][\text{s}^2]}{[\text{s}^2][\text{kg}][\text{m}]} \Rightarrow \text{m}.$ This result is reasonable.

14.34. **THINK:** This problem explores what occurs to two identical harmonic oscillators that have slightly different initial conditions. For well-behaved systems, the results should be similar for both situations. Systems whose solutions are vastly different for small changes in initial conditions are known as chaotic systems.

SKETCH:



RESEARCH:

(a) The position of the oscillation is given by $x = A\sin(\omega t+\phi)$, where the amplitude, *A*, and the phase, ϕ, are determined by the initial conditions. The phase is given by $\phi = \tan^{-1}(x_0\omega / v_0)$, and the amplitude is given by $A = x_0 / \sin\phi$.

SIMPLIFY:

(a) From the sketch, it can be seen that the amplitude is now given by:

$$A = \frac{x_0}{x_0\omega}\sqrt{x_0^2\omega^2+v_0^2} = \frac{\sqrt{x_0^2\omega^2+v_0^2}}{\omega}.$$

The position of an oscillator is then: $x(t) = A\sin(\omega t+\phi) = \dfrac{\sqrt{x_0^2\omega^2+v_0^2}}{\omega}\sin(\omega t+\phi)$. Using the double angle formula, $\sin(A+B) = \sin A\cos B + \sin B\cos A$, gives:

$$x(t) = \frac{\sqrt{x_0^2\omega^2+v_0^2}}{\omega}\left(\sin\omega t\cos\phi + \sin\phi\cos\omega t\right).$$

Noting the relation of the triangle:

$$x_1(t) = \frac{\sqrt{x_0^2\omega^2+v_0^2}}{\omega}\left[\sin(\omega t)\left(\frac{v_0}{\sqrt{x_0^2\omega^2+v_0^2}}\right) + \frac{x_0\omega}{\sqrt{x_0^2\omega^2+v_0^2}}\cos(\omega t)\right] = \frac{v_0}{\omega}\sin(\omega t) + x_0\cos(\omega t).$$

The positions of each oscillator are given by:

$$x_1(t) = \left(\frac{v_0}{\omega}\right)\sin(\omega t) + x_0\cos(\omega t), \text{ and } x_2(t) = \left(\frac{v_0+\delta v}{\omega}\right)\sin(\omega t) + (x_0+\delta x)\cos(\omega t).$$

Thus,

$$x_2(t) - x_1(t) = \left(\frac{v_0+\delta v}{\omega}\right)\sin(\omega t) + (x_0+\delta x)\cos(\omega t) - \frac{v_0}{\omega}\sin(\omega t) - x_0\cos(\omega t) = \frac{\delta v}{\omega}\sin(\omega t) + \delta x\cos(\omega t).$$

(b) The difference, $x_2(t) - x_1(t)$, can be written in the form $A\sin(\omega t+\phi)$. In terms of x_0, δx_0, v_0 and δv, the difference is:

$$x_2(t) - x_1(t) = \frac{\sqrt{\delta x^2\omega^2+\delta v^2}}{\omega}\sin\left[\omega t + \tan^{-1}\left(\frac{\delta x}{\delta v}\omega\right)\right].$$

Since the sine function has a range of $-1 \le \sin x \le 1$ for all *x*, the difference is bounded by:

$$\left|x_2(t) - x_1(t)\right| \le \frac{\sqrt{\delta x^2\omega^2+\delta v^2}}{\omega},$$

where $\omega = \sqrt{k/m}$.

CALCULATE: No calculations are necessary.

ROUND: There are no values to round.

DOUBLE-CHECK: This shows that the harmonic oscillator is not chaotic. A system obeying a linear equation of motion cannot be chaotic.

14.35. For a simple pendulum, the period is given by: $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g/l}} = 2\pi\sqrt{\frac{l}{g}}$. The value of g can change depending on the forces on the pendulum.

(a) In the physics lab, $g = 9.81 \text{ m/s}^2$, so $T = 2\pi\sqrt{1.00 \text{ m}/\left(9.81 \text{ m/s}^2\right)} = 2.01 \text{ s}$.

(b) In the elevator accelerating upwards, $g' = g + a = 9.81 \text{ m/s}^2 + 2.10 \text{ m/s}^2 = 11.9 \text{ m/s}^2$, and the period is $T = 2\pi\sqrt{1.0 \text{ m}/\left(9.81 \text{ m/s}^2 + 2.10 \text{ m/s}^2\right)} = 1.82 \text{ s}$.

(c) In the elevator accelerating downwards, $g' = g - a = 9.81 \text{ m/s}^2 - 2.1 \text{ m/s}^2 = 7.71 \text{ m/s}^2$, and the period is $T = 2\pi\sqrt{1.0 \text{ m}/\left(9.81 \text{ m/s}^2 - 2.1 \text{ m/s}^2\right)} = 2.26 \text{ s}$.

(d) In free fall, the pendulum experiences no tension in the string, thus there is no period, $T = \infty$.

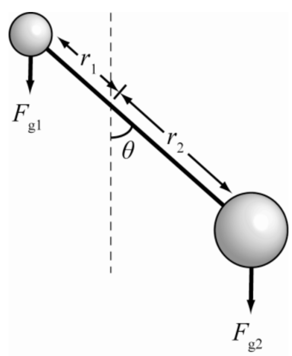
14.36. The period of the pendulum is given by: $T_1 = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g/l}} = 2\pi\sqrt{\frac{l}{g}}$. If the measured value of gravity varies by 0.16 %, then the new period is:

$$T_2 = 2\pi\sqrt{\frac{l}{1.0016g}} = \sqrt{\frac{1}{1.0016}}\left(2\pi\sqrt{\frac{l}{g}}\right) = \sqrt{\frac{1}{1.0016}}T_1.$$

The period therefore varies by: $\left(1 - \sqrt{\frac{1}{1.0016}}\right)(100\ \%) = 0.080\ \%$.

14.37. **THINK:** The two balls create a moment of inertia, *I*, which oscillates about the pivot point, *P*. Using the sum of the torques acting on the pendulum, the period can be determined. The masses are $m_1 = 1.00 \text{ kg}$ and $m_2 = 2.00 \text{ kg}$. The two masses are separated by 30.0 cm. The pivot point is 10.0 cm away from the 1.00 kg mass, so $r_1 = 10.0 \text{ cm}$ and $r_2 = 20.0 \text{ cm}$. A 'slight displacement' implies small values for θ.

SKETCH:



RESEARCH: The torque is given by $\tau = \vec{r} \times \vec{F} = rF\sin\theta$. The sum of the torques is $\tau = I\frac{d^2\theta}{dt^2} = \sum_i \tau_i$. The period is given by $T = 2\pi/\omega$.

SIMPLIFY: The torque equation gives $I\frac{d^2\theta}{dt^2} = -r_1F_{g1}\sin\theta + r_2F_{g2}\sin\theta$. For small angles, $\sin\theta \approx \theta$. The equation thus becomes: $I\frac{d^2\theta}{dt^2} = \left(-r_1m_1 + r_2m_2\right)g\theta$. The angular frequency is then:

$$\omega = \sqrt{\frac{\left(m_2r_2 - m_1r_1\right)g}{I}}.$$

The moment of inertia is $I = \sum_i m_ir_i^2 = m_1r_1^2 + m_2r_2^2$. Using the above equations, the period is:

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{I}{(m_2r_2-m_1r_1)g}}=2\pi\sqrt{\frac{m_1r_1^2+m_2r_2^2}{(m_2r_2-m_1r_1)g}}.$$

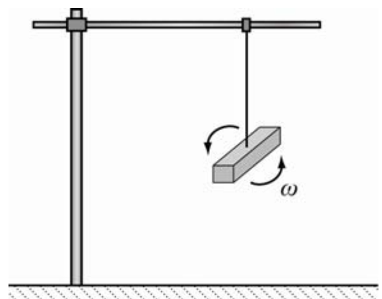
CALCULATE: $T=2\pi\sqrt{\frac{1.00\text{ kg}(0.100\text{ m})^2+2.00\text{ kg}(0.200\text{ m})^2}{\left(2.00\text{ kg}(0.200\text{ m})-1.00\text{ kg}(0.100\text{ m})\right)(9.81\text{ m/s}^2)}}=1.09876\text{ s}$

ROUND: Rounding to three significant figures, the period of oscillation is T= 1.10 s.

DOUBLE-CHECK: A simple pendulum of length 0.3 m will have a period of $2\pi\sqrt{\frac{0.3\text{ m}}{g}}=1.09876\text{ s},$ the same as that calculated above.

14.38. **THINK:** Knowing the form of a linear oscillator allows for the theory of angular oscillators and torsional pendulums to be inferred.

SKETCH:



RESEARCH: The equation of motion for a linear oscillator is $F=ma=-kx$. The frequency of such a system is $\omega=\sqrt{k/m}$. The torsional pendulum has a total torque of $\tau=I\frac{d^2\theta}{dt^2}=-\kappa\theta,$ where I is the moment of the oscillator, and κ is similar to the spring constant.

SIMPLIFY: The angular frequency of the torsional pendulum is then

$$\omega=\sqrt{\frac{\kappa}{I}}\ \text{ or }\ f=\frac{\omega}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{\kappa}{I}}.$$

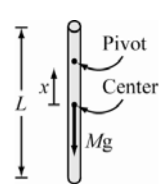
CALCULATE: No calculations are necessary.

ROUND: There are no values to round.

DOUBLE-CHECK: Note that the units of κ are N m/rad and the units of I are $\text{kg m}^2/\text{rad}$. Therefore, f has units $\sqrt{\frac{\text{N m/rad}}{\text{kg m}^2\text{/rad}}}=\sqrt{\frac{\text{kg m/s}^2(\text{m/rad})}{\text{kg m}^2\text{/rad}}}=\sqrt{\frac{1}{\text{s}^2}}=\frac{1}{\text{s}},$ as expected.

14.39. **THINK:** The usual pendulum equations cannot be used because the moment of inertia of the rod must be taken into account. The results of Solved Problem 14.2 can be used to determine the period and how it depends on *x*. Then the maximum and minimum values can be determined.

SKETCH:



RESEARCH: The period is given by: $T=2\pi\sqrt{\frac{I}{xMg}}$. Using the parallel axis theorem, the moment of inertia of the rod is given by: $I=\frac{1}{12}ML^2+Mx^2$.

SIMPLIFY: $T=2\pi\sqrt{\dfrac{ML^2/12+Mx^2}{xMg}}=2\pi\sqrt{\dfrac{L^2}{12xg}+\dfrac{x}{g}}$. To determine the extrema, set the derivative to zero:

$$\frac{dT}{dx}=0=\pi\left(\frac{L^2}{12xg}+\frac{x}{g}\right)^{-\frac{1}{2}}\left(\frac{-L^2}{12x^2g}+\frac{1}{g}\right)\Rightarrow 0=\frac{\left(\dfrac{-L^2}{12x^2g}+\dfrac{1}{g}\right)}{\sqrt{\dfrac{L^2}{12xg}+\dfrac{x}{g}}}\Rightarrow \frac{-L^2}{12x^2g}+\frac{1}{g}=0\Rightarrow \frac{L^2}{12x^2}=1\Rightarrow \frac{L^2}{12}=x^2.$$

Since x is a distance, the only solution is $x=L/\sqrt{12}$. Extrema can also occur where the second derivative is not defined, which is when $x=0$. When $x=0$, the tension T is not defined (as x approaches zero from the right, the tension goes to infinity). To verify that $x=L/\sqrt{12}$ is a minimum, take the second derivative of T with respect to x, and evaluate it at $x=L/\sqrt{12}$.

$$\frac{d^2T}{dx^2}=-\frac{\pi\left(-3\dfrac{L^2}{x^2g}+\dfrac{36}{g}\right)^2}{12\left(\dfrac{3L^2}{xg}+\dfrac{36x}{g}\right)^{3/2}}+\frac{\pi L^2}{x^3g\sqrt{3\dfrac{L^2}{xg}+\dfrac{36x}{g}}},$$

and $\left.\dfrac{d^2T}{dx^2}\right|_{x=L/\sqrt{12}}=12\dfrac{\pi}{Lg\sqrt{\dfrac{L\sqrt{3}}{g}}}>0$. By the second derivative test, $x=L/\sqrt{12}$ is a minimum. As x increases, T increases without bound. Therefore, there is no maximum.

(a) $x=0$

(b) $x=L/\sqrt{12}$

CALCULATE: This step is not necessary.

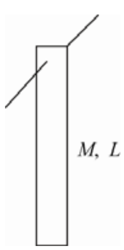
ROUND: Rounding is not necessary.

DOUBLE-CHECK: The results are reasonable. For example, at $x=0$, there is no restoring force, so there will be no oscillatory motion. The period should therefore diverge.

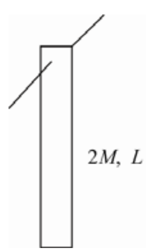
14.40. **THINK:** Ensure the use of the correct expressions for the period of a physical pendulum and an ordinary pendulum.

SKETCH:

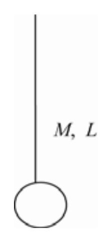
(a)



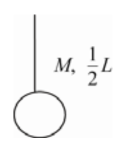
(b)



(c)



(d)



RESEARCH: The period of a bob pendulum is given by $T=2\pi\sqrt{\dfrac{L}{g}}$. The period of a pendulum in general is given by $T=2\pi\sqrt{\dfrac{I}{MgR}}$. The moment of inertia of a rod about one end is $I=ML^2/3$.

SIMPLIFY:

(a) $T=2\pi\sqrt{\dfrac{ML^2/3}{MgL/2}}=2\pi\sqrt{\dfrac{2L}{3g}}$

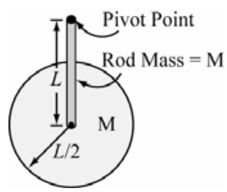
(b) $T=2\pi\sqrt{\dfrac{2ML^2/3}{2MgL/2}}=2\pi\sqrt{\dfrac{2L}{3g}}$

(c) $T=2\pi\sqrt{\dfrac{L}{g}}$

(d) $T=2\pi\sqrt{\dfrac{L/2}{g}}=2\pi\sqrt{\dfrac{L}{2g}}$

CALCULATE: This step is not necessary.
ROUND: This step is not necessary.
DOUBLE-CHECK: The initial angle and the masses play no role. However, the initial angle must be known, because it must be ensured that the angle is small enough for the small angle approximation to be valid, which it is in this case.

14.41. **THINK:** To determine the moment of inertia, the expressions for a thin rod and a sphere, along with the parallel axis theorem are needed. Using this result, the period of the pendulum can be determined by direct substitution into what we have derived in Solved Problem 14.2.
SKETCH:



RESEARCH: The moments of inertia are given by: $I_{\text{rod}}=\frac{1}{3}ML^2$ and $I_{\text{sphere}}=\frac{2}{5}Mr^2$. The parallel axis theorem is given by $I=I_{\text{center}}+Mx^2$, where x is the distance of the pivot point to the center of mass. The period is given by:

$$T=2\pi\sqrt{\frac{I}{M_{\text{t}}gR}},$$

where R is the distance from the pivot point to the center of gravity and M_{t} is the total mass, which is $2M$ in the present case.
SIMPLIFY:
(a) The total moment of inertia is $I=I_{\text{rod}}+\left(I_{\text{sphere}}+ML^2\right)$. Substituting the moments of inertia gives:

$$I=\frac{1}{3}ML^2+\frac{2}{5}M\left(\frac{L^2}{4}\right)+ML^2=\left(\frac{1}{3}+\frac{1}{10}+1\right)ML^2=\left(\frac{10+3+30}{30}\right)ML^2=\frac{43}{30}ML^2.$$

(b) The distance from the pivot point to the center of gravity is $R=3L/4$, so:

$$T=2\pi\sqrt{\frac{I}{M_{\text{t}}gR}}\Rightarrow T=2\pi\sqrt{\frac{43ML^2/30}{2Mg3L/4}}=2\pi\sqrt{\frac{4(43)L}{6(30)g}}=2\pi\sqrt{\frac{172L}{180g}}=2\pi\sqrt{\frac{43L}{45g}}.$$

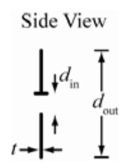
(c) $T^2=4\pi^2\left(\dfrac{43L}{45g}\right)\Rightarrow L=\dfrac{45gT^2}{43\left(4\pi^2\right)}$

CALCULATE:
(a) Not necessary.
(b) Not necessary.
(c) $T=2.0\text{ s}$, so, $L=\dfrac{45\left(9.81\text{ m/s}^2\right)\left(2.0\text{ s}\right)^2}{43\left(4\pi^2\right)}=1.0401915\text{ m}.$

ROUND:
(a) Not necessary.

(b) Not necessary.
(c) Since the desired period is given to two significant figures, $L = 1.0$ m.
DOUBLE-CHECK: The results are reasonable, because they work out similar to the case of the pendulum with a point mass at the end, for which we have calculated $T = 2\pi\sqrt{L/g}$. This implies that taking care of the proper distribution of the mass has a noticeable effect relative to what one obtains for a point mass pendulum, but it is not drastically different. [Since the entire exercise involving moments of inertia and center-of-mass coordinates only resulted in a correction factor of $\sqrt{43/45} \approx 0.98$, you may think that it was not worth the effort. However, if a clock is off by this factor, it is slow by approximately half an hour per day and therefore pretty much useless as a time-keeping device.]

14.42. **THINK:** Use the expression for the oscillation frequency of a physical pendulum. For the moment of inertia, use the parallel axis theorem.
SKETCH:



RESEARCH: $I = \frac{1}{2}m\left(r_{out}^{\ 2} + r_{in}^{\ 2}\right) + mr_{in}^{\ 2}$, $r = d/2$, $\omega = \sqrt{\frac{mgr_{in}}{I}}$, $f = \frac{1}{2\pi}\omega = \frac{1}{2\pi}\sqrt{\frac{mgr_{in}}{I}}$

SIMPLIFY: $I = \frac{m}{2}\left(\frac{d_{out}^{\ 2}}{4} + \frac{d_{in}^{\ 2}}{4}\right) + m\frac{d_{in}^{\ 2}}{4} = \frac{m}{4}\left[\frac{1}{2}d_{out}^{\ 2} + \frac{3}{2}d_{in}^{\ 2}\right] = \frac{m}{8}\left(d_{out}^{\ 2} + 3d_{in}^{\ 2}\right)$

$$f = \frac{1}{2\pi}\sqrt{\frac{mg\mathrm{d}_{in}/2}{m\left(d_{out}^{\ 2} + 3d_{in}^{\ 2}\right)/8}} = \frac{1}{2\pi}\sqrt{\frac{4gd_{in}}{d_{out}^{\ 2} + 3d_{in}^{\ 2}}}$$

CALCULATE: $f = \frac{1}{2\pi}\sqrt{\frac{4(1.5\text{ cm})(980\text{ cm/s}^2)}{(12\text{ cm})^2 + 3(1.5\text{ cm})^2}} = 0.994\text{ s}^{-1}$

ROUND: Rounding to two significant figures, $f = 0.99\text{ s}^{-1}$.

DOUBLE-CHECK: These results are reasonable. Note that the mass and the thickness of the CD are not relevant here.

14.43. (a) Substitute $t = 0$ and evaluate the expression: $x(0) = 2\sin(\pi/6) = 1.00$ m. To get the velocity function, take the derivative of the position function $x(t) = 2\sin((\pi/2)t + \pi/6)$:

$$v(t) = \frac{dx(t)}{dt} = \pi\cos\left(\frac{\pi}{2}t + \frac{\pi}{6}\right).$$

Again, substitute $t = 0$: $v(0) = \pi\cos(\pi/6) = 2.72$ m/s. Take the derivative of the velocity to get the acceleration:

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2} = -\frac{\pi^2}{2}\sin\left(\frac{\pi}{2}t + \frac{\pi}{6}\right).$$

Once again, substitute $t = 0$: $a(0) = -\frac{\pi^2}{2}\sin\left(\frac{\pi}{6}\right) = -2.47\text{ m/s}^2$.

(b) $K(t) = \frac{1}{2}mv^2 = (2.5)\pi^2\cos^2\left(\frac{\pi}{2}t + \frac{\pi}{6}\right)$

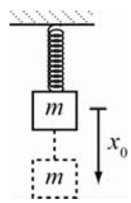
(c) The kinetic energy will be a maximum when $\cos^2(\pi t/2 + \pi/6) = 1$ or $\pi t/2 + \pi/6 = \pi$, which gives $t = 5/3\text{ s} = 1.67\text{ s}$.

14.44. (a) Once the mass has been displaced to the right and at the moment the mass is released there will be two forces acting on it. The forces are due to spring 1 and spring 2. They both point towards the left. The magnitude of the net force acting on the mass is $F_{\text{net}} = k_1x + k_2x = (k_1 + k_2)x = k_{\text{eff}}x.$ Therefore, the effective spring constant is $k_{\text{eff}} = k_1 + k_2 = 300.\text{ N/m}.$

(b) $f = \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{k_{\text{eff}}}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{k_1+k_2}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{300.\text{ N/m}}{10.0\text{ kg}}} = 0.872\text{ Hz}$

(c) The total energy is: $E = \dfrac{1}{2}(k_1+k_2)x^2 = \dfrac{1}{2}(300.\text{ N/m})(0.100\text{ m})^2 = 1.50\text{ J}.$ The maximum velocity is at: $\dfrac{1}{2}mv^2 = 1.50\text{ J} \Rightarrow v = \sqrt{\dfrac{2(1.50\text{ J})}{m}} = \sqrt{\dfrac{2(1.50\text{ J})}{10.0\text{ kg}}} = 0.548\text{ m/s}.$

14.45. **THINK:** For both parts, use the conservation of energy, along with the expressions for kinetic and potential energy. The mass is m = 2.00 kg, the displacement is $x_0 = 8.00\text{ cm}$ and the frequency is f = 4.00 Hz.
SKETCH:



RESEARCH:

(a) $\text{Total energy} = \text{constant} = \dfrac{1}{2}kx_0^2 = E_{\text{tot}},\quad k = m\omega^2 = 4\pi^2mf^2$

(b) $K = \dfrac{1}{2}mv^2,\quad U = \dfrac{1}{2}kx^2,\quad K + U = E_{\text{tot}}$

SIMPLIFY:

(a) The energy is constant throughout the oscillation:

$$E_{\text{tot}} = \frac{1}{2}kx_0^2 = \frac{1}{2}m\omega^2x_0^2 = \frac{1}{2}m(2\pi f)^2x_0^2 = 2\pi^2mf^2x_0^2.$$

(b) $\dfrac{1}{2}mv^2 + \dfrac{1}{2}kx^2 = E_{\text{tot}}$

$$\Rightarrow v^2 = \frac{2}{m}\left(E_{\text{tot}} - \frac{1}{2}kx^2\right) = \frac{2}{m}\left(\frac{1}{2}kx_0^2 - \frac{1}{2}kx^2\right) = \frac{k}{m}\left(x_0^2 - x^2\right) = \frac{4\pi^2mf^2}{m}\left(x_0^2 - x^2\right) \Rightarrow v = 2\pi f\sqrt{x_0^2 - x^2}$$

CALCULATE:

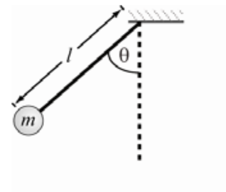
(a) $E_{\text{tot}} = 2\pi^2(2.00\text{ kg})(4.00\text{ Hz})^2(.0800\text{ m})^2 = 4.043\text{ J}$

(b) $v = 2\pi(4.00\text{ Hz})\sqrt{(8.00\text{ cm})^2 - (2.00\text{ cm})^2} = 194.7\text{ cm/s} = 1.947\text{ m/s}$

ROUND: Rounding to three significant figures, $E = 4.04\text{ J}$, v = 1.95 m/s

DOUBLE-CHECK: These results are reasonable.

14.46. **THINK:** The period can be determined directly from the given values. The maximum kinetic energy is equal to the maximum potential energy, which is also equal to the total energy. The maximum speed can be determined from the maximum kinetic energy. The pendulum has a length, l = 15 m, the bob has a mass of m = 110 kg, and the angle has an amplitude of $\theta = 3.5°$.
SKETCH:



RESEARCH:

(a) $T = 2\pi\sqrt{\frac{l}{g}}$

(b) $K_{max} = U_{max} = mgh_{max}$ and $h_{max} = l(1-\cos\theta)$.

(c) $K_{max} = \frac{1}{2}mv_{max}^{\ 2}$

SIMPLIFY:

(a) It is not necessary to simplify.

(b) $K_{max} = mgl(1-\cos\theta)$

(c) $v_{max} = \sqrt{\frac{2K_{max}}{m}}$

CALCULATE:

(a) $T = 2\pi\sqrt{\frac{15.0 \text{ m}}{9.81 \text{ m/s}^2}} = 7.769 \text{ s}$

(b) $K_{max} = 110. \text{ kg}(9.81 \text{ m/s}^2)(15.0 \text{ m})(1-\cos 3.50°) = 30.19 \text{ J}$

(c) $v_{max} = \sqrt{\frac{2(30.19 \text{ J})}{110 \text{ kg}}} = 0.7409 \text{ m/s}$

ROUND: Rounding to two significant figures:

(a) $T = 7.8 \text{ s}$

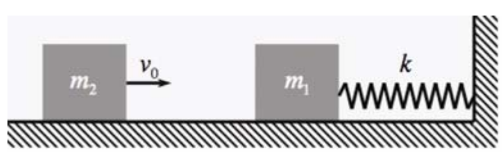
(b) $K_{max} = 30. \text{ J}$

(c) $v_{max} = 0.74 \text{ m/s}$

DOUBLE-CHECK: The results are reasonable for such a large pendulum. Each value has appropriate units.

14.47. THINK: By considering the conservation of momentum, the initial velocity of the mass/spring system can be determined. The amplitude can then be determined by considering the conservation of energy. The period can then be determined, which will yield the time needed to reach maximum compression. The given values are $m_1 = 8.00$ kg, $m_2 = 5.00$ kg, $k = 70.$ N/m and $v_0 = 17.0$ m/s.

SKETCH:



RESEARCH:

(a) The conservation of momentum gives: $m_2v_0 = (m_1+m_2)v$. The conservation of energy gives: $\frac{1}{2}(m_1+m_2)v^2 = \frac{1}{2}kx_{max}^{\ 2}$.

(b) $t = \frac{1}{4}T$, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m_1+m_2}{k}}$

SIMPLIFY:

(a) $x_{\max} = \sqrt{\dfrac{(m_1+m_2)v^2}{k}}, \quad v = \dfrac{m_2 v_0}{m_1+m_2} \Rightarrow x_{\max} = \sqrt{\dfrac{m_2^{\,2} v_0^{\,2}}{k(m_1+m_2)}}$

(b) $T = 2\pi\sqrt{\dfrac{m_1+m_2}{k}} \Rightarrow t = \dfrac{1}{2}\pi\sqrt{\dfrac{m_1+m_2}{k}}$

CALCULATE:

(a) $x_{\max} = \sqrt{\dfrac{(5.00 \text{ kg})^2(17.0 \text{ m/s})^2}{(70.0 \text{ N/m})(8.00 \text{ kg}+5.00 \text{ kg})}} = 2.818 \text{ m}$

(b) $t = \dfrac{1}{2}\pi\sqrt{\dfrac{8.00 \text{ kg}+5.00 \text{ kg}}{70.0 \text{ N/m}}} = 0.6769 \text{ s}$

ROUND: Rounding to three significant figures:

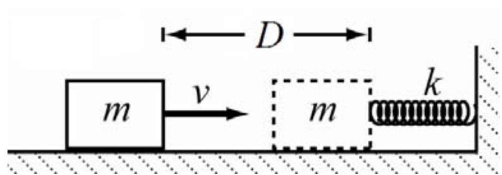
(a) $x_{\max} = 2.82 \text{ m}$

(b) $t = 0.677$ s

DOUBLE-CHECK: Dimensional analysis confirms the units are correct for the calculated quantities. The results are reasonable for this large initial speed.

14.48. **THINK:** The maximum compression can be determined from the conservation of energy. The elapsed time is twice the time taken to reach the spring, plus half the period of the mass/spring system. The given values are $m = 0.460$ kg, $D = 0.250$ m, $k = 840$ N/m and $v = 3.20$ m/s.

SKETCH:



RESEARCH:

(a) $\dfrac{1}{2}mv^2 = \dfrac{1}{2}kx_{\max}^{\;2}$

(b) $T = 2\pi\sqrt{m/k}$, $v = D/t_0$ and $t = 2t_0 + T/2$, where t_0 is the time it takes to cover the initial distance D to the spring, and t is the elapsed time to return to the starting point.

SIMPLIFY:

(a) $x_{\max} = \sqrt{\dfrac{mv^2}{k}} = v\sqrt{\dfrac{m}{k}}$

(b) $t = 2\dfrac{D}{v} + \pi\sqrt{\dfrac{m}{k}}$

CALCULATE:

(a) $x_{\max} = (3.20 \text{ m/s})\sqrt{\dfrac{0.460 \text{ kg}}{840 \text{ N/m}}} = 7.488 \cdot 10^{-2} \text{ m}$

(b) $t = 2\left(\dfrac{0.250 \text{ m}}{3.20 \text{ m/s}}\right) + \pi\sqrt{\dfrac{0.460 \text{ kg}}{840 \text{ N/m}}} = 0.2298 \text{ s}$

ROUND: Two significant figures:

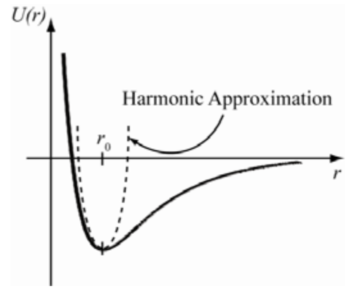
(a) The maximum distance that the spring is compressed is $x_{\max} = 7.5 \cdot 10^{-2}$ m.

(b) The total time taken for the block to get back to its original position is $t = 0.23$ s.

DOUBLE-CHECK: These are reasonable results for the given parameters.

14.49. **THINK:** The equilibrium separation will occur at the minimum of the potential. For small deviations from equilibrium, the potential can be expanded to second order about the minimum. The resulting potential will be formally equivalent to the harmonic potential.

SKETCH:



RESEARCH: The minimum is at r_0 : $\left[\dfrac{dU(r)}{dr}\right]_{r=r_0}=0.$ The Taylor series about $r=r_0$ is:

$$U(r)\approx U(r_0)+\frac{1}{1!}(r-r_0)^1\left[\frac{dU}{dr}\right]_{r_0}+\frac{1}{2!}(r-r_0)^2\left[\frac{d^2U}{dr^2}\right]_{r_0}+\ldots$$

$$\left[\frac{dU}{dr}\right]_{r_0}=0 \;\Rightarrow\; U(r)\approx U(r_0)+\frac{1}{2}(r-r_0)^2\left[\frac{d^2U}{dr^2}\right]_{r_0}$$

SIMPLIFY:

(a) $\dfrac{dU}{dr}=\dfrac{d}{dr}\left(\dfrac{A}{r^{12}}-\dfrac{B}{r^6}\right)=-\dfrac{12A}{r^{13}}+\dfrac{6B}{r^7}$

$$\left[\frac{dU}{dr}\right]_{r_0}=0:\; -\frac{12A}{r_0^{13}}+\frac{6B}{r_0^{7}}=0 \;\Rightarrow\; -\frac{12A}{r_0^{6}}+6B=0 \;\Rightarrow\; r_0^{\,6}=\frac{12A}{6B} \;\Rightarrow\; r_0=\left(\frac{2A}{B}\right)^{\frac{1}{6}}$$

(b) $\left[\dfrac{d^2U}{dr^2}\right]_{r_0}=\dfrac{12(13)A}{r_0^{14}}-\dfrac{6(7)B}{r_0^{8}}=\dfrac{156A}{r_0^{14}}-\dfrac{42B}{r_0^{8}}=k$

$$\Rightarrow\; U(r)\approx U(r_0)+\frac{1}{2}(r-r_0)^2k=U(r_0)+\frac{1}{2}k\left(r^2-2rr_0+r_0^{\,2}\right)=\text{constant}+kr_0r+\frac{1}{2}kr^2$$

$$F=-\frac{dU}{dr}\approx -kr_0-kr=\text{constant}-kr.$$

This is Hooke's law with a spring constant, *k*. The angular frequency is given by $\omega=\sqrt{k/m}$.

$$k=\frac{1}{r_0^{14}}\left[156A-42r_0^{\,6}B\right]=\left(\frac{B}{2A}\right)^{\frac{14}{6}}\left[156A-42B\left(\frac{2A}{B}\right)\right]=\left(\frac{B}{2A}\right)^{\frac{7}{3}}(156-84)A=72A\left(\frac{B}{2A}\right)^{\frac{7}{3}}=\frac{72}{2^{7/3}}\left(\frac{B^{7/3}}{A^{4/3}}\right)$$

$$\Rightarrow\; k=9\left(\frac{4B^7}{A^4}\right)^{1/3}$$

$$\omega=\sqrt{\frac{k}{m}}=\frac{3}{\sqrt{m}}\left(\frac{4B^7}{A^4}\right)^{1/6}$$

CALCULATE: This step is not necessary.
ROUND: Rounding is not necessary.
DOUBLE-CHECK: These oscillations yield the vibration spectra of the molecule. Most systems with an equilibrium configuration can exhibit simple harmonic motion for small perturbations from equilibrium.

14.50. (a) $b=10.0$ kg/s, $\sqrt{mk}=\sqrt{3.00\text{ kg}(140.\text{ N/m})}=\sqrt{420.\text{ kg}^2/\text{s}^2}=20.49$ kg/s. Note that $b<2\sqrt{mk}$, so this is small damping. The position is given by:

$$x(t)=Be^{-\omega_\gamma t}\cos(\omega' t)+Ce^{-\omega_\gamma t}\sin(\omega' t).$$

The amplitude will reach 1.00 % of its initial value when $e^{-\omega_\gamma t}=0.0100$ or $-\omega_\gamma t=\ln(0.0100)$.

Solving for *t*:

$$t=-\frac{1}{\omega_\gamma}\ln(0.0100)=-\frac{2m}{b}\ln(0.0100)=-\frac{2.00(3.00\text{ kg})}{10.0\text{ kg/s}}\ln(0.0100)=2.76\text{ s}.$$

(b) Solve for *b*, given that *t* = 1.00 s. Use the equation $-\omega_\gamma t=\ln(0.0100)$:

$$-\frac{bt}{2m}=\ln(0.0100)\Rightarrow b=-\frac{2m}{t}\ln(0.0100)=-\frac{2.00(3.00\text{ kg})}{1.00\text{ s}}\ln(0.0100)=27.6\text{ kg/s}.$$

Note that $b<2\sqrt{mk}$.

14.51. First, determine the damping region: $2\sqrt{mk}=2\sqrt{(0.3\text{ kg})(2.00\text{ N/m})}=1.549\text{ kg/s}$, $b=0.025\text{ kg/s}<2\sqrt{mk}$. This is the weak damping region. Therefore,

$$x(t)=Be^{-\omega_\gamma t}\cos(\omega' t)+Ce^{-\omega_\gamma t}\sin(\omega' t),\quad B=x_0,\text{ and } C=\frac{v_0+x_0\omega_\gamma}{\omega'}.$$

Here $\omega_\gamma=\frac{b}{2m}=\frac{0.025\text{ kg/s}}{2(0.3\text{ kg})}=4.167\cdot 10^{-2}\text{ s}^{-1}$ and $\omega_0=\sqrt{\frac{k}{m}}=\sqrt{\frac{2\text{ N/m}}{0.3\text{ kg}}}=2.582\text{ s}^{-1}$

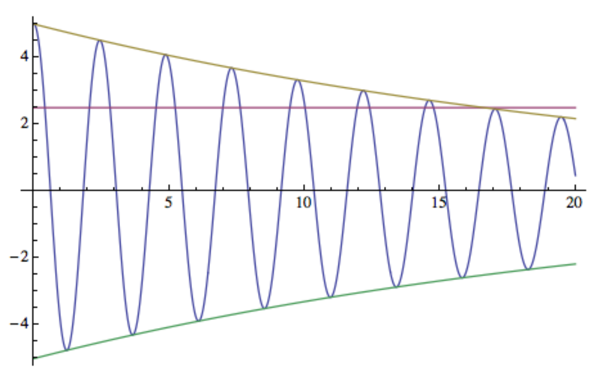
In the present case, $x_0=5.0\text{ cm}$ and $v_0=0$, so $C=\frac{x_0\omega_\gamma}{\omega'}$. The exponential envelope of the amplitude of the oscillation is $A=e^{-\omega_\gamma t}\sqrt{B^2+C^2}$. Substituting the expressions for *B* and *C* into this equation:

$$A=e^{-\omega_\gamma t}x_0\sqrt{1+\frac{\omega_\gamma^{\ 2}}{\omega'^2}}\approx e^{-\omega_\gamma t}x_0.$$

At $t=t_\text{f}$: $A_\text{f}=e^{-\omega_\gamma t_\text{f}}(5\text{ cm})=2.5\text{ cm}\Rightarrow e^{-\omega_\gamma t_\text{f}}=\frac{2.5\text{ cm}}{5.0\text{ cm}}=0.5\Rightarrow -\omega_\gamma t_\text{f}=\ln(0.5)$

$$\Rightarrow t_\text{f}=\frac{-\ln(0.5)}{4.16667\cdot 10^{-2}\text{ s}^{-1}}=16.6\text{ s}.$$

This plot shows the damped oscillation, its (+-) exponential amplitude envelopes, and the horizontal line of 2.5 cm as a function of time in the interval from 0 to 20 cm. The point at which the exponential envelope crosses the line *x* = 2.5 cm is the time we just derived.

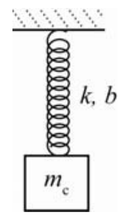


14.52. Determine the damping region: $2\sqrt{mk}=2\sqrt{(0.404\text{ kg})(206.9\text{ N/m})}=18.29\text{ kg/s}$, $b=14.5\text{ kg/s}<2\sqrt{mk}$. So, this is the small damping region. The angular oscillation frequency is therefore:

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\frac{206.9 \text{ N/m}}{0.404 \text{ kg}} - \left(\frac{14.5 \text{ kg/s}}{2(0.404 \text{ kg})}\right)^2} = 13.8 \text{ rad/s}.$$

14.53. **THINK:** From the critical damping condition, determine the value of the damping constant, b. When b is 60.7 % of its full value, use the expression for the underdamped oscillator to determine the period. Model the system as four independent oscillators, each supporting a quarter of the weight of the car. The mass of the car is $m_c = 851$ kg and the value of the spring constant is $k = 4005$ N/m.

SKETCH:



RESEARCH: For critical damping, $b_0 = 2\sqrt{mk}$. $b = 0.607 b_0 \equiv \alpha b_0 \Rightarrow \alpha = 0.607$, $m = \frac{m_c}{4}$. The period of underdamped motion is given by $T = 2\pi / \omega'$.

$$\omega' = \sqrt{\omega_0^2 - \omega_\gamma^2}, \quad \omega_\gamma = \frac{b}{2m}, \quad \omega_0 = \sqrt{\frac{k}{m}}$$

SIMPLIFY: $\omega_\gamma = \frac{\alpha b_0}{2m} = \frac{\alpha 2\sqrt{mk}}{2m} = \alpha\sqrt{\frac{k}{m}} = \alpha\omega_0, \quad \omega' = \sqrt{\omega_0^2 - \alpha^2\omega_0^2} = \omega_0\sqrt{1-\alpha^2}$

$$T = \frac{2\pi}{\omega_0\sqrt{1-\alpha^2}} = 2\pi\sqrt{\frac{m}{k(1-\alpha^2)}}$$

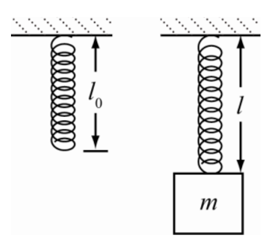
CALCULATE: $T = 2\pi\sqrt{\frac{851 \text{ kg/4}}{4005 \text{ N/m}\left(1-(0.607)^2\right)}} = 1.822 \text{ s}$

ROUND: Rounding to three significant figures, $T = 1.82$ s

DOUBLE-CHECK: This is a reasonable result for a car's shock absorbers. The value has seconds as units, which are appropriate for time.

14.54. **THINK:** From the change in length after the mass is hung on the spring, the spring constant can be determined and hence the undamped oscillation frequency. By the decrease of the amplitude after five cycles, the damping frequency and hence the period of oscillation when damping is included can be determined. $l_0 = 11.2$ cm, $l = 20.7$ cm, $m = 100.0$ g and $x \equiv l - l_0$.

SKETCH:



RESEARCH: When the mass is hung, the stretching of the spring by x yields a balance of forces: $kx = mg$.

The undamped frequency is given by: $\omega_0 = \sqrt{\frac{k}{m}}$. The undamped period is given by: $T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$. The damped frequency is given by $\omega' = \sqrt{\omega_0^2 - \omega_\gamma^2}$, where $\omega_\gamma = b/2m$. The position at time, *t*, is given by: $x(t) = A_0 e^{-\omega_\gamma t}\sin(\omega' t + \theta_0) = A(t)\sin(\omega' t + \theta_0)$. The amplitude at time, *t*, is given by $A(t) = A_0 e^{-\omega_\gamma t}$.

SIMPLIFY:

(a) Substitute $k = mg/x$ into the equation for T_0 to get: $T_0 = 2\pi\sqrt{\frac{m}{mg/x}} = 2\pi\sqrt{\frac{x}{g}}$.

(b) The amplitude at $t = 0$ is $A(0) = A_0$. The amplitude at $t = 5T$ is $A(5T) = A_0 e^{-\omega_\gamma 5T} = A_0/2$.

$$\Rightarrow e^{-\omega_\gamma 5T} = \frac{1}{2} \Rightarrow e^{5\omega_\gamma T} = 2 \Rightarrow 5\omega_\gamma T = \ln 2 \Rightarrow \omega_\gamma = \frac{\ln 2}{5T}$$

$$T = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega_0^2 - \omega_\gamma^2}}, \quad \omega_\gamma^2 = \frac{(\ln 2)^2}{25}\left[\frac{(\omega_0^2 - \omega_\gamma^2)}{4\pi^2}\right] = \left(\frac{\ln 2}{10\pi}\right)^2(\omega_0^2 - \omega_\gamma^2)$$

Let $\alpha = \ln 2/(10\pi)$:

$$\omega_\gamma^2 = \alpha^2(\omega_0^2 - \omega_\gamma^2) \Rightarrow \omega_\gamma^2(1+\alpha^2) = \omega_0^2\alpha^2 \Rightarrow \omega_\gamma^2 = \omega_0^2\frac{\alpha^2}{1+\alpha^2} \Rightarrow \omega_\gamma = \omega_0\frac{\alpha}{\sqrt{1+\alpha^2}}$$

$$T = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega_0^2 - \omega_\gamma^2}} \Rightarrow \omega_0^2 - \omega_\gamma^2 = \omega_0^2 - \omega_0^2\left(\frac{\alpha^2}{1+\alpha^2}\right)$$

$$\omega_0^2 - \omega_\gamma^2 = \omega_0^2\left(1 - \frac{\alpha^2}{1+\alpha^2}\right) = \omega_0^2\left(\frac{1+\alpha^2-\alpha^2}{1+\alpha^2}\right) = \omega_0^2\frac{1}{1+\alpha^2}$$

$$\Rightarrow T = 2\pi\frac{\sqrt{1+\alpha^2}}{\omega_0} = 2\pi\left(\sqrt{\frac{m}{k}}\right)\left(\sqrt{1+\alpha^2}\right) = T_0\sqrt{1+\alpha^2} \Rightarrow T - T_0 = T_0\left(\sqrt{1+\alpha^2} - 1\right)$$

CALCULATE:

(a) $T_0 = 2\pi\sqrt{\frac{0.207\text{ m} - 0.112\text{ m}}{9.81\text{ m/s}^2}} = 0.61831\text{ s}$

(b) $T - T_0 = T_0\left(\sqrt{1+\alpha^2} - 1\right) = 0.61831\left[\sqrt{1+\left(\frac{\ln 2}{10\pi}\right)^2} - 1\right] = 1.5048 \cdot 10^{-4}\text{ s}$. This is too small to be detected by the student.

ROUND:

(a) Rounding to three significant figures, $T_0 = 0.618\text{ s}$.

(b) Not applicable.

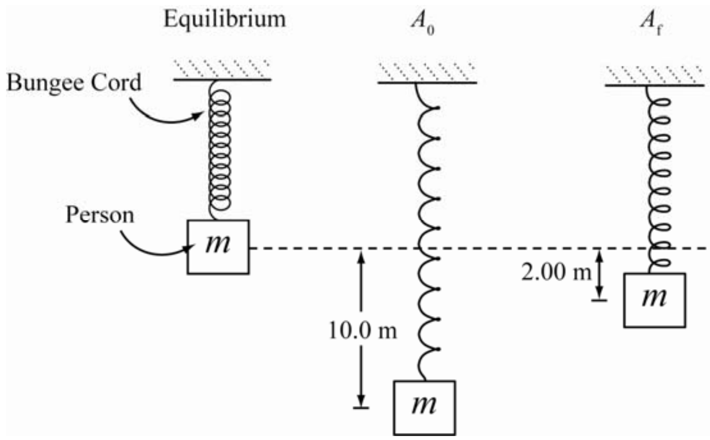
DOUBLE-CHECK: The calculation presented is essentially exact. Since the damping is so small, the approximate methods outlined in the text could have been used. The approximate method is:

$$\frac{|\Delta E|}{E} \approx \frac{4\pi\omega_\gamma}{\omega_0}.$$

This would have yielded very similar results.

14.55. **THINK:** For parts (a) and (b), the motion can be treated as undamped. For part (c), use the expressions for underdamped harmonic motion. The mass of the jumper is $m = 80.0$ kg, the initial amplitude is $A_0 = 10.0$ m, the final amplitude is $A_f = 2.00$ m, the initial period is $T_0 = 5.00$ s and the damping is $b = 7.50$ kg/s.

SKETCH:



RESEARCH:

(a) $T_0 = 2\pi / \omega_0$ and $\omega_0 = \sqrt{k/m}$.

(b) $E = \frac{1}{2} m v_{max}^2 = \frac{1}{2} k A_0^2$

(c) $A_0 e^{-\omega_\gamma t_f} = A_f$ and $\omega_\gamma = b/2m$.

SIMPLIFY:

(a) $\omega_0^2 = \frac{k}{m} \Rightarrow k = m\omega_0^2 \Rightarrow k = m\frac{4\pi^2}{T_0^2}$

(b) $v_{max} = \sqrt{\frac{kA_0^2}{m}} = \sqrt{\frac{4\pi^2 A_0^2}{T_0^2}} = \frac{2\pi A_0}{T_0}$

(c) $-\omega_\gamma t_f = \ln\frac{A_f}{A_0} \Rightarrow t_f = \frac{1}{\omega_\gamma}\ln\frac{A_0}{A_f} \Rightarrow t_f = \frac{2m}{b}\ln\frac{A_0}{A_f}$

CALCULATE:

(a) $k = (80.0 \text{ kg})\frac{4\pi^2}{(5.00 \text{ s})^2} = 126.33 \text{ N/m}$

(b) $v_{max} = \frac{2\pi(10.0 \text{ m})}{5.00 \text{ s}} = 4\pi \text{ m/s} = 12.566 \text{ m/s}$

(c) $t_f = \frac{2(80.0 \text{ kg})}{7.50 \text{ kg/s}}\ln\left(\frac{10.0 \text{ m}}{2.00 \text{ m}}\right) = 34.335 \text{ s}$

ROUND: Rounding to three significant figures:

(a) $k = 126$ N/m

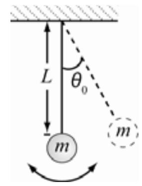
(b) $v_{max} = 12.6$ m/s

(c) $t_f = 34.3$ s

DOUBLE-CHECK: In each part the calculated value has appropriate units.

14.56. THINK: This is simple harmonic motion. For the damping case, the time taken for the oscillator to reach half its initial value must be determined. The motion is underdamped. $m = 50.0$ g, $L = 1.00$ m, $b = 0.0100$ kg/s, $\theta_0 = 10.0°$, $\theta_f = 5.00°$, the initial amplitude is $A_0 = L\theta_0$ and the final amplitude is $A_f = L\theta_f$.

SKETCH:



RESEARCH: $T_0 = \dfrac{2\pi}{\omega_0} = 2\pi\sqrt{\dfrac{L}{g}}$

For small damping, $x(t) = A_0 e^{-\omega_\gamma t}\cos(\omega' t + \varphi)$. $L\theta(t) = L\theta_0 e^{-\omega_\gamma t}\cos(\omega' t + \varphi)$, $\theta(0) = \theta_0$, $\omega_\gamma = \dfrac{b}{2m}$

SIMPLIFY: $T_0 = 2\pi\sqrt{\dfrac{L}{g}}$, $\theta_\text{f} = \theta_0 e^{-\omega_\gamma t_\text{f}} \Rightarrow e^{\omega_\gamma t_\text{f}} = \dfrac{\theta_0}{\theta_\text{f}} \Rightarrow \omega_\gamma t_\text{f} = \ln\dfrac{\theta_0}{\theta_\text{f}} \Rightarrow t_\text{f} = \dfrac{1}{\omega_\gamma}\ln 2 \Rightarrow t_\text{f} = \dfrac{2m\ln 2}{b}$

CALCULATE: $T_0 = 2\pi\sqrt{\dfrac{1.00 \text{ m}}{9.81 \text{ m/s}^2}} = 2.006 \text{ s}$, $t_\text{f} = \dfrac{2\left(50\cdot 10^{-3} \text{ kg}\right)\ln 2}{0.0100 \text{ kg/s}} = 6.931 \text{ s}$

ROUND: Rounding to three significant figures, $T_0 = 2.01 \text{ s}$ and $t_\text{f} = 6.93 \text{ s}$.

DOUBLE-CHECK: These are reasonable results with appropriate units for time. Note that the results are valid only because the angles are small enough to treat the motion as linear.

14.57. **THINK:** This is a straightforward exercise in inserting numbers into Eq. 14.33, which states that the amplitude of the damped driven oscillation is

$$A_\gamma = \frac{F_\text{d}}{m\sqrt{\left(\omega_0^2 - \omega_\text{d}^2\right)^2 + 4\omega_\text{d}^2\omega_\gamma^2}}$$

RESEARCH: The problem text specifies that $\omega_0 = 2.40$ rad/s, and that $\omega_\gamma = 0.140$ rad/s.

SIMPLIFY: No simplification necessary.

CALCULATE: Inserting values of (a) $\omega_\text{d} = \frac{1}{2}\omega_0$, (b) $\omega_\text{d} = \omega_0$, and (c) $\omega_\text{d} = 2\omega_0$ then yields:

(a) 1.20 rad/s: $A_\gamma = \dfrac{2}{3\sqrt{\left((2.40)^2 - (1.20)^2\right)^2 + 4(1.20)^2(0.140)^2}} \text{ m} = 0.1538 \text{ m}$

(b) 2.40 rad/s: $A_\gamma = \dfrac{2}{3\sqrt{\left((2.40)^2 - (2.40)^2\right)^2 + 4(2.40)^2(0.140)^2}} \text{ m} = 0.9921 \text{ m}$

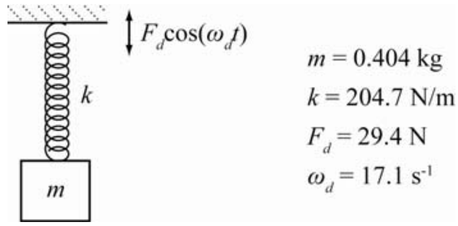
(c) 4.80 rad/s: $A_\gamma = \dfrac{2}{3\sqrt{\left((2.40)^2 - (4.80)^2\right)^2 + 4(4.80)^2(0.140)^2}} \text{ m} = 0.03846 \text{ m}$

ROUND: Rounding to three significant figures,
(a) $A_\gamma = 0.154$ m.
(b) $A_\gamma = 0.992$ m.
(c) $A_\gamma = 0.0385$ m.

DOUBLE-CHECK: The values seem reasonable. The largest value is for the case where the driving angular speed is the resonant speed of the system.

14.58. **THINK:** The maximum displacement will be dictated by the driving force. The maximum speed can be determined by taking the derivative of position with respect to time and using the maximum value. $m = 0.404$ kg, $k = 204.7$ N/m, $F_\text{d} = 29.4$ N and $\omega_\text{d} = 17.1 \text{ s}^{-1}$.

SKETCH:



RESEARCH: The driving force will cause motion of the mass: $A_\text{d}\cos(\omega_\text{d}t)$.

(a) The maximum displacement is given by: $A_\text{d} = \dfrac{F_\text{d}}{m\left(\omega_0^{\,2} - \omega_\text{d}^{\,2}\right)}$, where $\omega_0 = \sqrt{k/m}$.

(b) The maximum speed is given by $\omega_\text{d} A_\text{d}$.

SIMPLIFY: Simplification is not necessary.

CALCULATE:

(a) $\omega_0 = \sqrt{\dfrac{204.7 \text{ N/m}}{0.404 \text{ kg}}} = 22.51 \text{ s}^{-1}$, $A_\text{d} = \dfrac{29.4 \text{ N}}{0.404 \text{ kg}\left(\left(22.51 \text{ s}^{-1}\right)^2 - \left(17.1 \text{ s}^{-1}\right)^2\right)} = 0.3396 \text{ m}$

(b) $v_\text{max} = \omega_\text{d} A_\text{d} = \left(17.1 \text{ s}^{-1}\right)\left(0.3396 \text{ m}\right) = 5.807 \text{ m/s}$

ROUND: Three significant figures:

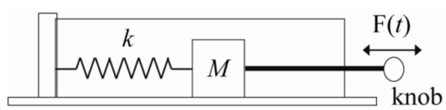
(a) The maximum displacement is $A_\text{d} = 0.340$ m.

(b) The maximum speed is $v_\text{max} = 5.81$ m/s.

DOUBLE-CHECK: The units are appropriate, and the orders of magnitude are reasonable for the given system.

14.59. **THINK:** The amplitude is approximately a maximum when the driving angular speed is equal to the natural angular speed. However, the maximum amplitude occurs for a driving frequency slightly lower than the natural frequency.

SKETCH:



RESEARCH: The amplitude is given by: $A_\gamma = \dfrac{F_\text{d}}{m\sqrt{\left(\omega_0^{\,2} - \omega_\text{d}^{\,2}\right)^2 + 4\omega_\text{d}^{\,2}\omega_\gamma^{\,2}}}$.

A_γ is maximized when $\omega_\text{d} = \sqrt{\omega_0^2 - 2\omega_\gamma^2}$.

SIMPLIFY: We can combine these equations to get

$$A_{\gamma,\max} = \frac{F_\text{d}}{m\sqrt{\left(\omega_0^{\,2} - \left(\omega_0^{\,2} - 2\omega_\gamma^{\,2}\right)\right)^2 + 4\left(\omega_0^{\,2} - 2\omega_\gamma^{\,2}\right)\omega_\gamma^{\,2}}}$$

$$A_{\gamma,\max} = \frac{F_\text{d}}{m\sqrt{\left(2\omega_\gamma^{\,2}\right)^2 + 4\omega_0^{\,2}\omega_\gamma^{\,2} - 8\omega_\gamma^{\,4}}} = \frac{F_\text{d}}{m\sqrt{4\omega_0^{\,2}\omega_\gamma^{\,2} - 4\omega_\gamma^{\,4}}} = \frac{F_\text{d}}{2m\omega_\gamma\sqrt{\omega_0^{\,2} - \omega_\gamma^{\,2}}}.$$

The amplitude is half this value when

$$A_\gamma = \frac{1}{2}A_{\gamma,\max}$$

$$\frac{F_\text{d}}{m\sqrt{\left(\omega_0^2-\omega_\text{d}^2\right)^2+4\omega_\text{d}^2\omega_\gamma^2}} = \frac{F_\text{d}}{4m\omega_\gamma\sqrt{\omega_0^2-\omega_\gamma^2}}$$

$$\frac{1}{\sqrt{\left(\omega_0^2-\omega_\text{d}^2\right)^2+4\omega_\text{d}^2\omega_\gamma^2}} = \frac{1}{4\omega_\gamma\sqrt{\omega_0^2-\omega_\gamma^2}}$$

$$\left(\omega_0^2-\omega_\text{d}^2\right)^2+4\omega_\text{d}^2\omega_\gamma^2 = \left(4\omega_\gamma\right)^2\left(\omega_0^2-\omega_\gamma^2\right)$$

$$\left(\omega_0^2\right)^2-2\omega_0^2\omega_\text{d}^2+\left(\omega_\text{d}^2\right)^2+4\omega_\text{d}^2\omega_\gamma^2 = 16\omega_\gamma^2\left(\omega_0^2-\omega_\gamma^2\right)$$

Let $x = \omega_\text{d}^2$

$$\left(\omega_0^2\right)^2-2\omega_0^2 x+x^2+4x\omega_\gamma^2 = 16\omega_\gamma^2\left(\omega_0^2-\omega_\gamma^2\right)$$

$$x^2+\left(4\omega_\gamma^2-2\omega_0^2\right)x+\left(\omega_0^4-16\omega_\gamma^2\omega_0^2+16\omega_\gamma^4\right)=0$$

Now we can solve a quadratic equation with

$$a=1,\ b=4\omega_\gamma^2-2\omega_0^2,\ \text{and } c=\omega_0^4-16\omega_\gamma^2\omega_0^2+16\omega_\gamma^4.$$

CALCULATE: $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{578\text{ N/m}}{1.60\text{ kg}}} = 19.007\text{ s}^{-1},\quad \omega_\gamma = \frac{b}{2m} = \frac{6.40\text{ kg/s}}{2(1.60\text{ kg})} = 2.000\text{ s}^{-1}$

The driving angular speed is

$$\omega_\text{d} = \sqrt{\left(19.007\text{ s}^{-1}\right)^2-2\left(2.000\text{ s}^{-1}\right)^2} = 18.795\text{ s}^{-1}.$$

The driving frequency is then

$$f_{\text{d,max}} = \frac{\omega_\text{d}}{2\pi} = \frac{18.795\text{ s}^{-1}}{2\pi} = 2.9913\text{ s}^{-1}.$$

The maximum amplitude is then

$$\Rightarrow A_{\gamma,\max} = \frac{F_\text{d}}{2m\omega_\gamma\sqrt{\omega_0^2-\omega_\gamma^2}} = \frac{52.0\text{ N}}{2(1.60\text{ kg})\left(2.000\text{ s}^{-1}\right)\sqrt{\left(19.007\text{ s}^{-1}\right)^2-\left(2.000\text{ s}^{-1}\right)^2}} = 0.4299\text{ m}.$$

Now for the driving angular speed at half amplitude

$$b=4\omega_\gamma^2-2\omega_0^2 = 4\left(2.000\text{ s}^{-1}\right)^2-2\left(19.007\text{ s}^{-1}\right)^2 = -706.5\text{ s}^{-2},$$

$$c=\omega_0^4-16\omega_\gamma^2\omega_0^2+16\omega_\gamma^4$$

$$=\left(19.007\text{ s}^{-1}\right)^4-16\left(19.007\text{ s}^{-1}\right)^2\left(2.000\text{ s}^{-1}\right)^2+16\left(2.000\text{ s}^{-1}\right)^4 = 107{,}638\text{ s}^{-4}.$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-\left(-706.5\text{ s}^{-2}\right)\pm\sqrt{\left(-706.5\text{ s}^{-2}\right)^2-4(1)\left(107{,}638\text{ s}^{-4}\right)}}{2}$$

$$x = 222.30\text{ s}^{-2},\ 484.20\text{ s}^{-1}$$

$$\omega_\text{d}=\sqrt{x}=14.910\text{ s}^{-1}\text{ or }22.006\text{ s}^{-1}.$$

It is necessary that $\omega_\text{d} < \omega_0 = 19.007\text{ s}^{-1}$, so $\omega_\text{d} = 14.910\text{ s}^{-1}$.

The driving frequency for half of the maximum amplitude is

$$f_{\text{d,half}} = \frac{\omega_{\text{d,half}}}{2\pi} = \frac{14.910\text{ s}^{-1}}{2\pi} = 2.3730\text{ s}^{-1}.$$

ROUND: We round to three significant digits: $f_{\text{d,max}} = 2.99\text{ s}^{-1}$, $A_{\gamma,\max} = 0.430\text{ m}$, $f_{\text{d,half}} = 2.37\text{ s}^{-1}$.

DOUBLE-CHECK: If there were no damping, then the maximum amplitude would occur at a driving angular speed equal to the natural angular speed, $\omega_d = \omega_0$, so $f_d = \frac{\omega_0}{2\pi} = \frac{19.007 \text{ s}^{-1}}{2\pi} = 3.03 \text{ s}^{-1}$, which is close to our result including damping. The amplitude at this angular speed would be infinite without any damping. It makes sense that we could decrease the amplitude by driving the system at a frequency below resonance and above resonance.

14.60. When the displacement of a mass on a spring is one half of the amplitude of its oscillation, determine the fraction of the energy that is kinetic energy. The total energy of a spring stretched to the full amplitude of its oscillation is $E_{tot} = K + U = kA^2/2$. When $\Delta x = \frac{A}{2}$, $U = \frac{1}{2}k\left(\frac{A}{2}\right)^2 = \frac{1}{4}\left(\frac{1}{2}kA^2\right)$. Therefore, $K = \frac{3}{4}E_{tot}$.

14.61. The conservation of energy gives $K_1 + U_1 = K_2 + U_2$. For a mass oscillating on a spring, $K_{max} = U_{max} = K_1 + U_1$, since $U = 0$, when $K = K_{max}$ and $K = 0$, when $U = U_{max}$. Therefore, when

$$K = \frac{1}{2}K_{max}, \quad U = \frac{1}{2}U_{max} = \frac{1}{2}\left(\frac{1}{2}kx^2{}_{max}\right) = \frac{1}{2}kx^2,$$

and this is when $x = \frac{x_{max}}{\sqrt{2}}$.

14.62. The oscillator has initial ($t = 0$) displacement of zero and initial velocity $v_0 = J_0/m$. The momentum imparted to the oscillator is equal to the impulse of the kick, the time integral of the force (the spring and damping forces have negligible effect during the infinitesimal duration of the kick). The solution of the weakly damped oscillator with these initial conditions is:

$$x(t) = \frac{J_0/m}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}} e^{-\frac{bt}{2m}} \sin\left(\left(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\right)t\right), \text{ for } t \geq 0, \text{ with } x(t) = 0 \text{ for } t \leq 0.$$

14.63. (a) The frequency is $\omega = \sqrt{k/m} = \sqrt{(1.00 \text{ N/m})/1.00 \text{ kg}} = 1.00 \text{ rad/s}$. If $x(t) = C_0 \sin(\omega t + \theta)$, then at time $t = 0$: $x = 0$, $0 \text{ m} = C_0 \sin\theta$ and $\theta = 0$. $v = \omega C_0 \cos(\omega t + \theta)$ and at $t = 0$: $1.00 \text{ m/s} = (1.00 \text{ rad/s})C_0 \Rightarrow C_0 = 1.00 \text{ m}$. The equation of motion is thus $x(t) = (1.00 \text{ m})\sin[(1.00 \text{ rad/s})t]$.

(b) The frequency is $\omega = \sqrt{k/m} = \sqrt{(1.00 \text{ N/m})/1.00 \text{ kg}} = 1.00 \text{ rad/s}$. Since $x(t) = C_0 \sin(\omega t + \theta)$, at time $t = 0$:

$$x = 0.500 \text{ m} = C_0 \sin\theta \qquad (1)$$

and

$$v = \omega C_0 \cos(\omega t + \theta) \Rightarrow 1.00 \text{ m/s} = (1.00 \text{ rad/s})C_0 \cos\theta. \qquad (2)$$

Dividing (1) by (2) gives $0.500 = \tan\theta \Rightarrow \theta = 26.6°$ or 0.464 rad. Using this in equation (1) yields $C_0 = 1.12 \text{ m}$. Thus, the equation of motion is $x(t) = (1.12 \text{ m})\sin((1.00 \text{ rad/s})t + 0.464 \text{ rad})$.

14.64. $x = A\sin(\omega t)$, $v = \omega A\cos(\omega t)$, $U_s = K \Rightarrow \frac{1}{2}kx^2 = \frac{1}{2}mv^2 \Rightarrow \frac{1}{2}kA^2\sin^2(\omega t) = \frac{1}{2}m\omega^2 A^2\cos^2(\omega t)$

$k = m\omega^2$ so the previous equation becomes:

$$\sin(\omega t) = \cos(\omega t) \Rightarrow \omega t = \frac{\pi}{4} \Rightarrow \omega = \frac{\pi}{4} \text{ rad/s}.$$

14.65. **THINK:** We model the hydrogen atom as two masses connected by a massless spring. Consider two masses m_1 and m_2 located at positions x_1 and x_2 respectively. These two masses are connected by a spring with spring constant k. The equilibrium length of the spring is L. Hooke's Law for a spring tells us that $F=-kx$ and Newton's Second Law tells us that $F=ma$.

RESEARCH: The equations describing the motion of the two masses are

$$m_1a_1 = k(x_2 - x_1 - L)$$
$$m_2a_2 = -k(x_2 - x_1 - L).$$

We can define the quantity $x = x_2 - x_1 - L$, which is the amount by which the spring is stretched or compressed from its equilibrium length. We can then write

$$m_1a_1 = kx$$
$$m_2a_2 = -kx.$$

If we add these two equations we get

$$m_1a_1 + m_2a_2 = kx + (-kx) \quad\Rightarrow\quad m_1a_1 + m_2a_2 = 0.$$

The mass of the system is $M = m_1 + m_2$. The *x*-coordinate of the center of mass of the system is

$$x_{\text{cm}} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}.$$

Because the center of mass of the system is at rest, we can write

$$Ma_{\text{cm}} = 0.$$

We can re-express our first two equations as

$$a_1 = \frac{kx}{m_1}$$
$$a_2 = \frac{-kx}{m_2}.$$

SIMPLIFY: We can subtract the second equation from the first to get

$$a_1 - a_2 = \frac{kx}{m_1} - \frac{-kx}{m_2} = kx\left(\frac{1}{m_1} + \frac{1}{m_2}\right).$$

Remembering that $x = x_2 - x_1 - L$, we can take the time derivative to get

$$\frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(x_2 - x_1 - L) = \frac{d^2}{dt^2}(x_2) - \frac{d^2}{dt^2}(x_2) = a_2 - a_1.$$

Now combining the last two equations we get

$$\frac{d^2x}{dt^2} = -kx\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = -kx\left(\frac{m_1 + m_2}{m_1m_2}\right).$$

We can define the reduced mass μ of our system as

$$\mu = \frac{m_1m_2}{m_1 + m_2}.$$

So we then have

$$\frac{d^2x}{dt^2} + \frac{k}{\mu}x = 0.$$

We recognize this differential equation as having the same form as simple harmonic motion with an angular speed of

$$\omega = \sqrt{\frac{k}{\mu}}.$$

Each hydrogen atom has $m_1 = m_2 = m_H$ so the reduced mass becomes

$$\mu_{\rm H} = \frac{m_{\rm H} m_{\rm H}}{m_{\rm H} + m_{\rm H}} = \frac{m_{\rm H}}{2}.$$

The angular speed then becomes

$$\omega = \sqrt{\frac{k}{\left(\frac{m_{\rm H}}{2}\right)}} = \sqrt{\frac{2k}{m_{\rm H}}}.$$

Solving for the spring constant gives us

$$k = \frac{m_{\rm H}\omega^2}{2}.$$

The period is related to the angular speed as

$$T = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{T}.$$

We can then write

$$k = \frac{m_{\rm H}\left(\frac{2\pi}{T}\right)^2}{2} = \frac{2\pi^2 m_{\rm H}}{T^2}.$$

CALCULATE: Putting in our numbers we get

$$k = \frac{2\pi^2\left(1.7\cdot 10^{-27}\text{ kg}\right)}{\left(8.0\cdot 10^{-15}\text{ s}\right)^2} = 524.3\text{ N/m}.$$

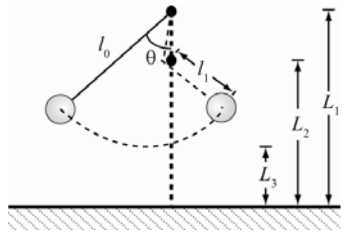
ROUND: Rounding to two significant digits gives us 520 N/m.
DOUBLE-CHECK: Our answer has the correct units. This answer is half what we would get if we incorrectly used $\omega = \sqrt{k/m_{\rm H}}$.

14.66. For critical damping: $x(t) = Be^{-\omega_\gamma t} + tCe^{-\omega_\gamma t}$, $B = x_0$, $C = v_0 + x_0\omega_\gamma$ and $\omega_\gamma = b/2m$. At t = 0: $x_0 = B = 6.41$ cm and $v_0 = 0 \Rightarrow C = x_0\omega_\gamma$. At $t = t_0 = 0.0247$ s:

$$x(t_0) = x_0 e^{-\omega_\gamma t_0} + t_0 x_0 \omega_\gamma e^{-\omega_\gamma t_0} = x_0\left(1 + t_0\omega_\gamma\right)e^{-\omega_\gamma t_0}$$
$$= 6.41\text{ cm}\left[1 + (0.0247\text{ s})\left(72.4\text{ s}^{-1}\right)\right]e^{-\left(72.4\text{ s}^{-1}\right)(0.0247\text{ s})} = 2.99\text{ cm}.$$

14.67. $T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow \left(\frac{T}{2\pi}\right)^2 = \frac{L}{g} \Rightarrow g = L\left(\frac{2\pi}{T}\right)^2 = (0.500\text{ m})\left(\frac{2\pi}{1.50\text{ s}}\right)^2 = 8.77\text{ m/s}^2$

14.68. **THINK:** The angle, $\theta = 14.2°$, is small enough to consider the motion harmonic. The weight of the child is irrelevant. The length values given for the problem are: $L_1 = 9.65$ m, $L_2 = 5.99$ m, $L_3 = 0.47$ m.
SKETCH:



RESEARCH: For half the oscillation period, the swing is a pendulum of length, l_0. For the other half, it is a pendulum of length, l_1. Thus,

$$T=\frac{1}{2}T_0+\frac{1}{2}T_1=\frac{1}{2}(2\pi)\sqrt{\frac{l_0}{g}}+\frac{1}{2}(2\pi)\sqrt{\frac{l_1}{g}}.$$

The length values are given by
$l_0=L_1-L_3,\ l_1=L_2-L_{3.}$

SIMPLIFY:

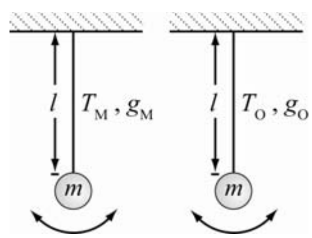
$$T=\pi\left(\sqrt{\frac{L_1-L_3}{g}}+\sqrt{\frac{L_2-L_3}{g}}\right).$$

CALCULATE:

$$T=\pi\left(\sqrt{\frac{9.65\text{ m}-0.47\text{ m}}{9.81\text{ m/s}^2}}+\sqrt{\frac{5.99\text{ m}-0.47\text{ m}}{9.81\text{ m/s}^2}}\right)=\ 5.395\text{ s}.$$

ROUND:
To three significant figures the period is $T = 5.40$ s.
DOUBLE-CHECK: The value is between $T_0 = 6.08$ s and $T_1 = 4.71$ s, so it seems reasonable.

14.69. **THINK:** The difference in the respective periods dictates how long they remain out of phase. The smaller this difference (but not zero!), the longer it takes them to come back into phase. $l = 1.000$ m, $g_\text{M}=9.784\text{ m/s}^2$ and $g_\text{O}=9.819\text{ m/s}^2$.
SKETCH:



RESEARCH: $T=2\pi\sqrt{\dfrac{l}{g}}$

SIMPLIFY: $\Delta T=T_\text{M}-T_\text{O}=2\pi\sqrt{l}\left(\dfrac{1}{\sqrt{g_\text{M}}}-\dfrac{1}{\sqrt{g_\text{O}}}\right)$

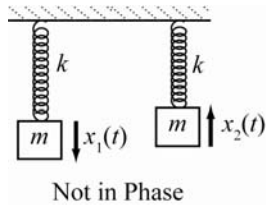
They will be in phase after n oscillations of the Manila pendulum, such that $nT_\text{M}=(n+1)T_\text{O}\Rightarrow n(T_\text{M}-T_\text{O})=T_\text{O}$ and so $n=T_\text{O}/\Delta T$. This will take a time of $t=nT_\text{M}$ to happen.

CALCULATE: $\Delta T=2\pi\sqrt{1.000\text{ m}}\left(\dfrac{1}{\sqrt{9.784\text{ m/s}^2}}-\dfrac{1}{\sqrt{9.819\text{ m/s}^2}}\right)=3.58327\cdot 10^{-3}\text{ s}$

$$n=\frac{2\pi\sqrt{1.000\text{ m}/9.819\text{ m/s}^2}}{3.58327\cdot 10^{-3}\text{ s}}=559.56,\ t=(559.56)2\pi\sqrt{1.000\text{ m}/\left(9.784\text{ m/s}^2\right)}=1124\text{ s}$$

ROUND: $n = 559.6$, $t = 1124$ s.
DOUBLE-CHECK: It takes approximately 19 minutes for the pendulums to come back into phase. This result is reasonable.

14.70. **THINK:** Both the amplitude and the initial velocity (zero) of each pendulum is given. The phase of both springs and their phase difference can be determined. k = 125 N/m, $x_1(0) = -5.00$ cm, $x_2(0.300\text{ s}) = -4.00$ cm, $v_1(0) = 0$, $v_1(0.300\text{ s}) = 0$ and $m = 1.00$ kg.
SKETCH:



RESEARCH: $x(t) = A(\cos\omega t + \gamma)$, $v(t) = -\omega A\sin(\omega t + \gamma)$

SIMPLIFY: $x_1(t) = A_1(\cos\omega t + \gamma_1) \;\Rightarrow\; x_1(0) = -5.00\text{ cm} = A_1\cos\gamma_1 \;\Rightarrow\; A_1 = 5.00\text{ cm}$ and $\gamma_1 = \pi$, so:

$$x_1(t) = 5.00\text{ cm}(\cos\omega t + \pi).$$

$x_2(t) = A_2(\cos\omega t + \gamma_2) \;\Rightarrow\; x_2(0.300\text{ s}) = -4.00\text{ cm} = A_2\cos((0.300\text{ s})\omega + \gamma_2)$

$\Rightarrow\; A_2 = 4.00\text{ cm},\; \gamma_2 = \pi - (0.300\text{ s})\omega,$ so:

$$x_2(t) = 4.00\text{ cm}(\cos\omega t + \pi - (0.300\text{ s})\omega).$$

$$\Delta\gamma = \gamma_1 - \gamma_2 = \pi - (\pi - (0.300\text{ s})\omega) = (0.300\text{ s})\omega = (0.300\text{ s})\sqrt{\frac{k}{m}} \quad \left(\omega = \sqrt{\frac{k}{m}}\right)$$

CALCULATE: $\Delta\gamma = (0.300\text{ s})\sqrt{\frac{125\text{ N/m}}{1.00\text{ kg}}} = 3.354\text{ rad} = 3.354\text{ rad}\left(\frac{180°}{\pi\text{ rad}}\right) = 192.2°$

$\gamma_1 = \pi = 180°,\; \gamma_2 = \gamma_1 - \Delta\gamma = 180° - 192.2° = -12.20°,\; \omega = \sqrt{\frac{125\text{ N/m}}{1.00\text{ kg}}} = 11.18\text{ s}^{-1}$

$x_1(t) = 5.00\text{ cm}\left(\cos(11.18\text{ s}^{-1})t + \pi\right),\; \gamma_2 = \pi - (0.300\text{ s})(11.18\text{ s}^{-1}) = -0.2125\text{ rad},$

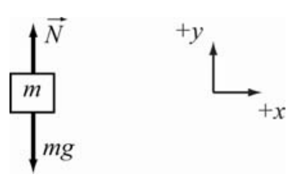
$x_2(t) = 4.00\text{ cm}\left(\cos(11.18\text{ s}^{-1})t - 0.2125\text{ rad}\right)$

ROUND: Rounding to three significant figures, $\Delta\gamma = 192°$, $x_1(t) = 5.00\text{ cm}\left(\cos(11.2\text{ s}^{-1})t + \pi\right)$, and $x_2(t) = 4.00\text{ cm}\left(\cos(11.2\text{ s}^{-1})t - 0.213\text{ rad}\right)$.

DOUBLE-CHECK: Note that the solutions fulfill the initial velocity conditions, as verified below.
$v_1(t) = -\omega A_1\sin(\omega t + \gamma_1),\; \gamma_1 = \pi;\; v_1(0) = -\omega A_1\sin(\pi) = 0;\; v_2(t) = -\omega A_2\sin(\omega t + \gamma_2),\; \gamma_2 = \pi - (0.3\text{ s})\omega$
$v_2(0.3\text{ s}) = -\omega A_2\sin((0.3\text{ s})\omega + \pi - (0.3\text{ s})\omega) = -\omega A_2\sin(\pi) = 0$

14.71. **THINK:** The only forces acting on the car are gravity and the normal force, due to the piston. The car will leave the piston when the piston's acceleration exceeds the acceleration due to gravity. The amplitude is $A = 0.0500$ m.

SKETCH:



RESEARCH: $x(t) = A\cos\omega t,\ v(t) = -\omega A\sin(\omega t),\ a(t) = -\omega^2 A\cos(\omega t)$

SIMPLIFY: $a_{\text{net}} = \omega^2 A - g.$ The car leaves the piston when $a_{\text{net}} = 0$:

$$a_{\text{net}} = 0 = \omega^2{}_{\text{max}} A - g \ \Rightarrow\ \omega_{\text{max}} = \sqrt{\frac{g}{A}}.$$

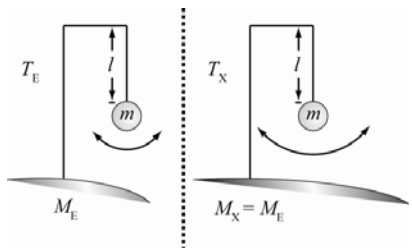
CALCULATE: $\omega_{\text{max}} = \sqrt{\dfrac{9.81 \text{ m/s}^2}{0.05 \text{ m}}} = 14.007 \text{ rad/s},\quad f = \dfrac{\omega}{2\pi} = \dfrac{14.007}{2\pi} = 2.23 \text{ Hz}$

ROUND: f = 2.23 Hz

DOUBLE-CHECK: This result is reasonable.

14.72. **THINK:** The period of the pendulum depends only on its length and the acceleration due to gravity. Knowing the period on planet X and on Earth is enough to determine the acceleration of gravity on planet X. If the acceleration due to gravity and the mass of planet X are known, the radius can be determined. $\text{T}_\text{E} = 0.24 \text{ s}$ and $\text{T}_\text{X} = 0.48 \text{ s}.$

SKETCH:



RESEARCH: $\text{T}_\text{E} = 2\pi\sqrt{\dfrac{L}{g_\text{E}}},\ \text{T}_\text{X} = 2\pi\sqrt{\dfrac{L}{g_\text{X}}},\ g_\text{E} = G\dfrac{M_\text{E}}{R_\text{E}^2},\ \text{g}_\text{X} = G\dfrac{M_\text{X}}{R_\text{X}^2} = G\dfrac{M_\text{E}}{R_\text{X}^2}$

SIMPLIFY:

(a) $\dfrac{\text{T}_\text{E}}{\text{T}_\text{X}} = \sqrt{\dfrac{g_\text{X}}{g_\text{E}}} \ \Rightarrow\ g_\text{X} = g_\text{E}\dfrac{\text{T}_\text{E}^2}{\text{T}_\text{X}^2}$

(b) $\dfrac{g_\text{E}}{g_\text{X}} = \dfrac{G\dfrac{M_\text{E}}{R_\text{E}^2}}{G\dfrac{M_\text{E}}{R_\text{X}^2}} = \dfrac{R_\text{X}^2}{R_\text{E}^2} \ \Rightarrow\ R_\text{X} = R_\text{E}\sqrt{\dfrac{g_\text{E}}{g_\text{X}}}$

CALCULATE:

(a) $g_\text{X} = \left(9.81 \text{ m/s}^2\right)\left(\dfrac{0.24 \text{ s}}{0.48 \text{ s}}\right)^2 = 2.4525 \text{ m/s}^2$

(b) $R_\text{X} = R_\text{E}\sqrt{\dfrac{9.81 \text{ m/s}^2}{2.4525 \text{ m/s}^2}} = 2.00R_\text{E}$

The radius of planet X is twice the radius of Earth.

ROUND:

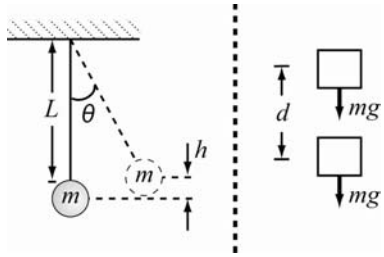
(a) $g_\text{X} = 2.5 \text{ m/s}^2$

(b) $R_\text{X} = 2.0R_\text{E}$

DOUBLE-CHECK: The period on planet X is longer, so it is expected that the gravitational acceleration is less than on Earth. Since the masses of the two planets are the same, it is expected that planet X is less dense or has a larger radius.

14.73. **THINK:** A pendulum has a period of 2.00 s and the mass of the bob is 0.250 kg. A weight slowly falls to provide the energy to overcome the frictional damping of the pendulum. The mass of the weight is 1.00 kg and it moves down 0.250 m every day. The Q factor of the pendulum must be determined.

SKETCH:



RESEARCH: The Q factor is defined by $Q = 2\pi E/|\Delta E|$, where E is the energy of the pendulum and ΔE is the energy loss. The energy of the pendulum is determined using the maximum height, h, by $E = mgh = mgL(1-\cos\theta)$. From the period of the pendulum, it is found that:

$$T^2 = 4\pi^2\frac{L}{g} \text{ or } L = \frac{gT^2}{4\pi^2}.$$

SIMPLIFY: Thus, the energy is given by: $E = mg\left(\frac{gT^2}{4\pi^2}\right)(1-\cos\theta) = \frac{mg^2T^2}{4\pi^2}(1-\cos\theta)$. The energy loss in one period of oscillation is: $\Delta E = Mgx = Mg\frac{d}{24(3600\text{ s})/(2.00\text{ s})}$. Thus, the Q factor is:

$$Q = 2\pi\frac{mg^2T^2}{4\pi^2}(1-\cos\theta)\left(\frac{43200}{Mgd}\right) = 43200\left(\frac{m}{M}\right)\frac{gT^2}{2\pi d}(1-\cos\theta).$$

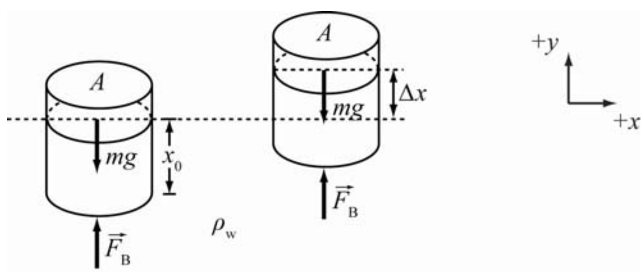
CALCULATE: $Q = 43200\left(\frac{0.250\text{ kg}}{1.00\text{ kg}}\right)\frac{(9.81\text{ m/s}^2)(2.00\text{ s})^2}{2\pi(0.250\text{ m})}(1-\cos 10.0°) = 4098.78$

ROUND: Keeping three significant figures, $Q = 4.10\cdot 10^3$.

DOUBLE-CHECK: The Q factor is a dimensionless quantity. $\frac{[\text{kg}]}{[\text{kg}]}\frac{[\text{m/s}^2][\text{s}^2]}{[\text{m}]} \Rightarrow \frac{[\text{kg}][\text{m}][\text{s}]}{[\text{kg}][\text{m}][\text{s}]}$. All the units cancel. The result is reasonable.

14.74. **THINK:** The restoring forces in this problem are the buoyant force and the gravitational force.

SKETCH:



RESEARCH: The buoyant force is given by $F_B = \rho_w gV$. Note that V is the volume of water displaced by the can, i.e. the fractional volume of the can that is below the water. Thus, $F_B = \rho_w gAx_0$. In equilibrium, the buoyant force must be equal to the gravitational force; $F_g = mg = \rho_w gAx_0$. If the can is lifted by a distance, Δx, the restoring force is $F = -mg + F_B = -mg + \rho_w gA(x_0 - \Delta x) = -mg + \rho_w gAx_0 - \rho_w gA\Delta x$.

SIMPLIFY: Since $mg = \rho_w gAx_0$, the restoring force is $F = -\rho_w gA\Delta x = -k\Delta x$. Thus, $k = \rho_w gA$. Using $\omega = \sqrt{k/m}$, the angular frequency is

$$\omega = \sqrt{\frac{\rho_w gA}{m}} \text{ and } A = \frac{\pi d^2}{4}.$$

Therefore, the equation for the displacement is $x = B\cos(\omega t)$, where B is the initial displacement.

CALCULATE: $\omega = \sqrt{\dfrac{\left(1000 \text{ kg/m}^3\right)\left(9.81 \text{ m/s}^2\right)\pi\left(0.100 \text{ m}\right)^2/4}{0.8000 \text{ kg}}} = 9.81 \text{ rad/s}.$

Therefore, $x(t) = (1.00 \text{ cm})\cos(9.81 \text{ rad/s})t$. The period of oscillation is:

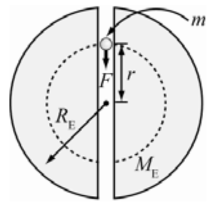
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{9.81 \text{ rad/s}} = 0.641 \text{ s}.$$

ROUND: Keeping two significant figures, $T = 0.64$ s.

DOUBLE-CHECK: This result is consistent with everyday observations of small objects bobbing in the water.

14.75. **THINK:** The restoring constant, k, must be determined in order to determine the period of oscillation.

SKETCH:



RESEARCH: The force due to gravity at a distance, r, from the center of the Earth is:

$$F = \frac{GmM_{\text{in}}}{r^2},$$

where M_{in} is the mass inside the spherical volume with radius, r. So,

$$F = \frac{Gm\rho_{\text{E}}}{r^2}\left(\frac{4}{3}\pi r^3\right) = Gm\rho_{\text{E}}\left(\frac{4}{3}\pi r\right).$$

SIMPLIFY: Since $\rho_{\text{E}} = M_{\text{E}} / \left(4\pi R_E{}^3/3\right)$ and $F = kr$, the restoring constant is:

$$k = \left(\frac{4\pi Gm}{3}\right)\left(\frac{3M_E}{4\pi R_E{}^3}\right) = G\frac{mM_E}{R_E^3}, \text{ thus}$$

$$T = 2\pi\sqrt{\frac{m}{\left(\dfrac{GmM_{\text{E}}}{R_E{}^3}\right)}} = 2\pi\sqrt{\frac{R_E{}^3}{GM_{\text{E}}}}.$$

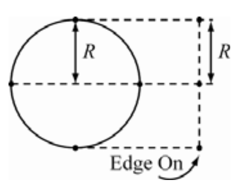
CALCULATE: $T = 2\pi\sqrt{\dfrac{\left(6.37\cdot 10^6 \text{ m}\right)^3}{6.67\cdot 10^{-11} \text{ Nm}^2/\text{kg}^2\left(5.98\cdot 10^{24} \text{ kg}\right)}} = 5057 \text{ s}$

ROUND: Rounding the result to three significant figures gives $T = 5060$ s.

DOUBLE-CHECK: The period T is proportional to the ratio of the radius to the acceleration due to gravity. That is, $T \propto R/\text{g}$. Since $\dfrac{R_{\text{E}}}{\text{g}_{\text{E}}} < \dfrac{R_{\text{M}}}{\text{g}_{\text{M}}}$, the period through the Earth is expected to be less than the period through the Moon.

14.76. **THINK:** The period of oscillation of an object is related to the spring constant, k.

SKETCH:



RESEARCH: From Kepler's third law:

$$T^2 = \frac{4\pi^2}{GM}R^3,$$

where R is the radius of revolution and M is the mass of an attracting body, which in this case is a star. The effective "spring constant" can be determined using an expression for a period of oscillation:

$$T = 2\pi\sqrt{\frac{m}{k}} \text{ or } T^2 = 4\pi^2\frac{m}{k}.$$

SIMPLIFY:

(a) $k = 4\pi^2\dfrac{m}{T^2} = 4\pi^2\dfrac{m}{\left(\dfrac{4\pi^2}{GM}R^3\right)} = \dfrac{GMm}{R^3}$

(b) The angular frequency of oscillation is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\sqrt{\dfrac{4\pi^2}{GM}R^3}} = \sqrt{\frac{GM}{R^3}}.$$

The amplitude of the oscillation is R. Therefore, the displacement of the observed planet is:

$$x(t) = R\sin(\omega t) = R\sin\left(\sqrt{\frac{GM}{R^3}}t\right).$$

Taking the first derivative of this equation yields the velocity:

$$v(t) = \frac{dx}{dt} = R\sqrt{\frac{GM}{R^3}}\cos\left(\sqrt{\frac{GM}{R^3}}t\right) = \sqrt{\frac{GM}{R}}\cos\left(\sqrt{\frac{GM}{R^3}}t\right).$$

The speed of the planet in orbit is equal to the maximum speed in the oscillatory motion. Therefore,

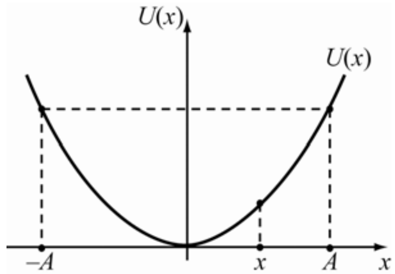
$$v = v_{\max} = \sqrt{\frac{GM}{R}}.$$

CALCULATE: Not necessary.
ROUND: Not necessary.
DOUBLE-CHECK: Using a centripetal acceleration equal to an acceleration of gravity, it is obtained:

$$\frac{v^2}{R} = \frac{GM}{R^2}.$$

This simplifies to $v = \sqrt{\dfrac{GM}{R}}$, which is equivalent to the above.

14.77. **THINK:** A restoring force of an oscillator is related to its potential energy.
SKETCH:



RESEARCH: The restoring force at a position, *x*, is given by $F(x)=-dU(x)/dx$. Thus, the potential is $U(x)=\int_{x_0}^{x}-F(x)dx$. The velocity of a mass *m* at the position *x* is determined using the conservation of energy. The energy at *x* is equal to the energy at *x* = *A*, so, $K_\text{f}+U_\text{f}=K_\text{i}+U_\text{i}$. Using $K_\text{i}=0$, the equation becomes $(mv^2/2)+U(x)=0+U(A)$.

SIMPLIFY: $v^2=2\frac{(U(A)-U(x))}{m}$ or $v=\frac{dx}{dt}=\sqrt{\frac{2}{m}(U(A)-U(x))}\Rightarrow dt=\left[\frac{2}{m}(U(A)-U(x))\right]^{-1/2}dx.$

Integrating both sides with intervals $x\in(-A,A)$ and $t\in(0,T/2)$ yields:

$$\int_0^{T/2}dt=\int_{-A}^{A}\left[\frac{2}{m}(U(A)-U(x))\right]^{-1/2}dx.$$

Thus, the period of oscillation is:

$$T=2\int_{-A}^{A}\left[\frac{2}{m}(U(A)-U(x))\right]^{-1/2}dx \qquad (1).$$

(a) Substituting $F(x)=-cx^3$ into $U(x)=\int_{x_0}^{x}-F(x)dx$ gives: $U(x)=\int_{x_0}^{x}cx^3dx=\frac{c}{4}x^4-\frac{c}{4}{x_0}^4$. For simplicity, it is assumed that $x_0=0$. The potential is therefore given by $U(x)=cx^4/4$. Thus, the expression for the period is:

$$T=2\int_{-A}^{A}\left[\frac{c}{2m}(A^4-x^4)\right]^{-1/2}dx.$$

(b) Changing the variable, *x*, with *x* = *Ay* yields: $T=2\int_{-1}^{1}\left[\frac{c}{2m}A^4(1-y^4)\right]^{-1/2}A\,dy$. Simplifying the previous expression gives: $T=2\sqrt{\frac{2m}{c}}\frac{1}{A}\int_{-1}^{1}(1-y^4)^{-1/2}dy=\frac{B}{A}$, where *B* is a constant. Therefore, the period is inversely proportional to *A*.

(c) Substituting $U(x)=\frac{\gamma}{\alpha}|x|^\alpha$ into equation (1) yields:

$$T=2\int_{-A}^{A}\left[\frac{2}{m}\left(\frac{\gamma}{\alpha}|A|^\alpha-\frac{\gamma}{\alpha}|x|^\alpha\right)\right]^{-1/2}dx.$$

Similarly as above, changing the variable, *x*, with *x* = *Ay* yields:

$$T=2\int_{-1}^{1}\left[\frac{2}{m}\frac{\gamma}{\alpha}|A|^\alpha\right]^{-1/2}(1-y^\alpha)^{-1/2}A\,dy.$$

Thus, $T=2\sqrt{\frac{\alpha m}{2\gamma}}A^{\left(1-\frac{\alpha}{2}\right)}\int_{-1}^{1}(1-y^\alpha)^{-1/2}dy=BA^{\left(1-\frac{\alpha}{2}\right)}$. The period is proportional to $A^{\left(1-\frac{\alpha}{2}\right)}$.

CALCULATE: Not necessary.

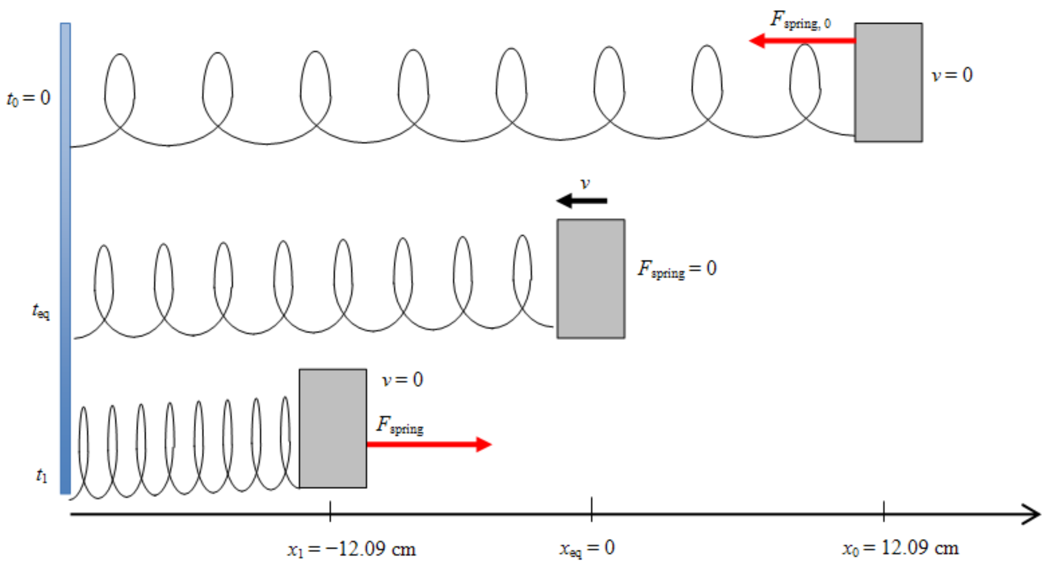
ROUND: Not necessary.

DOUBLE-CHECK: If $\alpha = 2$, T is constant and independent of A. If $\alpha = 4$, then $T = B/A$, which is the same as the result in part (b) .

Multi-Version Exercises

14.78. **THINK:** This problem involves a block and spring assembly, where the block is sliding back and forth on a frictionless surface. The block in this problem undergoes simple harmonic motion. The mass of the block, spring constant, and displacement are given, so it should be possible to find the displacement as a function of time, using the equations for the motion of a mass on a spring with no damping.

SKETCH: Show the displacement, velocity, and the force on the block due to the spring. Consider only the motion of the block in the x-direction, since there is no net force in the y-direction, and no friction force. Take the start time $t = 0$ to be the moment the block is released, and the equilibrium position of the spring to be the origin where $x = 0$.



RESEARCH: The equation of motion for a mass on a spring with no damping is $x(t) = A\sin(\omega_0 t + \theta_0)$, where A is the amplitude of the oscillation, ω_0 is the angular speed, and θ_0 is the phase angle. The angular speed can be found from the mass of the block and the spring constant using the equation $\omega_0 = \sqrt{\frac{k}{m}}$, where k is the spring constant and m is the mass. The amplitude of the oscillation is equal to the maximum stretch of the spring, which is the initial position of the block.

SIMPLIFY: The location of the block at time $t = 0$ is $A = 12.09$ cm, so $\theta_0 = \sin^{-1}(1) = \frac{\pi}{2}\text{rad}$. Substituting for the angular speed, the equation for the motion of the block becomes $x(t) = A\sin\left(\sqrt{\frac{k}{m}}t + \theta_0\right)$.

CALCULATE: The amplitude of the oscillation is $A = 12.09$ cm $= 0.1209$ m, the block has mass 1.605 kg, and the spring constant is 14.55 N/m. The phase angle is $\theta_0 = \frac{\pi}{2}\text{rad}$. The location of the block at time $t = 2.834 \cdot 10^{-1}$ s is:

$$x\left(2.834\cdot 10^{-1}\text{ s}\right)=A\sin\left(\sqrt{\frac{k}{m}}\left[2.834\cdot 10^{-1}\right]+\theta_0\right)$$
$$=0.1209\text{ m}\cdot\sin\left(\sqrt{\frac{14.55\text{ N/m}}{1.605\text{ kg}}}\cdot 2.834\cdot 10^{-1}\text{ s}+\frac{\pi}{2}\right)$$
$$=7.949320781\cdot 10^{-2}\text{ m}$$
$$=7.949320781\text{ cm}$$

ROUND: All of the numbers in this calculation had four significant figures, so the final answer will also have four figures. After $2.834\cdot 10^{-1}$ s, the block is 7.949 cm from the equilibrium position, in the same direction that the mass had been pulled at the start of the experiment.

DOUBLE-CHECK: The period of this motion is $\frac{2\pi}{\omega_0}=2.087\text{sec}$. Time $t=2.834\cdot 10^{-1}$ s is less than one fourth of the total period, so it is expected that the mass is between the fully stretched position and the equilibrium position. In fact, since the time is closer to one eighth of the period, it is expected that the block will be less than ½ way from the fully stretched position to the equilibrium position, somewhere close to (but more than) 6 cm from the equilibrium position. These estimates confirm that the calculated answer is reasonable.

14.79. **THINK:** The system is undergoing simple harmonic motion. We are given $m=1.833$ kg, $k=14.97$ N/m, and $d=13.37$ cm. We are asked for the time t so that $x(t)=4.990$ cm.

RESEARCH: The general equation for simple harmonic motion is $x(t)=A\sin(\omega_0 t+\theta_0)$, with $\omega_0=\sqrt{\frac{k}{m}}$.

SIMPLIFY: As in 14.78 we arrive at an equation for position: $x(t)=d\sin\left(\sqrt{\frac{k}{m}}t+\frac{\pi}{2}\right)=d\cos\left(\sqrt{\frac{k}{m}}t\right)$.

Solving for t yields $t=\sqrt{\frac{m}{k}}\cos^{-1}\left(\frac{x}{d}\right)$.

CALCULATE: $t=\sqrt{\frac{1.833\text{ kg}}{14.97\text{ N/m}}}\cos^{-1}\left(\frac{4.990\text{ cm}}{13.37\text{ cm}}\right)=4.1582\cdot 10^{-1}\text{ s}.$

ROUND: To four significant figures: $t=4.158\cdot 10^{-1}$ s.

DOUBLE-CHECK: The units of seconds are correct (recall that N/m = kg/s^2) and the value seems reasonable. It is a little less than half a second.

14.80. **THINK:** The system is undergoing simple harmonic motion. We are given $m=1.061$ kg, $k=15.39$ N/m, and $x\left(t=3.900\cdot 10^{-1}\text{ s}\right)=1.250$ cm. We are asked for the distance d that the block was initially displaced from equilibrium.

RESEARCH: The general equation for simple harmonic motion is $x(t)=A\sin(\omega_0 t+\theta_0)$, with $\omega_0=\sqrt{\frac{k}{m}}$.

SIMPLIFY: As in 14.78 we arrive at an equation for position: $x(t)=d\sin\left(\sqrt{\frac{k}{m}}t+\frac{\pi}{2}\right)=d\cos\left(\sqrt{\frac{k}{m}}t\right)$.

Solving for d yields $d=\frac{x}{\cos\left(\left(\sqrt{k/m}\right)t\right)}$.

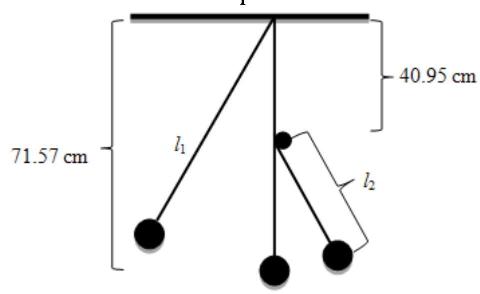
CALCULATE: $d = \dfrac{1.250 \text{ cm}}{\cos\left(\left(\sqrt{15.39 \text{ N/m}/1.061 \text{ kg}}\right)0.3900 \text{ s}\right)} = 14.645 \text{ cm}.$

ROUND: To four significant figures: $d = 14.65$ cm.

DOUBLE-CHECK: The units of d are correct and the value of the displacement seems reasonable.

14.81. **THINK:** This problem involves the motion of a pendulum, but the motion is restricted. For half of its motion, the pendulum will have a period corresponding to the whole length of the string. For the other half of its motion, it will have a period corresponding to the length of the string minus the distance of the peg from the ceiling.

SKETCH: The sketch shows the motion of the pendulum:



RESEARCH: The period of a pendulum hanging from a string of length l is $T_l = 2\pi\sqrt{\dfrac{l}{g}}$. The pendulum swings for half of the period corresponding the full length of the string l_1, with a half-period of $\dfrac{1}{2}T_1 = \dfrac{1}{2}2\pi\sqrt{\dfrac{l_1}{g}}$. Similarly, it swings for half of the period corresponding to the length of the string minus the distance of the peg from the ceiling, $\dfrac{1}{2}T_2 = \dfrac{1}{2}2\pi\sqrt{\dfrac{l_2}{g}}$. The total period of the pendulum is the sum of the swing corresponding to length l_1 and the swing corresponding to length l_2, for a total period of $T = \dfrac{1}{2}T_1 + \dfrac{1}{2}T_2$.

SIMPLIFY: The goal is to find the total period. Substitute the expressions for the periods corresponding to length l_1 and l_2 to find the total period $T = \dfrac{1}{2}2\pi\sqrt{\dfrac{l_1}{g}} + \dfrac{1}{2}2\pi\sqrt{\dfrac{l_2}{g}}$. This can be simplified to $T = \dfrac{\pi\left(\sqrt{l_1} + \sqrt{l_2}\right)}{\sqrt{g}}$, where l_1 is the full length of the string, and l_2 is the length of the string minus the distance from the peg to the ceiling.

CALCULATE: The question states that the length of the string is l_1 = 71.57 cm = 0.7157 m. The distance from the peg to the ceiling is 40.95 cm, so the length corresponding to the second period is l_2 = 71.57 cm − 40.95 cm = 30.62 cm = 0.3062 m. The gravitational acceleration near the surface of the Earth is 9.81 m/s^2. The total period is then

$$T = \frac{\pi\left(\sqrt{l_1} + \sqrt{l_2}\right)}{\sqrt{g}} = \frac{\pi\left(\sqrt{0.7157 \text{ m}} + \sqrt{0.3062 \text{ m}}\right)}{\sqrt{9.81 \text{ m/s}^2}} = 1.403588643 \text{ s}.$$

ROUND: The length of the string, the distance from the peg to the ceiling, and their difference all have four significant figures, so the final answer should have four significant figures. The pendulum has a period of 1.404 s.

DOUBLE-CHECK: Imagine that there were two pendulums. A pendulum on a string of length 0.7157 m has a period of 1.697 seconds, while a pendulum on a string of length 0.3062 m has a period of 1.110 seconds. The average of these two periods is 1.404 s. A pendulum swinging for half a cycle on a string of one length and half a cycle on a string of a second length will have a period equal to the average of the periods corresponding to the two lengths (it will NOT have the same period as a string of the average of the two lengths).

14.82.

$$T=\frac{\pi}{\sqrt{g}}\left(\sqrt{\ell_1}+\sqrt{\ell_2}\right)$$

$$T=\frac{\pi}{\sqrt{g}}\left(\sqrt{\ell_1}+\sqrt{\ell_1-h}\right)$$

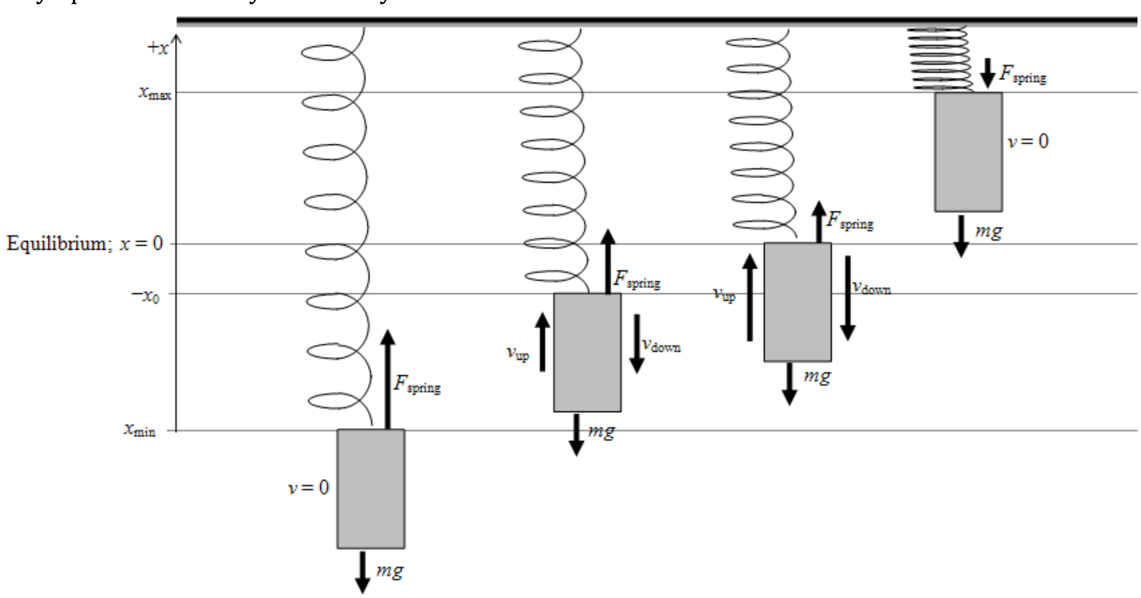
$$\frac{T\sqrt{g}}{\pi}=\sqrt{\ell_1}+\sqrt{\ell_1-h}$$

$$\sqrt{\ell_1-h}=\frac{T\sqrt{g}}{\pi}-\sqrt{\ell_1}$$

$$h=\ell_1-\left(\frac{T\sqrt{g}}{\pi}-\sqrt{\ell_1}\right)^2=0.7239\text{ m}-\left(\frac{(1.404\text{ s})\sqrt{9.81\text{ m/s}^2}}{\pi}-\sqrt{0.7239\text{ m}}\right)^2=0.4226\text{ m}=42.26\text{ cm}$$

14.83. **THINK:** The speed of the object attached to the spring depends only on the distance from the equilibrium position. Since it does not matter if the object is above the equilibrium, below the equilibrium, moving up, or moving down, it is easiest to solve this problem using conservation of energy.

SKETCH: The object is shown at four times: when the spring is stretched down before being released (x_{min}), 1.849 cm below equilibrium ($-x_0$), at equilibrium ($x = 0$), and at maximum height above equilibrium (x_{max}). The spring and gravitational forces are shown. The velocity of the object on the way up and its velocity on the way down are both shown.



RESEARCH: The potential energy stored in a spring is $U_\text{S}=\frac{1}{2}kx^2$, and the kinetic energy of the mass is $K=\frac{1}{2}mv^2$. The total mechanical energy of the mass on the spring is $E=\frac{1}{2}kA^2$. The total mechanical energy should also equal the sum of the kinetic energy and the potential energy ($E=U_\text{S}+K$). The spring constant is given in Newtons per meter and the displacements are given in centimeters, so the conversion 100 cm = 1 m will be needed.

SIMPLIFY: Since there are two expressions for the total mechanical energy, set them equal to one another to get $\frac{1}{2}kA^2=U_\text{S}+K$. Then, substitute in the expressions for the potential energy and kinetic energy to get $\frac{1}{2}kA^2=\frac{1}{2}kx^2+\frac{1}{2}mv^2$. Finally, solve for the speed of the mass:

$$\frac{1}{2}kA^2=\frac{1}{2}kx^2+\frac{1}{2}mv^2$$
$$mv^2=kA^2-kx^2$$
$$v^2=\frac{kA^2-kx^2}{m}$$
$$v=\sqrt{\frac{kA^2-kx^2}{m}}$$

CALCULATE: Since the spring was stretched and the object released from rest, the maximum distance from the equilibrium point (the amplitude) is equal to the distance at which it was released. So the amplitude A = 18.51 cm = 0.1851 m. The spring constant k is 23.31 N/m, and the object has a mass of 1.375 kg. The goal is to find the velocity when the mass is a distance of 1.849 cm = 0.01849 meters from the equilibrium point, so $x=\pm0.01849$ m. Using these values,

$$\begin{aligned}v&=\sqrt{\frac{kA^2-kx^2}{m}}\\&=\sqrt{\frac{23.31\text{ N/m}\cdot(0.1851\text{ m})^2-23.31\text{ N/m}\cdot(\pm0.01849\text{ m})^2}{1.375\text{ kg}}}\\&=0.7583130694\text{ m/s}\end{aligned}$$

ROUND: The values used in this calculation all have four significant figures, so the final answer should also have four figures. When it is 1.849 cm from the equilibrium point, the mass has a speed of 0.7583 m/s.

DOUBLE-CHECK: The speed of the mass is 0 m/s when it is at the bottom or top of its oscillations. The maximum speed of the mass occurs when the mass passes the equilibrium point ($x=0$). At this point, the mass achieves a speed of $0.1851\text{ m}\sqrt{\frac{23.31\text{ N/m}}{1.375\text{ kg}}}=0.7621\text{ m/s}$. This is slightly faster than the speed of the mass when it is 1.849 cm from the equilibrium point, but not by much, confirming that the answer of 0.7583 m/s is reasonable.

14.84.
$$v=\sqrt{\left(A^2-x^2\right)\frac{k}{m}}$$
$$v^2=\left(A^2-x^2\right)\frac{k}{m}$$
$$m=\left(A^2-x^2\right)\frac{k}{v^2}=\left([0.1979\text{ m}]^2-[0.07417\text{ m}]^2\right)\frac{23.51\text{ N/m}}{(0.7286\text{ m/s})^2}=1.491\text{ kg}$$

14.85.

$$v=\sqrt{\left(A^2-x^2\right)\frac{k}{m}}$$

$$v^2=\left(A^2-x^2\right)\frac{k}{m}$$

$$\frac{mv^2}{k}=A^2-x^2$$

$$A=\sqrt{\frac{mv^2}{k}+x^2}=\sqrt{\frac{(1.103\text{ kg})(0.4585\text{ m/s})^2}{23.73\text{ N/m}}+(0.04985\text{ m})^2}=0.1107\text{ m}$$

Chapter 15: Waves

Concept Checks

15.1. a **15.2.** c **15.3.** e **15.4.** a **15.5.** e **15.6.** b, c, a

Multiple-Choice Questions

15.1. a **15.2.** d **15.3.** c **15.4.** e **15.5.** d **15.6.** a **15.7.** a **15.8.** a **15.9.** c **15.10.** c

Conceptual Questions

15.11. (a) A transverse wave has a displacement perpendicular to the line of the Slinky. A simple motion up or down would produce this type of wave.
(b) A longitudinal wave travels along the line of the slinky. To create this type of motion, the Slinky should be pushed towards the friend.

15.12. The speed of the wave traveling along the wire is $v=\sqrt{T/\mu}$. A change in the area does not change the tension, but the linear mass does change. If the area increases, the linear mass decreases and thus the speed decreases. The wavelength is also related to the speed of the wave by the relation, $v=\lambda f$. As the speed changes, the frequency does not, so the wavelength must follow the change of speed. At the interface there will be a transmitted and reflected wave. The reflected wave will have a smaller amplitude than the incident wave and will be inverted (undergoes a 180° phase shift). A phase change will not occur for the transmitted wave, but a decrease in amplitude will occur when the wave crosses the boundary thin to thick.

15.13. For two waves to interfere with each other, the waves must have almost equal frequencies and constant phase. Since noise is composed of various frequencies, amplitudes and phases, it is unlikely that more than a few frequencies of the noise interfere. Thus, interference of noise between two sources is unlikely.

15.14. Since some of the sound is reflected off the walls, floor and ceiling, the intensity is the sum of the direct intensity and reflected sound. The sound falls off more slowly than $1/R^2$. If the sound is considered to be perfectly reflected, the sound intensity is constant except for points close to the source.

15.15. It is possible for two standing waves to create a traveling wave. If both standing waves have the same amplitude and frequency, $y_1(x,t)=A\cos wt\sin kx$ and $y_2(x,t)=A\cos(wt-\pi/2)\sin(kx+\pi/2)$, they can be added as follows:

$$\begin{aligned}y(x,t)=y_1(x,t)+y_2(x,t)&=A\cos wt\sin kx+A\cos(wt-\pi/2)\sin(kx+\pi/2)\\&=A\cos wt\sin kx-A\sin wt\cos kx=A\sin(kx-wt).\end{aligned}$$

This is a traveling wave propagating to the right. Standing waves and traveling waves are at the base of the wave phenomena.

15.16. The only ways for the ball to be stationary is for it to be at the node of a standing wave or at an interference point of two or more interfering waves. On a lake it is very unlikely that a standing wave is created. To create points of interference more than one wave is needed. There is no situation where one wave will allow the ball to stay in a stationary position.

15.17. There are two effects that contribute to the decrease of the amplitude. The first effect is due to the energy being spread over a greater line as the distance from the source increases. The energy of the wave is directly proportional to the square of the amplitude. Thus, as the energy decreases, so should the amplitude. The second effect is due to the damping of the wave. This also dissipates the mechanical energy, converting it to sound and heat, thus the amplitude decreases.

15.18. The speed of a point on a wave is given by $v_p(x,t) = -A\kappa v \sin(\kappa(x - vt) + \phi)$, where the wave the number is $\kappa = 2\pi / \lambda$. The maximum speed occurs when the cosine function is equal to negative one. Thus, the maximum speed is $v_{\text{max}} = A\kappa v = 2\pi A v / \lambda$.

15.19. The tightrope walker will fall as soon as the "signal" that the rope was cut reaches the walker. This takes a length of time given by $t = d/v$, where d is the distance to the walker, 0.5 mi, and v is the speed at which the signal travels. To know the speed, one would need to know the rope's linear mass density, μ, and also the rope tension, F; then the speed could be estimated as $v = \sqrt{F/\mu}$ using the assumption that the signal is a transverse wave. In reality the signal would be a more complicated motion, having both transverse and longitudinal components, which is why the calculation could only be an estimate.

15.20. The wave is longitudinal, meaning that it travels along the same axis as the cars—just as a sound wave travels through air along the same axis as the forward-backward motion of the air molecules. Also, just as a sound wave propagates through the action of air molecules with adjacent air molecules, the wave of moving cars propagates as each car begins to follow the one in front of it. The speed of propagation is set by the distance between cars and the time it takes for each car to start moving after the one in front of it has started moving.

15.21.

$$F_i = F_+ + F_- = -F\sin\theta_+ - F\sin\theta_-$$
$$ma_i = -F(\sin\theta_+ + \sin\theta_-)$$
$$\sin\theta_+ = \frac{y_i - y_{i+1}}{\Delta x}$$
$$\sin\theta_- = \frac{y_i - y_{i-1}}{\Delta x}$$
$$ma_i = \frac{F}{\Delta x}(y_{i+1} - 2y_1 + y_{i-1})$$
$$y_{i+1} - 2y_1 + y_{i-1} = \frac{ma_i(\Delta x)}{F}$$
$$\frac{ma_i(\Delta x)}{F(\Delta x)^2} = \frac{\partial^2}{\partial x^2} y(x,t)$$
$$ma_i = F(\Delta x)\frac{\partial^2}{\partial x^2} y(x,t)$$
$$\frac{\partial^2}{\partial t^2} y(x,t) = \frac{F(\Delta x)}{m}\frac{\partial^2}{\partial x^2} y(x,t) = \frac{F}{\mu}\frac{\partial^2}{\partial x^2} y(x,t)$$

So $v = \sqrt{F/\mu}$.

15.22. We will not derive the most general case but rather focus on the case of a wave traveling along a string. Therefore, assume a wave of the form $y(x,t) = A\sin(\kappa x - \omega t + \phi_0)$.

The kinetic energy is given by

$$K = \frac{1}{2}m\left(\frac{\partial y(x,t)}{\partial t}\right)^2 = \frac{1}{2}m\left(-\omega A\cos(\kappa x - \omega t + \phi_0)\right)^2$$

$$K=\frac{1}{2}m\omega^2A^2\cos^2\left(\omega t\right).$$

The time-average is

$$\Delta K_{\text{ave}}=\frac{1}{4}m\omega^2A^2.$$

The potential energy is given by the work done by the tension: F times the distance the mass moves in the same direction, $W=\Delta U=F\Delta l.$ We can see from the sketch in the preceding solution that $\left(\Delta s\right)^2=\left(\Delta x\right)^2+\left(\Delta y\right)^2$. The distance the mass moves is $\Delta l=\Delta s-\Delta x.$ We assume that the deflection of the string is small, which allows us to write

$$\left(\Delta s\right)^2=\left(\Delta x\right)^2+\left(\Delta y\right)^2=\left(\Delta x\right)^2\left(1+\left(\frac{\Delta y}{\Delta x}\right)^2\right).$$

Because $\Delta y/\Delta x\ll 1,$ we can write

$$\Delta s=\Delta x\sqrt{1+\left(\Delta y/\Delta x\right)^2}\approx\Delta x\left(1+\frac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\right)$$

$$\Delta s\approx\Delta x+\frac{\left(\Delta y\right)^2}{2\left(\Delta x\right)}$$

$$\Delta l=\Delta s-\Delta x=\frac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\left(\Delta x\right).$$

We can then write the potential energy as

$$U=F\Delta l=\frac{F}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\Delta x.$$

We can calculate $\Delta y/\Delta x$

$$\frac{\Delta y}{\Delta x}=\kappa A\cos\left(\kappa x-\omega t+\phi_0\right).$$

$$U=F\Delta l=\frac{F}{2}\left(\kappa A\cos\left(\kappa x-\omega t+\phi_0\right)\right)^2\Delta x$$

$$U=F\Delta l=\frac{F}{2}\kappa^2A^2\cos^2\left(\kappa x-\omega t+\phi_0\right)\Delta x.$$

The time-average is

$$U_{\text{ave}}=\frac{F}{4}\kappa^2A^2\Delta x.$$

Since $\kappa=\omega/v$, $\mu v^2=F$ and $\mu=m/\Delta x,$ we can write

$$U_{\text{ave}}=\frac{\mu v^2}{4}\left(\frac{\omega}{v}\right)^2A^2\frac{m}{\mu}=\frac{1}{4}m\omega^2A^2.$$

$$E=K+U=\frac{1}{2}m\omega^2A^2.$$

Exercises

15.23. The time resolution in the air is determined by the time it takes sound to travel 20.0 cm. At a speed of 343 m/s, the resolution time is $t_{\text{max}}=0.200\text{ m}/\left(343\text{ m/s}\right)=5.83\cdot 10^{-4}$ s. In the water, the speed of sound is $1.50\cdot 10^3$ m/s, corresponding to a resolution time of $t_{\text{max}}=0.200\text{ m}/(1.50\cdot 10^3\text{ m/s})=1.33\cdot 10^{-4}$ s. If an individual can only resolve a time difference of $5.83\cdot 10^{-4}$ s, they will not be able to distinguish a time difference of $1.33\cdot 10^{-4}$ s. Since our hearing is adapted to land conditions, a sound produced in water seems to reach the listener's ears at the same time regardless of direction. This is why it is impossible for the diver to detect the direction of a motor boat underwater.

15.24. In air, sound travels at a speed of 343 m/s. A time difference of 2.00 seconds gives a distance of $d = vt = (343 \text{ m/s})(2.00 \text{ s}) = 686 \text{ m},$ between the produced "hey" sound waves. Since the sound must travel to a surface before it is reflected, the total distance traveled by the sound is twice that of the distance from you to the mountain. The distance to the mountain is then $d = vt/2 = (343 \text{ m/s})(5.00 \text{ s})/2 = 858 \text{ m}.$

15.25. (a) The amplitude is the magnitude of the number outside of the sine function, $A = 0.00200$ m.

(b) The number of waves is given by the wave number, κ, times the distance traveled, divided by 2π. In this case, the number of waves is $\dfrac{\kappa x}{2\pi} = \dfrac{40.0 \text{ m}^{-1}(1.00 \text{ m})}{2\pi} = 6.37$ waves.

(c) The number of complete cycles in one second is the angular frequency, ω, times the time interval, t, divided by 2π. In this case, the number of complete cycles is $\dfrac{\omega t}{2\pi} = \dfrac{(800. \text{ s}^{-1})(1.00 \text{ s})}{2\pi} = 127$ cycles.

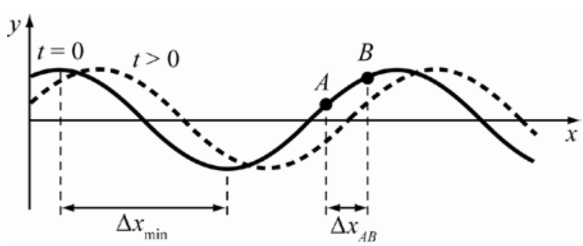
(d) The wavelength is 2π divided by the wave number, κ. The wavelength of this wave is $\lambda = 2\pi / 40.0 \text{ m}^{-1} = 0.157$ m.

(e) The speed of the wave is equal to the ratio of the angular frequency to the wave number:

$$v = \frac{\omega}{\kappa} = \frac{800. \text{ s}^{-1}}{40.0 \text{ m}^{-1}} = 20.0 \text{ m/s}.$$

15.26. **THINK:** The point of this question is to study the phases along a traveling wave. The wave is described by $y(x,t) = (5.00 \text{ mm})\sin\left((157.08 \text{ m}^{-1})x - (314.16 \text{ s}^{-1})t + 0.7854\right).$

SKETCH:



RESEARCH: A traveling wave is described by $y(x,t) = A\sin(\kappa x - \omega t + \phi).$

(a) The minimum separation, Δx_{min}, of two points that oscillate in perfect opposition is given by:

$$\kappa \Delta x_{\text{min}} = \pi \;\Rightarrow\; \Delta x_{\text{min}} = \pi / \kappa.$$

(b) The separation, Δx_{AB}, between two points A and B with a phase difference of $\phi_1 = 0.7854$ rad, can be determined from $\kappa \Delta x_{\text{AB}} = \phi_1 \;\Rightarrow\; \Delta x_{\text{AB}} = \phi_1 / \kappa.$

(c) The number of crests passing through the point A is equal to the number of troughs passing through the point in 10 seconds. The number, n, is determined from $\omega \Delta t = n2\pi \;\Rightarrow\; n = \omega \Delta t / 2\pi.$

(d) The point along the oscillation's trajectory is given by y at $x = 0$ and $t = 0$, $y = y(0,0) = A\sin(\kappa(0) - \omega(0) + \phi) = A\sin\phi.$

SIMPLIFY:

(a) $\Delta x_{\text{min}} = \pi / \kappa$

(b) $\Delta x_{\text{AB}} = \phi_1 / \kappa$

(c) $n = \omega \Delta t / 2\pi$

(d) $y = y(0,0) = A\sin(\kappa(0) - \omega(0) + \phi) = A\sin\phi$

CALCULATE:

(a) $\Delta x_{\text{min}} = \dfrac{\pi}{157.08\text{ m}^{-1}} = 0.019999953\text{ m}$

(b) $\Delta x_{\text{AB}} = \dfrac{0.7854}{157.08\text{ m}^{-1}} = 0.00500000\text{ m}$

(c) $n = \dfrac{\left(314.16\text{ s}^{-1}\right)\left(10.0\text{ s}\right)}{2\pi} = 500.0017$

(d) $y = y\left(0,0\right) = \left(5.00\text{ mm}\right)\sin\left(0.7854\right) = 3.53554\text{ mm}$

ROUND: Taking into account significant figures:

(a) The minimum distance between to points which are out of phase is $\Delta x_{\text{min}} = 0.019999\text{ m}.$

(b) The distance between two points which are out of phase by 0.7854 rad is $\Delta x_{AB} = 5.000\text{ mm}.$

(c) The number of crests and troughs that pass through the points *A* and *B* is 500. .

(d) The oscillator should be 3.54 mm above the zero line.

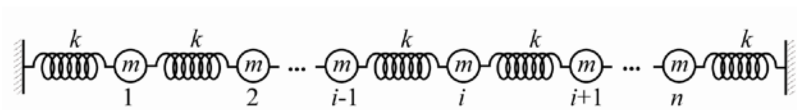
DOUBLE-CHECK: From the given parameters, the wavelength is $\lambda = 2\pi / \kappa = 0.04\text{ m}.$ Δx_{min} is half of the wavelength. Δx_{AB} is equal to $1/8$ of the wavelength, which makes sense since the phase shift $\phi = 2\pi / 8.$ Also from the given parameters, the velocity of the wave is $v = \omega / \kappa = 2\text{ m/s}.$ At this velocity, 50 waves of wavelength 0.04 m will pass a given point in one second, or equivalently, 500 waves in ten seconds. The y-value calculated is within the given amplitude of the wave. These values are reasonable.

15.27. THINK:

(a) The question asks for the equation of motion for the masses of an array of *n* masses each with mass m connected by with springs each with spring constant k. Each mass is located at $x_i = ia$ at equilibrium and we define the displacement of each mass as $x_i = ia + \psi_i$.

(b) The object is to determine the angular frequencies of the normal modes of the array of masses.

SKETCH:



RESEARCH:

(a) Let ψ_i denote the displacement of the i^{th} mass from its equilibrium position. The forces acting on the masses are due to the springs and have the form $F_s = -k\psi$. The angular frequency is given by $\omega_0^{\,2} = k / m.$

(b) For each normal mode, the whole system oscillates with an angular frequency of $\Omega,$ so the motion of each particle can be described as $\psi_i = A_i\cos\left(\Omega t\right)$. The left hand side is stationary, which implies $\psi_0 = 0.$ The right hand side is stationary, which means $\psi_{n+1} = 0.$ This suggests an *Ansatz* for the amplitudes $A_i = A\sin\left(i\phi\right)$ where A is an arbitrary amplitude, ϕ is a real number that is different for each normal mode, and $i = 1$ to *n*.

SIMPLIFY:

(a) The net force on the ith mass is

$$F_i = ma_i = m\frac{d^2\psi_i}{dt^2} = -k\left(\psi_i - \psi_{i-1}\right) - k\left(\psi_i - \psi_{i+1}\right) = k\left(\psi_{i-1} - 2\psi_i + \psi_{i+1}\right)$$

$$\frac{d^2\psi_i}{dt^2} = \omega_0^{\,2}\left(\psi_{i-1} - 2\psi_i + \psi_{i+1}\right).$$

All the masses and springs are identical, so this result describes the entire system.

(b) Insert $\psi_i = A_i \cos\Omega t$ into the result of part (a):

$$\frac{d^2\psi_i}{dt^2} = \frac{d^2}{dt^2}\left(A_i \cos\Omega t\right) = -A_i\Omega^2 \cos\Omega t$$
$$\omega_0^{\;2}\left(\psi_{i-1} - 2\psi_i + \psi_{i+1}\right) = \omega_0^{\;2}\left(A_{i-1}\cos\Omega t - 2A_i\cos\Omega t + A_{i+1}\cos\Omega t\right)$$
$$\Omega^2 A_i + \omega_0^{\;2}\left(A_{i-1} - 2A_i + A_{i+1}\right) = 0.$$

We have n equations of motion, one for each i from 1 to n. These normal modes look like standing waves. Each mass will oscillate with a sinusoidal form given by $\cos(\Omega t)$ and an amplitude that depends on the normal mode. We take $A_i = A\sin(i\phi)$ as an *Ansatz*. A is an arbitrary amplitude and ϕ is a real number determined by the boundary conditions. For $i = 0$, this form is clearly a solution. For $i = n+1$ this form is a solution if $A_{n+1} = A\sin\left((n+1)\phi\right) = 0$, which is only true for $(n+1)\phi = j\pi$, where j is an integer, which is true for

$$\phi_j = \frac{j\pi}{n+1}.$$

There are n normal modes so $1 \le j \le n$. We can write

$$A_i = A\left(\frac{j\pi}{n+1}i\right),\; 1 \le j \le n.$$

Now we put this result into our expression for the normal angular frequencies

$$\Omega^2 A\sin(i\phi) + \omega_0^{\;2}\left(A\sin\left((i-1)\phi\right) - 2A\sin(i\phi) + A\sin\left((i+1)\phi\right)\right) = 0$$
$$\Omega^2 \sin(i\phi) + \omega_0^{\;2}\left(\sin(i\phi - \phi) - 2\sin(i\phi) + \sin(i\phi + \phi)\right) = 0.$$

Remembering that

$$\sin(i\phi \pm \phi) = \cos(\phi)\sin(i\phi) \pm \cos(i\phi)\sin(\phi),$$

we can write

$$\Omega^2 \sin(i\phi) + \omega_0^{\;2}\left(\cos(\phi)\sin(i\phi) - \cos(i\phi)\sin(\phi) - 2\sin(i\phi) + \cos(\phi)\sin(i\phi) + \cos(i\phi)\sin(\phi)\right) = 0$$
$$\Omega^2 + \omega_0^{\;2}\left(\cos(\phi) - \cos(i\phi)\sin(\phi)/\sin(i\phi) - 2 + \cos(\phi) + \cos(i\phi)\sin(\phi)/\sin(i\phi)\right) = 0$$
$$\Omega^2 + \omega_0^{\;2}\left(\cos(\phi) - 2 + \cos(\phi)\right) = 0$$
$$\Omega^2 = 2\omega_0^{\;2}\left(1 - \cos\phi\right)$$
$$\Omega^2 = 2\omega_0^{\;2}\left(2\sin^2\left(\frac{\phi}{2}\right)\right)$$
$$\Omega^2 = 4\omega_0^{\;2}\sin^2\left(\frac{\phi}{2}\right)$$
$$\Omega = 2\omega_0 \sin\left(\frac{\phi}{2}\right).$$

Putting in out result for ϕ_j we get the angular frequencies for all the normal modes

$$\Omega_j = 2\omega_0 \sin\left(\frac{j\pi}{2(n+1)}\right),\; 1 \le j \le n.$$

CALCULATE: This step is not necessary.
ROUND: This step is not necessary.
DOUBLE-CHECK:
Let's double-check our result for $n = 2$ masses. We have two normal modes with angular frequencies given by

$$\Omega_1 = 2\omega_0 \sin\left(\frac{\pi}{6}\right) = \omega_0$$

$$\Omega_2 = 2\omega_0 \sin\left(\frac{\pi}{3}\right) = \sqrt{3}\omega_0.$$

Ω_1 corresponds to the two masses moving back and forth together while Ω_2 corresponds to the two masses vibrating opposite against each other. This result makes sense because when the two masses are moving together, the middle spring does not contribute and we have two masses with two springs so the angular frequency is just $\Omega_1 = \sqrt{2k/2m} = \omega_0$. For the two masses vibrating against each other, we essentially have three springs for each mass leading to $\Omega_2 = \sqrt{3k/m} = \sqrt{3}\omega_0$. So our result makes sense.

15.28. For a function to be a solution of the wave equation, the second time derivative of the function must be proportional to the second derivative of the function with respect to space. The second time derivative of this function is:

$$\frac{d}{dt}\left(\frac{d}{dt}A\ln\left(x+vt\right)\right) = \frac{d}{dt}\left(A\frac{v}{x+vt}\right) = -\frac{Av^2}{\left(x+vt\right)^2}.$$

The second position derivative is: $\frac{d}{dx}\left(\frac{d}{dx}A\ln\left(x+vt\right)\right) = \frac{d}{dx}\left(\frac{A}{x+vt}\right) = -\frac{A}{\left(x+vt\right)^2}.$

Inserting these values into the wave equation: $\frac{d^2}{dt}D = v^2\frac{d^2}{dx}D = -\frac{Av^2}{\left(x+vt\right)^2} = v^2\left(-\frac{A}{\left(x+vt\right)^2}\right)$, and it can be seen that the wave equation is satisfied.

15.29. The general equation of a wave is $y\left(x,t\right) = A\sin\left(\kappa x - \omega t + \phi\right)$. Thus, A, κ, ω, and ϕ must be determined using v = 30.0 m/s, f = 50.0 Hz, $y\left(0,0\right) = 4.00$ mm and $dy/dt\,|_{x=0,t=0} = 2.50$ m/s. The angular frequency is $\omega = 2\pi f = 2\pi\left(50.0 \text{ Hz}\right) = 100.\pi \text{s}^{-1} = 314 \text{s}^{-1}$. To determine the wave number:

$$\kappa = \frac{\omega}{v} = \frac{2\pi\left(50.0 \text{ Hz}\right)}{30.0 \text{ m/s}} = 10.5 \text{ m}^{-1}.$$

In order to determine A and ϕ, both of the initial conditions must be met. At x = 0 and t = 0:

$$y\left(0,0\right) = 4.00\cdot 10^{-3} \text{ m} = A\sin\phi. \quad (1)$$

The time derivative of the wave function is $dy/dt = -\omega A\cos\left(\kappa x - \omega t + \phi\right)$. At x = 0 and t = 0:

$$\frac{dy}{dt} = 2.50 \text{ m/s} = \left(-314 \text{ s}^{-1}\right)A\cos\phi \text{ or,}$$

$$\frac{2.50 \text{ m/s}}{-314 \text{ s}^{-1}} = A\cos\phi. \quad (2)$$

Square these two functions and add them:

$$\left(4.00\cdot 10^{-3} \text{ m}\right)^2 + \left(-\frac{2.50 \text{ m/s}}{314 \text{ s}^{-1}}\right)^2 = A^2\cos^2\phi + A^2\sin^2\phi.$$

Using the trigonometric identity, $\cos^2\phi + \sin^2\phi = 1$:

$$A = \pm\sqrt{\left(4.00\cdot 10^{-3} \text{ m}\right)^2 + \left(\left(-2.50 \text{ m/s}\right)/\left(314 \text{ s}^{-1}\right)\right)^2} = 0.008910 \text{ m} = 8.91 \text{ mm}.$$

The phase, ϕ, is determined by dividing equation (1) by equation (2):

$$\frac{4.00\cdot 10^{-3} \text{ m}}{-\left(\left(2.50 \text{ m/s}\right)/\left(314 \text{ s}^{-1}\right)\right)} = -0.5024 = \tan\phi \Rightarrow \phi = \tan^{-1}\left(-0.5024\right) = -0.466 \text{ rad}.$$

This is the wrong quadrant since it is the time derivative that is negative, not the displacement. This means $\phi = -0.466 \text{ rad} + \pi = 2.68 \text{ rad}$. It is best to keep extra significant figures in the equation at this point, and to only round after values of x and t have been substituted. The equation for the wave is:

$$y(x,t) = (8.91 \text{ mm})\sin\left(10.5 \text{ m}^{-1}x - 100.\pi \text{ s}^{-1}t + 2.68 \text{ rad}\right).$$

15.30. (a) The amplitude of the wave is 0.0200 m or 20.0 mm.

(b) The period of the wave is related to the angular frequency of the wave, $\omega = 2.63 \text{ s}^{-1}$. Calculating the period using the angular frequency gives:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.63 \text{ s}^{-1}} = 2.39 \text{ s}.$$

(c) The wavelength is related to a value given in the wave function, the wave number, κ. The wavelength is given by:

$$\lambda = \frac{2\pi}{\kappa} = \frac{2\pi}{6.35 \text{ m}^{-1}} = 0.989 \text{ m}.$$

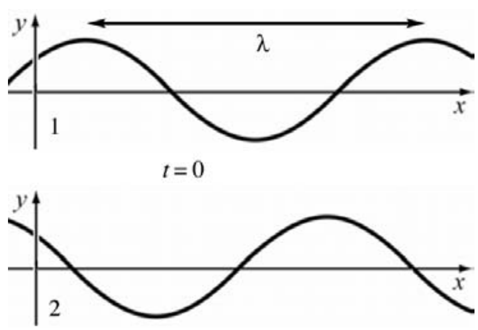
(d) The speed of the wave is the ratio of the angular frequency and the wave number:

$$v = \frac{\omega}{\kappa} = \frac{2.63 \text{ s}^{-1}}{6.35 \text{ m}^{-1}} = 0.414 \text{ m/s}.$$

(e) Since the functional dependence is $(\kappa x - \omega t + \phi)$, the wave travels in the negative x-direction.

15.31. **THINK:** Determine the equation for a wave traveling in the direction of the positive x-axis with $\lambda = 12.0$ cm, f = 10.0 Hz, A = 10.0 cm and $y(0,0) = 5.00$ cm. Note that during each complete wave oscillation the displacement assumes the same value twice. Since the displacement at (0,0) is given, find two solutions (unless the displacement corresponds to an extremum, which is not the case here). Sketch both cases, 1 and 2, in the next step of the solution.

SKETCH:



RESEARCH: The wave number and the angular frequency are given by $\kappa = 2\pi / \lambda$ and $\omega = 2\pi f$, respectively. The period is related to the frequency by T = 1/f. The speed of the wave is $v = \lambda f$. To determine the phase constant, use the point $y(0,0) = A\sin\phi$. The equation of motion is given by:

$$y(x,t) = A\sin(\kappa x - \omega t + \phi).$$

SIMPLIFY: $\sin\phi = \frac{y(0,0)}{A} \Rightarrow \phi = \pm\sin^{-1}\left(\frac{y(0,0)}{A}\right)$

CALCULATE:

(a) $\kappa = \frac{2\pi}{12.0 \text{ cm}} = \frac{2\pi}{0.120 \text{ m}} = 52.36 \text{ m}^{-1}$

(b) $T = \frac{1}{10.0 \text{ Hz}} = 0.100 \text{ s}$

(c) $\omega = 2\pi(10.0 \text{ Hz}) = 62.83 \text{ s}^{-1}$

(d) $v = 0.120 \text{ m}(10.0 \text{ Hz}) = 1.20 \text{ m/s}$

(e) $\phi = \pm\sin^{-1}\left(\dfrac{0.0500 \text{ m}}{0.100 \text{ m}}\right) = \pm\dfrac{\pi}{6}$

(f) $y(x,t) = (10.0 \text{ cm})\sin\left(52.36 \text{ m}^{-1}x - 62.83 \text{ s}^{-1}t \pm \dfrac{\pi}{6}\right)$

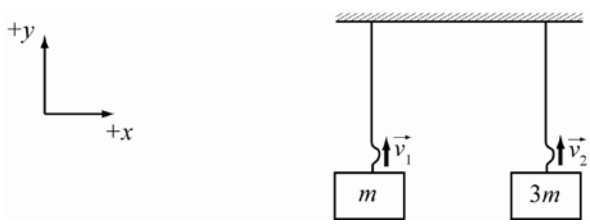
ROUND:

(a) The wave number is $\kappa = 52.4 \text{ m}^{-1}$.
(b) The period is $T = 0.100$ s.
(c) The angular frequency is $\omega = 62.8 \text{ s}^{-1}$.
(d) The velocity is $v = 1.20$ m/s.
(e) The phase is $\phi = \pm\pi/6$.
(f) It is better not to round the coefficients of the equation at this stage, and only round once particular values of x and t are substituted.

DOUBLE-CHECK: Each of the calculated values have the proper SI units.

15.32. **THINK:** The task is to compare the wave pulse in a rope with a mass, m, at its end versus a mass, 3.00 m, at its end. In each case the rope has the same linear density, but the tension will change. In each case the distance traveled by the wave pulse is the same. This means the speed of the wave pulses will be different in each case.

SKETCH:



RESEARCH: The speed of a wave is $v = \sqrt{T/\mu}$. For the hanging mass, the tension on the rope is equal to the force of gravity on the mass. In terms of the distance and velocity and distance, the time is given by $t = d/v$.

SIMPLIFY: $\dfrac{t_2}{t_1} = \dfrac{d/v_2}{d/v_1} = \dfrac{v_1}{v_2} = \dfrac{\sqrt{mg/\mu}}{\sqrt{3mg/\mu}} = \dfrac{1}{\sqrt{3}} \Rightarrow t_2 = \dfrac{t_1}{\sqrt{3}}$

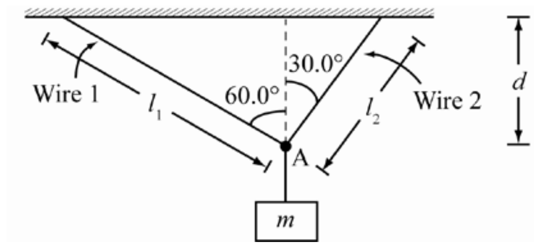
CALCULATE: $t_2 = 0.577t_1$

ROUND: Not required.

DOUBLE-CHECK: As expected, an increase in tension leads to a higher wave speed and thus to a smaller travel time.

15.33. **THINK:** Determine the time difference between a wave pulse that travels along wire 1 and one that travels along wire 2. To solve this problem, both the length of each wire and the speed of a wave on the wire must be determined. The tension in the wire is required to determine the speed along the wire. Wire 1 makes a $60.0°$ angle with the vertical line. Wire 2 makes a $30.0°$ angle with the vertical. The distance from the ceiling to point A is 0.300 m.

SKETCH:



RESEARCH: The time is given by $t = l/v$. The velocity is related to the tension by $v=\sqrt{T/\mu}$. The net force on a stationary body is zero. This can be used to determine the tension. The right angle triangle in the diagram is needed to determine the length and tension of each wire.

SIMPLIFY: The forces in the horizontal plane give:

$$F_{\text{net}\,x}=\sum F_x=T_{1x}-T_{2x}=T_1\sin\theta_1-T_2\sin\theta_2=0 \text{ or } T_1=T_2\sin\theta_2/\sin\theta_1.$$

The forces in the vertical plane give: $F_{\text{net}y}=\sum F_y=T_{1y}+T_{2y}-F_{\text{g}}=T_1\cos\theta_1+T_2\cos\theta_2-mg=0.$ Using these two equations to solve for T_1 and T_2:

$$T_1\cos\theta_1+T_2\cos\theta_2=T_2\frac{\sin\theta_2}{\sin\theta_1}\cos\theta_1+T_2\cos\theta_2=T_2\cos\theta_2\left(\frac{\tan\theta_2}{\tan\theta_1}+1\right)=mg$$

$$\Rightarrow\ T_2=\frac{mg}{\cos\theta_2\left(\dfrac{\tan\theta_2}{\tan\theta_1}+1\right)}=mg\frac{\sqrt{3}}{2}\quad \text{(Recall that } \theta_1=60° \text{ and } \theta_2=30°.)$$

Similarly, $T_1=\dfrac{T_2\sin\theta_2}{\sin\theta_1}=\left(mg\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{1/2}{\sqrt{3}/2}\right)=\dfrac{mg}{2}$. The lengths are given by $\cos\theta=d/l$ or $l_1=d/\cos\theta_1$ and $l_2=d/\cos\theta_2$. With the specific angles in mind:

$$l_2=\frac{d}{\cos 30°}=\frac{d}{\sqrt{3}/2}=\frac{2d}{\sqrt{3}} \text{ and } l_1=\frac{d}{\cos 60°}=\frac{d}{1/2}=2d.$$

The times are given by $t=l/v=l/\sqrt{T/\mu}$. Hence $t_1=l_1/\sqrt{T_1/\mu}$ and $t_2=l_2/\sqrt{T_2/\mu}$.

CALCULATE: $t_1=\dfrac{2d}{\left(\sqrt{\dfrac{mg}{2\mu}}\right)}=2d\sqrt{\dfrac{2\mu}{mg}},\quad t_2=\dfrac{\left(\dfrac{2d}{3}\right)}{\left(\sqrt{\dfrac{\sqrt{3}mg}{2\mu}}\right)}=\dfrac{2d}{3^{3/4}}\sqrt{\dfrac{2\mu}{mg}}$

$$\frac{t_1}{t_2}=\left(2d\sqrt{\frac{2\mu}{mg}}\right)\Bigg/\left(\frac{2d}{3^{3/4}}\sqrt{\frac{2\mu}{mg}}\right)=3^{3/4}\ \Rightarrow\ t_1=3^{3/4}t_2=2.2795t_2.$$

ROUND: $t_1=2.28t_2$

The trip along Wire 1 takes 2.30 times longer than the trip along Wire 2.

DOUBLE-CHECK: The tension is higher in the second wire than the first by a factor of $\sqrt{3}$. The second wire is shorter than the first by a factor of $\dfrac{1}{\sqrt{3}}$. Since the wave speed is proportional to the length, and inversely proportional to the square root of the tension, it makes sense for the pulse to take $\dfrac{1}{\sqrt{3}}\cdot\dfrac{1}{\sqrt{\sqrt{3}}}=\dfrac{1}{3^{3/4}}$ the time for the pulse to travel through Wire 2 than it takes to travel through Wire 1.

15.34. **THINK:** Determine the speed of a wave traveling along a guitar string. Also, determine the required tension to increase the speed of the wave by 1.00 %. The wire has a linear density of $\mu = 1.93 \text{ g/m} = 1.93 \cdot 10^{-3} \text{ kg/m}$ and starts with a tension of T = 62.2 N.

SKETCH: A sketch is not needed to solve the problem.

RESEARCH: The speed of a wave along a wire is $v = \sqrt{T / \mu}$.

SIMPLIFY: The speed of the wire in the first case is simply $v_1 = \sqrt{T_1 / \mu}$. For the second case, the speed is $v_2 = (1 + \alpha / 100) v_1$, where α is the percent increase. To determine the new tension, substitute $\sqrt{T / \mu}$ for v:

$$\sqrt{\frac{T_2}{\mu}} = \left(1 + \frac{\alpha}{100}\right)\sqrt{\frac{T_1}{\mu}}.$$

Squaring both sides gives: $\frac{T_2}{\mu} = \left(1 + \frac{\alpha}{100}\right)^2 \frac{T_1}{\mu}$ or $T_2 = \left(1 + \frac{\alpha}{100}\right)^2 T_1$.

CALCULATE: $v_1 = \sqrt{\frac{62.2 \text{ N}}{1.93 \cdot 10^{-3} \text{ kg/m}}} = 179.52 \text{ m/s}, \quad T_2 = \left(1 + \frac{1.00}{100}\right)^2 T_1 = 1.0201 T_1 = \left(1 + \frac{2.01}{100}\right) T_1$

ROUND: Round the results to three significant figures.

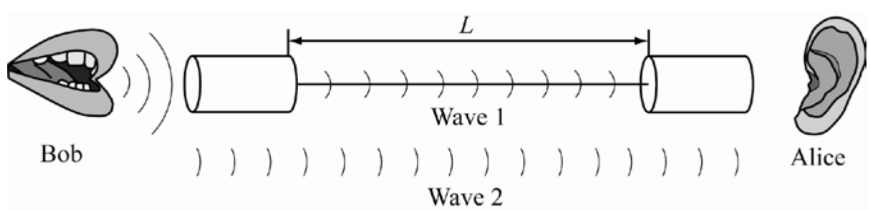
(a) The speed of the wave is 180. m/s.

(b) The tension should increase by 2.01 % for the speed to increase by 1.00 %.

DOUBLE-CHECK: Dimensional analysis confirms that the calculated values have the correct units. These values are reasonable.

15.35. **THINK:** Determine the time it takes for sound waves to reach Alice via two different media and compare these values. The first medium is air, in which sound travels at a speed of about 343 m/s. The second medium is a string with a tension of 25.0 N and a linear density of 6.13 g/m or 0.00613 kg/m. The distance between Alice and Bob is 20.0 m.

SKETCH:



RESEARCH: The time it takes for the sound to travel a distance, d, is given by $t = d / v_a$. The velocity of sound in a wire is given by $v_w = \sqrt{T / \mu}$.

SIMPLIFY: The time it takes to travel through air is $t_a = d / v_a$. The time it takes sound to travel through the wire is $t_w = d / v_w = d / \sqrt{T / \mu} = d\sqrt{\mu / T}$. The time difference is given by:

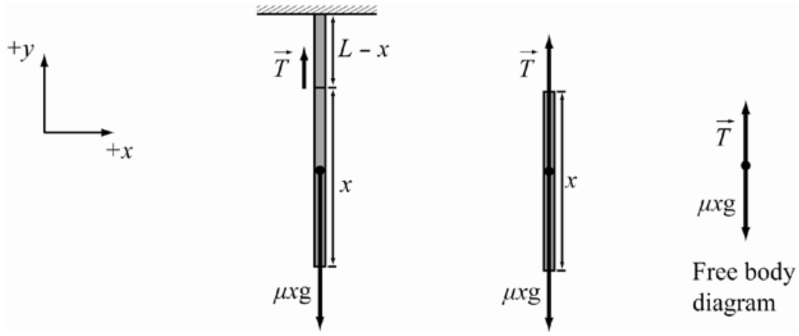
$$\Delta t = t_a - t_w = \frac{d}{v_a} - d\sqrt{\frac{\mu}{T}} = d\left(\frac{1}{v_a} - \sqrt{\frac{\mu}{T}}\right).$$

CALCULATE: $\Delta t = 20.0 \text{ m}\left(\frac{1}{343 \text{ m/s}} - \sqrt{\frac{6.13 \cdot 10^{-3} \text{ kg/m}}{25.0 \text{ N}}}\right) = -0.254868 \text{ s}$, indicating that the sound in air reaches Alice 0.245868 seconds before the sound in the wire.

ROUND: Rounding the result to three significant figures, the sound in the air reaches Alice 0.255 seconds before the sound from the wire does.

DOUBLE-CHECK: $t_a = \frac{20.0 \text{ m}}{343 \text{ m/s}} = 0.0583090$ s, and $t_w = 20.0 \text{ m}\sqrt{\frac{6.13\cdot 10^{-3} \text{ kg/m}}{25.0 \text{ N}}} = 0.313177$ s. The difference is −0.255 seconds, confirming the original result.

15.36. **THINK:** It is given that a wire hanging from the ceiling has a uniform linear mass, density, $\mu = M/L,$. It takes 1.00 second for a pulse to travel the length of the wire *L*. *L* is unknown and must be determined. In order to determine *L*, the speed of the pulse is needed.
SKETCH:



RESEARCH: The speed of the pulse in the wire is $v = \sqrt{T/\mu}$. Since *T* is not constant, the speed is also not constant, but depends on the height, *x*. Consider a position on the wire at a height, *x*, as shown in the above figure. Using Newton's second law, the tension in the wire is $T = \mu xg$. Therefore, the speed of the pulse at the position, *x*, is:

$$v = \sqrt{\frac{\mu x g}{\mu}} = \sqrt{xg}.$$

SIMPLIFY: Since $v = dx/dt$, the above equation simplifies to: $dx/\sqrt{x} = \sqrt{g}dt$. Integrating both sides:

$\int_0^L x^{-\frac{1}{2}}dx = \int_0^t \sqrt{g}dt \Rightarrow \left[2x^{\frac{1}{2}}\right]_0^L = \sqrt{g}t \Rightarrow 2L^{\frac{1}{2}} = \sqrt{g}t$. Therefore, the length of the wire is $L = \frac{1}{4}gt^2$.

CALCULATE: $L = \frac{1}{4}(9.81 \text{ m/s}^2)(1.00 \text{ s})^2 = 2.453$ m

ROUND: $L = 2.45$ m

DOUBLE-CHECK: The maximum speed on the wire is given by $v_{max} = \sqrt{\mu L g/\mu} = \sqrt{Lg}$, and the minimum speed is $v_0 = 0$. The average speed is $v_{ave} = (v_{max} + v_0)/2 = \left(\sqrt{Lg}\right)/2$. Using this speed, the length of the wire is $L \approx v_{ave}t = \left(\sqrt{Lg}\right)t/2$. After manipulation, it is determined that the length of the wire is $L = gt^2/4$. This result is the same as the previous "exact" result. This means that the speed is increasing linearly with time or the acceleration of the pulse is constant.

15.37. **THINK:** A Gaussian wave is represented by $y(x,t) = (5.00 \text{ m})e^{-0.1(x-5t)^2}$. Determine if the equation satisfies the wave equation.
SKETCH: A sketch is not necessary.
RESEARCH: The wave equation is given by: $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$.

SIMPLIFY:

(a) Differentiating $y(x,t)$ with respect to x yields:

$$\frac{\partial y}{\partial x}=(5.00\text{ m})\left[-0.1(2)(x-5t)\right]e^{-0.1(x-5t)^2}=-1.0(x-5t)e^{-0.1(x-5t)^2}.$$

Differentiating a second time with respect to x gives the second derivative of y with respect to x:

$$\frac{\partial^2 y}{\partial x^2}=\left[-1.0(x-5t)e^{-0.1(x-5t)^2}+0.2(x-5t)^2e^{-0.1(x-5t)^2}\right].$$

Differentiating $y(x,t)$ with respect to t yields:

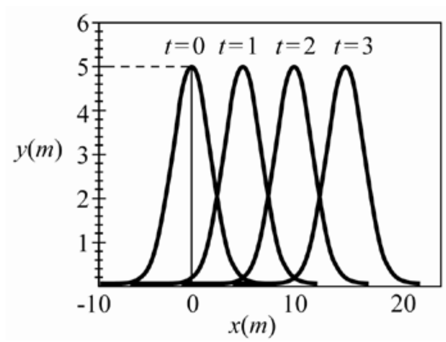
$$\frac{\partial y}{\partial t}=(5.00\text{ m})\left[-0.1(2)(x-5t)\right](-5)e^{-0.1(x-5t)^2}=5.0(x-5t)e^{-0.1(x-5t)^2}.$$

Differentiating a second time with respect to t gives the second derivative of y with respect to t:

$$\frac{\partial^2 y}{\partial t^2}=-25e^{-0.1(x-5t)^2}+5.0(x-5t)\left[-0.1(2)(x-5t)\right](-5)e^{-0.1(x-5t)^2}$$
$$=25\left[-1.0e^{-0.1(x-5t)^2}+0.2(x-5t)^2e^{-0.1(x-5t)^2}\right].$$

Careful examination of these two second partial derivatives reveals $\frac{\partial^2 y}{\partial x^2}=\frac{1}{25}\frac{\partial^2 y}{\partial t^2}$. This means that $y(x,t)$ satisfies the wave equation with a wave speed of v = 5 m/s.

(b)



(c) Consider a function, $f(x,t)=h(x\pm vt)$. Taking the first derivative of $f(x,t)$ with respect to t yields:

$$\frac{\partial f}{\partial t}=\frac{\partial h(x\pm vt)}{\partial t}=\frac{\partial h(y)}{\partial y}\frac{\partial(x\pm vt)}{\partial t}\ \Rightarrow\ \frac{\partial f}{\partial t}=\pm v\frac{\partial h(y)}{\partial y},\text{where } y=x\pm vt.$$

Differentiating the second time with respect to t gives:

$$\frac{\partial^2 f}{\partial t^2}=\pm v\frac{\partial}{\partial t}\left(\frac{\partial h(y)}{\partial y}\right)=\pm v\frac{\partial}{\partial y}\left(\frac{\partial h(y)}{\partial y}\right)=\pm v\frac{\partial^2 h(y)}{\partial y^2}\left(\frac{\partial y}{\partial t}\right).$$

Since $y=x\pm vt$, the above equation simplifies to: $\frac{\partial^2 f}{\partial t^2}=v^2\frac{\partial^2 h(y)}{\partial y^2}$. The second derivative of $f(x,t)$ with respect to x is: $\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 h(x\pm vt)}{\partial x^2}$, and using $y=x\pm vt$, the derivative is: $\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 h(y)}{\partial y^2}$, since dy = dx. Thus, $\frac{\partial^2 f}{\partial t^2}=v^2\frac{\partial^2 f}{\partial x^2}$, which is the wave equation.

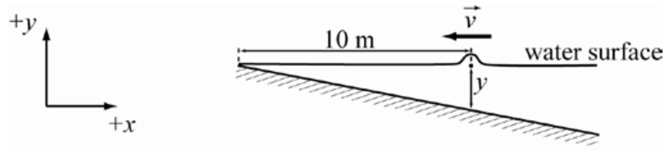
CALCULATE: This step is not necessary.

ROUND: This step is not necessary.

DOUBLE-CHECK: The proof is complete, and has verified what was to be shown.

15.38. **THINK:** The slope of the ground underneath the ocean at a beach is equal to $dy/dx = -12.0\text{ cm}/1.00\text{ m} = -0.120$. Note that because y is decreasing, dy/dx is negative. The acceleration of a wave at 10 m from the shore is required.

SKETCH:



RESEARCH: The speed of a water wave when its wavelength is much larger than the depth of the ocean is $v = \sqrt{gy}$. The acceleration of the wave is $a = dv/dt$. Using the chain rule for differentiation, the acceleration is given by:

$$a = \left(\frac{dv}{dy}\right)\left(\frac{dy}{dx}\right)\left(\frac{dx}{dt}\right) = v\left(\frac{dv}{dy}\right)\left(\frac{dy}{dx}\right).$$

SIMPLIFY: Therefore, the acceleration becomes: $a = \left(\sqrt{gy}\right)\left(\frac{1}{2}\sqrt{g}\frac{1}{\sqrt{y}}\right)\left(\frac{\partial y}{\partial x}\right) = \frac{1}{2}g\left(\frac{\partial y}{\partial x}\right).$

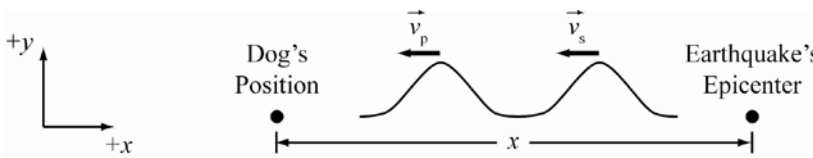
CALCULATE: The acceleration is: $a = \frac{1}{2}\left(9.81\text{ m/s}^2\right)\left(-0.120\right) = -0.5886\text{ m/s}^2.$

ROUND: $a = -0.589\text{ m/s}^2$

DOUBLE-CHECK: The negative value of the acceleration indicates that the wave is slowing down, as expected.

15.39. **THINK:** The speeds of S waves and P waves of an earthquake are $v_S = 4.0\text{ km/s},$ and $v_P = 7.0\text{ km/s},$ respectively. The distance of the dog from the location of the earthquake's epicenter must be determined. The time difference is $\Delta t = 30.0\text{ s}.$

SKETCH:



RESEARCH: The time taken for the P waves to reach the dog's position is $t_P = x/v_P$. Similarly, for S waves, the time is $t_S = x/v_S$. Therefore, the interval of time between S waves and P waves at the dog's position is:

$$\Delta t = \frac{x}{v_S} - \frac{x}{v_P}.$$

SIMPLIFY: After manipulation, the distance from the epicenter is $x = \frac{v_S v_P \Delta t}{v_P - v_S}.$

CALCULATE: Substituting the values:

$$x = \frac{(4.0\text{ km/s})(7.0\text{ km/s})(30.0\text{ s})}{7.0\text{ km/s} - 4.0\text{ km/s}} = 280.\text{ km}.$$

ROUND: The values in the question were given to two significant figures, so the final answer should be rounded so it also has two significant figures. $x = 280$ km.

DOUBLE-CHECK: Calculate the actual times for the different kinds of waves to arrive. The P waves travel at 7.0 km/s, so they will cover 280 km in 40. s. The S waves travel at 4.0 km/s, so they will cover 280 km in 70. s. The difference between these times is 30. s, which is consistent with what is given in the question.

15.40. It is given that the mass, the length and the tension of a string are $m = 30.0 \text{ g} = 3.00 \cdot 10^{-2}$ kg, $L = 2.00$ m, and $T = 70.0$ N, respectively. Therefore, the speed of a wave on the string is:

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{m/L}} = \sqrt{\frac{70.0 \text{ N}}{\left(3.00 \cdot 10^{-2} \text{ kg}/(2.00 \text{ m})\right)}} = 68.31 \text{ m/s}.$$

The power needed to generate a traveling wave with a frequency of 50.0 Hz and an amplitude of 4.00 cm is:

$$P = \frac{1}{2}\mu v \omega^2 A^2.$$

Substituting $\mu = m/L$ and $\omega = 2\pi f$ yields:

$$P = \frac{1}{2}\frac{m}{L}v(2\pi)^2 f^2 A^2 = \frac{1}{2}\left(\frac{3.00 \cdot 10^{-2} \text{ kg}}{2.00 \text{ m}}\right)(68.31 \text{ m/s})(2\pi)^2\left(50.0 \text{ s}^{-1}\right)^2(0.0400 \text{ m})^2 = 80.9 \text{ W}.$$

15.41. The power of a wave on a string is given by $P = \frac{1}{2}\mu v \omega^2 A^2$. Substituting $\omega = 2\pi f$ and $v = \sqrt{T/\mu}$ gives:

$$P = \frac{1}{2}\mu\sqrt{\frac{T}{\mu}}(2\pi)^2 f^2 A^2 = \frac{1}{2}\sqrt{\mu T}(2\pi)^2 f^2 A^2.$$

Substituting $\mu = 0.100$ kg/m, $T = 100.$ N, $f = 120.$ Hz and $A = 2.00 \cdot 10^{-2}$ m, the power is:

$$P = \frac{1}{2}\sqrt{(0.100 \text{ kg/m})(100. \text{ N})}(2\pi)^2\left(120. \text{ s}^{-1}\right)^2\left(2.00 \cdot 10^{-2}\right)^2 = 360. \text{ W}.$$

15.42. **THINK:** The equation of a wave on a string is $y = (0.100 \text{ m})\sin(0.750x - 40.0t)$, and the density of the string is $\mu = 10.0 \text{ g/m} = 1.00 \cdot 10^{-2}$ kg/m.

SKETCH: A sketch is not necessary.

RESEARCH: A sinusoidal wave on the string has an equation in the form $y = A\sin(\kappa x - \omega t + \phi)$, where A is the amplitude of the wave, κ is the angular wave number and ω is the angular frequency. The term in the bracket, $(\kappa x - \omega t + \phi)$, is the phase of the wave. ϕ is the phase constant.

SIMPLIFY:

(a) The phase constant is ϕ.

(b) The phase of the wave is given by $\kappa x - \omega t + \phi$.

(c) The speed of the wave is $v = \dfrac{\omega}{\kappa}$.

(d) The wavelength is $\lambda = \dfrac{2\pi}{\kappa}$.

(e) From $\omega = 2\pi f$, the frequency is $f = \dfrac{\omega}{2\pi}$.

(f) The power transmitted by the wave is $P = \dfrac{1}{2}\mu v \omega^2 A^2$.

CALCULATE: Comparing $y = A\sin(\kappa x - \omega t + \phi)$ with $y = (0.100 \text{ m})\sin(0.750x - 40.0t)$, $A = 0.100$ m, $\kappa = 0.750$ rad/s, $\omega = 40.0$ rad/s and $\phi = 0$.

(a) $\phi = 0$

(b) $\kappa x - \omega t + \phi = 0.750(0.0200) - 40.0(0.100) \text{ rad} = -3.985 \text{ rad}$

(c) $v = \dfrac{40.0 \text{ rad/s}}{0.750 \text{ rad/m}} = 53.33 \text{ m/s}$

(d) $\lambda = \dfrac{2\pi}{0.750 \text{ rad/m}} = 8.3776 \text{ m}$

(e) $f = \dfrac{40.0 \text{ rad/s}}{2\pi} = 6.366 \text{ Hz}$

(f) $P = \dfrac{1}{2}(0.0100 \text{ kg/m})(53.33 \text{ m/s})(40.0 \text{ rad/s})^2(0.100 \text{ m})^2 = 4.266 \text{ W}$

ROUND:

(a) $\phi = 0$

(b) The phase is -3.99 rad.

(c) $v = 53.3$ m/s

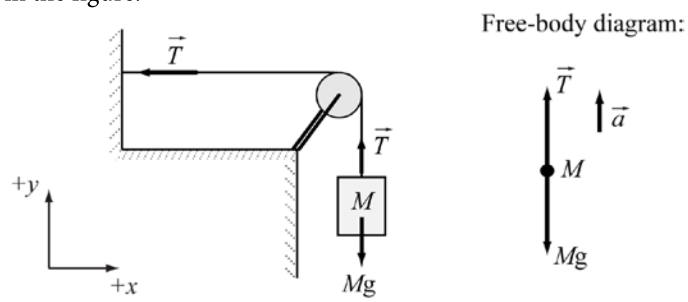
(d) $\lambda = 8.38$ m

(e) $f = 6.37$ Hz

(f) $P = 4.27$ W

DOUBLE-CHECK: The results have the correct units, and this supports the calculations as being reasonable.

15.43. The system is shown in the figure.



It is given that the mass of a string is $m_S = 5.00 \text{ g} = 5.00 \cdot 10^{-3} \text{ kg}$ and its length is L = 70.0 cm = 0.700 m. The mass of the weight is M = 250. kg. Using Newton's second law, the tension is given by $T - Mg = Ma \Rightarrow T = M(a+g)$. The fundamental frequency is given by:

$$f_1 = \frac{v}{2L} = \frac{\sqrt{T/\mu}}{2L}.$$

Substituting $\mu = m_S / L$ and $T = M(a+g)$: $f = \dfrac{\sqrt{T/(m_s/L)}}{2L} = \dfrac{1}{2}\sqrt{\dfrac{T}{m_sL}} = \dfrac{1}{2}\sqrt{\dfrac{M(a+g)}{m_sL}}.$

(a) Substituting the values of the variables and $a = 0$ yields:

$$f = \frac{1}{2}\sqrt{\frac{250. \text{ kg}\left(0+(9.81 \text{ m/s}^2)\right)}{(5.00 \cdot 10^{-3} \text{ kg})(0.700 \text{ m})}} = 418.5 \text{ Hz} \approx 419 \text{ Hz}.$$

(b) It was found previously that $f = \dfrac{1}{2}\sqrt{\dfrac{M(a+g)}{m_sL}}$. After rearrangement:

$$a = -g + \frac{4f^2 m_S L}{M} = -9.81 \text{ m/s}^2 + \frac{4(440. \text{ Hz})^2(5.00 \cdot 10^{-3} \text{ kg})0.700 \text{ m}}{250. \text{ kg}} = 1.03 \text{ m/s}^2.$$

Since a is positive, the elevator moves upward.

15.44. The fundamental mode of a string is given by $f = v/2L$. The speed of a wave on the string is given by $v = \sqrt{T/\mu}$. Therefore, the fundamental frequency is $f = \left(\sqrt{T/\mu}\right)/2L$. This means that the tension that produces the fundamental frequency, f_0, is:

$$\frac{T}{\mu} = 4f_0^2L^2 \Rightarrow T = 4\mu f_0^2L^2.$$

Substituting $\mu = 5.51\cdot 10^{-4}$ kg/m, L = 0.350 m and $f_0 = 660.$ Hz yields:

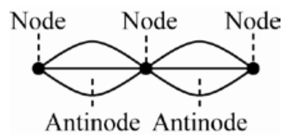
$$T = 4\left(5.51\cdot 10^{-4}\text{ kg/m}\right)\left(660.\text{ Hz}\right)^2\left(0.350\text{ m}\right)^2 = 117.61\text{ N} \approx 118\text{ N}.$$

15.45. (a) The linear density of the string is $\mu = \dfrac{m}{L} = \dfrac{0.0100\text{ kg}}{2.00\text{ m}} = 0.00500$ kg/m.

Using the tension, T = 150. N, the speed of a wave on the string is:

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{150.\text{ N}}{0.00500\text{ kg/m}}} = 173.2\text{ m/s} \approx 173\text{ m/s}.$$

(b)

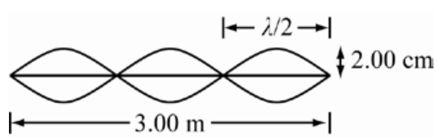


Since both ends of the string are fixed, there are two nodes at the ends of the string. To get two anti-nodes, there must be one node in the middle of the string as shown in the above figure. This means there is one wavelength on the string, that is, $L = \lambda$. Thus, the wavelength is $\lambda = L = 2.00$ m. The frequency is obtained from the relation $v = \lambda f \;\Rightarrow\; f = v/\lambda$. Thus,

$$f = \frac{173.2\text{ m/s}}{2.00\text{ m}} = 86.6\text{ Hz}.$$

15.46. **THINK:** It is given that there is a 3.00 m long string, fixed at both ends, that vibrates 15.0 times in a second (f = 15.0 Hz). A standing wave on the string has three anti-nodes of amplitude 2.00 cm. At time t = 0, $y(x,t)$ is equal to zero for all x. The equation of the standing wave and the speed of a pulse on the string are needed.

SKETCH:



RESEARCH: The equation of a standing wave is given by:

$$y(x,t) = A\sin\left(\kappa x + \omega t + \phi\right) + A\sin\left(\kappa x - \omega t - \phi\right) \text{ or } y(x,t) = 2A\sin\kappa x\cos\left(\omega t + \phi\right).$$

SIMPLIFY: It is given that the amplitude of the anti-nodes are equal to 2.00 cm. This means 2A = 2.00 cm, and therefore, A = 1.00 cm. From the value of frequency, f = 15.0 Hz, the angular frequency can be determined from $\omega = 2\pi f$. It is given that the standing wave has three anti-nodes and both ends of the string are fixed. This means that there are three halves of the wavelength, as shown above. Therefore,

$$3\left(\frac{1}{2}\lambda\right) = L \;\Rightarrow\; \lambda = \frac{2}{3}L.$$

The angular wave number is given by: $\kappa = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{(2/3)L} = \dfrac{3\pi}{L}.$

CALCULATE:

Substituting A = 1.00 cm, $\omega = 2\pi\left(15.0\text{ Hz}\right) = 30.0\pi$ rad/s and $\kappa = 3\pi/\left(3.00\text{ m}\right) = \left(\pi\text{ rad}\right)/\text{m}$ yields:

$$y(x,t) = 2\left(0.0100\text{ m}\right)\sin\left(\pi x\right)\cos\left(30.0\pi t + \phi\right) = \left(0.0200\text{ m}\right)\sin\left(\pi x\right)\cos\left(30.0\pi t + \phi\right),$$

where x is in meters and t is in seconds. It is given that at t = 0, y = 0. This means $\cos\phi = 0$ or $\phi = \pi/2$. Thus,

$$y(x,t) = \left(0.0200\text{ m}\right)\sin\left(\pi x\right)\cos\left(30.0\pi t + \pi/2\right).$$

Next, using the angular frequency, $\omega = 30.0\pi$ rad/s, and the angular wave number, $\kappa = \pi$ rad/m, the speed of a pulse on the string is:

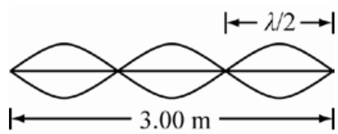
$$v = \frac{\omega}{\kappa} = \frac{30.0\pi \text{ rad/s}}{\pi \text{ rad/m}} = 30.0 \text{ m/s}.$$

ROUND: Three significant figures are required, so the answer is $v = 30.0$ m/s.

DOUBLE-CHECK: The standing wave equation is $y = (2.00 \text{ cm})\sin(\pi x)\cos(30.0\pi t + \pi/2)$. From the sketch, note that the nodes (not including the ends of the string) are x = 1.00 m and x = 2.00 m. Substituting these values into the wave equation gives y = 0 as expected.

15.47. **THINK:** It is known that a string has a length of 3.00 m and a mass of 6.00 g. Both ends of the string are fixed. A standing wave on the string has a frequency, f = 300. Hz and three anti-nodes. The tension in the string is to be determined.

SKETCH:



RESEARCH: The speed of a pulse on the string is given by $v = \sqrt{T/\mu}$ or $v = \omega/\kappa$.

SIMPLIFY: Combining these equations gives: $\sqrt{\frac{T}{\mu}} = \frac{\omega}{\kappa} \Rightarrow \frac{T}{\mu} = \left(\frac{\omega}{\kappa}\right)^2$. Using $\mu = m/L$, the tension is: $T = \left(\frac{m}{L}\right)\left(\frac{\omega}{\kappa}\right)^2$. Substituting $\omega/\kappa = \lambda f$, the tension becomes: $T = \left(\frac{m}{L}\right)(\lambda f)^2$. Because there are three anti-nodes, the wavelength of the standing wave is $\lambda = 2L/3$. Therefore,

$$T = \left(\frac{m}{L}\right)\left(\frac{2}{3}Lf\right)^2 = \frac{4}{9}mLf^2.$$

CALCULATE: Substituting $m = 6.00 \cdot 10^{-3}$ kg, L = 3.00 m and f = 300. Hz yields:

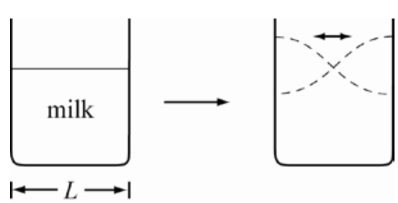
$$T = \left(\frac{4}{9}\right)(6.00 \cdot 10^{-3} \text{ kg})(3.00 \text{ m})(300. \text{ Hz})^2 = 720. \text{ N}.$$

ROUND: Rounding to three significant figures, $T = 720.$ N

DOUBLE-CHECK: The computed value has correct units for a force, and 720. N is a reasonable tension for a system such as the one given in the question.

15.48. **THINK:** It is given that a glass of milk has a diameter of 10.0 cm. For every step the cowboy takes, the milk sloshes in the glass once. The frequency of oscillation of the milk is the same as the number of steps the cowboy takes in one second. This means that the frequency is f = 2.00 Hz. If the sloshing amplitude is increasing with each step, this frequency must be near the resonant frequency for the standing wave shown in the sketch.

SKETCH:



RESEARCH: The speed of the wave is $v = \lambda f$ and $\lambda = 2L$.

SIMPLIFY: $v = \lambda f = 2Lf$

CALCULATE: $v = 2(0.100 \text{ m})(2.00 \text{ Hz}) = 0.400$ m/s.

ROUND: $v = 0.400$ m/s

DOUBLE-CHECK: The calculated speed has appropriate units and is of realistic magnitude.

15.49. **THINK:** A wave function is given by $y = (2.00 \text{ cm})\sin\left(20.0 \text{ m}^{-1}x\right)\cos\left(150. \text{ s}^{-1}t\right)$. This expression must be converted to the form of $y(x,t) = f(x,t) + g(x,t)$. The speed of the wave on the string is unknown.

SKETCH: A sketch is not necessary.

RESEARCH: Using the trigonometric relation, $\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$, a wave function of the form $y = 2A\sin(\kappa x)\cos(\omega t)$ can be written in the form:

$$y = A\sin(\kappa x + \omega t) + A\sin(\kappa x - \omega t).$$

SIMPLIFY: Simplification is not necessary.

CALCULATE: Use the formula found in RESEARCH, using the values: $2A$ = 2.00 cm, $\kappa = 20.0 \text{ m}^{-1}$ and $\omega = 150. \text{ s}^{-1}$. The wave function can be rewritten as:

$$y(x,t) = (1.00 \text{ cm})\sin\left(\left(20.0 \text{ m}^{-1}\right)x + \left(150. \text{ s}^{-1}\right)t\right) + (1.00 \text{ cm})\sin\left(\left(20.0 \text{ m}^{-1}\right)x - \left(150. \text{ s}^{-1}\right)t\right).$$

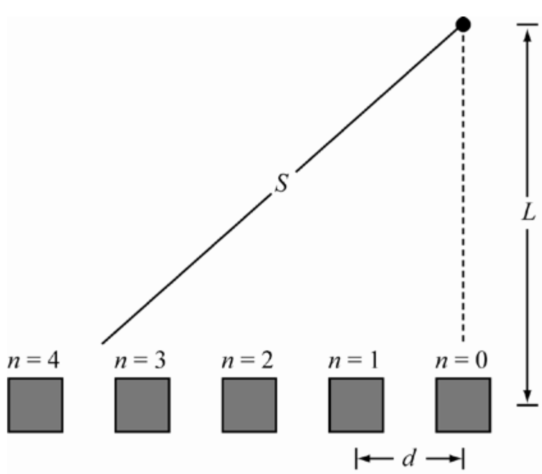
This means that $f(x,t) = (1.00 \text{ cm})\sin\left(\left(20.0 \text{ m}^{-1}\right)x + \left(150. \text{ s}^{-1}\right)t\right)$ and $g(x,t) = (1.00 \text{ cm})\sin\left(\left(20.0 \text{ m}^{-1}\right)x - \left(150. \text{ s}^{-1}\right)t\right)$. Using $\kappa = 20.0 \text{ m}^{-1}$ and $\omega = 150. \text{ s}^{-1}$, the speed of the wave is given by: $v = \frac{\omega}{\kappa} = \frac{150. \text{ s}^{-1}}{20.0 \text{ m}^{-1}} = 7.50$ m/s.

ROUND: The solution requires three significant figures. The answers are already expressed in the appropriate format.

DOUBLE-CHECK: The units for the speed are m/s, which are reasonable units for speed. The magnitude of the speed is reasonable for a wave. The functions *f*(*x*,*t*) and *g*(*x*,*t*) are both trigonometric functions, and are appropriate for left-moving and right-moving waves, respectively.

15.50. **THINK:** An array of wave emitters is shown in the figure in the Sketch step. It is known that *L* is much greater than *d*.

SKETCH:



RESEARCH: The interference of two waves from two sources at a detector is constructive when the path difference from the sources is $0, \lambda, 2\lambda, 3\lambda, \ldots$ and destructive when the path difference is $\lambda/2, 3\lambda/2, 5\lambda/2, \ldots$ The distance of the *n*th emitter to the detector is given by $s = \sqrt{L^2 + n^2d^2}$.

SIMPLIFY: Bringing L outside the square root yields: $s = L\sqrt{1+\frac{n^2d^2}{L^2}}$. Since $L >> d$ and using $\sqrt{1+x} \approx 1 + x/2$, the distance, *s*, is: $s \approx L\left(1+\frac{1}{2}\left(\frac{n^2d^2}{L^2}\right)\right)$.

(a) The extra distance for the *n*th emitter is: $\Delta s = s - L = \frac{n^2d^2}{2L}$.

(b) If $\lambda = d^2/2L$, the extra distance for the *n*th emitter is: $\Delta s = \frac{n^2d^2}{2L}\left(\frac{1}{d^2/2L}\right)\lambda = n^2\lambda$. Since Δs is a multiple of the wavelength, the interference at the detector will be constructive.

(c) Since $\lambda = d^2/2L$, the distance, *d*, between the emitters is $d = \sqrt{2\lambda L}$.

CALCULATE: (c) Substituting $\lambda = 10^{-3}$ m and $L = 1.00 \cdot 10^3$ m yields:

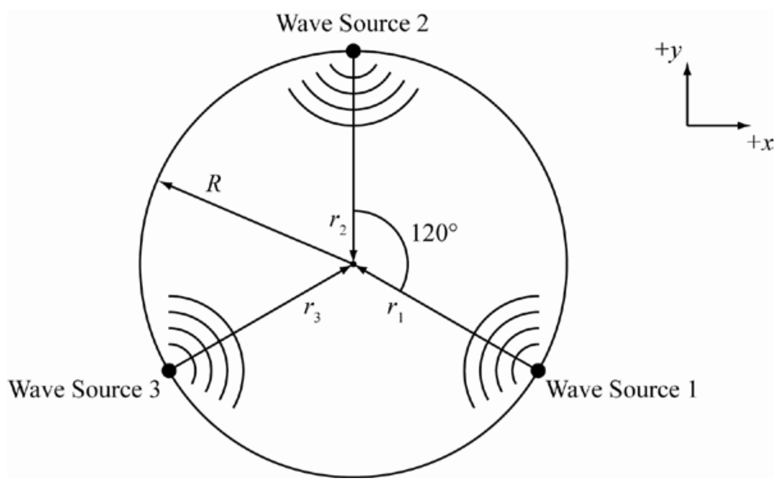
$d = \sqrt{2\left(10.0^{-3}\text{ m}\right)\left(1.00\cdot 10^3\text{ m}\right)} = \sqrt{2.00}\text{ m} = 1.414\text{ m}.$

ROUND: (c) $d = 1.41$ m

DOUBLE-CHECK: A distance of around 1 meter is much less than 1000 meters, as expected.

15.51. **THINK:** This problem is a superposition of three wave sources with three different frequencies, $\omega_1 = 2.00$ Hz, $\omega_2 = 3.00$ Hz and $\omega_3 = 4.00$ Hz. The speed of each wave is 5.00 m/s. The amplitudes of the waves are the same. The wave sources are located at the edges of a circular pool, as shown below. The displacement of a ball in the center of the pool must be plotted as a function of time.

SKETCH:



RESEARCH: From the principle of superposition, the displacement of the ball is given by the sum of all the displacements due to the wave sources, that is,

$$z(t) = z_1(t) + z_2(t) + z_3(t) = \frac{C}{\sqrt{r_1}}\sin\left(\kappa_1 r_1 - \omega_1 t + \phi_1\right) + \frac{C}{\sqrt{r_2}}\sin\left(\kappa_2 r_2 - \omega_2 t + \phi_2\right) + \frac{C}{\sqrt{r_3}}\sin\left(\kappa_3 r_3 - \omega_3 t + \phi_3\right).$$

SIMPLIFY: Since $\phi_1 = \phi_2 = \phi_3 = 0$ and $r_1 = r_2 = r_3 = R$, the displacement, $z(t)$ is:

$$z(t) = \frac{C}{\sqrt{R}}\left[\sin\left(\kappa_1 R - \omega_1 t\right) + \sin\left(\kappa_2 R - \omega_2 t\right) + \sin\left(\kappa_3 R - \omega_3 t\right)\right].$$

Note that when $t < R/v$, the displacement, $z(t)$, is zero, since the waves have not reached the center of the pool. Using the speeds of the waves and the frequencies, the angular frequencies are $\omega_1 = 2\pi f_1$, $\omega_2 = 2\pi f_2$ and $\omega_3 = 2\pi f_3$, and the wave numbers are $\kappa_1 = \omega_1/v$, $\kappa_2 = \omega_2/v$ and $\kappa_3 = \omega_3/v$. Assuming the amplitudes of the waves are all $C/\sqrt{R} = 1$ m, the displacement is given by:

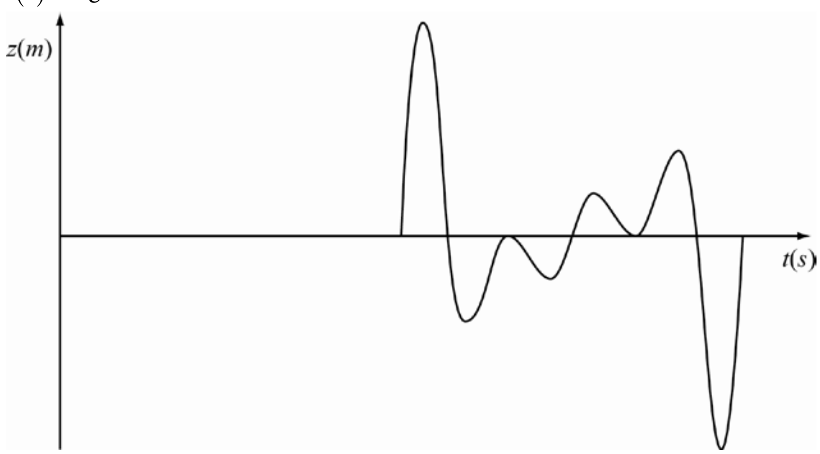
$$z(t)=\left[\sin\left(\frac{2\pi f_1}{v}(R-vt)\right)+\sin\left(\frac{2\pi f_2}{v}(R-vt)\right)+\sin\left(\frac{2\pi f_3}{v}(R-vt)\right)\right],$$

if $R-vt>0$, and $z(t)=0$ if $R-vt<0$. If a new time variable, $t_c=t-R/v$, is used, the above equation simplifies to: $z(t_c)=\sin(2\pi f_1 t_c)+\sin(2\pi f_2 t_c)+\sin(2\pi f_3 t_c)$, if $t_c>0$ and $z(t_c)=0$, if $t_c<0$.

CALCULATE: $z(t_c)=\sin\left(2\pi\left(2.00\text{ s}^{-1}\right)t_c\right)+\sin\left(2\pi\left(3.00\text{ s}^{-1}\right)t_c\right)+\sin\left(2\pi\left(4.00\text{ s}^{-1}\right)t_c\right)$

$$=\sin\left(4\pi\text{ s}^{-1}t_c\right)+\sin\left(6\pi\text{ s}^{-1}t_c\right)+\sin\left(8\pi\text{ s}^{-1}t_c\right)$$

The time for the waves to reach the center of the pool is: $t_0=\dfrac{R}{v}=\dfrac{5.00\text{ m}}{5.00\text{ m/s}}=1.00\text{ s}$. The plot of the displacement, $z(t)$, is given next.



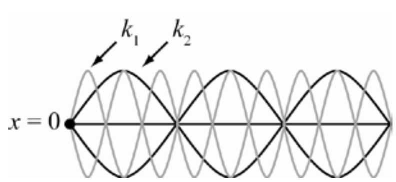
Note that $z(t)$ does not depend on the location of the wave sources at the edges of the pool. This is because the distance to the center of the pool is the same regardless of the location of the sources at the edges of the pool.

ROUND: Rounding is not necessary.

DOUBLE-CHECK: The superposition of three waves is also a wave, which is consistent with the plot.

15.52. **THINK:** Four waves are interfering. Two of the waves are traveling in the positive x direction with angular frequencies $\omega_1=\omega$ and $\omega_2=\omega/3$, and the other two waves are traveling in the negative x direction with angular frequencies $\omega_1=\omega$ and $\omega_2=\omega/3$. The amplitudes of the four waves are the same and equal to A. The first two nodes produced by these waves must be determined. The values $\mu=0.0250$ kg/m, $\omega=3000.$ rad/s and $F=250.$ N are given in the question.

SKETCH:



RESEARCH: Using the superposition principle, the displacement of the string is:

$$y=y_1(x,t)+y_2(x,t)+y_3(x,t)+y_4(x,t).$$

Since the first two waves travel in the positive x direction and the last two waves travel in the negative x direction, the displacement can be written as:

$$y=A\sin(\kappa_1 x-\omega_1 t)+A\sin(\kappa_2 x-\omega_2 t)+A\sin(\kappa_1 x+\omega_1 t)+A\sin(\kappa_2 x+\omega_2 t).$$

SIMPLIFY: Using the trigonometric relation

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),$$

the above expression can be rewritten as: $y = 2A\sin(\kappa_1 x)\cos(\omega_1 t) + 2A\sin(\kappa_2 x)\cos(\omega_2 t)$. The nodes can be determined by solving $y = 0$ for x at all time, t. This means that the conditions, $\sin(\kappa_1 x) = 0$ and $\sin(\kappa_2 x) = 0$ must be satisfied. Therefore, $\kappa_1 x = n\pi$ and $\kappa_2 x = m\pi$, where n and m are integers. Substituting $\kappa_1 = \omega_1 / v = \omega / v$ and $\kappa_2 = \omega_2 / v = \omega / 3v$ into the conditions, it is found that $\omega x / v = n\pi$ and $\omega x / v = 3m\pi$, or $x = nv\pi / \omega$ and $x = 3mv\pi / \omega$. Therefore, the condition for the nodes is:

$$\frac{nv\pi}{\omega} = \frac{3mv\pi}{\omega} \Rightarrow n = 3m.$$

Since m is an integer, the nodes are located at $x = \dfrac{3mv\pi}{\omega}$, $m = 1, 2, 3, \ldots$

CALCULATE: Using $v = \sqrt{\dfrac{F}{\mu}} = \sqrt{\dfrac{250.\text{ N}}{0.0250\text{ kg/m}}} = 100.\text{ m/s}$ the nodes are located at:

$$x = \frac{3m(100.\text{ m/s})\pi}{3000.\text{ rad/s}} = \frac{m\pi}{10}\text{ m}.$$

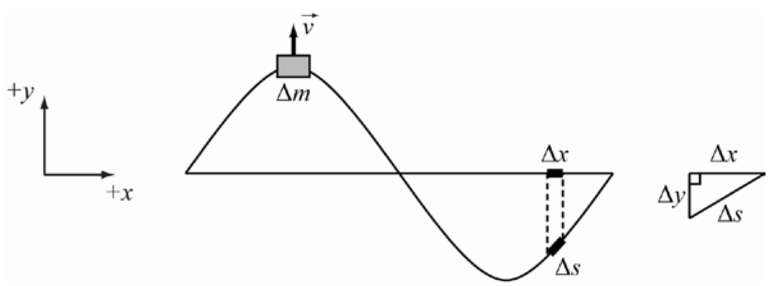
The first two nodes are located at $x_1 = \pi / 10.0\text{ m}$ and $x_2 = 2\pi / 10.0\text{ m} = \pi / 5.00\text{ m}$.

ROUND: $x_1 = 0.314\text{ m}$ and $x_2 = 0.628\text{ m}$.

DOUBLE-CHECK: The units of these calculated values are meters, which are the expected SI unit for distance.

15.53. **THINK:** The equation for a standing wave is $y = 2A\cos(\omega t)\sin(\kappa x)$. The mass density of a string is μ. The time-average kinetic energy and the time-average potential energy must be determined.

SKETCH:



RESEARCH: Consider a small element of the string with a length, Δx, and a mass, Δm. At a time, t, the transverse speed of the element Δm is $v = \partial y / \partial t$. Therefore, the kinetic energy is:

$$\Delta K = \frac{1}{2}\Delta m\left(\frac{\partial y}{\partial t}\right)^2.$$

The potential energy is given by the product of the tension F and the distance the mass Δm must move, Δl.

SIMPLIFY: The kinetic energy is given by

$$\Delta K = \frac{1}{2}\Delta m\omega^2\left(4A^2\right)\sin^2(\omega t)\sin^2(\kappa x) = 2A^2\omega^2\sin^2(\omega t)\sin^2(\kappa x)\Delta m.$$

The time-average of the kinetic energy is obtained by realizing that the time-average of $\sin^2(\omega t)$ is 1/2

$$\Delta K_{\text{ave}} = A^2\omega^2\sin^2(\kappa x)\Delta m.$$

Substituting $\Delta m = \mu\Delta x,$ the kinetic energy per unit of length is:

$$\left(\frac{\Delta K}{\Delta x}\right)_{\text{ave}} = \mu A^2\omega^2 \sin^2\left(\kappa x\right).$$

The potential energy is given by the work done by the tension F times the distance the mass moves in the same direction, $W = \Delta U = F\Delta l.$ We can see from the sketch that $\left(\Delta s\right)^2 = \left(\Delta x\right)^2 + \left(\Delta y\right)^2$. The distance the mass moves is $\Delta l = \Delta s - \Delta x.$ We assume that the deflection of the string is small, which allows us to write

$$\left(\Delta s\right)^2 = \left(\Delta x\right)^2 + \left(\Delta y\right)^2 = \left(\Delta x\right)^2\left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right).$$

Because $\Delta y / \Delta x \ll 1,$ we can write

$$\Delta s = \Delta x\sqrt{1 + \left(\Delta y / \Delta x\right)^2} \approx \Delta x\left(1 + \frac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\right)$$

$$\Delta s \approx \Delta x + \frac{\left(\Delta y\right)^2}{2\left(\Delta x\right)}$$

$$\Delta l = \Delta s - \Delta x = \frac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\left(\Delta x\right).$$

We can then write the potential energy as

$$\Delta U = F\Delta l = \frac{F}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\Delta x.$$

We can calculate $\Delta y / \Delta x$

$$\frac{\Delta y}{\Delta x} = 2A\kappa\cos\left(\omega t\right)\cos\left(\kappa x\right),$$

which gives us

$$\Delta U = \frac{F}{2}\left(2A\kappa\cos\left(\omega t\right)\cos\left(\kappa x\right)\right)^2\Delta x = 2F\kappa^2A^2\cos^2\left(\omega t\right)\cos^2\left(\kappa x\right)\Delta x.$$

Since the time-average of $\cos\left(\omega t\right)$ is 1/2, the time-averaged potential energy per unit length is then

$$\frac{\Delta U_{\text{ave}}}{\Delta x} = F\kappa^2A^2\cos^2\left(\kappa x\right) = F\left(\kappa A\right)^2\cos^2\left(\kappa x\right).$$

CALCULATE: This step is not necessary.
ROUND: Rounding is not necessary.
DOUBLE-CHECK: The average value of the kinetic energy per unit length is

$$\frac{\overline{K_{\text{ave}}}}{\Delta x} = \frac{1}{2}\mu A^2\omega^2.$$

The average value of the potential energy per unit length is

$$\frac{\overline{\Delta U_{\text{ave}}}}{\Delta x} = \frac{1}{2}F\left(\kappa A\right)^2.$$

Since $\kappa = \omega / v$ and $\mu v^2 = F$

$$\frac{\overline{\Delta U_{\text{ave}}}}{\Delta x} = \frac{1}{2}\left(\mu v^2\right)\left(\left(\frac{\omega}{v}\right)^2 A\right)^2 = \frac{1}{2}\mu A^2\omega^2.$$

So the average value per unit length of the kinetic energy is equal to the average value of the potential energy per unit length, which means our results are correct.

15.54. A sinusoidal wave is moving in the positive *x* direction. This means that the wave function is of the form $y = A\sin\left(\kappa x - \omega t + \phi\right)$. Note that the negative sign in from of ω means the wave travels in the positive *x* direction. The next step is to determine ω, κ and ϕ. Using the wavelength of the wave, the angular wave number, κ, is:

$$\kappa = \frac{2\pi}{\lambda} = \frac{2\pi}{4\text{ m}} = \frac{\pi}{2}\text{ rad/m} = 0.5\pi\text{ rad/m}.$$

Similarly, using the frequency of the wave, the angular frequency can be determined as $\omega = 2\pi f = 2\pi(50.0\text{ Hz}) = 100.\pi\text{ rad/s}$. The constant phase, ϕ, is an arbitrary value and it is assumed to be zero. The amplitude of the wave is A = 3.00 cm. Therefore, the wave equation is given by $y = (3.00\text{ cm})\sin(0.5\pi x - 100.\pi t)$.

15.55. (a) To determine the frequency of the fundamental note of a guitar string, the speed of a wave on the string is needed. Using the density of the string, $\mu = m / L_\text{s}$, the speed of wave is:

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{TL_\text{s}}{m}}.$$

The fundamental frequency is then: $f = \frac{v}{2L} = \sqrt{\frac{TL_\text{s}}{m}}\left(\frac{1}{2L}\right)$. Note that L is the distance between two fixed ends of the guitar string. Substituting m = 10.0 g = 0.0100 kg, L = 0.650 m, $L_\text{s} = 1.00\text{ m}$ and T = 81.0 N into the above equation yields:

$$f = \sqrt{\frac{81.0\text{ N}(1.00\text{ m})}{0.0100\text{ kg}}}\left(\frac{1}{2(0.650\text{ m})}\right) = 69.2\text{ Hz}.$$

(b) Replacing the mass with m = 16.0 g = 0.0160 kg, the frequency becomes:

$$f = \sqrt{\frac{81.0\text{ N}(1.00\text{ m})}{0.0160\text{ kg}}}\left(\frac{1}{2(0.650\text{ m})}\right) = 54.7\text{ Hz}.$$

15.56. A sinusoidal wave is propagating in the negative x direction with a speed of 120. m/s. This implies that the wave equation is of the form $y(x,t) = A\sin(\kappa x + \omega t + \phi)$. From the range of displacements, $A = \text{Range}/2 = 6.00\text{ cm}/2 = 3.00\text{ cm}$. The period of the oscillation is T = 4.00 s, since the particle swings back and forth in 4.00 seconds. Therefore, the angular frequency is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{4.00\text{ s}} = 0.500\pi\text{ rad/s}.$$

From the speed, v, and the angular frequency, the angular wave number is:

$$\kappa = \frac{\omega}{v} = \frac{0.500\pi\text{ rad/s}}{120.\text{ m/s}} = 0.00400\pi\text{ rad/m}.$$

At t = 0, the displacement is y = 0 and the particle moves to positive values of y immediately after t = 0. This means that $\partial y / \partial t$ is positive at t = 0. Substituting t = 0 and x = 0 into the wave equation, the phase constant, ϕ, is found to be: $y = 0 = A\sin(0+0+\phi) = 0 \Rightarrow \sin\phi = 0 \Rightarrow \phi = m\pi$, with $m = 0, 1, 2 \ldots$ Substituting t = 0 and x = 0 into the expression for $\partial y / \partial t$, the second condition for the phase constant is

$$\frac{\partial y}{\partial t} = \omega A\cos(0+0+\phi) = \text{a positive value}.$$

This means that $\cos\phi$ must be positive. Therefore, the phase constant is $\phi = m\pi$, with $m = 0, 2, 4 \ldots$, so choose $\phi = 0$. Collecting all parameters, the wave equation is $y = (3.00\text{ cm})\sin(0.00400\pi x + 0.500\pi t)$.

15.57. (a) Here, there is a half period of the oscillation. From the figure, $T/2 = 40.0\text{ ms}$, therefore the period is $T = 2(40.0\text{ ms}) = 80.0\text{ ms}$.

(b) Since this is a sinusoidal wave, $y(t) = A\sin(\omega t)$, where $\omega = 2\pi / T$. The maximum transverse speed is $v_\text{t} = dy/dt = \omega A\cos(\omega t)$, at t = 0, that is,

$$v_{\max} = \omega A = \frac{2\pi}{T}A = \frac{2\pi}{80.0\text{ ms}}\left(10.0\text{ cm}\right) = 0.785\text{ cm/ms}.$$

Converting to SI units, $v_{\max} = 7.85$ m/s.

(c) The maximum acceleration is obtained from $a = d^2y/dt^2$. Differentiating $y(t)$ twice gives $a = -\omega^2 A\sin(\omega t)$. Therefore, the maximum acceleration is:

$$a_{\max} = \omega^2 A = \left(\frac{2\pi}{T}\right)^2 A = \left(\frac{2\pi}{80.0\cdot 10^{-3}\text{ s}}\right)^2\left(0.100\text{ m}\right) = 616.8\text{ m/s}^2 \approx 617\text{ m/s}^2.$$

15.58. It is given that a wire has a mass, m = 10.0 g, and a length, L = 50.0 cm. The tension of the wire is $T = 50.0$ N and the two ends of the wire are held rigidly.

(a) The speed of a wave on the wire is given by $v = \sqrt{T/\mu}$. Using $\mu = m/L$, the speed is $v = \sqrt{TL/m}$. Substituting m = 10.0 g = 0.0100 kg, L = 0.500 m and T = 50.0 N gives:

$$v = \sqrt{\frac{50.0\text{ N}\left(0.500\text{ m}\right)}{0.0100\text{ kg}}} = 50.0\text{ m/s}.$$

(b) The fundamental frequency is: $f = \dfrac{v}{2L} = \dfrac{50.0\text{ m/s}}{2\left(0.500\text{ m}\right)} = 50.0$ Hz.

(c) The third harmonic frequency is: $f = n\dfrac{v}{2L}$, with n = 3: $f = 3\left(\dfrac{50.0\text{ m/s}}{2\left(0.500\text{ m}\right)}\right) = 150.$ Hz.

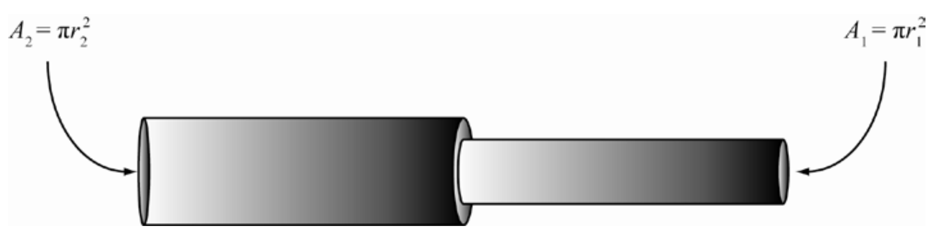
15.59. The wave speed on a brass wire is given by $v = \sqrt{T/\mu}$. The linear density, μ, is equal to $\mu = \rho A$, where $A = \pi r^2$ is the cross-sectional area of the wire. Therefore, the speed of the wave is:

$$v = \sqrt{\frac{T}{\rho A}} = \sqrt{\frac{T}{\rho\pi r^2}}.$$

Inserting $r = 0.500\text{ mm} = 0.500\cdot 10^{-3}$ m, T = 125 N and $\rho = 8.60\cdot 10^3\text{ kg/m}^3$ yields:

$$v = \sqrt{\frac{125\text{ N}}{\left(8.60\cdot 10^3\text{ kg/m}^3\right)\pi\left(0.500\cdot 10^{-3}\text{ m}\right)^2}} = 136.04\text{ m/s} = 136\text{ m/s}.$$

15.60. The wires are made of the same material, so assume that the densities of both wires are the same. Say the density is ρ. The linear density of a wire is related to the volume density, ρ, by $\mu = \rho A = \rho\pi r^2$.



The speed of a wave on the first wire is $v_1 = \sqrt{T/\rho A_1}$. From this, the tension is found to be $T = \rho A_1 {v_1}^2$. The speed of a wave on the second wire is $v_2 = \sqrt{T/\rho A_2}$. Substituting $T = \rho A_1 {v_1}^2$, gives:

$$v_2 = \sqrt{\frac{\rho A_1 {v_1}^2}{\rho A_2}} = v_1\sqrt{\frac{\pi {r_1}^2}{\pi {r_2}^2}} \Rightarrow v_2 = v_1\left(\frac{r_1}{r_2}\right) = 50.0\text{ m/s}\left(\frac{0.500\text{ mm}}{1.00\text{ mm}}\right) = 25.0\text{ m/s}.$$

15.61. The fundamental frequency of a string is given by $f = v/2L$. Substituting $v=\sqrt{T/\mu}$, gives:

$$f=\frac{1}{2L}\sqrt{\frac{T}{\mu}}=\frac{1}{2L\sqrt{\mu}}\sqrt{T}.$$

The ratio of two fundamental frequencies of the string is:

$$\frac{f_1}{f_2}=\frac{\sqrt{T_1}/\left(2L\sqrt{\mu}\right)}{\sqrt{T_2}/\left(2L\sqrt{\mu}\right)}=\sqrt{\frac{T_1}{T_2}}.$$

Therefore, the ratio of tensions in the two strings is: $\frac{T_1}{T_2}=\left(\frac{f_1}{f_2}\right)^2$. Substituting $f_1 = 262\text{ Hz}$ and $f_2 = 1046.5\text{ Hz}$ yields:

$$\frac{T_1}{T_2}=\left(\frac{262\text{ Hz}}{1046.5\text{ Hz}}\right)^2=0.06268\approx 0.0627.$$

15.62. A sinusoidal wave has an equation of the form $y=A\sin\left(\kappa x-\omega t\right)$. The maximum displacement of a point on the string is equal to the amplitude, A. The speed of the point is given by $v=\partial y/\partial t=-\omega A\cos\left(\kappa x-\omega t\right)$. This means that the maximum speed is $v_{\text{max}}=\omega A$. The acceleration of the point is:

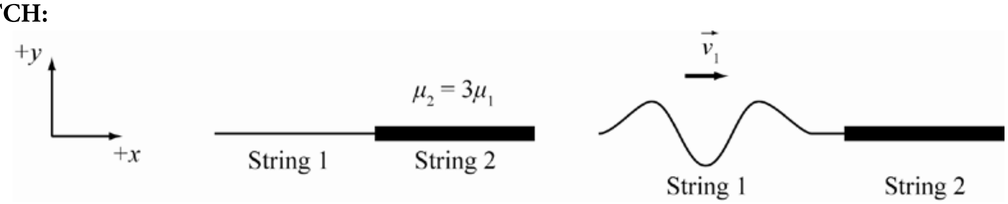
$$a=\frac{\partial^2 y}{\partial t^2}=-\omega^2 A\sin\left(\kappa x-\omega t\right).$$

This implies that the maximum acceleration is $a_{\text{max}}=\omega^2 A$. Using $v_{\text{max}}=\omega A$, the maximum acceleration becomes $a_{\text{max}}=\left(v_{\text{max}}/A\right)^2 A=v_{\text{max}}{}^2/A$. Inserting A = 2.00 cm = 0.0200 m and $v_{\text{max}}=1.00\text{ m/s}$ yields:

$$a_{\text{max}}=\frac{\left(1.00\text{ m/s}\right)^2}{0.0200\text{ m}}=50.0\text{ m/s}^2.$$

15.63. **THINK:** Two strings are connected and have the same tension, T. The linear mass density of string 2 is $\mu_2=3\mu_1$. If the speed, the frequency and the wavelength of a wave on string 1 are v_1, f_1 and λ_1, respectively. The corresponding variables for string 2, v_2, f_2 and λ_2, can be determined in terms of string 1's variables.

SKETCH:



RESEARCH: It is known that when a wave travels to a different material or medium, the frequency of the wave does not change. This means that the frequency of string 2, f_2, is equal to the frequency of string 1, that is, $f_2 = f_1$. The speeds of the wave on string 1 and string 2 are $v_1=\sqrt{T/\mu_1}$ and $v_2=\sqrt{T/\mu_2}$.

SIMPLIFY: Since v_1 and μ_1 are known, the tension is given by $T=\mu_1 v_1{}^2$. Substituting this expression into v_2, the speed of the wave on string 2 becomes:

$$v_2=\sqrt{\frac{\mu_1 v_1{}^2}{\mu_2}}=v_1\sqrt{\frac{\mu_1}{\mu_2}}.$$

Using $\mu_2 = 3\mu_1$, the above equation simplifies to $v_2 = v_1 / \sqrt{3}$. The wavelength of the wave on string 2 is determined using the relation $v_2 = \lambda_2 f_2$. Therefore, $\lambda_2 = v_2 / f_2$. Substituting $f_2 = f_1$ and $v_2 = v_1 / \sqrt{3}$, gives $\lambda_2 = v_1 / \sqrt{3} f_1$. Since $v_1 / f_1 = \lambda_1$, this becomes $\lambda_2 = \lambda_1 / \sqrt{3}$. Therefore, $f_2 = f_1$, $v_2 = v_1 / \sqrt{3}$ and $\lambda_2 = \lambda_1 / \sqrt{3}$.

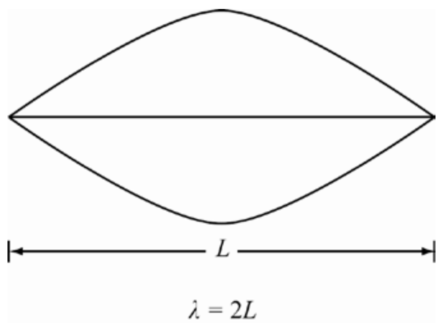
CALCULATE: This step is not necessary.

ROUND: This step is not necessary.

DOUBLE-CHECK: It is expected that as the speed of a wave decreases, the wavelength of the wave also decreases.

15.64. **THINK:** A cable has a length of 2.70 m, a diameter of 1.00 cm and a density of 7800 kg/m^3. The tension in the cable is 840. N. The fundamental frequency of vibration of the cable must be determined.

SKETCH:



RESEARCH: The fundamental frequency of vibration is given by $f = v/(2L)$. The speed of a wave on the cable is given by $v = \sqrt{T/\mu}$, where the linear mass density is equal to $\mu = \rho A$. A is the cross-sectional area of the cable and is given by $A = \pi r^2 = \pi D^2 / 4$.

SIMPLIFY: The fundamental frequency is given by: $f = \frac{1}{2L}\sqrt{\frac{T}{\rho \pi D^2 / 4}} = \frac{1}{L}\sqrt{\frac{T}{\rho \pi D^2}}$.

CALCULATE: Substituting $L = 2.70$ m, $D = 1.00 \cdot 10^{-2}$ m, $\rho = 7800 \text{ kg/m}^3$ and $T = 840.$ N yields:

$$f = \frac{1}{2.70 \text{ m}}\sqrt{\frac{840. \text{ N}}{\left(7800 \text{ kg/m}^3\right)\pi\left(1.00 \cdot 10^{-2} \text{ m}\right)^2}} = 6.857 \text{ Hz}.$$

ROUND: $f = 6.86$ Hz

DOUBLE-CHECK: This result is reasonable.

15.65. **THINK:** A wave function is given as $y(x,t) = 0.0200\sin(5.00x - 8.00t)$. The wavelength, the frequency and the velocity of the wave must be determined. If the mass density is $\mu = 0.10$ kg/m, the tension on the string can be determined.

SKETCH: A sketch is not necessary.

RESEARCH: The wave function can be expressed as $y = A\sin(\kappa x - \omega t)$.

SIMPLIFY: Comparing the above equation with the given equation yields $\kappa = 5.00 \text{ m}^{-1}$, $\omega = 8.00 \text{ s}^{-1}$ and A = 0.0200 m. The wavelength can be determine from $\lambda = 2\pi / \kappa$ and the frequency can be determined from $f = \omega / 2\pi$. The tension is determined from $v = \sqrt{T/\mu} \Rightarrow T = \mu v^2$.

CALCULATE:

(a) The wavelength is given by: $\lambda = \dfrac{2\pi}{5.00\text{ m}^{-1}} = 1.2566\text{ m}$. The frequency is given by: $f = \dfrac{8.00\text{ s}^{-1}}{2\pi} = 1.2732\text{ Hz}$.

(b) Using the values of λ and f, the speed of the wave is $v = \lambda f = 1.26\text{ m}(1.27\text{ Hz}) = 1.6002\text{ m/s}$.

(c) The tension in the string is $T = (0.10\text{ kg/m}^3)(1.60\text{ m/s})^2 = 0.256\text{ N}$.

ROUND:

(a) $\lambda = 1.26\text{ m}$, $f = 1.26\text{ Hz}$

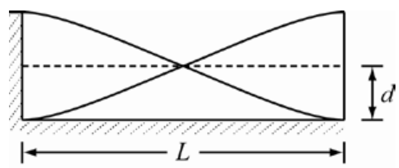
(b) $v = 1.60\text{ m/s}$

(c) $T = 0.256\text{ N}$

DOUBLE-CHECK: The units of the calculated results are standard SI units. This supports the results as reasonable.

15.66. **THINK:** A standing wave is produced in a bathtub of length $L = 1.50$ m, holding water with a depth of $d = 0.380$ m. The frequency of the standing wave can be determined from considering the surface speed of the wave and its wavelength.

SKETCH:



RESEARCH: Use $f = v/\lambda$ and $v_{\text{surface}} = \sqrt{gd}$. Assume it is an $n = 1$ standing wave, and $\lambda = 2L$.

SIMPLIFY: $f = \dfrac{v}{\lambda} = \dfrac{\sqrt{gd}}{2L}$

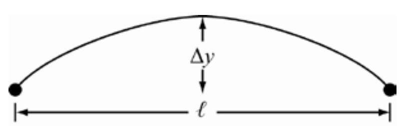
CALCULATE: $f = \dfrac{\sqrt{(9.81\text{ m/s}^2)(0.380\text{ m})}}{2(1.50\text{ m})} = 0.6436\text{ s}^{-1}$

ROUND: Since L and d have three significant figures, the result should be rounded to $f = 0.645\text{ s}^{-1}$.

DOUBLE-CHECK: The period is $T = 1/f = 1.6$ s and this is a reasonable period for a standing wave in a bathtub.

15.67. **THINK:** The guitar string has length, $l = 0.800$ m, and it is oscillating at its fundamental frequency, which means that it has one antinode in the middle, and thus the guitar string length is half of the wavelength (see sketch). The wave speed, v, can be determined from knowing the wavelength and the frequency, which is also given (261.6 Hz). But to find the maximum speed of the midpoint of the string, v_{max}, we have to take the derivative with respect to time.

SKETCH:



RESEARCH: The wave speed is given by $v = \lambda f$. For $n = 1$, $\lambda_1 = 2l$. To determine v_{max}, consider the standing wave equation $y(x,t) = 2A\sin(\kappa x)\cos(\omega t)$. For the midpoint $\sin(\kappa x) = 1$, and thus the midpoint oscillates in times according to $y_{\text{mid}}(t) = 2A\cos(\omega t)$. Taking the derivative with respect to time, we find $v_{\text{mid}}(t) = -\omega 2A\sin(\omega t)$.

SIMPLIFY: The velocity of the midpoint reaches its maximum value when the sine has a value of -1. We also use $\omega = 2\pi f$. And finally we use the fact that the initial displacement of the midpoint from equilibrium $\Delta y_0 = 2.00$ mm was specified, which means that $2A = \Delta y_0$.

$$v = \lambda_1 f_1 = 2l f_1$$
$$v_{\text{max,mid}} = 2A\omega = \Delta y_0 2\pi f_1$$

CALCULATE: $v = 2(0.800 \text{ m})(261.6 \text{ Hz}) = 418.56 \text{ m/s}$, $v_{\text{max,mid}} = 2\pi(2.00 \text{ mm})(261.6 \text{ Hz}) = 3.287 \text{ m/s}$

ROUND: All the given values have three significant figures, so the results should be rounded to v = 419. m/s and $v_{\text{max,mid}} = 3.29$ m/s.

DOUBLE-CHECK: $v_{\text{max,mid}}$ should be much less than v, because in the transverse direction the string is moving at this speed, but in the wave direction, no part of the string actually moves at this speed.

15.68. The standing wave is represented by $y(x,t) = 1.00 \cdot 10^{-2} \sin(25.0x)\cos(1200.t)$. The string has a linear mass density $\mu = 0.0100$ kg/m and a mass, m, hangs on one end of the string. In addition, n = 3. By comparison with the general standing wave equation, $y(x,t) = 2A\sin(\kappa x)\cos(\omega t)$, and taking units to be in meters and seconds for x and t, respectively: $\kappa_3 = \dfrac{2\pi}{\lambda_3} = 25 \text{ m}^{-1} \Rightarrow \lambda_3 = \dfrac{2\pi}{25}$ m,

$\omega_3 = 2\pi f_3 = 1200 \text{ Hz} \Rightarrow f_3 = \dfrac{1}{\pi}\left(6.00 \cdot 10^2\right)$ Hz, $\quad 2A = 0.0100 \text{ m} \Rightarrow A = 5.00 \cdot 10^{-3}$ m.

(a) The length, L, of the string can be determined from:

$$L = n\frac{\lambda_n}{2} = 3\left(\frac{\lambda_3}{2}\right) = \frac{3}{2}\left(\frac{2\pi}{25.0} \text{ m}\right) = \frac{3\pi}{25.0} \text{ m} \approx 0.377 \text{ m}.$$

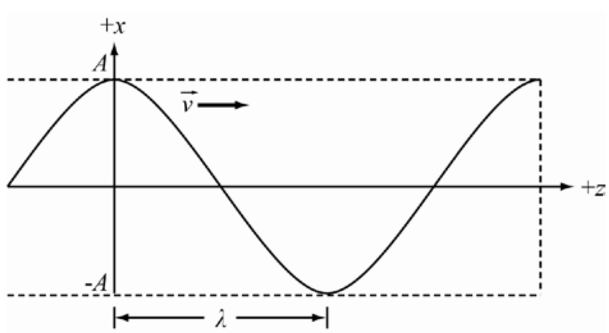
(b) The velocity is: $v = \lambda_3 f_3 = \dfrac{2\pi}{25.0}\left(\dfrac{6.00 \cdot 10^2}{\pi}\right) \text{ m/s} = 48.0 \text{ m/s}.$

(c) The mass is $m = T/g$, by Newton's third law. From $v = \sqrt{T/\mu}$, $T = v^2\mu$. Then:

$$m = \frac{v^2\mu}{g} = \frac{(48.0 \text{ m/s})^2 (0.0100 \text{ kg/m})}{9.81 \text{ m/s}^2} = 2.349 \text{ kg} \approx 2.35 \text{ kg}.$$

15.69. The known values for the transverse harmonic wave are $\lambda = 0.200$ m, f = 500. Hz, and A = 0.0300 m. It travels in the $+\hat{z}$ direction and its oscillations occur in the xz-plane. At t = 0 s, $(x_0, z_0) = (A, 0)$.

(a)



(b) $v = \lambda f = (0.200 \text{ m})(500. \text{ Hz}) = 100. \text{ m/s}$

(c) $\kappa = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{0.200 \text{ m}} = 10.0\pi \text{ rad/m} \approx 31.4 \text{ rad/m}$

(d) $\mu = 30.0 \text{ g/cm} = 0.0300 \text{ kg/m}$

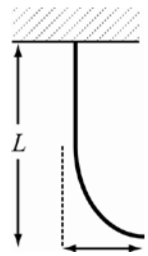
From $v=\sqrt{T/\mu}$, $T=v^2\mu=(100.\text{ m/s})^2(0.0300\text{ kg/m})=300.\text{ N}$.

(e) For a traveling wave, in general, $y(x,t)=A\sin(\kappa x-\omega t+\phi_0)$. By inspection, $\phi_0=\pi/2$. Then:

$$x=D(z,t)=(0.0300\text{ m})\sin\left((10.0\pi\text{ rad/m})x-2\pi\text{ rad}(500.\text{ Hz})t+\frac{\pi}{2}\text{ rad}\right)$$
$$=(0.0300\text{ m})\cos\left((10.0\pi\text{ rad/m})x-(1000.\pi\text{ rad/s})t\right).$$

15.70. **THINK:** A cable of mass, *M*, and length, *L* = 5.00 m, hangs from a support. Determine the time, t', for a small transverse displacement to propagate from the bottom to the top of the cable. Recall that tension is proportional to the mass of the cable.

SKETCH:



RESEARCH: As the wave travels upward, the tension in the cable at the location of the transverse displacement will increase as the mass below the displacement increases. In general, $\mu=m/L$, so the mass of the cable below the displacement at height *y*, is $m=\mu y$ (where *y* = 0 at the bottom of the cable). Then the tension as a function of height is $T=mg=\mu yg$. The traveling wave's velocity is $v=\sqrt{T/\mu}$, and $v=dy/dt$. By separation of variables, and integrating *y* over the length of the cable, the travel time, t', can be determined.

SIMPLIFY: $v=\sqrt{T/\mu}=\sqrt{\mu yg/\mu}=\sqrt{yg}$

Equating the velocity expressions gives:

$$\frac{dy}{dt}=\sqrt{yg}\ \Rightarrow\ (yg)^{-\frac{1}{2}}dy=dt\ \Rightarrow\ \int_0^L(yg)^{-\frac{1}{2}}dy=\int_0^{t'}dt\ \Rightarrow\ t'=g^{-\frac{1}{2}}\left[2y^{\frac{1}{2}}\right]_0^L=2\sqrt{\frac{L}{g}}.$$

CALCULATE: $t'=2\sqrt{\dfrac{5.00\text{ m}}{9.81\text{ m/s}^2}}=1.4278\text{ s}$

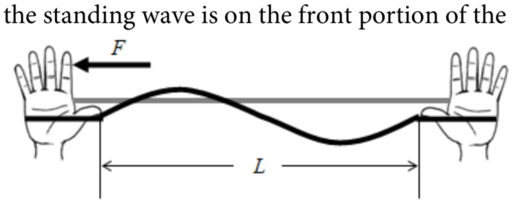
ROUND: Since *L* has three significant figures, $t'=1.43$ s.

DOUBLE-CHECK: The units of the result are the correct units of time. This is a reasonable travel time for a pulse along a 5.00 m cable.

Multi-Version Exercises

15.71. **THINK:** The mass and total length of the rubber band can be used to find its mass density. The mass density, tension, and the length of the standing wave can be used to find the lowest-frequency (fundamental frequency) vibration.

SKETCH: Imagine that the standing wave is on the front portion of the rubber band.



RESEARCH: Since this is a transverse wave on a rubber band, the velocity can be computed from the tension *F* and linear mass density μ using the equation $v=\sqrt{F/\mu}$. The linear mass density $\mu=m/l$ is computed from the total length and total mass of the rubber band. The resonance frequency of the fundamental frequency is given by $f_1 = \frac{v}{2L}$, where L is the length of the vibrating portion of the rubber band.

SIMPLIFY: Since the velocity $v = \sqrt{F/\mu}$, the fundamental frequency is given by $f_1 = \frac{v}{2L} = \frac{\sqrt{F/\mu}}{2L} = \frac{\sqrt{F}}{2L\sqrt{\mu}}$. Finally, express the mass density in terms of the total length and total mass of the rubber band to get $f_1 = \frac{\sqrt{F}}{2L\sqrt{m/l}}$.

CALCULATE: According to the problem statement, the tension on each side of the rubber band is F = 1.777 N, the total length of the rubber band is 20.27 cm = 0.2027 m, the length of the vibrating portion of the rubber band is 8.725 cm = 0.08725 m, and the mass of the rubber band is 0.3491 g = $3.491\cdot 10^{-4}$ kg. The fundamental frequency is

$$\begin{aligned} f_1 &= \frac{\sqrt{F}}{2L\sqrt{m/l}} \\ &= \frac{\sqrt{1.777 \text{ N}}}{2\cdot 0.08725 \text{ m}\sqrt{3.491\cdot 10^{-4} \text{ kg}/0.2027 \text{ m}}} \\ &= 184.0772987 \text{ Hz} \end{aligned}$$

ROUND: All of the numbers in this problem have four significant figures and the final answer will also have four figures. The lowest frequency of a vibration on this part of the rubber band is 184.1 Hz.

DOUBLE-CHECK: It is possible to stretch a rubber band with your hands so a low frequency sound is produced when it is plucked. A frequency of 184.1 Hz corresponds approximately to the F-sharp below middle C on a piano. It is possible to reproduce this with a rubber band at home, confirming that the answer is reasonable.

15.72.

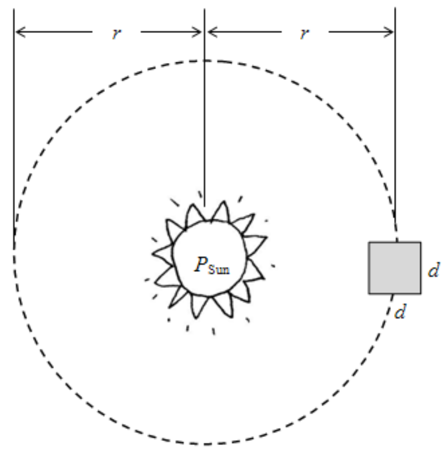
$$\begin{aligned} f_1 &= \frac{\sqrt{F}}{2L\sqrt{m/l}} \\ F &= \frac{4mL^2 f_1^2}{l} \\ &= \frac{4\left(0.4245\cdot 10^{-3} \text{ kg}\right)\left(0.08117 \text{ m}\right)^2\left(184.2 \text{ Hz}\right)^2}{0.2091 \text{ m}} \\ &= 1.825 \text{ N} \end{aligned}$$

15.73.

$$\begin{aligned} f_1 &= \frac{\sqrt{F}}{2L\sqrt{m/l}} \\ L &= \frac{\sqrt{F}}{2f_1\sqrt{m/l}} \\ &= \frac{\sqrt{1.851 \text{ N}}}{2\left(254.6 \text{ Hz}\right)\sqrt{\left(0.1701\cdot 10^{-3} \text{ kg}\right)/\left(0.2155 \text{ m}\right)}} \\ &= 9.510 \text{ cm} \end{aligned}$$

15.74. **THINK:** Assuming that Sun's power is emitted uniformly from every point on the spherical surface, the power per unit area can be computed for a given orbital radius. It is then possible to compute the power intercepted by the solar panel using the area and efficiency of the solar panel.

SKETCH: The sun and satellite are not shown to scale. Only the solar panel portion of the satellite is shown.



RESEARCH: The area of the solar panel is $A = d^2$. The surface area of a sphere or radius r is $4\pi r^2$. The power of the sun P_{Sun} is distributed evenly, so the power intercepted by the solar panel is $P = \frac{P_{\text{Sun}}}{4\pi r^2} \cdot A$. The solar panel is not 100% efficient, so the power delivered is $P_{\text{del}} = \varepsilon P$, where ε is the efficiency of the solar panel.

SIMPLIFY: First, use $A = d^2$ to find the power of intercepted by the solar panel, $P = \frac{P_{\text{Sun}}}{4\pi r^2} \cdot A = \frac{d^2 P_{\text{Sun}}}{4\pi r^2}$.

The power delivered is then $P_{\text{del}} = \varepsilon P = \varepsilon \frac{d^2 P_{\text{Sun}}}{4\pi r^2}$.

CALCULATE: According to the question, the efficiency of the solar panel is ε = 16.57% = 0.1657, the radius of the satellite's orbit is $r = 4.949 \cdot 10^7$ km $= 4.949 \cdot 10^{10}$ m, and the total power output of the Sun is $3.937 \cdot 10^{26}$ W. Since the edges of the square solar panel are each d = 1.459 m long, the power provided by the solar panel is

$$\begin{aligned} P_{\text{del}} &= \varepsilon \frac{d^2 P_{\text{Sun}}}{4\pi r^2} \\ &= 0.1657 \frac{(1.459 \text{ m})^2 \cdot 3.937 \cdot 10^{26} \text{ W}}{4\pi \left(4.949 \cdot 10^{10} \text{ m}\right)^2} \\ &= 4511.840469 \text{ W} \\ &= 4.511840469 \text{ kW} \end{aligned}$$

ROUND: The numbers in the problem all have four significant figures, so the final answer should also have four figures. The total power provided to the satellite by the solar panel is 4512 W.

DOUBLE-CHECK: The sunlight hitting the solar panel is about $\frac{d^2}{4\pi r^2} = 6.916 \cdot 10^{-23}$ of the total sunlight. The power of the solar panel should have an order of magnitude about 10^{-23} times the power output of the sun. $10^{26} \cdot 10^{-23} = 10^3$, so the final answer ($4.512 \cdot 10^3$) is indeed of an order about 10^{-23} times the power output of the sun ($3.937 \cdot 10^{26}$ W), confirming that the final answer is reasonable.

15.75. $P_{\text{del}} = \varepsilon \dfrac{d^2 P_{\text{Sun}}}{4\pi r^2}$

$$d = 2r\sqrt{\frac{\pi}{\varepsilon}\left(\frac{P_{\text{del}}}{P_{\text{Sun}}}\right)}$$

$$= 2\left(6.103 \cdot 10^{10} \text{ m}\right)\sqrt{\frac{\pi}{0.1687}\left(\frac{5.215 \cdot 10^3 \text{ W}}{3.937 \cdot 10^{26} \text{ W}}\right)}$$

$$= 1.917 \text{ m}$$

15.76. $P_{\text{del}} = \varepsilon \dfrac{d^2 P_{\text{Sun}}}{4\pi r^2}$

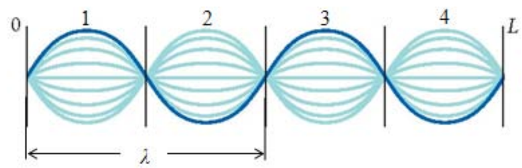
$$\varepsilon = \frac{4\pi r^2}{d^2}\left(\frac{P_{\text{del}}}{P_{\text{Sun}}}\right)$$

$$= \frac{4\pi\left(7.257 \cdot 10^{10} \text{ m}\right)^2}{\left(2.375 \text{ m}\right)^2}\left(\frac{5.768 \cdot 10^3 \text{ W}}{3.937 \cdot 10^{26} \text{ W}}\right).$$

$$= 0.1719 = 17.19\%$$

15.77. **THINK:** The mass density of the string and tension on the string are given in the problem. To find the frequency, it is necessary to determine at which harmonic the string is oscillating, which can be deduced from the image.

SKETCH: Use the image from the text for your sketch, labeling the wavelength and each spot where the amplitude of the wave is maximal:



RESEARCH: Looking at the picture, there are four places where the amplitude is maximal, so the string is oscillating in its fourth harmonic. This means that the frequency of the oscillation is given by the equation $f_4 = 4\dfrac{\sqrt{F}}{2L\sqrt{\mu}}$. Since the mass density and length of the string are given in grams and centimeters while the tension is given in Newtons, the conversion factors of 100 cm = 1 m and 1000 g = 1 kg will be needed.

SIMPLIFY: Using algebra, rewrite the equation $f_4 = 4\dfrac{\sqrt{F}}{2L\sqrt{\mu}}$ as $f_4 = \dfrac{2}{L}\sqrt{\dfrac{F}{\mu}}.$

CALCULATE: The problem statement includes the information that the total length of the string L = 116.7 cm = 1.167 m, the string's mass density is 0.2833 g/cm = 0.02833 kg/m, and the tension on the string is 18.25 N. The frequency is thus

$$f_4 = \frac{2}{L}\sqrt{\frac{F}{\mu}}$$

$$= \frac{2}{1.167 \text{ m}}\sqrt{\frac{18.25 \text{ N}}{2.833 \cdot 10^{-2} \text{ kg/m}}}$$

$$= 43.49779947 \text{ Hz.}$$

ROUND: The values in the question all have four significant figures. The harmonic number (an integer) is considered to have infinite precision, so the final answer should also have four figures. The string is vibrating at 43.50 Hz.

DOUBLE-CHECK: The velocity of the wave is given by $v = \sqrt{F/\mu} = 25.38$ m/s. Since $v = f\lambda$, this means that, if the frequency is 43.50 Hz and the velocity is 25.38 m/s, the wavelength is

58.35 cm. This agrees with the observation that the wavelength is half of the total length of the string (L / 2 = 58.35 cm), confirming that the calculated frequency was correct.

15.78. $f_4 = \frac{2}{L}\sqrt{\frac{F}{\mu}}$

$$F = \frac{1}{4}\mu f_4^2 L^2$$
$$= \frac{1}{4}\left(1.291\cdot 10^{-2}\text{ kg/m}\right)\left(93.63\text{ Hz}\right)^2\left(1.175\text{ m}\right)^2$$
$$= 39.06\text{ N}$$

15.79. $f_4 = \frac{2}{L}\sqrt{\frac{F}{\mu}}$

$$L = \frac{2}{f_4}\sqrt{\frac{F}{\mu}}$$
$$= \frac{2}{59.47\text{ Hz}}\sqrt{\frac{10.81\text{ N}}{1.747\cdot 10^{-2}\text{ kg/m}}}$$
$$= 83.66\text{ cm}$$

Chapter 16: Sound

Concept Checks

16.1. d **16.2.** a **16.3.** b **16.4.** e **16.5.** a **16.6.** b **16.7.** a **16.8.** d

Multiple-Choice Questions

16.1. b **16.2.** e **16.3.** c **16.4.** e **16.5.** c **16.6.** e **16.7.** a **16.8.** b **16.9.** b **16.10.** c **16.11.** b **16.12.** a **16.13.** a

16.14. c

Conceptual Questions

16.15. The speed of sound in a solid is much greater than the speed of sound in air, so the sound from the train in the tracks will be detected before the sound from the train in the air.

16.16. As mentioned in the chapter, sound requires a medium for propagation. By evacuating the jar to lower and lower pressures, the air molecules, whose motion transmits the sound, are removed, thus eliminating the medium in which the sound propagates. As the jar is evacuated, it will take longer and longer for the alarm sound to transmit to the glass walls of the jar until all air molecules are removed and no sound is heard anymore.

16.17. Presuming you are the same distance from each engine, the detection of audible beats, or recurring pulses, would suggest the engines are not perfectly synchronized. If one is 1% faster, or $f' = 5252/\text{min},$ then their beat frequency would be $f_{\text{beat}} = f' - f = 52/\text{min},$ or $f_{\text{beat}} = 0.87 \text{ Hz} \approx 1 \text{ Hz}.$ To detect this with a wrist watch, you would hear the engine noise increase in volume once every second. To get a more accurate beat frequency, you would need to count the beats for a longer time interval.

16.18. Sound travels faster under water, so it would be more difficult to discern any difference in time it takes sound to reach one ear over the other. Sound waves would therefore seem to be located more in the front or back than to the side of where the sound wave is actually coming from.

16.19. By considering the Doppler shift of the sound of the school bell, the velocity of the wind does not affect the frequency of the school bell. The velocity of the wind will affect the velocity of the sound as the medium in which the sound travels is itself moving. Given that $v = \lambda f,$ and *f* is unaffected, the wavelength must change since the velocity, *v,* changes.

16.20. As the car approaches the two frequencies, f and f', increase because of the Doppler shift:

$$f_{\text{o}} = f\left(\frac{v_{\text{sound}}}{v_{\text{sound}} - v_{\text{car}}}\right), \quad f_{\text{o}}' = f'\left(\frac{v_{\text{sound}}}{v_{\text{sound}} - v_{\text{car}}}\right).$$

In addition, the observed beat frequency, $f_{\text{beat,o}}$, increases:

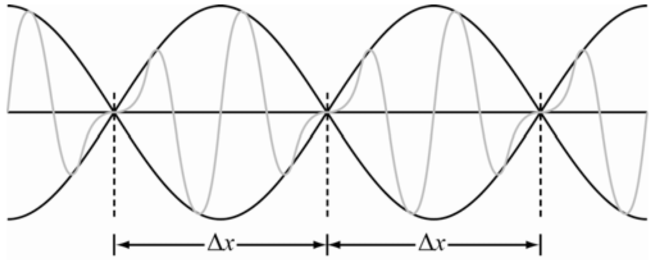
$$f_{\text{beat,o}} = f_{\text{o}} - f_{\text{o}}' = \left(\frac{v_{\text{sound}}}{v_{\text{sound}} - v_{\text{car}}}\right)(f - f').$$

As the car passes the pedestrian, assuming the pedestrian is directly beside the car's path, the frequencies and the beat frequency drop suddenly, and remain at this lower frequency while the car moves away, according to the Doppler shift:

$$f_{\text{o}} = f\left(\frac{v_{\text{sound}}}{v_{\text{sound}} + v_{\text{car}}}\right), \quad . f_{\text{o}}' = f'\left(\frac{v_{\text{sound}}}{v_{\text{sound}} + v_{\text{car}}}\right), \quad f_{\text{beat,o}} = \left(\frac{v_{\text{sound}}}{v_{\text{sound}} + v_{\text{car}}}\right)(f - f_{\text{o}}').$$

16.21. Sound cannot be generated in the vacuum that surrounds the Moon, as there is no medium for the sound to propagate. Sound waves can travel in the solid matter that makes up the Moon. These waves would be more "felt" than "heard".

16.22. Wave 1 is described by the equation $y_1(x,t) = A\sin(\kappa x - \omega t)$. Wave 2 is described by the equation $y_2(x,t) = A\sin((\kappa + \Delta\kappa)x - (\omega + \Delta\omega)t)$. The wave packet is shown below.



Δx is equal to $v_{\text{sound}} T_{\text{beat}}$. Consider f_{beat}: $f_{\text{beat}} = |f_1 - f_2| = v_{\text{sound}} \left| \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right| = v_{\text{sound}} (2\pi) |\kappa_1 - \kappa_2|$. Then,

$$\Delta\kappa = \frac{f_{\text{beat}}}{(2\pi) v_{\text{sound}}} \text{ and } \Delta x \Delta\kappa = (v_{\text{sound}} T_{\text{beat}}) \left(\frac{f_{\text{beat}}}{(2\pi) v_{\text{sound}}} \right) = \frac{v_{\text{sound}}}{f_{\text{beat}}} \left(\frac{f_{\text{beat}}}{(2\pi) v_{\text{sound}}} \right) = \frac{1}{2\pi}.$$

16.23. To the nearest orders of magnitude, assume you are a distance, $r = 10$ m from the convertible and you hear the music at a sound level of $\beta = 100$ dB. Then, $I = I_0 10^{\beta/10} = 10^{-12}\text{ W/m}^2 (10^{10}) = 10^{-2}\text{ W/m}^2$.

Assume the people in the car are a distance, $r_{\text{driver}} = 0.5$ m, from the speakers. Then:

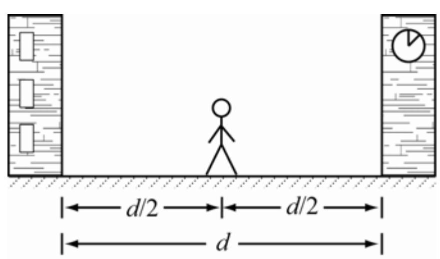
$$\frac{I_{\text{driver}}}{I} = \left(\frac{r}{r_{\text{driver}}} \right)^2 \Rightarrow I_{\text{driver}} = I \left(\frac{r}{r_{\text{driver}}} \right)^2 = 4\text{ W/m}^2.$$

This intensity translates into a sound level of $\beta_{\text{driver}} = 10\log(I_{\text{driver}} / I_0) = 126\text{ dB} \approx 130\text{ dB}$. This is as loud as a jet taking off from an aircraft carrier and the driver's ears will not be able to sustain this without damage.

16.24. By pouring water into the bottle, the wavelength of the fundamental frequency is decreased. The bottle is similar to a pipe with one open end and adding water is analogous to shortening the pipe. Since v_{sound} is constant, the frequency, or pitch, must increase.

Exercises

16.25. Assume $v_{\text{sound}} = 343$ m/s.



In $t = 0.500$ s, the sound travels a distance, d. It travels from your position, where you first hear it, to the tall building and back to your position. Note the distance between the clock tower and the building is also d. Now, solving gives $d = (343\text{ m/s})(0.500\text{ s}) = 172\text{ m}$.

16.26. The distance between the farmers is $d = 510$ m. The travel time for the sound is $t = 1.5$ s. Consider $v(T) = (331 + 0.6T / °\text{C})$ m/s. Then, $v = d / t = (331 + 0.6T / °\text{C})$ m/s. This implies that:

$$T = \frac{(d/t) - 331}{0.6}\ °\text{C} = \frac{(510/1.50) - 331}{0.6}\ °\text{C} = 15.0\ °\text{C}.$$

16.27. In this problem, the density of an air sample is given as $\rho = 1.205 \text{ kg/m}^3$, and the bulk modulus is $B = 1.42 \cdot 10^5 \text{ N/m}^2$. Determine (a) the speed of sound in the air sample and (b) the temperature of the air sample.

(a) The relationship between the speed of a wave, v, in a medium of density, ρ, and the bulk modulus, B, is given by:

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{1.42 \cdot 10^5 \text{ N/m}^2}{1.205 \text{ kg/m}^3}} = 343 \text{ m/s}.$$

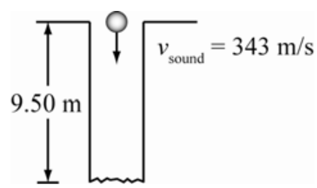
(b) The temperature dependent wave speed for sound is given by the relation, $v(T) = 331 + 0.6T$. Therefore, if the speed is known, the temperature can be determined by:

$$T = \frac{v - 331}{0.6} = \frac{343 - 331}{0.6} = \frac{12}{0.6} = 20.0 \text{ °C}.$$

The temperature of the air sample is $T = 20.0 \text{ °C}$.

16.28. **THINK:** Determine the amount of time it takes to hear the splash of a stone after dropping it down a well. It is given that the well is 9.50 m deep, and the speed of sound in air is 343 m/s. First, calculate the time it takes the stone to reach the water in the well (9.50 m down), and then the time it takes for the sound wave to make the trip up to your ear. The total time will be the sum of these two times.

SKETCH:



RESEARCH: To determine the time it takes for the stone to drop down to the well water, use the kinematic equation for constant acceleration, $d = v_0 t_1 + \frac{1}{2} a t_1^2$, where in this case, $a = g = 9.81 \text{ m/s}^2$. Since the stone is being dropped, it is clear that its initial speed, v_0, is zero. Therefore,

$$d = \frac{1}{2} g t_1^2 \Rightarrow t_1 = \sqrt{\frac{2d}{g}}.$$

Now, to determine the time it takes the sound wave to reach your ear, recall that the basic definition of speed is $v = d/t$. Therefore, $t_2 = d/v$.

SIMPLIFY: Putting all the expressions together: $t = t_1 + t_2 = \sqrt{\frac{2d}{g}} + \frac{d}{v}$.

CALCULATE: $t = \sqrt{\frac{2(9.50 \text{ m})}{9.81 \text{ m/s}^2}} + \frac{9.50 \text{ m}}{343 \text{ m/s}} = 1.419 \text{ s}$

ROUND: Since the values are given to three significant figures, the result should be rounded to $t =$ 1.42 s.

DOUBLE-CHECK: $\sqrt{\frac{[\text{m}]}{[\text{m/s}^2]}} + \frac{[\text{m}]}{[\text{m/s}]} = \sqrt{\frac{[\text{m}][\text{s}^2]}{[m]}} + \frac{[\text{m}][\text{s}]}{[\text{m}]} = \text{s}.$ Dimensional analysis confirms that the answer has the correct units. Note that the stone's speed during free-fall down the well is always quite significantly below the speed of sound. Therefore, it is expected that the bulk of the time in the answer will be taken by the stage in which the stone falls. The calculation reproduces this, giving a time of 0.03 s for the sound wave to travel up the well and 1.39 s for the stone to travel down the well.

16.29. Wave speed is given by $v = \sqrt{B/\rho}$. Solving for the elastic modulus gives:

$$B = v^2\rho = \left(2.0\cdot 10^8 \text{ m/s}\right)^2\left(2500 \text{ kg/m}^3\right) = 1.0\cdot 10^{20} \text{ N/m}^2.$$

This value is some nine orders of magnitude larger than the actual value. Indeed, light waves are electromagnetic oscillations that do not require the motion of glass molecules, or the hypothetical ether for transmission.

16.30. Determine the intensities at pain and whisper levels from:

$$120 \text{ dB} = 10\log\frac{I_{120}}{I_0} \Rightarrow I_{120} = I_0 10^{12} = \left(10^{-12} \text{ W/m}^2\right)10^{12} = 1 \text{ W/m}^2 \text{ and}$$

$$20 \text{ dB} = 10\log\frac{I_{20}}{I_0} \Rightarrow I_{20} = I_0 10^{2} = \left(10^{-12} \text{ W/m}^2\right)10^{2} = 10^{-10} \text{ W/m}^2.$$

Therefore, the pain level is 10^{10} times more intense than the whisper level.

16.31. The question is asking for the value of the sound pressure amplitude, *P*, from a rock concert with an intensity level of 110. dB. Use the expression for loudness and solve for *P*:

$$\beta = 110.\text{ dB} = 20\log\left(\frac{P}{P_0}\right)\text{ dB} \Rightarrow P = P_0\cdot 10^{5.50}.$$

Recall that the threshold pressure is given by $P_0 = 2.00\cdot 10^{-5}$ Pa, so $P = \left(2.00\cdot 10^{-5}\right)10^{5.50} = 6.32$ Pa.

Therefore, the sound pressure amplitude due to an intensity level of 110. dB is $P = 6.32$ Pa.

16.32. The question is asking for the intensity level due to 10,000 sources (people) at an equal distance from the detector (a person in the center of the field), when the intensity level due to one source is 50 dB. One way to solve this problem is to obtain the intensity due to one source from the equation for intensity level,

$$\beta = 10\log\left(\frac{I}{I_0}\right) \Rightarrow I = I_0 10^{\beta/10}.$$

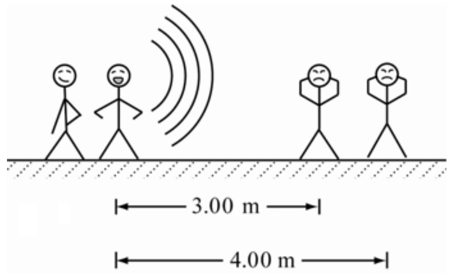
Since the threshold intensity for humans is $I_0 = 10^{-12} \text{ W/m}^2$:

$$I = I_0 10^{\beta/10} = \left(10^{-12}\right)10^{50/10} \text{ W/m}^2 = 10^{-7} \text{ W/m}^2.$$

Therefore, the intensity due to 10,000 sources is given by $I_{\text{all}} = (10000)\left(10^{-7} \text{ W/m}^2\right) = 10^{-3} \text{ W/m}^2$, and the intensity level is given by $\beta = 10\log\left(\frac{I_{\text{all}}}{I_0}\right) = 10\log\left(\frac{10^{-3}}{10^{-12}}\right) = 90$ dB. Another way to solve the problem is to realize that a multiple of 10 inside a $\log_{10}$ function will result in an addition of 1 outside. Therefore, $10{,}000 = 10^4$ inside the $\log_{10}$ function results in (1 + 1 + 1 + 1) = 4 outside. Therefore, the intensity level will increase from 50 dB to $50 + 10(4) = 90$ dB. Note also that every multiple of 1/10 inside a $\log_{10}$ function results in a subtraction of 1 outside.

16.33. **THINK:** The question asks for the sound intensity, I_2, measured by a detector $r_2 = 4.00$ m away from a source, when the sound intensity at $r_1 = 3.00$ m is $I_1 = 1.10\cdot 10^{-7} \text{ W/m}^2$.

SKETCH:



RESEARCH: Recall that the intensity is defined as $I = \text{Power/Area}$. The power emitted by the source can be determined from the intensity at the 3.00 m detector, and then this can be used to determine the intensity at 4.00 m.

SIMPLIFY: $I = \dfrac{\text{Power}}{\text{Area}} = \dfrac{P}{4\pi r^2} \Rightarrow P = I\left(4\pi r^2\right)$. Therefore, $I_2 = \dfrac{I_1 4\pi (r_1)^2}{4\pi (r_2)^2} = \dfrac{I_1 r_1^2}{r_2^2}$.

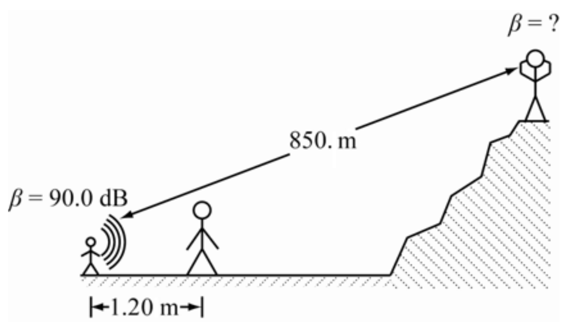
CALCULATE: $I_{4\text{ m}} = \dfrac{\left(1.10 \cdot 10^{-7}\text{ W/m}^2\right)(3.00\text{ m})^2}{(4.00\text{ m})^2} = 6.188 \cdot 10^{-8}\text{ W/m}^2$

ROUND: Since the given values have three significant figures, the result should be rounded to $I_{4\text{ m}} = 6.19 \cdot 10^{-8}\text{ W/m}^2$.

DOUBLE-CHECK: It is reasonable that the intensity will decrease as the distance increases.

16.34. **THINK:** The question asks for the intensity level heard by a climber 850. m away from a yelling child, when the parent hears it at 90.0 dB standing only 1.20 m away.

SKETCH:



RESEARCH: Let I_p be the intensity measured at the parent's position and I_c be the intensity at the climber's position. Let r_p be the distance between the parent and the child and r_c be the distance between the climber and the child. The intensity will fall off by a factor of $1/r^2$, that is,

$$I_\text{p} = \frac{P}{4\pi r_\text{p}^2},\; I_\text{c} = \frac{P}{4\pi r_\text{c}^2},\text{ where } P \text{ is the power of the sound.}$$

SIMPLIFY: Since the power of the sound is the same in both the parent's and the climber's equation for intensity, the intensities can be related as:

$$\frac{I_\text{c}}{I_\text{p}} = \frac{r_\text{p}^2}{r_\text{c}^2} \Rightarrow I_\text{c} = I_\text{p}\left(\frac{r_\text{p}^2}{r_\text{c}^2}\right).$$

Therefore, $\beta_\text{c} = 10\log\dfrac{I_\text{c}}{I_0} = 10\log\dfrac{I_p\left(r_\text{p}^2/r_\text{c}^2\right)}{I_0} = 10\log\left(\dfrac{I_\text{p}}{I_0}\right) + 10\log\left(\dfrac{r_\text{p}^2}{r_\text{c}^2}\right) = \beta_\text{p} + 10\log\left(\dfrac{r_\text{p}^2}{r_\text{c}^2}\right).$

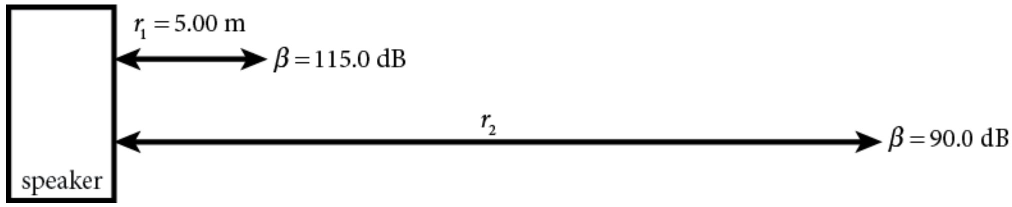
CALCULATE: $\beta_\text{c} = 90.0\text{ dB} + 10\log\left[\dfrac{(1.20\text{ m})^2}{(850.\text{ m})^2}\right] = 32.995\text{ dB}$

ROUND: Since all values are given to three significant figures, the result should be given as $\beta_c = 33.0$ dB.

DOUBLE-CHECK: It is expected that the farther you are from the source, the weaker the signal. The result supports this assertion.

16.35. **THINK:** The question asks for the distance from speakers at a rock concert where the intensity level is below or equal to 90.0 dB. It is known that the intensity level is 115.0 dB at 5.00 m.

SKETCH:



RESEARCH: Recall that the intensity level is defined as $\beta = 10\log(I/I_0)$, and the intensity drops as a square of the distance, $I_2/I_1 = r_1^2/r_2^2$. Therefore, for the 115.0 dB location and the 90.0 dB location, $\beta_1 = 10\log(I_1/I_0)$ and $\beta_2 = 10\log(I_2/I_0)$, which implies:
$\Delta\beta = \beta_1 - \beta_2 = 10\log\left(\frac{I_1}{I_2}\right) = 10\log\left(\frac{r_2^2}{r_1^2}\right) = 20\log\left(\frac{r_2}{r_1}\right).$

SIMPLIFY: The distance is given by $r_2 = r_1 10^{\Delta\beta/20}$.

CALCULATE: Since $\Delta\beta = 115.0 - 90.0 = 25.0$ dB and $r_1 = 5.00$ m,
$r_2 = (5.00 \text{ m})10^{25.0/20} = 88.91397$ m.

ROUND: The result should be rounded to $r_2 = 88.9$ m.

DOUBLE-CHECK: Considering an intensity level at the source of more than 115.0 dB, it is reasonable that a person has to move back almost 100 m to be safe from hearing damage.

16.36. In order to have constructive interference, the difference in path length must be equal to an integer number of wavelengths. Therefore, $\Delta s = d - 3\lambda = n\lambda$. The largest wavelength is for $n = 1$. Therefore, $d - 3\lambda = \lambda$ and $\lambda = d/4$. Recall the expression for the velocity of a wave is $v = f\lambda$. Use the fact that $v = 340.$ m/s and $d = 10.0$ m. The frequency is given by:

$$f = \frac{v}{\lambda} = v\left(\frac{4}{d}\right) = (340. \text{ m/s})\left(\frac{4}{10.0 \text{ m}}\right) = 136 \text{ Hz}.$$

16.37. The wavelength of the sound is $\lambda = v/f = (343 \text{ m/s})/(490. \text{ Hz}) = 0.700$ m. Since the speakers are in phase and are facing each other, their interference will yield a standing wave with an anti-node at the center between them. If she sits a half wavelength away from the center, then she will be at another anti-node. Therefore, the minimum distance away from the center that she can move on the straight line connecting the two speakers and again hear the loudest sound is: $d = \lambda/2 = 0.350$ m.

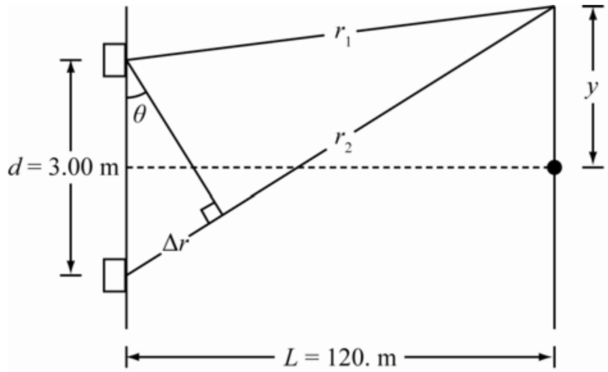
16.38. (a) Recall that the beat frequency is given by $f_{\text{beat}} = |f_1 - f_2|$, and as the string is tightened, the beat frequency and thus the difference between the fork and violin frequencies increases, it is clear that the frequency of the violin is greater than that of the fork. Therefore,

$$f_{\text{beat}} = |f_1 - f_2| = f_{\text{violin}} - f_{\text{fork}} \Rightarrow f_{\text{violin}} = f_{\text{fork}} + f_{\text{beat}} = 400. + 2 = 402 \text{ Hz}.$$

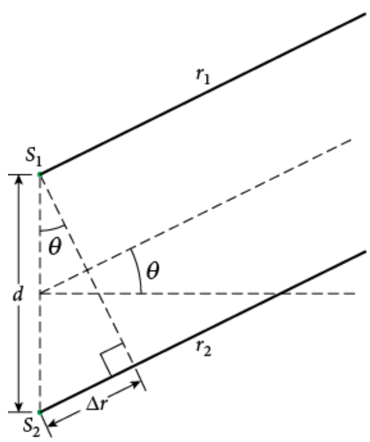
(b) Since it was found in part (a) that as the string is tightened, the difference in frequencies between the fork and the violin increases, to tune the violin, the string must be loosened (if the desired frequency for the string is 400. Hz).

16.39. **THINK:** The question asks for the locations of the spots of destructive interference along a far wall from two speakers in phase. The distance between the speakers is $d = 3.00$ m. The distance between both speakers and the far wall is $L = 120.$ m. The frequency of the sound wave is $f = 1372$ Hz. Take the speed of the wave to be $v = 343$ m/s.

SKETCH:



(not to scale)



RESEARCH: From the sketch above, it can be seen that with $d << L$ (which is the case), the difference in paths traveled by the two sound waves is given by $\Delta r = r_1 - r_2 \approx d\sin\theta,$ assuming that the wall is sufficiently far away that the two rays are nearly parallel. θ is the angle that the rays make relative to a perpendicular line joining the sources to the wall. The condition for destructive interference at the far wall is:

$$d\sin\theta = \left(m + \frac{1}{2}\right)\lambda,\ m = 0,1,2,3,...$$

For $d << L$, it can be seen that $\tan\theta \approx \sin\theta \approx \theta \approx y/L,$ where y is the distance from the point on the wall exactly opposite to the point centered between the speakers.

SIMPLIFY: The distances from the center of the far wall to the points of destructive interference are given by:

$$y = L\sin\theta \Rightarrow y = \frac{Ld\sin\theta}{d} = \frac{L\left(m + \frac{1}{2}\right)\lambda}{d},\ m = 0,1,2,3,\ldots$$

CALCULATE: The wavelength is determined from the equation for the wave speed, $v = f\lambda,$ which gives:

$$\lambda = \frac{v}{f} = \frac{1372 \text{ m/s}}{343 \text{ Hz}} = 0.250 \text{ m}.$$

Therefore, the first point of destructive interference occurs at: $y = \dfrac{(120.\text{ m})\left(\frac{1}{2}\right)(0.250)}{3.00\text{ m}} = 5.00\text{ m}.$

ROUND: $y = 5.00\text{ m}.$

DOUBLE-CHECK: Let's double-check that our small angle approximation is valid. We find that

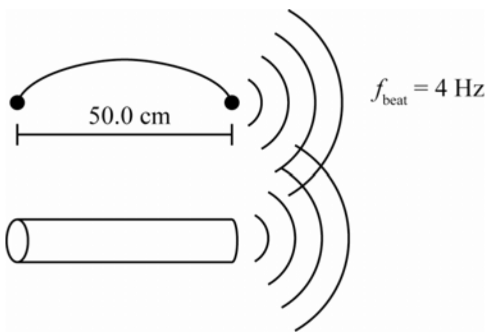
$$\tan\theta = \frac{5.00\text{ m}}{120.0\text{ m}}$$

$$\theta = \tan^{-1}\left(\frac{5.00\text{ m}}{120.0\text{ m}}\right) = 0.0416 = 2.39°.$$

So our assumed small angle approximation is valid.

16.40. **THINK:** A guitar and a pipe organ are played at the same time. The guitar frequency is $f_{\text{g}} = 400.\text{ Hz}$, the guitar string length is $l = 50.0\text{ cm} = 0.500\text{ m}$, the beat frequency is $f_{\text{b}} = 4\text{ Hz}$ and the mass per unit length of the string is $\mu = 2.00\text{ g/m}$. Determine (a) the possible frequencies of the open organ pipe, (b) the initial tension of the string if the beat frequency decreases when the string is tightened and (c) the length of the organ pipe.

SKETCH:



RESEARCH:

(a) The frequency of the pipe organ, f_{o}, can be determined from the definition of beat frequency, $f_{\text{b}} = \left|f_{\text{g}} - f_{\text{o}}\right|.$

(b) Recall that the wave speed on the string, $v = f_{\text{g}}\lambda_{\text{g}}$, can also be written as $v = \sqrt{T/\mu}$, where T is the tension and μ is the linear density. The tension can be determined by setting these expressions equal to each other.

(c) The length of the pipe can be determined from the equation for the fundamental frequency of an open pipe given by $f_{\text{o}} = v/2L.$

SIMPLIFY:

(a) Solving for the frequency of the organ gives $f_{\text{o}} = f_{\text{g}} \pm f_{\text{b}}.$

(b) Equating the expressions: $\sqrt{T/\mu} = f_{\text{g}}\lambda_{\text{g}} \Rightarrow T = \mu f_{\text{g}}^2\lambda_{\text{g}}^2.$ Recall that $\lambda_{\text{g}} = 2l$, and the expression for the tension becomes $T = \mu f_{\text{g}}^2 4l^2.$

(c) The length is given by $L = v/2f_{\text{o}}.$

CALCULATE:

(a) $f_{\text{o}} = 400.\text{ Hz} \pm 4\text{ Hz} \Rightarrow f_{\text{o1}} = 404\text{ Hz}$ and $f_{\text{o2}} = 396\text{ Hz}$

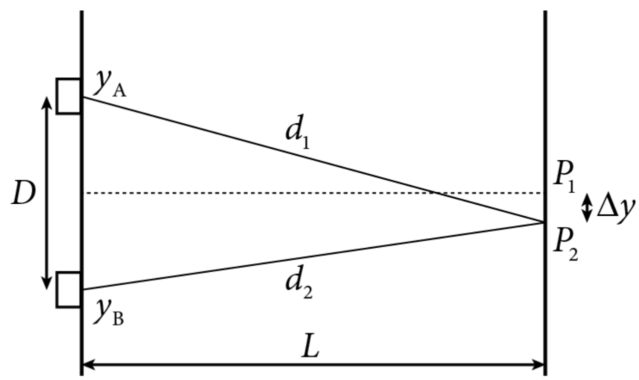
(b) $T = \left(2.00\cdot 10^{-3}\text{ kg/m}\right)\left(400.\text{ Hz}\right)^2 4\left(0.500\text{ m}\right)^2 = 320.\text{ N}$

(c) Since the beat frequency decreases as the string is tightened, $f_{\text{g}} < f_{\text{o}}$, therefore $f_{\text{o}} = 404\text{ Hz}$ and

$$L = \frac{343\text{ m/s}}{2(404\text{ Hz})} = 0.425\text{ m}.$$

ROUND:
(a) $f_{o1} = 404$ Hz and $f_{o2} = 396$ Hz
(b) $T = 320.$ N
(c) $L = 0.425$ m
DOUBLE-CHECK: Each of the calculated values has appropriate units and is of a reasonable order of magnitude for the given values.

16.41. **THINK:** The question asks for the intensity of the sound wave at the point P_1 in the sketch. The frequency of the sound wave is f= 10,000.0 Hz, and the coordinates at point P_1 are $x_1 = 4.50$ m and $y_1 = 0$ m. The distance between the speakers is D = 3.60 m, and the power delivered by the speakers is P = 100.0 W. The question next asks for the sound level due to speaker A at point P_1. Lastly, the question asks for the distance to the first maximum (constructive interference) from the center maximum.
SKETCH:



RESEARCH: The definition of intensity is $I = \text{Power / Area}$. Recall the equation for intensity level is $\beta = 10\log(I/I_0)$. With both speakers on, as one moves toward point P_2, the distances that the sound must travel from speakers A and B change. When the path difference is half a wavelength, the interference is destructive, but when the path difference increases to a full wavelength, the interference is constructive again. The distance from speaker A to P_2 is $d_1 = \sqrt{(D/2 + \Delta y)^2 + L^2}$. Similarly, the distance from speaker B to P_2 is $d_2 = \sqrt{(D/2 - \Delta y)^2 + L^2}$. The first constructive interference peak occurs at P_2 when $d_1 - d_2 = (1)\lambda$. From this, Δy can be determined.

SIMPLIFY: $I_1 = \dfrac{\text{Power}}{\text{Area}} = \dfrac{P}{4\pi r^2}$, $\beta = 10(\log I - \log I_0)$. In order to simplify the calculation of $d_1 - d_2$, let $a = D^2/4 + \Delta y^2 + L^2$ and $b = D\Delta y$. Rewrite the distances as:

$$d_1 = \sqrt{(D/2 + \Delta y)^2 + L^2} = \sqrt{D^2/4 + 2D\Delta y/2 + \Delta y^2 + L^2} = \sqrt{D^2/4 + D\Delta y + \Delta y^2 + L^2}$$
$$d_1 = \sqrt{(D/2 - \Delta y)^2 + L^2} = \sqrt{D^2/4 - 2D\Delta y/2 + \Delta y^2 + L^2} = \sqrt{D^2/4 - D\Delta y + \Delta y^2 + L^2}.$$

Assume that $\Delta y << D$ and $\Delta y << L$, then $a = D^2/4 + \Delta y^2 + L^2 \approx D^2/4 + L^2$. The value of *a* is much larger than the value of *b*, so we can apply the approximation given in the statement of the problem

$$\sqrt{a \pm b} \approx \sqrt{a} \pm \frac{b}{2\sqrt{a}}.$$

We can write $d_1 - d_2$ as

$$d_1 - d_2 = \sqrt{a+b} - \sqrt{a-b} = \frac{2b}{2\sqrt{a}} = \frac{D\Delta y}{\sqrt{D^2/4 + L^2}}.$$

Since $d_1 - d_2 = \lambda = v_{\text{sound}}/f$, substituting the equation from the previous line gives:

$$\frac{v_{\text{sound}}}{f} \approx \frac{D\Delta y}{\sqrt{D^2/4+L^2}},$$

which means, solving for Δy,

$$\Delta y \approx \left(\frac{v_{\text{sound}}}{f}\right)\frac{\sqrt{D^2/4+L^2}}{D}.$$

CALCULATE: (a) The intensity at point P_1 is given by:

$$I_1 = \frac{100.0 \text{ W}}{4\pi\left[\left(\frac{3.60 \text{ m}}{2}\right)^2 + (4.50 \text{ m})^2\right]} = 0.33877 \text{ W/m}^2.$$

(b) The sound level at point P_1

$$\beta_1 = 10\left(\log\left(0.33877\frac{\text{W}}{\text{m}^2}\right) - \log\left(10^{-12}\frac{\text{W}}{\text{m}^2}\right)\right) = 10\left(\log(0.33877)+12\right) \text{ dB} = 115.299 \text{ dB}.$$

(c) The distance to the first maximum is

$$\Delta y = \left(\frac{343 \text{ m/s}}{10{,}000.0 \text{ Hz}}\right)\frac{\sqrt{(3.60 \text{ m})^2/4+(4.50 \text{ m})^2}}{3.60 \text{ m}} = 0.046178 \text{ m}.$$

ROUND: Since the distances are given to three significant figures, the result is $I_1 = 0.339 \text{ W/m}^2$. Because the intensity is given to three significant figures, the result should be rounded to $\beta_1 = 115$ dB. Since there are values given with three significant figures, $\Delta y = 0.0462$ m.

DOUBLE-CHECK: The result is obtained directly from the definition of intensity and the units are correct. This sound level is high for a loud speaker but is possible. The value for Δy has appropriate units for a distance, and is consistent with $\Delta y << D$.

16.42. Recall the expression for the Doppler shift in frequency for a moving source is: $f_{\text{observer}} = f_{\text{source}}\frac{v_{\text{sound}}}{v_{\text{sound}} \pm v_{\text{source}}}$. In the approach to the policeman, the source is moving toward a stationary observer so we choose the negative sign in this equation, $f_{\text{observer}} = f_{\text{source}}\frac{v_{\text{sound}}}{v_{\text{sound}} - v_{\text{source}}}$.

At the instant the car passes the policemen, he will hear the sound of the horn with the original emitting frequency, f_{source}. We can then solve for the speed of the car

$$f_{\text{observer}}\left(v_{\text{sound}} - v_{\text{source}}\right) = f_{\text{source}}v_{\text{sound}}$$
$$f_{\text{observer}}v_{\text{sound}} - f_{\text{observer}}v_{\text{source}} = f_{\text{source}}v_{\text{sound}}$$
$$f_{\text{observer}}v_{\text{sound}} - f_{\text{source}}v_{\text{sound}} = f_{\text{observer}}v_{\text{source}}$$
$$v_{\text{source}} = v_{\text{sound}}\frac{f_{\text{observer}} - f_{\text{source}}}{f_{\text{observer}}}.$$

Putting in our values we get

$$v_{\text{source}} = (343 \text{ m/s})\frac{494 \text{ Hz} - 440. \text{ Hz}}{494 \text{ Hz}} = 37.5 \text{ m/s} = 83.87 \text{ mph}.$$

So the policeman gives the driver a speeding ticket for going 84 mph in a 40 mph zone.

16.43. From the Mach-angle of the cone expression, $\sin\theta = v_{\text{sound}}/v_{\text{source}}$.

(a) For $v_{\text{sound}} = 343$ m/s, the angle is solved as: $\theta = \sin^{-1}\left(\frac{v_{\text{sound}}}{v_{\text{source}}}\right) = \sin^{-1}\left(\frac{0.343 \text{ km/s}}{8.80 \text{ km/s}}\right) = 2.23°$.

(b) In water, $v_{\text{sound}} = 1560$ m/s. The angle is: $\theta = \sin^{-1}\left(\frac{v_{\text{sound}}}{v_{\text{source}}}\right) = \sin^{-1}\left(\frac{1.56 \text{ km/s}}{8.80 \text{ km/s}}\right) = 10.2°$.

16.44. The Doppler shift is given by: $f_{\text{d}} = f_{\text{s}}\dfrac{v_{\text{sound}} \pm v_{\text{d}}}{v_{\text{sound}} \pm v_{\text{s}}}$. Since $v_{\text{d}} = 0$ and f = 3000. Hz, the magnitude of the change in frequency when v = 343 m/s and u = 30.0 m/s is then:

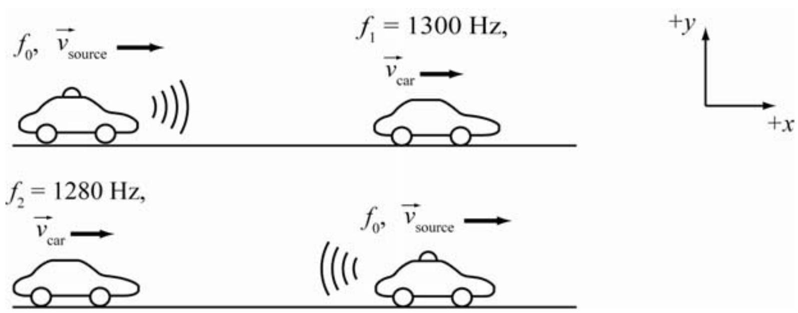
$$|\Delta f| = \frac{f}{(1-u/v)} - \frac{f}{(1+u/v)} = \frac{2f(u/v)}{1-(u/v)^2} = \frac{2(3000.\text{ Hz})\left(\dfrac{30.0 \text{ m/s}}{343 \text{ m/s}}\right)}{1-\left(\dfrac{30.0 \text{ m/s}}{343 \text{ m/s}}\right)^2} = 528.83 \text{ Hz.}$$

Since the given value for the speed has three significant figures, the result should be $|\Delta f| = 529$ Hz.

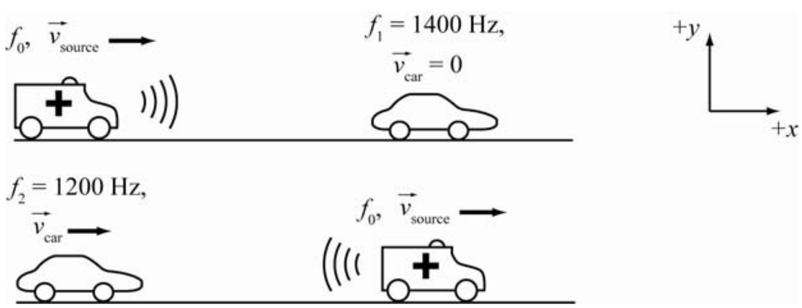
16.45. THINK:
(a) The question asks for the speed of a police car as it passes, if the frequency of the siren before it passes is $f_1 = 1300.$ Hz and the frequency after it passes is $f_2 = 1280.$ Hz.
(b) The question next asks for the actual frequency of an ambulance siren if before it passes the stopped car it has a shifted frequency of $f_1 = 1400.$ Hz, and after it passes the car it has a shifted frequency of $f_2 = 1200.$ Hz.

SKETCH:

(a)



(b)



RESEARCH:
Recall that the shift in frequency (Doppler shift) when both source and observer are moving is given by $f_{\text{observer}} = f_{\text{source}}\dfrac{v_{\text{sound}} \pm v_{\text{car}}}{v_{\text{sound}} \pm v_{\text{source}}}$. The speed of the source can be obtained by dividing the corresponding expressions for the before and after frequencies. The speed of the source can be obtained by dividing the corresponding expression for the before and after frequencies. Once the speed is determined, the actual frequency can be calculated.

SIMPLIFY:

(a) From the Doppler shift for the "before" frequency: $f_1 = f_0 \frac{v_{\text{sound}} - v_{\text{car}}}{v_{\text{sound}} - v_{\text{source}}}$. From the Doppler shift for the "after" frequency: $f_2 = f_0 \frac{v_{\text{sound}} + v_{\text{car}}}{v_{\text{sound}} + v_{\text{source}}}$. Therefore,

$$\frac{f_1}{f_2} = \frac{\left(\frac{v_{\text{sound}} - v_{\text{car}}}{v_{\text{sound}} - v_{\text{source}}}\right)}{\left(\frac{v_{\text{sound}} + v_{\text{car}}}{v_{\text{sound}} + v_{\text{source}}}\right)} = \left(\frac{v_{\text{sound}} - v_{\text{car}}}{v_{\text{sound}} - v_{\text{source}}}\right)\left(\frac{v_{\text{sound}} + v_{\text{source}}}{v_{\text{sound}} + v_{\text{car}}}\right) = \frac{v_{\text{sound}}^2 - v_{\text{sound}}v_{\text{car}} + v_{\text{source}}\left(v_{\text{sound}} - v_{\text{car}}\right)}{v_{\text{sound}}^2 + v_{\text{sound}}v_{\text{car}} - v_{\text{source}}\left(v_{\text{sound}} + v_{\text{car}}\right)}$$

$$\Rightarrow v_{\text{source}} = \frac{f_1\left(v_{\text{sound}}^2 + v_{\text{sound}}v_{\text{car}}\right) - f_2\left(v_{\text{sound}}^2 - v_{\text{sound}}v_{\text{car}}\right)}{f_1\left(v_{\text{sound}} + v_{\text{car}}\right) + f_2\left(v_{\text{sound}} - v_{\text{car}}\right)}.$$

(b) For the "before" frequency, the Doppler shift gives: $f_1 = f_0 \frac{v_{\text{sound}}}{v_{\text{sound}} - v_{\text{source}}}$. For the "after" frequency, the Doppler shift gives: $f_2 = f_0 \frac{v_{\text{sound}}}{v_{\text{sound}} + v_{\text{source}}}$. Therefore,

$$\frac{f_1}{f_2} = \frac{\left(\frac{v_{\text{sound}}}{v_{\text{sound}} - v_{\text{source}}}\right)}{\left(\frac{v_{\text{sound}}}{v_{\text{sound}} + v_{\text{source}}}\right)} \Rightarrow v_{\text{source}} = \frac{v_{\text{sound}}\left(f_1 - f_2\right)}{f_1 + f_2}.$$ The frequency is given by: $f_{\text{source}} = f_1\left(\frac{v_{\text{sound}} - v_{\text{source}}}{v_{\text{sound}}}\right)$.

CALCULATE:

(a)

$$v_{\text{source}} = \frac{1300.\text{ Hz}\left(\left(343.0\text{ m/s}\right)^2 + \left(343.0\text{ m/s}\right)\left(30.0\text{ m/s}\right)\right) - 1280.\text{ Hz}\left(\left(343.0\text{ m/s}\right)^2 - \left(343.0\text{ m/s}\right)\left(30.0\text{ m/s}\right)\right)}{1300.\text{ Hz}\left(343.0\text{ m/s} + 30.0\text{ m/s}\right) + 1280.\text{ Hz}\left(343.0\text{ m/s} - 30.0\text{ m/s}\right)}$$
$$= 32.64\text{ m/s}$$

(b) $$v_{\text{source}} = \frac{343.0\text{ m/s}\left(1400.\text{ Hz} - 1200.\text{ Hz}\right)}{1400.\text{ Hz} + 1200.\text{ Hz}} = 26.385\text{ m/s}$$

$$f_{\text{source}} = \left(1400.\text{ Hz}\right)\left(\frac{343.0\text{ m/s} - 26.385\text{ m/s}}{343.0\text{ m/s}}\right) = 1292.3\text{ Hz}$$

ROUND: The frequencies of the siren are given to four significant figures, while the speed is given to three significant figures.

(a) $v_{\text{source}} = 32.6\text{ m/s}$

(b) $f_{\text{source}} = 1292\text{ Hz}$

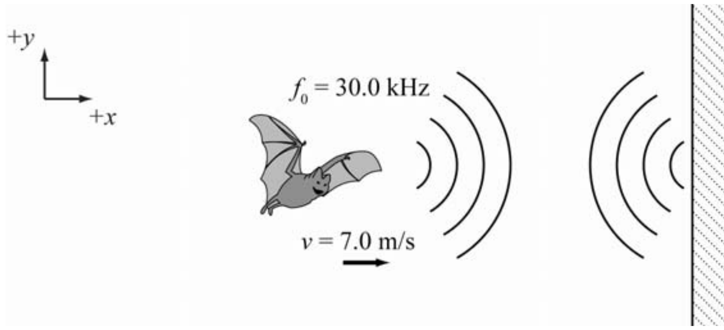
DOUBLE-CHECK:

(a) This is a reasonable speed for the police car as it passes.

(b) This is a reasonable value for the actual frequency, based on the given parameters.

16.46. **THINK:** The question asks about the shift in frequency of an ultrasound wave emitted by a bat, when it returns to the bat after bouncing from a wall. The given values are the speed of the bat toward the wall, v = 7.0 m/s, and the original wave frequency of $f_0 = 30.0$ kHz. As the bat flies towards the wall and emits the sound, the Doppler shift for the moving emitter applies. Since the bat moves towards the wall and the reflected sound, the process of receiving the sound is then governed by the Doppler shift for a moving observer. The two effects are superimposed, in the same way as was done in Solved Problem 16.4 in the textbook.

SKETCH:



RESEARCH: The equation for Doppler shift is: $f_{\text{observer}} = f_{\text{source}} \dfrac{v_{\text{sound}} \pm v_{\text{observer}}}{v_{\text{sound}} \pm v_{\text{source}}}$. First, consider the shift as the wave reaches the wall: $f_{\text{wall}} = f_0 \dfrac{v_{\text{sound}}}{v_{\text{sound}} - v_{\text{bat}}}$. Then consider the shift from the wall to the bat again, where the "unshifted" frequency is actually the frequency as the sound wave leaves the wall:

$$f_{\text{final}} = f_{\text{wall}} \left(1 + \frac{v_{\text{bat}}}{v_{\text{sound}}}\right).$$

SIMPLIFY: Putting both expressions together gives:

$$f_{\text{final}} = f_{\text{wall}} \left(1 + \frac{v_{\text{bat}}}{v_{\text{sound}}}\right) = f_0 \left(\frac{v_{\text{sound}}}{v_{\text{sound}} - v_{\text{bat}}}\right)\left(1 + \frac{v_{\text{bat}}}{v_{\text{sound}}}\right) = f_0 \frac{v_{\text{sound}} + v_{\text{bat}}}{v_{\text{sound}} - v_{\text{bat}}}.$$

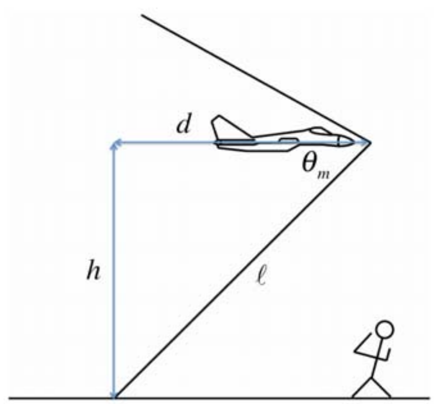
CALCULATE: The total shift is then given by: $f_{\text{final}} = (30.0 \text{ kHz})\left(\dfrac{343 + 7.00}{343 - 7.00}\right) = 31.25 \text{ kHz}.$

ROUND: Since the bat's speed is given to three significant figures, the result is reported as $f_{\text{final}} = 31.3$ kHz.

DOUBLE-CHECK: The expectation is for the frequency to increase if the bat is approaching the wall. Also, based on the given values, the result is reasonable.

16.47. **THINK:** A plane is flying at 1.30 the speed of sound. The Mach angle is simply related to the Mach number and requires almost no calculation. For part b) we need to realize that the sound propagates along the surface of the Mach cone, and the man on the ground will hear the sound when the surface of the Mach cone travels across him.

SKETCH:



RESEARCH:

(a) The Mach angle is given by $\theta_m = \sin^{-1}(v/1.30v) = \sin^{-1}(1/1.30)$.

(b) The altitude can be obtained from the relation: $\tan(\theta_m) = h/d$, where h is altitude and d is horizontal distance traveled by the plane during the time it takes the sound to reach the ear.

SIMPLIFY:

(a) It is not necessary to simplify.

(b) $\tan(\theta_m) = \dfrac{h}{d} \Rightarrow h = d\tan(\theta_m) = v_{plane}t\tan(\theta_m)$

CALCULATE:

(a) $\theta_{\text{Mach}} = \sin^{-1}\left(\dfrac{1}{1.30}\right) = 50.285°$

(b) $h = (1.30)(343.\text{ m/s})(3.14\text{ s})\tan(50.285°) = 1.6856\text{ km}$

ROUND: Rounding to three significant figures, the results are reported as:

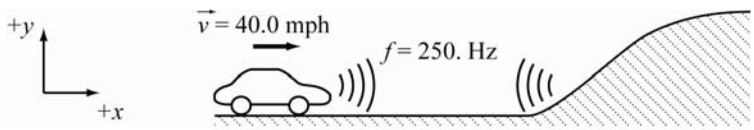
(a) $\theta_{\text{Mach}} = 50.3°$

(b) $h = 1.69\text{ km}$

DOUBLE-CHECK: We can calculate the time it takes sound to travel a distance of 1.69 km. This time is $t_0 = h/v = (1.69\text{ km})/(343\text{ m/s}) = 4.91\text{ s}$. As required, this time is larger than 3.14 s, the time that was given in the problem for the shock wave to arrive. Our solution for *h* at least passes this simple check.

16.48. **THINK:** The problem describes a car headed toward a hill. The car sounds the horn, and the hill reflects the sound wave, which later reaches the car. Determine (a) the shifted frequency as the sound waves reach the hill, (b) the frequency of the reflected wave as it reaches the car again, and (c) the beat frequency heard at the car between the emitted and reflected waves. The velocity of the car is $\vec{v} = 40.0\text{ mph} = 40.0\text{ mph}\left(\dfrac{1\text{ h}}{3600\text{s}}\right)\left(\dfrac{1609\text{ m}}{1\text{ mi}}\right) = 17.87\text{ m/s}.$

SKETCH:



RESEARCH:

(a) The Doppler frequency shift is given by: $f_{\text{observer}} = f_{\text{source}}\dfrac{v_{\text{sound}} \pm v_{\text{observer}}}{v_{\text{sound}} \pm v_{\text{source}}}$. Therefore, the shifted frequency at the hill is given by: $f_{\text{hill}} = f_0\dfrac{v_{\text{sound}}}{v_{\text{sound}} - v_{\text{car}}}$.

(b) The shift in frequency of the reflected wave is given by: $f_{\text{car}} = f_{\text{hill}}\dfrac{v_{\text{sound}} + v_{\text{car}}}{v_{\text{sound}}}$, where f_{hill} is the result from part (a).

(c) The beat frequency is given by $f_{\text{beat}} = |f_1 - f_2| = |f_{\text{car}} - f_0|$.

SIMPLIFY:

(a) Simplification is not necessary.

(b) $f_{\text{car}} = f_{\text{hill}}\dfrac{v_{\text{sound}} + v_{\text{car}}}{v_{\text{sound}}} = f_0\dfrac{v_{\text{sound}} + v_{\text{car}}}{v_{\text{sound}} - v_{\text{car}}}$

(c) Simplification is not necessary.

CALCULATE:

(a) $f_{\text{hill}} = (250.\text{ Hz})\left(\dfrac{340.\text{ m/s}}{340.\text{ m/s} - 17.87\text{ m/s}}\right) = 263.9\text{ Hz}$

(b) $f_{\text{car}} = (250.\text{ Hz})\dfrac{340.\text{ m/s} + 17.87\text{ m/s}}{340.\text{ m/s} - 17.87\text{ m/s}} = 277.7\text{ Hz}$

(c) $f_{\text{beat}} = |277.7\text{ Hz} - 250.\text{ Hz}| = 27.74\text{ Hz}$

ROUND: Rounding to three significant figures:

(a) $f_{\text{hill}} = 264 \text{ Hz}$

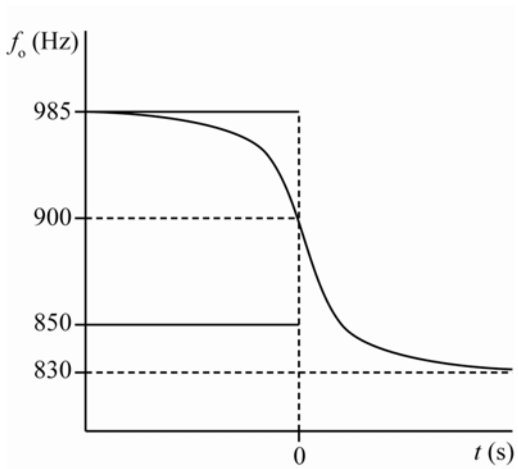
(b) $f_{\text{car}} = 278 \text{ Hz}$

(c) $f_{\text{beat}} = 27.7 \text{ Hz}$

DOUBLE-CHECK: It is expected that the frequency will shift up as the car moves toward the hill. Similarly, it is reasonable that the frequency shifts up again as the reflected wave reaches the car moving towards it. Also, based on the given values, the results are reasonable.

16.49. **(a, b) THINK:** The question presents a stationary car as a train passes by it. A sketch of the time dependent horn frequency heard at the car is shown. Determine (a) the frequency of the horn as it is emitted by the train and (b) the speed of the train.

SKETCH:



RESEARCH:

(a) From the given sketch, it is clear that the train passes the car at the zero time mark. From the curve, trace to the corresponding frequency at $t = 0$.

(b) From the result of part (a), the speed of the train can be determined by looking at the maximum frequency shift as shown in the sketch and using the Doppler frequency shift equation:

$$f_{\text{detector}} = f_{\text{source}} \frac{v_{\text{sound}} \pm v_{\text{detector}}}{v_{\text{sound}} \pm v_{\text{source}}}.$$

SIMPLIFY:

(a) Simplification is not necessary.

(b) Since the car is not moving, for the case where the train is moving away from the car:

$$f_{\text{car}} = f_{\text{train}} \left(\frac{v_{\text{sound}}}{v_{\text{sound}} + v_{\text{train}}} \right) \Rightarrow v_{\text{train}} = v_{\text{sound}} \left(\frac{f_{\text{train}} - f_{\text{car}}}{f_{\text{car}}} \right).$$

CALCULATE:

(a) From the sketch, the frequency corresponding to $t = 0$ is $f_0 = 900 \text{ Hz}$.

(b) Using the result of part (a), and the maximum frequency shift from the sketch, $f_{\text{car}} = 830 \text{ Hz}$:

$$v_{\text{train}} = (340 \text{ m/s}) \left(\frac{900 \text{ Hz} - 830 \text{ Hz}}{830 \text{ Hz}} \right) = 28.7 \text{ m/s}.$$

ROUND:

(a) Since the value was chosen from the sketch, the result remains $f_0 = 900 \text{ Hz}$.

(b) Since the values are all obtained by measuring a curve on a rough sketch, round the result to one significant figure, $v_{\text{train}} = 30 \text{ m/s}$.

DOUBLE-CHECK: It is reasonable that as the train moves away from the car, the frequency decreases. The results are also reasonable based on the inaccuracy of getting data from a rough sketch.

(c) THINK: In part (a), the value of f_0 is found to be $f_0 = 900$ Hz. In part (b), the speed of the train is found to be $v_{\text{train}} = 28.7$ m/s. The speed of sound is about $v_{\text{sound}} = 343$m/s. From the plot given in the question, the instantaneous slope of f_0 at $t = 0$ can be approximated.

SKETCH: A sketch is not needed.

RESEARCH: The question gives the hint to use the equation $\left.\dfrac{df_o}{dt}\right|_{t=0} = -\dfrac{fv^2}{bv_{\text{sound}}}$. Part (c) asks for the value of *b*. The rest of the quantities in the expression are known or can be found. By inspecting the graph, it appears that the function f_o passes through the points (-0.2, 920) and (0.2, 870). Thus, a reasonable approximation for the instantaneous slope at zero is $\left.\dfrac{df_o}{dt}\right|_{t=0} \approx \dfrac{870\text{ Hz} - 920\text{ Hz}}{0.2\text{ s} - (-0.2\text{ s})} = \dfrac{-50\text{ Hz}}{0.4\text{ s}} = -125\text{ s}^{-2}$.

SIMPLIFY: $b = -\dfrac{fv^2}{v_{\text{sound}} \left.\dfrac{df_o}{dt}\right|_{t=0}}$

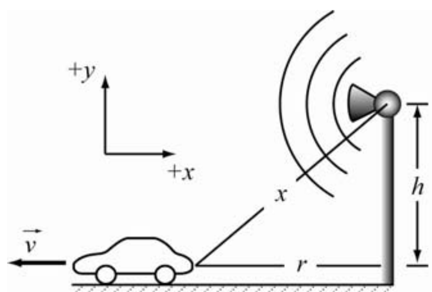
CALCULATE: $b = -\dfrac{(900\text{ Hz})(28.7\text{ m/s})^2}{(343\text{ m/s})(-125\text{ s}^{-2})} = 17.2903\text{ m}$

ROUND: Since the frequency can be read off the graph to two significant figures, the distance from the car to the train tracks should be rounded to two significant figures. This means $b = 17$ m.

DOUBLE-CHECK: Meters are appropriate units for a distance, and a distance of 17 m is a reasonable distance for a car to stop from train tracks while a train passes.

16.50. **THINK:** The question asks for the Doppler shifted frequency as a car moves at 100. km/h away from a siren elevated 100. m. The frequency of the siren is 440. Hz. Also, plot the frequency as a function of the car's position.

SKETCH:



RESEARCH: The frequency will be lower as the car drives away because of the Doppler shift. The car is moving from the base of the siren pole at 100. km/h, but the distance from the siren is slightly different. First, determine the velocity as the car moves from the siren and then use this velocity to determine the new Doppler shifted frequency. The distance from the source to the driver is given by $x = \sqrt{h^2 + r^2}$, where *r* is the distance of the driver from the base of the pole, and *h* is the height of the pole. The velocity of the driver from the source is:

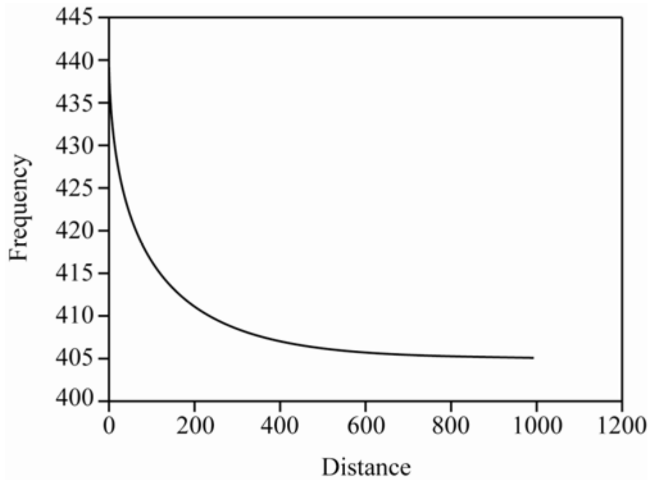
$$v_0 = \frac{dx}{dt} = \frac{dx}{dr}\frac{dr}{dt} = \frac{r}{\sqrt{h^2 + r^2}}\frac{dr}{dt} = \frac{r}{\sqrt{h^2 + r^2}}v_r,$$

where v_r is the ground velocity of the car (100. km/h or 27.8 m/s).

SIMPLIFY: Then, the Doppler shifted frequency heard by the driver is:

$$f' = f\left(1 - \frac{v_0}{v}\right) = f\left(1 - \frac{r}{\sqrt{h^2 + r^2}}\frac{v_r}{v}\right).$$

CALCULATE: The siren frequency as a function of distance is plotted.



As the car gets far away, it appears to be moving radially from the source, and the radius dependent term approaches unity, so the Doppler shift approaches what it would be if the car was moving radially at 100. km/h from the source.
ROUND: Rounding is not necessary.
DOUBLE-CHECK: The results of the plot are reasonable. As the car speeds away, the frequency decreases until the horizontal distance is much greater than the vertical distance to the siren.

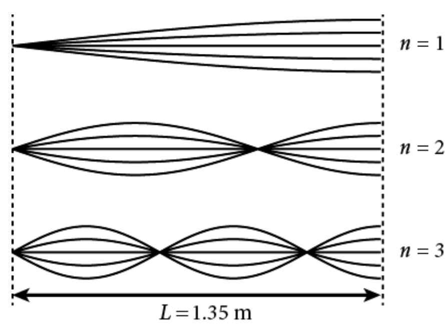
16.51. The problem presents a standing wave in a column of air with both ends open. The frequency is given as $f_\text{a} = 440.\text{ Hz},$ and the next higher harmonic as $f_\text{b} = 660.\text{ Hz}.$ Determine (a) the fundamental frequency and (b) the length of the air column.
(a) Recall that for standing waves, the harmonic frequencies are given with respect to the fundamental frequency, f_0, as $f_n = nf_0$. Therefore, $f_\text{a} = 440.\text{ Hz} = nf_0$ and $f_\text{b} = 660.\text{ Hz} = (n+1)f_0$. Therefore,

$$f_\text{b} - f_\text{a} = 660.\text{ Hz} - 440.\text{ Hz} = 220.\text{ Hz} = (n+1)f_0 - nf_0 = f_0.$$

Therefore, the fundamental frequency is $f_0 = 220.\text{ Hz}$.
(b) Now, to obtain the length of the air column, L, recall the expression for the wave speed is $v = f\lambda$. The wavelength is then given by $\lambda = v/f$, where for the fundamental wavelength, $\lambda = 2L$. Therefore,

$$L = \frac{v}{2f_0} = \frac{343\text{ m/s}}{2(220.\text{ Hz})} = 0.7795\text{ m} \approx 0.780\text{ m}.$$

16.52.



Because the tube is open at one end and closed at the other, the standing waves will have a node at one end and an antinode at the other end. Equation 16.19 tells us

$$\lambda_n = \frac{4L}{2n-1} \text{ for } n = 1,2,3,\ldots$$

So the shortest longest three wavelengths correspond to $n = 1,2,3$:

$$\lambda_1 = \frac{4L}{2-1} = 4(1.35 \text{ m}) = 5.40 \text{ m}$$
$$\lambda_2 = \frac{4L}{4-1} = (4/3)(1.35 \text{ m}) = 1.80 \text{ m}$$
$$\lambda_2 = \frac{4L}{6-1} = (4/6)(1.35 \text{ m}) = 1.08 \text{ m.}$$

Since these standing waves consist of traveling waves with the speed of sound, the wavelength of each node can be converted into a frequency as follows:

$$f_1 = \frac{v_{\text{sound}}}{\lambda_1} = \frac{343 \text{ m/s}}{5.40 \text{ m}} = 63.5 \text{ Hz}, \quad f_2 = \frac{v_{\text{sound}}}{\lambda_2} = \frac{343 \text{ m/s}}{1.80 \text{ m}} = 191 \text{ Hz}, \quad f_3 = \frac{v_{\text{sound}}}{\lambda_3} = \frac{343 \text{ m/s}}{1.08 \text{ m}} = 318 \text{ Hz.}$$

The wavelengths of the sound waves created by the bugle are the same as the wavelengths of the standing waves in the bugle.

16.53. Recall that the frequencies of standing waves in a pipe with an open end are given by:

$f_n = \frac{(2n-1)v}{4L}$. For $n = 1$ (fundamental node):

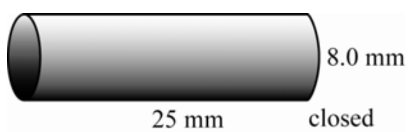
$f_1 = v/(4L) \Rightarrow 1047 \text{ Hz} = 343 \text{ m/s}/(4L) \Rightarrow L = 8.19 \text{ cm},$ so that the top of the liquid must be 8.19 cm from the top of the bottle.

16.54. Recall that the speed of the wave is given by $v = f\lambda$. The lowest resonant frequency occurs when there is a node at the center of the rod, and anti-nodes at each end, giving $L = \lambda/2$. Solving for the frequency gives:

$$f = \frac{v}{\lambda} = \frac{v}{2L} = \frac{5000. \text{ m/s}}{4.00 \text{ m}} = 1250 \text{ Hz}$$

16.55. **THINK:** The question asks for the resonant frequency of the ear canal when the temperature is 37 °C. The canal can be approximated by a tube with a diameter of 8.0 mm and a length of $L = 25$ mm, with one open end. Therefore this is a fairly straightforward problem of the lowest frequency of a standing wave in a half-open pipe.

SKETCH:



RESEARCH: Recall that the speed of sound can be calculated in terms of the temperature by $v(T) = 331 + 0.6T$. With this, the resonant frequency can be determined from the relation, $f = v/\lambda$, where λ is the wavelength.

SIMPLIFY: For the fundamental frequency of a standing wave in a half-open pipe, the wavelength is given by $\lambda = 4L$, where L is the length of the tube. Therefore,

$$f = \frac{v}{\lambda} = \frac{331 + 0.6T}{4L}.$$

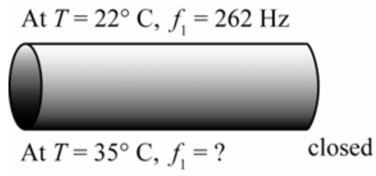
CALCULATE: Inserting the given values: $f = \frac{331 + 0.6(37)}{4(0.025)} \text{ Hz} = 3532 \text{ Hz.}$

ROUND: Since the values are given to two significant figures, the result should be rounded to $f = 3.5 \text{ kHz}$.

DOUBLE-CHECK: Based on the given values, the result is reasonable. In Figure 16.10 you can see that the ear of a teenager is most sensitive in the frequency range between 1 and 10 kHz, which gives us confidence that our result is in the right ballpark.

16.56. **THINK:** I want to determine the frequency produced by a pipe when the temperature is 35 °C. It is known that when the temperature is 22 °C, it produces a frequency of $f_1 = 262$ Hz.

SKETCH:



RESEARCH: Recall that the speed of sound can be calculated in terms of temperature by $v(T) = 331 + 0.6T$. With this, the wave speed for both temperatures can be determined from $v = f\lambda$, where λ is the wavelength. Since the length of the pipe is assumed to remain constant through the temperature change, it is clear that $f_{T_1} / f_{T_2} = v_{T_1} / v_{T_2}$.

SIMPLIFY: The frequency for the warmer temperature can be determined from:

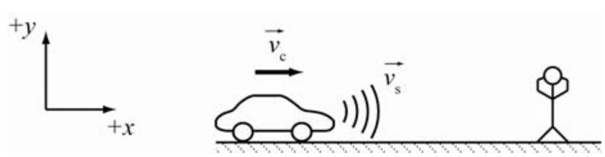
$$f_{T_2} = f_{T_1} \frac{v_{T_2}}{v_{T_1}} = f_{T_1} \frac{331 + 0.6T_2}{331 + 0.6T_1}.$$

CALCULATE: $f_{T_2} = (262 \text{ Hz}) \dfrac{331 + 0.6(35.0)}{331 + 0.6(22.0)} = 267.9 \text{ Hz}$

ROUND: Since the given values are accurate to three significant figures, the result is reported as $f_{T_2} = 268$ Hz.

DOUBLE-CHECK: It is expected that the frequency increases as the temperature increases. The result confirms this expectation.

16.57. The given quantities are the horn frequency, $f =$ 400.0 Hz, the car's speed, $v_c = 20.0$ m/s and the speed of sound, $v_s = 343$ m/s.

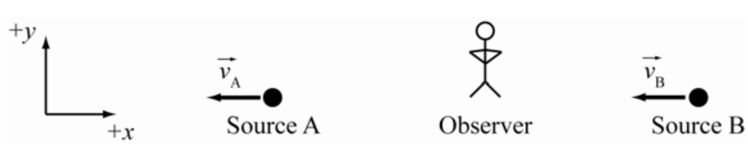


This problem can be solved using the Doppler effect. In this case, the equation is: $f_0 = f\left(\dfrac{v_s}{v_s - v_c}\right)$.

The negative sign indicates that the source is moving towards the observer. Inserting the values:

$$f_0 = (400.0 \text{ Hz})\left(\frac{343 \text{ m/s}}{343 \text{ m/s} - 20.0 \text{ m/s}}\right) = 425 \text{ Hz}.$$

16.58. Both sounds have the same frequency, and both sources are moving with the same velocity, $v = v_{\text{sound}} / 2$. Source A is moving away from the observer, and source B is moving toward the observer. The frequency observed for source A is f_A, the frequency observed for source B is f_B, the speed of source *A* is $v_A = v_s / 2$, the speed of source B is $v_B = v_s / 2$, and the speed of sound, $v_s = 343$ m/s.



The general equation that represents the Doppler effect is: $f_0 = f\left(\dfrac{v_s}{v_s \pm v}\right)$,

where f_0 is the observed frequency. Source A is moving away from the observer, so the observed frequency due to source A is given by:

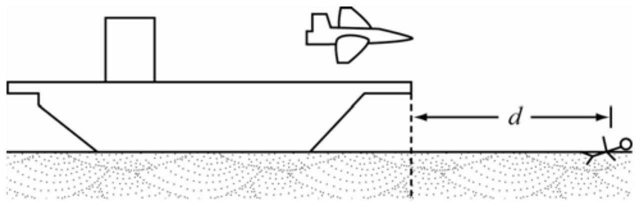
$$f_A = f\left(\frac{v_s}{v_s + v_A}\right) = f\left(\frac{v_s}{v_s + v_s/2}\right) = f\left(\frac{2v_s}{3v_s}\right) = \frac{2}{3}f.$$

Source B is moving towards the observer, so the observed frequency due to source B is:

$$f_B = f\left(\frac{v_s}{v_s - v_B}\right) = f\left(\frac{v_s}{v_s - v_s/2}\right) = f\left(\frac{2v_s}{v_s}\right) = 2f.$$

The ratio of the observed frequencies is: $\dfrac{f_A}{f_B} = \dfrac{2f/3}{2f} = \dfrac{1}{3}.$

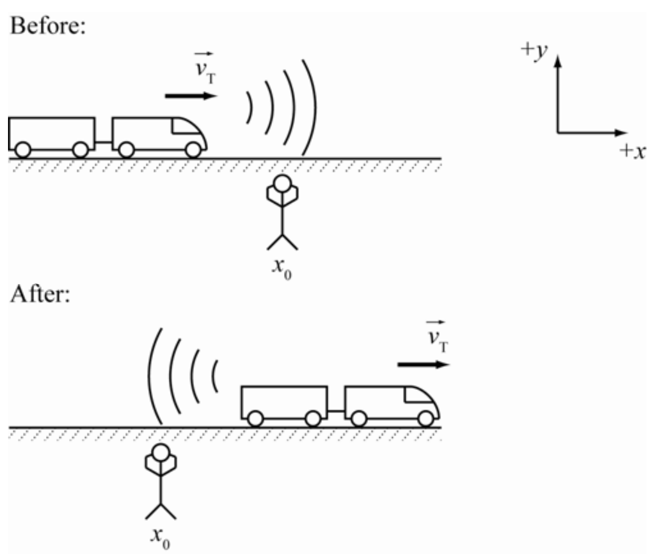
16.59. The quantities of interest for this problem are the distance from the aircraft carrier to the diver, $d = 1.00 \cdot 10^3$ m (assume the diver is stationary), the speed of sound in air, $v_s = 343$ m/s, and the speed of sound in seawater, $v_w = 1530$ m/s. The diver is a long distance from the jet and aircraft carrier, so assume that the distance between his ears can be neglected.



The general equation for speed is $v = d/t$, where t represents time. In this situation, the time difference, $\Delta t = t_s - t_w$, depends on the speed of sound through the two different mediums. The time it takes for the diver to hear the jet in his submerged ear is $t_w = d/v_w$. The time it takes for the diver to hear the jet in his ear that is above the water is $t_s = d/v_s$. The time difference is:

$$\Delta t = t_s - t_w = \frac{d}{v_s} - \frac{d}{v_w} = d\left(\frac{1}{v_s} - \frac{1}{v_w}\right) = \left(1.00 \cdot 10^3 \text{ m}\right)\left(\frac{1}{343 \text{ m/s}} - \frac{1}{1530 \text{ m/s}}\right) = 2.26 \text{ s}.$$

16.60. The given quantities are the frequency of the horn, $f = 311$ Hz, the speed of the train, $v_T = 22.3$ m/s, and the speed of sound in air, $v_s = 343$ m/s.



As the train approaches the stationary observer, the observed frequency is: $f_o = f\left(\frac{v_s}{v_s - v_T}\right)$. As the train moves away from the stationary observer, the observed frequency is: $f_o' = f\left(\frac{v_s}{v_s + v_T}\right)$. The frequency shift is $\Delta f = \left|f_o' - f_0\right| = \left|f\left(\frac{v_s}{v_s + v_T}\right) - f\left(\frac{v_s}{v_s - v_T}\right)\right| = \left|fv_s\left(\frac{1}{v_s + v_T} - \frac{1}{v_s - v_T}\right)\right|$.

$$\Delta f = \left|(311\text{ Hz})(343\text{ m/s})\left(\frac{1}{343\text{ m/s} + 22.3\text{ m/s}} - \frac{1}{343\text{ m/s} - 22.3\text{ m/s}}\right)\right| = 40.6\text{ Hz}$$

16.61. The given values are listed in the provided table. Determine the lengths of the pipes required to achieve the listed frequencies. The speed of sound in air is $v_s = 343\text{ m/s}$. A wind chime is made from pipes that are open at both ends. The general equation for standing waves in an open pipe is $L = n\lambda/2$, where λ is the wavelength and n = 1, 2, 3, ... The wavelength can be related to frequency using the equation $\lambda = v/f$. In this case, $\lambda = v_s/f$, and you can substitute this into the equation for L to get:

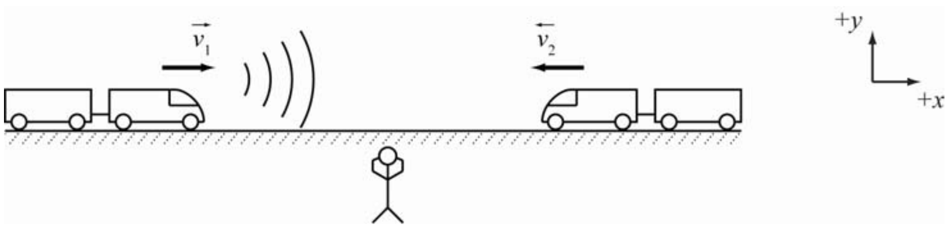
$$L = \frac{n}{2}\left(\frac{v_s}{f}\right).$$

For the fundamental frequency, $n = 1$. The expression used to fill in the table is $L = v_s/2f$.

Note	Frequency (Hz)	Length (m)
G4	392	0.438
A4	440	0.390
B4	494	0.347
F5	698	0.246
C6	1046	0.164

16.62. The given quantities are the speed of train 1, $v_1 = 25.0\text{ m/s}$, the speed of train 2, $v_2 = 25.0\text{ m/s}$, the frequency of the whistle, f = 300. Hz, and the speed of sound in air, $v_s = 343\text{ m/s}$.

(a)

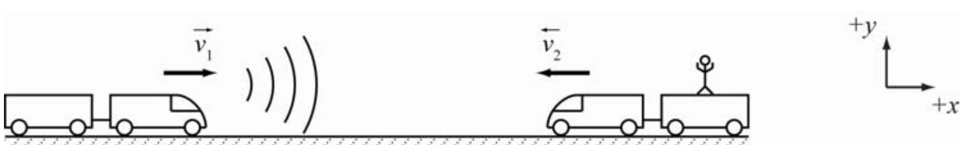


The choice of which train blows the whistle is arbitrary because they are both moving at the same speed. If train 1 blows the whistle while it is approaching the stationary observer, then the frequency that the observer detects is:

$$f_o = f\left(\frac{v_s}{v_s - v_1}\right) = (300.\text{ Hz})\left(\frac{343\text{ m/s}}{343\text{ m/s} - 25.0\text{ m/s}}\right) = 323.6\text{ Hz}.$$

Because the frequency provided in the question has three significant figures, the result should be rounded to $f_o = 324\text{ Hz}$.

(b)



The frequency that a man on train 2 would detect is:

$$f_o' = f\left(\frac{v_s + v_2}{v_s - v_1}\right),$$

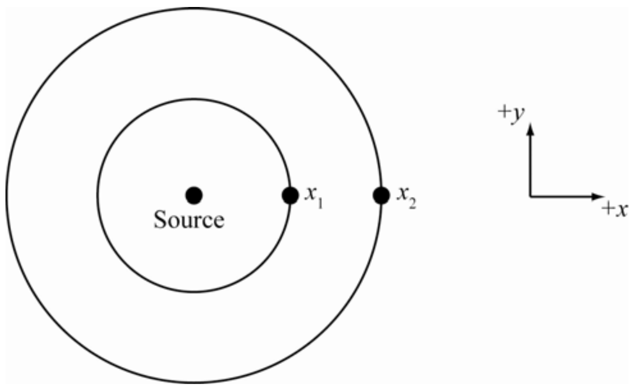
where the positive sign in the numerator and the minus sign in the denominator denote the observer on train 2 and the source (train 1) are approaching each other.

$$f_o' = 300.\text{ Hz}\left(\frac{343\text{ m/s} + 25.0\text{ m/s}}{343\text{ m/s} - 25.0\text{ m/s}}\right) = 347\text{ Hz}$$

Rounding to the appropriate number of significant figures, the result remains $f_o' = 347$ Hz.

16.63. **THINK:** The given quantities are the distance, $x_2 = 20.0$ m, the intensity level, $\beta_2 = 60.0$ dB, and the distance, $x_1 = 2.00$ m. The distances are relative to the source. The source radiates equally in all directions. Determine the intensity, I_1, at point x_1.

SKETCH:



RESEARCH: The source is assumed to be radiating equally in all directions, therefore the ratio of the two intensities is:

$$\frac{I_1}{I_2} = \frac{x_2^{\,2}}{x_1^{\,2}}.$$

The relationship between intensity, *I*, and sound level, β, is: $\beta = 10\log\frac{I}{I_0}$. The difference in sound levels is $\Delta\beta = \beta_2 - \beta_1$.

SIMPLIFY: I_1 can be solved for in terms of I_2: $I_1 = I_2\frac{x_2^{\,2}}{x_1^{\,2}}$. The difference in sound levels between points, x_2 and x_1 is $\Delta\beta = \beta_2 - \beta_1$.

$$\Delta\beta = \beta_2 - \beta_1 = 10\log I_2 - 10\log I_0 - 10\log I_1 + 10\log I_0 = 10\log\left(\frac{I_2}{I_1}\right)$$

Substituting $\Delta\beta$ into the sound level difference equation gives the simplified expression:

$$10\log\left(\frac{I_2}{I_1}\right) = \beta_2 - \beta_1 \;\Rightarrow\; \beta_1 = \beta_2 - 10\log\left(\frac{I_2}{I_1}\right).$$

Substitute the expression for I_1 into this equation to get:

$$\beta_1 = \beta_2 - 10\log\left(\frac{I_2}{I_2\frac{x_2^{\,2}}{x_1^{\,2}}}\right) = \beta_2 - 10\log\left(\frac{x_1^{\,2}}{x_2^{\,2}}\right).$$

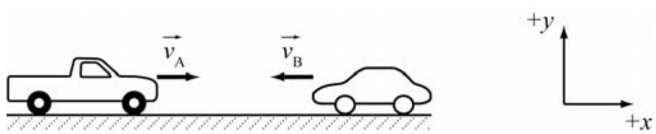
CALCULATE: $\beta_1 = 60.0 \text{ dB} - 10.0\log\left(\frac{(2.00 \text{ m})^2}{(20.0 \text{ m})^2}\right) = 60.0 \text{ dB} - (-20.0 \text{ dB}) = 80.0 \text{ dB}$

ROUND: The values in the question are given to three significant figures, so the result remains $\beta_1 = 80.0 \text{ dB}$.

DOUBLE-CHECK: It is reasonable that the sound intensity is greater when the distance to the source is less.

16.64. **THINK:** The frequencies emitted from the speakers in the two vehicles are $f = f_A = f_B = 1000.0 \text{ Hz}$. The vehicle speeds are $v_A = 10.00 \text{ m/s}$ and $v_B = 20.00 \text{ m/s}$. The vehicles are approaching each other. The speed of sound in air is $v_s = 343.0 \text{ m/s}$. Determine the detected frequencies for each vehicle, denoted f_{0A} and f_{0B}.

SKETCH:



RESEARCH: The general equation describing the Doppler effect when both the source and observer are moving is:

$$f_o = f\left(\frac{v_s \pm v_{\text{observer}}}{v_s \pm v_{\text{source}}}\right).$$

When the observer and the source are approaching each other, the sign in the numerator is positive and the sign in the denominator is negative.

SIMPLIFY: The frequency an observer in vehicle *A* detects is: $f_{0A} = f\left(\frac{v_s + v_A}{v_s - v_B}\right)$. The frequency an observer in vehicle *B* detects is: $f_{0B} = f\left(\frac{v_s + v_B}{v_s - v_A}\right)$.

CALCULATE: $f_{0A} = 1000.0 \text{ Hz}\left(\frac{343.0 \text{ m/s} + 10.00 \text{ m/s}}{343.0 \text{ m/s} - 20.00 \text{ m/s}}\right) = 1092.8793 \text{ Hz}$

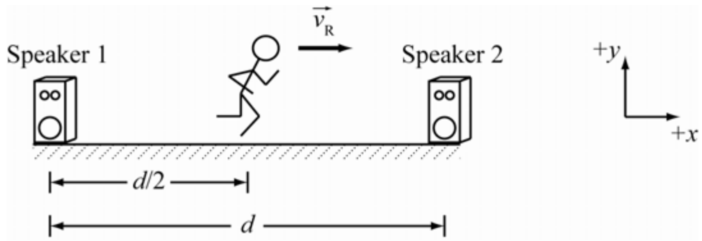
$f_{0B} = 1000.0 \text{ Hz}\left(\frac{343.0 \text{ m/s} + 20.00 \text{ m/s}}{343.0 \text{ m/s} - 10.00 \text{ m/s}}\right) = 1090.0901 \text{ Hz}$

ROUND: The speeds provided in the question have four significant figures, so the results should be rounded to $f_{0A} = 1093 \text{ Hz}$ and $f_{0B} = 1090. \text{ Hz}$.

DOUBLE-CHECK: The detected frequencies are different, but similar. This is expected because the detected frequencies depend on the speed of the sources and observers. The two vehicles travel at different speeds, so a difference is to be expected.

16.65. **THINK:** The separation distance is $d = 80.0$ m. The frequency of the speakers is $f = 286$ Hz. The beat frequency is $f_B = 10.0 \text{ Hz}$. Determine the speed, v_R, with which you are running toward one of the speakers. The speed of sound in air is $v_s = 343 \text{ m/s}$.

SKETCH:



RESEARCH: The equation for the beat frequency is $f_B = |f_1 - f_2|$. The frequencies you detect as you run from the position, $d/2$, towards speaker 2 are f_1 and f_2. These are the observed frequencies due to speaker 1 and speaker 2, respectively. The general equation for the Doppler effect is: $f_o = f\left(\frac{v_s \pm v_{\text{observer}}}{v_s}\right)$, where the minus sign denotes the observer is approaching the source.

SIMPLIFY: You are the observer, so $v_{\text{observer}} = v_R$. $f_1 = f\left(\frac{v_s - v_R}{v_s}\right)$, $f_2 = f\left(\frac{v_s + v_R}{v_s}\right)$. These expressions can be substituted into the beat frequency equation to get:

$$f_B = \left| f\left(\frac{v_s - v_R}{v_s}\right) - f\left(\frac{v_s + v_R}{v_s}\right)\right| = \left| f\left(\frac{v_s - v_R - (v_s + v_R)}{v_s}\right)\right| = f\left|-2\frac{v_R}{v_s}\right| \Rightarrow |v_R| = \frac{f_B v_s}{2f}.$$

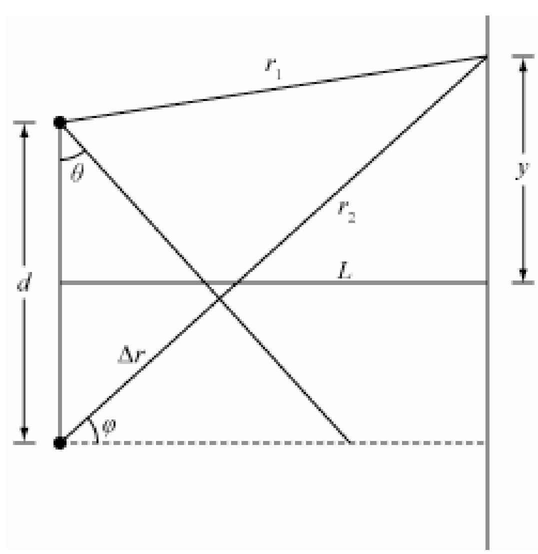
CALCULATE: $v_R = \dfrac{10.0 \text{ Hz}(343 \text{ m/s})}{2(286 \text{ Hz})} = 5.997 \text{ m/s}$

ROUND: The distance and the beat frequency each have three significant figures, so the result should be rounded to $v_R = 6.00$ m/s.

DOUBLE-CHECK: The calculated speed has the proper units. The speed of 6 m/s is reasonable, considering a world class sprinter can run at approximately 10 m/s (e.g. a world class athlete can run the 100 m dash in about 10 s).

16.66. **THINK:** The loud speakers are driven in phase at a frequency, $f = 3400.$ Hz, and are separated by a distance $d = 4.00$ m. A microphone mounted to the carrier detects sound at a distance, $L = 400.$ m, from the speakers. The speed of sound in air is $v_s = 340.$ m/s.

SKETCH:



RESEARCH: The condition for constructive interference is $\Delta r = r_2 - r_1 = n\lambda$, where Δr is the difference between the path lengths of the two sound wave fronts. The condition for destructive interference is $\Delta r = \left(n+\frac{1}{2}\right)\lambda$, where λ is the wavelength of the sound. For small values of n ($n = 0, \pm 1, \pm 2, \pm 3$) $\Delta r << d$, and $d << L$. Therefore the assumption can be made that $\Delta r \cong d\sin\theta$. From the sketch, it is clear that $\tan\phi = y/L$. By symmetry, and because the angles are small, $\phi \cong \theta$. Further, because the angles are small, the approximations can be made that $\tan\phi \approx \phi \approx \theta$, and $\sin\theta \approx \theta$ (in radians). The wavelength is given by $\lambda = v_s / f$. The distances in part (c) and (d) will be the approximate distances near the primary maximum. The approximation becomes very poor for very large values of *y*.

SIMPLIFY:

(a) $y = L\tan\theta \approx L\theta$, $\Delta r = d\sin\theta = d\theta \Rightarrow \theta = \Delta r / d$. Eliminate θ and solve for y: $y_{\text{max}} = \dfrac{L\Delta r}{d} = \dfrac{Ln\lambda}{d} = \dfrac{nLv_s}{fd}$.

(b) $y_{\text{min}} = \left(n+\frac{1}{2}\right)\dfrac{Lv_s}{fd}$

(c) Use the fact that $\lambda = v_s / f$. The separation between points of maximum intensity are given by

$$y_{\text{max},2} - y_{\text{max},1} = (n+1)\frac{\lambda L}{d} - n\frac{\lambda L}{d} = \frac{\lambda L}{d}.$$

(d) The separation between points of minimum intensity are given by

$$y_{\text{min},2} - y_{\text{min},1} = \left(n+1+\frac{1}{2}\right)\frac{\lambda L}{d} - \left(n+\frac{1}{2}\right)\frac{\lambda L}{d} = \frac{\lambda L}{d}.$$

(e) The waves would still interfere at the same positions (both constructively and destructively). However, at points of destructive interference, if the sound waves had different intensities, then you would not detect zero intensity. You would detect an intensity that corresponded to the difference between the speakers' intensities.

CALCULATE:

(a) $y_{\text{max}} = n\dfrac{\left(4.00\cdot 10^2\text{ m}\right)\left(340.\text{ m/s}\right)}{\left(3400.\text{ Hz}\right)\left(4.00\text{ m}\right)} = n\cdot 10.0\text{ m}$. The question asks where the sound intensity is at its maximum. Intensity is inversely proportional to the distance from the source. The shortest distance from the source is at the primary maximum, $n = 0$. So $y_{\text{max}} = (0)\cdot 10.0\text{ m} = 0\text{ m}$.

(b) $y_{\text{min}} = \left(n+\dfrac{1}{2}\right)\dfrac{\left(4.00\cdot 10^2\text{ m}\right)\left(340.\text{ m/s}\right)}{\left(3400.\text{ Hz}\right)\left(4.00\text{ m}\right)} = 10.0\left(n+\dfrac{1}{2}\right)\text{ m}$

(c) $\lambda = \dfrac{340.\text{ m/s}}{3400.\text{ Hz}} = 0.100\text{ m}$, so $\Delta y_{\text{max}} = \dfrac{\left(0.100\text{ m}\right)\left(4.00\cdot 10^2\text{ m}\right)}{4.00\text{ m}} = 10.0\text{ m}$.

(d) $\Delta y_{\text{min}} = \dfrac{\left(0.100\text{ m}\right)\left(4.00\cdot 10^2\text{ m}\right)}{4.00\text{ m}} = 10.0\text{ m}$

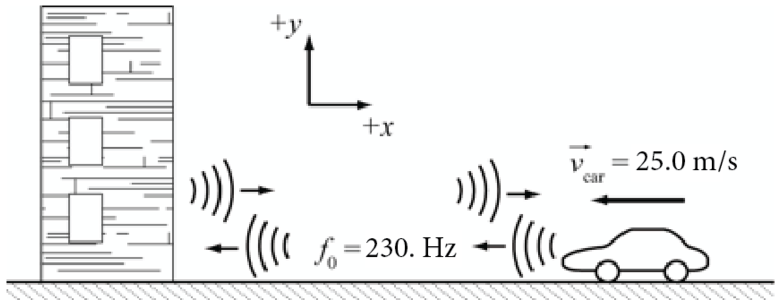
(e) does not apply

ROUND: Rounding is not necessary.

DOUBLE-CHECK: The calculated values are reasonable, considering the given quantities and approximations made.

16.67. **THINK:** The car is traveling towards the building at a speed of $v = 25.0$ m/s. The car horn emits sound with a frequency of $f_0 = 230.\text{ Hz}$. The sound is reflected off the building and back to the driver. The mixing of the emitted and reflected sound form a beat frequency, f_B, which must be determined. Assume the speed of sound is $v_s = 343\text{ m/s}$.

SKETCH:



RESEARCH: The beat frequency is given by $f_B = |f_0 - f_2|$. In this case, f_2 is the frequency of the reflected sound. To determine f_2, consider the situation in steps. The frequency observed at the wall, f_1, can be determined using the Doppler effect equation: $f_1 = f_0\left(\frac{v_s}{v_s - v}\right)$. This frequency, f_1, is now the frequency of the reflected sound. In this situation, treat the building as the source. Because the building is stationary, the observed frequency of the reflected wave is given by: $f_2 = f_1\left(\frac{v_s + v}{v_s}\right)$.

SIMPLIFY: Substitute the equation for f_2 into the equation for f_B to get:

$$f_B = \left| f_0 - f_1\left(\frac{v_s + v}{v_s}\right) \right|.$$

The equation can be further simplified by substituting the expression for f_1 to get:

$$\begin{aligned} f_B &= \left| f_0 - f_0\left(\frac{v_s}{v_s - v}\right)\left(\frac{v_s + v}{v_s}\right) \right| \\ &= \left| f_0 - f_0\left(\frac{v_s + v}{v_s - v}\right) \right|. \end{aligned}$$

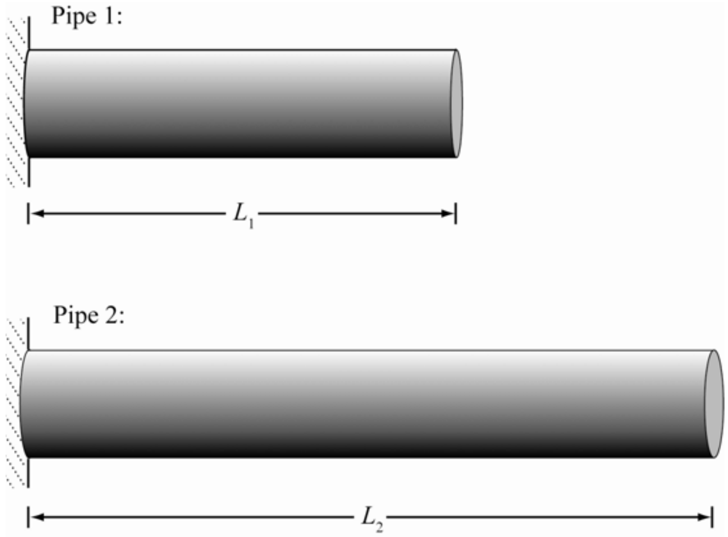
CALCULATE: $f_B = \left| 230.\text{ Hz} - (230.\text{ Hz})\left(\frac{343\text{ m/s} + 25.0\text{ m/s}}{343\text{ m/s} - 25.0\text{ m/s}}\right) \right| = 36.164\text{ Hz}$

ROUND: Three significant figures were provided in the problem so the result should be rounded to $f_B = 36.1$ Hz.

DOUBLE-CHECK: This is a reasonable beat frequency that could be detected by the driver.

16.68. **THINK:** The two pipes are identical half open pipes and have a fundamental frequency, $f_1 = 500.$ Hz. Determine the percent change in the length, *L*, of one of the pipes that will cause a beat frequency of $f_B = 10.0$ Hz when the pipes are sounded simultaneously. The speed of sound is $v = 343$ m/s.

SKETCH:



RESEARCH: The possible frequencies for a half open pipe are given by:

$$f_n = (2n-1)\frac{v}{4L},$$

for $n = 1, 2, 3, \ldots$ For a fundamental frequency, $n = 1$. The equation for the beat frequency is $f_B = f_1 - f_2$. In this case, f_1 is the fundamental frequency of pipe 1 and f_2 is the fundamental frequency of the elongated pipe 2. The percent change in the length, L, of the pipe is given by:

$$\%\,\text{change} = \left|\frac{L_2 - L_1}{L_1}\right|(100\%).$$

SIMPLIFY: $f_2 = f_1 - f_B$, $f_1 = \dfrac{v}{4L_1} \Rightarrow L_1 = \dfrac{v}{4f_1}$, $L_2 = \dfrac{v}{4f_2}$

Substitute the equations for L_1 and L_2 into the % length change equation to get:

$$\%\,\text{change} = \left|\frac{\frac{v}{4f_2} - \frac{v}{4f_1}}{\frac{v}{4f_1}}\right|(100\%) = \left|\frac{\frac{1}{f_2} - \frac{1}{f_1}}{\frac{1}{f_1}}\right|(100\%) = \left|\frac{f_1(f_1 - f_2)}{f_1 f_2}\right|(100\%) = \left|\frac{f_1 - f_2}{f_2}\right|(100\%).$$

Substitute $f_2 = f_1 - f_B$ into the above equation to get:

$$\%\,\text{length change} = \left|\frac{f_1 - f_B + f_1}{f_1 - f_B}\right|(100\%) = \frac{f_B}{f_1 - f_B}(100\%).$$

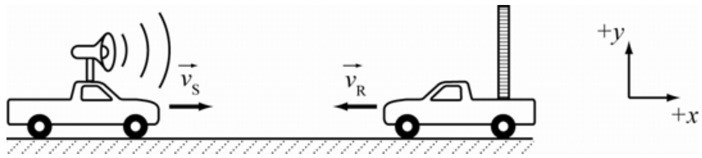
CALCULATE: $\%\,\text{length change} = \dfrac{10.0\text{ Hz}}{500.\text{ Hz} - 10.0\text{ Hz}}(100\%) = \dfrac{10.0\text{ Hz}}{490.\text{ Hz}} = 2.041\%$

ROUND: Three significant figures are provided in the question, so the result should be rounded to $\%\,\text{length change} = 2.04\%$.

DOUBLE-CHECK: A small change in length is expected for a small beat frequency. The change that was calculated is consistent with this expectation.

16.69. **THINK:** The source travels to the right at a speed of $v_s = 10.00$ m/s and emits a sound wave of frequency, $f_s = 100.0$ Hz. The reflector travels to the left at a speed of $v_R = 5.00$ m/s. Determine the frequency, f, of the reflected sound wave that is detected back at the source. The frequency of the reflected sound wave is f_R. Use $v = 343$ m/s as the speed of sound, the known value at 20 °C.

SKETCH:



RESEARCH: The general equation describing the Doppler effect when both the source and the observer are moving is:

$$f = f_{\text{source}}\left(\frac{v_{\text{sound}} \pm v_{\text{observer}}}{v_{\text{sound}} \pm v_{\text{source}}}\right).$$

When the observer and source approaching each other, the sign in the numerator is positive and the sign in the denominator is negative. In this problem, consider the frequency that is observed at the reflector. This frequency will be the source frequency for the sound wave that the observer moving to the right with the true source detects.

SIMPLIFY: $f_{\text{R}} = f_{\text{s}}\left(\frac{v+v_{\text{R}}}{v-v_{\text{s}}}\right)$, $f = f_{\text{R}}\left(\frac{v+v_{\text{s}}}{v-v_{\text{R}}}\right)$. Substitute the expression for f_{R} into the above equation to get:

$$f = f_{\text{s}}\left(\frac{v+v_{\text{R}}}{v-v_{\text{s}}}\right)\left(\frac{v+v_{\text{s}}}{v-v_{\text{R}}}\right).$$

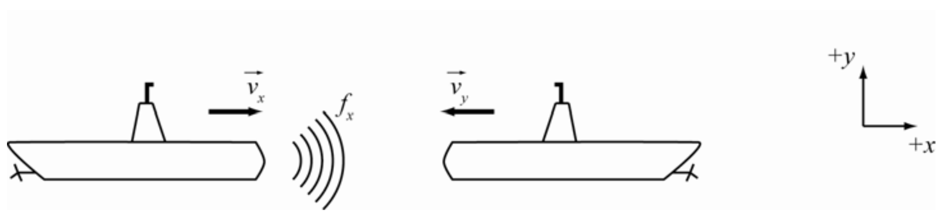
CALCULATE: $f = 100.0\text{ Hz}\left(\frac{343\text{ m/s}+5.00\text{ m/s}}{343\text{ m/s}-10.00\text{ m/s}}\right)\left(\frac{343\text{ m/s}+10.00\text{ m/s}}{343\text{ m/s}-5.00\text{ m/s}}\right) = 109.1423\text{ Hz}$

ROUND: The frequency should have three significant figures since the least precise value given in the question has three significant figures. Round the frequency to $f = 109$ Hz.

DOUBLE-CHECK: It is reasonable that the frequency of the reflected wave detected at the source is higher because the source and reflector are approaching each other. Also, the correct value has valid units for frequency.

16.70. **THINK:** Submarine *X* travels towards submarine *Y* at a speed of $v_x = 10.0$ m/s and emits a sonar wave of frequency, $f_x = 2000.0$ Hz. Submarine *Y* travels towards submarine *X* at a speed of $v_{\text{y}} = 15.0$ m/s. Clearly, in this case observer and source are both moving. However, when the sonar sound ping of submarine *X* bounces off the hull of submarine *Y* and is reflected back to the first submarine, submarine *Y* now emits the sound and sends it back; so the roles of emitter and observer are reversed.

SKETCH:



RESEARCH: The general equation describing the Doppler effect when both the source and observer are moving is:

$$f_0 = f\left(\frac{v_s \mp v_y}{v_s \pm v_x}\right).$$

When the observer and source are approaching each other, the sign in the numerator is positive and the sign in the denominator is negative. When the observer and source are moving away from each other, the numerator sign is negative and the denominator sign is positive.

SIMPLIFY:

(a) $f_y = f_x\left(\frac{v_s + v_y}{v_s - v_x}\right)$

(b) The sound coming back at *X* form *Y* $f_x' = f_y\left(\frac{v_s + v_x}{v_s - v_y}\right)$.

The frequency, f_y, is used as the source frequency because the sonar wave is being reflected off submarine *Y*. Substitute the expression for f_y from part (a) into the expression for f_x' to get:

$$f_x' = f_x\left(\frac{v_s + v_y}{v_s - v_x}\right)\left(\frac{v_s + v_x}{v_s - v_y}\right).$$

(c) In this part of the answer, both source and receiver are moving away from each other, and so we have to use the opposite signs that were used in part (a), result now in:

$$f_y' = f_x\left(\frac{v_s - v_y}{v_s + v_x}\right)$$

CALCULATE:

(a) $f_y = 2000.0 \text{ Hz}\left(\frac{1500.0 \text{ m/s} + 15.0 \text{ m/s}}{1500.0 \text{ m/s} - 10.0 \text{ m/s}}\right) = 2033.56 \text{ Hz}$

(b) $f_x' = 2033.56 \text{ Hz}\left(\frac{1500.0 \text{ m/s} + 10.0 \text{ m/s}}{1500.0 \text{ m/s} - 15.0 \text{ m/s}}\right) = 2067.795 \text{ Hz}$

(c) $f_y' = 2000.0 \text{ Hz}\left(\frac{1500.0 \text{ m/s} - 15.0 \text{ m/s}}{1500.0 \text{ m/s} + 10.0 \text{ m/s}}\right) = 1966.887 \text{ Hz},$

$\Delta f' = \left|1966.887 \text{ Hz} - 2000.0 \text{ Hz}\right| = 33.112 \text{ Hz}$

ROUND:

(a) $f_y = 2033.6 \text{ Hz}$

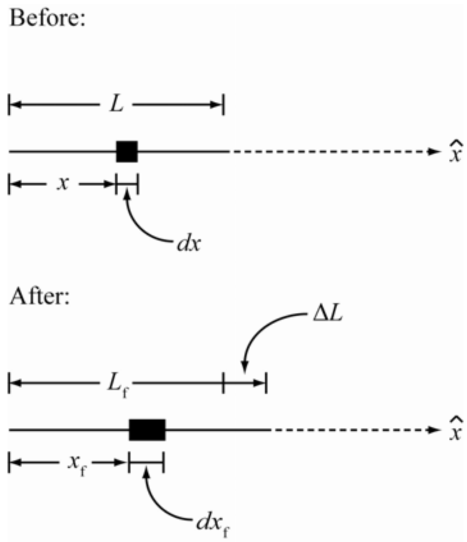
(b) $\Delta f = 2067.8 \text{ Hz}$

(c) $f_y' = 1966.9 \text{ Hz}, \ \Delta f' = 33.1 \text{ Hz}$

DOUBLE-CHECK: It makes sense that the detected frequency and frequency shift increase when the two submarines approach each other and decrease when the submarines are moving away from each other. This effect is experienced while approaching and moving away from a siren. The sound seems to increase in frequency and then decrease.

16.71. **THINK:** A sound wave is traveling in an elastic medium with either Young's modulus, *Y*, for a solid or bulk modulus, *B*, for a fluid. The function describing the wave is $\delta x(x,t)$, where δx denotes the displacement of a point in the medium from it equilibrium position, *x* is the position along the path of the wave, and *t* is time. The wave can also be described by a pressure wave function, $\delta p(x,t)$, where δp is a pressure change in the medium from its equilibrium value. Determine (a) the relationship between $\delta p(x,t)$ and $\delta x(x,t)$, and (b) the pressure wave function, $\delta p(x,t)$, given the wave function, $\delta x(x,t) = A\cos(\kappa x - \omega t)$. Also, determine the amplitude of the pressure wave.

SKETCH:



RESEARCH: Young's modulus is given by: $Y = \frac{F}{A}\left(\frac{L}{\Delta L}\right)$. The term *F*/*A* is called tensile stress. The term $\Delta L/L$ denotes the fractional length increase, also known as the strain. In the microscopic view, $\Delta L/L = \left(dx_{\mathrm{f}} - dx\right)/dx$. In this case, $\delta x = x_{\mathrm{f}} - x$, so $\frac{\Delta L}{L} = \frac{\partial}{\partial x}\delta x\left(x,t\right)$. Similarly to the discussion of Young's modulus, the bulk modulus, *B*, can be defined as: $B = \delta p \frac{v}{\Delta v}$. In one dimension: $\frac{\Delta v}{v} = \frac{\partial}{\partial x}\delta x\left(x,t\right)$.

SIMPLIFY:

(a) $\delta p = Y\frac{\Delta L}{L}$. Substituting $\frac{\Delta L}{L} = \frac{\partial}{\partial x}\delta x\left(x,t\right)$ into this equation gives $\delta p = Y\frac{\partial}{\partial x}\delta x\left(x,t\right)$. Similarly, for the bulk modulus: $\delta p = B\frac{\Delta v}{v}$, therefore, $\delta p\left(x,t\right) = B\frac{\partial}{\partial x}\delta x\left(x,t\right)$.

(b) $\delta x\left(x,t\right) = A\cos\left(\kappa x - \omega t\right)$, $\delta p\left(x,t\right) = Y\frac{\partial}{\partial x}\delta x\left(x,t\right) = Y\frac{\partial}{\partial x}A\cos\left(\kappa x - \omega t\right) = -Y\kappa A\sin\left(\kappa x - \omega t\right)$

For the bulk modulus case: $\delta p\left(x,t\right) = B\frac{\partial}{\partial x}A\cos\left(\kappa x - \omega t\right) = -B\kappa A\sin\left(\kappa x - \omega t\right)$. The maximum amplitude will occur when $\sin\left(\kappa x - \omega t\right) = 1$. Therefore, the pressure amplitude for the Young's modulus case is $\delta p_{\max} = Y\kappa A$ and is $\delta p_{\max} = B\kappa A$ in the bulk modulus case.

CALCULATE: This step is not necessary.

ROUND: Rounding is not necessary.

DOUBLE-CHECK: The derivations show that there is a relationship between Young's modulus and the bulk modulus, which is well known. This lends support to these derived equations.

16.72. **THINK:** Consider the sound wave described by the function $\delta x\left(x,t\right)$ or $\delta p\left(x,t\right)$ in the previous problem. Determine (a) the intensity, *I*, of the general wave in terms of $\delta x\left(x,t\right)$ and $\delta p\left(x,t\right)$, and (b) the intensity of the wave, $\delta x\left(x,t\right) = A\cos\left(kx - \omega t\right)$, in terms of the displacement and pressure amplitudes.

SKETCH: A sketch is not necessary.

RESEARCH:

(a) The wave can be thought of as a stack of rectangular slabs of infinitesimal thickness, compressed or stretched and set into oscillatory motion by the wave. Compression or stretching (rarefaction) of a slab imparts elastic potential energy to it, while its motion imparts kinetic energy. The sum of these two energy densities times the speed of the wave is the energy flux transported by the wave. The intensity, *I*, is the time averaged energy flux. The elastic force associated with the deformation of a slab of area, *A*, and thickness, ∂x, is:

$$F = -YA\frac{\partial \delta x}{\partial x} \text{ (solid)}, \; F = -BA\frac{\partial \delta x}{\partial x} \text{ (fluid)}.$$

The above expressions correspond to spring constants of: $k = \frac{YA}{\partial x}$ (solid), $k = \frac{BA}{\partial x}$ (fluid), respectively. The associated elastic potential energy per unit volume of the slab is then:

$$u_{\text{p}} = \frac{1}{2}Y\left(\frac{\partial \delta x}{\partial x}\right)^2 \text{ (solid)}, \; u_{\text{p}} = \frac{1}{2}B\left(\frac{\partial \delta x}{\partial x}\right)^2 \text{ (fluid)}.$$

The kinetic energy density associated with the wave is: $u_{\text{k}} = \frac{1}{2}\rho_0\left(\frac{\partial \delta x}{\partial t}\right)^2$. Hence, the energy flux associated with the wave is:

$$S = \frac{1}{2}v_{\text{s}}\left[Y\left(\frac{\partial \delta x}{\partial x}\right)^2 + \rho_0\left(\frac{\partial \delta x}{\partial t}\right)^2\right] \text{ (solid)}, \; S = \frac{1}{2}v_{\text{s}}\left[B\left(\frac{\partial \delta x}{\partial x}\right)^2 + \rho_0\left(\frac{\partial \delta x}{\partial t}\right)^2\right] \text{ (fluid)},$$

with the speed of sound as:

$$v_{\text{s}} = \sqrt{\frac{Y}{\rho_0}} \text{ (solid)}, \; v_{\text{s}} = \sqrt{\frac{B}{\rho_0}} \text{ (fluid)}.$$

This could also be written:

$$S = \frac{1}{2}v_{\text{s}}\left[\frac{1}{Y}(\delta p)^2 + \rho_0\left(\frac{\partial \delta x}{\partial t}\right)^2\right] \text{ (solid)}, \; S = \frac{1}{2}v_{\text{s}}\left[\frac{1}{B}(\delta p)^2 + \rho_0\left(\frac{\partial \delta x}{\partial t}\right)^2\right] \text{ (fluid)},$$

using the result of part (a) in the previous problem.
(b) This step is not applicable.
SIMPLIFY:
(a) The intensity of a sound wave, however, is not measured instantaneously, but over several or many periods of the wave. That is, the intensity is the time average of the energy flux over one or more periods, and is given by:

$$I = \frac{1}{2}v_{\text{s}}\left[Y\left\langle\left(\frac{\partial \delta x}{\partial x}\right)^2\right\rangle + \rho_0\left\langle\left(\frac{\partial \delta x}{\partial t}\right)^2\right\rangle\right] = \frac{1}{2}v_{\text{s}}\left[\frac{1}{Y}\left\langle(\delta p)^2\right\rangle + \rho_0\left\langle\left(\frac{\partial \delta x}{\partial t}\right)^2\right\rangle\right] \text{ (solid)}$$

$$I = \frac{1}{2}v_{\text{s}}\left[B\left\langle\left(\frac{\partial \delta x}{\partial x}\right)^2\right\rangle + \rho_0\left\langle\left(\frac{\partial \delta x}{\partial t}\right)^2\right\rangle\right] = \frac{1}{2}v_{\text{s}}\left[\frac{1}{B}\left\langle(\delta p)^2\right\rangle + \rho_0\left\langle\left(\frac{\partial \delta x}{\partial t}\right)^2\right\rangle\right] \text{ (fluid)}.$$

The angled brackets denote the time average for one or more periods of the wave.
(b) For a displacement wave with wave function $\delta x(x,t) = A\cos(\kappa x - \omega t)$, as in part (b) of the previous problem, the first formula above for the intensity becomes:

$$I = \frac{1}{2}v_{\text{s}}\left(\kappa^2 Y + \omega^2\rho_0\right)A^2\left\langle\sin^2\left(\kappa x - \omega t\right)\right\rangle = \frac{1}{4}v_{\text{s}}\left(\kappa^2 Y + \omega^2\rho_0\right)A^2.$$

Using the dispersion relation $\omega / \kappa = v_{\text{s}}$, the above equation becomes

$$I = \frac{1}{4}v_{\text{s}}\omega^2 A^2\left(\frac{Y}{v_{\text{s}}{}^2} + \rho_0\right),$$

The first of the earlier expressions for v_{s}, $v_{\text{s}} = \sqrt{\frac{Y}{\rho_0}}$, implies that the two terms in parentheses are equal. Hence, the intensity can be written:

$$I=\frac{1}{2}\rho_0\omega^2A^2v_s.$$

The expression for the intensity in a fluid medium yields the same result. In terms of the amplitude, *P*, of the corresponding pressure wave determined in part (b) of the previous problem, this result for the intensity for a solid medium takes the form:

$$I=\frac{1}{2}\rho_0\frac{\omega^2P^2}{k^2Y^2}v_s=\frac{1}{2}\rho_0\frac{P^2v_s^{\ 3}}{Y^2},$$

again using the dispersion relation for the waves. The above relationship for v_s then implies:

$$I=\frac{1}{2}\left(\frac{P^2}{\rho_0v_s}\right).$$

The corresponding expression for a fluid medium is identical.

CALCULATE: This step is not necessary.

ROUND: This step is not necessary.

DOUBLE-CHECK: In the computed formula the intensity is proportional to the square of the amplitude, as it should be.

16.73. **THINK:** Using the results of the previous problems, determine the displacement and pressure amplitudes of a pure tone of frequency $v=1.000\text{ kHz}$ in air (density $\rho_0=1.20\text{ kg/m}^3$ and speed of sound $v_s=343\text{ m/s}$) at $\beta=0.00\text{ db}$, and $\beta=120.\text{ db}$. Let $I_0=1.00\cdot10^{-12}\text{ W/m}^2$ be the standard reference intensity, and $f=\omega/(2\pi)$ be the frequency of the sound wave.

SKETCH: A sketch is not needed to solve this problem.

RESEARCH: From problem 16.72, we learned that $I=\frac{1}{2}\rho_0\omega^2A^2v_s$. We will use the formula $\beta=10\log\frac{I}{I_0}$ to eliminate *I*. The pressure amplitude is $P=\sqrt{2I\rho_0v_s}$.

SIMPLIFY: Solve the first formula in RESEARCH to get $A=A_\beta=\sqrt{\frac{2I}{p_0\omega^2v_s}}$, and substitute the expression $I=I_010^{\beta/10}$ to eliminate *I*. The result is $A_\beta=\sqrt{\frac{10^{\beta/10}I_0}{2\pi^2\rho_0f^2v_s}}$. The pressure amplitude is $P=P_\beta=\sqrt{2Ip_0v_s}=\sqrt{2\left(10^{\beta/10}\right)I_0p_0v_s}$.

CALCULATE: $A_{0.00}=\sqrt{\frac{10^{0.00/10}\cdot\left(1.00\cdot10^{-12}\text{ W/m}^2\right)}{2\pi^2\left(1.20\text{ kg/m}^3\right)\left(1.000\cdot10^3\text{ s}^{-1}\right)^2\left(343\text{ m/s}\right)}}=1.1094\cdot10^{-11}\text{ m}$ and

$A_{120.}=\sqrt{\frac{10^{120./10}\cdot\left(1.00\cdot10^{-12}\text{ W/m}^2\right)}{2\pi^2\left(1.20\text{ kg/m}^3\right)\left(1.000\cdot10^3\text{ s}^{-1}\right)^2\left(343\text{ m/s}\right)}}=1.1094\cdot10^{-5}\text{ m}$. At the threshold of hearing ($\beta=0.00$ decibels) the value of the pressure amplitude is given by the equation $P_{0.00}=\sqrt{2(1)\left(1.00\cdot10^{-12}\text{ W/m}^2\right)\left(1.20\text{ kg/m}^3\right)\left(343\text{ m/s}\right)}=2.8691\cdot10^{-5}\text{ Pa}$. On the other hand, the value of the pressure amplitude at the threshold of pain is given by the equation $P_{120.}=\sqrt{2\left(10^{120./10}\right)\left(1.00\cdot10^{-12}\text{ W/m}^2\right)\left(1.20\text{ kg/m}^3\right)\left(343\text{ m/s}\right)}=28.691\text{ Pa}$.

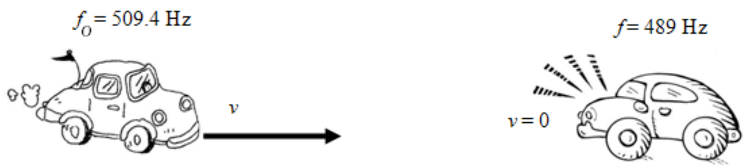
ROUND: The rounded values are: $A_{0.00}=1.11\cdot10^{-11}\text{ m}$, $A_{120.}=1.11\cdot10^{-5}\text{ m}$, $P_{0.00}=2.87\cdot10^{-5}\text{ Pa}$, and $P_{120.}=28.7\text{ Pa}$.

DOUBLE-CHECK: The calculated displacement of a pure tone frequency of 1.000 kHz at zero decibels is roughly a tenth of an atomic diameter. This is suprisingly small, but it is at least consistent with the fact that such a wave produces no sensation on the skin and only a barely discenible sensation at the eardrum. Increasing from 0 to 120 decibels corresponds to a 10^{12} factor increase in intensity. Since the displacement varies with the square root of the intensity, the calculated value for the displacement at 120 decibels should be larger by a factor of 10^6 , which it is. $1.11\cdot 10^{-5}$ m is one hundredth of a millimeter, and it is plausible that such an amplitude would begin to cause pain in the ear. The pressure amplitude at zero decibels is less than three ten-thousandths of an atmosphere, and this is consistent with what would be expected. Again, the pressure amplitude varies with the square root of the intensity, so the calculated value of the displacement at 120 decibels should be larger by a factor of 10^6, which it is. This shows that the calculated answers are reasonable.

Multi-Version Exercises

16.74. **THINK:** The frequency of the sound will be perceived differently by the drivers as a result of the Doppler effect. In this case, the source of the sound is the parked car, which is not moving, while the observer is in the moving car.

SKETCH: The car on the right is parked. The car on the left is moving towards the car on the left at a speed *v*.



RESEARCH: The source of the sound is stationary and the observer is moving towards the source, so the observed frequency is given by $f_O = f\left(1+\frac{v_{\text{observer}}}{v_{\text{sound}}}\right)$.

SIMPLIFY: The goal is to find the speed of the observer, so solve for that variable:

$$f\left(1+\frac{v_{\text{observer}}}{v_{\text{sound}}}\right) = f_O$$
$$1+\frac{v_{\text{observer}}}{v_{\text{sound}}} = \frac{f_O}{f}$$
$$\frac{v_{\text{observer}}}{v_{\text{sound}}} = \frac{f_O}{f}-1$$
$$v_{\text{observer}} = v_{\text{sound}}\left(\frac{f_O}{f}-1\right).$$

CALCULATE: The frequency of the horn is $f = 489$ Hz, but the driver in the approaching car hears a sound of frequency $f_0 = 509.4$ Hz. In this case, the speed of sound is 343 m/s, so the velocity of the observer must be

$$\begin{aligned} v_{\text{observer}} &= v_{\text{sound}}\left(\frac{f_O}{f}-1\right) \\ &= 343\text{ m/s}\left(\frac{509.4\text{ Hz}}{489\text{ Hz}}-1\right) \\ &= 14.30920245\text{ m/s}. \end{aligned}$$

ROUND: Though the frequency measured by the driver of the moving car has four significant figures, two other measured values, the frequency of the horn and the speed of sound, are given to only three significant figures, so the final answer can have only three significant figures. The car is going 14.3 m/s.

DOUBLE-CHECK: The car is going 14.3 m/s. Since 1 m = $6.214\cdot 10^{-4}$ miles and 3600 sec = 1 hour, the car is going $14.3\text{ m/s}\left(6.214\cdot 10^{-4}\,\frac{\text{miles}}{\text{meter}}\right)\left(3600\,\frac{\text{sec}}{\text{hour}}\right) = 32\text{ mph}$. For a car driving on a paved road, this is a perfectly reasonable speed, so the answer is reasonable.

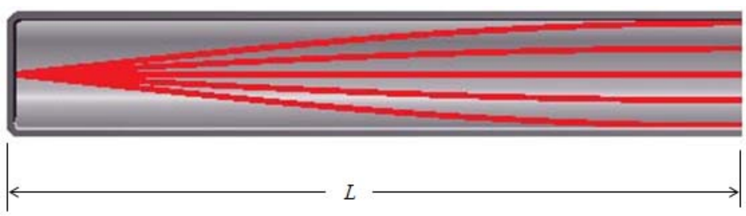
16.75. $$f_O = f\left(1+\frac{v_{\text{observer}}}{v_{\text{sound}}}\right)$$

$$f = \frac{f_O}{1+\frac{v_{\text{observer}}}{v_{\text{sound}}}} = \frac{579.4\text{ Hz}}{1+\frac{15.1\text{ m/s}}{343\text{ m/s}}} = 555.0\text{ Hz}$$

16.76. $$f_O = f\left(1+\frac{v_{\text{observer}}}{v_{\text{sound}}}\right) = (333\text{ Hz})\left(1+\frac{15.7\text{ m/s}}{343\text{ m/s}}\right) = 348\text{ Hz}$$

16.77. **THINK:** The tuba is brass instrument; it makes music by producing standing waves inside of the coiled tube. The lowest frequency that can be produced by the tuba corresponds to one quarter of a wavelength inside the tube.

SKETCH: Think of the tuba, uncoiled, as a pipe that is closed at one end and open at the other end. The lowest frequency corresponds to a wave with a node at the closed end and an antinode at the open end.



RESEARCH: The lowest frequency sound that can be produced corresponds to half of a wavelength inside the tuba, so the total wavelength is four times the length of the tuba, $\lambda = 4L$. The speed of the sound wave, wavelength, and frequency are related by the equation $v = \lambda f$.

SIMPLIFY: Solve $v = \lambda f$ for the frequency, $f = v/\lambda$. Now substitute $\lambda = 4L$ to get $f = \frac{v}{4L}$.

CALCULATE: The The length of the tuba is 7.373 m and the speed of the sound is 343.0 m/s, so the lowest frequency the tuba can produce is

$$f = \frac{v}{4L} = \frac{343.0\text{ m/s}}{4\cdot(7.373\text{ m})} = 11.63027\text{ Hz}.$$

ROUND: We round to four sigificant figures, $f = 11.63\text{ Hz}$.

DOUBLE-CHECK: The lowest tone that most humans can hear is about 20 Hz, so this sound will probably not be audible, although you can still feel it and hear various overtones. The next highest frequency is

$$f = \frac{(2\cdot 2-1)v}{4L} = \frac{3\cdot(343.0\text{ m/s})}{4\cdot(7.373\text{ m})} = 34.89\text{ Hz}.$$

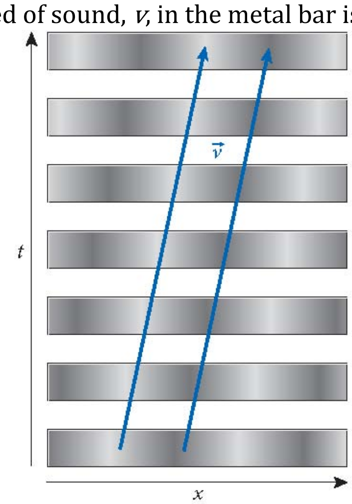
So the lowest solidly audible sound would be 34.89 Hz. This compares with A0 on a piano, which is

27.5 Hz.

16.78. $f = \dfrac{v}{2L}$

$$L = \frac{v}{2f} = \frac{343.0 \text{ m/s}}{2 \cdot 22.56 \text{ Hz}} = 7.602 \text{ m}$$

16.79. **THINK:** Since the metal bar is solid, the speed of sound will depend on the type and structure of the material. The speed of sound in a solid can be calculated using the density and Young's modulus.
SKETCH: The speed of sound, v, in the metal bar is shown (see Figure 16.2).



RESEARCH: In general, the speed of sound in a solid is given by $v = \sqrt{Y/\rho}$, where Y is the Young's modulus and ρ is the mass density.
SIMPLIFY: n/a
CALCULATE: The question states that the mass density of the bar is ρ = 3497 kg/m^3 and the Young's modulus is $266.3 \cdot 10^9$ N/m^2. The speed of sound in the metal bar is

$$v = \sqrt{Y/\rho} = \sqrt{\frac{266.3 \cdot 10^9 \text{ N/m}^2}{3497 \text{ kg/m}^3}} = 8726.453263 \text{ m/s}.$$

ROUND: The values used to calculate the speed of sound all have four significant figures, so the final answer should also have four figures. The speed of sound in the metal bar is 8726 m/s.
DOUBLE-CHECK: The speed of sound in air is 343 m/s, but air is less dense than a solid metal bar. Looking at Table 16.1, the speed of sound in this metal bar is somewhere between the speed of sound in Aluminum and the speed of sound in Diamond, confirming that the answer is physically realistic and of the correct order of magnitude.

16.80. $v = \sqrt{Y/\rho}$

$$\begin{aligned}\rho &= Y / v^2 \\ &= \left(112.1\cdot 10^9 \text{ N/m}^2\right) / \left(5628 \text{ m/s}\right)^2 \\ &= 3539 \text{ kg/m}^3\end{aligned}$$

16.81. $v = \sqrt{Y / \rho}$

$$\begin{aligned}Y &= \rho v^2 \\ &= \left(3579 \text{ kg/m}^3\right)\left(6642 \text{ m/s}\right)^2 \\ &= 157.9\cdot 10^9 \text{ N/m}^2\end{aligned}$$

Chapter 17: Temperature

Concept Checks

17.1. c **17.2.** a **17.3.** d **17.4.** c **17.5.** d

Multiple-Choice Questions

17.1. a **17.2.** a **17.3.** c **17.4.** b **17.5.** d **17.6.** d **17.7.** a **17.8.** d **17.9.** c **17.10.** e **17.11.** b **17.12.** b **17.13.** e

Conceptual Questions

17.14. Yes, this will still work. The lid will heat up more quickly if it is in direct contact with the warm water; the lid will expand more than the container, making it easier to open (not as easy as with a glass jar, but easier than before heating in the warm water).

17.15. As stated in Section 17.1, heat is defined as the transfer of a type of energy in the presence of a temperature gradient, and this energy is the random motion of the atoms and molecules that make up the material under study. In the conventional definition heat flows from high temperature to low temperature, indicating that the material with the higher temperature has more thermal energy. It is conceivable for heat to be defined in such a way as to represent the direction opposite to this energy transfer across a temperature gradient, in which case heat would flow from lower temperature to higher temperature.

In addition to defining heat flow, the methods of measuring temperature involving thermal expansion (Section 17.4) also depend upon temperature difference. Therefore it is entirely logical to define a temperature scale in such a way that the temperature difference between a cold material and a hot material is negative as well as it is to define a scale in such a way that the difference is positive; what is important is the magnitude of the difference. In fact, the original scale devised by Celsius had 0 as the boiling point of water and 100 as the freezing point. As thermodynamics became better understood, it was clear that there was a unique point on any temperature scale, which we now call *absolute zero*. To define a scale, as we now do, such that systems hotter than this have positive, rather than negative, temperatures (or even to label this point 0) is a matter of convenience.

17.16. Although the corona is very hot, it is not dense, so that the actual energy contained within the corona is small. This explains why a spaceship flying in the corona will not be burned up.

17.17. Different metals have different melting points and different heat capacities. Thus, one metal may liquefy easily while the other remains solid, making welding very difficult.

17.18. The volume of each object changes by the same amount during an identical ΔT. This implies that $\Delta V/\Delta T$ is the same for both objects. Therefore:

$$\frac{\Delta V}{\Delta T}=\frac{\Delta V_1}{\Delta T}=\beta_1 V_1=\frac{\Delta V_2}{\Delta T}=\beta_2 V_2 \;\Rightarrow\; \beta_1 V_1=\beta_2 V_2 \;\Rightarrow\; \beta_1 2V_2=\beta_2 V_2 \;\Rightarrow\; 2\beta_1=\beta_2 \Rightarrow\; \frac{\beta_1}{\beta_2}=\frac{1}{2}.$$

17.19. The only difference between these two temperature scales is where the zero point is. The units are of identical size. Hence, a temperature difference on the Kelvin scale is numerically equal to a temperature difference on the Celsius scale. The coefficient of linear expansion is used only in equations involving temperature differences, so it will take on the same value for either K^{-1} or $^\circ\text{C}^{-1}$.

17.20. (a) The system is not in equilibrium, so $T_0 > T_\text{d} > T_\text{i}$.

(b) A typical hot day is: $T_0 = 40$ °C (104 °F) The ice temperature is close to its melting point: $T_\text{i} = 0$ °C The drink temperature is somewhere in between, $40\,^\circ\text{C} > T_\text{d} > 0$ °C but hopefully closer to the ice temperature than the air temperature, such as $\text{T}_\text{d} \approx 10$ °C.

17.21. Rankine temperatures, T_R, differ from Fahrenheit temperatures, T_F, only in being measured from absolute zero. $0\text{ K} = -273.15\text{ °C} = -459.67\text{ °F}$. Thus, $T_R = T_F + 459.67\text{ °F} \Rightarrow T_F = T_R - 459.67\text{ °F}$. Rankine temperatures differ from Kelvin temperatures, T_K, only in being measured in Fahrenheit-degree increments. Thus,

$$T_R = \frac{9}{5}T_K \Rightarrow T_K = \frac{5}{9}T_R.$$

Finally, in terms of degrees Celsius, T_C:

$$T_R = \frac{9}{5}(T_C + 273.15\text{ °C}) \Rightarrow T_C = \frac{5}{9}T_R - 273.15\text{ °C}.$$

17.22. For such a two-level system, ordinary postive absolute temperatures correspond to the normal situation in which the lower energy level is more popluated than the higher. The lower the temperature, the more dominat is the lower level, as expected: the limit $T \to 0^+$ corresponds to a "ground state" in which all components of the system are in the lower level. The higher the temperature, the more components are excited into the upper level, but the lower level is always more populated; the limit $T \to +\infty$ corresponds to a limit in which both levels are equally populated.

Negative absolute temperatures correspond to a "population inversion," in which the higher energy level is more populated than the lower. The limit $T \to 0^-$ describes the limiting situation in which the entire system is in the higher level. Population inversions are real: the lasing medium of a laser, for example, must be driven ("pumped") into a population inversion for the laser to operate. The total energy of the system is higher at negative temperature than positive; negative absolute temperatures are not "colder than absolute zero," they are "hotter than infinity," in this context. It may be noted that the *time dependence* of a quantum state with energy E is given by the complex function:

$$\exp\left(-i\frac{Et}{\hbar}\right) = \cos\left(\frac{Et}{\hbar}\right) - i\sin\left(\frac{Et}{\hbar}\right),$$

where t is time, $\hbar$ is Planck's constant divided by 2π, and i the imaginary unit. Comparison of this with the temperature-dependent population factor (Maxwell-Boltzmann distribution) suggests that in a quantum context, (inverse) temperature can be interpreted as imaginary time!

17.23. Metal 1 expands and contracts more readily than does Metal 2.
(a) The strip will bend toward metal 1, since metal 1 will contract more than metal 2 does.
(b) The strip will bend toward metal 2, since metal 1 will expand more than metal 2 does.

17.24. The metal lid has a larger coefficient of thermal expansion than the glass. Hot food is placed in a hot glass jar and then a hot metal lid is screwed on top. As the glass and lid cools, the lid shrinks by a larger percentage than does the glass, making a tight seal.

17.25. Both will expand to the same outer radius. A cavity in a material will expand in the same manner as if it was filled with the same material.

Exercises

17.26. Use $T_C = \frac{5}{9}(T_F - 32\text{ °F})$ and $T_K = (T_C + 273.15\text{ °C})$

(a) -19 °F: $T_C = \frac{5}{9}(-19\text{ °F} - 32\text{ °F}) = -28\text{ °C}$; $T_K = (-28\text{ °C} + 273.15) = 245\text{ K}$. Rounding to two significant figures gives $T_K = 250\text{ K}$.

(b) 98.6 °F: $T_C = \frac{5}{9}(98.6\text{ °F} - 32.0\text{ °F}) = 37.0\text{ °C}$; $T_K = (37\text{ °C} + 273.15\text{ °C}) = 310.\text{ K}$

(c) 52 °F: $T_C = \frac{5}{9}(52\ °F - 32\ °F) = 11\ °C$; $\quad T_K = (11\ °C + 273.15\ °C) = 284$ K. Rounding to two significant figures gives $T_K = 280$ K.

17.27. The temperature $T_C = -21.8\ °C$ is three times its equivalent value $T_F = -7.3\ °F$. To check this, recall the conversion formula: $T_C = \frac{5}{9}(T_F - 32\ °F)$. $T_C = 3T_F \Rightarrow \frac{5}{9}(T_F - 32\ °F) = 3T_F \Rightarrow \frac{5}{9}(-32\ °F) = \left(\frac{27-5}{9}\right)T_F$

$\Rightarrow T_F = \frac{5(-32\ °F)}{22} = -7.2727\ °F$ and $T_C = \frac{5}{9}(-7.3\ °F - 32\ °F) = -21.8\ °C$

17.28. Using $T_C = \frac{5}{9}(T_F - 32\ °F)$ and $T_F = 134\ °F$, it is found that $134\ °F = 56.67\ °C$. Thus, $\Delta T = 56.67\ °C - 47\ °C = 9.67\ °C$. Rounding to two significant figures gives $\Delta T = 9.7\ °C$.

17.29. $T_C = \frac{5}{9}(T_F - 32.0\ °F) = \frac{5}{9}(-129\ °F - 32.0\ °F) = -89.4\ °C$

17.30. The perception of warmth or coolness is related to the temperature difference between a person and his environment. The temperature difference between an environment at a temperature of 0 °F and normal body temperature is 98.2 °F. So for the environment to feel twice as warm to a person, the temperature difference should be half of 98.2 °F, or 49.1 °F.

17.31. (a) $T_K = T_C + 273.15\ °C = -79\ °C + 273.15\ °C = 194$ K. Rounding to two significant figures gives $T_K = 190$ K.

(b) $T_F = \frac{9}{5}T_C + 32\ °C = \frac{9}{5}(-79\ °C) + 32\ °C = -110\ °F$

17.32. In the present-day Celsius scale, 77.0 °F corresponds to 25.0 °C, so that in the original Celsius scale, room temperature is $100.\ °C - 25.0\ °C = 75.0\ °C$.

17.33. $T_F = \frac{9}{5}(T_K - 273.15\text{ K}) + 32$. If $T_F = T_K = T$, then $T = \frac{9}{5}(T - 273.15\text{ K}) + 32 \Rightarrow$

$\frac{9}{5}T - T = \frac{9}{5}(273.15\text{ K}) - 32 \Rightarrow \frac{4}{5}T = 459.7$, so $T = \frac{5(459.67)}{4} = 574.59 \Rightarrow 574.59\text{ K} = 574.59\ °F$

17.34. At higher temperatures, the mass of copper remains constant, but its volume increases. Hence, the density is expected to decrease. The volume expansion is $\Delta V = \beta V_0 \Delta T$. For copper, $\beta = 5.1 \cdot 10^{-5}\ \text{K}^{-1}$. Taking room temperature to be 293 K, (ie. 20 °C), $\Delta T = 1356\text{ K} - 293\text{ K} = 1063$ K. Density is mass divided by volume. Thus, $\rho_0 = \rho(293\text{ K}) = M/V_0$, and $\rho_{mp} = \rho(1356\text{ K}) = M/(V_0 + \Delta V)$.

$$\frac{\rho_{mp}}{\rho_0} = \frac{M/(V_0+\Delta V)}{M/V_0} = \frac{V_0}{V_0+\Delta V} = \frac{1}{1+(\Delta V/V_0)} = \frac{1}{1+\beta\Delta T} = \frac{1}{1+(5.1\cdot 10^{-5}\text{K}^{-1})(1063\text{ K})} = 0.949$$

At $T = 1356$ K, copper is only 94.9% of the density of copper at $T = 293$ K.

17.35. We can calculate the bulk expansion coefficient for steel by multiplying the value of the linear expansion coefficient for steel from Table 17.2 by 3, which gives us

$\beta = 3\alpha = 3\left(13.0 \cdot 10^{-6}\ °C^{-1}\right) = 3.90 \cdot 10^{-5}\ °C^{-1}$.

$\rho =$ density at 20.0 °C = 7800. kg/m^3,

$\rho' =$ density at 100.0 °C $= \frac{M}{V_0 + \Delta V}$, $\quad V_0 =$ volume at 20.0 °C

$\Delta V = \beta V_0 \Delta T$;

$$\rho' = \frac{M}{V_0 + \beta V_0 \Delta T} = \frac{M}{V_0(1+\beta\Delta T)} = \frac{\rho}{1+\beta\Delta T} = \frac{7800.\text{ kg/m}^3}{1+(3.90\cdot 10^{-5}\text{ °C}^{-1})(80.0\text{ °C})} = 7776\text{ kg/m}^3.$$

17.36. The cubes will expand when they are heated and will have a total length of 201.0 mm when $L_{\text{al}} + L_{\text{br}} = 201.0$ mm or $\Delta L_{\text{al}} + \Delta L_{\text{br}} = 1.0$ mm. So, $\alpha_{\text{al}} L_{\text{al}} \Delta T + \alpha_{\text{br}} L_{\text{br}} \Delta T = 1.0$ mm, and:

$$\Delta T = \frac{1.0\text{ mm}}{(100.0\text{ mm})(22\cdot 10^{-6}\text{ K}^{-1} + 19\cdot 10^{-6}\text{ K}^{-1})} = 240\text{ K}.$$

17.37. The piston ring expands when it is heated, and the inner diameter must increase by 0.10 cm, so,

$$\alpha_{\text{brass}} L \Delta T = \Delta L \text{ and } \Delta T = \frac{10.10\text{ cm} - 10.00\text{ cm}}{(10.00\text{ cm})(19\cdot 10^{-6}\text{ °C}^{-1})} = 526.32\text{ °C}.$$

So the temperature to which the brass piston ring must be heated is

$$T = 20.\text{ °C} + 526.32\text{ °C} = 546.32\text{ °C}.$$

We round this result to two significant figures to get $T = 550$ °C. Note that this process cannot be reversed to remove a piston ring. Once seated, the piston ring is in thermal contact with the piston and cannot be heated without heating and expanding the piston.

17.38. To calculate the dimension change due to heat, the Kelvin scale should be used. $100.0\text{ °F} = 310.9\text{ K}$, and $200.0\text{ °F} = 366.5\text{ K}$. Therefore the temperature change is $\Delta T = 55.6$ K.

(a) The volume change = $3\alpha V \Delta T = (3)(22\cdot 10^{-6}\text{ K}^{-1})(4\pi/3)(10.0\text{ cm})^3(55.6\text{ K}) = 15\text{ cm}^3$

(b) The radius change = $\alpha R \Delta T = (22\cdot 10^{-6}\text{ K}^{-1})(10.0\text{ cm})(55.6\text{ K}) = 0.012$ cm

Note that the radius change could be found from $V = (4\pi/3)R^3$, $(dV) = 4\pi R^2 (dR)$. Therefore, $dR = dV/(4\pi R^2) = 0.012$ cm.

17.39. The cross-sectional area has no relevance-this depends on linear expansion, which is governed by $L_{\text{f}} = L(1+\alpha\Delta T)$, where in this case $L_{\text{f}} = 5.2000$ m, $\Delta T = 60.0$ °C and $\alpha = 13\cdot 10^{-6}$ per degree Celsius from Table 17.2. This gives $L = \frac{L_{\text{f}}}{(1+\alpha\Delta T)} = \frac{5.2000\text{ m}}{1+\left(13\cdot 10^{-6}\text{ °C}^{-1}\right)\left(60.0\text{°C}\right)} = 5.195947$ m at -10.0 °C. Thus there will be 4.1 mm between adjacent rails.

17.40. The track will be free of any built-in tension as long as the thermal expansion is less than the 10.0 mm gap. The expansion of the track at temperature T is: $\Delta L = \alpha_{\text{steel}} L(T - 20.0\text{ °C})$. At the maximum allowable temperature, $\Delta L = d_{\text{gap}}$. Therefore, $\alpha_{\text{steel}} L(T_{\max} - 20.0\text{ °C}) = d_{\text{gap}}$, so,

$$T_{\max} = \frac{d_{\text{gap}}}{\alpha_{\text{steel}} L} + 20.0\text{ °C} = \frac{1.00\cdot 10^{-2}\text{ m}}{\left(13\cdot 10^{-6}\text{ °C}^{-1}\right)\left(25.0\text{ m}\right)} + 20.0\text{ °C} = 51\text{ °C}.$$

For most places in a temperate climate, this is enough for secure operation of the tracks, although larger gaps (on the order of 13 mm) may also be used.

17.41. Suppose T_{f} is the temperature at which the two screws will touch. Use the equation $\Delta L = L_0 \alpha (T_{\text{f}} - T_0)$ to find the increase in length for both brass and the aluminum screws at this temperature and set

$$\Delta L_{\text{Brass}} + \Delta L_{\text{Aluminum}} = 1.00\text{ mm} \quad (1)$$

Now:

$$\Delta L_{\text{Brass}} = (20.0\text{ cm})(\alpha_{\text{Brass}})(T_{\text{f}} - 22.0\text{ °C}) \quad (2)$$

$$\Delta L_{\text{Aluminum}} = (30.0\text{ cm})\left(\alpha_{\text{Aluminum}}\right)\left(T_{\text{f}} - 22.0\text{ °C}\right) \quad (3)$$

Substituting equations (2) and (3) into equation (1) yields:

$$(20.0\text{ cm})(\alpha_{\text{Brass}})(T_{\text{f}} - 22.0\text{ °C}) + (30.0\text{ cm})(\alpha_{\text{Aluminum}})(T_{\text{f}} - 22.0\text{ °C}) = 0.100\text{ cm}.$$

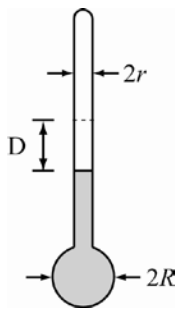
Given $\alpha_{\text{Brass}} = 18.9 \cdot 10^{-6}$ / °C and $\alpha_{\text{Aluminum}} = 23.0 \cdot 10^{-6}$ / °C:

$$(20.0\text{ cm})(18.9 \cdot 10^{-6}\text{ °C}^{-1})(T_{\text{f}} - 22.0\text{ °C}) + (30.0\text{ cm})(23.0 \cdot 10^{-6}\text{ °C}^{-1})(T_{\text{f}} - 22.0\text{ °C}) = 0.100\text{ cm}$$

$$T_{\text{f}} = \frac{0.100\text{ cm}}{(20.0\text{ cm})(18.9 \cdot 10^{-6}\text{ / °C}) + (30.0\text{ cm})(23.0 \cdot 10^{-6}\text{ / °C})} + 22.0\text{ °C} = 116\text{ °C}$$

17.42. **THINK:** From the change in volume, the total volume can be determined from the equation of volume expansion. It is necessary to find this volume V_0 and the radius R of the sphere that can hold it.

SKETCH:



RESEARCH: $dV = \beta V_0 dT$, $V_{\text{cylinder}} = \pi r^2 D$, $V_{\text{sphere}} = \frac{4}{3}\pi R^3$

SIMPLIFY: $dV = \beta V_0 dT = \pi r^2 D \Rightarrow V_0 = \frac{\pi r^2 D}{\beta dT}$

$$V_0 = \frac{4}{3}\pi R^3 = \frac{\pi r^2 D}{\beta dT},\ R^3 = \frac{3}{4\pi}\frac{\pi r^2 D}{\beta dT} = \frac{3r^2 D}{4\beta dT} \Rightarrow R = \left(\frac{3r^2 D}{4\beta dT}\right)^{1/3}$$

CALCULATE: $V_0 = \frac{\pi(0.100 \cdot 10^{-3}\text{ m})^2(1.00 \cdot 10^{-2}\text{ m})}{(1.81 \cdot 10^{-4}\text{ °C}^{-1})(1.00\text{ °C})} = 1.736 \cdot 10^{-6}\text{ m}^3$

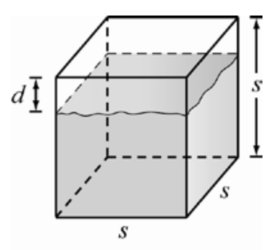
$$R = \left(\frac{3(0.100 \cdot 10^{-3}\text{ m})^2(1.00 \cdot 10^{-2}\text{ m})}{4(1.81 \cdot 10^{-4}\text{ °C}^{-1})(1.00\text{ °C})}\right)^{1/3} = 7.455 \cdot 10^{-3}\text{ m}$$

ROUND: Three significant figures: $V_0 = 1.74 \cdot 10^{-6}\text{ m}^3$, $R = 7.46 \cdot 10^{-3}\text{ m}$.

DOUBLE-CHECK: The radius at 7.5 mm for the sphere is considerably larger than the radius of 0.1 mm for the capillary. This is as expected since the capillary is much longer than the radius of the base.

17.43. **THINK:** Assume that at 37.0 °C, the pool just begins to overflow. The volume expansion equation can yield the depth of the pool. Let $d = 1.00$ cm = 0.0100 m.

SKETCH:



RESEARCH: $V_0 = S^2(S - d)$, $V = S^3$, $\Delta V = \beta V_0 \Delta T$

SIMPLIFY:

$$\Delta V = \beta V_0 \Delta T$$
$$V - V_0 = \beta V_0 \Delta T$$
$$S^3 - S^2(S-d) = \beta S^2(S-d)\Delta T$$
$$S - (S-d) = \beta(S-d)\Delta T$$
$$d = \beta S \Delta T - \beta d \Delta T$$
$$S = d\frac{1+\beta\Delta T}{\beta \Delta T}.$$

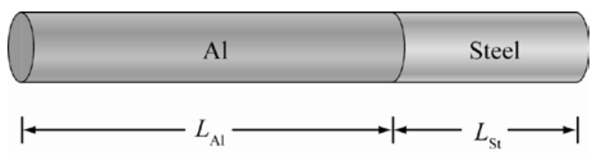
CALCULATE: $S = (0.0100 \text{ m})\left(\frac{1+(207\cdot 10^{-6} \text{ °C}^{-1})(16.0 \text{ °C})}{(207\cdot 10^{-6} \text{ °C}^{-1})(16.0 \text{ °C})}\right) = 3.029 \text{ m}$

ROUND: Three significant figures: S = 3.03 m.

DOUBLE-CHECK: 3.03 m is a realistic depth for a pool.

17.44. **THINK:** The change in length of the rods is equal to the sum of the change of the length of each rod separately.

SKETCH:



RESEARCH: $\Delta L = \alpha L \Delta T$, $\alpha_{\text{Steel}} = 1.30\cdot 10^{-5} \text{ K}^{-1}$, $\alpha_{\text{Al}} = 2.20\cdot 10^{-5} \text{ K}^{-1}$ and $L = L_{\text{Al}} + L_{\text{St}}$

SIMPLIFY: $L'_{\text{Al}} = L_{\text{Al}} + \Delta L_{\text{Al}} = L_{\text{Al}}(1+\alpha_{\text{Al}}\Delta T)$, $L'_{\text{St}} = L_{\text{St}} + \Delta L_{\text{St}} = L_{\text{St}}(1+\alpha_{\text{St}}\Delta T)$,

$$\Delta L = \left(L'_{\text{Al}} + L'_{\text{St}}\right) - \left(L_{\text{Al}} + L_{\text{St}}\right) = (L_{\text{Al}}\alpha_{\text{Al}} + L_{\text{St}}\,\alpha_{\text{Steel}})\Delta T$$

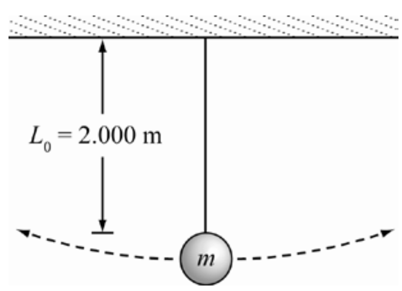
CALCULATE: $\Delta L = \left((2.00 \text{ m})(2.2\cdot 10^{-5} \text{ °C}^{-1}) + (1.00 \text{ m})(1.3\cdot 10^{-5} \text{ °C}^{-1})\right)\left(178 \text{ °C}\right) = 0.010146 \text{ m}$

ROUND: Rounding the change in length to two significant figures, $\Delta L = 1.0$ cm.

DOUBLE-CHECK: The increase of 1 cm is small but quite noticeable. This is a reasonable result.

17.45. **THINK:** Since the period of a pendulum is proportional to the square of length, I can compute the change of length due to thermal contraction and then compute the new period. Let t_0 be the period of the pendulum at a temperature of T_0, and let t_1 be the period of the pendulum at temperature T_1. The following quantities are given. $T_0 = 25.0$ °C, $T_1 = -20.0$ °C. When the temperature goes down, the pendulum will get shorter, the period of the pendulum will decrease, and the clock will run fast.

SKETCH:



RESEARCH: $\alpha_{\text{Brass}} = 1.9\cdot 10^{-5} \text{ K}^{-1}$, $L_1 = L_0 + \Delta L = L_0(1+\alpha_{\text{Brass}}\Delta T)$, $t = 2\pi\sqrt{\frac{L}{g}}$

SIMPLIFY: Time elapsed = $24 \text{ h}\left(\frac{t_0}{t_1}\right)$

$$t_0 = 2\pi\sqrt{\frac{L_0}{g}},\ t_1 = 2\pi\sqrt{\frac{L_1}{g}} \Rightarrow t_1 = 2\pi\sqrt{\frac{L_0(1+\alpha_{\text{Brass}}\Delta T)}{g}} = 2\pi\sqrt{\frac{L_0}{g}}(1+\alpha_{\text{Brass}}\Delta T)^{1/2} = t_0(1+\alpha_{\text{Brass}}\Delta T)^{1/2}$$

$$t_1 / t_0 = (1+\alpha_{\text{Brass}}\Delta T)^{1/2},\ \text{Time elapsed} = 24\text{ h}\left(\frac{t_0}{t_1}\right) = 24\text{ h} / \left(\frac{t_1}{t_0}\right) = (24\text{ h}) / (1+\alpha_{\text{Brass}}\Delta T)^{1/2}$$

CALCULATE: $\Delta T = T_1 - T_0 = -20.0 - 25.0 = -45.0$ K,

$$\text{Time elapsed} = (24\text{ h})(t_0 / t_1)$$

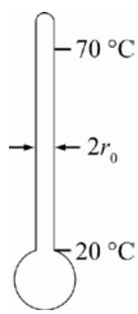
$$= (24\text{ h}) / \sqrt{1+(19.0\cdot 10^{-6}\text{ K}^{-1})(-45.0\text{ K})} = 24.0103\text{ h} = 24\text{ h and } 37\text{ s}.$$

ROUND: Taking 24 hours to be precise, have two significant figures. Subtract a precise 24 hours from our result, and report the seconds to two significant figures: 24 hours and 37 seconds.

DOUBLE-CHECK: The result shows the clock will run fast and gaining 37 seconds over 24 h is reasonable.

17.46. **THINK:** Both the capillary tube and the mercury will expand as the temperature increases. I can compute the height of the mercury at 70.0 °C for both the silica and quartz capillary. If the heights differ by more than 5%, then the quartz thermometers must be scrapped. The following quantities are given: $V_s = 1.00\text{ cm}^3$, $r_0 = 0.250$ mm, $\Delta T = T_1 - T_0 = 70.0\text{ °C} - 20.0\text{ °C} = 50.0\text{ °C} = 50.0$ K, $a_{\text{Si}} = 0.400\cdot 10^{-6}\text{ K}^{-1}$, $a_{\text{quartz}} = 12.3\cdot 10^{-6}\text{K}^{-1}$, and $\beta = \beta_{\text{Hg}} = 181\cdot 10^{-6}\text{ °C}^{-1} = 181\cdot 10^{-6}\text{ K}^{-1}$.

SKETCH:



RESEARCH: $\Delta V_{\text{Hg}} = \beta_{\text{Hg}} V_s \Delta T$, $C = 2\pi r =$ circumference of capillary, $\Delta C = \alpha C_0 \Delta T$, volume of a cylinder: $V_{\text{cyl}} = \pi r^2 h$, change in volume of spherical reservoir: $\Delta V_{\text{S}} = 3\alpha V_{\text{S}} \Delta T$.

SIMPLIFY: When the volume of Hg expands by ΔV_{Hg}, the excess Hg goes into the excess volume in the sphere, and the remainder goes up the capillary, up to a height *h*.

$$h = \frac{V_{\text{cyl}}}{\pi r^2},\ V_{\text{cyl}} = \Delta V_{\text{Hg}} - \Delta V_{\text{S}}$$

$$r = \frac{C}{2\pi} = \frac{C_0 + \Delta C}{2\pi} = \frac{C_0(1+\alpha\Delta T)}{2\pi}$$

$$= \frac{2\pi r_0(1+\alpha\Delta T)}{2\pi} = r_0(1+\alpha\Delta T)$$

$$h = \frac{\Delta V_{\text{Hg}} - \Delta V_{\text{S}}}{\pi r_0^2(1+\alpha\Delta T)^2}$$

$$= \frac{\beta_{\text{Hg}} V_{\text{S}} \Delta T - 3\alpha V_{\text{S}} \Delta T}{\pi r_0^2(1+\alpha\Delta T)^2}$$

For quartz:

$$h_{\text{quartz}} = \frac{\beta_{\text{Hg}} V_{\text{S}} \Delta T - 3\alpha_{\text{quartz}} V_{\text{S}} \Delta T}{\pi r_0^2 (1+\alpha_{\text{quartz}} \Delta T)^2} = \frac{V_{\text{S}} \Delta T (\beta_{\text{Hg}} - 3\alpha_{\text{quartz}})}{\pi r_0^2 (1+\alpha_{\text{quartz}} \Delta T)^2}.$$

The fractional change is height is:

$$f = 1 - \frac{h_{\text{quartz}}}{h_{\text{silica}}} = 1 - \frac{(\beta_{\text{Hg}} - 3\alpha_{\text{quartz}})}{(\beta_{\text{Hg}} - 3\alpha_{\text{silica}})} \left[\frac{(1+\alpha_{\text{silica}} \Delta T)}{(1+\alpha_{\text{quartz}} \Delta T)} \right]^2.$$

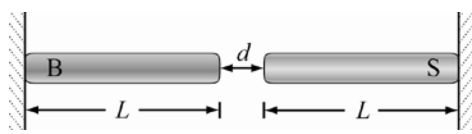
CALCULATE: $f = 1 - \dfrac{181 \cdot 10^{-6}\ \text{K}^{-1} - 3(12.3 \cdot 10^{-6}\ \text{K}^{-1})}{181 \cdot 10^{-6}\ \text{K}^{-1} - 3(0.400 \cdot 10^{-6}\ \text{K}^{-1})} \left\{ \dfrac{\left[1 + 0.400 \cdot 10^{-6}\ \text{K}^{-1}(50.0\ \text{K})\right]}{\left[1 + 12.3 \cdot 10^{-6}\ \text{K}^{-1}(50.0\ \text{K})\right]} \right\}^2 = 0.1995$

ROUND: To one significant figure, $f = 20\ \%$.

DOUBLE-CHECK: The quartz thermometers will give a maximum error of about 20% at 70 °C. They will have to be scrapped.

17.47. **THINK:** Assuming the brass and steel rods, L = 1.00 m each, do not sag, they will increase in length by ΔL_{B} and ΔL_{S}, respectively. The rods will touch when their combined extensions equals the separation, d = 5.00 mm. The linear expansion coefficients of the rods are $\alpha_{\text{B}} = 19 \cdot 10^{-6}$ / °C and $\alpha_{\text{S}} = 13 \cdot 10^{-6}$ / °C. The initial temperature of the rods is $T_{\text{i}} = 25.0$ °C.

SKETCH:



RESEARCH: The brass rod will increase by $\Delta L_{\text{B}} = L\alpha_{\text{B}} \Delta T$. The steel rod will increase by $\Delta L_{\text{S}} = L\alpha_{\text{S}} \Delta T$. The final temperature will be $\Delta T + T_{\text{i}}$. The rods touch when $\Delta L_{\text{B}} + \Delta L_{\text{S}} = d$.

SIMPLIFY: $d = \Delta L_{\text{B}} + \Delta L_{\text{S}} = L(\alpha_{\text{B}} + \alpha_{\text{S}}) \Delta T$. Thus, $\Delta T = \dfrac{d}{L(\alpha_{\text{B}} + \alpha_{\text{S}})} \Rightarrow T_{\text{f}} = \dfrac{d}{L(\alpha_{\text{B}} + \alpha_{\text{S}})} + T_{\text{i}}$.

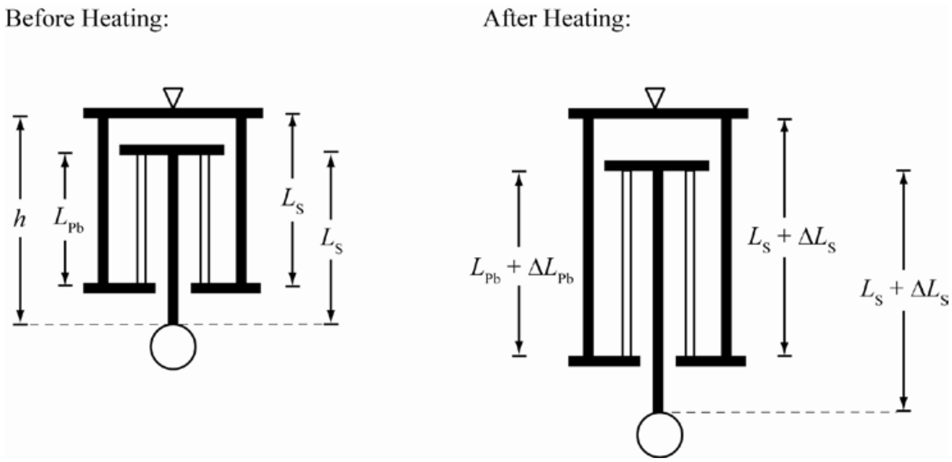
CALCULATE: $T_{\text{f}} = \dfrac{5.00 \cdot 10^{-3}\ \text{m}}{(1.00\ \text{m})(19+13) \cdot 10^{-6}\ /\text{°C}} + 25.0\ \text{°C} = 181.25\ \text{°C}$

ROUND: Two significant figures: $T_{\text{f}} = 180$ °C.

DOUBLE-CHECK: Given the long length, d, for the total expansion, such a high temperature is not unreasonable.

17.48. **THINK:** As the pendulum is heated, each bar increases in length. The steel bars, $L_{\text{S}} = 50.0$ cm and $\alpha_{\text{S}} = 13 \cdot 10^{-6}$/ °C, will increase in length such that the bob will move twice this distance from the pivot. The lead bars, $\alpha_{\text{Pb}} = 29 \cdot 10^{-6}$/ °C, will increase in length such that it will reduce the distance from the bob to the pivot. Determine the length, $L = L_{\text{Pb}}$, of each of the two lead bars.

SKETCH:



RESEARCH: The change in length of the steel rods is $\Delta L_S = \alpha_S L_S \Delta T$, while that of the lead rods is $\Delta L_{Pb} = \alpha_{Pb} L_{Pb} \Delta T$ .For the pendulum length , *h*, to remain unchanged, $2\Delta L_S = \Delta L_{Pb}$.

SIMPLIFY: $2\Delta L_S = \Delta L_{Pb} \Rightarrow 2\alpha_S L_S \Delta T = \alpha_{Pb} L_{Pb} \Delta T$. Thus, $L_{Pb} = \dfrac{2\alpha_S}{\alpha_{Pb}} L_S$

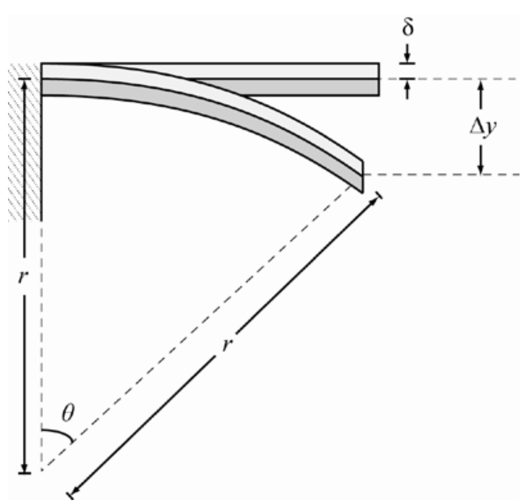
CALCULATE: $L_{Pb} = \dfrac{2(13 \cdot 10^{-6}\ /°\text{C})}{29 \cdot 10^{-6}\ /°\text{C}}(50.0\text{ cm}) = 44.83\text{ cm}$

ROUND: The values in Table 17.2 are given to two significant figures. This results in a final answer of $L_{Pb} = 45$ cm.

DOUBLE-CHECK: The value of $2\alpha_S$ is about 10% less than α_{Pb} (26 and 29, respectively), and this means the lead bars being 10% shorter is reasonable.

17.49. **THINK:** Since brass has a higher linear expansion coefficient than steel, $\alpha_B = 19 \cdot 10^{-6}\text{ K}^{-1}$ and $\alpha_S = 13 \cdot 10^{-6}\text{ K}^{-1}$, the brass will become larger in length. After being heated up, $\Delta T = 20.0$ K, the strip will arc. It is vital to focus on the radius of the midline of each part of the strip. Since each material has a thickness, $\delta = 0.500$ mm , each will be an arc of different radius, so the radius must be considered. The actual arc length of each strip will be its length after being heated up and each will share the same angle. The end of the strip lowers by $\Delta y = 3.00$ mm.

SKETCH:



RESEARCH: From the geometry of the system, $\Delta y = r(1-\cos\theta)$. Also, $(r+\delta/2)\theta = L_\text{B}'$ and $(r-\delta/2)\theta = L_\text{S}'$, because the arc length equals radius times angle. When the strips are heated up, their lengths are increased by $\Delta L_\text{B} = L\alpha_\text{B}\Delta T$ and $\Delta L_\text{S} = L\alpha_\text{S}\Delta T$. This means the final lengths of the strips are $L_\text{B}' = L(1+\alpha_\text{B}\Delta T)$ and $L_\text{S}' = L(1+\alpha_\text{S}\Delta T)$.

SIMPLIFY: Determine the radius, r, first:

$$\frac{L_\text{B}'}{L_\text{S}'} = \frac{(r+\delta/2)\theta}{(r-\delta/2)\theta} \Rightarrow \frac{r+\delta/2}{r-\delta/2} = \frac{1+\alpha_\text{B}\Delta T}{1+\alpha_\text{S}\Delta T}.$$

Thus,

$$r+\frac{\delta}{2} = \left(r-\frac{\delta}{2}\right)x \Rightarrow r = \frac{\delta}{2}\frac{(x+1)}{(x-1)}, \text{ where } x = \frac{1+\alpha_\text{B}\Delta T}{1+\alpha_\text{S}\Delta T}.$$

Now that the radius is known, the angle, θ, can be determined:

$$\Delta y = r(1-\cos\theta) \Rightarrow \cos\theta = 1-\frac{\Delta y}{r} \Rightarrow \theta = \cos^{-1}\left(1-\frac{\Delta y}{r}\right).$$

Next consider the difference in $L_\text{B}' - L_\text{S}'$,

$$L_\text{B}' - L_\text{S}' = L(1+\alpha_\text{B}\Delta T) - L(1+\alpha_\text{S}\Delta T) = L\alpha_\text{B}\Delta T - L\alpha_\text{S}\Delta T = L(\alpha_\text{B}-\alpha_\text{S})\Delta T.$$

However, $L_\text{B}' - L_\text{S}' = (r+\delta/2)\theta - (r-\delta/2)\theta = \delta\theta$. Therefore, $\delta\theta = L(\alpha_\text{B}-\alpha_\text{S})\Delta T \Rightarrow L = \dfrac{\delta\theta}{(\alpha_\text{B}-\alpha_\text{S})\Delta T}$.

Further algebraic simplification leads to $L = \dfrac{\delta}{(\alpha_\text{B}-\alpha_\text{S})\Delta T}\cos^{-1}\left(1-\left(\dfrac{2\Delta y}{\delta}\right)\left(\dfrac{x-1}{x+1}\right)\right)$.

CALCULATE: $x = \dfrac{1+(19\cdot 10^{-6}\text{ K}^{-1})(20.0\text{ K})}{1+(13\cdot 10^{-6}\text{ K}^{-1})(20.0\text{ K})} = 1.000119969$

$$L = \frac{0.500\cdot 10^{-3}\text{ m}}{(19-13)\cdot 10^{-6}\text{ K}^{-1}(20.0\text{ K})}\left(\cos^{-1}\left(1-\left(\frac{2(3.00\cdot 10^{-3}\text{ m})}{0.500\cdot 10^{-3}\text{ m}}\right)\left(\frac{1.000119969-1}{1.000119969+1}\right)\right)\right) = 0.1581\text{ m}$$

ROUND: One significant figure (subtraction rule applies to the difference of the two expansion coefficients): $L = 0.2$ m.

DOUBLE-CHECK: This length is a plausible length for a bimetallic strip of metal that deflects 3 mm when heated by 20 K.

17.50. **THINK:** Since the bulk modulus, $B = 160$ GPa, is the pressure per fractional change in volume, a change in temperature of $\Delta T = 1.0$ °C will cause a change in volume, and thus a change in pressure. The linear expansion coefficient of steel is $1.2\cdot 10^{-5}$ / °C.

SKETCH: A sketch is not needed to solve this problem.

RESEARCH: The bulk modulus is given by $B = \Delta P/(\Delta V/V)$. The change in volume is given by $\Delta V = 3\alpha V\Delta T$.

SIMPLIFY: $\Delta V = 3\alpha V\Delta T \Rightarrow \dfrac{\Delta V}{V} = 3\alpha\Delta T,\ B = \dfrac{\Delta P}{\Delta V/V} \Rightarrow \Delta P = B\left(\dfrac{\Delta V}{V}\right) = 3B\alpha\Delta T$

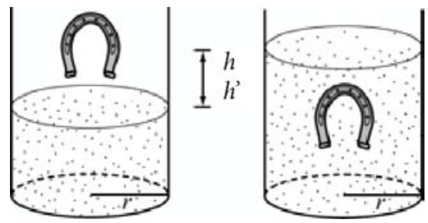
CALCULATE: $\Delta P = 3(160\text{ GPa})(1.2\cdot 10^{-5}\text{ / °C})(1.0\text{ °C}) = 5.76\text{ MPa}$

ROUND: Two significant figures: $\Delta P = 5.8$ MPa.

DOUBLE-CHECK: For comparison, atmospheric pressure is 0.10 MPa. The problem mentioned it could produce very large pressures, so this answer seems reasonable.

17.51. **THINK:** When the horseshoe is put in the tank, $r = 10.0$ cm, the water rises by $h = 0.250$ cm. The horseshoe, $T_{\text{i}} = 293.15$ K (room temperature, 20.0 °C) and $T_{\text{f}} = 700.$ K, will increase its volume. When it is put back in water, it will raise the water level by h'. The linear expansion coefficient of the horseshoe is $\alpha = 11.0 \cdot 10^{-6}\text{ K}^{-1}$.

SKETCH:



RESEARCH: When water rises by h or h', the volumes displaced are $V = \pi r^2 h$ and $V' = \pi r^2 h'$. The volume of the heated horseshoe is $V' = V(1 + 3\alpha\,\Delta T)$. The initial volume of the horseshoe, V_0, is the same as the volume of water it displaced before it was heated, $\pi r^2 h$. The volume of displaced heated water is equal to the volume of the heated horseshoe.

SIMPLIFY: $V' = V(1+3\alpha\,\Delta T) \Rightarrow \pi r^2 h' = \pi r^2 h(1+3\alpha\,\Delta T) \Rightarrow h' = h(1+3\alpha\,\Delta T).$

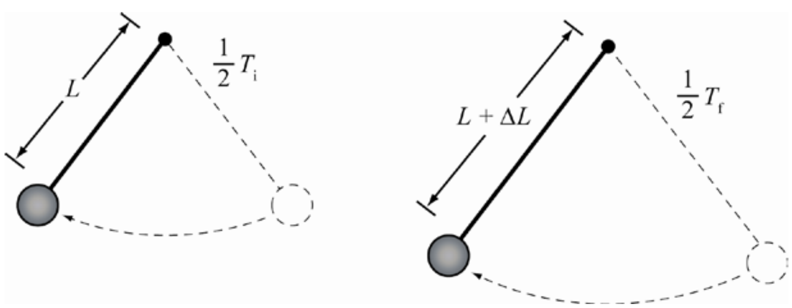
CALCULATE: $h' = (0.250\text{ cm})\left(1+3\left(11.0\cdot 10^{-6}\text{ K}^{-1}\right)(293.15\text{ K})\right) = 0.252418\text{ cm}.$

ROUND: The least precise value given in the question has three significant figures. Therefore the final answer should be rounded to $h' = 0.252$ cm.

DOUBLE-CHECK: The change in the water height is small, which seems reasonable since the change in volume is small and the cross sectional area $(3\cdot 10^4\text{ mm}^2)$ of the tank of water is relatively large.

17.52. **THINK:** Since the period of a pendulum is proportional to the length, an increase in temperature will increase the length and, hence, the period. If the pendulum makes n oscillations in one week when at 20.0 °C, it will take longer to go through n oscillations when the period is greater, making the week appear longer (the clock will run slow). The initial period of the pendulum is $T_{\text{i}} = 1.000$ s, and then increases while the temperature increases to 30.0 °C. Use $\alpha_{\text{Al}} = 22\cdot 10^{-6}$ / °C.

SKETCH:



RESEARCH: The period of the pendulum is given by $T = 2\pi\sqrt{L/g}$. The length of pendulum after being heated is $L' = L(1+\alpha_{\text{Al}}\Delta T)$. The number of oscillations that the pendulum makes over the period of time, *t*, is $n = t/T$.

SIMPLIFY:

(a) Period after temperature change:

$$T' = 2\pi\sqrt{L'/g} = 2\pi\sqrt{L(1+\alpha_{\text{Al}}\Delta T)/g} = 2\pi\sqrt{L/g}\sqrt{1+\alpha_{\text{Al}}\Delta T} = T_{\text{i}}\sqrt{1+\alpha_{\text{Al}}\Delta T}.$$

(b) Number of oscillations in one week at 20 °C is $n = t/T$. The amount of time for n oscillations at 30 °C is $t' = nT'$. The difference in time between the pendulum at 30 °C and the pendulum at 20 °C is:

$$\Delta t = n(T' - T) = t(T'/T - 1).$$

CALCULATE:

(a) $T' = (1.000 \text{ s})\sqrt{1 + (22 \cdot 10^{-6} / {}^\circ\text{C})(30.0 \text{ }^\circ\text{C} - 20.0 \text{ }^\circ\text{C})} = 1.00011 \text{ s}$

(b) $\Delta t = 1 \text{ week}\left(\dfrac{1.00011 \text{ s}}{1.000 \text{ s}} - 1\right) \cdot \dfrac{7 \text{ days}}{1 \text{ week}} \cdot \dfrac{24 \text{ hr}}{1 \text{ day}} \cdot \dfrac{3600 \text{ s}}{1 \text{ hr}} = 66.53 \text{ s}$

ROUND: The answers should be rounded to two significant figures.

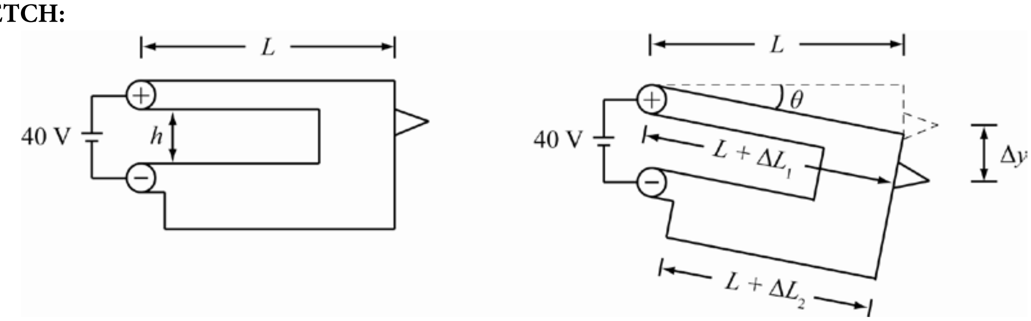
(a) $T' = 1.0 \text{ s}$

(b) $\Delta t = 67 \text{ s}$

DOUBLE-CHECK: Losing 67 seconds over a full week is a reasonable amount for a temperature change of 10. °C.

17.53. **THINK:** Use the subscript 1 to refer to the thin arm and the subscript 2 to refer to the thick arm. After the temperature change (using a room temperature of 20.0 °C), $\Delta T_1 = 380. \text{ K}$ and $\Delta T_2 = 180. \text{ K}$, for the upper and lower arms, respectively, each length will increase by a different amount. Since the ends are fixed in position, the device overall will begin to angle downward so the tip is pointing below its original position. The initial length and linear expansion coefficient of each arm is $L = 1800. \text{ μm}$ and $3.20 \cdot 10^{-6} \text{ K}^{-1}$. The separation of the electrical contacts is $h = 45.0 \text{ μm}$.

SKETCH:



RESEARCH: The upper and lower arms each increase in length by $\Delta L_1 = \alpha L \Delta T_1$ and $\Delta L_2 = \alpha L \Delta T_2$, respectively. The change in height of the tip is $\Delta y = (L + \Delta L_1)\sin\theta$. The difference in the extended length is $\Delta L_1 - \Delta L_2$ and this length is equal to $h\sin\theta$.

SIMPLIFY: Determine the angle: $h\sin\theta = \Delta L_1 - \Delta L_2 \Rightarrow \sin\theta = (\Delta L_1 - \Delta L_2)/h.$ The change in tip height: $\Delta y = (L + \Delta L_1)\sin\theta = (L + \Delta L_1)(\Delta L_1 - \Delta L_2)/h.$ Thus, $\Delta y = \alpha L^2(1 + \alpha \Delta T_1)(\Delta T_1 - \Delta T_2)/h.$

CALCULATE:

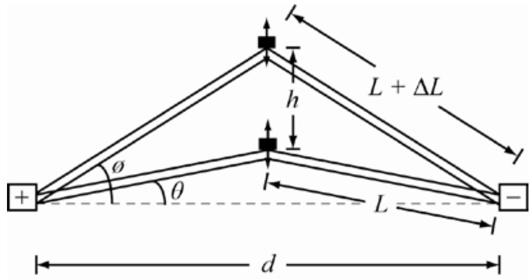
$$\Delta y = (3.20 \cdot 10^{-6} \text{ K}^{-1})(1800. \text{ μm})^2\left[1 + (3.20 \cdot 10^{-6} \text{ K}^{-1})(380. \text{ K})\right](380. \text{ K} - 180. \text{ K})/(45.0 \text{ μm}) = 46.14 \text{ μm}$$

ROUND: Three significant figures: $\Delta y = 46.1 \text{ μm}$ downwards

DOUBLE-CHECK: The change in the tip height is of the same order of magnitude as the separation between the contact points, so the result is reasonable.

17.54. **THINK:** The tip is located at the midpoint of the beam which is also midway between the contact points, which are separated by a distance *d*. The silicon beam, $\alpha_{\text{Si}} = 3.2 \cdot 10^{-6} \text{ K}^{-1}$, makes an angle of $\theta = 0.10 \text{ rad}$ from the horizontal when it is at a temperature of 20. °C. As the beam heats up to 500. °C, its length will increase, but since the tip must remain in the same position horizontally, the angle the beam makes must also increase, which in turn causes motion of the tip.

SKETCH:



RESEARCH: Before heating, the length, L, of the beam is given by $L = d/(2\cos\theta)$. After heating, the beam increases in length by $\Delta L = \alpha_{\text{Si}} L \Delta T$. Even after the length increases, the tip does not moves horizontally, so that $L + \Delta L = d/(2\cos\phi)$. The tip moves vertically by an amount $h = (L+\Delta L)\sin\phi - L\sin\theta = L\left[(1+\alpha_{\text{Si}}\Delta T)\sin\phi - \sin\theta\right]$.

SIMPLIFY: The initial length of the beam is $L = d/(2\cos\theta)$. Therefore, the length of the beam after heating is given by: $L + \Delta L = L(1+\alpha_{\text{Si}}\Delta T) = d(1+\alpha_{\text{Si}}\Delta T)/(2\cos\theta)$. Since the new angle of the beam after heating is $L + \Delta L = d/(2\cos\phi)$, this means that:

$$\frac{d}{2\cos\theta}(1+\alpha_{\text{Si}}\Delta T) = \frac{d}{2\cos\phi} \Rightarrow \cos\phi = \frac{\cos\theta}{1+\alpha_{\text{Si}}\Delta T} \Rightarrow \phi = \cos^{-1}\left(\frac{\cos\theta}{1+\alpha_{\text{Si}}\Delta T}\right).$$

The change in height of the tip is then:

$$h = L\left[(1+\alpha_{\text{Si}}\Delta T)\sin\phi - \sin\theta\right] = \frac{d}{2}\left((1+\alpha_{\text{Si}}\Delta T)\frac{\sin\phi}{\cos\theta} - \tan\theta\right)$$

$$= \frac{d}{2}\left((1+\alpha_{\text{Si}}\Delta T)\frac{\sin\left[\cos^{-1}\left(\dfrac{\cos\theta}{1+\alpha_{\text{Si}}\Delta T}\right)\right]}{\cos\theta} - \tan\theta\right).$$

CALCULATE:

$$h = \frac{1800\ \mu\text{m}}{2}\left(\left(1+(3.2\cdot 10^{-6}\ \text{K}^{-1})(480\ \text{K})\right)\frac{\sin\left[\cos^{-1}\left(\dfrac{\cos(0.10\ \text{rad})}{1+(3.2\cdot 10^{-6}\ \text{K}^{-1})(480\ \text{K})}\right)\right]}{\cos(0.10\ \text{rad})} - \tan(0.10\ \text{rad})\right)$$

$$= 12.993 \mu\text{m}$$

ROUND: The answer should be rounded to two significant figures: $h = 13\ \mu\text{m}$, upwards.

DOUBLE-CHECK: This value is the same order of magnitude as the beam width, meaning that it has a great sensitivity, which would be desired for such a device. This is sensible.

17.55. **THINK:** For simplicity, define $a = 1.00016$, $b = 4.52\cdot 10^{-5}$ and $c = 5.68\cdot 10^{-6}$. In part (a), a derivative can be used to determine the properties of the water. The volume, V, as a function of temperature, T, is given by $V = a - bT + cT^2$ when the temperature is in the range [0.00 °C, 50.0 °C). In part (b), evaluate β when $T = 20.0$ °C.

SKETCH: A sketch is not needed to solve this problem.

RESEARCH: The general function to evaluate the change in volume is $\Delta V = \beta V \Delta T$. The differences can be approximated as differentials, i.e. $\Delta Y / \Delta X \approx dy/dx$.

SIMPLIFY: $\frac{dV}{dT} = \frac{d}{dT}(a - bT + cT^2) = -b + 2cT$. Since $\Delta V = \beta V \Delta T$, it follows that:

$$\beta = \frac{1}{V}\left(\frac{\Delta V}{\Delta T}\right) \approx \frac{1}{V}\left(\frac{dV}{dT}\right) = \frac{-b + 2cT}{a - bT + cT^2}$$

CALCULATE:

(a) $\beta(T) = \frac{-4.52 \cdot 10^{-5} + 11.36 \cdot 10^{-6} T}{1.00016 - 4.52 \cdot 10^{-5} T + 5.68 \cdot 10^{-6} T^2}$

(b) $\beta(20\text{ °C}) = \frac{-4.52 \cdot 10^{-5} + (11.36 \cdot 10^{-6})(20.0\text{°C})}{1.00016 - (4.52 \cdot 10^{-5})(20.0\text{°C}) + (5.68 \cdot 10^{-6})(20.0\text{°C})^2}$

$= 1.8172 \cdot 10^{-4}\text{ / °C}$

ROUND:

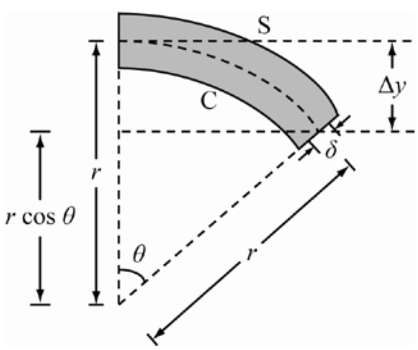
(a) Not necessary.

(b) Round to three significant figures: $\beta(T = 20.0\text{ °C}) = 1.82 \cdot 10^{-4}\text{ / °C}$

DOUBLE-CHECK: The value for β for water at 20.0 °C from Table 17.3 is $2.07 \cdot 10^{-4}$ / °C. Since the calculated value is close, this is a reasonable result.

17.56. **THINK:** Since copper has a higher linear expansion coefficient than steel ($\alpha_C = 17 \cdot 10^{-6}\text{ K}^{-1}$ and $\alpha_S = 13 \cdot 10^{-6}\text{ K}^{-1}$), the copper will become shorter in length than the steel. After a change in temperature of $\Delta T = -5.00$ K , the strip will arc. Since each material has a thickness of $\delta = 1.00$ mm each will be an arc with a different radius, so the radius to the midpoint of each strip must be considered. The actual arc length will be the length after being cooled and each will share the same angle or curvature. The initial length of each strip is $L = 25.0$ mm.

SKETCH:



RESEARCH: The lengths of each strip after cooling are $L_C' = L(1 + \alpha_C \Delta T)$ and $L_S' = L(1 + \alpha_S \Delta T)$. These lengths are the arc lengths of circles of radius $r - (\delta / 2)$ and $r + (\delta / 2)$, respectively, so:

$$\left(r - \frac{\delta}{2}\right)\theta = L_C' \text{ and } \left(r + \frac{\delta}{2}\right)\theta = L_S'.$$

The deflection of the strip is given by $\Delta y = r(1 - \cos\theta)$.

SIMPLIFY:

(a) Determine the radius of curvature, r:

$$\frac{L_S'}{L_C'}=\frac{(r+\delta/2)\theta}{(r-\delta/2)\theta}=\frac{L(1+\alpha_S\Delta T)}{L(1+\alpha_C\Delta T)} \Rightarrow \frac{r+\delta/2}{r-\delta/2}=\frac{1+\alpha_S\Delta T}{1+\alpha_C\Delta T}$$

$$\Rightarrow r+\delta/2=x(r-\delta/2) \Rightarrow r=\frac{\delta}{2}\left(\frac{x+1}{x-1}\right), \text{ where } x=\frac{1+\alpha_S\Delta T}{1+\alpha_C\Delta T}$$

$$\Rightarrow r=\frac{\delta}{2}\left(\frac{2+(\alpha_S+\alpha_C)\Delta T}{(\alpha_S-\alpha_C)\Delta T}\right)$$

(b) To find the deflection, Δy, I need θ: $(r-\delta/2)\theta=L_C'$, $(r+\delta/2)\theta=L_S'$. Thus,

$$\theta=\frac{L(1+\alpha_C\Delta T)}{r-\delta/2} \text{ or } \theta=\frac{L(1+\alpha_S\Delta T)}{r+\delta/2}.$$

So, the deflection is $\Delta y=r(1-\cos\theta)=r\left[1-\cos\left(\frac{L(1+\alpha_C\Delta T)}{r-\delta/2}\right)\right]=r\left[1-\cos\left(\frac{L(1+\alpha_S\Delta T)}{r+\delta/2}\right)\right]$.

CALCULATE:

(a) $r=\frac{1.00\cdot10^{-3}\text{ m}}{2}\left(\frac{2+(13+17)\cdot10^{-6}\text{ K}^{-1}(-5.00\text{ K})}{(13-17)\cdot10^{-6}\text{ K}^{-1}(-5.00\text{ K})}\right)=49.996\text{ m}$

(b) $\Delta y=49.996\text{ m}\left[1-\cos\left(\frac{25.0\cdot10^{-3}\text{ m}\left(1+17\cdot10^{-6}\text{ K}^{-1}(-5.00\text{ K})\right)}{49.996\text{ m}-\left((1.00\cdot10^{-3}\text{ m})/2\right)}\right)\right]=0.00625\text{ mm}$

ROUND: The results should be rounded to two significant figures.

(a) $r=50.\text{ m}$

(b) $\Delta y=6.3\ \mu\text{m}$

DOUBLE-CHECK: Since the expansion coefficients of each are close to each other and the change in temperature was small, it is reasonable that the strip barely curves (it has a large radius of curvature and a small dip).

17.57. Each side of the cube has length $l=40$ cm and its initial volume before heating is $V_i=l^3$. The change in temperature is $\Delta T=100.$ °C and linear expansion coefficient of copper is $\alpha_{Cu}=17\cdot10^{-6}$ / °C.

$$\Delta V=3\alpha V_i\Delta T=3\alpha l^3\Delta T=3(17\cdot10^{-6}\text{ / °C})(40.\text{ cm})^3(100.\text{ °C})=326.4\text{ cm}^3$$

Thus, change in volume is $\Delta V=330\text{ cm}^3$.

17.58. The initial length of the pipe is $L=50.0$ m, the change in temperature is $\Delta T=30.0$ °C, and the change in length is $\Delta L=2.85$ cm.

(a) $\Delta L=\alpha L\Delta T \Rightarrow \alpha=\frac{\Delta L}{L\Delta T}=\frac{0.0285\text{ m}}{(50.0\text{ m})(30.0\text{ °C})}=1.90\cdot10^{-5}$ /K

(b) This linear expansion coefficient matches that of brass.

17.59. When the aluminum container is filled with turpentine, the turpentine will have a volume of $V=5.00$ gal. The volume expansion coefficient of the turpentine, $\beta_{turp}=900.\cdot10^{-6}\text{ °C}^{-1}$. The volume expansion coefficient of aluminum is $\beta_{Al}=66.0\cdot10^{-6}\text{ °C}^{-1}$. The change in temperature is $\Delta T=12.0$ °C. The change in volume of the turpentine is given by:

$$\Delta V=\beta_{turp}V\Delta T=(900.\cdot10^{-6}\text{ °C}^{-1})(5.00\text{ gal})(12.0\text{ °C})\left(\frac{3.785\text{ L}}{1\text{ gal}}\right)=0.2044\text{ L}.$$

The change in volume of the is given by:

$$\Delta V = \beta_{\text{Al}} V \Delta T = (66.0 \cdot 10^{-6} \text{ / °C})(5.00 \text{ gal})(12.0 \text{ °C})\left(\frac{3.785 \text{ L}}{1 \text{ gal}}\right) = 0.01499 \text{ L}.$$

Thus, 0.189 L of turpentine spills out of the container.

17.60. The building has initial height of $L = 600.$ m. The change in temperature is $\Delta T = 45.0$ °C. The linear expansion coefficient of steel is $\alpha_{\text{S}} = 1.30 \cdot 10^{-5}$ / °C.

$$\Delta L = \alpha_{\text{S}} L \Delta T = (1.30 \cdot 10^{-5} \text{ / °C})(600. \text{ m})(45.0 \text{ °C}) = 0.351 \text{ m}$$

Thus, the building grows by 0.351 m.

17.61. The initial diameter of the rod at 20. °C is D_1, and after being cooled by a change in temperature of $\Delta T = \left[77.0 \text{ K} - (20. \text{ °C} + 273.15 \text{ K})\right] = -216.15$ K, it will have a diameter of $D_2 = 10.000$ mm. The linear expansion coefficient of aluminum is $\alpha_{\text{Al}} = 22 \cdot 10^{-6} \text{ K}^{-1}$.

$$\Delta D = \alpha_{\text{Al}} D_1 \Delta T \text{ , } D_2 = D_1 + \Delta D = D_1(1 + \alpha_{\text{Al}} \Delta T)$$

$$D_2 = (1 + \alpha_{\text{Al}} \Delta T) D_1 \Rightarrow D_1 = \frac{D_2}{1 + \alpha_{\text{Al}} \Delta T} \Rightarrow D_1 = \frac{10.000 \text{ mm}}{1 + (22 \cdot 10^{-6} \text{ K}^{-1})(-216.15 \text{ K})} = 10.0478 \text{ mm}$$

Thus, the maximum diameter the aluminum rod can have at 20. °C is $D_1 = 10.$ mm.

17.62. After the gas is heated up, its final volume is $V_{\text{f}} = 213$ L. The change in temperature is $\Delta T = 63$ °F. The volume expansion coefficient of gas is $950 \cdot 10^{-6} \text{ K}^{-1}$. Convert the change in temperature to Kelvin:

$$\Delta T_{\text{C}} = \frac{5}{9} \Delta T_{\text{f}} \text{ and } \Delta T_{\text{C}} = \Delta T_{\text{K}} \Rightarrow \Delta T = \frac{5}{9}(63 \text{ °F}) = 35 \text{ K}.$$

$$\Delta V = \beta_{\text{gas}} V_{\text{i}} \Delta T, \; V_{\text{f}} = V_{\text{i}} + \Delta V = V_{\text{i}}(1 + \beta_{\text{gas}} \Delta T) \Rightarrow V_{\text{i}} = \frac{V_{\text{f}}}{1 + \beta_{\text{gas}} \Delta T} = \frac{213 \text{ L}}{1 + (950 \cdot 10^{-6} \text{ K}^{-1})(35 \text{ K})} = 206.15 \text{ L}$$

Thus, the maximum amount of gasoline that should be put into the tank at 57 °F is 206.15 L. Rounding this value is dangerous, since the tank would overflow or possibly explode if 210 L is added.

17.63. The initial volume of the mercury is $V = 8.00$ mL, the cross-sectional area of the tube is $A = 1.00 \text{ mm}^2$ and the volume expansion coefficient of mercury is $\beta_{\text{Hg}} = 181 \cdot 10^{-6}$ / °C. Consider a change in temperature of $\Delta T = 1.00$ °C. Since the cross-sectional area remains closely the same, $\Delta V = A \Delta L$.

$$\Delta V = \beta_{\text{Hg}} V \Delta T = A \Delta L \Rightarrow \Delta L = \frac{\beta_{\text{Hg}} V \Delta T}{A} = \frac{(181 \cdot 10^{-6} \text{ / °C})(8.00 \text{ mL})(1.00 \text{ °C})}{1.00 \text{ mm}^2}\left(\frac{1000. \text{ mm}^3}{\text{mL}}\right) = 1.448 \text{ mm}$$

Thus, the 1.00 °C tick marks should be spaced about 1.45 mm apart.

17.64. The initial volume of gasoline is 14 gallons and the change in temperature is $\Delta T = 27$ °F. The volume expansion coefficient of gas is $9.6 \cdot 10^{-4}$ / °C. Convert the temperature change from Fahrenheit to Celsius:

$$\Delta T_{\text{C}} = \frac{5}{9} \Delta T_{\text{F}}. \text{ Thus } \Delta T = \frac{5}{9}(27 \text{ °F}) = 15 \text{ °C}.$$

Thus, $\Delta V = \beta_{\text{gas}} V \Delta T = (9.6 \cdot 10^{-4} \text{ / °C})(14 \text{ gal})(15 \text{ °C}) = 0.2016$ gal. So, 0.20 gallons of gasoline are lost.

17.65. The change in temperature is $\Delta T = 37.8$ °C. The initial length of the slabs is $L = 12.0$ m. The linear expansion coefficient of concrete is $\alpha_{\text{con}} = 15 \cdot 10^{-6}$ / °C.

$$\Delta L = \alpha_{\text{con}} L \Delta T = (15 \cdot 10^{-6} \text{ / °C})(12.0 \text{ m})(37.8 \text{ °C}) = 0.006804 \text{ m}$$

Since the slabs expand uniformly, each side will grow by $\Delta L / 2$. However, the slabs expand towards each other, so each can grow by $\Delta L / 2$. Thus, the gap must be $2(\Delta L / 2) = \Delta L = 6.8$ mm.

17.66. Since water and aluminum have similar volume expansion coefficients, both must be accounted for. The water has a volume of $V = 500.\text{ cm}^3$. Though the volume of the aluminum can is not known, it has a capacity to carry a volume V. For simplicity assume that the amount of water that it can hold is the same as the volume of the aluminum vessel after heating. The change in temperature is $\Delta T = 30.0$ °C, the volume expansion coefficient of the water is $\beta_{\text{w}} = 207 \cdot 10^{-6}$ / °C and the linear expansion coefficient of aluminum is $\alpha_{\text{Al}} = 22 \cdot 10^{-6}$ / °C. The change in volume of the water is given by: $\Delta V_{\text{w}} = \beta_{\text{w}} V \Delta T$. The change in volume of the aluminum vessel is given by: $\Delta V_{\text{Al}} = 3\alpha_{\text{Al}} V \Delta T$. The difference in the change in volumes is $V' = \Delta V_{\text{w}} - \Delta V_{\text{Al}} = V\Delta T(\beta_{\text{w}} - 3\alpha_{\text{Al}})$.

$$V' = (500.\text{ cm}^3)(30.0\text{ °C})(207 \cdot 10^{-6}\text{ / °C} - 3(22 \cdot 10^{-6}\text{ / °C})) = 2.115\text{ cm}^3$$

Thus, about 2.1 cm^3 of water spills out, since the volume change of the water is larger.

17.67. The volume expansion coefficient of kerosene is $\beta_{\text{k}} = 990. \cdot 10^{-6}$ / °C. If the volume increases by 1.00%, then $\Delta V / V = 0.0100$.

$$\Delta V = \beta_{\text{k}} V \Delta T \Rightarrow \Delta T = \frac{\Delta V}{V}\left(\frac{1}{\beta_{\text{k}}}\right) = \frac{0.0100}{990. \cdot 10^{-6}\text{ / °C}} = 10.1\text{ °C}$$

Thus, the kerosene must be heated up by at least 10.1 °C in order for its volume to increase by 1.00%.

17.68. The radius of the holes is $r_{\text{h}} = 1.99$ cm and the radius of the ball bearings is $r_{\text{bb}} = 2.00$ cm. The linear expansion coefficient of epoxy is $\alpha_{\text{e}} = 1.30 \cdot 10^{-4}$ / °C, the cross-sectional area of the ball bearings is $A_{\text{bb}} = \pi r_{\text{bb}}^2$ and the cross-sectional area of the holes is $A_{\text{h}} = \pi r_{\text{h}}^2$. The epoxy is heated so that $A_{\text{h}} = A_{\text{bb}}$.

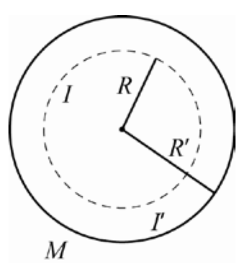
$$\Delta A = 2\alpha A \Delta T \Rightarrow A_{\text{bb}} = A_{\text{h}}(1 + 2\alpha_{\text{e}}\Delta T)$$

$$\frac{r_{\text{bb}}^2}{r_{\text{h}}^2} - 1 = 2\alpha_{\text{e}}\Delta T \Rightarrow \Delta T = \frac{\frac{r_{\text{bb}}^2}{r_{\text{h}}^2} - 1}{2\alpha_{\text{e}}} = \frac{\frac{(2.00\text{ cm})^2}{(1.99\text{ cm})^2} - 1}{2(1.30 \cdot 10^{-4}\text{ / °C})} = 38.752\text{ °C}$$

Thus, the epoxy needs to be heated up by about 38.8 °C.

17.69. **THINK:** When the disk (mass M, radius R and moment of inertia I) is heated up from $T_{\text{i}} = 20.0$ °C to $T_{\text{f}} = 100.$ °C, its radius, and hence area, will increase but the mass will stay the same. This allows us to determine the new moment of inertia and compare to the initial one.

SKETCH:



RESEARCH: Since the object is a disk, its moment of inertia, before and after heating, is:

$$I = \frac{1}{2}MR^2 \text{ and } I' = \frac{1}{2}MR'^2\text{, respectively.}$$

The area of the disk is $A = \pi R^2$ and the area changes by $\Delta A = 2\alpha A \Delta T$ upon heating. The linear expansion coefficient of the brass disk is $\alpha_{\text{B}} = 19 \cdot 10^{-6}$ / °C.

SIMPLIFY: Area after heating: $A_{\text{f}} = A_{\text{i}}(1+2\alpha_{\text{B}}\Delta T) \Rightarrow R'^2 = R^2(1+2\alpha_{\text{B}}\Delta T)$. The fractional change in moment of inertia given by:

$$\frac{\Delta I}{I} = \frac{I'-I}{I} = \frac{\frac{1}{2}MR'^2 - \frac{1}{2}MR^2}{\frac{1}{2}MR^2} = \frac{R'^2 - R^2}{R^2} = \frac{R^2(1+2\alpha_{\text{B}}\Delta T) - R^2}{R^2} = 2\alpha_{\text{B}}\Delta T.$$

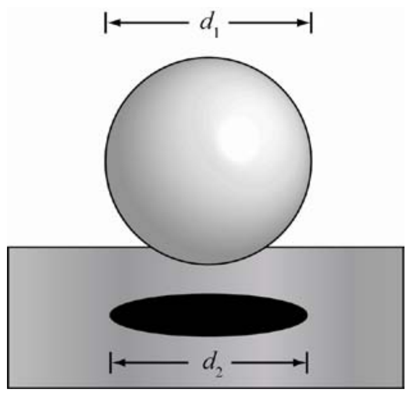
CALCULATE: $\dfrac{\Delta I}{I} = 2(19\cdot 10^{-6}\ /\ °\text{C})(100.\ °\text{C} - 20.0\ °\text{C}) = 0.00304$

ROUND: Two significant figures: the moment of inertia changes by 0.30%.

DOUBLE-CHECK: From our experience we would not expect the moment of inertia of a disk to change very dramatically for such a modest temperature change. A change of 0.30% is a reasonable result.

17.70. **THINK:** Initially, the brass sphere of diameter $d_{\text{B}} = 25.01$ mm is too big to fit through the hole, $d_{\text{Al}} = 25.00$ mm, in the aluminum plate. As both are heated up, both will expand. Since aluminum has a higher expansion coefficient, the hole will eventually become larger than the sphere.

SKETCH:



RESEARCH: The area of the hole and the cross-sectional area of the sphere increase with temperature as $\Delta A_{\text{Al}} = 2\alpha_{\text{Al}} A_{\text{Al}}\Delta T$ and $\Delta A_{\text{B}} = 2\alpha_{\text{B}} A_{\text{B}}\Delta T$, respectively, where the initial areas of the hole and sphere are $A_{\text{Al}} = \pi(d_{\text{Al}}/2)^2$ and $A_{\text{B}} = \pi(d_{\text{B}}/2)^2$, respectively. The sphere will fall into the hole when the final areas of the two are equal: $A_{\text{Al}}(1+2\alpha_{\text{Al}}\Delta T) = A_{\text{B}}(1+2\alpha_{\text{B}}\Delta T)$. The linear expansion coefficients of brass and aluminum are $\alpha_{\text{B}} = 19\cdot 10^{-6}\ /\ °\text{C}$ and $\alpha_{\text{A}} = 22\cdot 10^{-6}\ /\ °\text{C}$, respectively. The initial temperature of two objects is room temperature, $T_{\text{i}} = 20.\ °\text{C}$.

SIMPLIFY: $A_{\text{B}}' = \pi\left(\dfrac{d_{\text{B}}}{2}\right)^2(1+2\alpha_{\text{B}}\Delta T)$, $A_{\text{Al}}' = \pi\left(\dfrac{d_{\text{Al}}}{2}\right)^2(1+2\alpha_{\text{Al}}\Delta T)$

$$A_{\text{B}}' = A_{\text{Al}}' \Rightarrow d_{\text{B}}^{\,2}(1+2\alpha_{\text{B}}\Delta T) = d_{\text{Al}}^{\,2}(1+2\alpha_{\text{Al}}\Delta T)$$

$$d_{\text{B}}^{\,2} - d_{\text{Al}}^{\,2} = 2\Delta T(d_{\text{Al}}^{\,2}\alpha_{\text{Al}} - d_{\text{B}}^{\,2}\alpha_{\text{B}}) \Rightarrow \Delta T = \frac{d_{\text{B}}^{\,2} - d_{\text{Al}}^{\,2}}{2(d_{\text{Al}}^{\,2}\alpha_{\text{Al}} - d_{\text{B}}^{\,2}\alpha_{\text{B}})}$$

$$T_{\text{f}} = \frac{d_{\text{B}}^{\,2} - d_{\text{Al}}^{\,2}}{2(d_{\text{Al}}^{\,2}\alpha_{\text{Al}} - d_{\text{B}}^{\,2}\alpha_{\text{B}})} + T_{\text{i}}$$

CALCULATE:

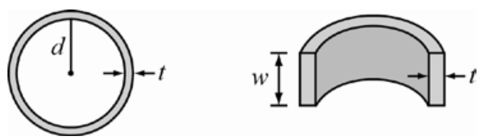
$$T_{\text{f}} = \frac{(25.01\text{ mm})^2 - (25.00\text{ mm})^2}{2\left[(25.00\text{ mm})^2(22\cdot 10^{-6}\ /\ °\text{C}) - (25.01\text{ mm})^2(19\cdot 10^{-6}\ /\ °\text{C})\right]} + 20.\ °\text{C} = 134.04\ °\text{C} + 20.\ °\text{C} = 154.04\ °\text{C}$$

ROUND: In the previous calculation, the quotient should be rounded to two significant figures, so the final answer is $T_{\text{f}} = 150\ °\text{C}$.

DOUBLE-CHECK: Since the expansion coefficients of the two materials are close in value, such a high temperature is expected.

17.71. **THINK:** The steel band has an initial diameter of $d_{\text{i}} = 4.40$ mm, width $w = 3.50$ mm, and thickness $t = 0.450$ mm. As the band cools from $T_{\text{i}} = 70.0$ °C to $T_{\text{f}} = 36.8$ °C its diameter will decrease. Since the circumference of the band is directly proportional to the diameter, both the circumference and the diameter have the same relative change with the decrease in temperature. The tension in the band can be found by considering the Young's modulus of the steel band. Effectively, the band is stretched from its diameter at T_{f} to the diameter of the tooth.

SKETCH:



RESEARCH: The change in the area of the band (i.e. area around the tooth) is $\Delta A = 2\alpha_{\text{S}} A \Delta T$ where $A = \pi (d/2)^2$. Young's modulus is the ratio of the stress to the strain where the stress is the force per unit area and the strain is the relative change in length,

$$Y = \frac{F/wt}{\Delta L/L} \quad \Rightarrow \quad \frac{\Delta L}{L} = \frac{F}{wtY}.$$

For steel, $Y = 200.\cdot 10^9$ N/m^2. The length of the band is the circumference, so $L = \pi d$. For this problem, use $|\Delta T|$ in place of ΔT. The linear expansion coefficient of steel is

$\alpha_{\text{S}} = 13.0 \cdot 10^{-6}$ °C^{-1}.

SIMPLIFY: The relative change in area is: $\dfrac{\Delta A}{A} = 2\alpha_{\text{S}} |\Delta T|$. Since the length is proportional to the diameter: $\dfrac{\Delta L}{L} = \dfrac{\Delta d}{d}$. Since $A = \pi d^2/4$, $\Delta A = (\pi d/2)\Delta d$. So we can write

$$\frac{\Delta A}{A} = \frac{(\pi d/2)\Delta d}{\pi d^2/4} = 2\frac{\Delta d}{d}.$$

We can combine these equations to get

$$2\alpha_{\text{S}} |\Delta T| = \frac{2F}{wtY} \quad \Rightarrow \quad F = \alpha_{\text{S}} |\Delta T| wtY.$$

CALCULATE: $F = (13.0 \cdot 10^{-6} \text{ °C}^{-1}) |70.0 \text{ °C} - 36.8 \text{ °C}| (3.50 \cdot 10^{-3} \text{ m})(0.450 \cdot 10^{-3} \text{ m})(200.\cdot 10^9 \text{ N/m}^2)$

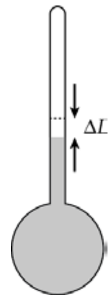
$F = 135.954$ N.

ROUND: The answer has to be rounded to three significant figures: $F = 136$ N.

DOUBLE-CHECK: Since a tooth is very strong, this large tension that is created will be able to act on the tooth without causing problems. The force must also be large in order to withstand the forces of biting food. Therefore, this is a reasonable result.

17.72. **THINK:** To find the spacing between tick marks, I must consider how high the mercury, of initial volume $V_{\text{i}} = 8.63$ cm^3, rises in the tube of diameter $d = 1.00$ mm when the temperature increases by $\Delta T = 1.00$ °C. I can assume that the cross-sectional area of the tube remains constant.

SKETCH:



RESEARCH: The cross-sectional area of the thermometer is $A=\pi\left(d/2\right)^2$. The change in volume of the mercury due to a temperature change is $\Delta V=\beta_{\text{Hg}}V\Delta T$. Since the expansion of the tube can be neglected, $\Delta V=A\Delta L$. The volume expansion coefficient for the mercury is $\beta_{\text{Hg}}=1.81\cdot 10^{-4}\ ^\circ\text{C}^{-1}$.

SIMPLIFY: $\Delta V=\beta_{\text{Hg}}V\Delta T=A\Delta L \ \Rightarrow\ \Delta L=\dfrac{\beta_{\text{Hg}}V\Delta T}{A}=\dfrac{\beta_{\text{Hg}}V\Delta T}{\pi\left(\dfrac{d}{2}\right)^2}.$

CALCULATE: $\Delta L=\dfrac{(1.81\cdot 10^{-4}\ ^\circ\text{C}^{-1})(8.63\text{ cm}^3)(1.00\ ^\circ\text{C})}{\pi\left(\dfrac{0.100\text{ cm}}{2}\right)^2}=0.19888\text{ cm}.$

ROUND: Three significant figures: $\Delta L=1.99$ mm.

DOUBLE-CHECK: This would indicate that a thermometer should be 20 cm long to allow reasonable temperature measurements from $0\ ^\circ\text{C}$ to $100\ ^\circ\text{C}$. This is a reasonable size for a thermometer, so this spacing size is logical.

17.73. **THINK:** The device has an initial volume of $V_{\text{i}}=0.0000500\text{ m}^3$, which will increase upon heating. Of course, this volume change is proportional to the volume expansion coefficient of the material, β. A change in temperature is proportional to a change in volume. This means that a temperature change rate ($\Delta T=200.\ ^\circ\text{C}$ in $\Delta t_{\text{T}}=3.00$ seconds) is also proportional to a volume change rate ($\Delta V=0.000000100\text{ m}^3$ in $\Delta t_{\text{V}}=5.00$ seconds).

SKETCH: A sketch is not needed to solve this problem.

RESEARCH: The change in volume is $\Delta V=\beta V_{\text{i}}\Delta T$. The maximum volume change rate is:

$$\left(\frac{\Delta V}{\Delta t_{\text{V}}}\right)_{\text{max}}=\beta V_{\text{i}}\frac{\Delta T}{\Delta t_{\text{T}}}.$$

SIMPLIFY: The value for β for the maximum volume change rate is when:

$$\beta=\frac{\Delta V}{\Delta t_{\text{V}}}\left(\frac{\Delta t_{\text{T}}}{\Delta T}\right)\left(\frac{1}{V_{\text{i}}}\right).$$

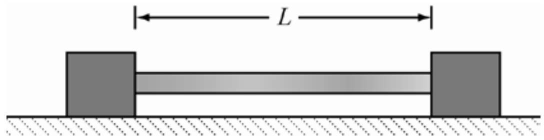
CALCULATE: $\beta=\dfrac{(0.000000100\text{ m}^3)(3.00\text{ s})}{(5.00\text{ s})(200.\ ^\circ\text{C})(0.0000500\text{ m}^3)}=6.0000\cdot 10^{-6}/\ ^\circ\text{C}$

ROUND: Three significant figures: $\beta=6.00\cdot 10^{-6}/\ ^\circ\text{C}$.

DOUBLE-CHECK: This value has the same order of magnitude that as has been seen for many volume expansion coefficients, so it is a reasonable answer.

17.74. **THINK:** The rod has a length of cross-sectional area of $L=1.0000$ m and $A=5.00\cdot 10^{-4}\text{ m}^2$. After an increase in temperature from $T_{\text{i}}=0.00\ ^\circ\text{C}$ to $T_{\text{f}}=40.0\ ^\circ\text{C}$, the rod will tend to expand. Since it cannot expand between the two end points, it will experience stress. The stress can then be determined by using Young's modulus. $Y=2.0\cdot 10^{11}\text{ N/m}^2$, $\alpha=13\cdot 10^{-6}/\ ^\circ\text{C}$.

SKETCH:



RESEARCH: Young's modulus (for steel is $Y = 20.\cdot 10^{10}\ \text{N/m}^2$) is the ratio of the stress to the strain where the stress is the force per unit area and the strain is the relative change in length,

$$\text{i.e. } Y = \frac{\text{stress}}{\Delta L / L}$$

Even though the rod does not actually extend because it is between to fixed points, we can then think of the stress as preventing the expansion, which still depends on $\Delta L / L$. The change in length of the rod is $\Delta L = \alpha_{\text{S}} L \Delta T$, where the linear expansion coefficient of steel is $\alpha_{\text{S}} = 13 \cdot 10^{-6}\ /\ ^\circ\text{C}$.

SIMPLIFY: $\text{stress} = Y \cdot \dfrac{\Delta L}{L} = Y \cdot \dfrac{\alpha_{\text{S}} L \Delta T}{L} = Y \alpha_{\text{S}} \Delta T$

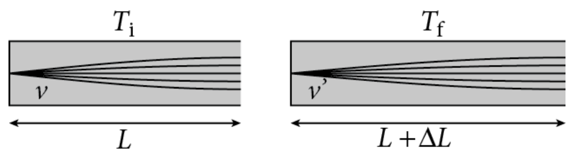
CALCULATE: $\text{stress} = (2.0 \cdot 10^{11}\ \text{N/m}^2)(13 \cdot 10^{-6}\ /\ ^\circ\text{C})(40.0\ ^\circ\text{C} - 0.00\ ^\circ\text{C}) = 1.04 \cdot 10^{8}\ \text{Pa}$

ROUND: Two significant figures: $\text{stress} = 1.0 \cdot 10^{8}\ \text{Pa}$

DOUBLE-CHECK: Even though the rod only wants to increase by $\Delta L = 0.52$ mm (which is small), the rod is made of steel which is very strong material, so a large stress is reasonable.

17.75. **THINK:** The bugle can be considered a half-closed pipe of length $L = 183.0$ cm. The speed of sound in air is dependent on temperature, as is the length of the bugle, so an increase in temperature from $T_{\text{i}} = 20.0\ ^\circ\text{C}$ to $T_{\text{f}} = 41.0\ ^\circ\text{C}$ will cause both to change.

SKETCH:



RESEARCH: The fundamental frequency of an open pipe is $f_1 = v / (4L)$, where v is the speed of sound. The speed of sound in air as a function of temperature is $v(T) = (331 + 0.6T)$ m/s, with T in units of °C. The length of the tube increases by $\Delta L = \alpha_{\text{B}} L \Delta T$, with a linear expansion coefficient for brass of $\alpha_{\text{B}} = 19.0 \cdot 10^{-6}\ ^\circ\text{C}^{-1}$.

SIMPLIFY:

(a) If only the change in air temperature is considered, $f_1 = \dfrac{v(T_{\text{f}})}{4L}$.

(b) If only the change in length of the bugle is considered, $f_1 = \dfrac{v(T_{\text{i}})}{4L(1 + \alpha_{\text{B}} \Delta T)}$.

(c) If both effects are taken into account, $f_1 = \dfrac{v(T_{\text{f}})}{4L(1 + \alpha_{\text{B}} \Delta T)}$.

CALCULATE:

(a) $f_1 = \dfrac{\left(331 + (0.6)(41.0)\right)\ \text{m/s}}{4(1.830\ \text{m})} = 48.579\ \text{Hz}.$

(b) $f_1 = \dfrac{\left(331 + (0.6)(20.0)\right)\ \text{m/s}}{4(1.830\ \text{m})\left(1 + (19.0 \cdot 10^{-6}\ ^\circ\text{C}^{-1})(41.0\ ^\circ\text{C} - 20.0\ ^\circ\text{C})\right)} = 46.839\ \text{Hz}.$

(c) $f_1 = \dfrac{\left(331 + (0.6)(41.0)\right)\ \text{m/s}}{4(1.830\ \text{m})\left(1 + (19.0 \cdot 10^{-6} /^\circ\text{C})(41.0\ ^\circ\text{C} - 20.0\ ^\circ\text{C})\right)} = 48.560\ \text{Hz}$

ROUND: Three significant figures:
(a) $f_1 = 48.6 \text{ Hz}$
(b) $f_1 = 46.8 \text{ Hz}$
(c) $f_1 = 48.6 \text{ Hz}$
DOUBLE-CHECK: The fundamental frequency of the bugle is fairly insensitive to changes in temperature if the changes in the speed of sound and the expansion of the brass are considered.

Multi-Version Exercises

Exercises 17.76–17.78 The change in length of the steel bar is given by $\Delta L_s = \alpha_s L_s \Delta T$. The change in length of the brass bar is given by $\Delta L_b = \alpha_b L_b \Delta T$. When the two bars have the same length, $L_s + \Delta L_s = L_b + \Delta L_b$. So we can write $L_s + \alpha_s L_s \Delta T = L_b + \alpha_b L_b \Delta T$. Rearranging and solving for the temperature difference gives us

$$\alpha_s L_s \Delta T - \alpha_b L_b \Delta T = L_b - L_s \quad \Rightarrow$$
$$\Delta T = \frac{L_b - L_s}{\alpha_s L_s - \alpha_b L_b}.$$

17.76. Applying the above result,

$$\Delta T = \frac{L_b - L_s}{\alpha_s L_s - \alpha_b L_b} = \frac{(2.6827 \text{ m}) - (2.6867 \text{ m})}{(13.00 \cdot 10^{-6} \text{ °C}^{-1})(2.6867 \text{ m}) - (19.00 \cdot 10^{-6} \text{ °C}^{-1})(2.6827 \text{ m})} = 249.311 \text{ °C}$$

So the temperature is $T = T_0 + \Delta T = 26.45 \text{ °C} + 249.311 \text{ °C} = 275.8 \text{ °C}$.

17.77. Again applying the above findings,

$$L_s + \alpha_s L_s \Delta T = L_b + \alpha_b L_b \Delta T$$
$$L_s(\alpha_s \Delta T + 1) = L_b(\alpha_b \Delta T + 1)$$
$$L_b = L_s \frac{\alpha_s \Delta T + 1}{\alpha_b \Delta T + 1}$$
$$\Delta T = T - T_0 = 214.07 \text{ °C} - 28.73 \text{ °C} = 185.34 \text{ °C}$$
$$L_b = (2.7073 \text{ m}) \frac{(13.00 \cdot 10^{-6} \text{ °C}^{-1})(185.34 \text{ °C}) + 1}{(19.00 \cdot 10^{-6} \text{ °C}^{-1})(185.34 \text{ °C}) + 1} = 2.704 \text{ m}$$

17.78. Using the same method as in the preceding problem,

$$L_s + \alpha_s L_s \Delta T = L_b + \alpha_b L_b \Delta T$$
$$L_s(\alpha_s \Delta T + 1) = L_b(\alpha_b \Delta T + 1)$$
$$L_s = L_b \frac{\alpha_b \Delta T + 1}{\alpha_s \Delta T + 1}$$
$$\Delta T = T - T_0 = 227.27 \text{ °C} - 31.03 \text{ °C} = 196.24 \text{ °C}$$
$$L_b = (2.7247 \text{ m}) \frac{(19.00 \cdot 10^{-6} \text{ °C}^{-1})(196.24 \text{ °C}) + 1}{(13.00 \cdot 10^{-6} \text{ °C}^{-1})(196.24 \text{ °C}) + 1} = 2.728 \text{ m}$$

17.79. Apply Equation 17.6:

$$\Delta A = 2\alpha A \Delta T = 2(3.749 \cdot 10^{-6} \text{ °C}^{-1})\left(\pi\left(\frac{5.093 \text{ m}}{2}\right)^2\right)(33.37 \text{ °C}) = 5.097 \cdot 10^{-3} \text{ m}^2.$$

17.80. From Equation 17.6,

$$\Delta A = 2\alpha A \Delta T$$

$$\alpha = \frac{\Delta A}{2A\Delta T} = \frac{4.253 \cdot 10^{-3}\ \text{m}^2}{2\left(\pi\left(\frac{4.553\ \text{m}}{2}\right)^2\right)(34.65\ {}^\circ\text{C})} = 3.769 \cdot 10^{-6}\ {}^\circ\text{C}^{-1}.$$

17.81. By virtue of Equation 17.6,

$$\Delta A = 2\alpha A \Delta T$$

$$\Delta T = \frac{\Delta A}{2\alpha A} = \frac{4.750 \cdot 10^{-3}\ \text{m}^2}{2\left(3.789 \cdot 10^{-6}\ {}^\circ\text{C}^{-1}\right)\left(\pi\left(\frac{4.713\ \text{m}}{2}\right)^2\right)} = 35.93\ {}^\circ\text{C}.$$

17.82. Apply Equation 17.5:

$$\Delta L = L\alpha\Delta T = (501.9\ \text{m})\left(13.89 \cdot 10^{-6}\ {}^\circ\text{C}^{-1}\right)(-15.91\ {}^\circ\text{C} - 28.09\ {}^\circ\text{C}) = -0.3067\ \text{m}.$$

The bar is 0.3067 m shorter.

17.83. By Equation 17.5,

$$\Delta L = L\alpha\Delta T$$

$$\alpha = \frac{\Delta L}{L\Delta T} = \frac{-0.4084\ \text{m}}{(599.7\ \text{m})(-18.95\ {}^\circ\text{C} - 28.31\ {}^\circ\text{C})} = 1.441 \cdot 10^{-5}\ {}^\circ\text{C}^{-1}.$$

17.84. From Equation 17.5,

$$\Delta L = L\alpha\Delta T$$

$$\Delta T = T_{\text{winter}} - T_{\text{summer}} = \frac{\Delta L}{L\alpha}$$

$$T_{\text{winter}} = T_{\text{summer}} + \frac{\Delta L}{L\alpha} = 28.51\ {}^\circ\text{C} + \frac{-0.3903\ \text{m}}{(645.5\ \text{m})\left(14.93 \cdot 10^{-6}\ {}^\circ\text{C}^{-1}\right)} = 28.51\ {}^\circ\text{C} + (-40.499\ {}^\circ\text{C}) = -11.99\ {}^\circ\text{C}$$

Chapter 18: Heat and the First Law of Thermodynamics

Concept Checks

18.1. d **18.2.** c **18.3.** b **18.4.** e **18.5.** c **18.6.** e **18.7.** (1) False (2) False (3) False (4) True

Multiple-Choice Questions

18.1. d **18.2.** b **18.3.** b **18.4.** d **18.5.** c **18.6.** b **18.7.** b **18.8.** d **18.9.** f **18.10.** e **18.11.** a **18.12.** d **18.13.** b **18.14.** a

Conceptual Questions

18.15. The average human, approximated as a cylinder, has a radius of about $R = 16.0$ cm to about $R = 30.0$ cm (upper and lower limit) with a minimum height of $h = 150$ cm to maximum $h = 200$ cm. The surface area of a cylinder of radius R and height h is $\text{A} = 2\pi Rh$. The temperature of a healthy human is $T = 37.0$ °C or $T = 310.$ K. Since we are assuming the person is black body, they will have an emissivity of $\varepsilon = 1$. The power radiated is given by the Stefan-Boltzmann equation, $P = \sigma\varepsilon \text{A} T^4$; therefore, the range of power emitted by an average person is given by:

$$P_{\text{min}} = \left(5.67 \cdot 10^{-8} \text{ W/}\left(\text{m}^2 \text{ K}^4\right)\right)(1)(2\pi)(0.160 \text{ m})(1.5 \text{ m})(310. \text{ K})^4 = 789.6 \text{ W}$$

$$P_{\text{max}} = \left(5.67 \cdot 10^{-8} \text{ W/}\left(\text{m}^2 \text{ K}^4\right)\right)(1)(2\pi)(0.300 \text{ m})(2.00 \text{ m})(310. \text{ K})^4 = 1974.1 \text{ W}$$

The average person radiates between 800 W and 2000 W.

18.16. The house that has snow on the roof is releasing less heat to the atmosphere; hence less snow is melting and therefore, the house would be better insulated.

18.17. A bathmat (most likely cotton or wool) would have a much lower thermal conductivity than the tile. Assuming the bathmat and tile are the same temperature (room temperature) then the tile will feel colder since its higher thermal conductivity takes thermal energy from your feet faster than a mat would, and hence feels colder. Even though the thermal energy flow is proportional to the temperature difference, when your feet are colder, the thermal energy flow would be more noticeable to you and hence seem colder.

18.18. A blackbody can be constructed by making a cavity out of this material designed such that any electromagnetic radiation that enters the cavity cannot escape (at least not until after many bounces off the walls of the cavity). After every bounce, half the energy is deposited. So after many bounces nearly all of the original energy has been deposited, giving the impression that it is an ideal black body since virtually no light is reflected.

18.19. After the eruption, the atmosphere was filled with increased amounts of ash and dust. This blocks out some sunlight and prevents the Earth from heating. As a result the temperature decreases until after many years the particles settle and the normal amount of sunlight makes it through to return the temperature to normal.

18.20. Even though the hot coals have a large temperature, they also have a very small thermal conductivity. This means that as long as the contact time between the coals and the person's feet is small (which is true when walking a steady pace) then the coals will not transfer enough heat and the person will not feel it.

18.21. When dry, the coat can be considered to be soaked in air. Air has a much lower thermal conductivity than water, so when it is dry, the air will insulate better than water. Also, when the coat is dry (and fluffy) it will be thicker; as opposed to when it is wet the fluffiness will disappear and flatten out. When the overall thickness decreases, it allows more heat to escape.

18.22. With an increased amount of dust in the atmosphere, more sunlight would be reflected back to space and this would prevent the heating of the Earth's atmosphere. This would overall drop the Earth's temperature.

18.23. (a) In order to push the piston down, the person must apply a force. This force applied to the area of the piston produces a pressure. Therefore, as the piston is pushed down, the pressure of the gas increases.

(b) Since the gas is thermally insulted from the environment, there is no thermal energy flow $(Q=0)$. The increase in pressure is a result of work done on the system. Since there is work done on the system and the volume decreases, the internal energy of the system increases $(\Delta E_{\text{int}}=-W=-P\Delta V)$. If the energy increases in a thermally insulated region, its temperature must increase.

(c) Other than pressure and temperature increasing and the volume decreasing, no other changes occur.

18.24. The thermal energy transferred per unit time is given by the equation $P=Q/t=kA(T_{\text{h}}-T_{\text{c}})/L.$ Consider a 10 cm long glass rod and a 10 m long aluminum rod where the cross sectional area and temperature difference are the same. The ratio of the thermal energy transferred is

$$\frac{P_{\text{Al}}}{P_{\text{glass}}}=\frac{k_{\text{Al}}A(T_{\text{h}}-T_{\text{c}})/L_{\text{Al}}}{k_{\text{glass}}A(T_{\text{h}}-T_{\text{c}})/L_{\text{glass}}}=\frac{k_{\text{Al}}L_{\text{glass}}}{k_{\text{glass}}L_{\text{Al}}}=\frac{(220\text{ W/(m K)})(0.10\text{ m})}{(0.8\text{ W/(m K)})(10\text{ m})}=2.75.$$

Therefore, despite the length difference, the aluminum transfers heat better than the glass rod by a factor of ≈ 3.

18.25. If the two canteens have a similar thickness, then the plastic bottle, which has a much lower thermal conductivity, will insulate the water better than the aluminum can.

18.26. The mass of the penny (copper) is about $3.0\text{ g},$ while that of the silver dollar is about $8.0\text{ g}.$ The temperature of the penny should be about 0 °C (273 K) since it was outside in the cold while the silver dollar should be 37 °C (310 K) since it was in the girl's hand. The specific heat of the penny and the dollar are $c_{\text{p}}=0.386\text{ KJ}/(\text{kg K})$ and $c_{\text{s}}=0.235\text{ KJ}/(\text{kg K}).$ Since the two coins are on wood (an insulator) they only exchange heat between each other. The final temperature T is then calculated using $m_{\text{p}}c_{\text{p}}(T-T_{\text{p}})=m_{\text{s}}c_{\text{s}}(T_{\text{s}}-T),$ which can be rewritten as $m_{\text{p}}c_{\text{p}}T_{\text{p}}+m_{\text{s}}c_{\text{s}}T_{\text{s}}=(m_{\text{p}}c_{\text{p}}+m_{\text{s}}c_{\text{s}})T;$ therefore, $T=(m_{\text{p}}c_{\text{p}}T_{\text{p}}+m_{\text{s}}c_{\text{s}}T_{\text{s}})/(m_{\text{p}}c_{\text{p}}+m_{\text{s}}c_{\text{s}}).$ Then,

$$T=\frac{(0.0030\text{ kg})(0.386\text{ kJ}/(\text{kg K}))(273\text{ K})+(0.0080\text{ kg})(0.235\text{ kJ}/(\text{kg K}))(310\text{ K})}{(0.0030\text{ kg})(0.386\text{ kJ}/(\text{kg K}))+(0.0080\text{ kg})(0.235\text{ kJ}/(\text{kg K}))}=296\text{ K}.$$

The final temperature of both coins is about $23\text{ °C}.$

Exercises

18.27. (a) The work to lift the elephant is $W=mgh$ or $W=(5.0\cdot 10^3\text{ kg})(2.0\text{ m})(9.81\text{m/s}^2)=9.8\cdot 10^4\text{ J}.$

(b) A food calorie is equal to $4.1868\cdot 10^3\text{ J}.$ The task of lifting the elephant consumes $W=9.8\cdot 10^4\text{ J}\left[1\text{ cal}/(4.1868\cdot 10^3\text{ J})\right]=23.4308$ food calories. Assuming that the body converts 100% of food energy into mechanical energy then the number of doughnuts needed is $23.4308\text{ cal}/(250\text{ cal}/\text{doughnut})=0.093723.$ It takes less than one doughnut to power its consumer to lift an elephant. The body usually converts only $30\ \%$ of the energy consumed. This corresponds to 0.31 of a doughnut.

18.28. The work done to expand the gas is given by $W=\int_{V_i}^{V_f} pdV$. The pressure is related to the volume by $P=\alpha V^3$. The work is then $W=\int_{V_i}^{V_f}\alpha V^3 dV=\alpha(1/4)V^4\big|_{V_i}^{V_f}=(\alpha/4)\left(V_f^4-V_i^4\right)$. If the final volume is 3 times larger, then the work done is:

$$W=(\alpha/4)\left((3V_i)^4-V_i^4\right)=(\alpha/4)\left(3^4-1\right)V_i^4=\left(4.00\text{ N/m}^{11}/4\right)(80)\left(2.00\text{ m}^3\right)^4=1280\text{ J}.$$

18.29. The work per cycle is given by the area enclosed by the pressure vs. volume graph. For this case the area is given by $W=(1/2)\left(2\cdot 10^2\text{ kPa}\right)\left(4\cdot 10^{-4}\text{ m}^3\right)=0.04\text{ KJ}=40\text{ J}.$

18.30. The process is adiabatic; thus, the change in internal energy is equal to the work done on the gas. The final internal energy is $\Delta E_{\text{int}}=-W=E_{\text{int, f}}-E_{\text{int, i}}=-PdV$, or

$$E_{\text{int, f}}=E_{\text{int, i}}-PdV=500.\text{ J}-\left[3.00\text{ atm}(101325\text{ Pa/atm})\right]\left(100.\text{ cm}^3(\text{m}/100.\text{ cm})^3\right)=470.\text{ J}.$$

18.31. The temperature of the material will be $Q=cm\Delta T=cm\left(T_f-T_i\right)$ or $T_f=T_i+Q/cm=T_i+\left(Q/cV\rho\right)$. Since the final temperature is inversely proportional to the specific heat, c, and the density, the material with the largest final temperature will be lead. A large specific heat will give a lower final temperature. The material with the largest specific heat and density, in this case, water, has the smallest final temperature. An example of the calculation for aluminum:

$$T_f^{\text{Al}}=22.0\text{ °C}+1.00\text{ J/}\left[1.00\text{ cm}^3\left(2.375\cdot 10^{-3}\text{ kg/cm}^3\right)\left(0.900\cdot 10^3\text{ J/(kg K)}\right)\right]=22.4678\text{ °C}.$$

Note that we need the density of the material.

Material	Specific Heat (KJ/kg K)	Density (g/cm^3)	Final Temperature °C
Lead	0.129	11.34	22.684
Copper	0.386	8.94	22.290
Steel	0.448	7.85	22.284
Aluminum	0.900	2.375	22.468
Glass	0.840	2.5	22.476
Water	4.19	1.00	22.239

18.32. The energy would only be transferred between the two bodies of water. This implies $m_1c\Delta T_1=m_2c\Delta T_2$ where $\Delta T_1,\Delta T_2$ indicate the water that starts at 20.0 °C and 32.0 °C, respectively. Further manipulation gives $m_1c\Delta T_1=\rho V_1c\Delta T_1=m_2c\Delta T_2=\rho V_2c\Delta T_2$ or $V_1\Delta T_1=V_2\Delta T_2$. The first volume of water will have a temperature increase but the temperature of the second volume of water decreases. Solving for the final temperature:

$$V_1\left(T_f-T_1\right)=-V_2\left(T_f-T_2\right)\Rightarrow V_1T_f+V_2T_f=V_1T_1+V_2T_2$$

$$\Rightarrow T_f=\left(V_1T_1+V_2T_2\right)/\left(V_1+V_2\right)=\left(7.00\text{ L}(20.0\text{ °C})+3.00\text{ L}(32.0\text{ °C})\right)/\left(7.00\text{ L}+3.00\text{ L}\right)=23.6\text{ °C}.$$

The final temperature of the water is 23.6 °C.

18.33. The heat transfer is equal for the aluminum and the water. The aluminum's temperature will decrease, but the water's temperature will increase. The temperature of the water is $T_w=10.0\text{ °C}$, and the temperature of the aluminum is $T_{\text{Al}}=85.0\text{ °C}$. The energy lost by the aluminum is transferred to the water. So we can write

$$m_{Al}c_{Al}\left(T_{AL}-T_f\right)=m_w c_w\left(T_f-T_w\right)$$
$$m_{Al}c_{Al}T_{AL}-m_{Al}c_{Al}T_f=m_w c_w T_f-m_w c_w T_w$$
$$m_{Al}c_{Al}T_{AL}+m_w c_w T_w=m_w c_w T_f+m_{Al}c_{Al}T_f$$
$$T_f=\frac{m_{Al}c_{Al}T_{AL}+m_w c_w T_w}{m_w c_w+m_{Al}c_{Al}}.$$

The mass of one liter of water is 1.00 kg.

$$\begin{aligned}T_f&=\frac{m_{Al}c_{Al}T_{Al}+m_w c_w T_w}{m_{Al}c_{Al}+m_w c_w}\\&=\frac{\left(25.0\cdot 10^{-3}\text{ kg}\right)\left(0.900\cdot 10^3\text{ J/}\left(\text{kg K}\right)\right)\left(85.0\text{ °C}\right)+\left(1.00\text{ kg}\right)\left(4.19\cdot 10^3\text{ J/}\left(\text{kg K}\right)\right)\left(10.0\text{ °C}\right)}{\left(25.0\cdot 10^{-3}\text{ kg}\right)\left(0.900\cdot 10^3\text{ J/}\left(\text{kg K}\right)\right)+\left(1.00\text{ kg}\right)\left(4.19\cdot 10^3\text{ J/}\left(\text{kg K}\right)\right)}\\&=10.4\text{ °C}\end{aligned}$$

The equilibrium temperature is 10.4 °C.

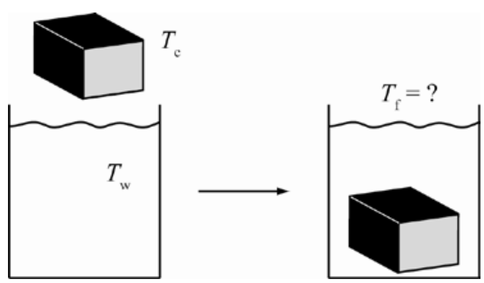
18.34. The kinetic energy of the bullet is $K=\left(1/2\right)mv^2$. If 75.0% of this energy is converted to heat then $Q=0.750K=\left(3/4\right)\left(1/2\right)mv^2=mc\Delta T$. The change in temperature is:

$$\Delta T=\left(\frac{3}{8}\right)\left(\frac{v^2}{c}\right)=\frac{\left(\frac{3}{8}\right)\left(250.\text{ m/s}\right)^2}{0.129\cdot 10^3\text{ J/}\left(\text{kg K}\right)}=181.69\text{ °C}.$$

Assuming the bullet is initially at room temperature of 20.0 °C, the final temperature of the bullet is $T_f=T_i+\Delta T=20.0\text{ °C}+181.67\text{ °C}=201.67\text{ °C}=202\text{ °C}$.

18.35. **THINK:** The problem calls for calculating the change in energy for the copper and the water. The volume of the materials does not change, so the process is isochoric. An isochoric process implies the change in internal energy is equal to the heat transferred. Therefore, to calculate the energies of the materials the final temperatures of the samples are needed. The magnitude of the heat transferred will be equal for both materials. Knowing this and the mass, initial temperatures and the specific heat it is possible to calculate the final temperature. Copper has a mass of $m_c=1.00$ kg, an initial temperature of $T_c=80.0$ °C, and a specific heat of $c_c=386\text{ J/}\left(\text{kg K}\right)$. The volume of water is 2.00 L which is equal to a mass of $m_w=2.00$ kg, an initial temperature of $T_w=10.0$ °C, and a specific heat of $c_w=4190\text{ J/}\left(\text{kg K}\right)$.

SKETCH:



RESEARCH: The heat transferred is equal to $Q=mc\Delta T$.

SIMPLIFY: The temperature change for the water and the copper will be positive and negative, respectively. The heat lost by the copper plus the heat gained by the water equals zero, so $Q_C+Q_W=0=m_c c_c\left(T_f-T_c\right)+m_w c_w\left(T_f-T_w\right)$. Solving this equation for T_f yields $T_f=\left(m_c c_c T_c+m_w c_w T_w\right)/\left(m_w c_w+m_c c_c\right)$. The magnitude of the change in energy is given by $\Delta E_{int}=Q=mc\Delta T$.

CALCULATE: $T_f = \dfrac{(1.00\text{ kg})(386\text{ J/kgK})(80.0\text{ °C}) + (2.00\text{ kg})(4190\text{ J/kgK})(10.0\text{ °C})}{(1.00\text{ kg})(386\text{ J/kgK}) + (2.00\text{ kg})(4190\text{ J/kgK})} = 13.0824\text{ °C}$

$\Delta E_{\text{int,w}} = (2.00\text{ kg})(4190\text{ J/kgK})(13.0824\text{ °C} - 10.0\text{ °C}) = 25830\text{ J}$

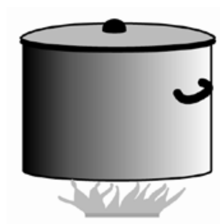
$\Delta E_{\text{int,Cu}} = (1.00\text{ kg})(386\text{ J/kgK})(13.0824\text{ °C} - 80.0\text{ °C}) = -25830\text{ J}$

ROUND: The energy should be rounded to three significant figures: $\Delta E_{\text{int}} = 25800$ J.

DOUBLE-CHECK: The magnitude of the change in energy for the water and copper must be equal since there are no other sources of change in energy. The signs must be opposite so energy is conserved. This is a reasonable amount of heat for a system of this size. Because copper has a much lower specific heat than water, it is expected that the copper will undergo a larger change in temperature.

18.36. **THINK:** The question asks how much heat must be added to bring an aluminum pot and water to 95.0 °C. The masses of the aluminum and water are 1.19 kg and 2.31 kg, respectively. Both materials start at a temperature of 19.7 °C and the temperature is kept uniform during the process.

SKETCH:



RESEARCH: The heat is given by $Q = mc\Delta T$.

SIMPLIFY: The total heat needed is the sum of the heat needed for each material respectively.

$Q_{\text{tot}} = Q_{\text{Al}} + Q_{\text{w}} = m_{\text{Al}}c_{\text{Al}}\Delta T + m_{\text{w}}c_{\text{w}}\Delta T = (m_{\text{Al}}c_{\text{Al}} + m_{\text{w}}c_{\text{w}})\Delta T = (m_{\text{Al}}c_{\text{Al}} + m_{\text{w}}c_{\text{w}})(T_f - T_i)$

CALCULATE:

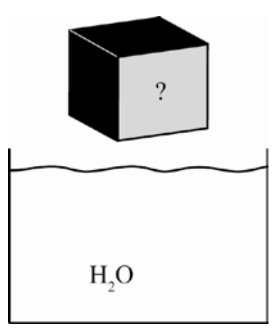
$Q_{\text{tot}} = \left[1.19\text{ kg}(0.900\text{ kJ/(kg K)}) + 2.31\text{ kg}(4.19\text{ kJ/(kg K)})\right](95.0\text{ °C} - 19.7\text{ °C}) = 809.5\text{ kJ}.$

ROUND: The heat is reported to 3 significant figures. The heat needed to bring the kettle and water is 810. kJ.

DOUBLE-CHECK: One calorie is defined as the energy required to heat one gram of water by one degree. $(2.31\cdot 10^3\text{ g})(95\text{ °C} - 19.7\text{ °C}) = 173943\text{ cal} \approx 730\text{ kJ}$. The low specific heat of aluminum ensures that the bulk of the energy goes toward heating the water. This is a reasonable answer.

18.37. **THINK:** Using a calorimeter, the type of material can be determined by calculating its specific heat. The heat transferred from brick is equal to the heat increase of the copper and water. The material has a mass of $m = 3.00$ kg and an initial temperature of $T = 300.$ °C. The copper and water have masses of 1.50 kg and 2.00 kg respectively and initial temperature of 20.0 °C. The equilibrium temperature of $T = 31.7$ °C.

SKETCH:



RESEARCH: The heat is given by $Q = mc\Delta T$.

SIMPLIFY: $Q_? = Q_w + Q_{Cu} = mc\Delta T = m_w c_w \Delta T_w + m_{Cu} c_{Cu} \Delta T_{Cu} = (m_w c_w + m_{Cu} c_{Cu})\Delta T_w$. Solving for c, the specific heat of the unknown material:

$$c = (m_w c_w + m_{Cu} c_{Cu})\Delta T_w / m\Delta T = (m_w c_w + m_{Cu} c_{Cu})(T_{eq} - T_w) / \left[-m(T_{eq} - T)\right].$$

CALCULATE:

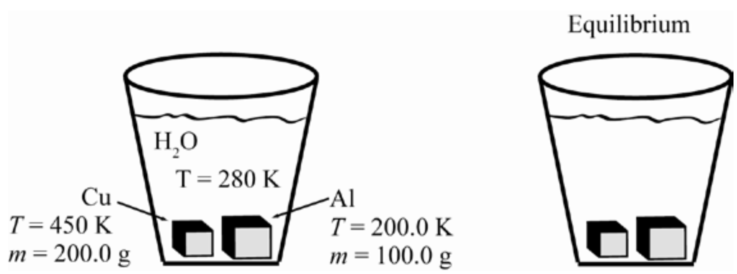
$$c = \frac{\left[(2.00 \text{ kg})(4190 \text{ J/(kg K)}) + (1.50 \text{ kg})(386 \text{ J/(kg K)})\right](31.7 \text{ °C} - 20.0 \text{ °C})}{-3.00 \text{ kg}(31.7 \text{ °C} - 300. \text{ °C})} = 130.228 \text{ J/(kg K)}$$

ROUND: The specific heat of the unknown material is reported to 3 significant figures: $c = 130.$ J/(kg K), which is very close to the specific heat of lead. The conclusion is that the material is lead.

DOUBLE-CHECK: The mass of the brick is similar to the combined mass of the calorimeter and water. The large change in temperature of the object compared to the small change in temperature of the calorimeter and water points to a material of low specific heat. The calculated value and conclusion are reasonable.

18.38. **THINK:** The equilibrium temperature of the three materials is desired after they are placed in thermal contact. The copper has a mass of $m_{Cu} = 2.00 \cdot 10^2$ g and an initial temperature of $T_{Cu} = 450.$ K. The aluminum has a mass of $m_{Al} = 1.00 \cdot 10^2$ g and an initial temperature of $T_{Al} = 2.00 \cdot 10^2$ K. The water has a mass of $m_w = 5.00 \cdot 10^2$ g and an initial temperature of $T_w = 280.$ K. The values are given to three significant figures.

SKETCH:



RESEARCH: The heat is given by $Q = mc\Delta T$. The copper will most likely decrease its temperature and the other two materials will increase their temperatures.

SIMPLIFY: $Q_{Cu} = Q_w + Q_{Al} \Rightarrow m_{Cu} c_{Cu} (T_{Cu} - T_{eq}) = m_w c_w (T_{eq} - T_w) + m_{Al} c_{Al} (T_{eq} - T_{Al})$

Solving for T_{eq}: $T_{eq}(m_w c_w + m_{Al} c_{Al} + m_{Cu} c_{Cu}) = m_{Cu} c_{Cu} T_{Cu} + m_w c_w T_w + m_{Al} c_{Al} T_{Al}$, or,

$$T_{eq} = \frac{m_{Cu} c_{Cu} T_{Cu} + m_w c_w T_w + m_{Al} c_{Al} T_{Al}}{m_w c_w + m_{Al} c_{Al} + m_{Cu} c_{Cu}}.$$

CALCULATE: Without units,

$$T_{eq} = \frac{(2.00 \cdot 10^2)(0.386)(450.) + (5.00 \cdot 10^2)(4.19)(280.) + (1.00 \cdot 10^2)(0.900)(200.)}{(2.00 \cdot 10^2)(0.386) + (5.00 \cdot 10^2)(4.19) + (1.00 \cdot 10^2)(0.900)}$$
$$= 282.619.$$

The units are:

$$\left[T_{eq}\right] = \frac{(\text{g})(\text{kJ/kg K})(\text{K}) + (\text{g})(\text{kJ/kg K})(\text{K}) + (\text{g})(\text{kJ/kg K})(\text{K})}{(\text{g})(\text{kJ/kg K}) + (\text{g})(\text{kJ/kg K}) + (\text{g})(\text{kJ/kg K})}$$
$$= \text{K}$$

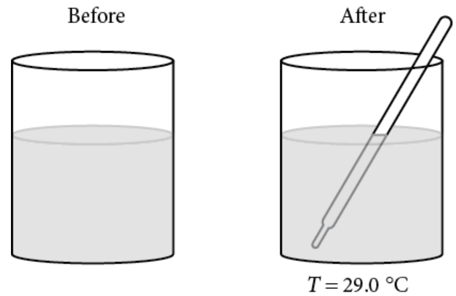
Altogether, $T_{eq} = 282.619$ K.

ROUND: The equilibrium temperature is reported to 3 significant figures, $T_{eq} = 283$ K.

DOUBLE-CHECK: This value is between the lowest and highest temperature. This value is also below the melting point of the metals and below the boiling point of water, so there is no need to worry about phase changes.

18.39. **THINK:** I want to find the original temperature of the water before the thermometer was placed in the water. The vial has a mass of 5.00 g and there is a volume of 6.00 mL of water. The thermometer is composed of 15.0 g of Pyrex and 4.00 g of mercury at a temperature of $T_0 = 20.0\ ^\circ\text{C}$. The thermometer reads $T_{\text{eq}} = 29.0\ ^\circ\text{C}$ when the thermometer is added to the water. The specific heat of Pyrex is given as 800. J/(kg K), and the specific heat of mercury is given as 140. J/(kg K). The mass of 6.00 mL of water is 6.00 g. The specific heat of water is 4190 J/(kg K).

SKETCH:



RESEARCH: The heat is given by $Q = mc\Delta T$.

SIMPLIFY: The water will decrease its temperature while the other materials increase their temperature. The heat transfer from the water and the vial must equal the heat transferred to the thermometer

$Q_{\text{water}} + Q_{\text{vial}} = Q_{\text{pyrex}} + Q_{\text{mercury}}$.

We can write this as

$$m_{\text{water}}c_{\text{water}}\left(T - T_{\text{eq}}\right) + m_{\text{vial}}c_{\text{vial}}\left(T - T_{\text{eq}}\right) = m_{\text{pyrex}}c_{\text{pyrex}}\left(T - T_0\right) + m_{\text{mercury}}c_{\text{mercury}}\left(T - T_0\right),$$

where T is the temperature of the water in the vial before the thermometer is inserted. Solving for T:

$$m_{\text{water}}c_{\text{water}}T - m_{\text{water}}c_{\text{water}}T_{\text{eq}} + m_{\text{vial}}c_{\text{vial}}T - m_{\text{vial}}c_{\text{vial}}T_{\text{eq}} = m_{\text{pyrex}}c_{\text{pyrex}}\left(T_{\text{eq}} - T_0\right) + m_{\text{mercury}}c_{\text{mercury}}\left(T_{\text{eq}} - T_0\right)$$

$$T\left(m_{\text{water}}c_{\text{water}} + m_{\text{vial}}c_{\text{vial}}\right) - T_{\text{eq}}\left(m_{\text{water}}c_{\text{water}} + m_{\text{vial}}c_{\text{vial}}\right) = m_{\text{pyrex}}c_{\text{pyrex}}\left(T_{\text{eq}} - T_0\right) + m_{\text{mercury}}c_{\text{mercury}}\left(T_{\text{eq}} - T_0\right)$$

$$T = \frac{m_{\text{pyrex}}c_{\text{pyrex}}\left(T_{\text{eq}} - T_0\right) + m_{\text{mercury}}c_{\text{mercury}}\left(T_{\text{eq}} - T_0\right) + T_{\text{eq}}\left(m_{\text{water}}c_{\text{water}} + m_{\text{vial}}c_{\text{vial}}\right)}{m_{\text{water}}c_{\text{water}} + m_{\text{vial}}c_{\text{vial}}}.$$

CALCULATE: Putting in our numerical values gives us

$$m_{\text{pyrex}}c_{\text{pyrex}}\left(T_{\text{eq}} - T_0\right) = (0.0150\text{ kg})(800.\text{ J/(kg }^\circ\text{C)})(29.0\ ^\circ\text{C} - 20.0\ ^\circ\text{C}) = 108\text{ J}$$

$$m_{\text{mercury}}c_{\text{mercury}}\left(T_{\text{eq}} - T_0\right) = (0.00400\text{ kg})(140.\text{ J/(kg }^\circ\text{C)})(29.0\ ^\circ\text{C} - 20.0\ ^\circ\text{C}) = 5.04\text{ J}$$

$$T_{\text{eq}}\left(m_{\text{water}}c_{\text{water}} + m_{\text{vial}}c_{\text{vial}}\right) = (29.0\ ^\circ\text{C})\big((0.00600\text{ kg})(4190.\text{ J/(kg }^\circ\text{C)}) + (0.00500\text{ kg})(800.\text{ J/(kg }^\circ\text{C)})\big) = 845.06\text{ J}$$

$$m_{\text{water}}c_{\text{water}} + m_{\text{vial}}c_{\text{vial}} = (0.00600\text{ kg})(4190.\text{ J/(kg }^\circ\text{C)}) + (0.00500\text{ kg})(800.\text{ J/(kg }^\circ\text{C)}) = 29.14\text{ J/}^\circ\text{C}$$

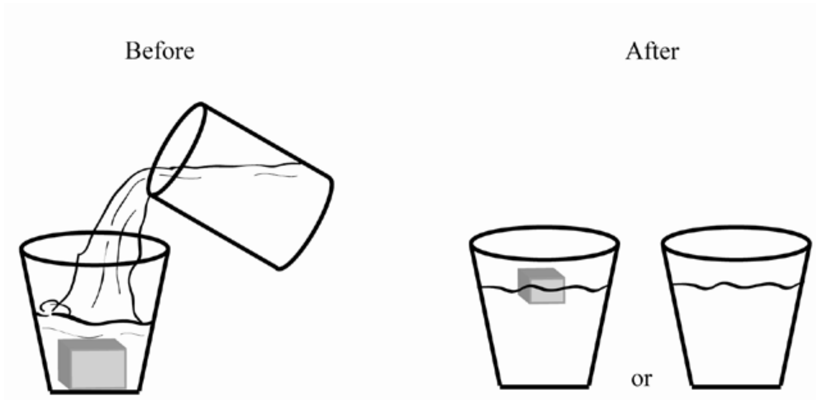
$$T = \frac{108\text{ J} + 5.04\text{ J} + 845.06\text{ J}}{29.14\text{ J/}^\circ\text{C}} = 32.879\ ^\circ\text{C}.$$

ROUND: The temperature is reported to three significant figures. The initial temperature of the water is $32.9\ ^\circ\text{C}$.

DOUBLE-CHECK: This is a reasonable answer.

18.40. **THINK:** When the water is poured on the ice, the heat transferred to the ice will warm it until it reaches a temperature of $0\ ^\circ\text{C}$ (the melting point of ice). If there is still heat left for transfer, it will start to convert the ice to water. Once all the ice melts into water, if there remains heat to be transferred than the melted water will increase its temperature from $T_{\text{m}} = 0\ ^\circ\text{C}$. The water has a mass of $m_1 = 400.$ g and an initial temperature of $T_1 = 30.0$ °C. The ice has a mass if $m_2 = 60.0$ g and an initial temperature of $T_2 = -5.00$ °C.

SKETCH:



RESEARCH: First, check the heat required to bring water from 30.0 °C to 0 °C, Q_1, and compare it to the heat required to bring ice from −5.00 °C to 0 °C and melt the ice completely, Q_2. If Q_1 is less than Q_2, the ice will only partially melt. If Q_1 is greater than or equal to Q_2, then the water will totally melt the ice. $Q_1 = m_1 c_{\text{w}}\left(T_1 - T_{\text{m}}\right)$ and $Q_2 = m_2 c_2\left(T_{\text{m}} - T_2\right) + m_2 L_m$. If $Q_1 < Q_2$, the ice left, ζ, must be calculated. Setting the transfer of heat of the water and ice equal gives $m_1 c_{\text{w}}\left(T_1 - T_{\text{m}}\right) = m_2 c_2\left(T_{\text{m}} - T_2\right) + \left(m_2 - \zeta\right)L_{\text{m}}$. The heating is given by $Q = mc\Delta T$ or $Q = mL$. If $Q_1 \geq Q_2$ then the temperature of the water is desired.

SIMPLIFY: Solving the equation for the transfer of heat for ζ gives $\zeta L_{\text{m}} = m_2 c_2\left(T_{\text{m}} - T_2\right) + m_2 L_m - m_1 c_{\text{w}}\left(T_1 - T_{\text{m}}\right) = Q_2 - Q_1 \Rightarrow \zeta = Q_2 - Q_1 / L_{\text{m}}$. Solve the following equation for the equilibrium temperature T_{eq}:

$$m_1 c_{\text{w}}\left(T_1 - T_{\text{eq}}\right) = m_2 c_{\text{i}}\left(T_{\text{m}} - T_2\right) + m_2 L_{\text{m}} + m_2 c_{\text{w}}\left(T_{\text{eq}} - T_{\text{m}}\right)$$
$$\Rightarrow m_2 c_{\text{w}} T_{\text{eq}} + m_1 c_{\text{w}} T_{\text{eq}} = m_1 c_{\text{w}} T_1 - m_2 c_{\text{i}}\left(T_{\text{m}} - T_2\right) - m_2 L_{\text{m}} + m_2 c_{\text{w}} T_{\text{m}}$$
$$\Rightarrow T_{\text{eq}} = \left(m_1 c_{\text{w}} T_1 - m_2 c_{\text{i}}\left(T_{\text{m}} - T_2\right) - m_2 L_{\text{m}} + m_2 c_{\text{w}} T_{\text{m}}\right) / \left(m_2 + m_1\right) c_{\text{w}}.$$

CALCULATE: $Q_1 = \left(0.400 \text{ kg}\right)\left(4190 \text{ J/}\left(\text{kg K}\right)\right)\left(30.0 \text{ °C} - 0 \text{ °C}\right) = 50280 \text{ J}$

$Q_2 = \left(0.0600 \text{ kg}\right)\left(2060 \text{ J/}\left(\text{kg K}\right)\right)\left(0 \text{ °C} - \left(-5.00 \text{ °C}\right)\right) + \left(0.0600 \text{ kg}\right)\left(334 \cdot 10^3 \text{ J/kg}\right) = 26220 \text{ J}$

Since $Q_1 > Q_2$ the equilibrium temperature is calculated as follows. Without units,

$$T_{\text{eq}} = \frac{(400.)(4.19)(30.0) - \left[(60.0)(2.06)\left(0 - (-5.00)\right)\right] - \left[(60.0)(334)\right] + \left[(60.0)(4.19)(0)\right]}{(400. + 60.0)(4.19)} = 15.369.$$ The

units for T_{eq} are:

$$\left[T_{\text{eq}}\right] = \frac{(\text{g})\left(\frac{\text{kJ}}{(\text{kg K})}\right)(\text{°C}) - \left[(\text{g})\left(\frac{\text{kJ}}{(\text{kg K})}\right)(\text{°C} - \text{°C})\right] - \left[(\text{g})(\text{kJ/kg})\right] + \left[(\text{g})\left(\frac{\text{kJ}}{(\text{kg K})}\right)(\text{°C})\right]}{(\text{g} + \text{g})\left(\text{kJ/}(\text{kg K})\right)} = \text{°C}$$

Altogether, $T_{\text{eq}} = 15.369$ °C.

ROUND: The temperature is reported to three significant figures. The water reaches 15.4 °C.

DOUBLE-CHECK: This temperature is in between −5.00 °C and 30.0 °C which makes sense.

18.41. The heat given off by the person is 180. kcal. This energy is consumed by converting water into steam. The amount of water is given by $m = Q / L_{\text{vap}} = \left(180. \text{ kcal}\right) / \left(539 \text{ cal/g}\right) = 0.33395 \text{ kg} = 334 \text{ g}$. The amount of water converted to steam is 334 g.

18.42. A block of aluminum of mass $m_{\text{Al}} = 1.30 \text{ kg}$ and temperature of 21.0 °C. The aluminum must be brought to a temperature of 932 K or 659 °C before it will melt. The heat required to bring the aluminum to this point is $Q_1 = mc(T_{\text{m}} - T_{\text{i}})$. At this temperature, more heat is needed to melt the aluminum equal to $Q_2 = mL_{\text{fus}}$. The total energy is then:

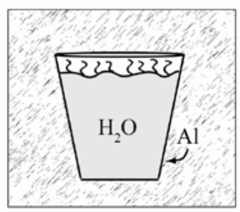
$$\begin{aligned} Q_{\text{tot}} &= Q_1 + Q_2 = mc(T_{\text{m}} - T_{\text{i}}) + mL_{\text{fus}} = m\left(c(T_{\text{m}} - T_{\text{i}}) + L_{\text{fus}}\right) \\ &= 1.30 \text{ kg}\left[(0.900 \text{ kJ/kgK})(932 \text{ K} - 294 \text{ K}) + 396 \text{ kJ/kg}\right] \\ &= 1261.26 \text{ kJ}. \end{aligned}$$

The heat required to melt 1.30 kg at a temperature of 21.0 °C is 1260 kJ.

18.43. The time needed to vaporize the liquid is given by $t = Q/P = mL_{\text{vap}}/P$. For liquid nitrogen the time of vaporization is calculated to be $t = (1.00 \text{ kg})(2.00 \cdot 10^5 \text{ J/kg})/(10.0 \text{ W}) = 2.00 \cdot 10^4 \text{ s}$. The time liquid helium takes to vaporize is given by $t = (1.00 \text{ kg})(2.09 \cdot 10^4 \text{ J/kg})/(10.0 \text{ W}) = 2090 \text{ s}$. Thus it takes about 10 times longer to vaporize liquid nitrogen than liquid helium.

18.44. **THINK:** The question asks for the equilibrium temperature inside the calorimeter after the steam is added. The steam will be converted to water and then the temperature will change to the equilibrium temperature. The steam has a mass of $m_{\text{s}} = 10.0 \text{ g}$ at a temperature of $T_{\text{s}} = 100.00 \text{ °C}$, and has the same mass after it has been converted to water at $T_{\text{w}} = 19.0 \text{ °C}$. The water in the cup has a mass of $m_{\text{w}} = 100. \text{ g}$, and the aluminum has a mass of 35.0 g, both at $T_{\text{w}} = 19.0 \text{ °C}$.

SKETCH:



RESEARCH: The heat from the change of temperature is $Q = cm\Delta T$. The heat from the change of state is $Q = mL_{\text{vap}}$.

SIMPLIFY: The heat lost by the steam is equal to the heat gained by the water and the aluminum. $-Q_{\text{s}} = Q_{\text{w}} + Q_{\text{Al}}$.

$$\begin{aligned} -\left(-m_{\text{s}}L_{\text{vap}} + m_{\text{s}}c_{\text{w}}(T_{\text{eq}} - T_{\text{s}})\right) &= m_{\text{w}}c_{\text{w}}(T_{\text{eq}} - T_{\text{w}}) + m_{\text{Al}}c_{\text{Al}}(T_{\text{eq}} - T_{\text{w}}) = (m_{\text{w}}c_{\text{w}} + m_{\text{Al}}c_{\text{Al}})(T_{\text{eq}} - T_{\text{w}}) \\ m_{\text{s}}L_{\text{vap}} - m_{\text{s}}c_{\text{w}}T_{\text{eq}} + m_{\text{s}}c_{\text{w}}T_{\text{s}} &= m_{\text{w}}c_{\text{w}}T_{\text{eq}} - m_{\text{w}}c_{\text{w}}T_{\text{w}} + m_{\text{Al}}c_{\text{Al}}T_{\text{eq}} - m_{\text{Al}}c_{\text{Al}}T_{\text{w}} \\ m_{\text{s}}c_{\text{w}}T_{\text{eq}} + m_{\text{w}}c_{\text{w}}T_{\text{eq}} + m_{\text{Al}}c_{\text{Al}}T_{\text{eq}} &= m_{\text{s}}L_{\text{vap}} + m_{\text{s}}c_{\text{w}}T_{\text{s}} + m_{\text{w}}c_{\text{w}}T_{\text{w}} + m_{\text{Al}}c_{\text{Al}}T_{\text{w}} \\ T_{\text{eq}}(m_{\text{s}}c_{\text{w}} + m_{\text{w}}c_{\text{w}} + m_{\text{Al}}c_{\text{Al}}) &= m_{\text{s}}L_{\text{vap}} + m_{\text{s}}c_{\text{w}}T_{\text{s}} + m_{\text{w}}c_{\text{w}}T_{\text{w}} + m_{\text{Al}}c_{\text{Al}}T_{\text{w}} \\ T_{\text{eq}} &= \frac{m_{\text{s}}L_{\text{vap}} + m_{\text{s}}c_{\text{w}}T_{\text{s}} + m_{\text{w}}c_{\text{w}}T_{\text{w}} + m_{\text{Al}}c_{\text{Al}}T_{\text{w}}}{m_{\text{s}}c_{\text{w}} + m_{\text{w}}c_{\text{w}} + m_{\text{Al}}c_{\text{Al}}} \end{aligned}$$

CALCULATE:

$$\begin{aligned} T_{\text{eq}} &= \frac{(10.0 \text{ g})(539) + (10.0)(1.00)(100.00 + 273) + (100.)(1.00)(19.0 + 273) + (35.0)(.215)(19.0 + 273)}{(10.0)(1.00) + (100.)(1.00) + (35.0)(.215)} \\ &= 344.75 \end{aligned}$$

$$\begin{aligned} \left[T_{\text{eq}}\right] &= \frac{(\text{g})(\text{cal/g}) + (\text{g})(\text{cal/g K})(\text{K}) + (\text{g})(\text{cal/g K})(\text{K}) + (\text{g})(\text{cal/g K})(\text{K})}{(\text{g})(\text{cal/g K}) + (\text{g})(\text{cal/g K}) + (\text{g})(\text{cal/g K})} \\ &= \text{K} \end{aligned}$$

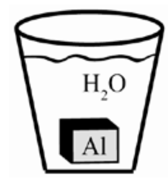
$$\therefore T_{\text{eq}} = 344.75 \text{ K} = 71.75 \text{ °C}$$

ROUND: The equilibrium temperature is reported to 3 significant figures since the masses are given to the least precise values (having three significant figures): $T_{\text{eq}} = 71.8\text{ °C}$.

DOUBLE-CHECK: The condensation of the steam alone provides 5390 calories of heat energy to the system. By definition, this is sufficient to raise the temperature of 100 g of water by about 54 °C. The low specific heat of aluminum ensures that the bulk of the energy goes toward raising the temperature of the water. The result is reasonable.

18.45. **THINK:** 0.100 kg of molten aluminum is dropped into 1.00 L of water. The temperatures of the aluminum and water are 932 K and 295 K, respectively. I want to determine how much water will boil away, how much aluminum will solidify, and what the equilibrium temperature will be. I also want to consider how the result would change if the temperature of the aluminum is increased to 1150 K. In order to determine the final state, the heat to solidify the aluminum, $Q_{\text{Al, s}}$ the heat to bring the water to its boiling point, $Q_{\text{w,b}}$, and the heat to vaporize the water, $Q_{\text{w,v}}$, are required. The aluminum has a mass of $m_{\text{Al}} = 0.100\text{ kg}$ at $T_{\text{Al}} = 932\text{ K} = 659\text{ °C}$. The water has mass of $m_{\text{w}} = 1.00\text{ kg}$ at $T_{\text{w}} = 22\text{ °C}$.

SKETCH:



RESEARCH: The heat is given by $Q = mc\Delta T$ and $Q = mL$.

SIMPLIFY: The heat required to solidify the aluminum is $Q_{\text{Al, s}} = m_{\text{Al}}\left(-L_{\text{fusion,Al}}\right)$. The heat required to bring the water to its boiling point $Q_{\text{w, b}} = m_{\text{w}}c_{\text{w}}\left(T_{\text{boiling}} - T_{\text{w}}\right)$. To find the equilibrium temperature use the following equation $m_{\text{Al}}L_{\text{Al, m}} + m_{\text{Al}}c_{\text{Al}}\left(T_{\text{Al}} - T_{\text{eq}}\right) = m_{\text{w}}c_{\text{w}}\left(T_{\text{eq}} - T_{\text{w}}\right)$. The equilibrium temperature is then:

$$T_{\text{eq}} = \left(m_{\text{Al}}L_{\text{Al, m}} + m_{\text{Al}}c_{\text{Al}}T_{\text{Al}} + m_{\text{w}}c_{\text{w}}T_{\text{w}}\right)/\left(m_{\text{Al}}c_{\text{Al}} + m_{\text{w}}c_{\text{w}}\right).$$

CALCULATE:

(a,b) $Q_{\text{Al, s}} = (0.100\text{ kg})(-396\text{ kJ/kg}) = -39.6\text{ kJ}.$

$Q_{\text{w, b}} = (4.19\text{ kJ/kgK})(1.00\text{ kg})(373\text{ K} - 295\text{ K}) = 326.82\text{ kJ}.$ Because $\left|Q_{\text{Al,s}}\right| < \left|Q_{\text{w,b}}\right|$, the molten aluminum does not supply enough heat to boil the water.

$$\text{(c) } T_{\text{eq}} = \frac{\left[(.100\text{ kg})\left(396\ \frac{\text{kJ}}{\text{kg}}\right)\right] + \left[(.100\text{ kg})\left(0.900\ \frac{\text{kJ}}{\text{kgK}}\right)(932\text{ K})\right] + \left[(1.00\text{ kg})\left(4.19\ \frac{\text{kJ}}{\text{kgK}}\right)(295\text{ K})\right]}{\left[(.100\text{ kg})\left(0.900\ \frac{\text{kJ}}{\text{kgK}}\right)\right] + \left[(1.00\text{ kg})\left(4.19\ \frac{\text{kJ}}{\text{kgK}}\right)\right]}$$

$$= 317.647\text{ K} = 44.647\text{ °C}$$

ROUND: The equilibrium temperature is reported to 2 significant figures.

(a) None of the water boils away.
(b) The aluminum will completely solidify.
(c) The final temperature is 44.6 °C.
(d) No. It is not possible to complete without knowing the specific heat of aluminum in its liquid phase.

DOUBLE-CHECK: The large specific heat value of water makes it a very efficient coolant. This is a reasonable answer.

18.46. **THINK:** The question asks for the loss of internal energy during a rigorous work out. The question also asks for the amount of nutritional calories required to replace the loss of internal energy. The total work done is $W = 1.80 \cdot 10^5$ J. The heat required to evaporate 150. g of water can be found using the latent heat of vaporization $L_{\text{vap}} = 2.42 \cdot 10^6$ J/kg. A nutritional calorie is equal to 4186 J.

SKETCH: A sketch is not needed to solve the problem.

RESEARCH: The first law of thermodynamics states $Q = \Delta E_{\text{int}} + W$. The heat loss is $Q = -mL_{\text{vap}}$.

SIMPLIFY: $\Delta E_{\text{int}} = Q - W = -mL_{\text{vap}} - W$.

CALCULATE: The change in internal energy is:

$$\Delta E_{\text{int}} = \left[-(0.150 \text{ kg})(2.42 \cdot 10^6 \text{ J/kg})\right] - (1.80 \cdot 10^5 \text{ J}) = -5.4300 \cdot 10^5 \text{ J}.$$

This energy is equivalent to the number of nutritional calories $(5.4300 \cdot 10^5 \text{ J})/(4186 \text{ J/kcal}) = 129.72$ kcal.

ROUND: The values will be reported to 3 significant figures.

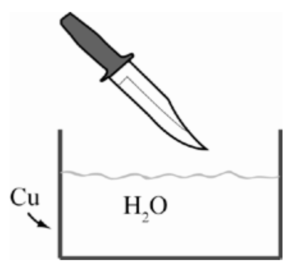
(a) The internal energy loses $5.43 \cdot 10^5$ J.

(b) You should consume 130. nutritional calories to compensate for this loss.

DOUBLE-CHECK: This is a reasonable number of calories to burn in a light workout.

18.47. **THINK:** The question asks for the amount of water necessary to cool the carbon steel. The steel has a mass of 0.500 kg and must go from a temperature of 1346 °F to 500. °F. These temperatures in Celsius are $T_{\text{h}} = (5/9)(1346 \text{ °F} - 32 \text{ °F}) = 730.$ °C and $T_{\text{c}} = (5/9)(500. \text{ °F} - 32 \text{ °F}) = 260.$ °C. The blade will be surrounded by an unknown quantity of water and 2.000 kg of copper both at room temperature 20.0 °C. The specific heat of copper is 386 J/(kg K). The table associated with the problem gives the specific heat of carbon steel at various temperature ranges.

SKETCH:



RESEARCH: The heat loss by the carbon steel is equal to the heat gained by the copper and water. The heat is given by $Q = mc\Delta T$.

SIMPLIFY: The heat loss by the carbon steel is:

$$Q_{\text{cs}} = \sum_{\text{i}} m_{\text{cs}} c_{\text{i}} \Delta T_{\text{i}},$$

where the summation goes over the different temperature ranges. The heat transferred to the copper is $Q_{\text{Cu}} = m_{\text{Cu}} c_{\text{Cu}} \Delta T_{\text{b}}$. The water will be brought to its boiling point $Q_{\text{w}} = m_{\text{w}} c_{\text{w}} \Delta T_{\text{b}}$. The water and the copper will reach a temperature of 100. °C. Equating the heat loss of the carbon steel to the heat gain of the water and copper:

$Q_{\text{cs}} = \sum_i m_{\text{cs}} c_i \Delta T_i = Q_{\text{Cu}} + Q_{\text{w}} = m_{\text{Cu}} c_{\text{Cu}} \Delta T_{\text{b}} + m_{\text{w}} c_{\text{w}} \Delta T_{\text{b}}$. Solving for the mass of the water:

$$m_{\text{w}} = \left(\sum_{\text{i}} m_{\text{cs}} c_{\text{i}} \Delta T_{\text{i}} - m_{\text{Cu}} c_{\text{Cu}} \Delta T_{\text{b}} \right) \Big/ (c_{\text{w}} \Delta T_{\text{b}}).$$

CALCULATE:

$$m_{\text{w}} = \frac{0.500\left[(846)(730.-650.)+(754+662+595)100.+553(350.-260.)\right]-2.000(386)(100.-20.0)}{4190(100.-20.0\ )}$$

$$=0.290916,$$

with units of

$$\left[m_{\text{w}}\right]=\frac{\text{kg}\left[(\text{J}/(\text{kg K}))(°\text{C})+(\text{J}/(\text{kg K}))(°\text{C})+(\text{J}/(\text{kg K}))(°\text{C})\right]-\text{kg}(\text{J}/(\text{kg K}))(°\text{C})}{(\text{J}/(\text{kg K}))(°\text{C})}=\text{kg}.$$

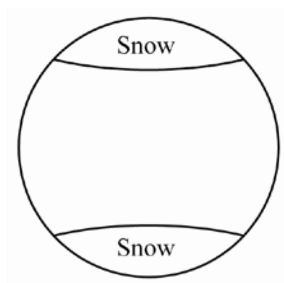
Hence, $m_{\text{w}}=0.290916$ kg.

ROUND: The least precise value is given to three significant figures. If the water does not convert to steam, it takes 291 g of water to cool off the carbon steel.

DOUBLE-CHECK: This is a reasonable answer. It takes relatively little water to accomplish the cooling due to the high specific heat of water. Note that the strength of the carbon steel also depends on the speed at which the steel hardens.

18.48. **THINK:** The question asks for the power dissipated in one Earth day by ethanol snowfall on Uranus. The polar regions are north of latitude $75.0°$ and south of latitude $-75.0°$. In the polar regions, the amount of snow fall is $d=1.00\text{ ft}=0.3048\text{ m}$. The thermodynamic variables of the ethanol are $c_{\text{v}}=1.70\text{ J}/(\text{g K})$, $c_{\text{l}}=2.44\text{ J}/(\text{g K})$, and $c_{\text{s}}=2.42\text{ J}/(\text{g K})$.

SKETCH:



RESEARCH: Since the minimum amount of energy that is lost to the atmosphere is required, assume that the ethanol starts at its boiling point, condenses to a liquid and then the liquid cools to the melting point and just freezes to become snow. The heat for a change in temperature is given by $Q=cm\Delta T$ and the heat for a change in phase is given by $Q=mL$. The area is given by $A=\int r^2 d\Omega$, where $d\Omega$ is the solid angle.

SIMPLIFY: The minimum amount of energy lost to the atmosphere is

$$Q=mL_{\text{v}}+mc_{\text{l}}\Delta T+mL_{\text{f}}=m\left(L_{\text{v}}+c_{\text{l}}\Delta T+L_{\text{f}}\right).$$

In terms of the volume and density of the ethanol snow,

$$Q=V\rho\left(L_{\text{v}}+c_{\text{l}}\Delta T+L_{\text{f}}\right)=(Ad)(1-0.9)\rho\left(L_{\text{v}}+c_{\text{l}}\Delta T+L_{\text{f}}\right).$$

The area covered by *one* pole is given by

$$A=\int r^2 d\Omega=2\pi r^2\int\sin\theta d\theta=2\pi r^2\cos\theta\,\Big|_{25°}^{0°}=2\pi r^2\left(1-\cos 25°\right).$$

The minimum amount of energy lost to the atmosphere for *both* poles is

$$Q=4\pi r^2\left(1-\cos 25.0°\right)d(1-0.9)\rho\left(L_{\text{v}}+c_{\text{l}}\Delta T+L_{\text{f}}\right).$$

The power dissipated is $P=Q/t$.

CALCULATE:

$$Q=4\pi\left(25559\cdot 10^3\text{ m}\right)^2\left(1-\cos 25.0°\right)(0.100)(0.3048\text{ m})\left(1.00\cdot 10^3\text{ kg/m}^3\right)$$
$$\cdot\left(858\cdot 10^3\text{ J/kg}+\left(2.44\cdot 10^3\text{ J}/(\text{kg K})\right)(351\text{ K}-156\text{ K})+104\cdot 10^3\text{ J/kg}\right)$$
$$=3.3707\cdot 10^{22}\text{ J}$$

$$P = \frac{\left(3.3707 \cdot 10^{22}\text{ J}\right)}{\left(86400\text{ s}\right)} = 3.901 \cdot 10^{17}\text{ W}$$

ROUND: To three significant figures, the minimum amount of energy lost to the atmosphere is $Q = 3.37 \cdot 10^{22}$ J, and the power dissipated in one Earth day is $P = 3.90 \cdot 10^{17}$ W.

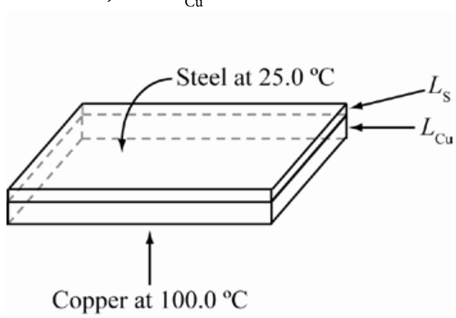
DOUBLE-CHECK: These large answers are reasonable for a planet like Uranus.

18.49. By finding the thermal conductivity of the metal, it is possible to identify it. The ice has a volume of $V = 100.\text{ mm}(100.\text{ mm})(5.00\text{ mm})$, a density of $920.\text{ kg/m}^3$, starts at a temperature of 0 °C and melts in 0.400 s. The metal disk is 10.0 mm thick. The temperature on the other side of the disk is 100. °C, the boiling point of water. The power of heat transfer is given by $P = Q/t = kA(\Delta T/d)$. The heat required to melt the ice is $Q = mL$, or in terms of volume and density $Q = mL = \rho VL$. $P = Q/t = \rho VL/t = kA(\Delta T/d)$. Solving for the thermal conductivity gives: $k = \rho VLd/tA\Delta T$.

$$k = \frac{920.\text{ kg/m}^3\left[0.100\text{ m}(0.100\text{ m})(5.00 \cdot 10^{-3}\text{ m})\right](334 \cdot 10^3\text{ J/kg})(0.0100\text{ m})}{0.400\text{ s}\left[0.100\text{ m}(0.100\text{ m})\right](100.\text{ °C} - 0\text{ °C})} = 384.1\text{ W/(m K)}.$$

Rounding to three significant figures, the thermal conductivity of the metal is 384 W/(m K) which is close to copper.

18.50. (a) The temperature at the copper-steel aluminum interface and the power flowing through the materials is desired. If the areas is a meter cubed, the copper has a thickness of 2.00 mm and one side is kept at a temperature of $T_{\text{Cu}} = 100.0$ °C. The steel on the other hand is 1.00 mm thick and is kept at $T_{\text{S}} = 25.0$ °C. Let $L_{\text{S}} = 1.00$ mm be the width of the steel, and $L_{\text{Cu}} = 2.00$ mm be the width of the copper.



The power of heat transfer is given by $P = Q/t = kA(\Delta T/L)$. At the interface, the power must be the same for both surfaces. Thus, $P = k_{\text{Cu}}A(T_{\text{Cu}} - T_{\text{int}})/L_{\text{Cu}} = k_{\text{S}}A(T_{\text{int}} - T_{\text{S}})/L_{\text{S}}$. Solving for the interface temperature, T_{int} gives:

$$\begin{aligned} T_{\text{int}} &= \left(k_{\text{Cu}}L_{\text{S}}T_{\text{Cu}} + k_{\text{S}}L_{\text{Cu}}T_{\text{S}}\right)/\left(k_{\text{S}}L_{\text{Cu}} + k_{\text{Cu}}L_{\text{S}}\right) \\ &= \frac{(386\text{ W/m K})(0.00100\text{ m})(100.0\text{ °C}) + (220.\text{ W/m K})(0.00200\text{ m})(25.0\text{ °C})}{(386\text{ W/m K})(0.00100\text{ m}) + (220.\text{ W/m K})(0.00200\text{ m})} \\ &= 60.048\text{ °C} \approx 60.0\text{ °C} \end{aligned}$$

(b) The result of part (a) can be used to calculate the power, $P = k_{\text{Cu}}A\left((T_{\text{Cu}} - T_{\text{int}})/L_{\text{Cu}}\right)$.

$$P = (386\text{ W/mK})(1.00\text{ m}^2)\left[(100.\text{ °C} - 60.048\text{ °C})/(0.00200\text{ m})\right] = 7710736\text{ W} \approx 7.71 \cdot 10^6\text{ W}$$

18.51. The question asks for the surface temperature of the Sun. This temperature can be determined by equating the power of a black body to the power reaching the Earth. $P = \sigma\varepsilon A_{\text{s}}T^4 = \int I d\text{A} = |I| A_{\text{Earth's orbit}}$. The sun is modeled as a black body, so $\varepsilon = 1$. The area of a sphere is $4\pi r^2$. Using these facts $\sigma 4\pi r_{\text{s}}^2 T^4 = I 4\pi r_{\text{ES}}^2$.

Solving for the temperature gives $T=\left(Ir_{\text{ES}}^2/\sigma r_{\text{s}}^2\right)^{1/4}$. Inputting the given values yields: $T=\left[\left(1370.\text{ W/m}^2\right)\left(1.496\cdot 10^8\text{ km}\right)^2/\left(5.67\cdot 10^{-8}\text{ W/K}^4\text{m}^2\right)\left(6.963\cdot 10^5\text{ km}\right)^2\right]^{1/4}=5778.99\text{ K}$ or 5506 °C.
The surface temperature of the Sun is 5780 K, or 5510 °C.

18.52. **THINK:** What is the equilibrium temperature of black and shiny engines? Then engine generates $P=11$ kW and has an area of $\text{A}=0.50\text{ m}^2$ and a temperature of $T_0=27$ °C or $300.$ K. The emissivity of the shiny engine is $\varepsilon_{\text{s}}=0.050$ and the black engine has $\varepsilon_{\text{b}}=0.95$.
SKETCH: A sketch is not needed to solve the problem.
RESEARCH: The power of the radiation is $P=\sigma\varepsilon AT^4$.
SIMPLIFY: If the engines can only dissipate heat via thermal radiation, the power is given by $P=\sigma\varepsilon\text{A}\left(T^4-T_0^4\right)$ or $P/\sigma\varepsilon A=\left(T^4-T_0^4\right)$. Solving for T gives $T=\left(P/\sigma\varepsilon A+T_0^4\right)^{1/4}$.
CALCULATE: The shiny engine has a equilibrium temperature of:

$$T=\left[\frac{\left(11\cdot 10^3\text{ W}\right)}{\left(5.67\cdot 10^{-8}\text{ W/K}^4\text{ m}^2\right)\left(0.050\right)\left(0.50\text{ m}^2\right)}+\left(300.\text{ K}\right)^4\right]^{1/4}=1669.478\text{ K}.$$

The black engine's temperature is:

$$T=\left[\frac{\left(11\cdot 10^3\text{ W}\right)}{\left(5.67\cdot 10^{-8}\text{ W/K}^4\text{ m}^2\right)\left(0.50\text{ m}^2\right)\left(0.95\right)}+\left(300.\text{ K}\right)^4\right]^{1/4}=803.362\text{ K}.$$

ROUND: The values will be reported to 2 significant figures. The shiny and black engine have temperature of $1400°$ C and $530°$ C respectively.
DOUBLE-CHECK: These temperatures are very high, which is expected. Other sources of heat dissipation will cool these engines further.

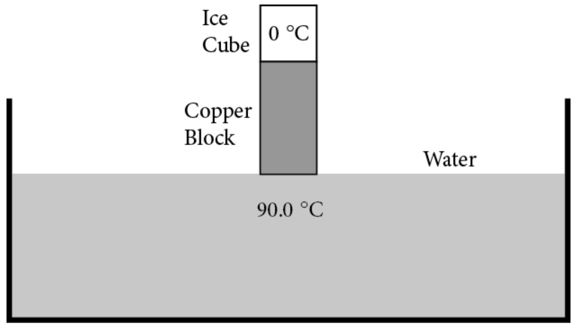
18.53. **THINK:** I want to know how long it will take to freeze the Popsicle. The values are not given in SI units. The volume of the juice is 8.00 oz $=$ $8.00\text{ oz}\left(0.0295735296\text{ L/oz}\right)=0.236588\text{ L}$. The temperature of the juice is $T_{\text{j}}=71.0$ °F or $\left(5/9\right)\left(71.0\text{ °F}-32\right)=21.667$ °C. The cooling power is: $P=4000\text{ BTU/hr}\left(\left(1055.06\text{ J/BTU}\right)/\left(3600\text{ s/hr}\right)\right)=1172.3\text{ W}$. Assume that the juice is similar to water.
SKETCH: A sketch is not needed to solve the problem.
RESEARCH: The power is given by $P=Q/t$. The heat is given by $Q=mc\Delta T+mL_{\text{fusion}}$. The mass is equal to the volume.
SIMPLIFY: The time is given by $t=Q/P$. The juice must first reach 0 °C before it can freeze. $t=\left[mc\left(T_{\text{j}}-T_{\text{f}}\right)+mL_{\text{fusion}}\right]/P=m\left[\left(c\left(T_{\text{j}}-T_{\text{f}}\right)+L_{\text{fusion}}\right)/P\right]$. The mass is given by the density times volume $t=\rho V\left[\left(c\left(T_{\text{j}}-T_{\text{f}}\right)+L\right)/P\right]$.
CALCULATE:

$$t=\left(1.00\text{ kg/L}\right)\left(0.236588\text{ L}\right)\left(\frac{\left(4190\text{ J/kg K}\right)\left(21.667\text{ °C}-0\text{ °C}\right)+334\cdot 10^3\text{ J/kg}}{1172.3\text{ W}}\right)=85.728\text{ s}.$$

ROUND: The time is reported to 3 significant figures. It will take the juice 85.7 s to freeze.
DOUBLE-CHECK: This is an unusually brief time to freeze an 8.00 oz Popsicle. The reasoning for this short time may be part of a classroom discussion: How efficient are freezers?

18.54. **THINK:** How long does it take for the ice to melt? The copper rod will transmit heat from the 90.0 °C water to the ice cube, melting it. The ice cube has dimensions $s=10.0$ cm, with a density of $\rho=0.917\text{ g/cm}^3=917\text{ kg/m}^3$ and its temperature is $T_{\text{i}}=0$ °C, its melting point. The area of the copper rod had a cross section of 10.0 cm and has a length of $\ell = 20.0$ cm. The other end of the copper rod is at $T_{\text{w}} = 90.0$ °C.

SKETCH:



RESEARCH: The heat required to melt the ice cube is $Q = mL_{\text{fusion}}$ or $\rho V L_{\text{fusion}}$. The rate of heat transfer of the rod is $P = kA\left(T_{\text{W}} - T_{\text{I}}\right)/\ell = Q/t$.

SIMPLIFY: Solving for the time gives $t = \dfrac{Q\ell}{kA\left(T_{\text{w}} - T_{\text{i}}\right)} = \dfrac{mL_{\text{fusion}}\ell}{kA\left(T_{\text{w}} - T_{\text{i}}\right)} = \dfrac{\rho V L_{\text{fusion}}\ell}{kA\left(T_{\text{w}} - T_{\text{i}}\right)}$.

CALCULATE: The time to melt the ice cube is:

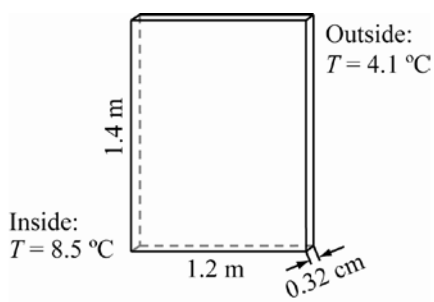
$$t = \frac{\left(917 \text{ kg/m}^3\right)\left(1.00 \cdot 10^{-3} \text{ m}^3\right)\left(334 \cdot 10^3 \text{ J/kg}\right)\left(0.200 \text{ m}\right)}{\left(390. \text{ W/m K}\right)\left(0.100 \text{ m}\right)^2\left(363 \text{ K} - 273 \text{ K}\right)} = 174.52 \text{ s}.$$

ROUND: The values are reported to three significant figures. It takes 175 seconds to melt the ice.

DOUBLE-CHECK: Because the copper rod is immersed in a large pool of water, it has an effectively infinite heat reservoir to draw on, so it will maintain its temperature of 90.0 °C.

18.55. **THINK:** I want to find the rate of heat flow through a window, which is 0.32 cm thick, and has an area of 1.2 m by 1.4 m. The inside and outside temperatures of the window are 8.5 °C and 4.1 °C, respectively.

SKETCH:



RESEARCH: The power is $P = kA\left(T_{\text{h}} - T_{\text{c}}\right)/L$.

SIMPLIFY: Not required.

CALCULATE: $P = \dfrac{\left(0.8 \text{ W/m K}\right)\left(1.2 \text{ m}\right)\left(1.4 \text{ m}\right)\left(281.5 \text{ K} - 277.1 \text{ K}\right)}{0.32 \cdot 10^{-2} \text{ m}} = 1848 \text{ W} = 1.848 \text{ kW}$

ROUND: The heat loss rate is reported to 2 significant figures, so $P = 1.8$ kW.

DOUBLE-CHECK: This is a fairly large amount of power being transmitted through the window, which would lead to significant heating costs. To reduce this power loss, double-pane windows with inert gas between the two panes are used rather than a single-pane window as in this example.

18.56. **THINK:** The rate of heat loss due to radiation and the boil-off rate are desired. The temperatures are $T_{\text{h}} = 3.00 \cdot 10^2$ K and $T_{\text{c}} = 4.22$ K. The area is $A = 0.500 \text{ m}^2$. The latent heat of vaporization of liquid helium is $L_{\text{v}} = 20.9$ kJ/kg, and its density is $\rho = 0.125$ kg/L.

SKETCH: A sketch is not needed to solve the problem.

RESEARCH: The heat loss due to radiation is given by $P = \sigma\varepsilon AT^4$, where $\varepsilon = 1$, since the dewar is treated as a black body. The net rate of heat transfer is the difference between the heat radiated by the black-body dewar, and the heat absorbed by it. The heat radiated is given by $P_{\text{rad}} = \sigma\varepsilon A{T_D}^4$, and the heat absorbed is given by $P_{\text{abs}} = \sigma\varepsilon A{T_H}^4$. The boil-off rate will be given by the equation $P = Q/t = dQ/dt$ $= dmL_{\text{vap}}/dt = (dm/dt)L_{\text{vap}}$. The volume is related to the mass by $m = \rho \text{V}$.

SIMPLIFY:

(a) $P = \sigma\varepsilon AT_D^4 - \sigma\varepsilon AT_H^4 = \sigma\varepsilon A\left(T_{\text{h}}^4 - T_{\text{c}}^4\right)$

(b) The boil-off rate is $P = (dm/dt)L = (d\rho V/dt)L = (dV/dt)\rho L$. Solving for the rate of volume boil-off, dV/dt, gives $dV/dt = P/\rho L.$

CALCULATE:

(a) $P = \left(5.67\cdot 10^{-8}\text{ W/K}^4\text{m}^2\right)(1)\left(0.500\text{ m}^2\right)\left[(300.\text{ K})^4 - (4.22\text{ K})^4\right] = 229.635\text{ W}$

(b) $dV/dt = \dfrac{229.635\text{ W}}{(0.125\text{ kg/L})(20.9\text{ kJ/kg})} = 0.087899\text{ L/s}$

ROUND: The values will be reported to 3 significant figures.

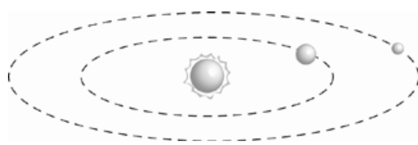
(a) The rate of heat loss due to radiation is 0.230 kW.

(b) The volume boil-off rate of the dewar is 0.0879 L/s.

DOUBLE-CHECK: The dewar loses heat at a rapid rate. This heat is transferred to the liquid helium contained in the dewar, causing a rapid boil-off rate. It is for this reason that cryogenic dewars are manufactured of reflective materials.

18.57. **THINK:** The solar irradiance on the surface of Mars and its temperature are desired. Mars is 1.52 times farther from the Sun than the Earth. Mars has a diameter of 0.532 times that of Earth.

SKETCH:



RESEARCH: The solar irradiance on the surface of Earth is approximately 1400 W/m^2. The average surface temperature on Earth is approximately 288 K. The intensity of the light reaching the planets is given by the equation $P = \int |I|dA = |I|4\pi r^2$. The temperature can be found using the rate of heat loss for radiation $P = Q/t = \sigma\varepsilon AT^4$.

SIMPLIFY: To find the irradiance note that the power of the Sun is equal for both Earth and Mars: $I_{\text{M}}4\pi r_{\text{MS}}^2 = I_{\text{E}}4\pi r_{\text{ES}}^2$ or $I_{\text{M}} = I_{\text{E}}r_{\text{ES}}^2/r_{\text{MS}}^2$. Using this irradiance, the temperature of Mars can be calculated assuming it is a blackbody, $I_{\text{M}}\pi r_{\text{M}}^2 = \sigma\varepsilon 4\pi r_{\text{M}}^2 T_{\text{M}}^4$. Solving for the temperature of the surface of Mars: $T_{\text{M}} = \sqrt[4]{\dfrac{I_{\text{M}}}{4\sigma}}$.

CALCULATE: $I_{\text{M}} = \left(1368\text{ W/m}^2\right)\left(r_{\text{ES}}^2\right)/\left(1.52r_{\text{ES}}\right)^2 = \left(1368\text{ W/m}^2\right)/(1.52)^2 = 592.105\text{ W/m}^2$

$$T_{\text{M}} = \sqrt[4]{\frac{592.105\text{ W/m}^2}{4\left(5.67\cdot 10^{-8}\text{ W/K}^4\text{m}^2\right)}} = 226.04\text{ K}.$$

ROUND: The values are given to 3 significant figures.

(a) The solar irradiance is 592 W/m^2 at the surface of Mars.

(b) The temperature on the surface of Mars is 226 K.

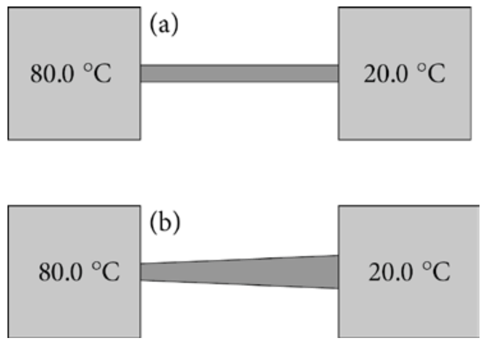
DOUBLE-CHECK: The published value for the solar irradiance on the surface of Mars is 590 W/m^2. The temperature on the surface of Mars is around 210 K. Our answers are reasonable.

18.58. THINK:

(a) The question asks for the rate of heat flow if the copper bar has a length of $L = 2.00$ m, a square cross section with sides $s = 0.100$ m and is bordered by reservoirs 80.0 °C and 20.0 °C.

(b) If the bar from part (a) has an area that varies as $A = \left(0.0100 \text{ m}^2\right)\left(1 + x/\left(2.00 \text{ m}\right)\right)$ where x is the distance along the bar from the warm end to the cold end, what is the rate of heat flow? What is the rate of change of the temperature with distance at hot end, the middle and the cool end?

SKETCH:



RESEARCH:

(a) The rate is given by $P = Q/t = kA\left(T_\text{h} - T_\text{c}\right)/L.$

(b) The heat flow through the bar will be the same everywhere in the bar independent of the area of the bar and is a constant P. The rate of change of temperature will depend on the distance. We start by generalizing equation 18.15 to state

$$P = -kA(x)\frac{dT}{dx},$$

where $A(x)$ is the cross sectional area of the bar at position x. We can rewrite this equation as

$$Pdx = -kA(x)dT \;\;\Rightarrow\;\; P\frac{dx}{A(x)} = -kdT.$$

We can integrate between $x = x_1$ and $x = x_2$ for the left side of the equation and between $T = T_1$ and $T = T_2$ on the right side of the equation to obtain

$$P\int_{x_1}^{x_2} \frac{dx}{A(x)} = -k\int_{T_1}^{T_2} dT.$$

In this case, the area as a function of distance is given by $A(x) = a\left(1 + \frac{x}{b}\right)$ where $a = 0.0100 \text{ m}^2$ and $b = 2.00$ m.

SIMPLIFY:

(a) The area of the bar is $A = s^2$ so $P = \frac{Q}{t} = \frac{ks^2\left(T_\text{h} - T_\text{c}\right)}{L}.$

(b) Inserting our definition of $A(x)$ into the integral gives us

$$P\int_{x_1}^{x_2} \frac{dx}{a\left(1 + \frac{x}{b}\right)} = -k\int_{T_1}^{T_2} dT.$$

Carrying out the integrals we get

$$\frac{Pb}{a}\left(\ln(b+x)\right)\Big|_{x_1}^{x_2} = -kT\Big|_{T_1}^{T_2} \quad \Rightarrow \quad \frac{Pb}{a}\left(\ln(b+x_2)-\ln(b+x_1)\right) = -k(T_2-T_1).$$

We can determine P

$$P = \frac{ka}{b}\frac{(T_1-T_2)}{\ln(b+x_2)-\ln(b+x_1)}.$$

The rate of change of the temperature is

$$\frac{dT}{dx} = -\frac{P}{kA(x)} = -\frac{P}{ka\left(1+\frac{x}{b}\right)}.$$

CALCULATE:

(a) $P = (390.\text{ W / m K})(0.100\text{ m})^2\left[(80.0\text{ °C}-20.0\text{ °C})/2.00\text{ m}\right] = 117\text{ W}$

(b) The heat flow is

$$P = \frac{(390\text{ W/(m K)})(0.0100\text{ m}^2)}{2.00\text{ m}}\frac{(80.0\text{ °C}-20.0\text{ °C})}{\ln(2.00\text{ m}+2.00\text{ m})-\ln(2.00\text{ m})} = 168.795\text{ W}.$$

The rate of change of the temperature at $x=0$ is

$$\frac{dT}{dx} = -\frac{168.795\text{ W}}{(390\text{ W/(m K)})(0.0100\text{ m}^2)(1+0)} = -43.281\text{ K/m}.$$

The rate of change of the temperature at $x=L$ is

$$\frac{dT}{dx} = -\frac{168.795\text{ W}}{(390\text{ W/(m K)})(0.0100\text{ m}^2)\left(1+\frac{2.00\text{ m}}{2.00\text{ m}}\right)} = -21.640\text{ K/m}.$$

The rate of change of the temperature at $x=L/2$ is

$$\frac{dT}{dx} = -\frac{168.795\text{ W}}{(390\text{ W/(m K)})(0.010\text{ m}^2)\left(1+\frac{1.00\text{ m}}{2.00\text{ m}}\right)} = -28.853\text{ K/m}.$$

ROUND:

(a) The rate is reported to 3 significant figures. The rate of heat transfer is $P = 117\text{ W}$.

(b) The values are reported to 3 significant figures. The rate of heat flow is the same throughout the bar and is 169 W. At the warm end, the middle and the cool end the rate of temperature change with distance is -43.3 °C/m, -21.6 °C/m, and -28.8 °C/m, respectively.

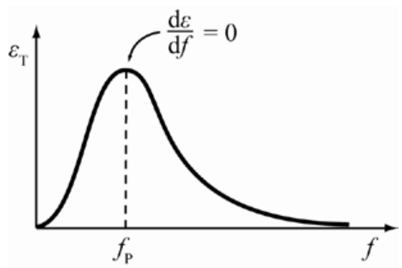
DOUBLE-CHECK:

(a) This value of a reasonable order of magnitude.

(b) The rate of heat flow is higher for the larger bar, as expected. The change in temperature per unit length is larger for the smaller cross sectional area. That makes sense because the rate of heat transfer is constant everywhere in the bar.

18.59. **THINK:** The Planck spectrum distribution is given by $\varepsilon_{\text{T}}(f) = \left(2\pi h/c^2\right)\left(f^3/(e^{hf/k_{\text{B}}T}-1)\right)$, where h is Planck's constant and c is the speed of light. The frequency of the peak of this distribution is needed. The Boltzmann constant is $k_{\text{B}} = 1.38\cdot 10^{-23}\text{ m}^2\text{ kg s}^{-2}\text{ K}^{-1}$.

SKETCH:



RESEARCH: The frequency of the peak of the Planck distribution is determined from solving $d\varepsilon / df = 0$. The derivative of the Planck distribution is given by:

$$\frac{d\varepsilon}{df} = \left(\frac{2\pi h}{c^2}\right)\left(3f^2\left(e^{hf/k_BT}-1\right)^{-1} - f^3\left(\frac{h}{k_BT}\right)\left(e^{hf/k_BT}-1\right)^{-2}\right).$$

SIMPLIFY: Solving $d\varepsilon / df = 0$, it is found that:

$$3f^2\left(e^{hf/k_BT}-1\right)^{-1} - f^3\left(h/k_BT\right)\left(e^{hf/k_BT}-1\right)^{-2} = 0 \Rightarrow f^2\left(e^{hf/k_BT}-1\right)^{-2}\left[3\left(e^{hf/k_BT}-1\right)-\left(hf/k_BT\right)e^{hf/k_BT}\right]=0.$$

This leads to $3e^x - 3 - xe^x = 0$, where $x = hf/k_BT$. Simplifying yields $3 - x = 3e^{-x}$ or $x = 3 - 3e^{-x} = 3\left(1-e^{-x}\right)$. Solving x iteratively with a starting value $x_0 = 3$, it is found that $x_0 = 3$, $x_1 = 3\left(1-e^{-x_0}\right) = 3\left(1-e^{-3}\right) = 2.8506$, $x_2 = 3\left(1-e^{-x_1}\right) = 3\left(1-e^{-2.8506}\right) = 2.8266$ and after few iterations, the most is $x = 2.8215$. This means that $hf/k_BT = 2.8215$ or $f = 2.8215\left(k_B/h\right)T$.

CALCULATE:

(a) Substituting $k_B = 1.38\cdot 10^{-23}$ m^2 kg s^{-2} K^{-1} and $h = 6.626\cdot 10^{-34}$ J s gives $f = \left(5.8792\cdot 10^{10} \text{ Hz/K}\right)T$.

(b) At $T = 6.00\cdot 10^3$ K, the frequency of the peak is:

$$f = \left(5.8792\cdot 10^{10}\text{ Hz/K}\right)\left(6.00\cdot 10^3\text{ K}\right) = 3.5275\cdot 10^{14}\text{ Hz}.$$

(c) At $T = 2.735$ K, the frequency of the peak is:

$$f = \left(5.8792\cdot 10^{10}\text{ Hz/K}\right)\left(2.735\text{ K}\right) = 1.60796\cdot 10^{11}\text{ Hz}.$$

(d) At $T = 300.$ K, the frequency of the peak is:

$$f = \left(5.8792\cdot 10^{10}\text{ Hz/K}\right)\left(300.\text{ K}\right) = 1.7638\cdot 10^{13}\text{ Hz} \approx 1.76\cdot 10^{13}\text{ Hz}.$$

ROUND:

(a) $f = \left(5.88\cdot 10^{10}\text{ Hz/K}\right)T$.

(b) The temperature of the Sun is given to three significant figures in the question: $f = 3.53\cdot 10^{14}$ Hz.

(c) Boltzmann's constant is given to three significant figures: $f = 1.61\cdot 10^{11}$ Hz. (d) The Earth's temperature is given only to three significant figures in the question: $f = 1.76\cdot 10^{13}$ Hz..

DOUBLE-CHECK: The result in (a), where $f = \text{constant}(T)$ is known as Wien's displacement law. The rest of the calculated frequencies have Hertz as their units, which is appropriate. As one might expect, the frequencies increase with the temperatures.

18.60. Energy required to raise the temperature of an object by ΔT is given by $Q = mc\Delta T$. Substituting the specific heat of aluminum, $c = 900.$ J/(kg K), the mass $m = 0.300$ kg and

$\Delta T = \left(\left(100.0+273\right)-\left(20.0+273\right)\right)$ K $= 80.0$ K , hence,

$Q = \left(900.\text{ J/kg K}\right)\left(0.300\text{ kg}\right)\left(80.0\text{ K}\right) = 21600\text{ J} = 21.6\text{ kJ}.$

18.61. The R value of an object is equal to $R = L/k$, where L is the thickness of the object and k is its thermal conductivity. Inserting the thermal conductivity of fiberglass batting $k = 8.00 \cdot 10^{-6}$ BTU/(ft °F s) and its thickness $L = 4.00$ in or $L = 0.333$ ft yields:

$$R = \left(\frac{0.3333 \text{ ft}}{8.00 \cdot 10^{-6} \text{ BTU/}\left(\text{ft °F s}\right)} \right) \left(\frac{1 \text{ hr}}{3600. \text{ s}} \right) = 11.574 \text{ ft}^2 \text{ °F hr/BTU} \approx 11.6 \text{ ft}^2 \text{ °F hr/BTU}.$$

18.62. The amount of heat required to change the temperature of 10.0 kg of water by 10.0 K is $Q = cm\Delta T = (4190 \text{ J/kg K})(10.0 \text{ kg})(10.0 \text{ K}) = 4.19 \cdot 10^5$ J. The kinetic energy of a car with $m = 1.00 \cdot 10^3$ kg and a speed of 27.0 m/s is:

$$K = (1/2)mv^2 = (1/2)\left(1.00 \cdot 10^3 \text{ kg}\right)(27.0 \text{ m/s})^2 = 3.645 \cdot 10^5 \text{ J} \approx 3.65 \cdot 10^5 \text{ J}.$$

It should be noted that Q and K are about the same.

18.63. The conduction rate through a spherical glass is given by $P_{\text{cond}} = kA(\Delta T / L)$ where ΔT is the temperature difference, k is the thermal conductivity of the glass and A and L are the area and thickness of the glass. Simplifying gives $\Delta T = P_{\text{cond}} L / kA$. Using the area of sphere, it is found that $\Delta T = P_{\text{cond}} L / \left(k 4\pi r^2\right)$. Substituting $P_{\text{cond}} = 0.95(100.0 \text{ W}) = 95$ W, $L = 0.50 \cdot 10^{-3}$ m, $r = 3.0 \cdot 10^{-2}$ m and $k = 0.80$ W/(m K) gives

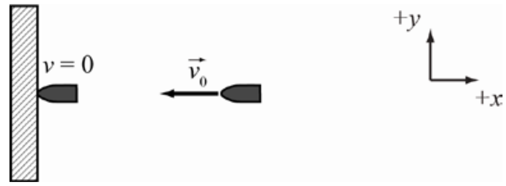
$$\Delta T = \frac{(95 \text{ W})\left(0.50 \cdot 10^{-3} \text{ m}\right)}{(0.80 \text{ W/mK})(4\pi)\left(3.0 \cdot 10^{-2} \text{ m}\right)^2} = 5.2 \text{ K}.$$

18.64. "Calories" here refers to food calories. The amount of heat transferred to the soft drink is $Q = cm\Delta T$. Inserting $c = 4190$ J/(kg K), $m = 0.355$ kg and $\Delta T = (37.0 - 10.0)$ K $= 27.0$ K yields the expression $Q = (4190 \text{ J/(kg K)})(0.355 \text{ kg})(27.0 \text{ K}) = 40.16 \cdot 10^3 \text{ J} = 9594$ cal. Recall that the energy content of food given in calories on the label is actually in food calories. The net energy content is $E = E_0 = 150.$ kcal $+ 9594$ cal $= 159{,}594$ cal. Rounding the value of E to 3 significant figures gives $E = 1.60 \cdot 10^5$ food calories.

18.65. The conduction rate of the skin is given by $P_{\text{cond}} = kA(\Delta T / L)$. After rearrangement, $k = P_{\text{cond}} L / A\Delta T$. Substituting $L = 3.00 \cdot 10^{-3}$ m, $P_{\text{cond}} = 100.$ W, $A = 1.50 \text{ m}^2$ and $\Delta T = (37.0 - 27.0)$ K $= 10.0$ K gives the thermal conductivity, $k = \dfrac{(100. \text{ W})\left(3.00 \cdot 10^{-3} \text{ m}\right)}{\left(1.50 \text{ m}^2\right)(10.0 \text{ K})} = 2.00 \cdot 10^{-2}$ W/(m K).

18.66. **THINK:** A lead bullet is fired at a wall. It is assumed that the bullet receives 75.0% of the work done on it by the wall as it stops. It is assumed the bullet has an initial temperature of 293 K (room temperature). Before the bullet starts to melt, the bullet's temperature needs to be increased to the melting point which is 601 K. The heat of fusion for lead is $L = 23.2$ kJ/kg, and the specific heat of lead is $c_{\text{Pb}} = 0.129$ kJ/(kg K).

SKETCH:



RESEARCH: The amount of energy absorbed by the bullet is given by the work done by the wall times 0.75. Therefore, $Q_{\text{absorbed}} = (0.750)(\Delta K) = (0.750)(1/2)(mv_0^2 - 0) = (0.375)(mv_0^2)$. Before the bullet begins to melt, its temperature must be raised to the melting point of 601 K. The heat needed for this is given by $Q_1 = cm(T_f - T_i)$. After the melting point has been reached, the heat needed to completely melt the bullet is $Q_2 = mL_{\text{fusion}}$.

SIMPLIFY:

(a) Thus, the minimum speed needed to start melting the bullet is found by equating the heat absorbed with the heat required to raise the temperature of the bullet Q_1. $Q_{\text{absorbed}} = Q_1 \Rightarrow (0.375)(mv_0^2) = cm(T_f - T_i)$, which implies that $v_0 = \sqrt{c(T_f - T_i)/0.375}$.

(b) In order to completely melt the bullet, the heat absorbed must be equal to the heat required to raise the temperature and to melt the bullet, that is $Q_{\text{absorbed}} = Q_1 + Q_2 \Rightarrow (0.375)(mv_0^2) = cm(T_f - T_i) + mL_{\text{fusion}}$.

Thus, the speed required is $v_0 = \sqrt{\left(c(T_f - T_i) + L_{\text{fusion}}\right)/0.375}$.

CALCULATE:

(a) $v_0 = \sqrt{129 \text{ J/(kg K)}(601 \text{ K} - 293 \text{ K})/0.375} = 325.5 \text{ m/s}$

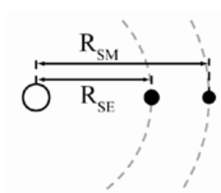
(b) $v_0 = \sqrt{\left[129 \text{ J/(kg K)}(601 \text{ K} - 293 \text{ K}) + 23200 \text{ J/kg}\right]/0.375} = 409.7 \text{ m/s}$

ROUND: Rounding to three significant figures, (a) $v_0 = 326$ m/s and (b) $v_0 = 410.$ m/s.

DOUBLE-CHECK: The results in (a) and (b) are typical speeds of bullets.

18.67. **THINK:** Solar radiation reaches the Earth's surface at about 1.4 kW/m^2. It is assumed here that the Earth is a black body. This means that there is no reflection due to Earth's atmosphere, and all solar radiation that reaches the Earth is absorbed by the Earth's surface.

SKETCH:



RESEARCH: The Sun emits radiation uniformly in all directions. This means that if the total power of solar radiation is P, then the intensity of radiation at a distance r from the Sun is distributed uniformly over a spherical surface area. Thus, $I = \dfrac{P}{\text{Spherical Area}} = \dfrac{P}{4\pi r^2}$. Therefore, the intensity of radiation that reaches the Earth is $I_E = P/4\pi r_{SE}^2$. Similarly for Mars, the intensity that reaches Mars is $I_M = P/4\pi r_{SM}^2$.

SIMPLIFY: Since I_E is known, the power of radiation P can be eliminated from the above equations giving the intensity on Mars as $I_M = \left(4\pi r_{SE}^2 / 4\pi r_{SM}^2\right) I_E = \left(r_{SE}^2 / r_{SM}^2\right) I_E$.

CALCULATE: Substituting $I_E = 1.4 \text{ kW/m}^2$, $R_{SE} = 1.496 \cdot 10^{11}$ m and $R_M = 2.28 \cdot 10^{11}$ m yields:

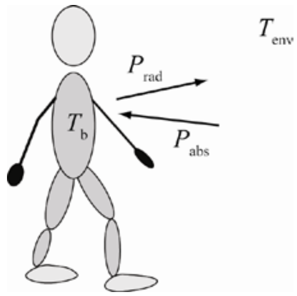
$$I_M = \left(\frac{1.496 \cdot 10^{11} \text{ m}}{2.28 \cdot 10^{11} \text{ m}}\right)^2 1.4 \cdot 10^3 \text{ W/m}^2 = 6.03 \cdot 10^2 \text{ W/m}^2.$$

ROUND: Keeping only two significant figures gives $I_M = 6.0 \cdot 10^2 \text{ W/m}^2$.

DOUBLE-CHECK: Since R_{SM} is larger than R_{SE} it is expected that I_M is less than I_E.

18.68. **THINK:** The radiated power of a body depends on its surface area, temperature and emissivity. Assume that your body has an emissivity of $\varepsilon = 1.00$, that is, that you are a black body.

SKETCH:



RESEARCH: The radiated power of an object is given by $P = \sigma\varepsilon A T_b^4$. The power radiated by my body is $P_{rad} = \sigma\varepsilon A T_b^4$. If the surrounding environment has a temperature T_{env}, the power absorbed by my body is $P_{abs} = \sigma\varepsilon A T_{env}^4$. Since my body radiates energy to the environment and at the same time absorbs energy, the net power is $P_{net} = P_{rad} + P_{abs}$.

SIMPLIFY: $P_{net} = \sigma\varepsilon A\left(T_b^4 - T_{env}^4\right)$

CALCULATE:

(a) The radiated power from my body is:

$$P_{rod} = \sigma\varepsilon A T_b^4 = \left(5.6703\cdot 10^{-8}\text{ W/m}^2\text{ K}^4\right)(1.00)\left(2.00\text{ m}^2\right)\left((273+33.0)\text{ K}\right)^4 = 994.3\text{ W}.$$

(b) The net body radiated power is:

$$P_{net} = \left(5.6703\cdot 10^{-8}\text{ W/m}^2\text{ K}^4\right)(1.00)\left(2.00\text{ m}^2\right)\left[\left((273+33.0)^4 - (273+20.0)^4\right)(\text{K})^4\right] = 158.5\text{ W}.$$

(c) The net body radiated power is:

$$P_{net} = \left(5.6703\cdot 10^{-8}\text{ W/m}^2\text{ K}^4\right)(1.00)\left(2.00\text{ m}^2\right)\left[\left((273+27.0)^4 - (273+20.0)^4\right)(\text{K})^4\right] = 82.8\text{ W}.$$

ROUND: The final answers should be rounded to three significant figures.

(a) $P_{rod} = 994\text{ W}$

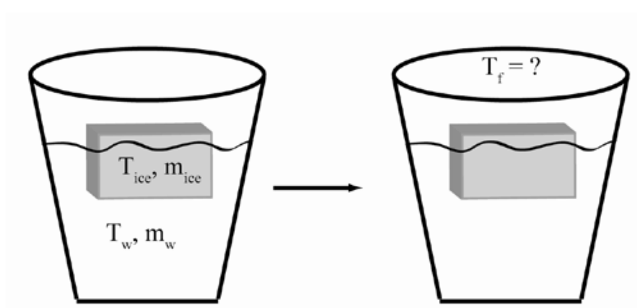
(b) $P_{net} = 159\text{ W}$

(c) $P_{net} = 82.8\text{ W}$

DOUBLE-CHECK: The answers are reasonable. It is expected that the body radiates less heat when its surface temperature is lower.

18.69. **THINK:** A 10.0 g ice cube has an initial temperature of -10.0 °C. The ice cube is dropped into 40.0 g of water at 30.0 °C. The equilibrium temperature of the water must be calculated. Then, a second similar ice cube is added. The new equilibrium temperature is required.

SKETCH:



RESEARCH: It is noted that the ice temperature is below the freezing point. Therefore, any heat transferred to the ice initially can only increase the temperature of the ice. The heat Q_1 required to raise

the temperature of the ice is $Q_1 = c_I m_I \Delta T_1 = c_I m_I (0\text{ °C} - T_I)$. After the temperature of the ice reaches 0 °C, the heat Q_2 needed to melt the ice is $Q_2 = L_F m_I$. Then, after all ice has melted, the heat Q_3 needed to increase the temperature by ΔT_2 is $Q_3 = c_W m_I \Delta T_2 = c_W m_I (T_f - 0\text{ °C})$. The amount of heat Q_4 transferred from the water to the ice is $Q_4 = c_W m_W \Delta T_3 = c_W m_W (T_f - T_W)$. Since the system is isolated, the net heat must be zero. $Q_1 + Q_2 + Q_3 + Q_4 = 0$. When a second ice cube is added to the system, the process must be repeated with the new starting temperature, and taking into account the added mass of the first ice cube.
SIMPLIFY: Adding all heats, it is found that:

$$c_I m_I (0\text{°C} - T_I) + L_F m_I + c_W m_I (T_f - 0\text{ °C}) + c_W m_W (T_f - T_W) = 0.$$

Solving for T_f gives:

$$T_f = \frac{c_W m_W T_W + c_I m_I T_I - L_F m_I}{c_W (m_W + m_I)}.$$

CALCULATE:
(a) The final temperature of the water is:

$$T_f = \frac{\left(4190\ \frac{\text{J}}{\text{kg °C}}\right)(0.0400\text{ kg})(30.0\text{ °C}) + \left(2060\ \frac{\text{J}}{\text{kg °C}}\right)(0.0100\text{ kg})(-10.0\text{ °C}) - \left(334\cdot 10^3\ \frac{\text{J}}{\text{kg}}\right)(0.0100\text{ kg})}{(4190\text{ J/kg °C})(0.0400\text{ kg} + 0.0100\text{ kg})}$$
$$= 7.074\text{ °C}.$$

(b) The final temperature using $m_w = 50.0\text{ g}$ and $T_w = 7.074\text{ °C}$ is:

$$T_f = \frac{\left(4190\frac{\text{J}}{\text{kg °C}}\right)(0.0500\text{ kg})(7.074\text{ °C}) - \left(2060\frac{\text{J}}{\text{kg °C}}\right)(0.0100\text{ kg})(10.0\text{ °C}) - \left(334\cdot 10^3\ \frac{\text{J}}{\text{kg}}\right)(0.0100\text{ kg})}{\left(4190\text{ J/}(\text{kg °C})\right)(0.0500\text{ kg} + 0.0100\text{ kg})}$$

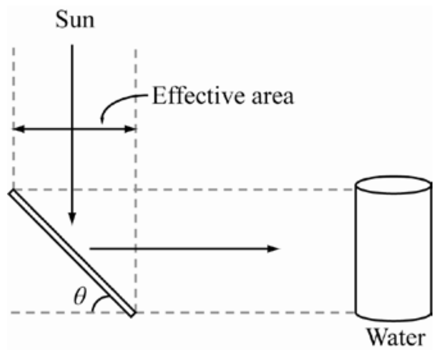
$T_f = -8.210\text{ °C}$.
Since T_f is negative, this means that the ice has not melted completely or has not reached the melting point. Let us calculate the value of Q_1, Q_2 and Q_4 assuming $T_f = 0$ in order to determine the state of the ice. The heat is:

$$Q_1 = \left(2060\text{ J/}(\text{kg °C})\right)(0.0200\text{ kg})\left(0\text{ °C} - (-10\text{ °C})\right) = 412\text{ J}$$
$$Q_2 = \left(334\cdot 10^3\text{ J/kg}\right)(0.0200\text{ kg}) = 6680\text{ J}$$
$$Q_4 = \left(4190\text{ J/}(\text{kg °C})\right)(0.0400\text{ kg})(0\text{ °C} - 30\text{ °C}) = -5028\text{ J}.$$

Since $Q_1 + Q_2$ is larger than the magnitude of Q_4 and Q_1 is less than the magnitude of Q_4, this means that the ice has not melted completely and the final temperature is therefore, $T_f = 0\text{ °C}$.
ROUND: Rounding to three significant figures, (a) $T_f = 7.07\ °\text{ C}$. (b) $T_f = 0.00\text{ °C}$.
DOUBLE-CHECK: The largest part of the heat required for the process is that which causes the phase change from solid to liquid. Doubling the mass of the ice effectively doubles this heat requirement. It is not surprising that the system does not contain sufficient heat to complete the process.

18.70. **THINK:** The mass of water is 60.0 kg and the temperature is 35.0 °C. It is assumed the Sun gives out $1.00\cdot 10^3\text{ W/m}^2$ and the dimension of a mirror is 25.0 cm by 25.0 cm. The mirrors are held at an angle of $45.0°$. The total power received by the cylinder of water depends on the amount of power reflected by the mirrors.

SKETCH:



RESEARCH: The amount of power reflected by a mirror is given by $P_1 = I(\text{Effective Area})$, I is intensity of solar radiation. $P_1 = IA\cos\theta$. The total power reflected is $P = NP_1 = NIA\cos\theta$. For an interval of time t, the energy absorbed by the water is $E = Pt = NIA\cos\theta t$. The amount of heat to raise the temperature of the water is $Q = cm(T_f - T_i)$.

SIMPLIFY: Equating E with Q gives the time required is $t = \dfrac{cm(T_f - T_i)}{NIA\cos\theta}$.

CALCULATE: Substituting $I = 1.00\cdot 10^3$ W/m^2, $c = 4.20\cdot 10^3$ J/kg °C, $m = 60.0$ kg, $T_f = 100.$ °C, $T_i = 35.0$ °C,
$A = (0.250)(0.250)$ m^2 $= 0.0625$ m^2 and $\theta = 45.0°$ yields:

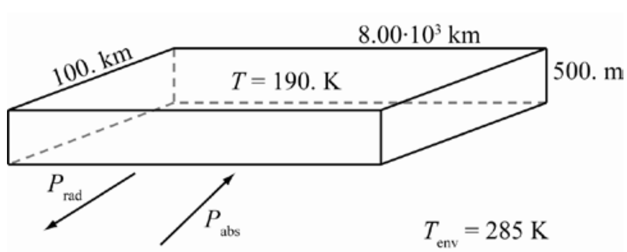
$$t = \frac{(4.20\cdot 10^3 \text{ J/kg °C})(60.0 \text{ kg})(100. \text{ °C} - 35.0 \text{ °C})}{(50000)(1.00\cdot 10^3 \text{ W/m}^2)(0.0625 \text{ m}^2)\cos(45.0°)} = 7.41 \text{ s}.$$

ROUND: Using three significant figures gives $t = 7.41$ s.

DOUBLE-CHECK: 50,000 mirrors each with an area of 0.0625 m^2 is equivalent to a single mirror with an area of more than 3 km^2. Considering the fact that all of the power reflected by this massive mirror is focused on a single point, the result is not surprising.

18.71. **THINK:** Assume the Gulf Stream is a box shaped object with a length of $8.00\cdot 10^3$ km, a width of 100. km and a depth of 500. m. Assume also that the temperature inside the box is uniform.

SKETCH:



RESEARCH: The power radiated by an object is $P = \sigma\varepsilon AT^4$. The surface area of the Gulf Stream is equal to $A = 2LW + 2Ld + 2Wd$. Here L, W and D are the length, width and depth of the Gulf Stream. The absorbed power is from the Sun, which corresponds to receiving 1400. W/m^2 for half the day on the surface of the water, $P_{abs} = (1/2)(1400. \text{ W/m}^2)(LW) = (700.0 \text{ W/m}^2)(LW)$.

SIMPLIFY: $P_{net} = P_{rad} - P_{abs} = \sigma\varepsilon AT^4 - (700.0 \text{ W/m}^2)(LW)$.

CALCULATE: To make the equation fit on the page, compute P_{rad} first without units:

$$P_{rad} = (5.6703\cdot 10^{-8})(0.930)(2)\left[(8.00\cdot 10^6)(1.00\cdot 10^5) + (8.00\cdot 10^6)(500.) + (1.00\cdot 10^5)(500.)\right](290)^4$$
$$= 5.9978\cdot 10^{14}.$$

Then, the units of P_{rad} are: $\left[P_{\text{rad}}\right]=\left(\text{W}/\left(\text{K}^4\ \text{m}^2\right)\right)\left[(\text{m})(\text{m})+(\text{m})(\text{m})+(\text{m})(\text{m})\right](\text{K})^4=\text{W}$, so altogether

$P_{\text{rad}}=5.9978\cdot 10^{14}$ W.

$P_{\text{abs}}=\left(700.\ \text{W/m}^2\right)\left[\left(8.00\cdot 10^6\ \text{m}\right)\left(1.00\cdot 10^5\ \text{m}\right)\right]$

$=5.60\cdot 10^{14}$ W.

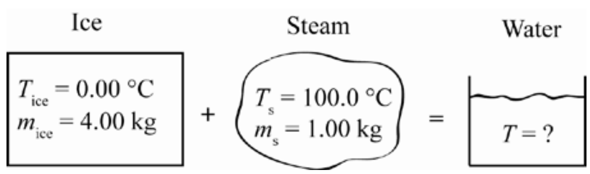
$P_{\text{net}}=5.9978\cdot 10^{14}\ \text{W}-5.60\cdot 10^{14}\ \text{W}=3.978\cdot 10^{14}$ W.

ROUND: Rounding to three significant figures gives $P_{\text{net}}=3.98\cdot 10^{13}$ W.

DOUBLE-CHECK: It is well known that the Gulf Stream is responsible for significantly warming the waters in the North Atlantic, so one would expect its radiated power to be very large.

18.72. **THINK:** 1.00 kg of steam at 100.0 °C is poured over a 4.00 kg block of ice at 0.00 °C. After the system reaches equilibrium, the final temperature is needed.

SKETCH:



RESEARCH: Since the ice is already at the freezing point, the heat needed to raise the ice to a final temperature T_{f} is equal to the heat needed to melt the ice and the heat needed to increase the temperature of the ice, that is, $Q_1=L_{\text{I}}m_{\text{I}}+c_{\text{W}}m_{\text{I}}\left(T_{\text{f}}-0\ °\text{C}\right)$. The heat released from the steam to decrease its temperature to the equilibrium temperature is equal to the heat released during condensation and the heat released when its temperature is reduced, that is, $Q_2=-L_{\text{S}}m_{\text{S}}+c_{\text{W}}m_{\text{S}}\left(T_{\text{f}}-T_{\text{S}}\right)$. Since the system is isolated, this means there is no heat entering or leaving the system. Therefore, $Q_1+Q_2=0$.

SIMPLIFY: $L_{\text{I}}m_{\text{I}}+c_{\text{W}}m_{\text{I}}\left(T_{\text{f}}-0\ °\text{C}\right)-L_{\text{S}}m_{\text{S}}+c_{\text{W}}m_{\text{S}}\left(T_{\text{f}}-T_{\text{s}}\right)=0$. Solving for T_{f} yields: $T_{\text{f}}=\left(L_{\text{S}}m_{\text{S}}+c_{\text{W}}m_{\text{S}}T_{\text{S}}-L_{\text{I}}m_{\text{I}}\right)/c_{\text{W}}\left(m_{\text{I}}+m_{\text{S}}\right)$.

CALCULATE: The final temperature is:

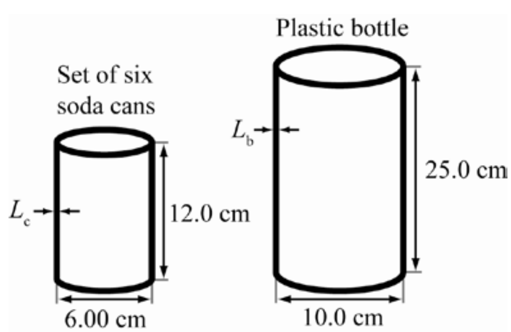
$$T_{\text{f}}=\frac{\left(2.26\cdot 10^6\ \text{J/kg}\right)(1.00\ \text{kg})+\left(4186\ \text{J}/\left(\text{kg}\ °\text{C}\right)\right)(1.00\ \text{kg})(100.0\ °\text{C})-\left(3.33\cdot 10^5\ \text{J/kg}\right)(4.00\ \text{kg})}{\left(4186\ \text{J}/\left(\text{kg}\ °\text{C}\right)\right)(1.00\ \text{kg}+4.00\ \text{kg})}=64.338\ °\text{C}.$$

ROUND: $T_{\text{f}}=64.3$ °C

DOUBLE-CHECK: Because $L_{\text{S}}>L_I$ by a factor of 10, it is reasonable that the equilibrium temperature is slightly closer to the initial temperature of the steam.

18.73. **THINK:** The heat transfer through an object depends on its thermal conductivity, surface area and thickness, and the temperature difference between the heat reservoirs (in this case, the environment and the soda).

SKETCH:



RESEARCH: First, the surface area of the soda cans and the plastic bottle are needed. The surface area of a cylindrical object is $A = 2\pi(D/2)^2 + \pi Dh = (1/2)\pi D^2 + \pi Dh,$ where D is the diameter and h is the height. The heat transfer for the 6 soda cans is $P_c = 6k_c A_c \Delta T_c / L_c$. For the plastic bottle, the heat transfer is $P_B = k_B A_B \Delta T_B / L_B$.

SIMPLIFY: The ratio of the initial heat current into all six cans to the initial heat current into the bottle is: $\text{ratio} = P_c / P_B = (6k_c A_c \Delta T_c / L_c)/(k_B A_B \Delta T_B / L_B)$. Since $\Delta T_p = \Delta T_c$ and $L_c = L_B$, the ratio becomes: $\text{ratio} = (6k_c A_c)/(k_B A_B)$. Substituting $A_c = (1/2)\pi D_c^2 + \pi D_c h_c$ and $A_B = (1/2)\pi D_B^2 + \pi D_B h_B$ gives:

$$\text{ratio} = \frac{6k_c\left((1/2)\pi D_c^2 + (\pi D_c h_c)\right)}{k_B\left((1/2)\pi D_B^2 + (\pi D_B h_B)\right)} = \frac{6k_c\left((1/2)D_c^2 + (D_c h_c)\right)}{k_B\left((1/2)D_B^2 + (D_B h_B)\right)}.$$

CALCULATE: Putting in the numerical values yields:

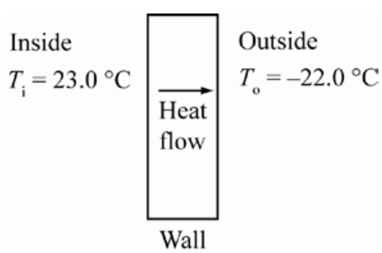
$$\text{ratio} = \frac{6(205\text{ W/(m K)})\left[(1/2)(0.0600\text{ m})^2 + (0.0600\text{ m})(0.120\text{ m})\right]}{(0.100\text{ W/(m K)})\left[(1/2)(0.100\text{ m})^2 + (0.100\text{ m})(0.250\text{ m})\right]} = 3690.$$

ROUND: To three significant figures, the ratio is 3690:1.

DOUBLE-CHECK: The cans have a larger combined surface area than the bottle. This, in combination with the larger thermal conductivity of aluminum leads to the expected result that the cans warm up more quickly than the bottle.

18.74. **THINK:** A 10.0 in. thick piece of fiberglass has an R-factor of $40.0\text{ ft}^2(°\text{F}\cdot\text{hr/BTU})$. I want to convert this value to SI units. I also want to determine the heat transfer through a wall that is insulated with this fiberglass, when the outdoor and indoor temperatures are $-22.0°$ C and $23.0°$ C, respectively.

SKETCH:



RESEARCH: The conversion values for the units are $1\text{ ft} = 0.3048\text{ m}$, $1\text{ °F} = 5/9\text{ K}$, $1\text{ h} = 3.60\cdot 10^3\text{ s}$ and $1\text{ Btu} = 1.055\cdot 10^3\text{ J}$. The heat transfer is given by : $P_{\text{cond}} = kA(T_i - T_0)/L.$

SIMPLIFY: Therefore, the heat transfer is $P_{\text{cond}} = kA(T_i - T_0)/L = A(T_i - T_0)/R,$ since $R = L/k$. The heat transfer per m^2 is $P_{\text{cond}}/\text{m}^2 = (T_i - T_0)/R.$

CALCULATE:

(a) The thermal resistance R in SI unit is:

$$R = 40.0\text{ ft}^2\ \text{°F}\cdot\text{h/BTU}(0.3048\text{ m/ft})^2((5/9)\text{ K/°F})(3.60\cdot 10^3\text{ s/h})/(1.055\cdot 10^3\text{ J/BTU})$$
$$= 7.045\text{ m}^2\text{ K/W}.$$

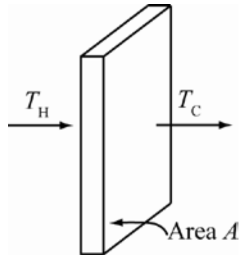
(b) The heat transfer per m^2 is $P_{\text{cond}}/\text{m}^2 = \dfrac{(273 + 23.0\text{ K}) - (273 - 22.0\text{ K})}{7.045\text{ m}^2\text{ K/W}} = 6.388\text{ W/m}^2.$

ROUND: Rounding to three significant figures, $P_{\text{cond}} = 7.05\text{ m}^2\text{ K/W},\ P_{\text{cond}}/\text{m}^2 = 6.39\text{ W/m}^2$.

DOUBLE-CHECK: This is within the range of R-values for commercial fiberglass insulation.

18.75. **THINK:** The conduction rate of insulation, P_{cond}, depends on the area and the R value of the material, and the temperature difference on either side of the insulation. Let $T_{\text{H}} = 294$ K, and $T_{\text{C}} = 277$ K.

SKETCH:



RESEARCH: The conduction rate is given by $P_{\text{cond}} = A\left(T_{\text{H}} - T_{\text{C}}\right)/R.$ The increase in the conduction rate due to the change in the R value is $\Delta P = P_{\text{cond1}} - P_{\text{cond2}} = \left[A\left(T_{\text{H}} - T_{\text{C}}\right)/R_1\right] - \left[A\left(T_{\text{H}} - T_{\text{C}}\right)/R_2\right].$

SIMPLIFY: $\Delta P = A\left(T_{\text{H}} - T_{\text{C}}\right)\left[\left(1/R_1\right) - \left(1/R_2\right)\right]$

(a) The change in heat that exits the room in an interval of time is:

$Q = t\Delta P = tA\left(T_{\text{H}} - T_{\text{C}}\right)\left[\left(1/R_1\right) - \left(1/R_2\right)\right].$

(b) In three months, the extra heat that exits the room is $90Q$.

CALCULATE:

(a) Substituting $A = (5.0\text{ m})(5.0\text{ m}) = 25\text{ m}^2,$

$R_1 = 19\left(0.176\text{ m}^2\text{ K/W}\right) = 3.344\text{ m}^2\text{ K/W},\; R_2 = 30\left(0.176\text{ m}^2\text{ K/W}\right) = 5.28\text{ m}^2\text{ K/W}$ and

$t = 24(3600\text{ s}) = 86400\text{ s}:$

$Q = 86400\text{ s}\left(25\text{ m}^2\right)(294\text{ K} - 277\text{ K})\left[\left(1/\left(3.344\text{ m}^2\text{ K/W}\right)\right) - \left(1/\left(5.28\text{ m}^2\text{ K/W}\right)\right)\right] = 4.03\cdot 10^6\text{ J}.$

(b) $Q = 90\left(4.03\cdot 10^6\text{ J}\right) = 3.62\cdot 10^8\text{ J}.$ Since the electrical energy for heating costs 12 cents/kWh or $3.33\cdot 10^{-6}$ cents/J, the increase in cost of electrical heating is $\text{cost} = 3.62\cdot 10^8\text{ J}\left(3.33\cdot 10^{-6}\text{ cents/J}\right)$ $= 1206.7$ cents.

ROUND: Keep only two significant figures.

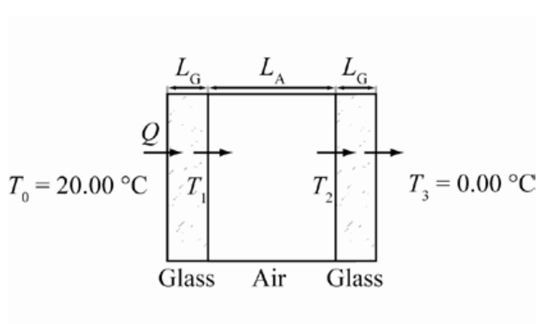
(a) $Q = 4.0\cdot 10^6$ J

(b) The extra cost is 1200 cents, or 12 dollars.

DOUBLE-CHECK: The units of Q are Joules, and the cost is in cents. These units are expected units for these values. 4 million Joules is a reasonable amount of energy, corresponding to a cost of $12 dollars.

18.76. **THINK:** This is a problem of heat flow through three layers of material; two panes of glass and an air gap. Each glass pane has a thickness of 3.00 mm and the air gap is 1.00 cm thick. The thermal conductivities of glass and air are 1.00 W/m K and 0.0260 W/m K, respectively.

SKETCH:



RESEARCH: The heat flow through all three layers must be the same. The conduction rate P_{cond} is given by $P_{\text{cond}} = kA(T_{\text{H}} - T_{\text{C}})/L$. Therefore, the conduction rate through the layers is $P_{\text{cond}} = k_{\text{G}}A(T_0 - T_1)/L_{\text{G}} = k_{\text{A}}A(T_1 - T_2)/L_{\text{A}} = k_{\text{G}}A(T_2 - T_3)/L_{\text{G}}$. The rate of heat energy lost through the window per m^2 is $P_{\text{cond}} / A = k_{\text{G}}(T_0 - T_1)/L_{\text{G}}$. If the window had no air layer between the panes, the rate of heat loss will become $P_{\text{cond}} / A = k_{\text{G}}(T_0 - T_3)/L_{\text{G}}$

SIMPLIFY: Therefore, two equations are obtained:

$$(1)\ (T_0 - T_1) = (T_2 - T_3) \text{ and } (2)\ (T_1 - T_2) = (k_{\text{G}}L_{\text{A}}/k_{\text{A}}L_{\text{G}})(T_2 - T_3).$$

Adding (1) and (2) yields $(T_0 - T_2) = (1 + k_{\text{G}}L_{\text{A}}/k_{\text{A}}L_{\text{G}})(T_2 - T_3)$. Solving for T_2 gives

$$T_2 = \frac{T_0 k_A L_G + T_3 k_A L_G + T_3 k_G L_A}{2k_A L_G + k_G L_A}$$

The temperature T_1 is calculated using $T_1 = T_0 - T_2 + T_3$.

CALCULATE:

(a) Substituting the numerical values gives:

$$T_2 = \frac{(293)(0.0260)(3.00\cdot 10^{-3}) + (273)(0.0260)(3.00\cdot 10^{-3}) + (273)(1.00)(1.00\cdot 10^{-2})}{2(0.0260)(3.00\cdot 10^{-3}) + (1.00)(1.00\cdot 10^{-2})} = 273.154,$$

with units of $[T_2] = \dfrac{(\text{K})(\text{W/m K})(\text{m}) + (\text{K})(\text{W/m K})(\text{m}) + (\text{K})(\text{W/m K})(\text{m})}{(\text{W/m K})(\text{m}) + (\text{W/m K})(\text{m})} = \text{K}$. Therefore $T_2 = 273.154$ K. $T_1 = 293.0\text{ K} - 273.154\text{ K} + 273.0\text{ K} = 292.846\text{ K} = 19.846\text{ °C}$. So, the temperatures of the four air-glass interfaces are $T_0 = 20.00$ °C, $T_1 = 19.846$ °C, $T_2 = 0.154$ °C, and $T_3 = 0.00$ °C.

(b) $P_{\text{cond}} / A = 1.00\text{ W/mK}(20.00\text{ °C} - 19.846\text{ °C}) / 3.00\cdot 10^{-3}\text{ m} = 51.33\text{ W/m}^2$

(c) $P_{\text{cond}} / A = 1.00\text{ W/mK}(20.00\text{ °C} - 0.00\text{ °C}) / 6.00\cdot 10^{-3}\text{ m} = 3333\text{ W/m}^2$

(d) Window glass is not manufactured in this way because there is a pressure difference between the vacuum and the outside air. This pressure difference produces enough force to break the glass. If the glass can withstand the pressure difference, the glass must be very thick and the production of the glass with the vacuum gap would be very expensive

ROUND: Keeping three significant figures;

(a) $T_0 = 20.0$ °C, $T_1 = 19.8$ °C, $T_2 = 0.154$ °C, $T_3 = 0.00$ °C

(b) $P_{\text{cond}} / A = 51.3\text{ W/m}^2$

(c) $P_{\text{cond}} / A = 3330\text{ W/m}^2$

DOUBLE-CHECK: The temperatures decrease with each interface, as expected.

Multi-Version Exercises

18.77. Power = heat /time; solve this for the time:

$$P = Q/t \Rightarrow t = Q/P$$

$$Q = c_{\text{granite}} m_{\text{granite}} \Delta T = c_{\text{granite}} V_{\text{granite}} \rho_{\text{granite}} \Delta T$$

$$t = c_{\text{granite}} V_{\text{granite}} \rho_{\text{granite}} \Delta T / P$$

$$= \frac{(790\text{ J/kg/°C})(0.669\cdot 10^9\text{ m}^3)(2750\text{ kg/m}^3)(64.8\text{°C})}{13.9\cdot 10^6\text{ W}}$$

$$= 6.78\cdot 10^9\text{ s} = 215\text{ years}$$

18.78. By reasoning similar to that in the preceding problem,

$$P = Q/t \Rightarrow Q = Pt$$
$$Q = c_{\text{granite}} V_{\text{granite}} \rho_{\text{granite}} \Delta T = c_{\text{granite}} V_{\text{granite}} \rho_{\text{granite}} \left(T_{\text{f}} - T_{\text{i}}\right)$$
$$Pt = c_{\text{granite}} V_{\text{granite}} \rho_{\text{granite}} \left(T_{\text{i}} - T_{\text{f}}\right)$$
$$T_{\text{i}} = T_{\text{f}} + \frac{Pt}{c_{\text{granite}} V_{\text{granite}} \rho_{\text{granite}}}$$
$$= 104.5\,^\circ\text{C} + \frac{(15.7 \cdot 10^6 \text{ W})(5.194 \cdot 10^9 \text{ s})}{(790 \text{ J/kg/}^\circ\text{C})(0.581 \cdot 10^9 \text{ m}^3)(2750 \text{ kg/m}^3)}$$
$$= 169.1\,^\circ\text{C}$$

18.79.

$$P = Q/t$$
$$Q = c_{\text{granite}} V_{\text{granite}} \rho_{\text{granite}} \Delta T = c_{\text{granite}} V_{\text{granite}} \rho_{\text{granite}} \left(T_{\text{f}} - T_{\text{i}}\right)$$
$$P = \frac{c_{\text{granite}} V_{\text{granite}} \rho_{\text{granite}} \left(T_{\text{i}} - T_{\text{f}}\right)}{t}$$
$$= \frac{(790 \text{ J/kg/}^\circ\text{C})(0.493 \cdot 10^9 \text{ m}^3)(2750 \text{ kg/m}^3)(64.4^\circ)}{3.942 \cdot 10^9 \text{ s}}$$
$$= 17.5 \text{ MW}$$

18.80.

$$P = A\frac{T_{\text{h}} - T_{\text{c}}}{R_{\text{SI}}}$$
$$A = LW$$
$$R_{\text{SI}} = R/5.678$$
$$P = \frac{LW\left(T_{\text{h}} - T_{\text{c}}\right)}{R/5.678} = \frac{5.678\left(LW\right)\left(T_{\text{h}} - T_{\text{c}}\right)}{R} = \frac{5.678\left(5.183 \text{ m}\right)\left(3.269 \text{ m}\right)\left(23.37\ ^\circ\text{C} - 1.073\ ^\circ\text{C}\right)}{29} = 74 \text{ W}.$$

18.81.

$$P = A\frac{T_{\text{h}} - T_{\text{c}}}{R_{\text{SI}}}$$
$$A = LW$$
$$R_{\text{SI}} = R/5.678$$
$$P = \frac{LW\left(T_{\text{h}} - T_{\text{c}}\right)}{R/5.678} = \frac{5.678\left(LW\right)\left(T_{\text{h}} - T_{\text{c}}\right)}{R}$$
$$R = \frac{5.678\left(LW\right)\left(T_{\text{h}} - T_{\text{c}}\right)}{P} = \frac{5.678\left(5.869 \text{ m}\right)\left(3.289 \text{ m}\right)\left(24.21\ ^\circ\text{C} - 3.857\ ^\circ\text{C}\right)}{69.71 \text{ W}} = 32$$

The insulation has a rating of R-32.

18.82.

$$P = A\frac{T_{\text{h}} - T_{\text{c}}}{R_{\text{SI}}}$$
$$A = LW$$
$$R_{\text{SI}} = R/5.678$$
$$P = \frac{LW\left(T_{\text{h}} - T_{\text{c}}\right)}{R/5.678} = \frac{5.678\left(LW\right)\left(T_{\text{h}} - T_{\text{c}}\right)}{R}$$
$$T_{\text{h}} - T_{\text{c}} = \frac{PR}{5.678\left(LW\right)}$$
$$T_{\text{h}} = \frac{PR}{5.678\left(LW\right)} + T_{\text{c}} = \frac{PR}{5.678\left(LW\right)} + T_{\text{c}} = \frac{\left(63.10 \text{ W}\right)\left(34\right)}{5.678\left(6.555 \text{ m}\right)\left(3.311 \text{ m}\right)} + 2.641\ ^\circ\text{C} = 20\ ^\circ\text{C}.$$

Note that the R factor is known only to two significant figures, and with typical wall insulation greater precision would be unrealistic.

Chapter 19: Ideal Gases

Concept Checks

19.1. b **19.2.** c **19.3.** c **19.4.** b **19.5.** a **19.6.** c **19.7.** c **19.8.** a **19.9.** a

Multiple-Choice Questions

19.1. a **19.2.** a **19.3.** c **19.4.** c **19.5.** a **19.6.** c **19.7.** b **19.8.** d **19.9.** d **19.10.** b **19.11.** e **19.12.** c **19.13.** b **19.14.** e

Conceptual Questions

19.15 As the hot air rises, its volume expands due to a decrease in pressure. If we assume that there is no heat exchanged between the hot air and the environment (an adiabatic process) then the temperature of the hot air decreases since the air molecules do work to expand its volume. From the First Law of Thermodynamics, it is known that for an adiabatic process, $\Delta E_{\text{int}} = -W.$ Since the work done is positive, the change in the internal energy, and consequently its temperature, is negative. This is known as adiabatic cooling.

19.16 If the gas molecules do not exchange energy with the walls of their container or with each other, then they will never reach equilibrium, unless they are already in equilibrium. In the kinetic theory derivations in the text, it is assumed that the gas is already in equilibrium; that is, the speeds of the gas molecules are already distributed according to the Maxwell speed distribution. Yes, it is true that if all gas molecules had the same speed initially, then due to collisions and interactions between the molecules, the speed would be redistributed according to the Maxwell speed distribution to put the gas in equilibrium.

19.17 The surface of our skin loses heat mostly by evaporative cooling and convection. The rate of heat loss of the surface depends on the speed of the air passing over the surface. This means that as you blow hard on your skin, the evaporation from your skin increases and therefore the rate of heat loss also increases. This is why you feel a cool sensation. When you breathe softly, you feel a warm sensation because the rate of heat loss is reduced and because your breath is also warmer than your skin.

19.18 The velocity of an air molecule has a magnitude and a direction. The auditorium is closed, so there is no net flow of molecules into or out of the room. For a molecule moving with a certain speed in a certain direction in the auditorium, there is another molecule moving with the same speed in the opposite direction. In this way, the average velocity is zero. However, since the root-mean-square speed is a scalar, the direction of the molecules is not considered. Therefore, the average speed is greater than zero since all of the molecules are in motion.

19.19 This is an adiabatic process $(Q = 0)$, so $\Delta E_{\text{int}} = -W,$ where W represents the work done by the system. Since the fuel-air mixture is compressed, there is work done on the system, so the work done by the system is negative: $\Delta E_{\text{int}} = -W = -(-W) = W.$ Thus, the internal energy, or temperature, of the mixture increases causing the fuel to ignite. The speed of this compression is irrelevant since no heat flows into or out of the system.

19.20 By the First Law of Thermodynamics, the change in internal energy is $\Delta E = Q - W$, where Q is the heat flow into the gas and W is the work done by the gas. Under condition 1, the piston is blocked to prevent it from moving. Therefore, no work is done by the gas and the change in internal energy is due only to the heat added, $\Delta E = Q.$ As a result the temperature of the gas increases. Under condition 2, some of the heat energy transferred to the gas is used to move the piston (e.g. work is done by the gas on the piston). Therefore, the change in internal energy is $\Delta E = Q - W.$ Since the change in temperature is proportional to the change in internal energy, the final temperature of the gas under condition 1 is larger. The only way for the final temperature to be the same is if the space behind the piston head is under vacuum. Then the gas under condition 2 undergoes a free expansion and no work is done by the gas to move the piston.

19.21 The adiabatic bulk modulus is defined as $B = -V(dp/dV)$. For an adiabatic process, $pV^\gamma = \text{constant}$. Taking the full derivative of both sides of this equation gives:

$$V^\gamma dp + p\gamma V^{\gamma-1}dV = 0 \Rightarrow \frac{dp}{dV} = -\frac{p\gamma V^{\gamma-1}}{V^\gamma} = -\frac{\gamma p}{V}.$$

Therefore, the adiabatic bulk modulus for an ideal gas is given by:

$$B = -V\frac{dp}{dV} = -V\left(-\frac{\gamma p}{V}\right) = \gamma p,$$

as required.

19.22 (a) The monatomic ideal gas undergoes three processes: (1) (p_1, V_1, T_1) to (p_2, V_2, T_1), (2) (p_2, V_2, T_1) to (p_1, V_2, T_2) and (3) (p_1, V_2, T_2) to (p_1, V_1, T_1). The First Law of Thermodynamics states that the change in internal energy is $\Delta E = Q - W$, where Q is the heat flow into the gas and W is the work done by the gas. In process (1), the pressure and volume change but the temperature stays the same. Since this step is isothermal, $\Delta E_1 = 0$, so $Q_1 = W_1 = \int_{V_1}^{V_2} p\,dV$. Using the Ideal Gas Law, the integral can be written as

$$W_1 = \int_{V_1}^{V_2} \frac{nRT_1}{V}dV = nRT_1 \ln\left(\frac{V_2}{V_1}\right).$$

However, the answer can be simplified further by noting that $nRT_1 = p_1V_1 = p_2V_2$. Therefore, the heat flow into the gas during process (1) is

$$Q_1 = W_1 = p_1V_1 \ln\left(\frac{V_2}{V_1}\right).$$

In process (2), the volume is constant so no work is done and the heat flow is

$$Q_2 = \Delta E_2 = \frac{3}{2}nR\Delta T = \frac{3}{2}nR(T_2 - T_1).$$

Since $T_1 = \frac{p_2V_2}{nR}$ and $T_2 = \frac{p_1V_2}{nR}$, the heat flow into the gas during process (2) is

$$Q_2 = \frac{3}{2}nR\left(\frac{p_1V_2}{nR} - \frac{p_2V_2}{nR}\right) = \frac{3}{2}V_2(p_1 - p_2).$$

In process (3) the pressure remains constant so,

$$\Delta E_3 = Q_3 - W_3 \Rightarrow Q_3 = \Delta E_3 + W_3 = \frac{3}{2}nR(T_1 - T_2) + \int_{V_2}^{V_1} p_1 dV = \frac{3}{2}nR(T_1 - T_2) + p_1(V_1 - V_2).$$

Substituting $nRT_1 = p_1V_1$ and $nRT_2 = p_1V_2$ gives:

$$Q_3 = \frac{3}{2}(p_1V_1 - p_1V_2) + p_1(V_1 - V_2) = \frac{3}{2}p_1V_1 - \frac{3}{2}p_1V_2 + p_1V_1 - p_1V_2 = \frac{5}{2}(p_1V_1 - p_1V_2).$$

(b) The total heat flow into the gas is

$$Q_{\text{total}} = Q_1 + Q_2 + Q_3 = p_1V_1 \ln\left(\frac{V_2}{V_1}\right) + \frac{3}{2}p_1V_2 - \frac{3}{2}p_2V_2 + \frac{5}{2}p_1V_1 - \frac{5}{2}p_1V_2 \Rightarrow Q_{\text{total}} = p_1V_1\left(1 + \ln\left(\frac{V_2}{V_1}\right)\right) - p_1V_2,$$

using $p_2V_2 = p_1V_1$.

19.23 Consider two atomic gasses reacting to form a diatomic gas that proceeds as $\mathrm{A} + \mathrm{B} \rightarrow \mathrm{AB}$, where A is one of the atomic gases and B is the other. There is 1 mole of each gas and the reaction happens in a thermally isolated chamber. Conservation of energy can be applied to solve this problem. Initially the chamber is at temperature T_i, so the sum of the initial energies E_i of the two monatomic gases is

$$E_i = E_A + E_B = \frac{3}{2}RT_i + \frac{3}{2}RT_i = 3RT_i \quad \text{(with } n=1 \text{ for each gas).}$$

The energy E_f of the diatomic gas at temperature T_f is $E_f = (5/2)RT_f$, using $n=1$ and $C_V = (5/2)R$ for a diatomic gas. By conservation of energy,

$$E_i = E_f \Rightarrow 3RT_i = \frac{5}{2}RT_f \Rightarrow T_f = \frac{6}{5}T_i.$$

Therefore, the temperature of the system increases since $T_f > T_i$.

19.24 The Ideal Gas Law is $pV = nRT$, where p is the pressure, V is the volume, n is the number of moles, R is the gas constant and T is the temperature. Rearranging to solve for p gives $p = nRT/V$. The number of moles, n, is equal to the mass of the gas, m, divided by the molar mass, M: $n = m/M$. The mass of the gas can be written as $m = \rho V$, where ρ is the density, so $n = \rho V/M$. Therefore, the equation of state form of the Ideal Gas Law is $p = \rho RT/M$.

19.25 The compression and rarefaction of a sound wave in gas can be treated as an adiabatic process.

(a) The speed of sound v_S in an ideal gas of molar mass M is $v_S = \sqrt{B/\rho}$, where B is the bulk modulus and ρ is the density. For an ideal gas, the adiabatic bulk modulus is defined as $B = -V(dp/dV)$. In an adiabatic process, $pV^\gamma = \text{constant}$. Taking the full derivative of this equation on both sides gives:

$$\frac{d}{dp}\left(pV^\gamma\right)dp + \frac{d}{dV}\left(pV^\gamma\right)dV = 0 \Rightarrow V^\gamma dp + p\gamma V^{\gamma-1}dV = 0 \Rightarrow \frac{dp}{dV} = -\frac{\gamma p}{V}.$$

Substituting this into the equation for the bulk modulus gives $B = -V(-\gamma p/V) = \gamma p$. For an ideal gas with density ρ where the number of moles can be expressed as $n = \rho V/M$,

$$pV = nRT = \frac{\rho VRT}{M} \Rightarrow \rho = \frac{pM}{RT}.$$

Therefore, the speed of sound is

$$v_S = \sqrt{\frac{\gamma pRT}{pM}} = \sqrt{\frac{\gamma RT}{M}}.$$

(b) The speed of sound cannot exceed the speed of light: $v_S \le c$. Since the speed of sound is directly proportional to the temperature, the maximum temperature occurs at the maximum sound speed, c. Therefore,

$$v_{S,max} = c = \sqrt{\frac{\gamma RT_{max}}{M}} \Rightarrow T_{max} = \frac{Mc^2}{\gamma R}.$$

(c) For a monatomic gas, the ratio of the molar specific heats is $\gamma = 5/3$. Therefore, the maximum temperature is

$$T_{max} = \frac{\left(1.008\cdot 10^{-3} \text{ kg/mol}\right)\left(2.998\cdot 10^{8} \text{ m/s}\right)^2}{\frac{5}{3}\left(8.314 \text{ J/}\left(\text{mol K}\right)\right)} = 6.538\cdot 10^{12} \text{ K}.$$

(d) At this maximum temperature, the equations used would not properly describe the situation. As a particle approaches the speed of light, Newtonian mechanics no longer applies and this situation requires quantum mechanics for an accurate description.

19.26 For a monatomic gas, the internal energy is $E_m = (3/2)k_B T_m$, where k_B is Boltzmann's constant. The factor of three corresponds to the translational degrees of freedom the monatomic gas has. Monatomic species do not undergo rotational motion. A diatomic gas has three translational degrees of freedom and two rotational degrees of freedom. The internal energy for a diatomic gas is $E_d = ((3+2)/2)k_B T_d = (5/2)k_B T_d$. If the same energy is added to the monatomic gas and the diatomic gas then: $(3/2)k_{\text{B}}T_{\text{m}}=(5/2)k_{\text{B}}T_{\text{d}} \;\Rightarrow\; T_{\text{d}}=(3/5)T_{\text{m}}$. Therefore, the temperature increase for the diatomic gas is less than the temperature increase for the monatomic gas. This is logical since temperature is a measure of the translational motion of atoms or molecules and some of the energy added to a diatomic gas is used for rotational motion.

19.27 Since the temperature in the thermosphere is high at 1500 °C, the speed of the molecules is high. However, at such an extremely high altitude the atmospheric pressure is very low, and so there would not be very many gas molecules in this region. Because there are a low number of gas molecules, the frequency of molecule collisions with the skin would be small. Therefore, the transfer of energy to one's skin would be very small and the skin would feel very cold.

19.28 The speed distribution of the water molecules is similar to the Maxwell speed distribution for a gas. Most of the molecules are unable to escape through the surface of the water because they have insufficient speed. However, some very fast molecules in the high speed tail of the distribution are able to do so. This is the process that allows evaporation. As fast molecules escape, the speed distribution is maintained by heat from the surroundings and collisions that lead to other fast molecules.

Exercises

19.29 A tire has an initial gauge pressure of $p_{\text{ig}}=300.\text{ kPa}$, an initial temperature of $T_{\text{i}}=15.0\text{ °C}=288\text{ K}$, and a final temperature of $T_{\text{f}}=45.0\text{ °C}=318\text{ K}$. The volume change of the tire is negligible, so $V_{\text{i}}=V_{\text{f}}=V$. The number of moles of gas in the tire is constant as well: $n_{\text{i}}=n_{\text{f}}=n$. The final gauge pressure p_{fg} can be found by using Gay-Lussac's Law, $p_{\text{i}}/T_{\text{i}}=p_{\text{f}}/T_{\text{f}}$, with $p_{\text{i}}=p_{\text{ig}}+p_{\text{atm}}$ and $p_{\text{f}}=p_{\text{fg}}+p_{\text{atm}}$. Rearranging to solve for p_{fg} gives

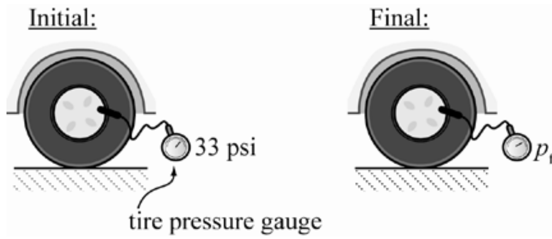
$$p_{\text{fg}}=\left(p_{\text{ig}}+p_{\text{atm}}\right)\frac{T_{\text{f}}}{T_{\text{i}}}-p_{\text{atm}}=\left(300.\text{ kPa}+101.3\text{ kPa}\right)\frac{(318\text{ K})}{(288\text{ K})}-101.3\text{ kPa}=342\text{ kPa}.$$

19.30 The given quantities are the helium tank gauge pressure, $p_{\text{T}}=2400.\text{ psi}$ and the volume the gas would occupy at atmospheric pressure $V_{\text{atm}}=244\text{ ft}^3$. Atmospheric pressure is $p_{\text{atm}}=101.3\text{ kPa}=14.7\text{ psi}$. For a constant temperature, Boyle's law can be used to calculate the tank volume, V_{T}. Boyle's Law gives: $\left(p_{\text{T}}+p_{\text{atm}}\right)V_{\text{T}}=p_{\text{atm}}V_{\text{atm}}$. Solving for V_{T}:

$$V_{\text{T}}=\frac{p_{\text{atm}}}{p_{\text{T}}+p_{\text{atm}}}V_{\text{atm}}=\frac{(14.7\text{ psi})}{(2400.\text{ psi})+(14.7\text{ psi})}\left(244\text{ ft}^3\right)=1.49\text{ ft}^3.$$

19.31 **THINK:** Initially, the gas inside the tires is at a pressure of $p_{\text{i}}=33.0\text{ psi}$ and a temperature of $T_{\text{i}}=25.0\text{ °F}$. The final temperature is $T_{\text{f}}=72.0\text{ °F}$. The valve caps are airtight so it can be assumed that no air leaked out of the tires during the trip. Since it can be assumed that the volume of the tires stays constant, the final tire pressure can be calculated by using Gay-Lussac's Law. Since the mass of the gas is directly proportional to the number of moles, the Ideal Gas Law can be used to find the percentage of the original mass that is released.

SKETCH:



RESEARCH:

(a) Gay-Lussac's law applies, so $p_i / T_i = p_f / T_f$. The number of moles of gas, *n*, in the tire will not change during the trip assuming the tires are airtight. Note that the tire pressure is a gauge pressure rather than an absolute pressure. The absolute pressure is given by $p = p_g + p_{atm}$, where p_g is the gauge pressure and p_{atm} is atmospheric pressure.

(b) The number of moles of gas in the tire is given by $n = m / M$, where *M* is the molar mass and *m* is the total mass of the gas. The initial number of moles, n_i, and number of moles after releasing air from the tire, n_f, can be found by using the Ideal Gas Law: $n = pV / RT$. In general, the percentage of a final value, X_f with respect to an initial value, X_i can be evaluated using the formula:

$$\left(\frac{X_f - X_i}{X_i}\right)100\%.$$

The SI unit for pressure is the Pascal (Pa). The conversion from psi to (kilo) Pa is 6.89473 kPa/psi.

SIMPLIFY:

(a) $p_{fg} = \left(p_{ig} + p_{atm}\right)\dfrac{T_f}{T_i} - p_{atm}$

(b) Since the volume is constant:

$$V = \frac{\left(m_i / M\right)RT_i}{p_i} \text{ and } V = \frac{\left(m_f / M\right)RT_f}{p_i}.$$

Note that p_i is used in the second equation because it is the pressure of the tire in Florida after it is deflated back to the original pressure. Equating these two expressions gives:

$$\frac{m_i RT_i}{Mp_i} = \frac{m_f RT_f}{Mp_i} \Rightarrow m_i T_i = m_f T_f \Rightarrow m_f = \frac{T_i}{T_f} m_i.$$

The percentage of the original mass of gas that is released is:

$$\frac{m_f - m_i}{m_i}(100\%) = \frac{m_i\left(\dfrac{T_i}{T_f} - 1\right)}{m_i}(100\%) = \left(\frac{T_i - T_f}{T_f}\right)(100\%).$$

CALCULATE:

$T_i = \dfrac{5}{9}\left(25.0\ °F - 32.0\ °F\right) + 273.15\ °C = 269.26\ K$ and $T_f = \dfrac{5}{9}\left(72.0\ °F - 32.0\ °F\right) + 273.15\ °C = 295.37\ K$

(a) $p_{fg} = \left(\left(33.0\ psi\right)\left(6.89473\ kPa/psi\right) + \left(101.3\ kPa\right)\right)\left(\dfrac{295.37\ K}{269.26\ K}\right) - \left(101.3\ kPa\right) = 259.41\ kPa$

(b) $\left(\dfrac{\left(269.26\ K\right) - \left(295.37\ K\right)}{\left(295.37\ K\right)}\right)\left(100\%\right) = -8.84\%$

The negative sign denotes that the final mass of air is less than the initial mass of air.

ROUND: The answers should be given to three significant figures.

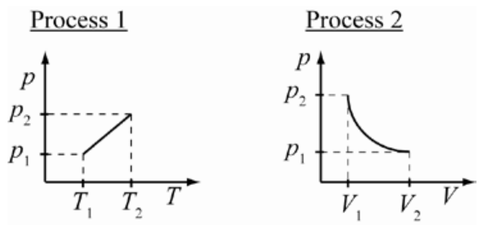
(a) $p_{fg} = 259$ kPa

(b) 8.84% of the original mass was released.

DOUBLE-CHECK: (a) It is known from the Ideal Gas Law that pressure is proportional to temperature. Since the temperature of the air inside the tires increases in the trip from Michigan to Florida, it is reasonable for the pressure to increase. (b) It is known from the Ideal Gas Law that pressure is proportional to the amount of gas enclosed. Since the pressure must decrease, air must be released, as indicated by the negative sign.

19.32 **THINK:** A gas with volume $V_1 = 1.00$ L undergoes an isochoric (or constant volume) process until its pressure doubles: $p_2 = 2p_1$. This process is followed by an isothermal (or constant temperature) process until the original pressure p_1 has been reached. Boyle's Law can be used for the constant temperature process to find the final volume.

SKETCH:



RESEARCH: In the second process, the temperature is constant so Boyle's Law is used: $p_2V_1 = p_1V_2$.

SIMPLIFY: The final volume is

$$V_2 = \frac{p_2}{p_1}V_1 = \frac{(2p_1)}{p_1}V_1 = 2V_1.$$

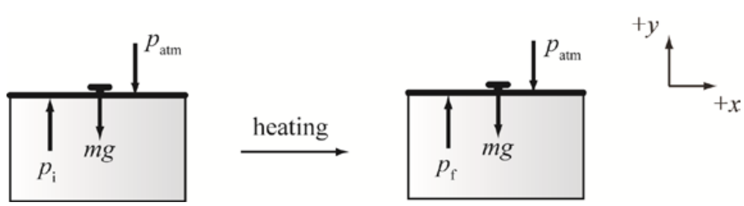
CALCULATE: $V_2 = 2(1.00 \text{ L}) = 2.00 \text{ L}$

ROUND: To three significant figures, the final volume is $V_2 = 2.00$ L.

DOUBLE-CHECK: For an isothermal expansion it is reasonable for the volume to double as the pressure is halved.

19.33 **THINK:** The pot is filled with steam at a pressure of $p_i = 1.00$ atm and a temperature of $T_i = 100.0$ °C $= 373.2$ K. The mass of the pot's lid is $m = 0.500$ kg. The pot's diameter is $d = 0.150$ m and its height is $h = 0.100$ m. In order for the lid to off of the pot, the force upward from the pressure created by the steam must be greater than the force downward from the weight of the lid. Gay-Lussac's Law can be used to relate the initial and final states of the steam under constant volume.

SKETCH:



RESEARCH: Pressure is defined as force per unit area: $p = F / A$. The minimum force required to lift the lid off of the pot can be found when the sum of the forces in the y-direction equals zero:

$$\sum_{j=1}^{n} F_{y,j} = 0.$$

Up until the moment when the lid lifts, the volume that the steam occupies is constant. Thus, pressure and temperature can be related using Gay-Lussac's Law: $p_{\text{i}} / T_{\text{i}} = p_{\text{f}} / T_{\text{f}}$. The area of the lid is $A = \pi (d/2)^2$.

SIMPLIFY: $\sum_{j=1}^{n} F_{y,j} = p_{\text{f}} A - p_{\text{atm}} A - mg = 0 \Rightarrow p_{\text{f}} = p_{\text{atm}} + \frac{mg}{A}$

$$\frac{p_{\text{i}}}{T_{\text{i}}} = \frac{p_{\text{f}}}{T_{\text{f}}} \Rightarrow T_{\text{f}} = \frac{p_{\text{f}}}{p_{\text{i}}} T_{\text{i}} = \frac{p_{\text{atm}} + mg/A}{p_{\text{i}}} T_{\text{i}}$$

CALCULATE:

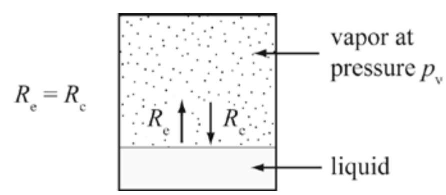
$$T_{\text{f}} = \frac{\left(101.3 \cdot 10^3 \text{ Pa}\right) + \left(0.500 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right) / \left(\pi \left(\left(0.150 \text{ m}\right)/2\right)^2\right)}{\left(101.3 \cdot 10^3 \text{ Pa}\right)} \left(373.2 \text{ K}\right) = 374.22 \text{ K} = 101.02 \text{ °C}$$

ROUND: Rounding to three significant figures, $T_{\text{f}} = 374 \text{ K} = 101 \text{ °C}$.

DOUBLE-CHECK: It is reasonable that heating the steam slightly above the boiling temperature of water would cause the lid to be lifted off of the pot. You have probably noticed before that a lid on top of a pot rattles as water boils inside.

19.34 **THINK:** The water tends to evaporate at a rate R_{e} and the vapor tends to condense back to its liquid form at a rate R_{c}. The vapor pressure of a liquid varies with temperature and occurs when $R_{\text{e}} = R_{\text{c}}$. The vapor pressure of water at temperature $T = 25.0 \text{ °C}$ is $p_{\text{V}} = 3.1690 \text{ kPa}$. The Ideal Gas Law can be used to find the number of moles. Then the mass can be found by using the molar mass of water.

SKETCH:



RESEARCH: The Ideal Gas Law is $pV = nRT$, where p is the pressure, V is the volume, n is the number of moles of gas, R is the gas constant and T is the temperature. For this problem, it is convenient to write the Ideal Gas Law in terms of density. The number of moles is $n = m/M$, where the molar mass of water is $M = 18.0153$ g/mol.

SIMPLIFY:

$$p_{\text{V}} = \frac{nRT}{V} = \frac{mRT}{MV} \Rightarrow m = \frac{p_{\text{V}} MV}{RT}$$

CALCULATE: $T = 25.0 \text{ °C} = 298.15 \text{ K}$

$$m = \frac{\left(3.1690 \cdot 10^3 \text{ Pa}\right)\left(18.0153 \cdot 10^{-3} \text{ kg/mol}\right)\left(1 \text{ L}\right)\left(1 \cdot 10^{-3} \text{ m}^3\text{/L}\right)}{\left(8.3145 \text{ J/}\left(\text{mol K}\right)\right)\left(298.15 \text{ K}\right)} = 2.3030 \cdot 10^{-2} \text{ g}$$

ROUND: To three significant figures, the mass of the vapor is $m = 0.0230$ g.

DOUBLE-CHECK: Water vapor is not very dense. Therefore, a small amount of mass in a one liter volume is a reasonable result.

19.35 **THINK:** Given the equation $\rho T = \text{consant},$ results of Chapter 16 can be used to estimate the speed of sound $v_{\text{S},2}$ in air at $40.0\ ^\circ\text{C},$ knowing that the speed at $0.00\ ^\circ\text{C}$ is $v_{\text{S},1} = 331\ \text{m/s}.$

SKETCH: A sketch is not needed to solve the problem.

RESEARCH: From Chapter 16, the equation for the speed of sound in a gas is $v_\text{S} = \sqrt{B/\rho},$ where B is the bulk modulus and ρ is the density of the air. Let ρ_1 be the density at temperature $T_1 = 0.00\ ^\circ\text{C}$ and ρ_2 be the density at $T_2 = 40.0\ ^\circ\text{C}.$

SIMPLIFY: $v_{\text{S},1} = \sqrt{\dfrac{B}{\rho_1}} \Rightarrow B = v_{\text{S},1}^2\rho_1,\ v_{\text{S},2} = \sqrt{\dfrac{B}{\rho_2}} \Rightarrow B = v_{\text{S},2}^2\rho_2.$ Therefore, the speeds are related by,

$$v_{\text{S},2}^2\rho_2 = v_{\text{S},1}^2\rho_1 \Rightarrow v_{\text{S},2} = \sqrt{\frac{\rho_1}{\rho_2}}v_{\text{S},1}.$$

Since $\rho T = \text{constant},$

$$\rho_1 T_1 = \rho_2 T_2 \text{ or } \frac{\rho_1}{\rho_2} = \frac{T_2}{T_1}.$$

Substituting this into the above expression gives

$$v_{\text{S},2} = \sqrt{\frac{T_2}{T_1}}v_{\text{S},1}.$$

CALCULATE: $v_{\text{S},2} = \sqrt{\dfrac{(40.0\ ^\circ\text{C} + 273.15\ ^\circ\text{C})}{(0.00\ ^\circ\text{C} + 273.15\ ^\circ\text{C})}}(331\ \text{m/s}) = 354.41\ \text{m/s}$

ROUND: Rounding to three significant figures, the speed of sound in air at $40.0\ ^\circ\text{C}$ is $354\ \text{m/s}.$

DOUBLE-CHECK: Molecules move faster in air that is at a higher temperature since the molecules have a higher kinetic energy. Since sound is carried through the air by the motion of air molecules, it is reasonable that the speed is greater for a higher temperature.

19.36 There are $n = 2.00$ moles of an ideal gas enclosed in a container of volume $V = 1.00 \cdot 10^{-4}\ \text{m}^3.$ The container is heated to a temperature of $T = 400.\ \text{K}.$ It can be assumed that the volume of the container remains constant. The Ideal Gas Law can be used to find the pressure of the gas after the increase in temperature.

$$pV = nRT \Rightarrow p = \frac{nRT}{V} = \frac{(2.00\ \text{mol})(8.314\ \text{J/(mol K)})(400.\ \text{K})}{(1.00 \cdot 10^{-4}\ \text{m}^3)} = 6.65 \cdot 10^4\ \text{kPa}$$

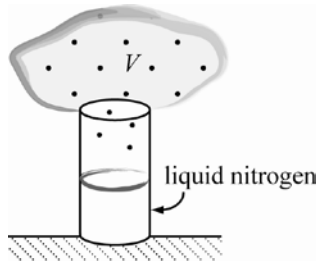
19.37 The given quantities are the number of moles, $n = 1.00\ \text{mol},$ the volume, $V = 2.00\ \text{L} = 2.00 \cdot 10^{-3}\ \text{m}^3$ and the temperature change, $\Delta T = 100.\ ^\circ\text{C} = 100.\ \text{K}.$ The Ideal Gas Law can be used to find the change in pressure. Since the volume is constant, a change in temperature must cause a change in pressure.

$$\Delta p V = nR\Delta T \Rightarrow \Delta p = \frac{nR\Delta T}{V} = \frac{(1.00\ \text{mol})(8.314\ \text{J/(mol K)})(100.\ \text{K})}{(2.00 \cdot 10^{-3}\ \text{m}^3)} = 416\ \text{kPa}$$

19.38 Work, $W = 2.00\ \text{kJ},$ is performed by an ideal gas in an isothermal process. There is no change in internal energy when a gas undergoes an isothermal process; that is, $\Delta E_\text{int} = 0.$ The First Law of Thermodynamics states $\Delta E_\text{int} = Q - W,$ so for an isothermal process, $Q = W,$ where Q is the heat added to the gas. Therefore, if the work performed by the ideal gas is $W = 2.00\ \text{kJ}$ then the heat added is $Q = 2.00\ \text{kJ}.$

19.39 **THINK:** A quantity of liquid nitrogen of density $\rho = 808 \text{ kg/m}^3$ and volume $V_\text{L} = 1.00 \text{ L} = 1.00 \cdot 10^{-3} \text{ m}^3$ evaporates and comes into equilibrium with air at a temperature $T = 21.0 \text{ °C} = 294.2 \text{ K}$ and pressure $p = 101 \text{ kPa}$. The volume V that the evaporated nitrogen occupies can be calculated with the Ideal Gas Law.

SKETCH:



RESEARCH: The number of moles of N_2 is $n = (\rho V_\text{L}) / M$, where $M = 28.0 \cdot 10^{-3}$ kg/mol is the molar mass of N_2. To find the volume that the evaporated nitrogen occupies, use the Ideal Gas Law: $pV = nRT$.

SIMPLIFY: $V = \dfrac{nRT}{p} = \dfrac{\rho RT}{pM} V_\text{L}$

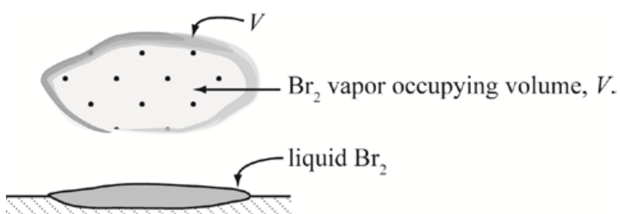
CALCULATE: $V = \dfrac{(808 \text{ kg/m}^3)(8.314 \text{ J/(mol K)})(294.2 \text{ K})}{(28.0 \cdot 10^{-3} \text{ kg/mol})(101 \cdot 10^3 \text{ Pa})}(1.00 \text{ L}) = 698.85 \text{ L}$

ROUND: To three significant figures, the volume occupied is $V = 699$ L.

DOUBLE-CHECK: It is reasonable that the volume of the gas is much greater than the volume of the liquid since the density of the gas is much less than the density of the liquid.

19.40 **THINK:** A quantity of liquid bromine has evaporated in a laboratory. Assuming that the vapor behaves like an ideal gas with temperature $T = 20.0 \text{ °C} = 293.2 \text{ K}$ and pressure $p = 101.0 \text{ kPa}$, the Ideal Gas Law can be used to find the density, ρ, of the vapor.

SKETCH:



RESEARCH: The number of moles of bromine is $n = \rho V / M$, where V is the volume of the gas and M is the molar mass. Because bromine occurs as Br_2, the molar mass of the gas is $M = 159.81 \cdot 10^{-3}$ kg/mol. To find the density of the evaporated bromine, use the Ideal Gas Law: $pV = nRT$.

SIMPLIFY: $pV = nRT = \dfrac{\rho VRT}{M} \Rightarrow \rho = \dfrac{pM}{RT}$

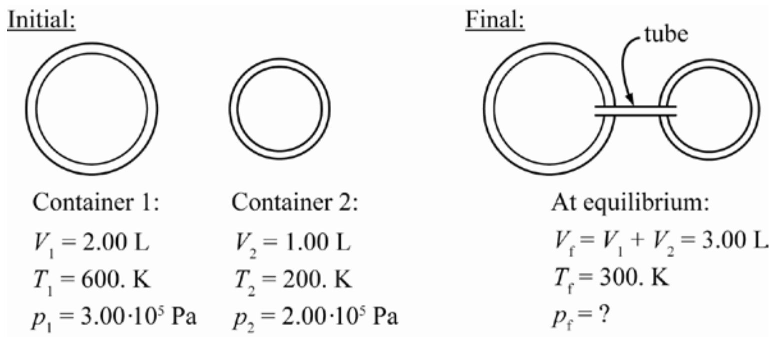
CALCULATE: $\rho = \dfrac{(101.0 \cdot 10^3 \text{ Pa})(159.81 \cdot 10^{-3} \text{ kg/mol})}{(8.314 \text{ J/(mol K)})(293.2 \text{ K})} = 6.6214 \text{ kg/m}^3$

ROUND: Rounding to three significant figures, the density of the evaporated bromine is $\rho = 6.62 \text{ kg/m}^3$.

DOUBLE-CHECK: It is reasonable that the density of the evaporated bromine is greater than the density of air (1.2 kg/m^3) because it has a higher molar mass than the constituent gases that make up air (mainly nitrogen and oxygen).

19.41 **THINK:** Two containers with the volumes, temperatures and pressures shown in the sketch are connected by a tube and allowed to come to equilibrium at pressure p_{f}. At equilibrium, the final temperature of the system is $T_{\text{f}} = 300.$ K. The tube that connects the two containers has a negligible volume so the final volume is $V_{\text{f}} = V_1 + V_2$. The total number of moles in the system remains constant so $n_{\text{f}} = n_1 + n_2$. The Ideal Gas Law can be used to find the final temperature.

SKETCH:



RESEARCH: Ideal Gas Law: $pV = nRT$

SIMPLIFY: The number of moles in the system at equilibrium is:

$$n_1 = \frac{p_1 V_1}{RT_1},\ n_2 = \frac{p_2 V_2}{RT_2} \Rightarrow n_{\text{f}} = n_1 + n_2 = \frac{1}{R}\left(\frac{p_1 V_1}{T_1} + \frac{p_2 V_2}{T_2}\right).$$

Therefore, the final pressure is:

$$p_{\text{f}} = \frac{n_{\text{f}} R T_{\text{f}}}{V_{\text{f}}} = \frac{1}{R}\left(\frac{p_1 V_1}{T_1} + \frac{p_2 V_2}{T_2}\right)\left(\frac{RT_{\text{f}}}{V_1 + V_2}\right) \Rightarrow p_{\text{f}} = \left(\frac{p_1 V_1}{T_1} + \frac{p_2 V_2}{T_2}\right)\left(\frac{T_{\text{f}}}{V_1 + V_2}\right).$$

CALCULATE:

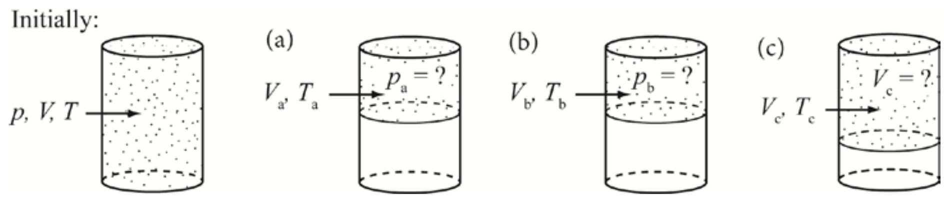
$$p_{\text{f}} = \left(\frac{\left(3.00\cdot 10^5\ \text{Pa}\right)\left(2.00\cdot 10^{-3}\ \text{m}^3\right)}{\left(600.\ \text{K}\right)} + \frac{\left(2.00\cdot 10^5\ \text{Pa}\right)\left(1.00\cdot 10^{-3}\ \text{m}^3\right)}{\left(200.\ \text{K}\right)}\right)\left(\frac{\left(300.\ \text{K}\right)}{\left(3.00\cdot 10^{-3}\ \text{m}^3\right)}\right) = 2.00\cdot 10^5\ \text{Pa}$$

ROUND: $p_{\text{f}} = 200.$ kPa

DOUBLE-CHECK: Since initially $p_1 V_1 / T_1 = p_2 V_2 / T_2$, the two containers start out with the same number of gas molecules. Therefore, the average initial kinetic energy per molecule corresponds to an average temperature of $(600 + 200)/2 = 400$ K. Because of higher initial pressure in container 1, gas at first flows from left to right. However, as the temperatures equalize the flow reverses, so that in the end container 1, being twice as large as container 2, holds twice as many molecules. Since the final temperature is less than 400 K, heat has been dissipated into the surrounding environment during the process.

19.42 **THINK:** The sample of gas confined in a cylinder is initially at pressure $p = 1000.$ Pa, volume $V = 1.00\ \text{L} = 1.00\cdot 10^{-3}\ \text{m}^3$ and temperature $T = 300.$ K. The Ideal Gas Law can be used to find the new pressure when (a) the volume is reduced to $V_{\text{a}} = (1/2)V$, (b) the volume is reduced to $V_{\text{b}} = (1/2)V$ and the temperature is increased to $T_{\text{b}} = 400.$ K, and it can be used to (c) find the new volume if the gas is at a temperature of $T_{\text{c}} = 600.$ K and pressure of $p_{\text{c}} = 3000.$ Pa. The number of moles remains constant for each process since gas does not enter or exit the cylinder.

SKETCH:



RESEARCH: Ideal Gas Law: $pV = nRT$

SIMPLIFY:

(a) If the temperature remains the same, $pV = nRT = p_a V_a \Rightarrow p_a = \frac{pV}{V_a} = \frac{pV}{(1/2)V} = 2p.$

(b) If the temperature and volume change, $\frac{pV}{T} = nR = \frac{p_b V_b}{T_b} \Rightarrow p_b = \frac{pVT_b}{TV_b} = \frac{pVT_b}{T(1/2)V} = 2p\frac{T_b}{T}.$

(c) If the temperature and pressure change, $\frac{pV}{T} = nR = \frac{p_c V_c}{T_c} \Rightarrow V_c = \frac{pT_c}{p_c T}V.$

CALCULATE:

(a) $p_a = 2(1000.\text{ Pa}) = 2000.\text{ Pa}$

(b) $p_b = 2(1000.\text{ Pa})\left(\frac{400.\text{ K}}{300.\text{ K}}\right) = 2667\text{ Pa}$

(c) $V_c = \frac{(1000.\text{ Pa})(600.\text{ K})}{(3000.\text{ Pa})(300.\text{ K})}(1.00\text{ L}) = 0.6667\text{ L}$

ROUND: The answers to three significant figures are:

(a) $p_a = 2.00 \cdot 10^3\text{ Pa}$

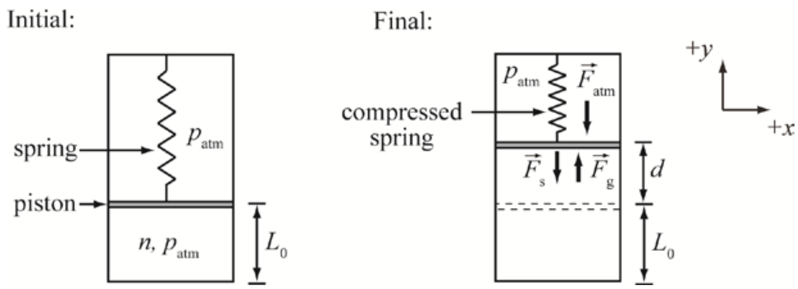
(b) $p_b = 2670\text{ Pa}$

(c) $V_c = 0.667\text{ L}$

DOUBLE-CHECK: (a) The volume decreased so the pressure is expected to increase. (b) The volume decreased and the temperature increased so it is expected that $p_b > p_a$. (c) Since the increase in pressure is greater than the increase in temperature (the pressure triples and the temperature doubles) and $V \propto T/p$, the volume is expected to decrease.

19.43 **THINK:** Assumes that the gas behind the cylinder head is initially at atmospheric pressure, and that it stays at atmospheric pressure as the other gas is heated. The cylinder has cross-sectional area of $A = 12.0\text{ cm}^2 = 1.20 \cdot 10^{-3}\text{ m}^2$. The piston is connected to a spring of spring constant $k = 1000.\text{ N/m}$. The cylinder is filled with $n = 0.005000$ moles of gas that occupies a length L_0 of the cylinder. When the gas is at temperature $T_i = 23.0\text{ °C} = 296\text{ K}$, the spring is at its equilibrium position. If the temperature of the gas is raised to $T_f = 150.\text{ °C} = 423\text{ K}$, the higher pressure p_f can be determined with the Ideal Gas Law. There are three forces acting on the piston after the temperature increase: the force F_g due to the pressure p_g of the gas, the force F_{atm} due to atmospheric pressure p_{atm}, and the spring force F_s. The spring force is given by Hooke's Law. Newton's Second Law can be used to relate these forces.

SKETCH:



RESEARCH: When the spring is compressed, the sum of the forces in the y-direction are zero:

$$\sum_{i=1}^{n} F_{yi} = 0.$$

The force on the piston due to an applied pressure is $F = pA$. The initial volume that the gas occupies is $V_i = AL_0$ and the final volume the gas occupied is $V_f = A(L_0 + d)$. The Ideal Gas Law is $pV = nRT$. The force of the spring obeys Hooke's law: $F_s = kd$.

SIMPLIFY: Using Newton's Second Law, the pressure of the gas is:

$$\sum_{i=1}^{n} F_{y,i} = F_g - F_s - F_{atm} = p_g A - kd - p_{atm} A = 0 \Rightarrow p_g = \frac{kd}{A} + p_{atm}.$$

By the Ideal Gas Law, the pressure of the gas is

$$p_g = \frac{nRT_f}{V_f} = \frac{nRT_f}{A(L_0 + d)}.$$

Substituting for p_g gives:

$$\frac{kd}{A} + p_{atm} = \frac{nRT_f}{A(L_0 + d)}$$

$$(kd + p_{atm}A)(L_0 + d) = nRT_f$$

$$kd^2 + kdL_0 + p_{atm}Ad + p_{atm}AL_0 - nRT_f = 0$$

$$kd^2 + (kL_0 + p_{atm}A)d + p_{atm}AL_0 - nRT_f = 0$$

To solve for L_0, use the Ideal Gas Law for the initial conditions:

$$p_{atm}V_i = p_{atm}(AL_0) = nRT_i \Rightarrow L_0 = \frac{nRT_i}{p_{atm}A}.$$

Substituting this into the quadratic equation above gives:

$$kd^2 + \left(\frac{knRT_i}{p_{atm}A} + p_{atm}A\right)d + p_{atm}A\frac{nRT_i}{p_{atm}A} - nRT_f = 0$$

$$kd^2 + \left(\frac{knRT_i}{p_{atm}A} + p_{atm}A\right)d + nR(T_i - T_f) = 0.$$

Let $a = k$, $b = \left(\frac{knRT_i}{p_{atm}A} + p_{atm}A\right)$ and $c = nR(T_i - T_f)$. Then the quadratic formula can be used to solve for the amount of compression: $d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

CALCULATE: $a = 1000.\text{ N/m}$

$$b = \frac{(1000.\text{ N/m})(0.005000\text{ mol})(8.314\text{ J/(mol K)})(296\text{ K})}{(101.3\cdot 10^3\text{ Pa})(1.20\cdot 10^{-3}\text{ m}^2)} + (101.3\cdot 10^3\text{ Pa})(1.20\cdot 10^{-3}\text{ m}^2) = 222.78\text{ N}$$

$$c = (0.005000\text{ mol})(8.314\text{ J/(mol K)})((296\text{ K}) - (423\text{ K})) = -5.2794\text{ N m}$$

$$d = \frac{-(222.78\text{ N}) \pm \sqrt{(222.78\text{ N})^2 - 4(1000.\text{ N/m})(-5.2794\text{ N}\cdot\text{m})}}{2(1000.\text{ N/m})} = 0.02160\text{ m or } -0.2444\text{ m}$$

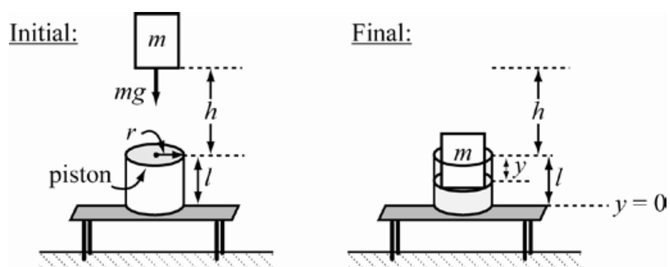
The correct answer is $d = 0.02160$ m, as the negative sign on the other value obtained from the quadratic formula indicates that the spring would have been stretched which is not consistent with the physical situation.

ROUND: To three significant figure, the spring is compressed by $d = 2.16$ cm.

DOUBLE-CHECK: Since the spring is rather stiff with a spring constant of 1000 N/m a small amount of compression might have been expected for the given temperature increase.

19.44 **THINK:** The cylinder of air with radius $r = 0.200$ m and length $l = 0.200$ m is at a pressure of $p_\text{i} = 1.00\text{ atm} = 101.3\cdot 10^3$ Pa. A block of mass $m = 20.0$ kg is dropped onto the piston. The principle of conservation of energy and the Ideal Gas Law can be used to calculate the height, h, from which the block must be dropped to compress the piston by $y_1 = 1.00\text{ mm} = 1.00\cdot 10^{-3}$ m, $y_2 = 2.00\text{ mm} = 2.00\cdot 10^{-3}$ m and $y_3 = 1.00\text{ cm} = 1.00\cdot 10^{-2}$ m.

SKETCH:



RESEARCH: The initial energy is the gravitational potential energy of the block, $E_\text{i} = mg(h+l)$. The reference point for the gravitational potential energy is the table top, at $y = 0$. The final energy of the system is $E_\text{f} = mg(l-y) + \Delta E_\text{int}$, where ΔE_int is the increase in internal energy of the air. Assuming that the compression is rapid so that there is no heat flow out of the cylinder, ΔE_int is due to the work the block does in compressing the air and is given by:

$$\Delta E_\text{int} = -W = nRT_\text{i}\ln\left(\frac{V_\text{i}}{V_\text{f}}\right),$$

where the negative sign is used because work is done on the air. The Ideal Gas Law can be used to relate the initial temperature, T_i, to the initial pressure, p_i, and volume, $V_\text{i} = \pi r^2 l$. The final volume of the cylinder is $V_\text{f} = \pi r^2(l-y)$.

SIMPLIFY: Conservation of energy gives:

$$E_\text{i} = E_\text{f} \Rightarrow mg(h+l) = mg(l-y) + nRT_\text{i}\ln\left(\frac{V_\text{i}}{V_\text{f}}\right) \Rightarrow h = \frac{nRT_\text{i}}{mg}\ln\left(\frac{V_\text{i}}{V_\text{f}}\right) - y.$$

The Ideal Gas Law gives $nRT_{\text{i}} = p_{\text{i}}V_{\text{i}}$. Substituting this and the equations for the initial and final volumes into the equation for h gives:

$$h = \frac{p_{\text{i}}\pi r^2 l}{mg}\ln\left(\frac{\pi r^2 l}{\pi r^2 (l-y)}\right) - y = \frac{p_{\text{i}}\pi r^2 l}{mg}\ln\left(\frac{l}{l-y}\right) - y.$$

CALCULATE: For $y_1 = 1.00 \cdot 10^{-3}$ m,

$$h_1 = \frac{\left(101.3\cdot 10^3 \text{ Pa}\right)\pi\left(0.200 \text{ m}\right)^2\left(0.200 \text{ m}\right)}{\left(20.0 \text{ kg}\right)\left(9.81 \text{ m/s}^2\right)}\ln\left(\frac{\left(0.200 \text{ m}\right)}{\left(0.200 \text{ m}\right)-\left(1.00\cdot 10^{-3} \text{ m}\right)}\right) - \left(1.00\cdot 10^{-3} \text{ m}\right) = 0.0640 \text{ m}$$

Similarly, for $y_2 = 2.00\cdot 10^{-3}$ m, $h_2 = 0.1284$ m and for $y_3 = 1.00\cdot 10^{-2}$ m, $h_3 = 0.6556$ m.

ROUND: To three significant figures the answers are: $h_1 = 6.40$ cm, $h_2 = 12.8$ cm and $h_3 = 65.6$ cm.

DOUBLE-CHECK: It is expected that the block must be dropped from a greater height $(h_3 > h_2 > h_1)$ in order for the piston to be compressed by a larger amount $(y_3 > y_2 > y_1)$.

19.45 The number density of atomic hydrogen (H) is

$$\frac{N}{V} = \frac{1.00 \text{ atom}}{\text{cm}^3}\left(\frac{100 \text{ cm}}{1 \text{ m}}\right)^3 = 1\cdot 10^6 \text{ atoms / m}^3.$$

The temperature is $T = 2.73$ K.

(a) The pressure can be found with the Ideal Gas Law: $pV = Nk_{\text{B}}T$.

$$p = \frac{N}{V}k_{\text{B}}T = \left(1.00\cdot 10^6 \text{ m}^{-3}\right)\left(1.381\cdot 10^{-23} \text{ J/K}\right)\left(2.73 \text{ K}\right) = 3.77\cdot 10^{-17} \text{ Pa}$$

This is a very high vacuum!

(b) The rms speed is $v_{\text{rms}} = \sqrt{3k_{\text{B}}T / m_{\text{H}}}$ and $m_{\text{H}} = 1.008$ u, where u $= 1.661\cdot 10^{-27}$ kg.

$$v_{\text{rms}} = \sqrt{\frac{3\left(1.381\cdot 10^{-23} \text{ J/K}\right)\left(2.73 \text{ K}\right)}{\left(1.008\right)\left(1.661\cdot 10^{-27} \text{ kg}\right)}} = 260. \text{ m/s}.$$

(c) The energy of a monatomic gas is $E_{\text{tot}} = NK_{\text{ave}} = (3/2)Nk_{\text{B}}T$. Given that the number density is $N/V = 10^6$ atoms/m^3, the energy density is:

$$\frac{E_{\text{tot}}}{V} = \frac{3}{2}\frac{N}{V}k_{\text{B}}T \Rightarrow \frac{E_{\text{tot}}}{V} = \frac{3}{2}\left(1.00\cdot 10^6 \text{ m}^{-3}\right)\left(1.381\cdot 10^{-23} \text{ J/K}\right)\left(2.73 \text{ K}\right) = 5.6552\cdot 10^{-17} \text{ J/m}^3.$$

An energy of $E = 1.00$ J would require a cube edge length of:

$$L = \left(\frac{E}{E_{\text{tot}} / V}\right)^{1/3} = \left(\frac{1.00 \text{ J}}{5.6552\cdot 10^{-17} \text{ J/m}^3}\right)^{1/3} = 260{,}525.76 \text{ m} = 261 \text{ km}.$$

19.46 The rms speed is given by $v_{\text{rms}} = \sqrt{\frac{3k_{\text{B}}T}{m}}$.

(a) The mass of ^{4}He is $m = 4.003 \text{ u} = \left(4.003\right)\left(1.661\cdot 10^{-27} \text{ kg}\right)$. At $T = 300.$ K,

$$v_{\text{rms},\,^4\text{He}} = \sqrt{\frac{3\left(1.381\cdot 10^{-23} \text{ J/K}\right)\left(300. \text{ K}\right)}{\left(4.003\right)\left(1.661\cdot 10^{-27} \text{ kg}\right)}} = 1.37 \text{ km/s}.$$

(b) The mass of ^{3}He is $m = 3.016 \text{ u} = \left(3.016\right)\left(1.661\cdot 10^{-27} \text{ kg}\right)$. At $T = 300.$ K,

$$v_{\text{rms},\,^3\text{He}} = \sqrt{\frac{3\left(1.381\cdot 10^{-23} \text{ J/K}\right)\left(300. \text{ K}\right)}{\left(3.016\right)\left(1.661\cdot 10^{-27} \text{ kg}\right)}} = 1.58 \text{ km/s}.$$

It is reasonable that ^3He has a greater v_{rms} than ^4He at the same temperature because it has less mass.

19.47 The mass of $^{235}\text{UF}_6$ is $m_{235} = 349.03$ amu. The mass of $^{238}\text{UF}_6$ is $m_{238} = 352.04$ amu. The ratio of rms speeds for these molecules is:

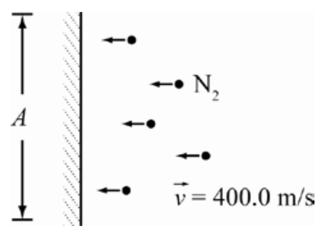
$$\frac{v_{\text{rms},235}}{v_{\text{rms},238}} = \frac{\sqrt{\dfrac{3k_{\text{B}}T}{m_{235}}}}{\sqrt{\dfrac{3k_{\text{B}}T}{m_{238}}}} = \sqrt{\frac{m_{238}}{m_{235}}} = \sqrt{\frac{352.04 \text{ amu}}{349.03 \text{ amu}}} = 1.004302694.$$

Therefore, the rms speed of $^{235}\text{UF}_6$ is 1.0043 times that of $^{238}\text{UF}_6$.

19.48 The electrons are like an ideal gas, so their rms speed is $v_{\text{rms}} = \sqrt{\dfrac{3k_{\text{B}}T}{m_{\text{e}}}}$ with $m_{\text{e}} = 9.109 \cdot 10^{-31}$ kg and $T = 300.0$ K. The rms speed is: $v_{\text{rms}} = \sqrt{\dfrac{3\left(1.381 \cdot 10^{-23} \text{ J/K}\right)\left(300.0 \text{ K}\right)}{\left(9.109 \cdot 10^{-31} \text{ kg}\right)}} = 1.168 \cdot 10^5$ m/s.

19.49 **THINK:** In a period of $\Delta t = 6.00$ s, $N = 9.00 \cdot 10^{23}$ nitrogen molecules strike a wall with an area of $A = 2.00 \cdot 10^{-4} \text{ m}^2$. The molecules move with a speed of $v = 400.0$ m/s and strike the wall head-on in elastic collisions. Find the pressure p exerted on the wall. Using the principle of conservation of linear momentum for elastic collisions, the pressure on the wall can be calculated.

SKETCH:



RESEARCH: The pressure is $p = F_{\text{ave}} / A$, where $F_{\text{ave}} = \left|\Delta \vec{p}_{\text{tot}}\right| / \Delta t$ is the average force that acts on the wall, $\left|\Delta \vec{p}_{\text{tot}}\right|$ is the absolute value of the change in the linear momentum of all of the molecules, and A is the area over which the force is acting. In the head-on elastic collisions that the nitrogen molecules undergo, the final velocity has reversed direction with respect to the initial velocity, so

$$\Delta \vec{p} = m\Delta v = m\left(v_{\text{f}} - v_{\text{i}}\right) = m\left(v - \left(-v\right)\right) = 2mv.$$

For the total change in linear momentum of N particles, $\Delta \vec{p}_{\text{tot}} = N\Delta p.$ The mass of one N_2 molecule is $m = 4.65 \cdot 10^{-26}$ kg.

SIMPLIFY: $p = \dfrac{F_{\text{ave}}}{A} = \dfrac{\left|\Delta \vec{p}_{\text{tot}}\right|}{A\Delta t} = \dfrac{2Nmv}{A\Delta t}$

CALCULATE: $p = \dfrac{2\left(9.00 \cdot 10^{23}\right)\left(4.65 \cdot 10^{-26} \text{ kg}\right)\left(400.0 \text{ m/s}\right)}{\left(2.00 \cdot 10^{-4} \text{ m}^2\right)\left(6.00 \text{ s}\right)} = 2.790 \cdot 10^4$ Pa

ROUND: To three significant figures, the pressure on the wall is $p = 27.9$ kPa.

DOUBLE-CHECK: This pressure is about one quarter of atmospheric pressure so it is reasonable considering the momentum of the N_2 molecules.

19.50 **THINK:** The equation for rms speed can be used to find the temperature T_{He} at which the rms speed of a helium atom is equal to the rms speed of an air molecule at $T = 273.15$ K.

SKETCH: A sketch is not needed to solve the problem.

RESEARCH: The rms speed for an ideal gas is given by $v_{\text{rms}} = \sqrt{3k_{\text{B}}T/m}$. The molar masses of helium and air are $M_{\text{He}} = 4.00$ g/mol and $M_{\text{air}} = 28.97$ g/mol.

SIMPLIFY: $v_{\text{rms,He}} = v_{\text{rms,air}} \Rightarrow \sqrt{\dfrac{3k_{\text{B}}T_{\text{He}}}{m_{\text{He}}}} = \sqrt{\dfrac{3k_{\text{B}}T}{m_{\text{air}}}} \Rightarrow T_{\text{He}} = T\dfrac{m_{\text{He}}}{m_{\text{air}}} = T\dfrac{M_{\text{He}}}{M_{\text{air}}}$

CALCULATE: $T_{\text{He}} = (273.15\text{ K})\dfrac{(4.00\text{ g/mol})}{(28.97\text{ g/mol})} = 37.715\text{ K}$

ROUND: To three significant figures the helium temperature needs to be $T_{\text{He}} = 37.7$ K.

DOUBLE-CHECK: The temperature at which $v_{\text{rms,He}} = v_{\text{rms,air}}$ (with $v_{\text{rms,air}}$ at STP) is greater than the boiling point of He (4 K), so this is possible. It is expected that the temperature of helium will be much lower to have the same rms speed since helium is lighter than an air molecule (mainly oxygen and nitrogen).

19.51 The cylinders are made of copper, which has a high thermal conductivity, so the temperature of the gases is equal to the temperature of the water, $T = 50.0\text{ °C} = 323$ K. The fact that the volume of He is twice the volume of N_2 is irrelevant.

(a) Helium is monatomic. The average kinetic energy K_{ave} of a monatomic gas is $K_{\text{ave}} = 3k_{\text{B}}T/2$:

$$K_{\text{ave, He}} = \frac{3}{2}\left(1.381\cdot 10^{-23}\text{ J/K}\right)(323\text{ K}) = 6.69\cdot 10^{-21}\text{ J}.$$

Nitrogen is diatomic. The average kinetic energy K_{ave} of a diatomic gas is $K_{\text{ave}} = 5k_{\text{B}}T/2$:

$$K_{\text{ave, N}_2} = \frac{5}{2}\left(1.381\cdot 10^{-23}\text{ J/K}\right)(323\text{ K}) = 1.12\cdot 10^{-20}\text{ J}.$$

(b) The molar specific heats at constant volume and constant pressure for helium (a monatomic gas) are:

$$C_{V,\text{He}} = \frac{3}{2}R = \frac{3}{2}\left(8.31\text{ J/}(\text{mol K})\right) = 12.5\text{ J/}(\text{mol K})$$

$$C_{p,\text{He}} = C_{V,\text{He}} + R = \frac{5}{2}R = \frac{5}{2}\left(8.31\text{ J/}(\text{mol K})\right) = 20.8\text{ J/}(\text{mol K})$$

The molar specific heats at constant volume and constant pressure for nitrogen (a diatomic gas) are:

$$C_{V,\text{N}_2} = \frac{5}{2}R = \frac{5}{2}\left(8.31\text{ J/}(\text{mol K})\right) = 20.8\text{ J/}(\text{mol K})$$

$$C_{p,\text{N}_2} = C_{V,\text{N}_2} + R = \frac{7}{2}R = \frac{7}{2}\left(8.31\text{ J/}(\text{mol K})\right) = 29.1\text{ J/}(\text{mol K})$$

(c) The adiabatic coefficient is $\gamma = C_p / C_V$. Therefore,

$$\gamma_{\text{He}} = \frac{C_{p,\text{He}}}{C_{V,\text{He}}} = \frac{5}{3}\text{ and }\gamma_{\text{N}_2} = \frac{C_{p,\text{N}_2}}{C_{V,\text{N}_2}} = \frac{7}{5}.$$

19.52 Room temperature is $T = 293$ K. One cylinder contains $n_{\text{N}_2} = 10$ mol of N_2 gas and the other contains $n_{\text{Ar}} = 10$ mol of argon gas. For an ideal monatomic gas, the internal energy is $E_{\text{int}} = (3/2)nRT$, while for an ideal diatomic gas, the internal energy is $E_{\text{int}} = (5/2)nRT$. The ratio of energies is then

$$\frac{E_{\text{N}_2}}{E_{\text{Ar}}} = \frac{\frac{5}{2}n_{\text{N}_2}RT}{\frac{3}{2}n_{\text{Ar}}RT} = \frac{5n_{\text{N}_2}}{3n_{\text{Ar}}} = \frac{5}{3},\text{ since } n_{\text{N}_2} = n_{\text{Ar}}.$$

19.53 The temperature of $n = 1.00$ mol of a diatomic ideal gas is increased by $\Delta T = 2.00$ K. Assuming a constant volume, the change in internal energy is $\Delta E_{\text{int}} = nC_V \Delta T$ where $C_V = (5/2)R$ for a diatomic gas:

$$\Delta E_{\text{int}} = \frac{5}{2} nR\Delta T = \frac{5}{2}(1.00 \text{ mol})(8.314 \text{ J/(mol K)})(2.00 \text{ K}) = 41.6 \text{ J}.$$

19.54 Assume air is an ideal gas of diatomic molecules. The volume of the room does not change, the initial temperature is $T_\text{i} = 293$ K, the final temperature is $T_\text{f} = 295$ K, and the initial pressure is $p_\text{i} = 101$ kPa. The number of moles of air in the room does not change. The heat required to raise the temperature of the air in the room is $Q = nC_V \Delta T = (5/2)nR\Delta T$ for a diatomic gas at constant volume. By the Ideal Gas Law, the number of moles is $n = p_\text{i} V/(RT_\text{i})$. The amount of heat required is

$$Q = \frac{5}{2}\frac{p_\text{i} V}{RT_\text{i}} R\Delta T = \frac{5}{2}\frac{p_\text{i} V}{T_\text{i}}(T_\text{f} - T_\text{i}) = \frac{5}{2}\frac{(1.01 \cdot 10^5 \text{ Pa})(8.00 \text{ m})(10.0 \text{ m})(3.00 \text{ m})(295 \text{ K} - 293 \text{ K})}{(293 \text{ K})} = 414 \text{ kJ}.$$

19.55 **THINK:** A volume $V = 1.00$ L of air is heated by $\Delta T = 100.$ K. The process is isochoric (volume is constant). The amount of heat energy required can be calculated by determining the number of moles of each major constituent in the air and using the Ideal Gas Law.

SKETCH: A sketch is not needed to solve the problem.

RESEARCH: Air is composed approximately of 78.08% N_2, 20.95% O_2 and 0.93% Ar. The number of moles of each constituent can be found from the Ideal Gas Law. The total energy required to raise the temperature of the air will be the sum of the energies required to raise the temperature of each constituent, where the energy is the heat that is required: $Q = nC_V \Delta T$. The molar specific heats at constant volume are: $C_{V,\text{N}_2} = 20.7$ J/(mol K), $C_{V,\text{O}_2} = 21.0$ J/(mol K), and $C_{V,\text{Ar}} = 12.5$ J/(mol K). Assume the air is initially at STP.

SIMPLIFY: $n_{\text{tot}} = \dfrac{pV}{RT}$, $n_{\text{N}_2} = 0.7808 n_{\text{tot}}$, $n_{\text{O}_2} = 0.2095 n_{\text{tot}}$, $n_{\text{Ar}} = 0.0093 n_{\text{tot}}$, $Q_{\text{N}_2} = n_{\text{N}_2} C_{V,\text{N}_2} \Delta T$, $Q_{\text{O}_2} = n_{\text{O}_2} C_{V,\text{O}_2} \Delta T$, $Q_{\text{Ar}} = n_{\text{Ar}} C_{V,\text{Ar}} \Delta T$, $Q_{\text{tot}} = Q_{\text{N}_2} + Q_{\text{O}_2} + Q_{\text{Ar}}$.

CALCULATE:

$$n_{\text{tot}} = \frac{(1.013 \cdot 10^5 \text{ Pa})(1.00 \cdot 10^{-3} \text{ m}^3)}{(8.314 \text{ J/(mol K)})(273 \text{ K})} = 0.04463 \text{ mol}$$

$$Q_{\text{N}_2} = 0.7808(0.04463 \text{ mol})(20.7 \text{ J/(mol K)})(100 \text{ K}) = 72.14 \text{ J}$$

$$Q_{\text{O}_2} = 0.2095(0.04463 \text{ mol})(21.0 \text{ J/(mol K)})(100 \text{ K}) = 19.64 \text{ J}$$

$$Q_{\text{Ar}} = 0.0093(0.04463 \text{ mol})(12.5 \text{ J/(mol K)})(100 \text{ K}) = 0.5188 \text{ J}$$

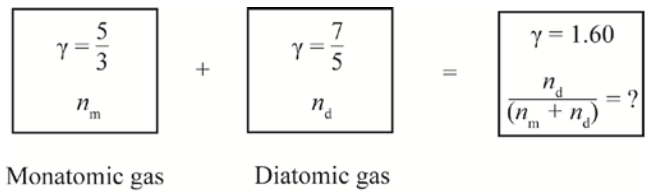
$$Q_{\text{tot}} = 92.29 \text{ J}$$

ROUND: To three significant figures, the approximate energy required to heat the air is $Q_{\text{tot}} = 92.3$ J.

DOUBLE-CHECK: Since $n_{\text{N}_2} > n_{\text{O}_2} > n_{\text{Ar}}$, it is reasonable that $Q_{\text{N}_2} > Q_{\text{O}_2} > Q_{\text{Ar}}$. The value for Q_{tot} is reasonable for 1.00 L of air.

19.56 **THINK:** The ratio of the molar specific heats of the gas that is to be made is $\gamma = 1.60$. No pure gas has such a γ value, but by mixing monatomic and diatomic gases one can obtain a gas with such a γ value. The fraction of diatomic molecules in the mixture is just the number of diatomic moles divided by the total number of moles.

SKETCH:



RESEARCH: The change in internal energy of a monatomic gas is $\Delta E_{\text{int,m}} = (3/2)n_{\text{m}}R\Delta T$, where n_{m} is the number of moles of the monatomic gas. The change in internal energy of a diatomic gas is $\Delta E_{\text{int,d}} = (5/2)n_{\text{d}}R\Delta T$, where n_{d} is the number of moles of the diatomic gas. The total change in thermal energy of the mixture is $\Delta E_{\text{int}} = \Delta E_{\text{int,m}} + \Delta E_{\text{int,d}}$. The total thermal energy is also given by $\Delta E_{\text{int}} = (n_{\text{m}} + n_{\text{d}})C_V \Delta T$. The problem requires that $\gamma = C_p / C_V = (C_V + R)/C_V = 1.60$, or $R/C_V = 0.60$.

SIMPLIFY: $\Delta E_{\text{int}} = \Delta E_{\text{int,m}} + \Delta E_{\text{int,d}} = (n_{\text{m}} + n_{\text{d}})C_V \Delta T$

$$\frac{1}{2}(3n_{\text{m}} + 5n_{\text{d}})R\Delta T = (n_{\text{m}} + n_{\text{d}})C_V\Delta T$$

$$(3n_{\text{m}} + 5n_{\text{d}})(R/C_V) = 2n_{\text{m}} + 2n_{\text{d}}$$

$$\left(3(R/C_V) - 2\right)n_{\text{m}} = \left(2 - 5(R/C_V)\right)n_{\text{d}}$$

$$n_{\text{m}} = \frac{2 - 5(R/C_V)}{3(R/C_V) - 2}n_{\text{d}}$$

Therefore, $\dfrac{n_{\text{d}}}{n_{\text{d}} + n_{\text{m}}} = \dfrac{n_{\text{d}}}{n_{\text{d}} + \dfrac{2-5(R/C_V)}{3(R/C_V)-2}n_{\text{d}}} = \dfrac{1}{1 + \dfrac{2-5(R/C_V)}{3(R/C_V)-2}}.$

CALCULATE: $\dfrac{n_{\text{d}}}{n_{\text{d}} + n_{\text{m}}} = \dfrac{1}{1 + \dfrac{2-5(0.60)}{3(0.60)-2}} = 0.16667$

ROUND: To two significant figures, the fraction of diatomic molecules required for this mixture is 0.17.

DOUBLE-CHECK: Since the given γ value is closer to that for monatomic gases, it is reasonable that a smaller fraction of diatomic molecules is required.

19.57 An initial volume $V_{\text{i}} = 15.0$ L of an ideal monatomic gas at a pressure of $p_{\text{i}} = 1.50 \cdot 10^5$ kPa is expanded adiabatically until the volume **is** doubled, $V_{\text{f}} = 2V_{\text{i}}$. The ratio of the molar specific heats for a monatomic gas is $\gamma = 5/3$.

(a) For an adiabatic gas, pV^γ is constant. The pressure of the gas after the adiabatic expansion is:

$$p_{\text{f}}V_{\text{f}}^{\gamma} = p_{\text{i}}V_{\text{i}}^{\gamma} \Rightarrow p_{\text{f}} = p_{\text{i}}\left(\frac{V_{\text{i}}}{V_{\text{f}}}\right)^{\gamma} = (1.50 \cdot 10^5 \text{ kPa})\left(\frac{V_{\text{i}}}{2V_{\text{i}}}\right)^{5/3} = (1.50 \cdot 10^5 \text{ kPa})\left(\frac{1}{2}\right)^{5/3} = 4.72 \cdot 10^4 \text{ kPa}.$$

(b) Suppose the initial temperature of the gas was $T_{\text{i}} = 300.$ K. For an adiabatic expansion, $TV^{\gamma-1}$ is constant. The temperature of the gas after the adiabatic expansion is:

$$T_{\text{f}}V_{\text{f}}^{\gamma-1} = T_{\text{i}}V_{\text{i}}^{\gamma-1} \Rightarrow T_{\text{f}} = T_{\text{i}}\left(\frac{V_{\text{i}}}{V_{\text{f}}}\right)^{\gamma-1} = T_{\text{i}}\left(\frac{V_{\text{i}}}{2V_{\text{i}}}\right)^{\gamma-1} = (300. \text{ K})\left(\frac{1}{2}\right)^{(5/3)-1} = 189 \text{ K}.$$

19.58 The compression ratio (ratio of volumes) of a specific diesel engine is 20.0 to 1.00. Air enters a cylinder at pressure $p_{\text{i}} = 1.00$ atm and temperature $T_{\text{i}} = 298$ K, and it is compressed adiabatically. The final pressure is $p_{\text{f}} = 66.0$ atm. Air is dominantly a diatomic molecule, so a good approximation of the ratio of molar specific heats is $\gamma = 7/5$. Assuming the air acts like an ideal gas, the final temperature of the compressed air is:

$$T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1} \Rightarrow T_f = T_i \left(\frac{V_i}{V_f}\right)^{\gamma-1} = (298 \text{ K})(20.0)^{(7/5)-1} = 988 \text{ K}.$$

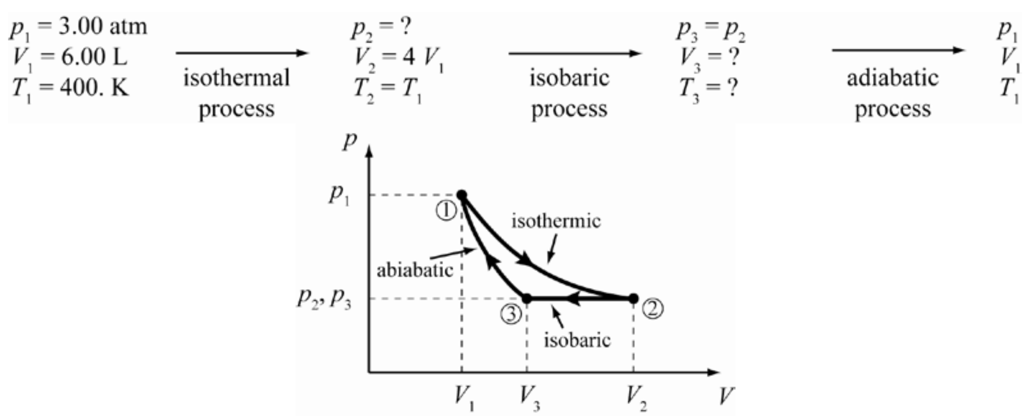
19.59 Air in an engine cylinder is quickly compressed from initial temperature $T_i = 293$ K, pressure $p_i = 1.00$ atm and volume of $V_i = 600.$ cm^3 to a final volume of $V_f = 45.0$ cm^3. Since the compression is rapid, it can be assumed that no heat flow takes place such that the process is adiabatic. For an adiabatic process pV^γ and $TV^{\gamma-1}$ are constant. For an ideal diatomic gas, the ratio of molar specific heats is $\gamma = 7/5$. The final pressure and temperature are

$$p_f V_f^\gamma = p_i V_i^\gamma \Rightarrow p_f = p_i \left(\frac{V_i}{V_f}\right)^\gamma = (1.00 \text{ atm})\left(\frac{600. \text{ cm}^3}{45.0 \text{ cm}^3}\right)^{7/5} = 37.6 \text{ atm},$$

and

$$T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1} \Rightarrow T_f = T_i \left(\frac{V_i}{V_f}\right)^{\gamma-1} = (293 \text{ K})\left(\frac{600. \text{ cm}^3}{45.0 \text{ cm}^3}\right)^{(7/5)-1} = 826 \text{ K}.$$

19.60 **THINK:** The initial volume of the monatomic ideal gas is $V_1 = 6.00$ L, the initial temperature is $T_1 = 400.$ K, and the initial pressure is $p_1 = 3.00$ atm. The gas undergoes an isothermal (constant temperature) expansion to $V_2 = 4V_1$, then an isobaric (constant pressure) compression to pressure p_2 and volume V_2, then an adiabatic compression (no heat transfer) to its original state. Since the gas undergoes an isothermal expansion from state 1 to state 2, $T_2 = T_1$, and since the gas undergoes an isobaric compression from state 2 to state 3, $p_3 = p_2$. The values p_2, V_3, T_3, and the number of moles n can be found by using the Ideal Gas Law and the equation for an adiabatic process.
SKETCH:



RESEARCH: To find the values at state 2 following an isothermal process, the Ideal Gas Law can be used:

$$p_1 V_1 = nRT_1 = nRT_2 = p_2 V_2.$$

To find the values at state 3 following an isobaric process, the Ideal Gas Law can be used:

$$\frac{V_2}{T_2} = \frac{nR}{p_2} = \frac{nR}{p_3} = \frac{V_3}{T_3}.$$

Here, since both V_f and T_f are unknown, another equation is needed. It is provided by the adiabatic process where pV^γ is constant, so $p_3 V_3^\gamma = p_1 V_1^\gamma$. For a monatomic ideal gas, $\gamma = 5/3$. The number of moles n is found by using the Ideal Gas Law: $n = pV/(RT)$.

SIMPLIFY: State 2: $p_1V_1 = p_2V_2 \;\Rightarrow\; p_2 = p_1\left(\dfrac{V_1}{V_2}\right) = p_1\left(\dfrac{V_1}{4V_1}\right) = \dfrac{p_1}{4}$

State 3: From above, $p_1 / p_2 = 4.$ From the adiabatic process back to state 1, the volume is:

$$p_3V_3^{\gamma} = p_1V_1^{\gamma} \;\Rightarrow\; V_3 = V_1\left(\frac{p_1}{p_3}\right)^{1/\gamma} = V_1\left(\frac{p_1}{p_2}\right)^{1/\gamma} = 4^{1/\gamma}V_1$$

The temperature is:

$$T_3 = \frac{V_3}{V_2}T_2 = \frac{4^{1/\gamma}V_1}{V_2}T_2 = 4^{1/\gamma}\frac{V_1}{(4V_1)}T_1 = 4^{(1/\gamma)-1}T_1.$$

To find the number of moles, the values for state 1 can be used,

$$n = \frac{p_1V_1}{RT_1}.$$

CALCULATE: $p_2 = \dfrac{(3.00 \text{ atm})}{4} = 0.750 \text{ atm}$

$V_2 = 4(6.00 \text{ L}) = 24.0 \text{ L}$

$V_3 = 4^{3/5}(6.00 \text{ L}) = 13.78 \text{ L}$

$T_3 = 4^{(3/5)-1}(400. \text{ K}) = 229.7 \text{ K}$

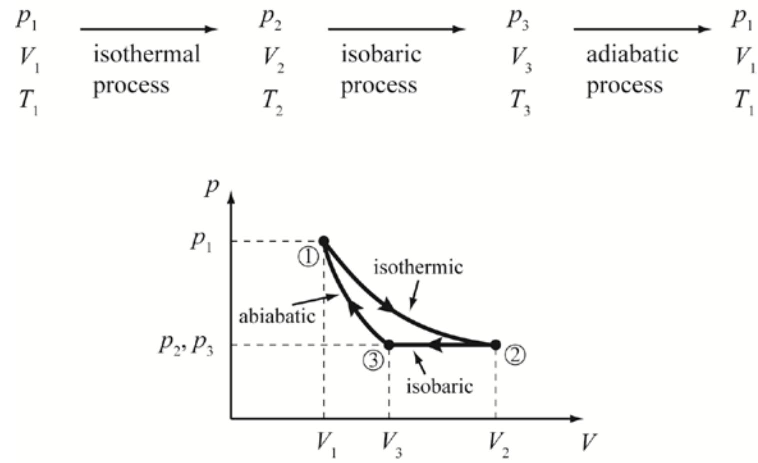
$n = \dfrac{(3.00 \text{ atm})(1.013\cdot 10^5 \text{ Pa/atm})(6.00\cdot 10^{-3} \text{ m}^3)}{(8.314 \text{ J/(mol K)})(400. \text{ K})} = 0.5483 \text{ mol}$

ROUND: To three significant figures the answers are: $p_2 = 0.750$ atm, $V_2 = 24.0$ L, $T_2 = 400.$ K, $p_3 = 0.750$ atm, $V_3 = 13.8$ L, $T_3 = 230.$ K, and $n = 0.548$ mol.

DOUBLE-CHECK: In the isothermal expansion (state 1 to state 2), p_2 should be less than p_1 since V_2 is greater than V_1. In the isobaric compression (state 2 to state 3), V_3 should be less than V_2 since the compression must lead to a smaller volume. In an adiabatic compression (state 3 to state 1), the temperature always increases, so it is expected that T_1 is greater than T_3.

19.61 **THINK:** In problem 19.60, the initial volume of the monatomic ideal gas was $V_1 = 6.00$ L, the initial temperature was $T_1 = 400.$ K and the initial pressure was $p_1 = 3.00$ atm. The gas underwent an isothermal (constant temperature) expansion to $V_2 = 4V_1$, then an isobaric (constant pressure) compression, and then an adiabatic compression (no heat transfer) back to its original state. The number of moles of gas was $n = 0.5483$ mol and the pressure, volume and temperature of state 2 and state 3 were found to be: $p_2 = 0.750$ atm, $V_2 = 24.0$ L, $T_2 = 400.$ K, $p_3 = 0.750$ atm, $V_3 = 13.78$ L, $T_3 = 229.7$ K (all accurate to three significant figures).

SKETCH:



RESEARCH: For an isothermal process, the work done by the gas and the heat flow into the gas is $Q = W = nRT\ln\left(V_{\text{f}}/V_{\text{i}}\right)$. For an isobaric process, the work done by the gas is $W = p\Delta V$ and the heat flow into the gas is $Q = nC_p\Delta T$. For an adiabatic process, the work done is $W = nR\Delta T/\left(1-\gamma\right)$ and the heat flow into the gas is $Q = 0$. For a monatomic ideal gas, $C_p = \left(5/2\right)R$ and the ratio of the molar specific heats is $\gamma = 5/3$. To find the number of moles *n*, use the Ideal Gas Law: $pV = nRT$.

SIMPLIFY: For the isothermal process (state 1 to state 2),

$$Q_{12} = W_{12} = nRT_1\ln\left(\frac{V_2}{V_1}\right).$$

For the isobaric process (state 2 to state 3),

$$W_{23} = p_2\left(V_3 - V_2\right) \text{ and } Q_{23} = \frac{5}{2}nR\left(T_3 - T_2\right).$$

For the adiabatic process (state 3 to state 1),

$$W_{31} = \frac{nR\left(T_1 - T_3\right)}{1-\gamma} \text{ and } Q_{31} = 0.$$

CALCULATE: For the isothermal process (state 1 to state 2),

$$Q_{12} = W_{12} = \left(0.5483\text{ mol}\right)\left(8.314\text{ J/}\left(\text{mol K}\right)\right)\left(400.\text{ K}\right)\ln\left(\frac{\left(24.0\text{ L}\right)}{\left(6.00\text{ L}\right)}\right) = 2.5278\text{ kJ}.$$

For the isobaric process (state 2 to state 3),

$$W_{23} = \left(0.750\text{ atm}\right)\left(1.013\cdot 10^5\text{ Pa/atm}\right)\left(13.78\text{ L} - 24.0\text{ L}\right)\left(\frac{1\text{ m}^3}{1000\text{ L}}\right) = -0.7765\text{ kJ, and}$$

$$Q_{23} = \frac{5}{2}\left(0.5483\text{ mol}\right)\left(8.314\text{ J/}\left(\text{mol K}\right)\right)\left(229.7\text{ K} - 400.\text{ K}\right) = -1.9408\text{ kJ}.$$

For the adiabatic process (state 3 to state 1),

$$W_{31} = \frac{\left(0.5483\text{ mol}\right)\left(8.314\text{ J/}\left(\text{mol K}\right)\right)\left(400.\text{ K} - 229.7\text{ K}\right)}{1-\left(5/3\right)} = -1.1645\text{ kJ, and}$$

$$Q_{31} = 0.$$

ROUND: To three significant figures the answers are: $Q_{12} = W_{12} = 2.53$ kJ, $W_{23} = -0.776$ kJ, $Q_{23} = -1.94$ kJ, $W_{31} = -1.16$ kJ and $Q_{31} = 0$.

DOUBLE-CHECK: The First Law of Thermodynamics is $\Delta E_{\text{int}} = Q - W$. Since the total process is cyclical (starts and ends at state 1), the total change in internal energy must be zero:

$$\Delta E_{\text{int},12} + \Delta E_{\text{int},23} + \Delta E_{\text{int},31} = \left(Q_{12} - W_{12}\right) + \left(Q_{23} - W_{23}\right) + \left(Q_{31} - W_{31}\right)$$
$$= \left(2.5278 \text{ kJ} - 2.5278 \text{ kJ}\right) + \left(-1.9408 \text{ kJ} - \left(-0.7765 \text{ kJ}\right)\right) + \left(0 - \left(-1.1645 \text{ kJ}\right)\right) = 0.$$

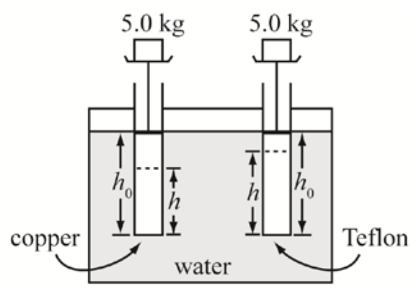
Since the total change in internal energy is zero, the answers are reasonable.

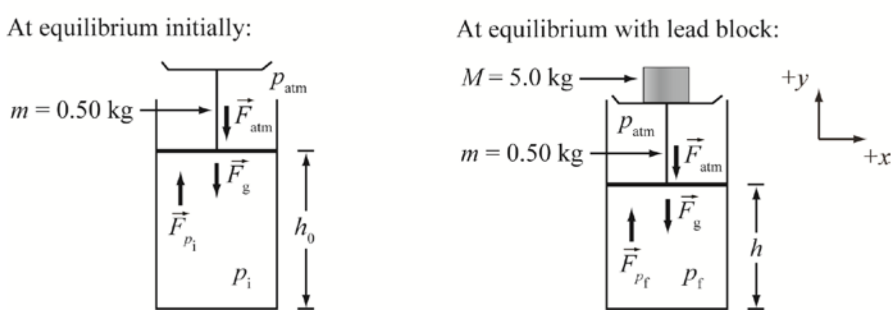
19.62 **THINK:** Two geometrically identical cylinders, one made of copper and the other made of Teflon, are immersed in a large-volume water tank at temperature $T = 20.0$ °C. The diameter of each cylinder is $d = 0.0500$ m. The mass of the piston-rod-platter assembly is $m = 0.500$ kg. The cylinders are filled with helium gas such that in their initial state, both pistons are at equilibrium at $h_0 = 0.200$ m from the bottom of their respective cylinders.

(a) A block of mass $M = 5.00$ kg is placed gently on each platter and the pistons are allowed to slowly reach their equilibrium positions. Since copper has a high thermal conductivity, the helium will stay in thermal equilibrium with the water in the tank and the compression of the helium is an isothermal process. On the other hand, since Teflon has a low thermal conductivity (good insulator) it can be assumed that the compression of the helium is an adiabatic process since there is no heat flow. The change in height can be calculated by using Newton's Second Law and the Ideal Gas Law. The initial and final pressures will be the same in each cylinder at equilibrium since the pressure of the helium gas in each cylinder is supporting the same weight.

(b) When the lead block is dropped suddenly on the platters, the compression of the helium in the copper cylinder will no longer be an isothermal process since there is not time for heat to flow into the helium gas. So the process in both cases is adiabatic.

SKETCH:





RESEARCH:

Initially the three forces acting on each piston are the downward force F_{atm} due to atmospheric pressure p_{atm}, the downward force F_g due to the weight of the piston-rod-platter assembly, and the upward force $F_{p_{\text{i}}}$ due to the initial pressure p_{i} of the compressed helium. The net force on the piston is zero, so by Newton's Second Law:

$$F_{p_i} - F_g - F_{\text{atm}} = p_i A - mg - p_{\text{atm}} A = 0 \Rightarrow p_i = p_{\text{atm}} + \frac{mg}{A}.$$

At equilibrium after the lead block is placed on the platter, there are three forces acting on each piston: the force F_{atm} due to atmospheric pressure p_{atm}, the force F_g due to the weight of the lead block and the piston-rod-platter assembly, and the force F_{p_f} due to the final pressure p_f of the compressed helium. The net force on the piston is zero, so by Newton's Second Law:

$$F_{p_f} - F_g - F_{\text{atm}} = p_f A - (m+M)g - p_{\text{atm}} A = 0 \Rightarrow p_f = p_{\text{atm}} + \frac{(m+M)g}{A}.$$

For the copper cylinder in part (a), the process is isothermal, so the Ideal Gas Law gives: $p_i V_i = nRT = p_f V_f$. For the Teflon cylinder, the process is adiabatic so: $p_i V_i^\gamma = p_f V_f^\gamma$. Helium is a monatomic gas, so the ratio of molar specific heats is $\gamma = 5/3$.

SIMPLIFY:

(a) For each cylinder, $p_i = p_{\text{atm}} + \dfrac{mg}{\pi(d/2)^2}$ and $p_f = p_{\text{atm}} + \dfrac{(m+M)g}{\pi(d/2)^2}$.

Copper: $p_i V_i = p_f V_f \Rightarrow p_i(Ah_0) = p_f(Ah) \Rightarrow h = \dfrac{p_i}{p_f} h_0$

Teflon: $p_i V_i^\gamma = p_f V_f^\gamma \Rightarrow p_i(Ah_0)^\gamma = p_f(Ah)^\gamma \Rightarrow h = \left(\dfrac{p_i}{p_f}\right)^{1/\gamma} h_0$

CALCULATE:

(a) $p_i = (1.013 \cdot 10^5 \text{ Pa}) + \dfrac{(0.500 \text{ kg})(9.81 \text{ m/s}^2)}{\pi((0.0500 \text{ m})/2)^2} = 1.03798 \cdot 10^5 \text{ Pa}$

$p_f = (1.013 \cdot 10^5 \text{ Pa}) + \dfrac{((0.500 \text{ kg}) + (5.00 \text{ kg}))(9.81 \text{ m/s}^2)}{\pi((0.0500 \text{ m})/2)^2} = 1.28779 \cdot 10^5 \text{ Pa}$

Copper: $h = \dfrac{(1.03798 \cdot 10^5 \text{ Pa})}{(1.28779 \cdot 10^5 \text{ Pa})}(0.200 \text{ m}) = 0.1612 \text{ m},$

Teflon: $h = \left(\dfrac{(1.03798 \cdot 10^5 \text{ Pa})}{(1.28779 \cdot 10^5 \text{ Pa})}\right)^{3/5}(0.200 \text{ m}) = 0.1757 \text{ m}$

(b) If the blocks are dropped *suddenly* on their platters, the change undergone by the helium in both the copper and Teflon cylinders is adiabatic. Therefore, the final equilibrium heights will be the same as that calculated in part (a) for the Teflon cylinder.

ROUND: (a) To three significant figures, the final height of the piston in the copper cylinder is 16.1 cm and the final height of the piston in the Teflon cylinder is 17.6 cm.

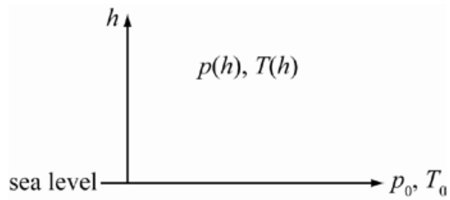
(b) To three significant figures, the final height of the piston in the copper cylinder is 17.6 cm and the final height of the piston in the Teflon cylinder is 17.6 cm.

DOUBLE-CHECK: It is reasonable that the final height of the piston in the Teflon cylinder is larger since the helium gas in the Teflon cylinder retains all of the energy from the compression. On the other hand, the helium gas in the copper cylinder loses energy in the form of heat to the water reservoir, so its piston falls farther.

19.63 **THINK:** To a good approximation, the pressure variation with altitude in the Earth's atmosphere can be treated as adiabatic. Air has an effective molar mass of $M_{\text{air}} = 0.02897$ kg/mol. The pressure at sea level is $p_0 = 101.0$ kPa and the temperature at sea level is $T_0 = 293.2$ K.

(a) Find the air pressure and temperature as functions of altitude: $p(h)$ and $T(h)$.

(b) Find the altitude at which the air pressure and the density are half their sea-level values, and the corresponding temperature.
(c) Compare the results with the isothermal model of Chapter 13.
SKETCH:



RESEARCH: The Ideal Gas Law will be useful: $pV = nRT$. Since $n = m/M = \rho V/M$, the pressure can be expressed as $p = \rho RT/M$, where ρ is the density of the air and M is the effective molar mass of the air. Since V is inversely proportional to ρ and, for an adiabatic process, pV^γ is constant, the density ρ is proportional to $p^{1/\gamma}$: $\rho/\rho_0 = (p/p_0)^{1/\gamma}$. Since air can be treated as a diatomic gas, $\gamma = 7/5$. The variation of pressure with altitude h is determined by the equation of hydrostatic equilibrium, $dp/dh = -\rho g$, where g is taken to be constant within the Earth's atmosphere. From this, the pressure variation with altitude $p(h)$ can be found. The isothermal model has pressure and density varying exponentially with altitude as the temperature remains constant:

$$p_{\text{iso}}(h) = p_0 \exp\left(-\frac{Mgh}{RT_0}\right),\ \rho_{\text{iso}}(h) = \rho_0 \exp\left(-\frac{Mgh}{RT_0}\right),\ \text{and } T_{\text{iso}} = T_0.$$

SIMPLIFY:

(a) Since $\rho = \rho_0\left(\dfrac{p}{p_0}\right)^{1/\gamma}$, $\dfrac{dp}{dh} = -\rho g$, and from the Ideal Gas Law $\dfrac{\rho_0}{p_0} = \dfrac{M}{RT_0}$:

$$\frac{d(p/p_0)}{dh} = -\frac{M}{RT_0} g\left(\frac{p}{p_0}\right)^{1/\gamma}.$$

Hence,

$$\int_1^{p/p_0} \frac{d(p'/p_0)}{(p'/p_0)^{1/\gamma}} = -\frac{Mg}{RT_0}\int_0^h dh' \Rightarrow \frac{\gamma}{\gamma-1}\left[\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] = -\frac{Mgh}{RT_0}.$$

Therefore, the pressure as a function of altitude is:

$$p(h) = p_0\left(1 - \left(\frac{\gamma-1}{\gamma}\right)\frac{Mgh}{RT_0}\right)^{\frac{\gamma}{\gamma-1}}.$$

Since $\rho = \rho_0\left(\dfrac{p}{p_0}\right)^{1/\gamma}$, the air density as a function of altitude is:

$$\rho(h) = \rho_0\left(1 - \left(\frac{\gamma-1}{\gamma}\right)\frac{Mgh}{RT_0}\right)^{\frac{1}{\gamma-1}}.$$

To find the temperature as a function of altitude, the Ideal Gas Law can be applied:

$$p = \frac{\rho RT}{M} \Rightarrow T = \frac{Mp}{R\rho} \Rightarrow \frac{T}{T_0} = \frac{p/p_0}{\rho/\rho_0}$$

$$T(h) = T_0 - \left(\frac{\gamma-1}{\gamma}\right)\frac{Mgh}{R}.$$

(b) When $p(h)/p_0 = 1/2$ and since $\gamma = 7/5$:

$$\frac{1}{2} = \left(1 - \left(\frac{\gamma-1}{\gamma}\right)\frac{Mgh_{p/2}}{RT_0}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 - \left(\frac{2}{7}\right)\frac{Mgh_{p/2}}{RT_0}\right)^{7/2} \Rightarrow h_{p/2} = \frac{7}{2}\left(1 - \left(\frac{1}{2}\right)^{2/7}\right)\frac{RT_0}{Mg}.$$

When $\rho(h)/\rho_0 = 1/2$ and since $\gamma = 7/5$:

$$\frac{1}{2} = \left(1 - \left(\frac{\gamma-1}{\gamma}\right)\frac{Mgh_{\rho/2}}{RT_0}\right)^{\frac{1}{\gamma-1}} = \left(1 - \left(\frac{2}{7}\right)\frac{Mgh_{\rho/2}}{RT_0}\right)^{5/2} \Rightarrow h_{\rho/2} = \frac{7}{2}\left(1 - \left(\frac{1}{2}\right)^{2/5}\right)\frac{RT_0}{Mg}.$$

(c) In the isothermal model, the altitude $h_{\text{iso},1/2}$ at which $p_{\text{iso}}(h)/p_0 = 1/2$ and $\rho_{\text{iso}}(h)/\rho_0 = 1/2$ is:

$$h_{\text{iso},1/2} = -\frac{RT_0}{Mg}\ln\left(\frac{p_{\text{iso}}}{p_0}\right) = -\frac{RT_0}{Mg}\ln\left(\frac{\rho_{\text{iso}}}{\rho_0}\right) = \frac{RT_0}{Mg}\ln 2.$$

CALCULATE:

(b) $h_{p/2} = \frac{7}{2}\left(1 - \left(\frac{1}{2}\right)^{2/7}\right)\frac{(8.314 \text{ J/(mol K)})(293.2 \text{ K})}{(0.02897 \text{ kg/mol})(9.81 \text{ m/s}^2)} = 5393.7 \text{ m}$

$T_{p/2} = (293.2 \text{ K}) - \frac{((7/5)-1)}{(7/5)}\frac{(0.02897 \text{ kg/mol})(9.81 \text{ m/s}^2)(5393.7 \text{ m})}{(8.314 \text{ J/(mol K)})} = 240.52 \text{ K}$

$h_{\rho/2} = \frac{7}{2}\left(1 - \left(\frac{1}{2}\right)^{2/5}\right)\frac{(8.314 \text{ J/(mol K)})(293.2 \text{ K})}{(0.02897 \text{ kg/mol})(9.81 \text{ m/s}^2)} = 7269.3 \text{ m}$

$T_{\rho/2} = (293.2 \text{ K}) - \frac{((7/5)-1)}{(7/5)}\frac{(0.02897 \text{ kg/mol})(9.81 \text{ m/s}^2)(7269.3 \text{ m})}{(8.314 \text{ J/(mol K)})} = 222.20 \text{ K}$

(c) $h_{\text{iso},1/2} = \frac{(8.314 \text{ J/(mol K)})(293.2 \text{ K})}{(0.02897 \text{ kg/mol})(9.81 \text{ m/s}^2)}\ln 2 = 5945.4 \text{ m}$

$T_{\text{iso},1/2} = T_0 = 293.2 \text{ K}$

ROUND:

(b) The answers for the adiabatic model are:

$h_{p/2} = 5.39 \text{ km},\ T_{p/2} = 241 \text{ K},\ h_{\rho/2} = 7.27 \text{ km, and } T_{\rho/2} = 222 \text{ K}.$

(c) The answers for the isothermal model are:

$h_{\text{iso},1/2} = 5950 \text{ m}$ and $T_{\text{iso},1/2} = 293 \text{ K}.$

DOUBLE-CHECK: It is expected that the temperature, pressure, and density decrease with increasing height.

19.64 The mass of the nitrogen molecule is $m = 28.0$ amu. The root-mean-square speed is: $v_{\text{rms}} = \sqrt{3k_{\text{B}}T/m}.$ The most probable speed is: $v_{\text{mp}} = \sqrt{2k_{\text{B}}T/m}.$ At temperature $T = 293$ K,

$$v_{\text{rms}} = \sqrt{\frac{3(1.381\cdot 10^{-23} \text{ J/K})(293 \text{ K})}{(28.0)(1.661\cdot 10^{-27} \text{ kg})}} = 511 \text{ m/s and } v_{\text{mp}} = \sqrt{\frac{2(1.381\cdot 10^{-23} \text{ J/K})(293 \text{ K})}{(28.0)(1.661\cdot 10^{-27} \text{ kg})}} = 417 \text{ m/s}.$$

For the most probable speed, the Maxwell speed distribution is given by

$$f(v_{\text{mp}}) = 4\pi\left(\frac{m}{2\pi k_{\text{B}}T}\right)^{3/2} v_{\text{mp}}^2 e^{-\frac{mv_{\text{mp}}^2}{2k_{\text{B}}T}} = 4\sqrt{\frac{m}{2\pi k_{\text{B}}T}}e^{-1}.$$

The probability that a molecule has a most probable speed between v_{mp} and $v_{\text{mp}}+dv$ is given by $f\left(v_{\text{mp}}\right)dv$. For a speed within 1.00 m/s, the variation in the speed is $\Delta v=2.00$ m/s (plus or minus one meter per second). Since 2.00 m/s is small, the approximation $dv \approx \Delta v = 2.00$ m/s is valid. Therefore, the percentage of N_2 molecules within 1.00 m/s of the most probable speed is:

$$f\left(v_{\text{mp}}\right)\Delta v = 4\sqrt{\frac{(28.0)\left(1.661\cdot 10^{-27}\text{ kg}\right)}{2\pi\left(1.381\cdot 10^{-23}\text{ J/K}\right)(293\text{ K})}}e^{-1}(2.00\text{ m/s}) = 0.398\%$$

19.65 In general, the average speed is $v_{\text{ave}}=\sqrt{\frac{8k_{\text{B}}T}{\pi m}}$, and does not depend on the pressure.

(a) The mass of an N_2 molecule is 28.01 amu. At $T=291$ K, the average speed is:

$$v_{\text{ave, N}_2}=\sqrt{\frac{8\left(1.381\cdot 10^{-23}\text{ J/K}\right)(291\text{ K})}{\pi(28.01)\left(1.661\cdot 10^{-27}\text{ kg}\right)}}=469\text{ m/s.}$$

(b) The mass of an H_2 molecule is 2.016 amu. At $T=291$ K, the average speed is:

$$v_{\text{ave, H}_2}=\sqrt{\frac{8\left(1.381\cdot 10^{-23}\text{ J/K}\right)(291\text{ K})}{\pi(2.016)\left(1.661\cdot 10^{-27}\text{ kg}\right)}}=1750\text{ m/s.}$$

19.66 One mole of neon gas is at STP. The mass of a neon atom is 20.2 amu. The Maxwell speed distribution is:

$$f(v)=4\pi\left[\frac{m}{2\pi k_{\text{B}}T}\right]^{3/2}v^2e^{-\frac{mv^2}{2k_{\text{B}}T}}.$$

Take the speed to be the average speed, $v=201.00$ m/s. The probability that an atom has a speed between v and $v+dv$ is given by $f(v)dv$. Since the probability of finding the neon atoms with a speed between 200.00 m/s and 202.00 m/s is constant, the approximation $dv\approx\Delta v=2.00$ m/s is valid. The fraction of neon atoms having a speed between $v+\Delta v$ for $v=201.00$ m/s is:

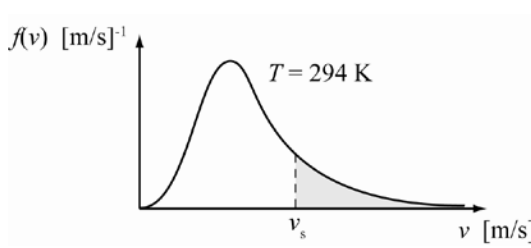
$$f(201.00\text{ m/s})\Delta v=4\pi\left(\frac{(20.2)\left(1.661\cdot 10^{-27}\text{ kg}\right)}{2\pi\left(1.381\cdot 10^{-23}\text{ J/K}\right)(293.15\text{ K})}\right)^{3/2}(201.00\text{ m/s})^2e^{\left(-\frac{(20.2)\left(1.661\cdot 10^{-27}\text{ kg}\right)(201.00\text{ m/s})^2}{2\left(1.381\cdot 10^{-23}\text{ J/K}\right)(293.15\text{ K})}\right)}(2.00\text{ m/s})$$
$$=1.3011\cdot 10^{-3}$$

Therefore, the number of neon atoms with speeds between 200.00 m/s and 202.00 m/s is then:

$$N_{\text{Ne}}=f(201.00\text{ m/s})\Delta v n N_{\text{A}}=\left(1.3011\cdot 10^{-3}\right)(1.00\text{ mol})\left(6.02\cdot 10^{23}\text{ atoms/mol}\right)=7.83\cdot 10^{20}.$$

19.67 **THINK:** The temperature and pressure of air in a room are $T=294$ K and $p=1.00$ atm. The Maxwell speed distribution can be used to find an expression for the fraction of molecules having speeds greater than the speed of sound. The equations for the average speed and rms speed will be used as well. The gas consists of uniform particles with a mass of 15.0 amu.

SKETCH:



RESEARCH: The Maxwell speed distribution is:

$$f(v) = 4\pi\left[\frac{m}{2\pi k_B T}\right]^{3/2} v^2 e^{-\frac{mv^2}{2k_B T}}.$$

The fraction of molecules with a speed greater than the speed of sound v_S is given by:

$$F = \frac{\int_{v_S}^{\infty} f(v)dv}{\int_0^{\infty} f(v)dv} = \int_{v_S}^{\infty} f(v)dv \qquad \left(\text{Recall } \int_0^{\infty} f(v)dv = 1\right).$$

The average speed of the molecules is:

$$v_{\text{ave}} = \sqrt{\frac{8k_B T}{\pi m}}.$$

The rms speed of the molecules is:

$$v_{\text{rms}} = \sqrt{\frac{3k_B T}{m}}.$$

SIMPLIFY: The expression for the fraction of air molecules having speeds greater than the speed of sound is

$$F = \int_{v_S}^{\infty} f(v)dv = 4\pi\left[\frac{m}{2\pi k_B T}\right]^{3/2} \int_{v_S}^{\infty} v^2 e^{-\frac{mv^2}{2k_B T}} dv.$$

CALCULATE: $v_{\text{ave}} = \sqrt{\frac{8\left(1.381\cdot 10^{-23}\text{ J/K}\right)\left(294\text{ K}\right)}{\pi\left(15.0\right)\left(1.661\cdot 10^{-27}\text{ kg}\right)}} = 644.2\text{ m/s}$

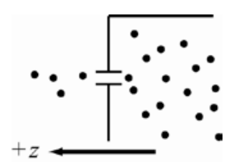
$v_{\text{rms}} = \sqrt{\frac{3\left(1.381\cdot 10^{-23}\text{ J/K}\right)\left(294\text{ K}\right)}{15.0\left(1.661\cdot 10^{-27}\text{ kg}\right)}} = 699.2\text{ m/s}$

ROUND: To three significant figures, $v_{\text{ave}} = 644\text{ m/s}$ and $v_{\text{rms}} = 699\text{ m/s}.$

DOUBLE-CHECK: These are reasonable speeds for gas molecules.

19.68 **THINK:** The volume of the cabin is $V = (5.00\text{ m})^3 = 125\text{ m}^3$, the temperature in the cabin remains constant at $T = 294$ K, and the initial pressure is $p_i = 1.00$ atm. The mass of each diatomic molecule of air is $m = 15$ amu. The radius of the hole created by the meteor is $r = 5.00\cdot 10^{-3}$ m. To find the time t it takes for the cabin's pressure to become $p_f = p_i / 2$, it can be assumed that only the molecules moving in the direction that is perpendicular to the plane of the hole exit through the hole – call this the z direction. Assume that once a molecule leaves the hole, it never returns.

SKETCH:



RESEARCH: In one dimension, $f(v) \propto \exp\left\{-mv^2 / (2k_B T)\right\}$. Take the average speed of the molecules as an estimate of the rate at which molecules are leaving. The average speed v is given by:

$$v_{\text{ave}} = \frac{\int_0^{\infty} v f(v)dv}{\int_0^{\infty} f(v)dv}.$$

Consider the number of molecules dN leaving a hole of cross-sectional area A in a given time dt and define the rate at which particles are leaving in terms of $v_{\text{ave},z}$:

$$\frac{dN}{dt}=\frac{d(NAz/V)}{dt}=\frac{N}{V}A\frac{dz}{dt}=\frac{N}{V}Av_{\text{ave},z}.$$

The rate of change in pressure is found from the Ideal Gas Law, $pV=Nk_BT$. Specifically,

$$\frac{dp}{dt}=\frac{d}{dt}\left(\frac{Nk_BT}{V}\right)=\frac{k_BT}{V}\frac{dN}{dt},$$

since the volume of the shuttle and the temperature inside the shuttle are constant.

SIMPLIFY: The average speed $v_{\text{ave},z}$ is:

$$v_{\text{ave},z}=\frac{\int_0^\infty v_z\exp\left\{\frac{-mv_z^2}{2k_BT}\right\}dv_z}{\int_0^\infty \exp\left\{\frac{-mv_z^2}{2k_BT}\right\}dv_z}=\frac{\frac{k_BT}{m}}{\frac{1}{2}\sqrt{\frac{2k_BT\pi}{m}}}=\sqrt{\frac{2k_BT}{\pi m}},$$

where the following integral identities were used:

$$\int_0^\infty e^{-ax^2}dx=\frac{1}{2}\sqrt{\frac{\pi}{a}}\text{ and }\int_0^\infty xe^{-ax^2}dx=\frac{1}{2a}.$$

Next,

$$\frac{dN}{dt}=\frac{N}{V}Av_{\text{ave},z}=\frac{N}{V}A\sqrt{\frac{2k_BT}{\pi m}}.$$

Therefore,

$$\frac{dp}{dt}=\frac{Nk_BT}{V^2}A\sqrt{\frac{2k_BT}{\pi m}}.$$

Since $p=Nk_BT/V$, this becomes

$$\frac{dp}{dt}=\frac{p}{V}A\sqrt{\frac{2k_BT}{\pi m}}\Rightarrow\frac{dp}{p}=\frac{A}{V}\sqrt{\frac{2k_BT}{\pi m}}dt.$$

The change in pressure over time is:

$$\int_{p_i}^{p_f}\frac{dp}{p}=\ln(p)\Big|_{p_i}^{p_f}=\ln\left(\frac{p_f}{p_i}\right)=\frac{A}{V}\sqrt{\frac{2k_BT}{\pi m}}t.$$

Therefore, the time taken is given by:

$$t=\left|\ln\left(\frac{p_f}{p_i}\right)\frac{V}{r^2}\sqrt{\frac{m}{2\pi k_BT}}\right|.$$

CALCULATE: The time taken for the cabin's pressure to become $p_f=p_i/2$ is:

$$t=\left|\ln\left(\frac{1}{2}\right)\frac{(125\text{ m}^3)}{(5.00\cdot10^{-3}\text{ m})^2}\sqrt{\frac{(15)(1.661\cdot10^{-27}\text{ kg})}{2\pi(1.381\cdot10^{-23}\text{ J/K})(294\text{ K})}}\right|=3425\text{ s}=57.1\text{ min}.$$

ROUND: To two significant figures, it takes 57 minutes for the pressure inside the cabin of the shuttle to be reduced to half its original value.

DOUBLE-CHECK: This time is reasonable for the pressure to come to half its original value, considering the small size of the hole and the large volume of the cabin.

19.69 This is an adiabatic process since no heat flow occurs. The temperature of the $n = 1.00$ mol of gas drops from $T_1 = 295$ K to $T_2 = 291$ K. For a diatomic molecule $\gamma = 7/5$, so the work done on the environment is:

$$W = \frac{nR}{1-\gamma}\left(T_\text{f} - T_\text{i}\right) \Rightarrow W = \frac{(1.00 \text{ mol})(8.314 \text{ J/mol K})}{1-(7/5)}(291 \text{ K} - 295 \text{ K}) = 83.1 \text{ J}$$

19.70 Since $k_\text{B} = R/N_\text{A}$ and $M = mN_\text{A}$, the root-mean-square speed is given by:

$$v_\text{rms} = \sqrt{\frac{3k_\text{B}T}{m}} = \sqrt{\frac{3RT}{M}}.$$

Using an effective molar mass for air of $M = 28.97$ g/mol, the root-mean-square speed is:

$$v_\text{rms} = \sqrt{\frac{3(8.314 \text{ J/(mol K)})(295 \text{ K})}{(0.02897 \text{ kg/mol})}} = 504 \text{ m/s}.$$

19.71 The balloon is at temperature $T = 293$ K and pressure $p = 1.013 \cdot 10^5$ Pa. Assume the balloon is spherical. Its radius is $r = 0.200$ m and its volume is $V = (4/3)\pi r^3$.

(a) The number of helium atoms N in the balloon is

$$\frac{N}{N_\text{A}} = n = \frac{pV}{RT} \Rightarrow N = \frac{pVN_\text{A}}{RT} = \frac{(1.013 \cdot 10^5 \text{ Pa})\left(\frac{4}{3}\pi\right)(0.200 \text{ m})^3 (6.02 \cdot 10^{23} \text{ atoms/mol})}{(8.314 \text{ J/(mol K)})(293 \text{ K})} = 8.39 \cdot 10^{23} \text{ atoms}.$$

(b) For a monatomic gas, the kinetic energy of the atoms is:

$$K_\text{ave} = \frac{3}{2}k_\text{B}T = \frac{3}{2}(1.381 \cdot 10^{-23} \text{ J/K})(293 \text{ K}) = 6.07 \cdot 10^{-21} \text{ J}.$$

(c) Since $k_\text{B} = R/N_\text{A}$ and $M = mN_\text{A}$, the root-mean-square speed is given by:

$$v_\text{rms} = \sqrt{\frac{3k_\text{B}T}{m}} = \sqrt{\frac{3RT}{M}}.$$

The molar mass of a helium atom is $M = 4.00$ g/mol. Then,

$$v_\text{rms} = \sqrt{\frac{3(8.314 \text{ J/(mol K)})(293 \text{ K})}{(0.00400 \text{ kg/mol})}} = 1350 \text{ m/s}.$$

19.72 Air is about 20.9% O_2. In a volume $V = 1.00 \text{ m}^3$, at a temperature $T = 298$ K and pressure $p = 1.01 \cdot 10^5$ Pa, the number of moles of oxygen is approximately:

$$n_{\text{O}_2} = (0.209)\frac{pV}{RT} = (0.209)\frac{(1.01 \cdot 10^5 \text{ Pa})(1.00 \text{ m}^3)}{(8.314 \text{ J/(mol K)})(298 \text{ K})} = 8.5200 \text{ mol}.$$

The molecular mass of O_2 is $M_{\text{O}_2} = 32.0$ g/mol, so the actual mass of O_2 in one cubic meter of air is:

$$m = n_{\text{O}_2}M_{\text{O}_2} = (8.5200 \text{ mol})(32.0 \text{ g/mol}) = 273 \text{ g}.$$

19.73 The tires of the $3.00\cdot 10^3$ lb car on a lift have a pressure $p_\text{i} = 1\text{ atm} + 32.0\text{ lb/in}^2$. The car is then lowered to the ground.

(a) If the volume of the tires does not change appreciably, the weight of the car may deform the tires, but it cannot change the pressure inside the tires. Pressure must vary with either volume or temperature. The pressure in pascals inside the tires after the car has been lowered to the ground is then

$$p_\text{i} = 1\text{ atm} + 32.0\text{ lb/in}^2 = 1.013\cdot 10^5\text{ Pa} + 32.0\text{ lb/in}^2\left(\frac{6.894\cdot 10^3\text{ Pa}}{\text{lb/in}^2}\right) = 3.219\cdot 10^5\text{ Pa}$$

$$p_\text{i} = 3.22\cdot 10^5\text{ Pa}.$$

(b) The force that supports the weight of the car comes from the tire pressure acting against the ground. The gauge pressure supports the weight of the car. The gauge pressure is

$$p = 32.0\text{ lb/in}^2\left(\frac{6.894\cdot 10^3\text{ Pa}}{\text{lb/in}^2}\right) = 2.206\cdot 10^5\text{ Pa}.$$

The contact area is:

$$A = F/p = \frac{(3.00\cdot 10^3\text{ lb})(4.4482\text{ N/lb})}{(2.206\cdot 10^5\text{ Pa})} = 0.06049229\text{ m}^2$$

$$A = 605\text{ cm}^2.$$

19.74 The gas expands from $V_\text{i} = 3.00$ L at $T_\text{i} = 288$ K to $V_\text{f} = 4.00$ L, at constant pressure. Assume the number of moles remains constant. By the Ideal Gas Law, $V_\text{i}/T_\text{i} = nR/p = V_\text{f}/T_\text{f}$:

$$T_\text{f} = \frac{V_\text{f}}{V_\text{i}}T_\text{i} = \frac{(4.00\text{ L})}{(3.00\text{ L})}(288\text{ K}) = 384\text{ K}.$$

19.75 At $T = 273.15$ K and $p = 1.01325\cdot 10^5$ Pa, the unknown gas has a density of

$$\rho = (0.0899\text{ g/L})\left(\frac{1\text{ kg}}{1000\text{ g}}\right)\left(\frac{1000\text{ L}}{\text{m}^3}\right) = 0.0899\text{ kg/m}^3.$$

The number of moles is $n = m/M$, where *m* is the total mass of the gas and *M* is the molar mass. From the Ideal Gas Law, $pV = nRT \Rightarrow pV = \frac{mRT}{M} \Rightarrow M = \frac{mRT}{pV} = \frac{\rho RT}{p}$.

$$M = \frac{(0.0899\text{ kg/m}^3)(8.314462\text{ J/(mol K)})(273.15\text{ K})}{(1.01325\cdot 10^5\text{ Pa})} = 0.00201502\text{ kg/mol} = 2.02\text{ g/mol}.$$

Since the gas has a molar mass of 2.02 g/mol, the gas is most likely hydrogen (H_2).

19.76 The tank has 30.0% O_2 and 70.0% Ar in a volume $V = 1.00\text{ m}^3$. The initial temperature is $T_\text{i} = 293$ K, the initial gauge pressure is $p_\text{ig} = 1000.$ psi, and the final gauge pressure is $p_\text{fg} = 1500.$ psi. Recall that the absolute pressure is the sum of the gauge pressure and the atmospheric pressure, $p_\text{atm} = 14.7$ psi. From the Ideal Gas Law for constant volume, $p_\text{i}/T_\text{i} = nR/V = p_\text{f}/T_\text{f}$. Then,

$$T_\text{f} = \frac{p_\text{f}}{p_\text{i}}T_\text{i} = \frac{p_\text{fg}+p_\text{atm}}{p_\text{ig}+p_\text{atm}}T_\text{i} = \frac{(1500.\text{ psi})+(14.7\text{ psi})}{(1000.\text{ psi})+(14.7\text{ psi})}(293\text{ K}) = 437\text{ K}.$$

19.77 At constant temperature of $T = 295$ K, $n = 5.00$ mol of an ideal monatomic gas expands from $V_\text{i} = 2.00\text{ m}^3$ to $V_\text{f} = 8.00\text{ m}^3$.

(a) For an isothermal process (at constant temperature), the work done by the gas is:

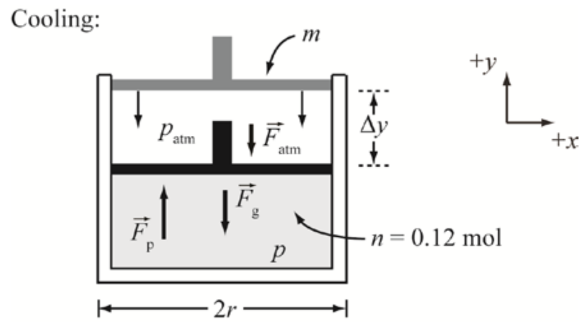
$$W = nRT\ln\left(\frac{V_\text{f}}{V_\text{i}}\right) = (5.00\text{ mol})(8.314\text{ J/(mol K)})(295\text{ K})\ln\left(\frac{(8.00\text{ m}^3)}{(2.00\text{ m}^3)}\right) = 17.0\text{ kJ}.$$

(b) The final pressure is found from the Ideal Gas Law:

$$p_\text{f}V_\text{f} = nRT \Rightarrow p_\text{f} = \frac{nRT}{V_\text{f}} = \frac{(5.00\text{ mol})(8.314\text{ J/(mol K)})(295\text{ K})}{(8.00\text{ m}^3)} = 1.53\text{ kPa}.$$

19.78 **THINK:** A cylinder with radius $r = 0.0350$ m contains $n = 0.120$ mol of an ideal gas. The piston at the top of the cylinder has a mass $m = 0.450$ kg. The temperature of the gas cools by $\Delta T = -15.0$ K, while the pressure and the number of moles of the gas stay constant. The Ideal Gas Law and Newton`s Second Law can be used to find the change in the piston's height.

SKETCH:



RESEARCH: The change in the enclosed cylindrical volume is a function of the change in height y of the piston: $\Delta V = \pi r^2 \Delta y$. The Ideal Gas Law is $pV = nRT$. The pressure p can be determined by using Newton's Second Law. When the piston comes to rest, the net force on the piston is zero.

SIMPLIFY: Since the number of moles n and the pressure p remain constant, a change in volume must be due to a change in temperature:

$$p\Delta V = nR\Delta T \Rightarrow \Delta V = \frac{nR\Delta T}{p}.$$

There are three forces acting on the piston: the downward force F_atm on the piston due to atmospheric pressure p_atm, the downward force F_g of gravity due to the mass m of the piston, and the upward force F_p due to the pressure p of the gas. By Newton's Second Law,

$$F_p - F_g - F_\text{atm} = pA - mg - p_\text{atm}A = 0 \Rightarrow p = p_\text{atm} + \frac{mg}{\pi r^2}$$

Therefore, the change in height of the piston is:

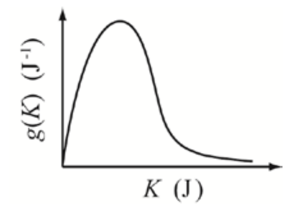
$$\Delta y = \frac{\Delta V}{\pi r^2} = \frac{nR\Delta T}{\pi r^2 p} = \frac{nR\Delta T}{\pi r^2\left(p_\text{atm} + \dfrac{mg}{\pi r^2}\right)} = \frac{nR\Delta T}{\pi r^2 p_\text{atm} + mg}.$$

CALCULATE: $\Delta y = \dfrac{(0.120\text{ mol})(8.314\text{ J/(mol K)})(-15.0\text{ K})}{\pi(0.0350\text{ m})^2(1.013\cdot 10^5\text{ Pa}) + (0.450\text{ kg})(9.81\text{ m/s}^2)} = -0.03796\text{ m},$ where the negative sign indicates that the piston falls to a lower height.

ROUND: To three significant figures, the piston ends up $\Delta y = 3.80$ cm below its initial position.

DOUBLE-CHECK: Since the gas is cooling, the piston should fall to reduce the volume.

19.79 **THINK:** Find the most probable kinetic energy for a molecule of gas at temperature $T = 300.$ K. The Maxwell kinetic energy distribution for a gas can be used to find the most probable kinetic energy for a molecule at this temperature.
SKETCH:



RESEARCH: The Maxwell kinetic energy distribution is

$$g(K) = \frac{2}{\sqrt{\pi}}\left(\frac{1}{k_B T}\right)^{3/2}\sqrt{K}e^{\frac{-K}{k_B T}}$$

The most probable kinetic energy K_{mp} occurs at the maximum value of $g(K)$, and is found by taking the derivative of $g(K)$ with respect to K and setting it to zero.
SIMPLIFY:

$$\frac{d}{dK}g(K) = \frac{2}{\sqrt{\pi}}\left(\frac{1}{k_B T}\right)^{3/2}\left[\frac{1}{2}\frac{1}{\sqrt{K}}e^{-\frac{K}{k_B T}} - \frac{\sqrt{K}}{k_B T}e^{-\frac{K}{k_B T}}\right]_{K=K_{mp}} = 0 \Rightarrow \frac{1}{2\sqrt{K_{mp}}} = \frac{\sqrt{K_{mp}}}{k_B T} \Rightarrow K_{mp} = \frac{k_B T}{2}$$

CALCULATE: $K_{mp} = \frac{(1.381\cdot 10^{-23}\text{ J/K})(300.\text{ K})}{2} = 2.072\cdot 10^{-21}$ J.

ROUND: To three significant figures, the most probably kinetic energy is $K_{mp} = 2.07\cdot 10^{-21}$ J.

DOUBLE-CHECK: It is reasonable that the energy depends only on temperature. The answer contains the proper units.

19.80 **THINK:** The auditorium has volume $V = 2.50\cdot 10^4\text{ m}^3$, temperature $T_i = 293$ K, and pressure $p_i = 1.013\cdot 10^5$ Pa. It contains 2000 people who have an average metabolism of 70.0 W each. Find the change in temperature after two hours have passed.
SKETCH: A sketch is not needed to solve the problem.
RESEARCH: The amount of heat added to the auditorium due to the audience's collective metabolism can be determined from $Q = N_P P\Delta t$ where P is the average metabolic rate and N_P is the number of people. The temperature increase can be found from $Q = nC_V\Delta T$ (since the volume does not change). The number of moles of air can be found from the Ideal Gas Law: $pV = nRT$.

SIMPLIFY: Since it is a closed auditorium, the number of moles of air is constant: $n = p_i V / RT_i$. The change in temperature is given by:

$$\Delta T = \frac{Q}{nC_V} = \frac{QRT_i}{p_i V C_V} = \frac{N_P P\Delta t R T_i}{p_i V C_V}.$$

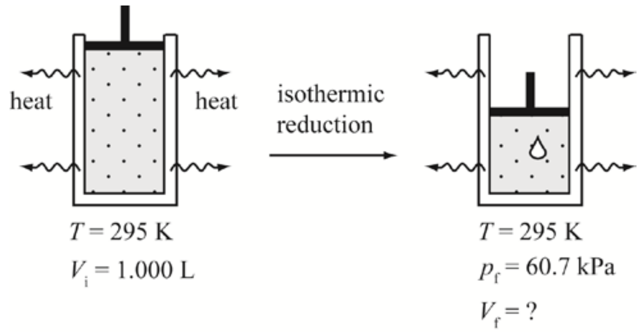
CALCULATE: Air is predominantly composed of N_2, so use $C_V = 20.7$ J/(mol K) for N_2.

$$\Delta T = \frac{(2000\text{ people})(70.0\text{ W/person})(7200.\text{ s})(8.314\text{ J/(mol K)})(293\text{ K})}{(1.013\cdot 10^5\text{ Pa})(2.50\cdot 10^4\text{ m}^3)(20.7\text{ J/(mol K)})} = 46.84016\text{ K}$$

ROUND: To three significant figures, the temperature increase in the auditorium will be $\Delta T = 46.8$ K.
DOUBLE-CHECK: This is a reasonable temperature increase for a closed room with 2000 people in it.

19.81 **THINK:** Gaseous pentane turns to vapor when the pressure increases to $p_{\text{f}} = 60.7$ kPa. The temperature of the pentane of mass $m = 1.000$ g is constant at $T = 295$ K, while the volume is isothermally reduced from $V_{\text{i}} = 1.000$ L. The first drop of liquid pentane will appear when the pressure of liquid pentane in the cylinder is equal to its vapor pressure at the given temperature. By using the Ideal Gas Law, the volume V_{f} for which liquid pentane first appears can be found.

SKETCH:



RESEARCH: The Ideal Gas Law is given by: $pV = nRT$. For pentane, the molar mass is $M = 72.15$ g/mol.

SIMPLIFY: $p_{\text{f}}V_{\text{f}} = nRT = \dfrac{m}{M}RT \Rightarrow V_{\text{f}} = \dfrac{mRT}{Mp_{\text{f}}}$

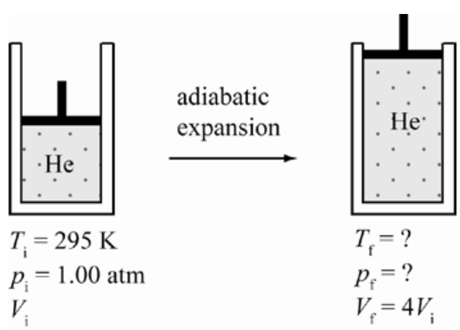
CALCULATE: $V_{\text{f}} = \dfrac{(1.000\text{ g})(8.314\text{ J/(mol K)})(295\text{ K})}{(72.15\text{ g/mol})(60.7\cdot 10^3\text{ Pa})} = 5.6002\cdot 10^{-4}\text{ m}^3$

ROUND: To three significant figures, the volume at which liquid pentane occurs is $V_{\text{f}} = 0.560$ L.

DOUBLE-CHECK: It is expected that the final volume is less than the initial volume since the pressure must increase in order for the gaseous pentane to condense into a liquid.

19.82 **THINK:** Helium gas is in a well-insulated cylinder, at a temperature of $T_{\text{i}} = 295$ K and a pressure of $p_{\text{i}} = 1.00$ atm. Under adiabatic expansion, the volume increases to four time its original volume: $V_{\text{f}} = 4V_{\text{i}}$. The equations for adiabatic processes can be used to find (a) the final pressure p_{f}, and (b) the final temperature, T_{f}.

SKETCH:



RESEARCH: For an adiabatic process, $p_{\text{i}}V_{\text{i}}^{\gamma} = p_{\text{f}}V_{\text{f}}^{\gamma}$ and $T_{\text{i}}V_{\text{i}}^{\gamma-1} = T_{\text{f}}V_{\text{f}}^{\gamma-1}$. Since helium is a monatomic gas, $\gamma = 5/3$.

SIMPLIFY:

(a) $p_{\text{f}} = \left(\dfrac{V_{\text{i}}}{V_{\text{f}}}\right)^{\gamma} p_{\text{i}} = 4^{-\gamma} p_{\text{i}}$

(b) $T_f = T_i\left(\dfrac{V_i}{V_f}\right)^{\gamma-1} = 4^{1-\gamma}T_i$

CALCULATE:

(a) $p_f = 4^{-(5/3)}(1.00 \text{ atm}) = 0.099213 \text{ atm}$

(b) $T_f = 4^{1-(5/3)}(295 \text{ K}) = 117.07 \text{ K}$

ROUND:

(a) To three significant figures, the final pressure is $p_f = 0.0992$ atm.

(b) To three significant figures, the final temperature is $T_f = 117$ K.

DOUBLE-CHECK: Since the volume increased adiabatically, there was no heat flow. Since the helium gas did work as the volume increased, the internal energy of the gas must decrease (since $Q = 0$, the First Law of Thermodynamics gives $\Delta E_{int} = -W$). Therefore, it is expected that $T_f < T_i$. This is known as adiabatic cooling. Since the volume increases and the temperature decreases, it is expected that $p_f < p_i$.

Multi-Version Exercises

Exercises 19.83–19.85 At constant pressure *p*, the work done is

$$W = p\Delta V = p(V_f - V_i) = p\left[(1-f)V_i - V_i\right] = -fpV_i.$$

From the Ideal Gas Law, $V_i = nRT_i / p$. Therefore, the work done by the gas is: $W = -fp\left(\dfrac{nRT_i}{p}\right) = -fnRT_i$.

19.83 $W = -fnRT_i = -(0.4711)(0.05839 \text{ mol})(8.314 \text{ J/mol/K})(273.15 \text{ K}) = -62.47 \text{ J}.$

19.84 $W = -fnRT_i$

$$n = -\frac{W}{fRT_i} = -\frac{-75.40 \text{ J}}{(0.4753)(8.314 \text{ J/mol/K})(273.15 \text{ K})} = 0.06985 \text{ mol}.$$

19.85 $W = -fnRT_i$

$$f = -\frac{W}{nRT_i} = -\frac{-34.04 \text{ J}}{(0.03127 \text{ mol})(8.314 \text{ J/mol/K})(273.15 \text{ K})} = 0.4793 = 47.93\%$$

19.86 The volume and temperature remain constant. The Ideal Gas Law gives us

$$pV = nRT$$
$$\frac{p}{n} = \frac{RT}{V} = \text{constant}$$
$$\frac{p_f}{n_f} = \frac{p_i}{n_i}$$
$$p_f = p_i\frac{n_f}{n_i}.$$

Initially there is only air in the bottle, so $n_i = V / 22.414 \text{ L mol}^{-1}$. After the reaction, there is 1.393 mole of CO_2, so the final number of moles is $n_f = n_i + n_{CO_2}$. So the final pressure is

$p_f = (1.013 \cdot 10^5 \text{ Pa})\dfrac{n_i + n_{CO_2}}{n_i}$. The initial number of moles is $n_i = (2.869 \text{ L}) / 22.414 \text{ L mol}^{-1} = 0.1280 \text{ mol}$.

The final number of moles is

$$p_f = (1.013 \cdot 10^5 \text{ Pa})\frac{0.1280 \text{ mol} + 1.393 \text{ mol}}{0.1280 \text{ mol}} = 1.204 \cdot 10^6 \text{ Pa}.$$

(Note that we assumed STP and, to attain four significant figures as suggested by the problem data, used 22.414 L/mol rather than 22.4 L/mol.)

19.87 As noted in the preceding problem, $p_f = p_i \dfrac{n_i + n_{CO_2}}{n_i}$.

$$n_i p_f = n_i p_i + n_{CO_2} p_i$$
$$n_i \left(p_f - p_i \right) = n_{CO_2} p_i$$
$$n_i = \frac{n_{CO_2} p_i}{\left(p_f - p_i \right)} = V / \left(22.414 \text{ L/mol} \right)$$
$$V = \frac{\left(22.414 \text{ L/mol}\right)\left(1.413 \text{ mol}\right)\left(1.013 \cdot 10^5 \text{ Pa}\right)}{\left(1.064 \cdot 10^6 \text{ Pa} - 1.013 \cdot 10^5 \text{ Pa}\right)} = 3.333 \text{ L}$$

19.88 As noted in the preceding problems, $p_f = p_i \dfrac{n_i + n_{CO_2}}{n_i}$.

$$n_i p_f = n_i p_i + n_{CO_2} p_i$$
$$n_i \left(p_f - p_i \right) = n_{CO_2} p_i$$
$$n_{CO_2} = \frac{n_i \left(p_f - p_i \right)}{p_i}$$
$$n_i = \left(3.787 \text{ L}\right) / \left(22.414 \text{ L/mol}\right) = 0.1690 \text{ mol}$$
$$n_{CO_2} = \frac{\left(0.1690 \text{ mol}\right)\left(9.599 \cdot 10^5 \text{ Pa} - 1.013 \cdot 10^5 \text{ Pa}\right)}{1.013 \cdot 10^5 \text{ Pa}} = 1.433 \text{ mol}$$

Chapter 20: The Second Law of Thermodynamics

Concept Checks

20.1. e **20.2.** e **20.3.** c **20.4.** c **20.5.** b **20.6.** e

Multiple-Choice Questions

20.1. d **20.2.** c **20.3.** a **20.4.** a **20.5.** c **20.6.** e **20.7.** d **20.8.** b **20.9.** a **20.10.** b e **20.11.** a **20.12.** c
20.13. c **20.14.** a

Conceptual Questions

20.15. One possible reason why the Second Law of Thermodynamics is a benefit is a scenario that is best described by the ideal gas in a box scenario as explained on page 635. If the Second Law did not exist, it would be possible for all of the particles to migrate to a small corner of the box. Therefore, a person sitting in a room could find all of the oxygen collecting in a corner, and thus suffocate.

20.16. It is not a Nobel-winning discovery. Even though the scientist is looking at a very small system, there is the possibility that it is an open system. For an open system, a decrease in entropy is perfectly reasonable.

20.17. Typical heat pumps have a coefficient of performance greater than 1. This means that for every unit of electrical power that it uses, more than one unit of power in the form of heat is produced. This is done by extracting energy from the environment. An electric heater uses the current supplied to produce heat and the coefficient of performance is typically 1. Therefore, the heat pump has a higher rate of heat produced per unit electrical energy.

20.18. The most likely arrangement of the 4 particles is 2 particles in each partition. If particle 1 is in partition A, then there are 3 possible cases where only one particle (2, 3 or 4) is with particle 1 and the remaining two are in partition B. If particle 2 is in partition A, then there are 2 new possible cases where only one particle (3 or 4) is with particle 2 and the remaining two are in partition B. A similar argument for putting particle 3 in partition A gives only 1 new configuration for a total of 6 possible ways to place the four particles in a box with 2 in each partition. The entropy of this system is then given as $S = k_{\text{B}}\ln(w)$ where $w = 6,$ so $S = \left(1.38\cdot 10^{-23}\text{ J/K}\right)\ln\left(6\right) = 2.47\cdot 10^{-23}$ J/K. The next most likely arrangement is when 1 particle is in one partition and the remaining 3 are in the other. If only 1 particle can be in a partition than there are 4 possible configurations, one for each particle. Using $w = 4$ to find the entropy of this system: $S = \left(1.38\cdot 10^{-23}\text{ J/K}\right)\ln\left(4\right) = 1.91\cdot 10^{-23}$ J/K.

20.19. The equations under consideration are $dE_{\text{int}} = TdS - pdV$, $dS = dE_{\text{int}}/T + pdV/T$. From the first equation, it is clear E_{int} is a function of both S and V $\left(E_{\text{int}} \to E_{\text{int}}\left(S,V\right)\right)$ and from the second equation, S is a function of E_{int} and V $\left(S \to S\left(E_{\text{int}},V\right)\right)$. This means a partial differential can be taken of each variable so that $\partial E_{\text{int}}/\partial S = T$, $\partial E_{\text{int}}/\partial V = -p$, $\partial S/\partial E_{\text{int}} = 1/T$, $\partial S/\partial V = p/T$. In general, the mixed second partial differentials of a continuous function are equal, i.e. $\dfrac{\partial^2 f\left(x,y\right)}{\partial x\partial y} = \dfrac{\partial^2 f\left(x,y\right)}{\partial y\partial x}$. This means then that $\dfrac{\partial^2 E_{\text{int}}}{\partial V\partial S} = \dfrac{\partial T}{\partial V} = -\dfrac{\partial p}{\partial S}$ and $\dfrac{\partial^2 \text{S}}{\partial V\partial E_{\text{int}}} = \dfrac{\partial\left(1/T\right)}{\partial V} = \dfrac{\partial\left(\text{p}/T\right)}{\partial E_{\text{int}}}$. Relationships such as these are known as *Maxwell relations*. Note that in each case a partial derivative is taken with its companion independent variable fixed. Hence, e.g., $\partial/\partial V$ in the first Maxwell relation is not the same as $\partial/\partial V$ in the second.

20.20. (a) The equations under consideration are $H = E_{\text{int}} + pV$, $A = E_{\text{int}} - TS$ and $G = E_{\text{int}} + pV - TS$. After taking the differential of each equation, keeping in mind that no variable is always constant and using the First Law of Thermodynamics, $dE_{\text{int}} = TdS - pdV$, three equations arise.

$$\text{(1)} \quad dH = dE_{\text{int}} + pdV + Vdp = TdS + Vdp$$

$$\text{(2)} \quad dA = dE_{\text{int}} - TdS - SdT = -pdV - SdT$$

$$\text{(3)} \quad dG = dE_{\text{int}} + pdV + Vdp - TdS - SdT = Vdp - SdT$$

(b) From the three equations, it is clear that each energy (H, A, G) are actually functions of other variables whereby $H \to H(S, p), A \to A(V, T), G \to G(p, T)$. This means a partial differential can be taken of each variable so that $\partial H/\partial S = T$, $\partial H/\partial p = V$, $\partial A/\partial V = -p$, $\partial A/\partial T = -S$, $\partial G/\partial p = V$ and $\partial G/\partial T = -S$. In general, if the mixed second-order partial derivatives are continuous, then they are equal, i.e. $\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x}$. This yields three further Maxwell relations: $\frac{\partial^2 H}{\partial S \partial p} = \frac{\partial T}{\partial p} = \frac{\partial V}{\partial S}$, $\frac{\partial^2 A}{\partial V \partial T} = -\frac{\partial p}{\partial T} = -\frac{\partial S}{\partial V}$ and $\frac{\partial^2 G}{\partial p \partial T} = \frac{\partial V}{\partial T} = -\frac{\partial S}{\partial p}$, where again each partial derivative is taken with its companion independent variable fixed.

20.21. In general, the Boltzmann definition of entropy is given as $S = k_B \ln(w)$. If there are two systems, A and B, it can be seen that the system A has w_A possible states and system B has w_B possible states. Therefore, for the given system A and B, the total number of possible states is $w = w_A \cdot w_B$, so then the entropy is separable by logarithm rules:

$$S = k_B \ln(w) = k_B \ln(w_A \cdot w_B) = k_B \ln(w_A) + k_B \ln(w_B) = S_A + S_B.$$

20.22. The heat pump is doing work to move heat energy from the cold outside into the warm house. For each 6.28 kJ of electricity used by the heat pump, 21.98 kJ of heat is moved from the outside to the inside. Adding the heat energy from the heat pump to the heat energy extracted from the cold reservoir gives the total heat energy released to the hot reservoir.

20.23. The wind chill factor from Canada is given by $T_{\text{wc}} = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$ where T is the surface temperature in °C and v is the wind speed in km/h at a point 10 m above the ground. Of course, this formula for wind chill was never intended to be applied for the temperatures and wind speed present in the clouds of Saturn. But supposing a temperature of 150 K (–123 °C), and a wind speed of 600. km/h, the wind chill is 42 K (–231 °C), which is still above absolute zero. The definition of absolute zero is when a particle is completely without motion. As long as the particle is moving in the wind, it will have some kinetic energy and thus the temperature is greater than absolute zero.

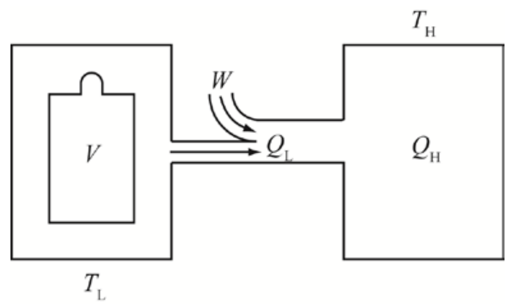
20.24. The process of turning water into steam is an irreversible process. The Second Law of Thermodynamics states that entropy must increase in an irreversible process, so it would require energy to convert the steam back into water to use over again, resulting in a net energy loss. Eventually, the engine will not have enough energy to convert the steam back to water.

20.25. To increase the entropy, heat the water up. To decrease the entropy, cool the water down. As long as the water is not a closed system, the entropy can decrease.

Exercises

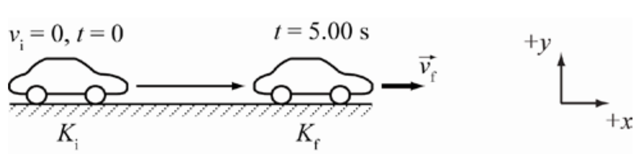
20.26. The hot reservoir has $Q_{\text{H}} = 2100.$ J extracted from it while the cold reservoir has $Q_{\text{C}} = 1500.$ J put into it. The work done by this engine is then $W = Q_{\text{H}} - Q_{\text{C}} = 2100.\text{ J} - 1500.\text{ J} = 600.\text{ J}.$ The efficiency of the engine is $\varepsilon = W / Q_{\text{H}} = 600.\text{ J} / 2100.\text{ J} = 0.300.$ If the engine does $600.$ J of work in one cycle and operates at a power of $P = 2500.$ W, the time taken for one cycle is $W / P = 600.\text{ J}/2500.\text{ W} \approx 0.200$ s.

20.27. **THINK:** As the water (specific heat $c = 4.19\text{ kJ/(kg}\cdot\text{K)},$ density $\rho = 1.00\text{ g/cm}^3 = 1.00\text{ kg/L},$ and volume $V = 2.00$ L) is cooled from $T_{\text{H}} = 25.0$ °C to $T_{\text{L}} = 4.00$ °C, a quantity of heat, $Q_{\text{L}},$ is extracted from it by the refrigerator of power $P = 480.$ W. The coefficient of performance, $K = 3.80,$ relates the heat extracted to the work, $W,$ done by the fridge.
SKETCH:



RESEARCH: The mass of the water is $m = \rho V.$ The heat then extracted from the water is $Q_{\text{L}} = mc\Delta T.$ The coefficient of performance of the fridge is $K = Q_L / W$ and the work done by the fridge to cool the water is $W = P\Delta t.$
SIMPLIFY: The heat extracted from water is $Q_{\text{L}} = \rho V c\left(T_{\text{f}} - T_{\text{i}}\right).$ The work done by the fridge is $W = Q_{\text{L}} / K = P\Delta t;$ therefore, $\Delta t = Q_{\text{L}} / (PK) = \rho V c\left(T_{\text{f}} - T_{\text{i}}\right) / (PK).$
CALCULATE: $\Delta t = \left(1.00\text{ kg/L}\right)\left(2.00\text{ L}\right)\left(4.19\cdot 10^3\text{ J/kg K}\right)\left(25.0\text{ °C} - 4.00\text{ °C}\right)\Big/\left(\left(480.\text{ W}\right)\left(3.80\right)\right) = 96.48\text{ s}$
ROUND: Three significant figures: $\Delta t = 96.5$ s.
DOUBLE-CHECK: A K-value of 3.80 is a bit higher than most conventional fridges, so a relatively short time of cooling is reasonable.

20.28. **THINK:** The efficiency $\left(\varepsilon = 0.250\right)$ is the ratio of work the engine does to the energy (heat) supplied to it. Given an interval of $t = 5.00$ s, the energy supplied to the engine is related to the power supplied to the engine, $P = 4.00\cdot 10^5$ W. The change in kinetic energy of the car, whose mass is $M = 2000.$ kg, should be equal to the work done by the engine.
SKETCH:



RESEARCH: The efficiency of the engine is $\varepsilon = W/Q_H,$ where W is the work done by the engine and Q_H is the energy supplied to the engine. The power supplied to the engine is $P = Q_H/t.$ The power the engine uses to increase the vehicle's velocity is $W/t.$ It is assumed that the work the engine does is transformed entirely into kinetic energy $W = \Delta K = (1/2)M\left(v_{\text{f}}^2 - v_i^2\right).$

SIMPLIFY: $\varepsilon = W/Q_{\text{H}} = (W/t)/(Q_{\text{H}}/t) = (W/t)/P \Rightarrow W = \varepsilon Pt.$ The velocity is then $W = \Delta K = \frac{1}{2}M\left(v_{\text{f}}^2 - 0^2\right) = \varepsilon Pt \Rightarrow v_{\text{f}} = \sqrt{2\varepsilon Pt/M}.$

CALCULATE: $v_{\text{f}} = \sqrt{2(0.250)\left(4.00\cdot 10^5 \text{ W}\right)(5.00 \text{ s})/2000. \text{ kg}} = 22.36 \text{ m/s}$

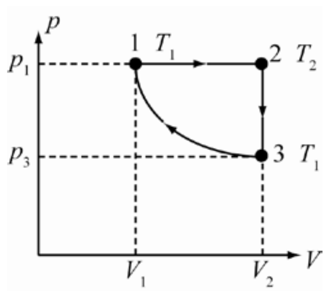
ROUND: Three significant figures: $v_{\text{f}} = 22.4$ m/s.

DOUBLE-CHECK: Most car companies boast about their cars being able to go from 0 mph to 60 mph in a few seconds. This vehicle goes to about 50 mph in 5 seconds, which is very reasonable.

20.29. THINK: Initially, the pressure, volume and temperature are $p_1 = 300.$ kPa, $V_1 = 150.$ cm^3 and $T_1 = 20.0$ °C. An isobaric expansion causes the gas to do work, W_{12}, by absorbing heat, Q_{12}, so that the gas now has volume, $V_2 = 450.$ cm^3, and temperature, T_2. It then isochorically reduces its pressure and temperature to p_3 and $T_3 = T_1$ by expelling heat, Q_{23}. It then returns to its original pressure and volume isothermally by having work, W_{23}, done on it by the piston as the system relaxes back to its original state. Since the gas is monatomic, $C_V = (3/2)R$ and $C_p = (5/2)R$.

SKETCH:

(a)



RESEARCH: For the first step, the work done *on* the gas is $W_{12} = -p_1(V_2 - V_1)$ and the heat flow out of the gas is $Q_{12} = -nC_p(T_2 - T_1)$. During the isobaric process $T_1/T_2 = V_1/V_2$ (from ideal gas law, $pV = nRT$, where p is constant). For the second step, the work on the gas and the heat flow out of the gas are $W_{23} = 0$ and $Q_{23} = -nC_V(T_1 - T_2)$. For the third step, $\Delta E = 0$; therefore, $W_{31} = Q_{31}$ where $W_{31} = -nRT_1 \ln(V_1/V_2)$. A negative sign is used here because the work done on a gas undergoing compression ($\ln(V_1/V_2) < 0$) must be positive. Similarly, Q_{31} must also be positive since $W_{31} = Q_{31}$. The efficiency is given by $\varepsilon = W/Q_{\text{H}}$, where W is the useful work extracted done by the piston, and Q_{H} is the heat absorbed by the gas.

SIMPLIFY:

(b) Determine T_2 first: $\frac{T_1}{T_2} = \frac{V_1}{V_2} \Rightarrow T_2 = \frac{V_2 T_1}{V_1} \Rightarrow T_2 - T_1 = \left(\frac{V_2}{V_1} - 1\right)T_1 = \left[\frac{(V_2 - V_1)}{V_1}\right]T_1$; therefore, $T_2 - T_1 = \left(\frac{(V_2 - V_1)}{V_1}\right)T_1$ and $Q_{12} = -nC_p(T_2 - T_1) = -\left(\frac{5}{2}\right)\left(\frac{nRT_1}{V_1}\right)(V_2 - V_1)$. From the ideal gas law, $p_1V_1 = nRT_1 \Rightarrow p_1 = \frac{nRT_1}{V_1}$; so, for the first leg $W_{12} = -p_1(V_2 - V_1)$ and $Q_{12} = -\left(\frac{5}{2}\right)p_1(V_2 - V_1)$. For the second leg of the cycle $Q_{23} = -nC_V(T_1 - T_2) = \left(\frac{3}{2}\right)\left(\frac{nRT_1}{V_1}\right)(V_2 - V_1) = \left(\frac{3}{2}\right)p_1(V_2 - V_1)$ and $W_{23} = 0$ J. For the third leg of the cycle $W_{31}=-nRT_1\ln\left(\frac{V_1}{V_2}\right)=-p_1V_1\ln\left(\frac{V_1}{V_2}\right)=Q_{31}$.

(c) The total work on the system is $W=W_{12}+W_{23}+W_{31}$. The only time heat was added to the system from the hot reservoir was during the isobaric process; therefore, $Q_\text{H}=Q_{12}$ and $\varepsilon=\frac{\left(W_{12}+W_{23}+W_{31}\right)}{Q_{12}}$.

CALCULATE:

(b) Leg 1: $W_{12}=-\left(300.\cdot 10^3\text{ Pa}\right)\left(450.\text{ cm}^3-150.\text{ cm}^3\right)\left(1\text{ m}/100\text{ cm}\right)^3=-90.00\text{ J}$

$$Q_{12}=-\left(\frac{5}{2}\right)\left(300.\cdot 10^3\text{ Pa}\right)\left(450.\text{ cm}^3-150.\text{ cm}^3\right)\left(\frac{1\text{ m}}{100\text{ cm}}\right)^3=-225.0\text{ J}$$

Leg 2: $W_{23}=0\text{ J}$, $Q_{23}=\left(\frac{3}{2}\right)\left(300.\cdot 10^3\text{ Pa}\right)\left(450.\text{ cm}^3-150.\text{ cm}^3\right)\left(\frac{1\text{ m}}{100\text{ cm}}\right)^3=135.0\text{ J}$

Leg 3: $W_{31}=Q_{31}=-\left(300.\cdot 10^3\text{ Pa}\right)\left(150.\text{ cm}^3\right)\ln\left(\frac{150.\text{ cm}^3}{450.\text{ cm}^3}\right)\left(\frac{1\text{ m}}{100\text{ cm}}\right)^3=49.44\text{ J}$

(c) $\varepsilon=\frac{\left(-90\text{ J}+0\text{ J}+49.44\text{ J}\right)}{\left(-225\text{ J}\right)}=0.1803$

ROUND: Three significant figures:

(b) $W_{12}=-90.0\text{ J}$, $Q_{12}=-225\text{ J}$, $W_{23}=0\text{ J}$, $Q_{23}=135\text{ J}$, $W_{31}=49.4\text{ J}$ and $Q_{31}=49.4\text{ J}$.

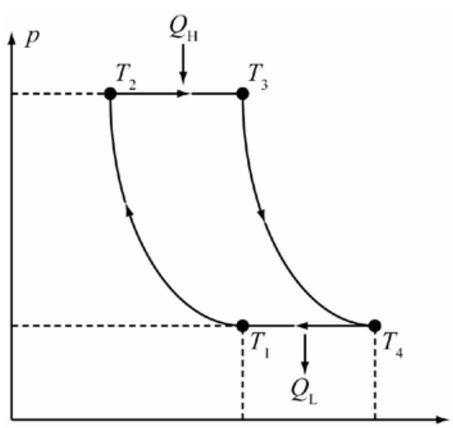
(c) $\varepsilon=0.180$

DOUBLE-CHECK: The efficiency seems reasonable for an engine, and therefore the values of the work and the heat flow are also reasonable. They have appropriate units, which helps to support the calculations.

20.30. **THINK:** During the two adiabatic processes (points $1\rightarrow 2$ and $3\rightarrow 4$), the heat flow $Q_{12}=Q_{34}=0\text{ J}$. Heat flows to and from the system, respectively, during the two isobaric processes (points $2\rightarrow 3$ and $4\rightarrow 1$) where $Q_{23}=Q_\text{H}$ and $Q_{41}=Q_\text{L}$. Since this a closed path process, the total work, W, is equal to the total heat flow, Q. The efficiency can then be calculated solely from the heat flow.

SKETCH:

(a)



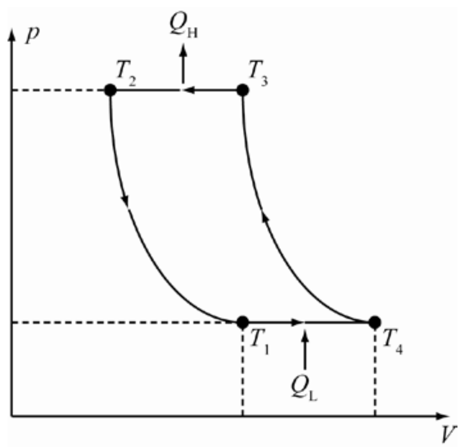
RESEARCH: The heat flowing into the system is $Q_\text{H}=Q_{23}=nC_p\left(T_3-T_2\right)$. The heat flowing out of the system is $Q_\text{L}=Q_{41}=-nC_p\left(T_1-T_4\right)$. Since it is a closed path $Q=Q_\text{H}-Q_\text{L}=W$. The efficiency is $\varepsilon=W/Q_\text{H}$.

SIMPLIFY:

(b) $\varepsilon=W/Q_\text{H}=\left(Q_\text{H}-Q_\text{L}\right)/Q_\text{H}=1-Q_\text{L}/Q_\text{H}=1+nC_p\left(T_1-T_4\right)/nC_p\left(T_3-T_2\right)=1-\left(T_4-T_1\right)/\left(T_3-T_2\right)$

CALCULATE: Not required.
ROUND: Not required.
DOUBLE-CHECK: The efficiency should be a function of the temperatures at the vertices, as found. The efficiency will not have any units, since 1 has no units, and the units of the T will cancel.

20.31. **THINK:** During the two adiabatic processes (points $4 \to 3$ and $2 \to 1$), the heat flow $Q_{43} = Q_{21} = 0$ J. Heat flows to and from the system, respectively, during the two isobaric processes (points $1 \to 4$ and $3 \to 2$) where $Q_{14} = Q_L$ and $Q_{32} = Q_H$. Since this a closed path process, the total work, W, is equal to the total heat flow, Q. The coefficient of performance can then be calculated solely from the heat flow.
SKETCH:
(a)



RESEARCH: The heat flowing into the system is $Q_L = Q_{14} = nC_p(T_4 - T_1)$. The heat flowing out of the system is $Q_H = Q_{32} = -nC_p(T_2 - T_3)$. Since it is a closed path $Q = Q_H - Q_L = W$. The coefficient of performance is $K = Q_L / W$.
SIMPLIFY:
(a) Not necessary.
(b) $K = Q_L / W = Q_L / (Q_H - Q_L) = nC_p(T_4 - T_1) / \left[-nC_p(T_2 - T_3) - nC_p(T_4 - T_1)\right]$
Therefore, $K = (T_4 - T_1)/(T_3 - T_2 - T_4 + T_1)$.
CALCULATE: Not required.
ROUND: Not required.
DOUBLE-CHECK: The coefficient of performance will not have any units, since the units of the T values will cancel. The correct coefficient of performance has been derived.

20.32. The coefficient of performance for a heat pump is given by equation 20.8: $K = \dfrac{T_H}{T_H - T_L}$ where $T_H = 23.0\ ^\circ\text{C} = 296.15$ K.

(a) $T_L = -10.0\ ^\circ\text{C} = 263.15\ \text{K} \Rightarrow K = \dfrac{296.15\ \text{K}}{296.15\ \text{K} - 263.15\ \text{K}} = 8.97$

(b) $T_L = 9.00\ ^\circ\text{C} = 282.15\ \text{K} \Rightarrow K = \dfrac{296.15\ \text{K}}{296.15\ \text{K} - 282.15\ \text{K}} = 21.2$

20.33. For this Carnot engine, the temperature of the hot and cold reservoirs are $T_H = 1000.0$ K and $T_L = 300.0$ K, respectively.
(a) The efficiency of such an engine is $\varepsilon = 1 - (T_L / T_H) = 1 - (300.0\ \text{K}/1000.0\ \text{K}) = 0.7000$.

(b) The efficiency is also related to the work the engine does, $W = 1.00$ kJ, and the heat, Q_H, extracted from the hot reservoir: $\varepsilon = W / Q_H \Rightarrow Q_H = W / \varepsilon = 1.00 \cdot 10^3 \text{ J} / 0.7000 = 1430 \text{ J}$.

(c) According to the First Law of Thermodynamics, $Q_\text{H} = Q_\text{L} + W$, so the heat delivered to the colder reservoir, Q_L, is $Q_\text{L} = Q_\text{H} - W = 1430 \text{ J} - 1.00 \cdot 10^3 \text{ J} = 430.$ J.

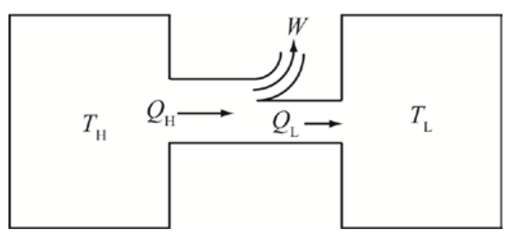
20.34. For the Carnot fridge, the hot and cold reservoirs are at a temperature of $T_\text{H} = 27.0\ °\text{C} = 300.15$ K and $T_\text{L} = 0.00\ °\text{C} = 273.15$ K, respectively.
(a) The coefficient of performance of the fridge is related to both temperatures and the heat extracted from the cold reservoir, $Q_\text{L} = 10.0$ J, and the work, W, it does:

$$K = T_\text{L} / (T_\text{H} - T_\text{L}) = Q_\text{L} / W \Rightarrow W = Q_\text{L}(T_\text{H} - T_\text{L}) / T_\text{L}.$$
$$W = (10.0 \text{ J})(300.15 \text{ K} - 273.15 \text{ K}) / 273.15 \text{ K} = 0.988 \text{ J}$$

(b) If instead $T_\text{L} = -20.0\ °\text{C} = 253.15$ K, the work needed is then calculated to be $W = (10.0 \text{ J})(300.15 \text{ K} - 253.15 \text{ K}) / 253.15 \text{ K} = 1.86 \text{ J}$.

20.35. If such a thermal-energy plant could be designed to operate at maximum efficiency, it would act as an ideal Carnot engine so that the efficiency would be $\varepsilon = (T_\text{H} - T_\text{L}) / T_\text{H}$. A reasonable values for the temperature at sea level would be $T_\text{H} = 10.0\ °\text{C}\ (283 \text{ K})$, and the temperature at the bottom of sea would be $T_\text{L} = 4.00\ °\text{C}\ (277 \text{ K})$, and since $T_\text{H} - T_\text{L} = 6.00\ °\text{C}\ (6.00 \text{ K})$, the maximum efficiency of the plant would be $\varepsilon = 6.00 \text{ K} / 283 \text{ K} = 0.0212 = 2.12$ %.

20.36. **THINK:** The efficiency, ε, of the Carnot engine can be calculated by the temperatures of the hot and cold reservoirs, $T_\text{H} = 1000\ °\text{C} = 1273.15$ K and $T_\text{L} = 10\ °\text{C} = 283.15$ K, respectively. The work is then determined by comparing this with the heat removed from the hot reservoir, $Q_\text{H} = 100.$ J.
SKETCH:



RESEARCH: The efficiency of the engine is given by $\varepsilon = 1 - (T_\text{L} / T_\text{H})$ and $\varepsilon = W / Q_\text{H}$.

SIMPLIFY: $\varepsilon = 1 - \dfrac{T_\text{L}}{T_\text{H}} = \dfrac{W}{Q_\text{H}} \Rightarrow W = \left(1 - \dfrac{T_\text{L}}{T_\text{H}}\right) Q_\text{H}$

CALCULATE: $W = \left(1 - \dfrac{283.15 \text{ K}}{1273.15 \text{ K}}\right)(100. \text{ J}) = 77.7599 \text{ J}$

ROUND: Three significant figures: $W = 77.8$ J.
DOUBLE-CHECK: The efficiency of the engine is about 80% because the difference in the temperatures of the reservoirs is quite large. Hence, the work obtained from this process is 80% of the heat that is input (80 J output from 100 J input).

20.37. **THINK:** This problem is essentially an exercise in algebra. The efficiencies, ε_1 and ε_2, are both related to the temperatures of the hot and cold reservoirs, T_1 and T_2. When T_1 is doubled, the efficiency is doubled.
SKETCH: Not required.

RESEARCH: The initial efficiency is $\varepsilon_1 = 1-(T_2/T_1)$. The efficiency when $T_1 \to 2T_1$ is $\varepsilon_2 = 1-(T_2/2T_1)$. The efficiency is doubled for this temperature increase of the hot reservoir, $2\varepsilon_1 = \varepsilon_2$. The ratio of the temperatures is T_2/T_1 and the efficiency of the initial engine is ε_1

SIMPLIFY: Solve for the ratio of T_2/T_1: $2\varepsilon_1 = \varepsilon_2 \Rightarrow 2-2(T_2/T_1) = 1-(T_2/2T_1)$; therefore, $1 = -(T_2/2T_1)+(2T_2/T_1) = -(T_2/2T_1)+(4T_2/2T_1) = 3T_2/2T_1$ and $T_2/T_1 = 2/3$.

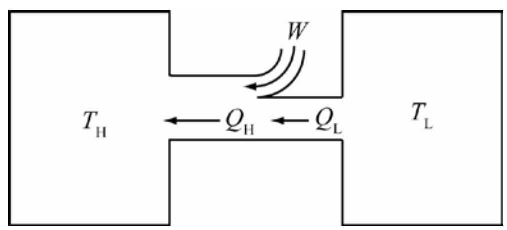
CALCULATE: There are no values that need to be substituted.

ROUND: The ratio of the temperatures is $T_2 : T_1 = 2:3$. The efficiency of the engine is $\varepsilon_1 = 0.33$.

DOUBLE-CHECK: These efficiencies are both less than 1 and are therefore reasonable.

20.38. **THINK:** A Carnot fridge has an ideal coefficient of performance, K. This refrigerator operates with a coefficient that is 32.0% of an ideal refrigerator. The coefficient of performance of an ideal refrigerator can be calculated using the temperatures of the two reservoirs, $T_H = 22.0\text{ °C} = 295.15\text{ K}$ and $T_L = 0.00\text{ °C} = 273.15\text{ K}$. The work is then determined by comparing this with the heat removed from the cold reservoir, $Q_L = 100.\text{ J}$.

SKETCH:



RESEARCH: The ideal coefficient of performance of a refrigerator is $K = T_L/(T_H - T_L)$. The actual coefficient of performance the refrigerator is $K' = 0.320\text{ K}$. The actual coefficient is also calculated as $K' = Q_L/W$.

SIMPLIFY: The actual coefficient of performance is $K' = 0.320 \cdot T_L/(T_H - T_L)$. The work done on the refrigerator is then $W = Q_L/K' = Q_L(T_H - T_L)/(0.320T_L)$.

CALCULATE: $W = (100.\text{ J})(295.15\text{ K} - 273.15\text{ K})/(0.320 \cdot 273.15\text{ K}) = 25.169\text{ J}$

ROUND: Three significant figures: $W = 25.2\text{ J}$.

DOUBLE-CHECK: This is a reasonable value for the work needed to remove the heat, as you expect $W < Q_L$.

20.39. $K = 5.00$ and $Q_L = 40.0\text{ cal}$. In cooling mode:

$$K = \frac{Q_L}{W} = \frac{Q_L}{Q_H - Q_L} = \frac{1}{\frac{Q_H}{Q_L} - 1} \Rightarrow \frac{Q_H}{Q_L} - 1 = \frac{1}{K} \Rightarrow \frac{Q_H}{Q_L} = \frac{1}{K} + 1 \Rightarrow Q_H = Q_L\left(\frac{1}{K} + 1\right).$$

So, $Q_H = 40.0\text{ cal}\left(\frac{1}{5.00} + 1\right) = 48.0\text{ cal}.$

20.40. $K = 5.00$ and $Q_L = 40.0\text{ cal}$. In heating mode:

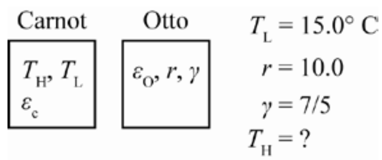
$$K = \frac{Q_H}{W} = \frac{Q_H}{Q_H - Q_L} = \frac{1}{1 - \frac{Q_L}{Q_H}} \Rightarrow \frac{1}{K} = 1 - \frac{Q_L}{Q_H} \Rightarrow \frac{Q_L}{Q_H} = 1 - \frac{1}{K} \Rightarrow Q_H = \frac{Q_L}{1 - \frac{1}{K}}.$$

So, $Q_H = 40.0\text{ cal}\left(\frac{1}{1-(1/5.00)}\right) = 50.0\text{ cal}.$

20.41. For an Otto engine, the efficiency is given as: $\varepsilon = 1 - r^{1-\gamma}$. For a diatomic gas $\gamma = 7/5 = 1.4$. The efficiency of this engine is $\varepsilon = 0.200$. Therefore, $r^{1-\gamma} = 1 - \varepsilon \Rightarrow r = (1-\varepsilon)^{1/(1-\gamma)} = (1-0.200)^{1/(1-1.4)} = 1.75.$

20.42. **THINK:** From the given data, one can compute the efficiency of the Otto cycle. The Carnot efficiency is given in terms of the temperatures. By comparing these two efficiencies the inequality for the engine temperature can be obtained since the Carnot efficiency is always greater than the Otto efficiency.

SKETCH:



RESEARCH: $\varepsilon_C = 1 - (T_L / T_H)$, $\varepsilon_O = 1 - r^{1-\gamma}$, $\varepsilon_C > \varepsilon_O$. The maximum temperature for T_H is for $\varepsilon_C = \varepsilon_O$: $1 - (T_L / T_{H,\max}) = 1 - r^{1-\gamma}$. The temperature of the cool reservoir is $T_L = 15.0\text{ °C} = 288.15\text{ K}$ and the compression ratio is $r = 10.0$. For a diatomic gas, $\gamma = 7/5$.

SIMPLIFY:

(a) $\varepsilon_O = 1 - r^{1-\gamma}$

(b) $T_L / T_{H,\max} = 1 - (1 - r^{1-\gamma}) \Rightarrow T_{H,\max} = T_L / r^{1-\gamma}$

CALCULATE:

(a) $\varepsilon_O = 1 - 10.0^{1-(7/5)} = 0.60189$

(b) $T_H = 288.15\text{ K} / 10^{1-(7/5)} = 723.800\text{ K} = 450.65\text{ °C}$

ROUND: Three significant figures:

(a) $\varepsilon_O = 0.602$

(b) $T_H = 451\text{ °C}$

DOUBLE-CHECK: The efficiency has no units, and lies between zero and one. The temperatures are reasonable values for an outboard motor on a boat.

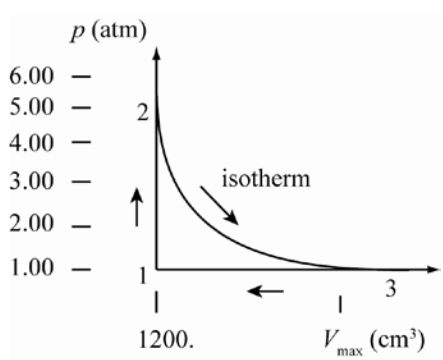
20.43. **THINK:** The thermal efficiency of an engine is the ratio of how much work is output each cycle to how much heat is input.

(a) The pressure and volume can be read off the graph. The temperature can be calculated using the ideal gas law.

(b) The efficiency can be found by computing the total work done and the total heat.

(c) In part (a) the maximum and minimum temperatures were calculated. From this, the maximum efficiency can be computed.

SKETCH:



RESEARCH: Ideal gas law: $pV = nRT$. Work done by the gas: $W = 0$ if $\Delta V = 0$; $W = nRT\ln(V_f / V_i)$ if $\Delta T = 0$; $W = p\Delta V$ if $\Delta p = 0$. Helium is a monatomic gas, so $C_V = (3/2)R$ and $C_p = (5/2)R$. Heat flow:

$Q=(3/2)nR\Delta T$ if $\Delta V=0$; $Q=W$ if $\Delta T=0$; $Q=(5/2)nR\Delta T$ if $\Delta p=0$. Thermal efficiency: $\varepsilon=W_{\text{net}}/Q_{\text{H}}$. The maximum thermal efficiency can be calculated as: $\varepsilon_{\max}=1-\left(T_{\min}/T_{\max}\right)$.

SIMPLIFY:

(a) At point 1: $V_1=1200.\text{ cm}^3$, $p_1=1\text{ atm}$, $p_1V_1=nRT_1 \Rightarrow T_1=p_1V_1/nR$. At point 2: $V_2=1200.\text{ cm}^3$, $p_2=5.00\text{ atm}$, and $T_2=p_2V_2/nR$. At point 3: $p_3V_3=nRT_3$, $p_3=1\text{ atm}$. Since $T_3=T_2$ (isotherm), $p_3V_3=p_2V_2 \Rightarrow V_3=p_2V_2/p_3$.

(b) $\varepsilon=W_{\text{net}}/Q_{\text{H}}$, $W_{\text{net}}=W_{12}+W_{23}+W_{31}$ and $Q_{\text{H}}=Q_{12}+Q_{23}+Q_{31}$ only for $Q>0$ (only for heat input) since the thermal efficiency is being computed.

$$\left.\begin{aligned} &W_{12}=0\\ &W_{23}=nRT_2\ln\left(\frac{V_3}{V_2}\right)\\ &W_{31}=p_3\left(V_1-V_3\right)\\ &Q_{12}=\frac{3}{2}nR\left(T_2-T_1\right)\\ &Q_{23}=W_{23}\\ &Q_{31}=\frac{5}{2}nR\left(T_1-T_3\right)\end{aligned}\right\}\ \varepsilon=\frac{W_{12}+W_{23}+W_{31}}{Q_{12}+Q_{23}+Q_{31}},\text{ only for } Q>0$$

(c) $\varepsilon_{\max}=1-T_{\min}/T_{\max}$

CALCULATE:

(a) First, calculate the number of moles of helium present: $n=\dfrac{0.100\text{ g}}{4.003\text{ g/mol}}=0.02498\text{ mol}.$

For point 1 the values are: $V_1=\left(1200.\text{ cm}^3\right)\left(10^{-6}\text{ m}^3\cdot\text{cm}^{-3}\right)=1.200\cdot10^{-3}\text{ m}^3$, $p_1=1\text{ atm}=1.013\cdot10^5\text{ Pa}$,

$$T_1=\frac{\left(1.013\cdot10^5\text{ Pa}\right)\left(1.200\cdot10^{-3}\text{ m}^3\right)}{\left(0.02498\text{ mol}\right)\left(8.31\text{ J/(mol}\cdot\text{K)}\right)}=585.59\text{ K}.$$

For point 2 the values are: $V_2=1.200\cdot10^{-3}\text{ m}^3$, $p_2=5.00\text{ atm}=5.065\cdot10^5\text{ Pa}$,

$$T_2=\frac{\left(1.200\cdot10^{-3}\text{ m}^3\right)\left(5.065\cdot10^5\text{ Pa}\right)}{\left(0.02498\text{ mol}\right)\left(8.31\text{ J/(mol}\cdot\text{K)}\right)}=2927.97\text{ K}.$$

For point 3 the values are: $p_3=1\text{ atm}=1.013\cdot10^5\text{ Pa}$, $T_3=T_2=2927.97\text{ K}$,

$$V_3=\frac{p_2V_2}{p_3}=\frac{\left(5.00\text{ atm}\right)\left(1.200\cdot10^{-3}\text{ m}^3\right)}{1\text{ atm}}=6.0000\cdot10^{-3}\text{ m}^3.$$

(b) The work done by the gas along each leg of the cycle is:

$$W_{12}=0\text{ J},\ W_{23}=\left(0.02498\text{ mol}\right)\left(8.31\text{ J/(mol}\cdot\text{K)}\right)\left(2927.97\text{ K}\right)\ln\left(\frac{6.000\cdot10^{-3}\text{ m}^3}{1.200\cdot10^{-3}\text{ m}^3}\right)=978.2150\text{ J, and}$$

$W_{31}=\left(1.013\cdot10^5\text{ Pa}\right)\left(1.200\cdot10^{-3}\text{ m}^{-3}-6.000\cdot10^{-3}\text{ m}^{-3}\right)=-486.24\text{ J}$. The heat that provided to the engine (absorbed by the gas) along each leg of the cycle is:

$$Q_{12}=\frac{3}{2}\left(0.02498\text{ mol}\right)\left(8.31\text{ J/(mol}\cdot\text{K)}\right)\left(2927.97\text{ K}-585.59\text{ K}\right)=729.36\text{ J},\ Q_{23}=W_{23}=978.2150\text{ J, and}$$

$$Q_{31}=\frac{5}{2}\left(0.02498\text{ mol}\right)\left(8.31\text{ J/(mol}\cdot\text{K)}\right)\left(585.59\text{ K}-2927.97\text{ K}\right)=-1215.60\text{ J}.$$

Since $Q_{31} < 0$, ignore it, because this is heat that is taken from the engine. The efficiency only depends on the heat input to the gas in the engine. Therefore, $\varepsilon = \frac{0\text{ J} + 978.2150\text{ J} - 486.24\text{ J}}{729.36\text{ J} + 978.2149\text{ J}} = 0.288113.$

(c) $\varepsilon_{\max} = 1 - \frac{T_{\min}}{T_{\max}} = 1 - \frac{585.59}{2927.97} = 0.8000$

ROUND: Three significant figures:

(a) $p_1 = 101$ kPa, $V_1 = 1.20 \cdot 10^{-3}\text{ m}^3$, $T_1 = 586$ K, $p_2 = 507$ kPa, $V_2 = 1.20 \cdot 10^{-3}\text{ m}^3$, $T_2 = 2930$ K, $p_3 = 101$ kPa, $V_3 = 6.00 \cdot 10^{-3}\text{ m}^3$, $T_3 = 2930$ K

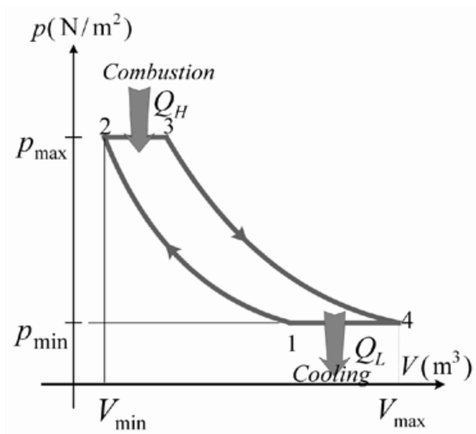
(b) $\varepsilon = 0.288$

(c) $\varepsilon_{\max} = 0.800$

DOUBLE-CHECK: Of course, it must be that $\varepsilon < \varepsilon_{\max}$ and this is true here. These are reasonable results for the given system

20.44. **THINK:** The efficiency can be expressed in terms of the heat transfers Q_H and Q_L. These heat transfers can be expressed in terms of temperatures. Using the ideal gas law and the fact that this is an adiabatic process, the temperatures can be related to the processes.

SKETCH:



RESEARCH: $\varepsilon = \frac{W_{\text{net}}}{Q_H} = \frac{Q_H - Q_L}{Q_H} = 1 - \frac{Q_L}{Q_H}$, isobaric process: $Q = nC_p \Delta T$, adiabatic process: $pV^\gamma = \text{constant}$, and ideal gas law: $pV = nRT$. For a diatomic gas $\gamma = \frac{7}{5} = 1.4$.

SIMPLIFY:

(a) $\varepsilon = 1 - \frac{Q_L}{Q_H}$, heat flow into the engine: $Q_H = nC_p(T_3 - T_2)$, heat flow out of the engine $Q_L = -nC_p(T_1 - T_4)$, $\varepsilon = 1 - \frac{T_4 - T_1}{T_3 - T_2}$. Next, express ε in terms of pressure. $pV = nRT \Rightarrow V = nRT/p$, $V^\gamma = (nRT/p)^\gamma \Rightarrow$ adiabatic: $pV^\gamma = p\left(\frac{nRT}{p}\right)^\gamma = \frac{(nRT)^\gamma}{p^{\gamma-1}} = \text{constant}$. Since n and R are also constants $\Rightarrow T^\gamma p^{1-\gamma} = \text{constant}$.

Process $1 \rightarrow 2$: $T_1^\gamma p_1^{1-\gamma} = T_2^\gamma p_2^{1-\gamma} \Rightarrow T_1^\gamma = T_2^\gamma \left(\frac{p_2}{p_1}\right)^{1-\gamma} \Rightarrow T_1 = T_2 \left(\frac{p_2}{p_1}\right)^{\left(\frac{1}{\gamma}-1\right)}$

Process $3 \to 4$: $T_3^\gamma p_3^{1-\gamma} = T_4^\gamma p_4^{1-\gamma} \;\Rightarrow\; T_3 = T_4\left(\frac{p_4}{p_3}\right)^{\left(\frac{1}{\gamma}-1\right)} = T_4\left(\frac{p_1}{p_2}\right)^{\left(\frac{1}{\gamma}-1\right)} \;\Rightarrow\; T_4 = T_3\left(\frac{p_2}{p_1}\right)^{\left(\frac{1}{\gamma}-1\right)}$

Therefore, $\frac{T_1}{T_2} = \frac{T_4}{T_3} = \left(\frac{p_2}{p_1}\right)^{\left(\frac{1}{\gamma}-1\right)} \;\Rightarrow\; T_4 = \frac{T_1 T_3}{T_2}.$

Efficiency: $\varepsilon = 1 - \frac{T_4 - T_1}{T_3 - T_2},$

$$\varepsilon = 1 - \left(\frac{\frac{T_1 T_3}{T_2} - T_1}{T_3 - T_2}\right) = 1 - \frac{T_1\left(\frac{T_3}{T_2} - 1\right)}{T_2\left(\frac{T_3}{T_2} - 1\right)} = 1 - \frac{T_1}{T_2} = 1 - \left(\frac{p_2}{p_1}\right)^{\left(\frac{1}{\gamma}-1\right)}, \quad \frac{p_2}{p_1} \equiv r_{\text{p}} \;\Rightarrow\; \varepsilon = 1 - r_p^{\left(\frac{1}{\gamma}-1\right)}$$

(b) $\varepsilon = 1 - r_p^{\left(\frac{1}{\gamma}-1\right)}, \quad r_{\text{p}} = 10.0$

CALCULATE:

(a) The requested relation for the efficiency is given as: $\varepsilon = 1 - r_p^{\left(\frac{1}{\gamma}-1\right)}$.

(b) $\varepsilon = 1 - (10.0)^{\left(\frac{5}{7}-1\right)} = 0.4821$

ROUND: (b) Three significant figures: $\varepsilon = 0.482$.

DOUBLE-CHECK: It was shown that the efficiency of the Brayton cycle only depends on the pressure ratio, as requested. An efficiency of approximately 50% is a reasonable value for part (b).

20.45. (a) $\Delta S_{\text{H}} = Q/T_{\text{H}} = -8500.\text{ J}/700.\text{ K} = -12.1\text{ J/K}$ and $\Delta S_L = Q/T_L = 8500.\text{ J}/100.\text{ K} = 85.0\text{ J/K}$.

(b) $\Delta S_{\text{rod}} = 0$ J/K. There is no change in entropy for the rod since there is no net heat flow into or out of the rod.

(c) $\Delta S_{\text{system}} = \Delta S_{\text{H}} + \Delta S_{\text{L}} + \Delta S_{\text{rod}} = -8500.\text{ J}/700.\text{ K} + 8500.\text{ J}/100.\text{ K} + 0.00\text{ J/K} = 72.9\text{ J/K}$

20.46. To find the entropy change, examine the number of microstates of the system before and after the expansion. The number of gas molecules is $N = nN_A$, where n is the number of moles and N_A is Avogadro's constant. Before the expansion there were w_i microstates for the gas molecules that were in half of the box. After the barrier is removed the gas molecules are free to expand into the other half of the box. The number of microstates after the expansion is $w_{\text{f}} = 2^N w_{\text{i}}$. The change in entropy is then $\Delta S = k_{\text{B}}\ln(w_{\text{f}}/w_{\text{i}}) = k_{\text{B}}\ln(2^N) = Nk_{\text{B}}\ln 2 = nR\ln(2)$.

20.47. (a) The theoretical maximum efficiency is provided by a Carnot engine, whose efficiency depends reservoir temperatures. $\varepsilon = 1 - (T_{\text{L}}/T_{\text{H}}) = 1 - (300.\text{ K}/400.\text{ K}) = 0.250$.

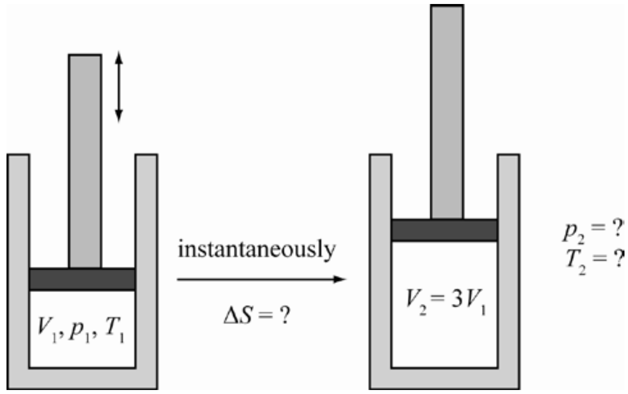
(b) A Carnot engine consists of two isothermal processes and two isentropic processes. Therefore, after a cycle the total entropy change is zero.

20.48. It is assumed that the block lost all of its kinetic energy to heat so that $Q = K = (1/2)mv^2$. Then, $\Delta S = (1/2)mv^2/T = (1/2)(10.0\text{ kg})(10.0\text{ m/s})^2/(300.15\text{ K}) = 1.67\text{ J/K}$.

20.49. The initial number of states available is $w_{\text{i}} = V/V_A$. When the volume is doubled the final number of states available is $w_{\text{f}} = 2V/V_A$. Therefore, the change in entropy is $\Delta S = k_{\text{B}}\ln(w_{\text{f}}/w_{\text{i}}) = k_{\text{B}}\ln\left((2V/V_{\text{A}})/(V/V_{\text{A}})\right) = k_{\text{B}}\ln 2$.

20.50. **THINK:** Due to the insulation, no heat is exchanged, $Q=0$, so the temperature of the gas remains constant. Since the piston moves very quickly, no work is done by the gas. Therefore, by the First Law of Thermodynamics the internal energy of the gas must remain the same as well. The problem can be solved by using the ideal gas law and the definition of entropy.

SKETCH:



RESEARCH: $\Delta E_{\text{int}} = Q - W$, $Q=0$, $W=0 \Rightarrow \Delta E_{\text{int}} = 0 \Rightarrow \Delta T = 0 \Rightarrow T_2 = T_1$. Ideal gas law: $p_2 V_2 = nRT_2$, $\Delta S = \int \frac{dQ}{T}$, $\Delta E_{\text{int}} = 0 \Rightarrow dQ = dW = pdV$.

SIMPLIFY: $T_2 = T_1$, $p_2 = nRT_2 / V_2 \Rightarrow p_2 = nRT_1 / \left(3V_1\right) = (1/3)nRT_1 / V_1 \Rightarrow p_2 = \frac{1}{3}p_1$ $\quad \Delta S = \int_{V_1}^{3V_1} \frac{pdV}{T}$.

Since $pV = nRT \Rightarrow \frac{1}{T} = \frac{nR}{pV}$ Therefore,

$$\Delta S = \int_{V_1}^{3V_1} \frac{nRpdV}{pV} = nR\int_{V_1}^{3V_1} \frac{dV}{V} = nR\left[\ln V\right]\Big|_{V_1}^{3V_1} = nR\ln\left(\frac{3V_1}{V_1}\right) = nR\ln 3$$

CALCULATE: Not required.

ROUND: Not required.

DOUBLE-CHECK: It is logical that the pressure should decrease if the volume increases at a constant temperature. Similarly, an increase of entropy is reasonable since the molecules of the gas now have a larger volume to occupy.

20.51. **THINK:** The entropy is related to the number of states shared by the specified property. This problem can be solved by counting the possibilities.

SKETCH: Possible spin states (5 spins in total, "+"= spin-up, "-" = spin-down).

+ + + + +
+ + + + -
+ + + - +
+ + + - -
⋮
- - - - -

$2^5 = 32$ possibilites

RESEARCH: $S = k_{\text{B}} \ln w$

SIMPLIFY: 5-up: $w=1$ (there is only 1 way to put 5 up). 3-up: w can be determined by listing the possibilities:

+++--			
++-+-	++--+		
+-++-	+-+-+	+--++	
-+++-	-++-+	-+-++	--+++

10 possibilities. $\Rightarrow w = 10$

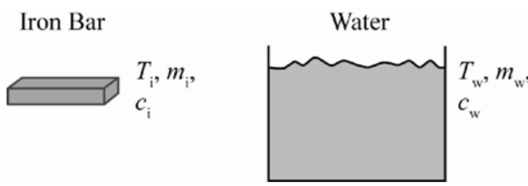
Alternate method: Choosing 3 of 5 spin-up: $\Rightarrow w = {}_5C_3 = 10$ (as before). $S_{3\text{ up}} = k_{\text{B}} \ln 10$.

CALCULATE: $S_{5\text{ up}} = k_B \ln w = k_B \ln 1 = 0$

$S_{3\text{ up}} = k_B \ln w = k_B \ln 10 = \left(1.38 \cdot 10^{-23}\text{ J/K}\right)\ln 10 = 3.178 \cdot 10^{-23}\text{ J/K}$

ROUND: $S_{5\text{ up}} = 0$ (exactly). For $S_{3\text{ up}}$, round to three significant figures: $S_{3\text{ up}} = 3.18 \cdot 10^{-23}$ J/K.
DOUBLE-CHECK: By definition the entropy of one microstate is exactly 0. As the number of states increase the entropy of the system increases so it must be that $S_{3\text{ up}} > S_{5\text{ up}}$, as it has been found.

20.52. **THINK:** The entropy change is the sum of the entropy changes for the water and for the iron. The final temperature is the same for the both iron and water. This temperature can be found by noting that the heat lost by the iron is the heat gained by the water. The question gives the values: $m_i = 0.545$ kg, $c_i = 448\text{ J/}\left(\text{kg K}\right)$, $T_i = 1000.0\text{ °C} = 1273.2\text{ K}$, $m_W = 10.00$ kg, $c_W = 4186\text{ J/}\left(\text{kg K}\right)$, $T_W = 22.0\text{ °C} = 295.2\text{ K}$.
SKETCH:



RESEARCH: Both the iron and the water reach the same final temperature, T_f. The entropy change is $\Delta S = \Delta S_W + \Delta S_i$, $\Delta S_W = \int_i^f dQ_W / T$, $\Delta S_i = \int_i^f dQ_i / T$, $Q_W = -Q_i$, $dQ_W = -dQ_i$ where $dQ = mc\,dT$.

SIMPLIFY: $\Delta S = \int_{T_W}^{T_f} m_W c_W dT / T + \int_{T_i}^{T_f} m_i c_i dT / T = m_W c_W \ln\left(T_f / T_W\right) + m_i c_i \ln\left(T_f / T_i\right)$. In order to calculate the entropy change T_f needs to be obtained from: $Q_W = -Q_i \Rightarrow m_W c_W \Delta T_W = -m_i c_i \Delta T_i$, $m_W c_W\left(T_f - T_W\right) = -m_i c_i\left(T_f - T_i\right)$, $\left(m_W c_W + m_i c_i\right)T_f = m_i c_i T_i + m_W c_W T_W \Rightarrow T_f = \dfrac{\left(m_i c_i T_i + m_W c_W T_W\right)}{\left(m_W c_W + m_i c_i\right)}$.

CALCULATE:

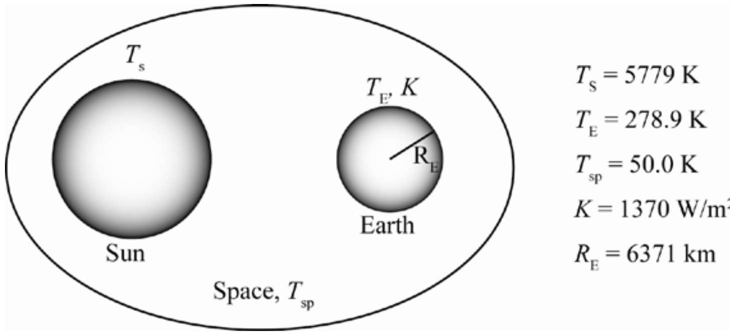
$$T_f = \frac{\left(0.545\text{ kg}\right)\left(448\text{ J/}\left(\text{kg K}\right)\right)\left(1273.2\text{ K}\right) + \left(10.00\text{ kg}\right)\left(4186\text{ J/}\left(\text{kg K}\right)\right)\left(295.2\text{ K}\right)}{\left(10.00\text{ kg}\right)\left(4186\text{ J/}\left(\text{kg K}\right)\right) + \left(0.545\text{ kg}\right)\left(448\text{ J/}\left(\text{kg K}\right)\right)} = 300.87\text{ K}$$

$$\Delta S = \left(10.00\text{ kg}\right)\left(4186\text{ J/}\left(\text{kg K}\right)\right)\ln\left(300.7\text{ K}/295.2\text{ K}\right) + \left(0.545\text{ kg}\right)\left(448\text{ J/}\left(\text{kg K}\right)\right)\ln\left(300.7\text{ K}/1273.2\text{ K}\right)$$
$$= 420.4\text{ J/K}$$

ROUND: The least precise value given in the question has three significant figures, so the final answer is rounded to three significant figures: $\Delta S = 420.$ J/K.
DOUBLE-CHECK: The entropy change has the units of J/K, which is appropriate.

20.53. **THINK:** Consider the rate of change in entropy of the Earth as it absorbs energy from Sun and space, and as it radiates energy back into space. The Earth's entropy increases when it absorbs energy and decreases when it radiates energy.

SKETCH:



RESEARCH: The rate of change of entropy is given by: $\dfrac{\Delta S}{\Delta t} = \dfrac{1}{T}\left(\dfrac{\Delta Q}{\Delta t}\right)$.

SIMPLIFY: The rate of change of Earth's entropy is given by:

$$\left(\frac{\Delta S}{\Delta t}\right)_{\text{tot}} = \frac{1}{T_{\text{S}}}\left(\frac{\Delta Q}{\Delta t}\right)_{\text{S}} + \frac{1}{T_{\text{E}}}\left(\frac{\Delta Q}{\Delta t}\right)_{\text{E}} + \frac{1}{T_{\text{sp}}}\left(\frac{\Delta Q}{\Delta t}\right)_{\text{sp}}.$$

The Earth absorbs radiation from the Sun at a rate of $\left(\dfrac{\Delta Q}{\Delta t}\right)_{\text{S}} = \pi R_{\text{E}}^2 K$, where K is the solar constant. The Earth radiates energy into space at a rate given by the Stefan-Boltzmann law:

$$\left(\frac{\Delta Q}{\Delta t}\right)_{\text{E}} = -4\pi R_{\text{E}}^2 \sigma T_{\text{E}}^4,$$

The Earth absorbs energy from space at a rate given by the Stefan-Boltzmann law:

$$\left(\frac{\Delta Q}{\Delta t}\right)_{\text{sp}} = 4\pi R_{\text{E}}^2 \sigma T_{\text{sp}}^4.$$

Therefore, the total rate of change of Earth's entropy is given by:

$$\left(\frac{\Delta S}{\Delta t}\right)_{\text{tot}} = \frac{\pi R_{\text{E}}^2 K}{T_{\text{S}}} - 4\pi R_{\text{E}}^2 \sigma T_{\text{E}}^3 + 4\pi R_{\text{E}}^2 \sigma T_{\text{sp}}^3 = \pi R_{\text{E}}^2\left(\frac{K}{T_{\text{S}}} + 4\sigma\left(T_{\text{sp}}^3 - T_{\text{E}}^3\right)\right)$$

CALCULATE:

$$\left(\frac{\Delta S}{\Delta t}\right)_{\text{tot}} = \pi\left(6.371\cdot 10^6 \text{ m}\right)^2\left(\frac{\left(1370.\text{ W/m}^2\right)}{\left(5779\text{ K}\right)} + 4\left(5.670\cdot 10^{-8}\text{ W m}^{-2}\text{ K}^{-4}\right)\left(\left(50.0\text{ K}\right)^3 - \left(278.9\text{ K}\right)^3\right)\right)$$

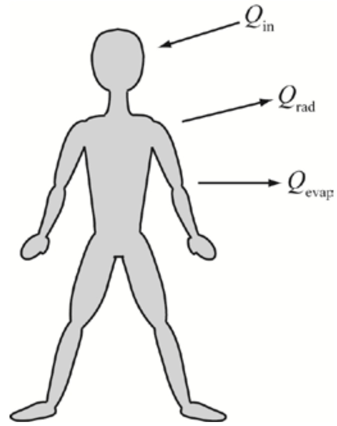
$$= -5.93569\cdot 10^{14}\text{ W/K}$$

ROUND: To three significant figures: $\left(\dfrac{\Delta S}{\Delta t}\right)_{\text{tot}} = -5.94\cdot 10^{14}\text{ W/ K}.$

DOUBLE-CHECK: The negative sign indicates that the entropy of the Earth is decreasing. At first, one may hesitate to accept this answer since the entropy of a system must never decrease. However, in this case, the system is the Sun, the Earth and space, so the entropy of Earth (only one part of the system) may decrease. This result is justified because in order for Earth to have life, Earth must be organized (i.e. it must decrease its entropy). Since the Earth uses the Sun's energy to create and sustain life, the Sun acts to reduce the entropy of the Earth.

20.54. **THINK:** Parts (a), (b) and (d) can be computed directly. Part (c) can be calculated with the Stefan-Boltzmann radiation law. For part (e), the lower-bound can be computed by considering the sum of the entropy production rates for the energy consumed and the amount lost to radiation and evaporation.

SKETCH:



RESEARCH: $\Delta Q_{in}/\Delta t = 2000.\text{ kcal/day}$, $T_{core} = 37.0\text{ °C}$, $T_{skin} = 27.0\text{ °C}$, $A_{skin} = 1.50\text{ m}^2$, $T_{air} = 20.0\text{ °C}$, $e = 0.600$ (emissivity), $L_{vap} = 575\text{ cal/g}$ and $\sigma = 5.670\cdot 10^{-8}\text{ W m}^{-2}\text{ K}^{-4}$, specific heat for water is 4186 J/kcal.

(a) The efficiency of a Carnot engine operating between these temperatures would be: $\varepsilon_{max} = 1-(T_L/T_H)$.

(b) The rate of heat generated as waste would be: $\Delta Q_{waste}/\Delta t = (1-\varepsilon_{max})\Delta Q_{in}/\Delta t$.

(c) Stefan-Boltzmann yields: $\Delta Q_{rad}/\Delta t = \sigma eA\left(T_{skin}^4 - T_{air}^4\right)$.

(d) The transpiration rate must be: $\Delta M/\Delta t = (1/L_{vap})(\Delta Q_{vap}/\Delta t) = (1/L_{vap})(\Delta Q_{waste}/\Delta t - \Delta Q_{rad}/\Delta t)$.

(e) The processes described are not reversible, but we can calculate the lower bound of the entropy rate:
$\Delta S/\Delta t \geq -(1/T_{core})(\Delta Q_{in}/\Delta t) + \sigma eA\left(T_{skin}^3 - T_{air}^3\right) + (1/T_{core})(\Delta Q_{vap}/\Delta t)$.

SIMPLIFY: Not required.

CALCULATE:

(a) $\varepsilon_{max} = 1-(20.0\text{ °C}+273.15\text{ °C})/(37.0\text{ °C}+273.15\text{ °C}) = 0.05481$

(b) $\Delta Q_{waste}/\Delta t = (1-0.05481)(2000.\text{ kcal/day})(4186\text{ J/kcal})/(86400\text{ s/day}) = 91.59\text{ W}$

(c) $$\frac{\Delta Q_{rad}}{\Delta t} = \left(5.670\cdot 10^{-8}\text{ W m}^{-2}\text{ K}^{-4}\right)(0.600)\left(1.50\text{ m}^2\right)\left((27.0\text{ °C}+273.15\text{ °C})^4 - (20.0\text{ °C}+273.15\text{ °C})^4\right)$$
$$= 37.31\text{ W}$$

(d) $$\Delta M/\Delta t = \frac{(91.59\text{ W} - 37.31\text{ W})(3600\text{ s/h})}{(575\text{ cal/g})(4.186\text{ J/cal})} = 81.18\text{ g/h}$$

(e) $$\Delta S/\Delta t \geq -\frac{(2000.\text{ kcal/day})(4186\text{ J/k cal})}{(86400\text{ s/day})(37.0\text{ °C}+273.15\text{ °C})}$$
$$+\left(5.670\cdot 10^{-8}\text{ Wm}^{-2}\text{K}^{-4}\right)(0.600)\left(1.50\text{ m}^2\right)\left((27.0\text{ °C}+273.15\text{ °C})^3 - (20.0\text{ °C}+273.15\text{ °C})^3\right)$$
$$+\frac{(91.59\text{ W} - 37.31\text{ W})}{(37.0\text{ °C}+273.15\text{ °C})}$$
$$\geq -0.04310\text{ W/K}$$

ROUND: The values need to be rounded to three significant figures.

(a) $\varepsilon_{max} = 0.0548$

(b) $\Delta Q_{waste}/\Delta t = 91.6\text{ W}$

(c) $\Delta Q_{rad}/\Delta t = 37.3\text{ W}$

(d) $\Delta M / \Delta t = 81.2$ g/h

(e) $\Delta S / \Delta t \geq -0.0431$ W/K

DOUBLE-CHECK: These are all reasonable values for this particular person. The fact that the entropy production rate is close to zero is consistent with the fact that the person is working with Carnot efficiency, as the net entropy production rate for a Carnot engine is zero.

20.55. $\varepsilon = 1 - \dfrac{T_\text{L}}{T_\text{H}} = 1 - \dfrac{288\text{ K}}{5700\text{ K}} = 0.95.$

20.56. For an isothermal process, $pV = \text{constant}$. For an adiabatic process, $pV^\gamma = \text{constant}$. Differentiate both sides of both equations with respect to V.

Isothermal process: $\left(\dfrac{dp}{dV}\right)V + p\left(\dfrac{dV}{dV}\right) = \left(\dfrac{dp}{dV}\right)V + p = 0 \Rightarrow \left(\dfrac{dp}{dV}\right) = -\dfrac{p}{V}$

Adiabatic process: $\left(\dfrac{dp}{dV}\right)V^\gamma + p\left(\dfrac{d\left(V^\gamma\right)}{dV}\right) = \left(\dfrac{dp}{dV}\right)V^\gamma + p\gamma V^{\gamma-1} = 0 \Rightarrow \left(\dfrac{dp}{dV}\right) = -\dfrac{p\gamma V^{\gamma-1}}{V^\gamma} = -\gamma\dfrac{p}{V}$

Therefore, $\left(\dfrac{dp}{dV}\right)_\text{adiabatic} = \gamma\left(\dfrac{dp}{dV}\right)_\text{isotherm}$. An adiabatic curve is steeper than an isotherm by a factor of γ.

20.57. Gasoline in vapor form when ignited is diatomic with $\gamma = 7/5 = 1.4$.

$$\varepsilon_\text{max, 91} = 1 - r_\text{max, 91}^{1-\gamma} = 1 - 8.5^{1-1.4} = 0.5752$$

$$\varepsilon_\text{max, 93} = 1 - r_\text{max, 93}^{1-\gamma} = 1 - 9.0^{1-1.4} = 0.5848$$

$$\varepsilon_\text{max, 95} = 1 - r_\text{max, 95}^{1-\gamma} = 1 - 9.8^{1-1.4} = 0.5987$$

$$\varepsilon_\text{max, 97} = 1 - r_\text{max, 97}^{1-\gamma} = 1 - 10.5^{1-1.4} = 0.6096$$

So, in going from a gasoline with octane number 91 (in an engine with 8.5 compression ratio) to a gasoline with octane number 97 (in an engine with compression ratio 10.5), the percentage increase in efficiency is: $(0.6096 - 0.5752)/0.5752 = 6.0\%$.

20.58. (a) The reservoir temperatures are $T_\text{H} = 35\text{ °C} + 273\text{ °C} = 308\text{ K}$ and $T_\text{L} = 18\text{ °C} + 273\text{ °C} = 291\text{ K}$. The coefficient of performance of the air conditioner is $K = Q_\text{L}/W$ and the for an ideal engine: $K_\text{max} = T_\text{L}/\left(T_\text{H} - T_\text{L}\right)$. Solving these equations for $W: W = \left[\left(T_\text{H} - T_\text{L}\right)/T_\text{L}\right]Q_\text{L}$ and using the First Law of Thermodynamics gives, $Q_\text{H} = W + Q_\text{L} = Q_\text{L}\left(\dfrac{T_\text{H} - T_\text{L}}{T_\text{L}} + 1\right) = \left(1.00\text{ J}\right)\left(\dfrac{308\text{ K} - 291\text{ K}}{291\text{ K}} + 1\right) = 1.06\text{ J}.$

(b) Since 1.00 J of heat flows out of the room (negative Q with respect to the indoors) and the heat transfers are isothermal, $\Delta S = \dfrac{Q}{T} = -\dfrac{1.00\text{ J}}{291\text{ K}} = -3.44 \cdot 10^{-3}$ J/K.

(c) Since 1.06 J of heat flows into the outdoors (positive Q with respect to the outdoors) and the heat transfers are isothermal, $\Delta S = \dfrac{Q}{T} = \dfrac{1.06\text{ J}}{308\text{ K}} = 3.44 \cdot 10^{-3}$ J/K.

20.59. Heat capacity: $C = \Delta E / \Delta T$. Entropy:

$$\Delta S = \int_i^f \frac{dQ}{T} = C\int_{T_i}^{T_f} \frac{dT}{T} = \frac{\Delta E}{\Delta T}\ln\left(\frac{T_f}{T_i}\right) = \frac{0.0700 \text{ J}}{0.500 \text{ K}}\ln\left(\frac{100.\ ^\circ\text{C} + 273.15\ ^\circ\text{C}}{10.\ ^\circ\text{C} + 273.15\ ^\circ\text{C}}\right) = 0.0386 \text{ J/K}.$$

20.60. A Carnot engine is the most efficient engine possible that operates between two temperature reservoirs. The efficiency of the Carnot cycle between these two temperature reservoirs would be

$$\varepsilon = 1 - \left(T_L / T_H\right) = 1 - \left(4.0\ ^\circ\text{C} + 273.15\ ^\circ\text{C}\right) / \left(20.0\ ^\circ\text{C} + 273.15\ ^\circ\text{C}\right) = 0.055.$$

Therefore, the claim of a water-driven engine with an efficiency of 0.200 is invalid since 0.0546 is the maximum efficiency possible.

20.61. The entropy of the system of dice will increase if more dice are added. The entropy can be calculated by the formula $S = k_B \ln w$, where w is the number of available states. For one six-sided dice $w = 6$ so $S_1 = k_B \ln 6$. For n six-sided die, $w = 6^n$ so $S_n = k_B \ln\left(6^n\right) = nk_B \ln 6 \Rightarrow S_n / S_1 = n$. Therefore, for n dice, the entropy increases by a factor of n. This means that for two dice the entropy is doubled and for three dice the entropy is tripled.

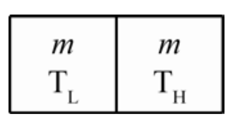
20.62. The change in entropy is $\Delta S = \Delta Q / T$. By the First Law of Thermodynamics, $\Delta Q = \Delta E_{\text{int}} + \Delta W$. When the moving car hits the wall all of the kinetic energy *K* is converted into work (the car destroys the wall and the wall twists the metal of the car), that is $\Delta K = \Delta W$. For the purpose of answering the question, it can be said that the internal energy of the car does not change $(\Delta E_{\text{int}} = 0)$, since a temperature increase is only generated from the work that is being done. Therefore, $\Delta Q = \Delta K$ and the change in entropy of the car is:

$$\Delta S = \frac{\Delta Q}{T} = \frac{\Delta K}{T} = \frac{\frac{1}{2}mv^2}{T} = \frac{\frac{1}{2}\left(1200 \text{ kg}\right)\left(30.0 \text{ m/s}\right)^2}{27\ ^\circ\text{C} + 273\ ^\circ\text{C}} = 1800 \text{ J/K}.$$

20.63. The entropy change for this process is given by $\Delta S = \Delta Q / T$. Dividing both sides by the number of moles, *n,* yields $\left(\Delta S / n\right) = \left(\Delta Q / n\right) / T = L_{\text{vap}} / T$. So, at a pressure of 100.0 kPa, the boiling temperature is $T = L_{\text{vap}} / \left(\Delta S / n\right) = \left(5.568 \cdot 10^3 \text{ J/mol}\right) / \left(72.1 \text{ J/}\left(\text{mol K}\right)\right) = 77.2 \text{ K}.$

20.64. **THINK:** If heat cannot escape, then the final equilibrium temperature is $50.0\ ^\circ\text{C}$ since there are equal amounts of water at $0\ ^\circ\text{C}$ and at $100.\ ^\circ\text{C}$. The cold water increases its entropy and the hot water decreases its entropy.

SKETCH:



RESEARCH: $\Delta S = \int dQ / T$ and $dQ = mcdT$

SIMPLIFY: $\Delta S = \Delta S_L + \Delta S_H = \int_{T_L}^{T_f} mc\frac{dT}{T} + \int_{T_H}^{T_f} mc\frac{dT}{T} = mc\left[\ln\left(\frac{T_f}{T_L}\right) + \ln\left(\frac{T_f}{T_H}\right)\right] = mc\ln\left(\frac{T_f^2}{T_L T_H}\right)$

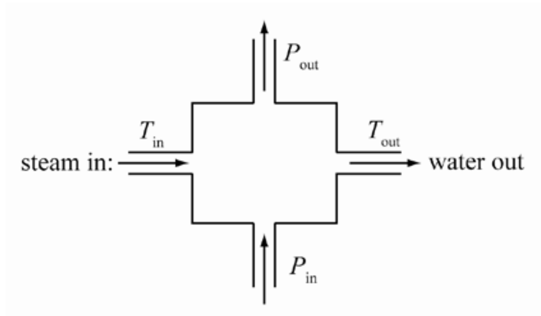
CALCULATE: $\Delta S = \left(100. \cdot 10^{-3} \text{ kg}\right)\left(4186 \text{ J/(kg K)}\right)\ln\left(\frac{\left(50.0\ ^\circ\text{C} + 273.15\ ^\circ\text{C}\right)^2}{(273.15\ ^\circ\text{C})(100.\ ^\circ\text{C} + 273.15\ ^\circ\text{C})}\right) = 10.14 \text{ J/K}$

ROUND: Three significant figures: $\Delta S = 10.1$ J/K.

DOUBLE-CHECK: The change in entropy for 100 g of water going from $0\ ^\circ\text{C}$ to $50\ ^\circ\text{C}$ is $\Delta S_1 = cm\ln\left(T_f / T_i\right) = cm\ln\left(323 \text{ K} / 273 \text{ K}\right) = 70.40$ J/K. Similarly, the change in entropy for 100 g of water going from $100\ ^\circ\text{C}$ to $50\ ^\circ\text{C}$ is $\Delta S_2 = cm\ln\left(323 \text{ K} / 373 \text{ K}\right) = -60.25$ J/K. $\Delta S_1 + \Delta S_2 = 10.15$ J/K.

20.65. THINK: A Carnot engine is the most efficient engine possible that operates between two temperature reservoirs. The efficiency of the Carnot cycle between these two temperature reservoirs depends on the ratio of the temperatures. The question gives the power input and the power output; a ratio of the two will provide the actual efficiency. The unit gal is taken to be American gallons (1 gal = 3.785 L).

SKETCH:



RESEARCH:

(a) The maximum thermal efficiency of an engine (Carnot) is $\varepsilon_{max} = 1 - \left(T_{out} / T_{in}\right)$.

(b) The actual efficiency is simply $\varepsilon_{act} = P_{out} / P_{in}$.

(c) The heat given to the river water during condensation of the steam is $Q = \left(P_{in} - P_{out}\right)t = \Delta Pt$ over time t and $Q = mc\Delta T$. The volumetric flow rate of the water can be written as $f = (m/\rho)/t$, where m is the mass, ρ is the density and t is time. Therefore, $m = f\rho t$.

SIMPLIFY:

(a) Already in simplified form.

(b) Already in simplified form.

(c) $Q = mc\Delta T = \Delta Pt = f\rho tc\Delta T \Rightarrow \Delta T = \dfrac{\Delta P}{f\rho c} \Rightarrow T_f = T_i + \dfrac{\Delta P}{f\rho c}$

CALCULATE:

(a) $\varepsilon_{max} = 1 - \dfrac{30.0\text{ °C} + 273.15\text{ °C}}{300.\text{ °C} + 273.15\text{ °C}} = 0.47108$

(b) $\varepsilon = \dfrac{1000.\text{ MW}}{3000.\text{ MW}} = 0.3333$

(c) $T_f = 20.0\text{ °C} + \dfrac{\left(3000.\text{ MW} - 1000.\text{ MW}\right)\left(10^6\text{ W/MW}\right)}{\left(4.00\cdot 10^7\text{ gal/h}\right)\left(1\text{ h/3600 s}\right)\left(3.785\text{ L/gal}\right)\left(1\text{ kg/L}\right)\left(4186\text{ J/}\left(\text{kg °C}\right)\right)} = 31.361\text{ °C}$

ROUND: Three significant figures:

(a) $\varepsilon_{max} = 0.471$

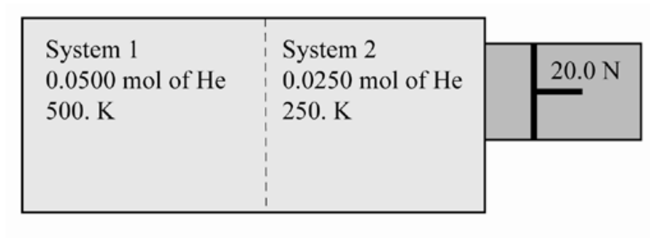
(b) $\varepsilon = 0.333$

(c) $T_f = 31.4\text{ °C}$

DOUBLE-CHECK: These are reasonable results for the parameters given.

20.66. THINK: When the two systems are allowed to combine, the combined system will be under constant pressure due to the force of the piston. Heat will be transferred from System 1 to System 2 since System 1 is at a higher temperature, but the net heat flow will be zero.

SKETCH:



RESEARCH:

(a) $Q_1 + Q_2 = 0,\ Q_1 = n_1 C_p \Delta T_1$ and $Q_2 = n_2 C_p \Delta T_2$.

(b) $Q_2 = n_2 C_p \Delta T_2$

(c) To determine the displacement of the piston we need to find the amount of work that System 1 does on the piston. By the First Law of Thermodynamics, $W = Q - \Delta E_{\text{int}}$. The work done by the heat transfer pushes on the piston with force F, moving it by Δx: $W = F\Delta x$. The change in internal energy is $\Delta E_{\text{int}} = \frac{3}{2} nR\Delta T_2$.

(d) The work required to move the piston must come from the heat transfer within the system. The fraction of the heat transferred that is converted into work is given by W / Q_2.

SIMPLIFY:

(a) $-n_1 C_p \left(T_{\text{f}} - T_1\right) = n_2 C_p \left(T_{\text{f}} - T_2\right) \Rightarrow n_1 T_{\text{f}} + n_2 T_{\text{f}} = n_1 T_1 + n_2 T_2$

$$\left(n_1 + n_2\right) T_{\text{f}} = n_1 T_1 + n_2 T_2 \Rightarrow T_{\text{f}} = \frac{n_1 T_1 + n_2 T_2}{n_1 + n_2}$$

(b) $Q_2 = n_2 C_p \left(T_{\text{f}} - T_2\right)$

(c) $$F\Delta x = Q_2 - \frac{3}{2} R n_2 \left(T_{\text{f}} - T_2\right) \Rightarrow \Delta x = \frac{n_2 C_p \left(T_{\text{f}} - T_2\right) - \frac{3}{2} R n_2 \left(T_{\text{f}} - T_2\right)}{F} = \left(C_p - \frac{3}{2} R\right) \frac{n_2 \left(T_{\text{f}} - T_2\right)}{F}$$

(d) $$\frac{W}{Q_2} = \frac{F\Delta x}{Q_2} = \frac{\left(C_p - \frac{3}{2} R\right) n_2 \left(T_{\text{f}} - T_2\right)}{Q_2}$$

CALCULATE:

(a) $$T_{\text{f}} = \frac{(0.0500 \text{ mol})(500.\text{ K}) + (0.0250 \text{ mol})(250.\text{ K})}{0.0500 \text{ mol} + 0.0250 \text{ mol}} = 416.7 \text{ K}$$

(b) $$Q_2 = \left(0.0250 \text{ mol}\right)\left(\frac{5}{2}\right)\left(8.31 \text{ J/(mol K)}\right)\left(416.7 \text{ K} - 250.\text{ K}\right) = 86.5798 \text{ J}$$

(c) $$\Delta x = \left(8.31 \text{ J/(mol K)}\right) \frac{\left(0.0250 \text{ mol}\right)\left(416.7 \text{ K} - 250.\text{ K}\right)}{20.0 \text{ N}} = 1.732 \text{ m}$$

(d) $$\frac{W}{Q_2} = \frac{\left(8.31 \text{ J/(mol K)}\right)\left(0.0250 \text{ mol}\right)\left(416.7 \text{ K} - 250.\text{ K}\right)}{86.5798 \text{ J}} = 0.40000$$

ROUND: Three significant figures:

(a) $T_{\text{f}} = 417$ K

(b) $Q_2 = 86.6$ J

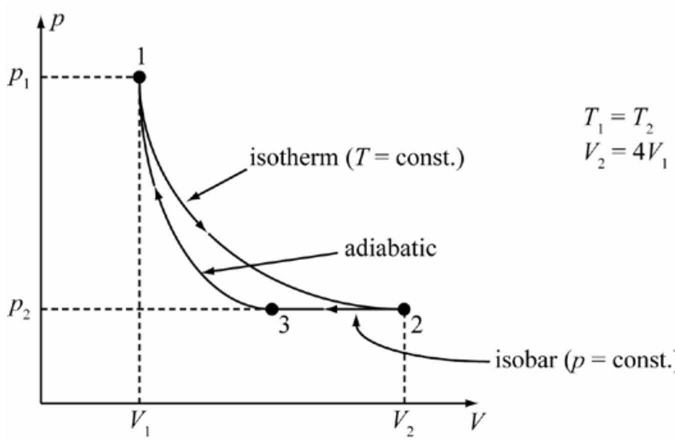
(c) $\Delta x = 1.73$ m

(d) The fraction of the heat transferred that is converted to heat is 40.0%.

DOUBLE-CHECK: The final temperature is an increase from the average temperature of 375 K, as would be expected from the average force. The change of 86.6 J is of an appropriate value for the system. The piston being displaced by 1.73 meters is realistic. If the engine had been an ideal Carnot engine, it would have a work to heat ratio of $\frac{500-250}{500}=\frac{1}{2}$. The calculated ratio is lower than the ideal value, as would be expected.

20.67. **THINK:** For reversible processes, the entropy changes can be computed directly from the defining equations. The net change is entropy is zero for the full cycle.

SKETCH:



RESEARCH: $T_1 = T_2$ and $V_2 = 4V_1$. The entropy change is given by $dS = dQ/T$. For an adiabatic process: $dQ = 0,\ \ pV^\gamma = \text{constant}$ with $\gamma = 5/3$ for a monatomic gas. For an isobaric process: $dQ = nC_p dT$. For an ideal gas: $pV = nRT$.

SIMPLIFY: An adiabatic curve is a curve of constant entropy since there is not heat flow $(dQ = 0)$, so $\Delta S_{\text{adrabatic}} = 0.$ $\Delta S_{\text{isobar}} = \int_{T_2}^{T_3} nC_\text{P}\frac{dT}{T} = nC_\text{P}\ln\left(\frac{T_3}{T_2}\right)$. By the ideal gas law, $\frac{T_3}{T_2} = \frac{p_3V_3}{p_2V_2} = \frac{V_3}{V_2}$ since $p_3 = p_2$.

Finding, $\frac{V_3}{V_2}$: $pV^\gamma = \text{constant}$ along $3 \to 1$ (the adiabatic process), so $p_3V_3^\gamma = p_1V_1^\gamma$, but $p_3 = p_2$ and $V_1 = \frac{V_2}{4}$, so $p_2V_3^\gamma = p_1\left(\frac{V_2}{4}\right)^\gamma$. Using the ideal gas law, since $T_1 = T_2$: $p_1V_1 = p_2V_2$. Since $V_2 = 4V_1$: $p_1V_1 = p_2(4V_1) \Rightarrow p_1 = 4p_2$. Back to the adiabatic process: $p_2V_3^\gamma = p_1\left(\frac{V_2}{4}\right)^\gamma \Rightarrow p_2V_3^\gamma = (4p_2)\left(\frac{V_2}{4}\right)^\gamma \Rightarrow \frac{V_3}{V_2} = 4^{\frac{1}{\gamma}-1} = \frac{T_3}{T_2}$.

$\Delta S_{\text{isobar}} = nC_p\ln\left(\frac{T_3}{T_2}\right) = nC_p\ln\left(4^{\frac{1}{\gamma}-1}\right) = \left(\frac{1}{\gamma}-1\right)nC_p\ln 4$. To find $\Delta S_{\text{isotherm}}$, use the fact that the sum of the entropy changes is zero: $\Delta S_{\text{tot}} = 0 = \Delta S_{\text{isotherm}} + \Delta S_{\text{isobar}} + \Delta S_{\text{adiabatic}} \Rightarrow \Delta S_{\text{isotherm}} = -\Delta S_{\text{isobar}}$.

CALCULATE:

$$nR = \frac{P_1V_1}{T_1} = \frac{(3.00\text{ atm})(1.013\cdot 10^5\text{ Pa/atm})(6.00\text{ L})(10^{-3}\text{ m}^3/\text{L})}{400.\text{ K}} = 4.559\text{ J/K}$$

$$\Delta S_{\text{adiabatic}} = 0$$

$$\Delta S_{\text{isobar}} = \left(\frac{3}{5}-1\right)n\left(\frac{5}{2}R\right)\ln 4 = -2nR\ln 2 = -2(4.559\text{ J/K})\ln 2 = -6.320\text{ J/K}$$

$$\Delta S_{\text{isotherm}} = 2nR\ln 2 = 2(4.559\text{ J/K})\ln 2 = 6.320\text{ J/K}$$

ROUND: Three significant figures:

$\Delta S_{\text{adiabatic}} = 0$ (exact)

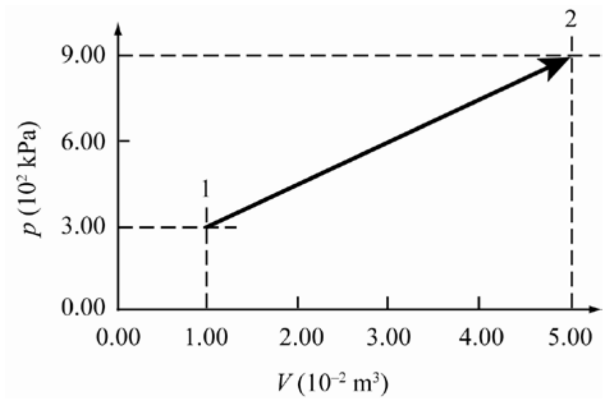
$\Delta S_{\text{isobar}} = -6.32$ J/K

$\Delta S_{\text{isotherm}} = 6.32$ J/K

DOUBLE-CHECK: The entropy changes are reasonable for the values that are given. These results are expected for a reversible cycle.

20.68. **THINK:** To compute the heat input, use the First Law of Thermodynamics and calculate the work done and the change in internal energy of the gas.

SKETCH:



RESEARCH: Use the values: $V_{\text{i}} = 1.00 \cdot 10^{-2}$ m^3, $V_{\text{f}} = 5.00 \cdot 10^{-2}$ m^3, $n = 3.00$ mol, $p_{\text{i}} = 3.00 \cdot 10^5$ Pa and $p_{\text{f}} = 9.00 \cdot 10^{-2}$ Pa. The ideal gas law is: $pV = nRT$, and the First Law of Thermodynamics is: $Q = W + \Delta E_{\text{int}}$, $W = \int_{V_{\text{i}}}^{V_{\text{f}}} p dV$ and $\Delta E_{\text{int}} = nC_V \Delta T = \frac{3}{2} nR\left(T_{\text{f}} - T_{\text{i}}\right)$.

SIMPLIFY: First, find an equation for the work:

$$W = \int_{V_{\text{i}}}^{V_{\text{f}}} PdV = \text{area under the curve on the graph} = \frac{1}{2}\left(V_{\text{f}} - V_{\text{i}}\right)\left(p_{\text{f}} - p_{\text{i}}\right) + p_{\text{i}}\left(V_{\text{f}} - V_{\text{i}}\right)$$

$$= \left(V_{\text{f}} - V_{\text{i}}\right)\left(\frac{1}{2}\left(p_{\text{f}} - p_{\text{i}}\right) + p_{\text{i}}\right) = \frac{1}{2}\left(p_{\text{f}} + p_{\text{i}}\right)\left(V_{\text{f}} - V_{\text{i}}\right).$$

Next, find an equation for the change in internal energy:

$$T_{\text{f}} = \frac{p_{\text{f}} V_{\text{f}}}{nR}, \; T_{\text{i}} = \frac{p_{\text{i}} V_{\text{i}}}{nR} \Rightarrow T_{\text{f}} - T_{\text{i}} = \frac{p_{\text{f}} V_{\text{f}} - p_{\text{i}} V_{\text{i}}}{nR}, \; \Delta E_{\text{int}} = \frac{3}{2} nR\left(\frac{p_{\text{f}} V_{\text{f}} - p_{\text{i}} V_{\text{i}}}{nR}\right) = \frac{3}{2}\left(p_{\text{f}} V_{\text{f}} - p_{\text{i}} V_{\text{i}}\right)$$

Therefore, $Q = W + \Delta E_{\text{int}} = \frac{1}{2}\left(\left(p_{\text{f}} + p_{\text{i}}\right)\left(V_{\text{f}} - V_{\text{i}}\right) + 3\left(p_{\text{f}} V_{\text{f}} - p_{\text{i}} V_{\text{i}}\right)\right)$.

CALCULATE:

$$Q = \frac{1}{2}\left[\left(12.0 \cdot 10^5 \text{ Pa}\right)\left(4.00 \cdot 10^{-2} \text{ m}^3\right) + 3\left(\left(9.00 \cdot 10^5 \text{ Pa}\right)\left(5.00 \cdot 10^{-2} \text{ m}^3\right) - \left(3.00 \cdot 10^5 \text{ Pa}\right)\left(1.00 \cdot 10^{-2} \text{ m}^3\right)\right)\right]$$

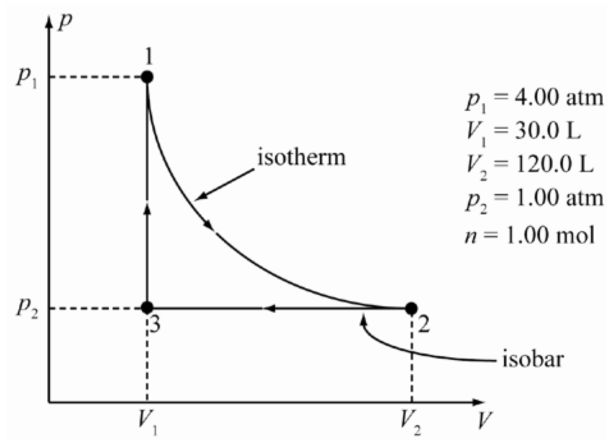
$$= 8.7000 \cdot 10^4 \text{ J}$$

ROUND: Three significant figures: $Q = 8.70 \cdot 10^4$ J.

DOUBLE-CHECK: This is a reasonable result. Note that the result is independent of the number of moles.

20.69. **THINK:** Since there is no internal energy change during isothermal expansion, the heat absorbed is equal to the work done. Along the isobar, it is only necessary to compute the work done. During the constant volume process, the work is zero, and the heat is equal to the change in internal energy.

SKETCH:



RESEARCH: The ideal gas law: $pV = nRT$. For an isothermal process there is no change in internal energy so the work done by the gas is equal to the heat flow into the gas: $W = Q_H = nRT_2 \ln\left(V_2 / V_1\right) \equiv W_{\text{T}}$. For an isobaric process: $W = p_2 \Delta V \equiv W_{\text{P}}$. For a constant-volume process: $\Delta E_{\text{int}} = nC_{\text{V}} \Delta T \equiv Q_{\text{V}}$. The efficiency of the engine is equal to the ratio of the net work done by the gas to the heat that is input: $\varepsilon = W_{\text{net}} / Q_{\text{in}}$.

SIMPLIFY: The efficiency of the engine in terms of the work and heat flow over the cycle is: $\varepsilon = \dfrac{W_{\text{T}} + W_{\text{P}}}{Q_H + Q_{\text{V}}} = \dfrac{W_{\text{T}} + W_{\text{P}}}{W_{\text{T}} + Q_{\text{V}}}$.

$$T_2 = \frac{p_2 V_2}{nR} \Rightarrow W_{\text{T}} = nRT_2 \ln\left(\frac{V_2}{V_1}\right) = p_2 V_2 \ln\left(\frac{V_2}{V_1}\right)$$

$$W_{\text{p}} = p_2\left(V_1 - V_2\right)$$

$$Q_{\text{V}} = \frac{3}{2} nR\left(T_1 - T_3\right) = \frac{3}{2} nR\left(\frac{p_1 V_1}{nR} - \frac{p_2 V_1}{nR}\right) = \frac{3}{2} V_1\left(p_1 - p_2\right)$$

CALCULATE:

$$W_{\text{T}} = \left(1.00 \text{ atm}\right)\left(1.013 \cdot 10^5 \text{ Pa/atm}\right)\left(120.0 \text{ L}\right)\left(10^{-3} \text{ m}^3\text{/L}\right)\ln\left(\frac{120.0 \text{ L}}{30.0 \text{ L}}\right) = 16852 \text{ J}$$

$$W_{\text{P}} = \left(1.00 \text{ atm}\right)\left(1.013 \cdot 10^5 \text{ Pa/atm}\right)\left(30.0 \text{ L} - 120.0 \text{ L}\right)\left(10^{-3} \text{ m}^3\text{/L}\right) = -9117 \text{ J}$$

$$Q_{\text{V}} = \frac{3}{2}\left(30.0 \text{ L}\right)\left(10^{-3} \text{ m}^3\text{/L}\right)\left(4.00 \text{ atm} - 1.00 \text{ atm}\right)\left(1.013 \cdot 10^5 \text{ Pa/atm}\right) = 13676 \text{ J}$$

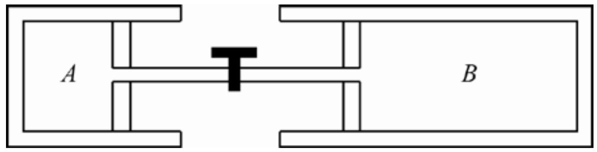
Therefore the efficiency is $\varepsilon = \dfrac{16852 \text{ J} - 9117 \text{ J}}{16852 \text{ J} + 13676 \text{ J}} = 0.2534$.

ROUND: Three significant figures: $\varepsilon = 0.253$.

DOUBLE-CHECK: This is a reasonable efficiency for an engine.

20.70. **THINK:** Cylinder A is at a higher pressure than cylinder B. When the piston is released the argon will expand and move the piston. This will in turn compress the oxygen in cylinder B since the pistons are connected by a rigid rod. Cylinder A is insulated and so processes occurring are adiabatic. Cylinder B is in contact with a thermostat that maintains the cylinder at a constant temperature and so the oxygen in this cylinder therefore expands or contracts isothermally. (Assuming the cylinders are thermally isolated initially.) The cylinders have the same internal diameters and their pistons are connected by a rigid rod so, when free to move, the pistons will move until the pressures are equal in each cylinder. Also, since the pistons are connected by a rigid rod, the *total* volume in the two cylinders must remain constant. In the final part of the question, if the valve is opened, the gases will mix and heat will flow from the hotter cylinder to the cooler one (2nd Law Thermodynamics) until the equilibrium temperature is reached. The piston will move to maintain equal pressure in each cylinder. When the valve is opened to connect the two cylinders, the equilibrium temperature will be 300. K since the thermostat will maintain the now connected cylinders at this temperature.

SKETCH:



RESEARCH:

(a) For cylinder A changes are adiabatic (Q = 0) and so (by the First Law Thermodynamics) $\Delta E_{\text{int}} = -W$. For adiabatic process:

$$pV^{\gamma} = \text{constant} \tag{1}$$

or equivalently

$$TV^{\gamma-1} = \text{constant} \tag{2}$$

Work, W, done by the argon is at the expense of its internal energy. Work done during an adiabatic expansion as temperature changes from T_{Ai} to T_{Af} is:

$$W = n_{\text{A}}R\left(T_{\text{Af}} - T_{\text{Ai}}\right)/(1-\gamma). \tag{3}$$

Argon is a monotonic gas. According to kinetic theory of gases considering the elastic collisions of molecules with the walls of the container (see Chapter 19 for details) it can be shown that for a monotonic gas: $\gamma = 5/3$.

(b) For cylinder B changes occur isothermally so (by the First Law Thermodynamics) $\Delta E_{\text{int}} = 0$ and $Q = W$. For a gas undergoing compression isothermally:

$$pV = \text{constant}. \tag{4}$$

Work done on the gas during an isothermal compression as volume changes from V_{Bi} to V_{Bf} is:

$$W = n_{\text{B}}RT\ln(V_{\text{Bf}}/V_{\text{Bi}}), \tag{5}$$

and this is expelled as heat to the thermostat to maintain the temperature.

(c) After the piston is released, the connected pistons move so that the argon expands, while the oxygen contracts until the pressures are equal in each cylinder:

$$p_{\text{Af}} = p_{\text{Bf}} \quad . \tag{6}$$

For cylinder A using equation (1): $p_{\text{Ai}}V_{\text{Ai}}{}^{\gamma} = p_{\text{Af}}V_{\text{Af}}{}^{\gamma}$ so that

$$p_{\text{Af}} = p_{\text{Ai}}V_{\text{Ai}}{}^{\gamma}/V_{\text{Af}}{}^{\gamma}. \tag{7}$$

For cylinder B using equation (4): $p_{\text{Bi}}V_{\text{Bi}} = p_{\text{Bf}}V_{\text{Bf}}$ so that

$$p_{\text{Bf}} = p_{\text{Bi}}V_{\text{Bi}}/V_{\text{Bf}}. \tag{8}$$

SIMPLIFY:

(a) For cylinder A using equation (2): $TV^{\gamma-1}$ = constant and $\gamma = 5/3$ for argon. Therefore $\gamma - 1 = 2/3$, and $TV^{\gamma-1} = TV^{2/3}$, as required. For cylinder A it is given that when the piston is released the volume expands from V_{Ai} to $V_{\text{Af}} = 8V_{\text{Ai}}$. This means that $T_{\text{Ai}}V_{\text{Ai}}{}^{2/3} = T_{\text{Af}}V_{\text{Af}}{}^{2/3} = T_{\text{Af}}\left(8V_{\text{Ai}}\right)^{2/3}$ or equivalently (since $8^{2/3} = 4$) that

$$T_{\text{Af}} = \frac{1}{4}T_{\text{Ai}}. \tag{9}$$

(b) For cylinder B it is given that the density, ρ_{Bf}, of the oxygen after the compression is twice the original density, ρ_{Bi}. Density is $\rho = m/V$ and this means that the volume of oxygen in cylinder B after the compression is $V_{\text{Bf}} = m/\rho_{\text{Bf}} = m/2\rho_{\text{Bi}}$. That is,

$$V_{\text{Bf}} = \frac{1}{2}V_{\text{Bi}}. \quad (10)$$

CALCULATE:

(a) For cylinder A: Mass of Argon = 0.320 kg = 320. g. Number of moles of Argon n_A = mass in grams/molar mass = 320./39.95 = 8.0100.

(b) Using equations (6), (7) and (8):

$$p_{\text{Af}} = p_{\text{Bf}} \Rightarrow p_{\text{Ai}}V_{\text{Ai}}^{\gamma} / V_{\text{Af}}^{\gamma} = p_{\text{Bi}}V_{\text{Bi}} / V_{\text{Bf}} \quad (11)$$

Now using result (10) and the given fact that for cylinder A the volume increases from V_{Ai} to $V_{Af} = 8V_{Ai}$ implies that in equation (11): $p_{\text{Af}} = p_{\text{Ai}}(1/8)^{5/3} = p_{\text{Bf}} = 2p_{\text{Bi}} = 2\text{ atm} = 202.65\text{ kPa}$, since it is given that the original pressure in cylinder B is normal atmospheric pressure (101.325 kPa). Therefore the original pressure of the argon in cylinder A is $p_{\text{Ai}} = 8^{5/3}\cdot(2\text{ atm}) = 64\text{ atm} = 6484.8\text{ kPa}$. The final pressure after expansion is p_{Af} = 2 atm = 202.65 kPa.

(c) Work done during the adiabatic expansion using equations (3) and (9) is:

$$W = n_A R(T_{\text{Af}} - T_{\text{Ai}})/(1-\gamma) = (8.0100\text{ mol})\ (8.31\text{ J mol}^{-1}\text{K}^{-1})\left(\frac{1}{4}T_{\text{Ai}} - T_{\text{Ai}}\right)/\left(1 - \frac{5}{3}\right) = 74.88\ T_{\text{Ai}}\text{ JK}^{-1}$$

This can be equated to the heat received by the thermostat $7.479\cdot10^4$ J. This means that for the argon: T_{Ai} = 1000. K. Using equation (9): $T_{\text{Af}} = ¼T_{\text{Ai}}$ = 250. K.

(d) For an ideal gas pV = nRT and so for the argon, using the above results for $T_{\text{Ai}}, T_{\text{Af}}, p_{\text{Ai}}, p_{\text{Af}}$:

$$V_{\text{Ai}} = n_A RT_{\text{Ai}} / p_{\text{Ai}} = (8.00\text{ mol})(8.31\text{ mol}^{-1}\text{ J K}^{-1})(1000.\text{ K})/(64\cdot101.325\cdot10^3\text{ Pa}) = 0.01025\text{ m}^3$$

$$V_{\text{Af}} = n_A RT_{\text{Af}} / p_{\text{Af}} = (8.01\text{ mol})(8.31\text{ mol}^{-1}\text{ J K}^{-1})(250.\text{ K})/(2\cdot101.325\cdot10^3\text{Pa}) = 0.08201\text{ m}^3$$

(check: $V_{\text{Af}} = 0.08201\text{ m}^3 = 8V_{\text{Ai}} = 8\cdot0.01025\text{ m}^3$)

(e) Since the total volumes of cylinders A and B must remain constant it follows that:

$$V_{\text{Ai}} + V_{\text{Bi}} = V_{\text{Af}} + V_{\text{Bf}} = 8V_{\text{Ai}} + \frac{1}{2}V_{\text{Bi}};\ \frac{1}{2}V_{\text{Bi}} = 7V_{\text{Ai}};\ V_{\text{Bi}} = 14V_{\text{Ai}} = 14(0.01025\text{ m}^3) = 0.1435\text{ m}^3.$$

So the total volume at any time is:

$$0.01025 + 0.1435 = 0.15375\text{ m}^3. \quad (12)$$

(check: $V_{\text{Af}} + V_{\text{Bf}} = 0.08201\text{ m}^3 + (0.1435/2)\text{ m}^3 = 0.15375\text{ m}^3$).

(f) For cylinder B: It is in contact with a thermostat that maintains the cylinder at a constant temperature of 27°C = 273 + 27 = 300. K. Work done during an isothermal compression using equation (5) is:

$$W = n_B RT\ln(V_{\text{Bf}} / V_{\text{Bi}}) = n_B(8.31\text{ mol}^{-1}\text{ J K}^{-1})(300.\text{ K})\ln(½V_{\text{Bi}} / V_{\text{Bi}}) = -n_B(1728\text{ J mol}^{-1}) \quad (13)$$

where n_B is the number of moles of oxygen in cylinder B.

(g) The oxygen, maintained at 300. K by the thermostat, is warmer than the argon (250. K) after the pressures equal in the cylinders. When the valve is opened to connect the two cylinders, the equilibrium temperature will be 300. K since the thermostat will maintain the now connected cylinders at this temperature. The final pressure in the connected cylinders can be found using $pV = nRT$ with T = 300. K and the known n, the total number of moles of oxygen and argon. Additionally, equation (13) can be used to find the number of moles of oxygen: $1728\ n_B = 7.479\cdot10^4$; n_B = 43.28. The total number of moles of gas is: 8.010 + 43.28 = 51.29, so that the pressure in the connected cylinders is:

$$\begin{aligned} p &= nRT / V = (51.29\text{ mol})(8.31\text{ mol}^{-1}\text{ J K}^{-1})(300.\text{ K})\ /(0.15375\text{ m}^3) \\ &= 831.648\text{ kPa} = 8.208\text{ atm} \end{aligned}$$

ROUND: Quote all values to three significant figures:
$p_{Ai} = 64.0$ atm = 6.48 MPa
$p_{Af} = 2.00$ atm = 0.203 MPa
$T_{Ai} = 1.00 \cdot 10^3$ K
$T_{Af} = 250.$ K
$V_{Ai} = 0.0103$ m^3
$V_{Af} = 0.0820$ m^3
Final pressure when connected is: $p = 0.831$ MPa = 8.21 atm.

DOUBLE CHECK: The final pressure when pistons were connected, after the valve was opened is more than it was when they were not connected. This is as expected since heat flows into the cylinders to heat the argon to 300 K. The final pressure is less than it initially was when the piston was fixed, again this seems reasonable as at that time the argon was much hotter.

Multi-Version Exercises

Exercises 20.71–20.73 The work that the air conditioner is required to do is

$$W = \frac{Q_\text{L}}{K_\text{air conditioner}}.$$

We can relate the coefficient of performance to the energy efficiency rating by

$$K_\text{air conditoner} = \frac{\text{EER}_\text{air conditioner}}{3.41}.$$

So the power required to cool the house is

$$P = \frac{W}{t} = \frac{Q_\text{L}/t}{K_\text{air conditioner}} = \frac{3.41(Q_\text{L}/t)}{\text{EER}_\text{air conditioner}}.$$

The cost to cool the house for a day is

$$\text{cost} = \text{hourly rate} \cdot P \cdot (24 \text{ hr}) = \frac{3.41(24 \text{ hr})(Q_\text{L}/t)(\text{hourly rate})}{\text{EER}_\text{air conditioner}}.$$

20.71. From the above,

$$\text{daily cost} = \frac{3.41(24 \text{ hr})(5.375 \text{ kW})(0.1285 \text{ \$/(kW}\cdot\text{hr}))}{10.47} = \$5.399.$$

20.72. From the above, $\text{daily cost} = \dfrac{3.41(24 \text{ hr})(Q_\text{L}/t)(\text{hourly rate})}{\text{EER}_\text{air conditioner}}.$

$$\begin{aligned}\text{hourly rate} &= \frac{(\text{daily cost})\text{EER}_\text{air conditioner}}{3.41(24 \text{ hr})(Q_\text{L}/t)} \\ &= \frac{(\$5.605)10.71}{3.41(24 \text{ hr})(5.437 \text{ kW})} \\ &= 13.49 \text{ cents/(kW hr)}.\end{aligned}$$

20.73. From the above,

$$\text{EER}_\text{air conditioner} = \frac{3.41(24 \text{ hr})(Q_\text{L}/t)(\text{rate})}{\text{cost}} = \frac{3.41(24 \text{ hr})(5.499 \text{ kW})(0.1413 \text{ dollars/(kW hr)})}{5.818 \text{ dollars}} = 10.93.$$

Exercises 20.74–20.76 This system acts like a refrigerator. The maximum coefficient of performance of a refrigerator is

$$K_\text{max} = \frac{T_\text{L}}{T_\text{H} - T_\text{L}}.$$

The minimum work that must be done is then

$$W_{\text{min}} = \frac{Q_{\text{L}}}{K_{\text{max}}} = \frac{Q_{\text{L}}\left(T_{\text{H}} - T_{\text{L}}\right)}{T_{\text{L}}}.$$

20.74. From the above,

$$W_{\text{min}} = \frac{Q_{\text{L}}\left(T_{\text{H}} - T_{\text{L}}\right)}{T_{\text{L}}} = \frac{(288.1 \text{ J})(195.3 \text{ °C} - 24.93 \text{ °C})}{24.93 \text{ °C} + 273.15 \text{ °C}} = 164.7 \text{ J}.$$

20.75. From the above,

$$W_{\text{min}} = \frac{Q_{\text{L}}\left(T_{\text{H}} - T_{\text{L}}\right)}{T_{\text{L}}}$$

$$W_{\text{min}} T_{\text{L}} = Q_{\text{L}} T_{\text{H}} - Q_{\text{L}} T_{\text{L}}$$

$$T_{\text{H}} = \frac{W_{\text{min}} T_{\text{L}} + Q_{\text{L}} T_{\text{L}}}{Q_{\text{L}}}$$

$$= T_{\text{L}} \frac{W_{\text{min}} + Q_{\text{L}}}{Q_{\text{L}}} = (25.05 \text{ °C} + 273.15 \text{ °C}) \frac{(118.5 \text{ J}) + (425.5 \text{ J})}{(425.5 \text{ J})}$$

$$= 381.25 \text{ K} = 108.1 \text{ °C}.$$

20.76. From the above,

$$W_{\text{min}} = \frac{Q_{\text{L}}\left(T_{\text{H}} - T_{\text{L}}\right)}{T_{\text{L}}}$$

$$W_{\text{min}} T_{\text{L}} = Q_{\text{L}} T_{\text{H}} - Q_{\text{L}} T_{\text{L}}$$

$$W_{\text{min}} T_{\text{L}} + Q_{\text{L}} T_{\text{L}} = Q_{\text{L}} T_{\text{H}}$$

$$T_{\text{L}} = T_{\text{H}} \frac{Q_{\text{L}}}{W_{\text{min}} + Q_{\text{L}}}$$

$$= (120.9 \text{ °C} + 273.15 \text{ °C}) \frac{562.9 \text{ J}}{180.7 \text{ J} + 562.9 \text{ J}}$$

$$= 298.29 \text{K} = 25.14 \text{ °C}$$

Exercises 20.77–20.79 The efficiency is given by $\varepsilon = W / Q_{\text{H}}$. The first law of thermodynamics tells us that $Q_{\text{H}} = W + Q_{\text{L}}$. We can relate Q_{L} to the change in temperature of the water $Q_{\text{L}} = mc\Delta T$. The flow rate in terms of volume is

$$f = \frac{V}{t} = \frac{(m / \rho)}{t}.$$

So we can write the mass as $m = f \rho t$. So the efficiency is

$$\varepsilon = \frac{W}{Q_{\text{H}}} = \frac{W}{W + Q_{\text{L}}} = \frac{W}{W + mc\Delta T} = \frac{W}{W + f \rho t c \Delta T}.$$

We can write $W = Pt$, so

$$\varepsilon = \frac{Pt}{Pt + f \rho t c \Delta T} = \frac{P}{P + f \rho c \Delta T}.$$

20.77. From the above,

$$\varepsilon = \frac{1833 \text{ W}}{1833 \text{ W} + (132.3 \text{ L/h})(10^{-3} \text{ m}^3/\text{L})\left(\frac{1 \text{ h}}{3600 \text{ s}}\right)(1000 \text{ kg/m}^3)(4186 \text{ J/(kg °C)})(26.69\text{°C} - 11.25\text{°C})}$$

$$= 0.4356.$$

20.78. From the preceding,

$$\varepsilon = \frac{P}{P + f\rho c\Delta T}$$

$$P + f\rho c\Delta T = \frac{P}{\varepsilon}$$

$$f = \frac{P/\varepsilon - P}{\rho c\Delta T} = \frac{P - \varepsilon P}{\varepsilon\rho c\Delta T}$$

$$= \frac{(1061\text{ W}) - (0.3591)(1061\text{ W})}{(0.3591)(1000\text{ kg/m}^3)(4186\text{ J/(kg °C)})(27.33\text{°C} - 11.35\text{°C})}$$

$$= 2.8308 \cdot 10^{-5}\text{ m}^3\text{/s}$$

$$f = \left(2.8308 \cdot 10^{-5}\ \frac{\text{m}^3}{\text{s}}\right)\left(\frac{L}{1.0 \cdot 10^{-3}\text{ m}^3}\right)\left(\frac{3600\text{ s}}{1\text{ h}}\right) = 101.9\text{ L/h.}$$

20.79. From the above,

$$\varepsilon = \frac{P}{P + f\rho c\Delta T}$$

$$\varepsilon(P + f\rho c\Delta T) = P$$

$$\varepsilon P + \varepsilon f\rho c\Delta T = P$$

$$P = \frac{\varepsilon f\rho c\Delta T}{1 - \varepsilon}$$

$$= \frac{(0.2815)(171.5\text{ L/h})\left(\frac{1.0 \cdot 10^{-3}\text{ m}^3}{1\text{ L}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right)(1000\text{ kg/m}^3)(4186\text{ J/(kg °C)})(27.97\text{°C} - 11.45\text{°C})}{1 - 0.2815}$$

$$= 1291\text{ W.}$$

Chapter 21: Electrostatics

Concept Checks

21.1. d **21.2.** a **21.3.** e **21.4.** e **21.5.** c **21.6.** b **21.7.** a **21.8.** a **21.9.** c **21.10.** b **21.11.** a

Multiple-Choice Questions

21.1. b **21.2.** b **21.3.** b **21.4.** d **21.5.** b **21.6.** b **21.7.** a **21.8.** a **21.9.** c **21.10.** b **21.11.** a **21.12.** b **21.13.** a **21.14.** e

Conceptual Questions

21.15. The given quantities are the charge of the two particles, $Q_1 = Q$ and $Q_2 = Q$. They are separated by a distance d. The Coulomb force between the changed particles is $F = k\frac{Q_1Q_2}{d^2} = k\frac{Q^2}{d^2}$. If the charge on each particle is doubled so that $Q_1' = 2Q = Q_2'$ and the separation distance is $d' = 2d$ the then the Coulomb Force is given by: $F' = k\frac{4Q^2}{4d^2} = k\frac{Q^2}{d^2}$ so the force is the same as it was in the initial situation.

21.16. The gravitational force between the Sun and the Earth is $F_\text{g} = G\frac{M_\text{S}M_\text{E}}{r^2}$ where G is the gravitational constant and is equal to $6.67\cdot 10^{-11}$ N m^2 / kg^2, M_S is the mass of the Sun ($1.989\cdot 10^{30}$ kg) and M_E is the mass of the Earth ($5.974\cdot 10^{24}$ kg). The Coulomb force is given by the equation $F_\text{C} = k\frac{Q_1Q_2}{r^2}$ where k is Coulomb's constant ($k = 8.99\cdot 10^9$ N m^2 / C^2). In this question $Q_1 = Q_2 = Q$ and is the charge given to the Earth and Sun to cancel out the gravitational force.

$$F_\text{C} = F_\text{g} \Rightarrow \frac{kQ^2}{r^2} = \frac{GM_\text{S}M_\text{E}}{r^2} \Rightarrow Q = \sqrt{\frac{GM_\text{S}M_\text{E}}{k}}$$

Therefore,

$$Q = \sqrt{\frac{(6.67\cdot 10^{-11}\text{ N m}^2/\text{kg}^2)(1.989\cdot 10^{30}\text{ kg})(5.974\cdot 10^{24}\text{ kg})}{8.99\cdot 10^9\text{ N m}^2/\text{C}^2}} = 2.97\cdot 10^{17}\text{ C}.$$

I can get the number of elementary charges, n, by dividing Q by $1.602\cdot 10^{-19}$ C (the charge of one electron):

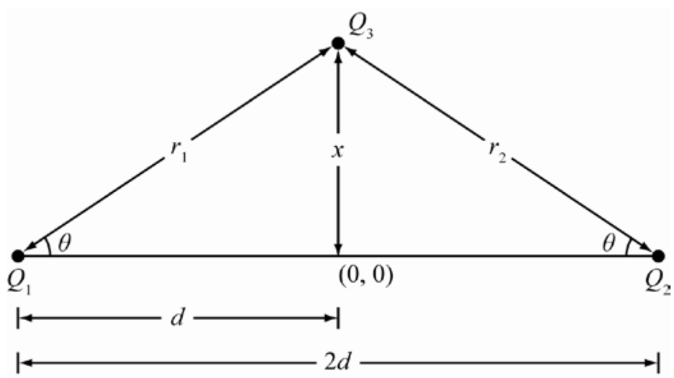
$$n = \frac{2.97\cdot 10^{17}\text{ C}}{1.602\cdot 10^{-19}\text{ C}} = 1.85\cdot 10^{36}.$$

To estimate the number of elementary change of either sign for the Earth I can assume the mass of the Earth is due to the mass of the protons, neutrons and electrons of which it is primarily composed. If I assume that the Earth's mass is due to the proton and neutron masses primarily (became an electrons mass is much smaller than a protons) and I assume that there are an equal number of protons and neutrons than I can get the number of protons by dividing the Earth's mass by two times the mass of a proton. The mass of a proton is $m_\text{p} \approx 1.6726\cdot 10^{-27}$ kg, so you can estimate the number of elementary charges on the Earth, n_E by: $n_\text{E} = \frac{m_\text{E}}{m_\text{p}} = \frac{5.97\cdot 10^{24}\text{ kg}}{1.67\cdot 10^{-27}\text{ kg}} = 3.57\cdot 10^{51}$. So the percentage of the Earth's changes that would be required to cancel out the gravitational force is $(n/n_\text{E})\cdot 100\% = 5.18\cdot 10^{-14}\%$, a very small percentage.

21.17. One reason that it took such a long time to understand the electrostatic force may have been because it was not observed as frequently as the gravitational force. All massive objects are acted on by the gravitational force; however, only objects with a net charge will experience an electrostatic force.

21.18. The accumulation of static charge gives the individual hairs a charge. Since like charges repel and because the electrostatic force is inversely proportional to the charges separation distance squared, the hairs arrange themselves in a manner in which they are as far away from each other as possible. In this case that configuration is when the hairs are standing on end.

21.19. The given quantities are the charge which is $Q_1 = Q_2 = Q$ and the separation distance of $2d$. The third charge is $Q_3 = -0.2Q$ and it is positioned at d. Charge Q_3 is then displaced a distance x perpendicular to the line connecting the positive charges. The displacement $x \ll d$. The question asks for the force, F, on charge Q_3. For $x \ll d$ the question also asks for the approximate motion of the negative charge.



$\vec{F} = \vec{F}_{13} + \vec{F}_{23}$, where $\vec{F}_{13}$ is the force Q_3 feels due to Q_1 and $\vec{F}_{23}$ is the force Q_3 feels due to charge Q_2. Because Q_1 and Q_2 have the same sign and are of equal charge there is no net force in the horizontal direction. The forces due to Q_1 and Q_2 in the vertical direction are given by:

$$F_{13} = k\frac{Q_1Q_3}{r_1^2}\sin\theta \text{ and } F_{23} = k\frac{Q_2Q_3}{r_2^2}\sin\theta,$$

where $r_1 = \sqrt{d^2 + x^2}$ and $r_2 = \sqrt{d^2 + x^2}$. To simplify we can substitute $\sin\theta_1 = x / r_1$ and $\sin\theta_2 = x / r_2$ into force equations. So we can write the force equation as:

$$F = F_{13} + F_{23} = \frac{kQ_1Q_3}{\left(d^2 + x^2\right)}\left(\frac{x}{\sqrt{d^2 + x^2}}\right) + \frac{kQ_2Q_3}{\left(d^2 + x^2\right)}\left(\frac{x}{\sqrt{d^2 + x^2}}\right) = \left(Q_1 + Q_2\right)\frac{kxQ_3}{\left(d^2 + x^2\right)^{3/2}}$$

Substituting $Q_1 = Q_2 = Q$ and $Q_3 = -0.2Q$ gives:

$$F = \left(Q + Q\right)\frac{kx\left(-0.2Q\right)}{\left(d^2 + x^2\right)^{3/2}} = -\frac{2k\left(0.2Q^2\right)x}{\left(d^2 + x^2\right)^{3/2}} = -\frac{0.4kQ^2x}{\left(d^2 + x^2\right)^{3/2}}$$

The negative sign indicates that the force is downward. Since $x \ll d$, it is reasonable to use the approximation $(d^2 + x^2)^{3/2} = (d^2)^{3/2} = d^3$. Hence, $F \approx -\frac{0.4kQ^2x}{d^3}$. This solution is similar in form to Hooke's law which describes the restoring force due to the compression or expansion of a spring, $F_{\text{spring}} = -kx$ where k is the spring constant. The motion of the negative charge can therefore be approximated using simple harmonic motion.

21.20. As the garment is dried it acquires a charge from tumbling in the dryer and rubbing against other clothing. When I put the charged garment on it causes a redistribution of the charge on my skin and this causes the attractive electric force between the garment and my skin.

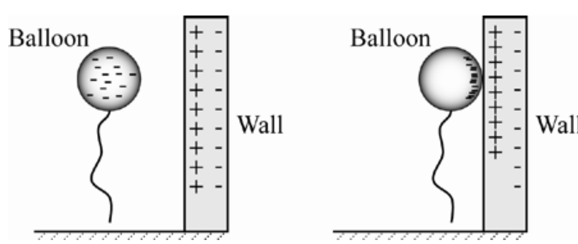
21.21. The initial separation of the spheres is x_1. The magnitude of the force on each of the spheres at separation x_1 is $F_1 = k\dfrac{Q_1Q_2}{x_1^2}$. The force after the distance change is $F_2 = k\dfrac{Q_1Q_2}{x_2^2}$, where the new distance is x_2. Because the charge is conserved I can equate the forces F_1 and F_2. $F_1 = k\dfrac{Q_1Q_2}{x_1^2}$ and $F_2 = k\dfrac{Q_1Q_2}{x_2^2}$, so $kQ_1Q_2 = F_1x_1^2 = F_2x_2^2$, or $x_2^2 = \left(F_1/F_2\right)x_1^2$. Substituting $F_2 = 9F_1$ into the equation gives: $x_2^2 = \dfrac{F_1}{9F_1}x_1^2 \Rightarrow x_2 = \sqrt{\dfrac{1}{9}x_1^2} = \dfrac{1}{3}x_1$. Therefore the distance would have to decrease to a factor of a third of its original value to achieve nine times the original force.

21.22. An electrically neutral atom can exert electrostatic force on another electrically neutral atom if they do not have symmetric charge distribution. In the case of two atoms where one atoms electron or electrons were closer to the proton of the other atom. This type of situation can occur when atoms undergo polar bonding to form a molecule.

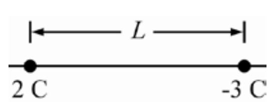
21.23. The scientist could convince themselves that the electrostatic force was not a variant of the gravitational force in various ways. One distinction is that gravitating objects attract but in the electric force like charged objects repel. For Earth bound experiments the scientists may observe that massive objects are pulled towards the ground by the gravitational force at a constant acceleration. If they performed careful experiments with objects of the same charge they would observe that the gravitational force downward on one of the charged objects could be diminished or balanced by the electrostatic force that object felt due to the second like charged object that was placed underneath it.

21.24. The electrostatic force is an inverse square force, of the same form as the Newtonian gravitational force. As long as the bodies are not moving too rapidly (i.e., not at speeds near the speed of light), the problem of determining their motion is the same as the Kepler problem. The motion of the two particles decomposes into a center of mass motion with constant velocity, and a relative motion which traces out a trajectory which can be either a portion of a straight line (for zero angular momentum, i.e., head on collisions) or a Keplerian ellipse (including a circle), parabola, or hyperbola, in the case of opposite charges. For charges of the same sign, for which the force is repulsive, the relative motion must be either a straight line or a hyperbola, an open orbit.

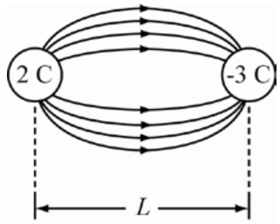
21.25. The wall does not have to be positively charged. The negatively charged balloon induces charges on the wall. The repulsive force between electrons in the balloon and those in the wall cause the electrons in the wall to redistribute. This leaves the portion of the wall that is closest to the balloon with a positive charge. The negatively charged balloon will be attached to the positively charged region of the wall even though the net charge of the wall is neutral.



21.26.

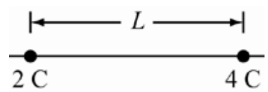


The electric lines flow from the positive charge to the negative charge as is shown in the sketch below.



There is nowhere on the line between the charged particles that I could place a test charge without it moving. This is due to the electric charges on the line having opposite charge, so a test charge (of either sign) that is placed between these two charges would be attracted by one and repelled by the other.

21.27.



In order for the test charge to feel no net force it would have to be at a location where the force it felt due to the charge $Q_2 = 4\text{ C}$ is equal and opposite to the force felt due to the charge $Q_1 = 2\text{ C}$. For convenience I can say that the charge $Q_1 = 2\text{ C}$ is located at $x_1 = 0$, and charge $Q_2 = 4\text{ C}$ is located at $x_2 = L$ and charge Q_3 is located at a position, x_3 which is between 0 and L. I can equate the expressions for the electric force on Q_3 due to Q_1 and the electric force on Q_3 due to Q_2 to solve for x_3 as these forces would have to balance for the charge Q_3 to feel no net force.

$$F_{13} = F_{23}$$
$$\frac{kQ_1Q_3}{x_3^{\,2}} = \frac{kQ_2Q_3}{(L-x_3)^2}$$
$$Q_1(L-x_3)^2 = Q_2x_3^{\,2}$$
$$Q_1(x_3^{\,2} - 2x_3L + L^2) - Q_2x_3^{\,2} = 0$$
$$(Q_1 - Q_2)x_3^{\,2} - 2Q_1x_3L + Q_1L^2 = 0$$

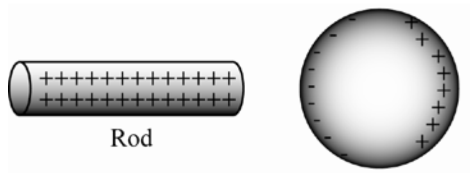
Note that in the second step of the calculation above, it is shown that the sign and magnitude of Q_3 will not impact the answer. I can solve using the quadratic equation:

$$x_3 = \frac{2Q_1L \pm \sqrt{4Q_1^{\,2}L^2 - 4(Q_1 - Q_2)(Q_1L^2)}}{2(Q_1 - Q_2)} = \frac{2(2\text{ C})L \pm \sqrt{4(2\text{ C})^2L^2 + 4(4\text{C}^2L^2)}}{-4\text{C}} = 0.414L, -2.414L$$

The correct answer is $x_3 = 0.414L$ because this point is between Q_1 and Q_2. One can also see from the second step of the algebraic manipulation that the magnitude and charge of Q_3 is irrelevant to the position of x_3, as it drops out of the equation. Intuitively, this makes sense, since whatever magnitude and charge of Q_3 is placed between the two existing charges, it will experience opposite forces from Q_1 and Q_2, since they have the same sign.

21.28. When a positively charged rod is brought near to an isolated neutral conductor without touching it the rod will experience an attractive force. The electric charge on the rod induces a redistribution of charge in the conductor.

The net effect of this distribution is that electrons move to the side of the conductor nearest to the rod. The positively charged rod is attracted to this region.



21.29. Using a metal key to touch a metal surface before exiting the car will discharge any charge I carry. When I begin to fuel a car, I can touch the gas pump and the car before pumping the gas, discharging myself. If I get back into the car, I can re-charge myself, and when I again get out of the car and touch the fuel nozzle without grounding myself first, I can get a spark, which might ignite the gasoline.

Exercises

21.30. Since charge is quantized, the number of electrons, when summed, yields the given charge: $n \cdot e = Q$. The charge of each electron is $1.602 \cdot 10^{-19}$ C. The total number n of electrons required to give a total charge of 1.00 C is obtained by dividing the total charge by the charge per electron:

$$n = \frac{Q}{e} = \frac{(1.00\text{ C})}{(1.602 \cdot 10^{-19}\text{ C/electron})} = 6.18 \cdot 10^{18}\text{ electrons.}$$

21.31. The number of atoms or molecules in one mole of a substance is given by Avogadro's number, $N_\text{A} = 6.022 \cdot 10^{23}\text{ mol}^{-1}$. The faraday unit is $F = N_\text{A}e$, where e is the elementary charge of an electron or proton and is equal to $1.602 \cdot 10^{-19}$ C. To calculate the number of coulombs in 1.000 faraday you can multiply N_A by the elementary charge:

$$1.000\text{ F} = N_\text{A}e = (6.022 \cdot 10^{23}\text{ atoms/mol})(1.602 \cdot 10^{-19}\text{ C}) = 96470\text{ C.}$$

21.32. $1\text{ dyne} = 1\text{ g cm/s}^2 = 1 \cdot 10^{-5}\text{ N}$ and it is a unit of force. An electrostatic unit or esu is defined as follows: Two point charges, each of 1 esu and separated by one centimeter exert a force of exactly one dyne on each other. Coulomb's law gives the magnitude of the force on one charge due to another, which is $F = k|q_1q_2|/r^2$ (where $k = 8.99 \cdot 10^9\text{ N m}^2/\text{C}^2$, q_1 and q_2 are electric charges and r is the separation distance between charges.)

(a) By substituting the values given in the question into Coulomb's law, the relationship between the esu and the Coulomb can be determined:

$$1 \cdot 10^{-5}\text{ N} = \frac{k(1\text{ esu})^2}{(0.01\text{ m})^2} \Rightarrow 1\text{ esu} = \sqrt{\frac{(0.01\text{ m})^2(1 \cdot 10^{-5}\text{ N})}{8.99 \cdot 10^9\text{ N m}^2/\text{C}^2}} = 3.34 \cdot 10^{-10}\text{ C}$$

(b) The result of part (a) shows that $1\text{ esu} = 3.34 \cdot 10^{-10}\text{ C}$. The elementary charge on an electron or proton is $e = 1.602 \cdot 10^{-19}$ C. To get the relationship between the esu and elementary charge, divide 1 esu by the charge per electron (or proton).

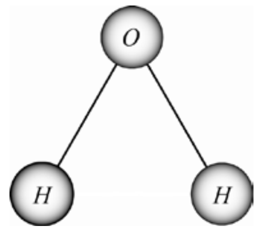
$$1\text{ esu} = \frac{3.34 \cdot 10^{-10}\text{ C}}{1.602 \cdot 10^{-19}\text{ C}/e} = 2.08 \cdot 10^9 e$$

21.33. The given quantities are the current, $I = 5.00 \cdot 10^{-3}$ A and the exposure time, $t = 10.0$ s. One coulomb is equal to 1 A s. To calculate the number of electrons that flow through your skin at this current and during this time, multiply I by t to yield the quantity of charge in coulombs. Then divide by the elementary charge per electron, which is $1.602 \cdot 10^{-19} C$.

$$I\cdot t=\left(5.00\cdot 10^{-3}\text{ A}\right)\left(10.0\text{ s}\right)=0.0500\text{ A s}=0.0500\text{ C};$$
$$\frac{0.0500\text{ C}}{1.602\cdot 10^{-19}\text{ C}/e}=3.12\cdot 10^{17}\text{ electrons.}$$

21.34. **THINK:** Consider a mass, $m=1.00\text{ kg}$ of water. To calculate how many electrons are in this mass, a relationship must be found between mass, the number of water atoms presents and their charge. Let η denote the number of electrons.

SKETCH:



RESEARCH: The molecular mass of water (H_2O), $m_W=18.015$ g/mol. The number of moles of water can be found by dividing the mass of water by its molecular mass. The number of electrons present in the water can be found from the atomic numbers, Z, for hydrogen and oxygen ($Z=1$ and $Z=8$ respectively). The total number of water molecules can be found by multiplying the number of moles of water present by Avogadro's number, $N_A=6.022\cdot 10^{23}\text{ mol}^{-1}$.

SIMPLIFY: $\eta=\dfrac{m}{m_W}\cdot N_A\cdot\dfrac{10\text{ electrons}}{H_2O\text{ atom}}$

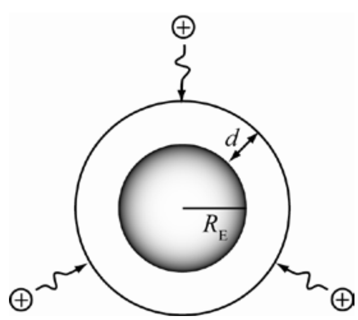
CALCULATE: $\eta=\left(\dfrac{1.00\cdot 10^3\text{ g}}{18.015\text{ g/mol}}\right)\left(6.022\cdot 10^{23}\text{ mol}^{-1}\right)\left(10\text{ electrons}\right)=3.34277\cdot 10^{26}\text{ electrons}$

ROUND: The values in the question were provided to 3 significant figures, so the answer is $3.34\cdot 10^{26}$ electrons.

DOUBLE-CHECK: Considering that there are approximately 55 moles of H_2O per kilogram of water and there are 10 electrons per H_2O atom, it makes sense that the answer is approximately 550 times greater than Avogadro's number.

21.35. **THINK:** Protons are incident on the Earth from all directions at a rate of $n=1245.0\text{ protons}/\left(\text{m}^2\text{ s}\right)$. Assuming that the depth of the atmosphere is $d=120\text{ km}=120{,}000\text{ m}$ and that the radius of the Earth is $r=6378\text{ km}=6{,}378{,}000\text{ m}$, I want to determine the total charge incident upon the Earth's atmosphere in 5.00 minutes.

SKETCH:



RESEARCH: Modeling the Earth like a sphere, the surface area A can be approximated as $A=4\pi r^2$. The total number of protons incident on the Earth in the time t can be found by multiplying the rate, n

by the surface area of the Earth and the time, t. The total charge Q can be found by multiplying the total number of protons, P by the charge per proton. The elementary charge of a proton is $1.602\cdot 10^{-19}$ C.

SIMPLIFY: $P = nAT = n4\pi r^2 t, \quad Q = P\left(1.602\cdot 10^{-19} \text{ C}/P\right)$

CALCULATE:

$P = 1245.0 \text{ protons}/(\text{m}^2\text{s})\left[4\pi(6{,}378{,}000 \text{ m} + 120{,}000 \text{ m})^2\right](300. \text{ s}) = 1.981800\cdot 10^{20} \text{ protons},$

$Q = 1.981800\cdot 10^{20} \text{ protons}\cdot\left(1.602\cdot 10^{-19} \text{ C/protons}\right) = 31.74844 \text{ C}$

ROUND: To three significant figures 31.7 C

DOUBLE-CHECK: The calculated answer has the correct units of charge. The value seems reasonable considering the values that were provided in the question.

21.36. The charges obtained by the student performing the experiment are listed here: $3.26\cdot 10^{-19}$ C, $6.39\cdot 10^{-19}$ C, $5.09\cdot 10^{-19}$ C, $4.66\cdot 10^{-19}$ C, $1.53\cdot 10^{-19}$ C. Dividing the above values by the smallest measured value will give the number of electrons, n_e found in each measurement.

Observed charge	n_e	Integer value	Observed charge (integer value)
$3.26\cdot 10^{-19}$ C	2.13	2	$1.63\cdot 10^{-19}$ C
$6.39\cdot 10^{-19}$ C	4.17	4	$1.60\cdot 10^{-19}$ C
$5.09\cdot 10^{-19}$ C	3.32	3	$1.69\cdot 10^{-19}$ C
$4.66\cdot 10^{-19}$ C	3.04	3	$1.55\cdot 10^{-19}$ C
$1.53\cdot 10^{-19}$ C	1	1	$1.53\cdot 10^{-19}$ C

The number of electrons, n_e, must be rounded to their closest integer value because charge is quantized. Dividing the observed charge by the integer number of electrons gives the charge per electron. Taking the average of the observed charge/integer value data the average charge on an electron is calculated to be $(1.60 \pm 0.03)\cdot 10^{-19}$ C.

Optional Error analysis: Given a set of n measured values a_i, there exists a mean value, μ. Then the standard deviation σ of the data is given by the relation:

$$\sigma^2 = \frac{\sum_{i=1}^{n} a_i^2}{n} - \frac{n}{n-1}\mu^2$$

$$\sigma = \sqrt{\frac{\left(1.63^2 + 1.60^2 + 1.69^2 + 1.55^2 + 1.53^2\right)}{5} - \frac{5}{4}\left(\frac{1.63 + 1.60 + 1.69 + 1.55 + 1.53}{5}\right)^2} = 0.06403$$

The error in a repeated measurement of the same quantity is:

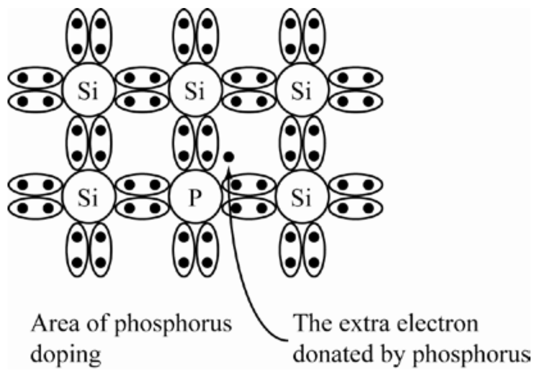
$$\text{Error} = \frac{\text{standard deviation}}{\sqrt{\text{number of measurements}}} = \frac{\sigma}{\sqrt{n}}.$$

$$\text{Error} = \frac{0.06403}{\sqrt{5}} = 0.0286$$

The measurements have an error of 2.8%.

21.37. **THINK:** An intrinsic silicon sample is doped with phosphorous. The level of doping is 1 phosphorous atom per one million silicon atoms. The density of silicon is $\rho_S = 2.33 \text{ g/cm}^3$ and its atomic mass is $m_S = 28.09 \text{ g/mol}$. The phosphorous atoms act as electron donors. The density of copper is $\rho_C = 8.96 \text{ g/cm}^3$ and its atomic mass is $m_C = 63.54 \text{ g/mol}$.

SKETCH:



RESEARCH: Avogadro's number is $N_A = 6.022 \cdot 10^{23} \text{ mol}^{-1}$. It gives the number of atoms or molecules per mole of a substance. Density, $\rho = m/V$, where $m = \text{mass}$ and $V = \text{volume}$.

SIMPLIFY:

(a) There will be 1 conduction electron per $1.00 \cdot 10^6$ silicon atoms. The number of silicon atoms per cm^3 is $n_S = (\rho_S / m_S) \cdot N_A$. The number of conduction electrons per cm^3 is $n_e = n_S / (1.00 \cdot 10^6)$.

(b) The number of copper atoms is $n_C = (\rho_C / m_C) \cdot N_A$. The number of conduction electrons in the copper is n_C. The ratio of conduction electrons in silicon to conduction electrons in copper is n_e / n_C.

CALCULATE:

(a) $n_C = \left(\frac{2.33 \text{ g/cm}^3}{28.09 \text{ g/mol}} \right) 6.022 \cdot 10^{23} \text{ mol}^{-1} = 4.995 \cdot 10^{22} \text{ /cm}^3$

$$n_e = \frac{4.995 \cdot 10^{22}}{1.00 \cdot 10^6} = 4.995 \cdot 10^{16} \text{ conduction electrons / cm}^3$$

(b) $n_C = \left(\frac{8.96 \text{ g/cm}^3}{63.54 \text{ g/mol}} \right) 6.022 \cdot 10^{23} \text{ mol}^{-1} = 8.4918 \cdot 10^{22} \text{ /cm}^3$

$$\frac{n_e}{n_C} = \frac{4.995 \cdot 10^{16}}{8.4918 \cdot 10^{22}} = 5.88215 \cdot 10^{-7}$$

ROUND: There were three significant figures provided in the question so the answers should be:

(a) $n_e = 5.00 \cdot 10^{16} \text{ conduction electrons / cm}^3$

(b) There are $5.88 \cdot 10^{-7}$ conduction electrons in the doped silicon sample for every conduction electron in the copper sample.

DOUBLE-CHECK: It is reasonable that there are approximately $5 \cdot 10^{-7}$ less conduction electrons in the doped silicon sample compared to the copper sample.

21.38. The force between the two charged spheres is $F_1 = k\dfrac{q_a q_b}{d_1^2}$ initially. After the spheres are moved the force is $F_2 = k\dfrac{q_a q_b}{d_2^2}$. Taking the ratio of the force after to the force before gives: $F_2 / F_1 = \left(k\dfrac{q_a q_b}{d_2^2}\right) / \left(k\dfrac{q_a q_b}{d_1^2}\right) = d_1^2 / d_2^2 = 4$. The new distance is then $d_2 = \sqrt{d_1^2 / 4} = d_1 / 2 = 4 \text{ cm}$.

21.39. The charge on each particle is q. When the separation distance is $d = 1.00$ m, the electrostatic force is $F = 1.00$ N. The charge q is found from $F = kq_1q_2 / d^2 = kq^2 / d^2$. Then,

$$q = \sqrt{\frac{Fd^2}{k}} = \sqrt{\frac{(1.00 \text{ N})(1.00 \text{ m})^2}{8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2}} = 1.05 \cdot 10^{-5} \text{ C}.$$

The sign does not matter, so long as each particle has a charge of the same sign, so that they repel.

21.40. In order for two electrons to experience an electrical force between them equal to the weight of one of the electrons, the distance *d* separating them must be such that. $F_\text{g} = F_\text{Coulomb} \Rightarrow m_e g = ke^2 / d^2$. Then,

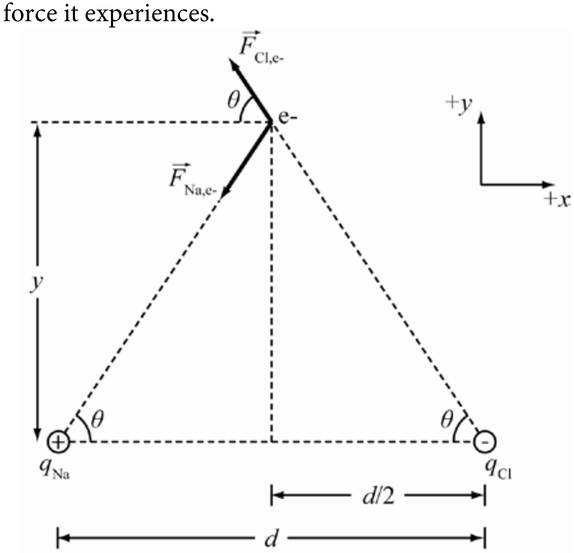
$$d = \sqrt{\frac{ke^2}{m_e g}} = \sqrt{\frac{\left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(1.602 \cdot 10^{-19} \text{ C}\right)^2}{(9.109 \cdot 10^{-31} \text{ kg})(9.81 \text{ m/s}^2)}} = 5.08 \text{ m}$$

21.41. In solid sodium chloride, chloride ions have a charge $q_\text{Cl} = -e = -1.602 \cdot 10^{-19}$ C, while sodium ions have a charge $q_\text{Na} = e = 1.602 \cdot 10^{-19}$ C. These ions are separated by about $d = 0.28$ nm. The Coulomb force between the ions is

$$F = \frac{kq_\text{Cl}q_\text{Na}}{d^2} = \frac{-\left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)(1.602 \cdot 10^{-19} \text{ C})^2}{(0.28 \cdot 10^{-9} \text{ m})^2} = -2.94285 \cdot 10^{-9} \text{ N} \approx -2.9 \cdot 10^{-9} \text{ N}.$$

The negative sign indicates that the force is attractive.

21.42. In gaseous sodium chloride, chloride ions have a charge $q_\text{Cl} = -e = -1.602 \cdot 10^{-19}$ C, while sodium ions have a charge $q_\text{Na} = e = 1.602 \cdot 10^{-19}$ C. These ions are separated by about $d = 0.24$ nm. Another electron is located $y = 0.48$ nm above the midpoint of the sodium chloride molecule. Find the magnitude and the direction of the Coulomb force it experiences.



The x-component of the force is

$$F_x = F_{\text{Cl, e}^- x} + F_{\text{Na, e}^- x}$$

$$= \frac{-ke^2\cos\theta}{\left(\frac{d}{2}\right)^2 + y^2} - \frac{ke^2\cos\theta}{\frac{d^2}{4}+y^2} = -\frac{2ke^2\cos\theta}{\frac{d^2}{4}+y^2} = -\frac{2ke^2}{\frac{d^2}{4}+y^2}\cdot\frac{d/2}{\sqrt{\frac{d^2}{4}+y^2}} = -\frac{ke^2 d}{\left(\frac{d^2}{4}+y^2\right)^{3/2}}$$

$$= -\frac{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)(1.602\cdot 10^{-19}\text{ C})^2(0.24\cdot 10^{-9}\text{ m})}{\left[\frac{(0.24\cdot 10^{-9}\text{ m})^2}{4} + (0.48\cdot 10^{-9}\text{ m})^2\right]^{3/2}}$$

$$= -4.5717\cdot 10^{-10}\text{ N} \approx -4.6\cdot 10^{-10}\text{ N}$$

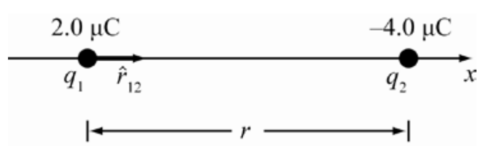
By symmetry, the y-components cancel; that is $F_{\text{Cl, e}^- y} = F_{\text{Na, e}^- y}$. The magnitude is therefore $F = 4.6\cdot 10^{-10}$ N; The electron is pulled in the $-\hat{x}$ direction (in this coordinate system).

21.43. The two up quarks have identical charge $q = (2/3)e = (2/3)\left(1.602\cdot 10^{-19}\text{ C}\right)$. They are $d = 0.900\cdot 10^{-15}$ m apart. The magnitude of the electrostatic force between them is

$$F = \frac{kq^2}{d^2} = \frac{\left(8.99\cdot 10^9\text{ N m}^2/\text{C}^2\right)\left[\frac{2}{3}(1.602\cdot 10^{-19}\text{ C})\right]^2}{(0.900\cdot 10^{-15}\text{ m})^2} = 127\text{ N}.$$

This is large, however the proton does not 'break apart' because of the strength of the strong nuclear force which binds the quarts together to form the proton. A proton is made of 2 up quarks, each with charge $(2/3)e$, and one down quark with charge $-(1/3)e$. The net charge of the proton is e.

21.44. Coulomb's Law can be used to find the force on $q_1 = 2.0\ \mu\text{C}$ due to $q_2 = -4.0\ \mu\text{C}$, where q_2 is $r = 0.200$ m to the right of q_1.

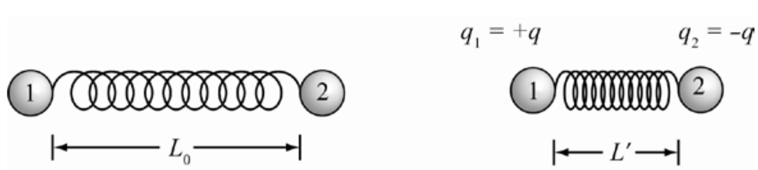


$$\vec{F}_{2\to 1} = -k\frac{q_1q_2}{r^2}\hat{r}_{21} = -k\frac{q_1q_2}{r^2}\hat{x} = -\left(8.99\cdot 10^9\text{ N m}^2/\text{C}^2\right)\frac{(2.0\ \mu\text{C})(-4.0\ \mu\text{C})}{(0.200\text{ m})^2}\hat{x} = 1.8\text{ N }\hat{x}$$

The $-4.0\ \mu\text{C}$ charge pulls the $2.0\ \mu\text{C}$ charge to the right.

21.45. **THINK:** The two identical spheres are initially uncharged. They are connected by an insulating spring of equilibrium length $L_0 = 1.00$ m and spring constant $k = 25.0$ N/m. Charges $+q$ and $-q$ are then placed on metal spheres 1 and 2, respectively. Because the spring is insulating, the charges cannot neutralize across the spring. The spring contracts to new length $L' = 0.635$ m, due to the attractive force between the charges spheres. Determine the charge q. If someone coats the spring with metal to make it conducting, find the new length of the spring.

SKETCH:



RESEARCH: The magnitude of the spring force is $F_S = k_S \Delta x$. The magnitude of the electrostatic force is $F = kq_1q_2 / r^2$. For this isolated system, the two forces must be in balance, that is $F_S = F$. From this balance, the charge q can be determined. The spring constant is denoted by k_S to avoid confusion with the Coulomb constant, *k*.

SIMPLIFY: $F_S = F \Rightarrow k_S \Delta x = \frac{kq_1q_2}{r^2} \Rightarrow k_S(L_0 - L') = \frac{kq^2}{(L')^2} \Rightarrow q = \sqrt{\frac{k_S (L')^2 (L_0 - L')}{k}}$

CALCULATE: $q = \sqrt{\frac{(25.0 \text{ N/m})(0.635 \text{ m})^2 (1.00 \text{ m} - 0.635 \text{ m})}{(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2)}} = 2.02307 \cdot 10^{-5}$ C

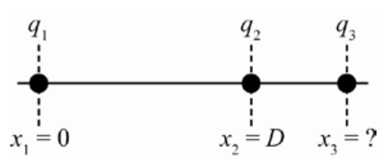
If someone were to coat the spring such that it conducted electricity, the charge on the two spheres would distribute themselves evenly about the system. If the charges are equal in magnitude and opposite in sign, as they are in this case, the net charge in the system would be zero. Then the electrostatic force between the two spheres would be zero, and the spring would return to its equilibrium length of 1.00 m.

ROUND: To three significant figures, $q = 2.02 \cdot 10^{-5}$ C.

DOUBLE-CHECK: Dimensional analysis confirms that the answer is in coulombs, the appropriate unit for charge.

21.46. **THINK:** A point-like charge of $q_1 = +3q$ is located at $x_1 = 0$, and a point-like charge of $q_2 = -q$ is located on the *x*-axis at $x_2 = D$, where $D = 0.500$ m. Find the location on the *x*-axis x_3 where will a third charge $q_3 = q_0$ experiences no net force from the other two charges.

SKETCH:



RESEARCH: The magnitude of the electrostatic force is $F = kq_1q_2 / r^2$. The net force on the third charge q_3 is zero when the sum of the forces from the other two charges is zero: $F_{\text{net},3} = F_{13} + F_{23} = 0 \Rightarrow F_{13} = -F_{23}$. The two forces F_{13} and F_{23} must be equal in magnitude, but opposite in direction. Consider the following three possible locations for the charge q_3. Note that this analysis is independent of the charge of q_3. In the case $x_3 < x_1 = 0$, the two forces F_{13} and F_{23} will be opposite in direction but they cannot be equal in magnitude: the charge q_1 at x_1 is greater in magnitude than the charge q_2 at x_2 and x_3 would be closer to x_1. (Remember that the electrostatic force increases as the distance between the charges decreases.) This makes the magnitude of F_{13} greater than that of F_{23}. In the case $0 \text{ m} < x_3 < D$, the two forces are in the same direction and therefore cannot balance. In the case $x_3 > x_2 = D$, the two forces are opposite in direction, and in direct opposition to the first situation, the force F_{13} and F_{23} can now be balanced. The solution will have a positive *x* position, or more accurately, the third charge q_3 must be placed near the smaller fixed charge, q_2, without being between the two fixed charges q_1 and q_2

SIMPLIFY:

Since $x_3 > x_2$, consider only the magnitudes of the forces. Since only the magnitudes of the forces are compared, only the magnitudes of the charges need be considered.

$$|F_{13}| = |F_{23}| \Rightarrow \left|\frac{kq_1q_3}{x_3^2}\right| = \left|\frac{kq_2q_3}{(x_3 - x_2)^2}\right| \Rightarrow |q_1|(x_3 - x_2)^2 = |q_2| x_3^2 \Rightarrow 3q(x_3 - D)^2 = qx_3^2$$

$$3(x_3 - D)^2 - x_3^2 = 0 \Rightarrow 2x_3^2 - 6x_3D + 3D^2 = 0$$

Solving for x_3 : $x_3 = \dfrac{6D \pm \sqrt{36D^2 - 4(2)(3D^2)}}{4}$

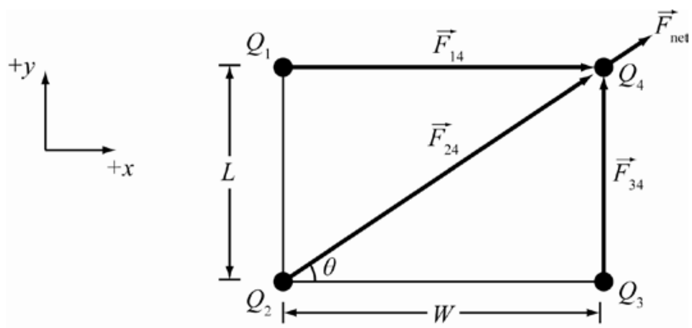
CALCULATE: $x_3 = \dfrac{6(0.500 \text{ m}) \pm \sqrt{36(0.500 \text{ m})^2 - 24(0.500 \text{ m})^2}}{4} = 1.1830 \text{ m}, 0.3170 \text{ m}$

ROUND: Since $x_3 > x_2$, $x_3 = 1.18$ m.

DOUBLE-CHECK: The solution fits the expected location that was determined above (where $x_3 > x_2$).

21.47. **THINK:** Identical point charges $Q = 32 \cdot 10^{-6} C$ are placed at each of the four corners of a rectangle of dimensions $L = 2.0$ m by $W = 3.0$ m. Find the magnitude of the electrostatic force on any one of the charges. Note that by symmetry the magnitude of the net force on each charge is equal. Choose to compute the net electrostatic force on Q_4.

SKETCH:



RESEARCH: The magnitude of the force between two charges is $\vec{F}_{12} = \left(kq_1q_2 / \left|\vec{r}_{21}\right|^2\right)\hat{r}_{21}$. The total force on a charge is the sum of all the forces acting on that charge. The magnitude of the force is found from $F = \left(F_x^{\ 2} + F_y^{\ 2}\right)^{1/2}$, where the components F_x and F_y can be considered one at a time.

SIMPLIFY: x-component: $F_x = F_{14,x} + F_{24,x} + F_{34,x} = \dfrac{kQ^2}{W^2} + \dfrac{kQ^2}{W^2+L^2}\cos\theta + 0 = kQ^2\left(\dfrac{1}{W^2} + \dfrac{W}{\left(W^2+L^2\right)^{3/2}}\right)$

y-component: $F_y = F_{14,y} + F_{24,y} + F_{34,y} = 0 + \dfrac{kQ^2}{W^2+L^2}\sin\theta + \dfrac{kQ^2}{L^2} = kQ^2\left(\dfrac{W}{\left(W^2+L^2\right)^{3/2}} + \dfrac{1}{L^2}\right)$

$F_{\text{net}} = \sqrt{F_x^{\ 2} + F_y^{\ 2}}$

CALCULATE: $F_x = \left(8.99 \cdot 10^9 \text{ N m}^2 / C^2\right)(32 \cdot 10^{-6} \text{ C})^2\left(\dfrac{1}{\left(3.0 \text{ m}\right)^2} + \dfrac{3.0 \text{ m}}{\left[\left(3.0 \text{ m}\right)^2 + \left(2.0 \text{ m}\right)^2\right]^{3/2}}\right) = 1.612 \text{ N}$

$F_y = \left(8.99 \cdot 10^9 \text{ N m}^2 / C^2\right)(32 \cdot 10^{-6} \text{ C})^2\left(\dfrac{2.0 \text{ m}}{\left[\left(3.0 \text{ m}\right)^2 + \left(2.0 \text{ m}\right)^2\right]^{3/2}} + \dfrac{1}{\left(2.0 \text{ m}\right)^2}\right) = 2.694 \text{ N}$

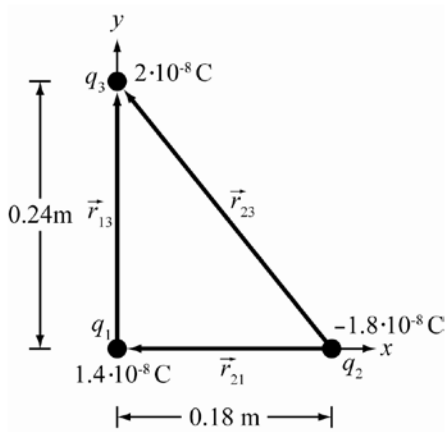
$F_{\text{net}} = \sqrt{\left(1.612 \text{ N}\right)^2 + \left(2.694 \text{ N}\right)^2} = 3.1397 \text{ N}$

ROUND: Since each given value has 2 significant figures, $F_{\text{net}} = 3.1$ N

DOUBLE-CHECK: Since L is less than W, the y-component of F_{net} should be greater than the x-component.

21.48. **THINK:** Charge $q_1 = 1.4\cdot 10^{-8} C$ is at $r_1 = (0,0)$. Charge $q_2 = -1.8\cdot 10^{-8} C$ is at $r_2 = (0.18 \text{ m}, 0 \text{ m})$, and charge $q_3 = 2.1\cdot 10^{-8} C$ is at $r_3 = (0 \text{ m}, 0.24 \text{ m})$. Determine the net force (magnitude and direction) F_3 on charge q_3.

SKETCH:



RESEARCH: The magnitude of the force between two charges is $\overrightarrow{F_{12}} = kq_1q_2\hat{r}_{12} / \left|\overrightarrow{r_{12}}\right|^2 = kq_1q_2\overrightarrow{r_{12}} / r_{12}^{\ 3}$. The total force on charge q_3 is the sum of all the forces acting on it. The magnitude of F_3 is found from $F_3 = \left(F_1^{\ 2} + F_2^{\ 2}\right)^{1/2}$, and the direction θ is found from $\theta = \tan^{-1}\left(F_y / F_x\right)$.

SIMPLIFY: $\vec{F}_{\text{net, 3}} = \vec{F}_{13} + \vec{F}_{23}$

$$= \frac{kq_1q_3\overrightarrow{r_{13}}}{r_{13}^{\ 3}} + \frac{kq_2q_3\overrightarrow{r_{23}}}{r_{23}^{\ 3}}$$

$$= \frac{kq_1q_3\left[(x_3-x_1)\hat{x} + (y_3-y_1)\hat{y}\right]}{\left[(x_3-x_1)^2 + (y_3-y_1)^2\right]^{3/2}} + \frac{kq_2q_3\left[(x_3-x_2)\hat{x} + (y_3-y_2)\hat{y}\right]}{\left[(x_3-x_2)^2 + (y_3-y_2)^2\right]^{3/2}}$$

$$= \frac{kq_1q_3}{y_3^{\ 3}}\left(y_3\hat{y}\right) + \frac{kq_2q_3}{(x_2^{\ 2} + y_3^{\ 2})^{3/2}}\left(-x_2\hat{x} + y_3\hat{y}\right)$$

CALCULATE: $$\vec{F}_{\text{net, 3}} = \frac{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)(1.4\cdot 10^{-8} \text{ C})(2.1\cdot 10^{-8} \text{ C})\left(0.24 \text{ m}\right)}{\left(0.24 \text{ m}\right)^3}\hat{y}$$

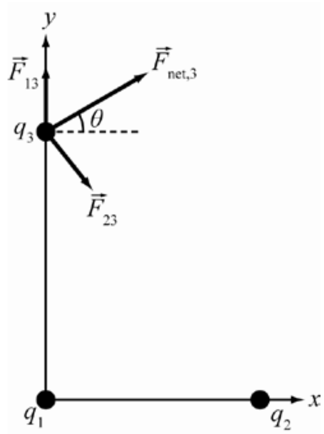
$$+ \frac{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)(-1.8\cdot 10^{-8} \text{ C})(2.1\cdot 10^{-8} \text{ C})\left(-0.18\hat{x} \text{ m} + 0.24\hat{y} \text{ m}\right)}{\left[\left(0.18 \text{ m}\right)^2 + \left(0.24 \text{ m}\right)^2\right]^{3/2}}$$

$$= (4.5886\cdot 10^{-5} \text{ N})\hat{y} + (2.265\cdot 10^{-5} \text{ N})\hat{x} - (3.0206\cdot 10^{-5} \text{ N})\hat{y}$$

$$= \left(2.265\cdot 10^{-5} \text{ N}\right)\hat{x} + \left(1.568\cdot 10^{-5} \text{ N}\right)\hat{y}$$

$$F_{\text{net, 3}} = \sqrt{F_x^{\ 2} + F_y^{\ 2}} = \sqrt{(2.265\cdot 10^{-5} \text{ N})^2 + (1.568\cdot 10^{-5} \text{ N})^2} = 2.755\cdot 10^{-5} \text{ N}$$

$$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{1.568\cdot 10^{-5} \text{ N}}{2.265\cdot 10^{-5} \text{ N}}\right) = 34.69° \text{ above the horizontal}$$

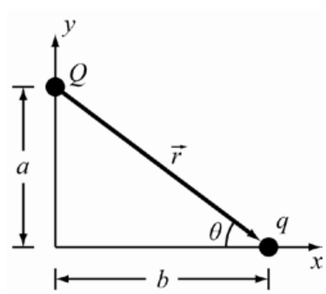


ROUND: With 2 significant figures in each given value, the final answers should be rounded to $\vec{F}_{\text{net, 3}} = \left(2.265 \cdot 10^{-5} \text{ N}\right)\hat{x} + \left(1.568 \cdot 10^{-5} \text{ N}\right)\hat{y} = 2.8 \cdot 10^{-5} \text{ N}$ and $\theta = 35°$.

DOUBLE-CHECK: Due to the attraction between q_2 and q_3 and that q_1 is directly underneath q_3, the x component of $\vec{F}_{\text{net, 3}}$ has to be positive.

21.49. **THINK:** A positive charge Q is on the y-axis at a distance a from the origin and another positive charge q is on the x-axis at a distance b from the origin. (a) Find the value(s) of b for which the x-component of the force on q is a minimum. (b) Find the value(s) of b for which the x-component of the force on q is a maximum.

SKETCH:



RESEARCH: The electrostatic force is $F = kqQr / \left|r\right|^2$. The x-component of this force is $F_x = (kqQ / r^2)\cos\theta$. The values of b for which F_x is a minimum can be determined by inspection; the values of b for which F_x is a maximum can be found by calculating the extrema of F_x, that is, taking the derivative of F_x with respect to b, setting it to zero, and solving for b.

SIMPLIFY: $F_x = \dfrac{kqQ}{r^2}\cos\theta = \dfrac{kqQb}{r^3} = \dfrac{kqQb}{\left(a^2+b^2\right)^{3/2}}$

a) Minima: By inspection, the least possible value of F_x is zero, and this is attained only when $b = 0$.

b) Maxima: $\dfrac{dF_x}{db} = 0 \;\Rightarrow\; \dfrac{kqQ}{\left(a^2+b^2\right)^{3/2}} - \dfrac{3}{2}kqQ\left(a^2+b^2\right)^{-5/2} 2b = 0 \;\Rightarrow\; \dfrac{kqQ\left(a^2+b^2\right) - 3kqQb^2}{\left(a^2+b^2\right)^{5/2}} = 0$

$$\Rightarrow \left(a^2+b^2\right) - 3b^2 = 0 \;\Rightarrow\; b = \pm\frac{a}{\sqrt{2}}$$

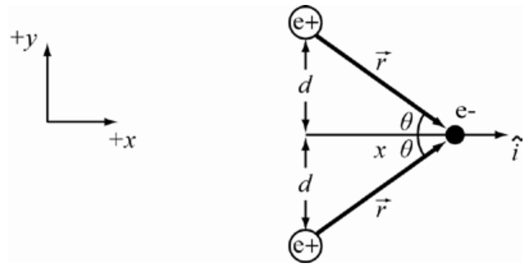
CALCULATE: Reject the negative solution, since distances have to be positive: $b = \dfrac{a}{\sqrt{2}}$.

ROUND: Not applicable

DOUBLE-CHECK: It makes sense that the possible values of b should be symmetrically distributed about the origin (above which lies the charge Q).

21.50. **THINK:** Two protons are placed near one electron as shown in the figure provided. Determine the electrostatic force on the electron. The charge of the electron is $q_{\text{e}} = -e$ and the charge of each proton is $q_{\text{p}} = e$, where $e = 1.602 \cdot 10^{-19}$ C.

SKETCH:



RESEARCH: By symmetry the forces in the vertical direction cancel. The force is therefore due solely to the horizontal contribution $F\cos\theta$ in the $\hat{x}$ direction: the Coulomb force is $F_{21} = kq_1q_2 / r_{21}^{\ 2}$.

SIMPLIFY: By symmetry, and with the two protons, $\vec{F} = 2F_{\text{pe}}\cos\theta\hat{x} = -2\frac{ke^2}{r^2}\frac{x}{r}\hat{x} = -2\frac{ke^2 x}{\left(x^2+d^2\right)^{3/2}}\hat{x}.$

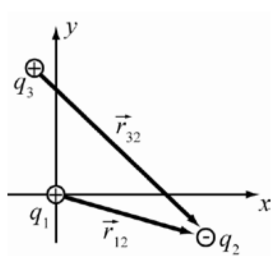
CALCULATE: $\vec{F} = -2\frac{\left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(1.602 \cdot 10^{-19} \text{ C}\right)^2\left(0.0700 \text{ m}\right)}{\left[\left(0.0700 \text{ m}\right)^2 + \left(0.0500 \text{ m}\right)^2\right]^{3/2}}\hat{x} = (-5.0742 \cdot 10^{-26} \text{ N})\hat{x}$

ROUND: $\vec{F} = \left(-5.07 \cdot 10^{-26} \text{ N}\right)\hat{x}$

DOUBLE-CHECK: This is a reasonable force as the charges are as small as they can possibly be and the separation is large.

21.51. **THINK:** The positions of the three fixed charges are $q_1 = 1.00$ mC at $r_1 = (0,0)$, $q_2 = -2.00$ mC at $r_2 = (17.0 \text{ mm}, -5.00 \text{ mm})$, and $q_3 = +3.00$ mC at $r_3 = (-2.00 \text{ mm}, 11.0 \text{ mm})$. Find the net force on the charge q_2.

SKETCH:



RESEARCH: The magnitude force is $\overrightarrow{F_{12}} = kq_1q_2\hat{r}_{12} / \left|\overrightarrow{r_{12}}\right|^2 = kq_1q_2\overrightarrow{r_{12}} / r_{12}^{\ 3}$. The net force on q_2 is the sum of all the forces acting on q_2.

SIMPLIFY: $\vec{F}_{\text{net},2} = \vec{F}_{12} + \vec{F}_{32} = kq_2\left(\frac{q_1\left[(x_2-x_1)\hat{x}+(y_2-y_1)\hat{y}\right]}{\left[(x_2-x_1)^2+(y_2-y_1)^2\right]^{3/2}} + \frac{q_3\left[(x_2-x_3)\hat{x}+(y_2-y_3)\hat{y}\right]}{\left[(x_2-x_3)^2+(y_2-y_3)^2\right]^{3/2}}\right)$

CALCULATE: Without units,

$$\vec{F}_{\text{net, }2} = \left(8.99\cdot10^9\right)(-2.00)\left[\frac{(1.00)\left(17.0\hat{x}-5.00\hat{y}\right)}{\left[\left(17.0\right)^2+\left(-5.00\right)^2\right]^{3/2}}+\frac{(3.00)(19.0\hat{x}-16.0\hat{y})}{\left[\left(19.0\right)^2+\left(-16.0\right)^2\right]^{3/2}}\right]$$
$$= -1.2181\cdot10^8\,\hat{x}+7.2469\cdot10^7\,\hat{y}.$$

Then, the units of $\vec{F}_{\text{net, }2}$ are:

$$\left[\vec{F}_{\text{net, }2}\right]=\left(\text{N m}^2/\text{C}^2\right)(\text{mC})\left[\frac{(\text{mC})(\text{mm}-\text{mm})}{\left[\left(\text{mm}\right)^2+\left(\text{mm}\right)^2\right]^{3/2}}+\frac{(\text{mC})(\text{mm}-\text{mm})}{\left[\left(\text{mm}\right)^2+\left(\text{mm}\right)^2\right]^{3/2}}\right]=\text{N}$$

Altogether , $\vec{F}_{\text{net, }2}=\left(-1.2181\cdot10^8\text{ N}\right)\hat{x}+\left(7.2469\cdot10^7\text{ N}\right)\hat{y}.$ The magnitude of the force is

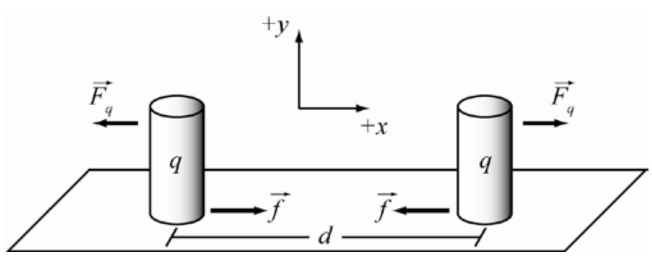
$$F_{\text{net, }2}=\sqrt{F_x^{\,2}+F_y^{\,2}}=\sqrt{\left(-1.2181\cdot10^8\text{ N}\right)^2+\left(7.2469\cdot10^7\text{ N}\right)^2}=1.4174\cdot10^8\text{ N}$$

ROUND: $\vec{F}_{\text{net, }2}=\left(-1.22\cdot10^8\text{ N}\right)\hat{x}+\left(7.25\cdot10^7\text{ N}\right)\hat{y}$ and $\left|\vec{F}_{\text{net, }2}\right|=1.42\cdot10^8\text{ N}.$

DOUBLE-CHECK: The charges are large and the separation distance are small, so $F_{\text{net, }2}$ should be very strong.

21.52. **THINK:** the masses of the beads are $m=10.0\text{ mg}=1.00\cdot10^{-5}\text{ kg}$ and they have identical charge. They are a distance $d=0.0200\text{ m}$ apart. The coefficient of static friction between the beads and the surface is $\mu=0.200.$ Find the minimum charge q needed for the beads to start moving.

SKETCH:



RESEARCH: Assume the surface is parallel to the surface of the Earth. The frictional force is $f=\mu N,$ where $N=mg$. The electrostatic force is $F=kq^2/d^2$. The beads will start to move as soon as F is greater than f, enabling one bead to move away from the other. Then the minimum charge q can be found by equating f and F.

SIMPLIFY: $F=f\ \Rightarrow\ \dfrac{kq^2}{d^2}=\mu mg\ \Rightarrow\ q=\sqrt{d^2\mu mg/k}$

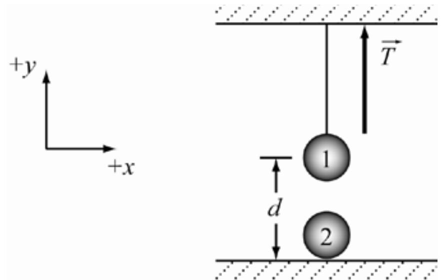
CALCULATE: $q=\sqrt{\dfrac{\left(0.0200\text{ m}\right)^2(0.200)(1.00\cdot10^{-5}\text{ kg})(9.81\text{ m/s}^2)}{\left(8.99\cdot10^9\text{ N m}^2/\text{C}^2\right)}}=9.3433\cdot10^{-10}\text{ C}$

ROUND: All of the given values have three significant figures, so $q=9.34\cdot10^{-10}$ C.

DOUBLE-CHECK: The units of the solution are those of charge. This is a reasonable charge required to overcome the frictional force.

21.53. **THINK:** The ball's mass is $m_1=0.0300$ kg; its charge is $q_1=-0.200\ \mu C.$ The ball is suspended a distance of $d=0.0500$ m above an insulating floor. The second small ball has mass $m_2=0.0500$ kg and a charge $q_2=0.400\ \mu\text{C}.$ Determine if the second ball leaves the floor. Find the tension T in the string when the second ball is directly beneath the first ball. Because the balls are small, we will treat them as point charges with radius zero.

SKETCH:



RESEARCH: The electrostatic force between two charges is $F = kq_1q_2 / r^2$. The force of gravity is $F_g = mg$. The ball will leave the floor if the electrostatic force between the two balls is greater that the force of gravity, that is if $F > F_g$, and if the charges are opposite. The tension in the rope can be found by considering all of the vertical forces acting on the first ball.

SIMPLIFY: The electrostatic force is: $F = kq_1q_2 / d^2$. The gravitational force is: $F_g = m_2(-g)$. The forces acting on m_1 in the y-direction sum to: $0 = T - F_{\text{coulomb}} - m_1g$. So the tension is $T = F_{\text{coulomb}} + m_1g$.

CALCULATE: $F = \dfrac{(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2)(-0.200 \cdot 10^{-6} \text{ C})(0.400 \cdot 10^{-6} \text{ C})}{(0.0500 \text{ m})^2} = -0.28768 \text{ N},$

$F_g = (0.0500 \text{ kg})(-9.81 \text{ m/s}^2) = -0.4905 \text{ N}, \qquad T = -0.28768 \text{ N} + (0.0300 \text{ kg})(-9.81 \text{ m/s}^2) = -0.58198 \text{ N}.$

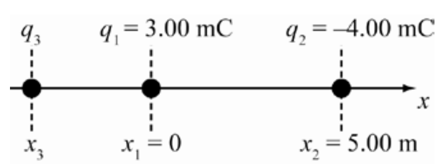
Since $F_g > F$, the second ball does not leave the ground.

ROUND: With all given values containing three significant figures, round the tension to $T = -0.582$ N.

DOUBLE-CHECK: The balls are not quite close enough to overcome the force of gravity, but the magnitude of F_{coulomb} is comparable to F_g, despite the small charges (on the order of 10^{-7} C).

21.54. **THINK:** A $q_1 = +3.00$ mC charge and a $q_2 = -4.00$ mC charge are fixed in position and separated by $d = 5.00$ m. Take the position of q_1 to be at $x_1 = 0$, and position of q_2 to be at $x_2 = 5.00$ m. (a) Find the location, x_3, of a $q_3 = +7.00$ mC charge so that the net force on it is zero. (b) Find the location, x_3', of a $q_3 = -7.00$ mC charge so that the net force on it is zero.

SKETCH:



RESEARCH: The electrostatic force between two charges is $F = kq_1q_2 / r^2$. The net force on a third charge is zero: $F_{\text{net},3} = F_{13} + F_{23} = 0 \Rightarrow F_{13} = -F_{23}$. The two forces must be equal in magnitude, but opposite in direction. Consider the following three possible locations for the charge q_3. Note that this analysis is independent of the charge of q_3: At $x_3 > 5.00$ m, the two forces F_{13} and F_{23} will be opposite in direction but they cannot be equal in magnitude: the charge q_2 at $x_2 = 5.00$ m is greater in magnitude than the charge q_1 at $x_1 = 0$ and x_3 would be closer to x_2. (Remember that the electrostatic force increases as the distance between the charges decreases.) This makes the magnitude of F_{23} greater than that of F_{13}. Next, consider values of x_3 satisfying: $0 \text{ m} < x_3 < 5.00$ m. The two forces are in the same direction and therefore cannot balance. At $x_3 < 0$ m, the two forces are opposite in direction, and in direct opposition to the first situation, the force F_{13} and F_{23} can now be balanced. The solution will have a negative position, or more accurately, the third charge q_3 must be placed near the smaller fixed charge, q_1, without being between the two fixed charges q_1 and q_2. This answer is independent of the charge of q_3, therefore the numeric answer to parts a and b is the same.

SIMPLIFY: With $x_3 < 0$, and $\hat{F}_{13}$ opposite in direction to $\hat{F}_{23}$, the force are balanced when

$$F_{13} = -F_{23} \Rightarrow \frac{kq_1q_3}{|x_3|^2} = -\frac{kq_2q_3}{\left(x_2 - x_3\right)^2} \Rightarrow q_1\left(x_2 - x_3\right)^2 = -q_2x_3^{\ 2} \Rightarrow \left(q_1 + q_2\right)x_3^{\ 2} - 2q_1x_2x_3 + q_1x_2^{\ 2} = 0.$$

Solving for x_3:

$$x_3 = \frac{2q_1x_2 \pm \sqrt{4q_1^{\ 2}x_2^{\ 2} - 4(q_1 + q_2)q_1x_2^2}}{2(q_1 + q_2)}.$$

CALCULATE:

$$x_3 = \frac{2(3.00\text{ mC})(5.00\text{ m}) \pm \text{mC m}\sqrt{4(3.00)^2(5.00)^2 - 4(3.00 - 4.00)(3.00)(5.00)^2}}{2(3.00\text{ mC} - 4.00\text{ mC})} = -32.321\text{ m, } 2.305\text{ m}$$

By the convention established in this solution, x_3 is negative. (The second solution places q_3 a between q_1 and q_2, a possibility which has been ruled out.)

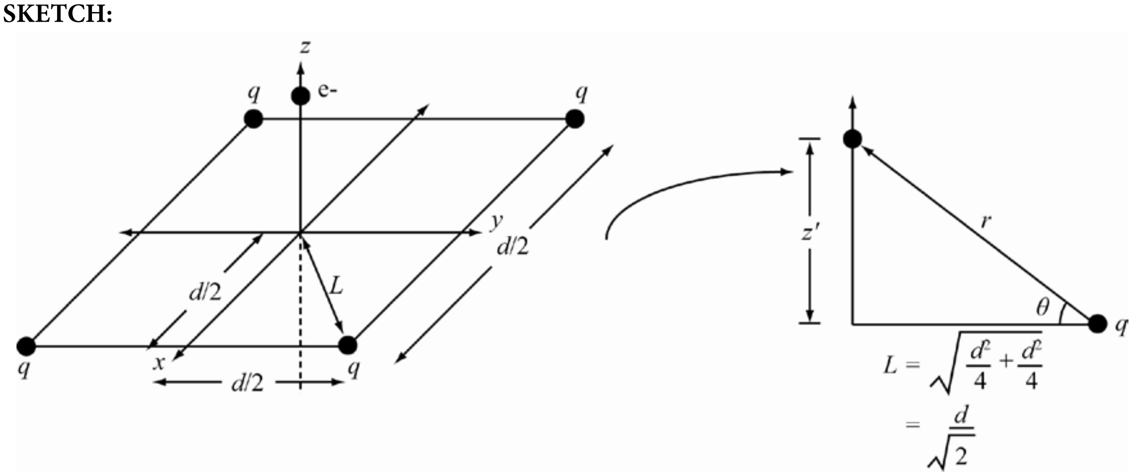
ROUND: All given values have three significant figures, so $x_3 = -32.3$ m.

DOUBLE-CHECK: Inserting the calculated value of x_3 back into the expressions for the Coulomb force:

$$F_{13} = \frac{kq_1q_3}{x_3^2} = \frac{k\left(3.00\text{ mC}\right)\left(7.00\text{ mC}\right)}{\left(-32.3\text{ m}\right)^2} = 181\text{ N} \text{ and } F_{23} = \frac{kq_2q_3}{\left(x_2 - x_3\right)^2} = \frac{k\left(-4.00\text{ mC}\right)\left(7.00\text{ mC}\right)}{\left(5.00\text{ m} + 32.3\text{ m}\right)^2} = -181\text{ N.}$$

21.55. THINK: Four point charges, each with charge q, are fixed to the four corners of a square with a sides of length $d = 1.00$ cm. An electron is suspended above a point at which its weight is balanced by the electrostatic force due to the four electrons: $z' = 15.0$ nm above the center of the square. The mass of an electron is $m_e = 9.109 \cdot 10^{-31}$ kg, and the charge is $q_e = -e = -1.602 \cdot 10^{-19}$ C. Find the value of q of the fixed charges, in Coulombs and as a multiple of the electron charge.

SKETCH:



RESEARCH: The electrostatic force between two charges is $F = kq_1q_2 / r^2$. By symmetry, the net force in the horizontal direction is zero, and the problem reduces to a balance of the forces in the vertical direction, with one fixed charge balancing a quarter of the electron's weight. The vertical component of the electrostatic force is $F\sin\theta$. The weight of the electron is $W = m_e g$.

SIMPLIFY: Balancing the forces in the vertical (z) direction yields $F_{\text{coulomb}} = \frac{1}{4}W \Rightarrow \frac{kqq_e}{r^2}\sin\theta = \frac{1}{4}m_e g.$

Solving for q: $q=\frac{1}{4}\frac{m_e gr^2}{kq_e\sin\theta}=\frac{m_e gr^3}{4kq_e z'}=\frac{m_e g(L^2+z'^2)^{3/2}}{4kq_e z'}=\frac{-m_e g\left(\frac{d^2}{2}+z'^2\right)^{3/2}}{4kez'}.$

CALCULATE: $q=\frac{-\left(9.109\cdot 10^{-31}\text{ kg}\right)(9.81\text{ m/s}^2)\left[\frac{(0.100\text{ m})^2}{2}+(15.0\cdot 10^{-9}\text{ m})^2\right]^{3/2}}{4\left(8.99\cdot 10^9\text{ N m}^2/\text{C}^2\right)(1.602\cdot 10^{-19}\text{ C})(15.0\cdot 10^{-9}\text{ m})}$

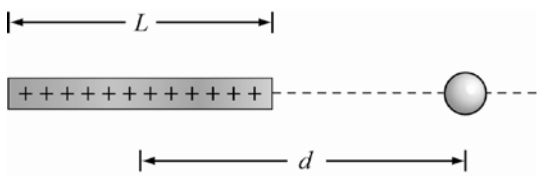
$=-3.6561\cdot 10^{-17}\text{ C, or }-228.22e.$

ROUND: With three significant figures in z', $q=-3.66\cdot 10^{-17}\text{ C}=-228e.$

DOUBLE-CHECK: The gravitational force on an electron is small. Each charge q needs to be a few hundred electron charges to balance the gravitational force on the electron.

21.56. **THINK:** A uniformly charged thin rod of length L has a total charge Q. Find an expression for the electrostatic force strength acting on an electron, whose magnitude of charge is e, is positioned on the axis of the rod at distance d from the center.

SKETCH:



RESEARCH: The electrostatic force between two charges is $F=kqQ/r^2$. The net electrostatic force acting on a charge q is the sum of all the electrostatic forces acting on q. In the event of a continuous and linear distribution of charge of length L and total charge Q, the force due to an infinitesimal amount of charge dq' from the distribution acting on the charge q is: $dF=kq\,dq'/x^2$, where $dq'=(Q/L)dx=\lambda dx$. (λ is the linear charge density.) In this case, the total force on the electron is then

$$F=\int_{d-L/2}^{d+L/2}\frac{ke\lambda}{x^2}dx,$$

where the integration runs over the length of the rod, starting from the point closest to the electron $\left(d-L/2\right)$ and ending with the point farthest from the electron $\left(d+L/2\right)$.

SIMPLIFY:

$$F=\int_{d-L/2}^{d+L/2}\frac{ke\lambda}{x^2}dx=ke\lambda\int_{d-L/2}^{d+L/2}\frac{1}{x^2}dx=ke\lambda\left(-x\Big|_{d-L/2}^{d+L/2}\right)=2ke\lambda\left(\frac{1}{2d-L}-\frac{1}{2d+L}\right)=\frac{4ke\lambda L}{4d^2-L^2}=\frac{4keQ}{4d^2-L^2}$$

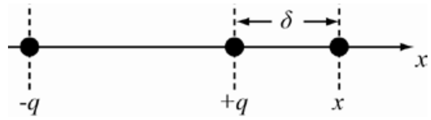
CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: The answer is in the correct units of force: $\left[F\right]=\frac{\left(\frac{\text{N m}^2}{\text{C}^2}\right)(\text{C})(\text{C})}{\text{m}^2}=\text{N}.$

21.57. **THINK:** A negative charge $-q$ is located and fixed at $(0,\,0)$. A positive charge $+q$ is initially at $(x,\,0)$. The positive charge will accelerate towards the negative charge. Use the binomial expansion to show that when the positive charge moves a distance $\delta\ll x$ closer to the negative charge, the force on it increases by $\Delta F=2kq^2\delta/x^3$.

SKETCH:



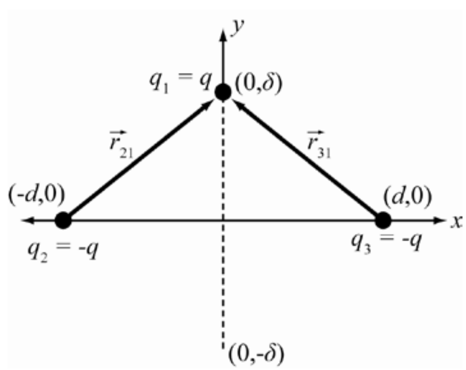
RESEARCH: The Coulomb force is $\overrightarrow{F_{21}} = kq_1q_2\hat{r}_{21} / r_{21}^2$, where $\hat{r}_{21}$ is the unit vector that points from charge 2 to charge 1. To first order, the binomial expansion is $(1+x)^n \approx 1+nx$ for $|x| << 1$.

SIMPLIFY: The initial force on $+q$ (when it was at $(x,\ 0)$ was $\vec{F} = -\frac{kq^2}{x^2}\hat{x}$. After moving closer to $-q$ by δ_1 the new force on $+q$ is $\vec{F}' = -\frac{kq^2}{(x-\delta)^2}\hat{x} = -\frac{kq^2}{x^2\left(1-\frac{\delta}{x}\right)^2}\hat{x} = -\frac{kq^2}{x^2}\left(1-\frac{\delta}{x}\right)^{-2}\hat{x}$. Using the binomial expansion, $\vec{F}' = -\frac{kq^2}{x^2}\left(1-(-2)\frac{\delta}{x}+...\right)\hat{x} \approx -\frac{kq^2}{x^2}\left(1+2\frac{\delta}{x}\right)\hat{x}$ (to first order in δ). Then, $\Delta\vec{F} = \vec{F}' - \vec{F} \approx -\frac{2kq^2\delta}{x^3}\hat{x}$ and $\Delta F = \frac{2kq^2\delta}{x^3}$, as desired.

CALCULATE: Not applicable.
ROUND: Not applicable.

DOUBLE-CHECK: The charge in force has the correct units for force: $\left[\Delta F\right] = \frac{\frac{\text{N m}^2}{\text{C}^2}\text{C}\frac{\text{C}}{\text{m}}\text{m}}{\text{m}^2} = \text{N}.$

21.58. **THINK:** Two charges, both $-q$, are located and fixed at coordinates $(-d,0)$ and $(d,0)$ in the x-y plane. A positive charge of the same magnitude q and of mass m is placed at coordinate $(0,0)$. The positive charge is then moved a distance $\delta \ll d$ along the +y direction and then released. It will oscillate between co-ordinates $(0,\delta)$ and $(0,-\delta)$. Find the net force F_{net} acting on the positive charge when it is moved to $(0,\delta)$ and use the binomial expansion to find an expression for the frequency of the resulting oscillation.
SKETCH:



RESEARCH: The Coulomb force is $\overrightarrow{F_{21}} = kq_1q_2\hat{r}_{21} / r_{21}^2$, where $\overrightarrow{F_{21}}$ is the force on the charge 1 by charge 2, and $\hat{r}_{21}$ points from charge 2 to charge 1. To first order, the binomial expansion is, in general, $(1+x)^n \approx 1+nx$ for $x \ll 1$. The restoring force of a simple harmonic oscillator obeys Hooke's Law, $F = -\omega^2 mx$, where ω is the characteristic angular frequency, and $f = \omega/(2\pi)$.

SIMPLIFY: $\vec{F}_{\text{net}} = \frac{kq_1q_2\vec{r}_{21}}{r_{21}^{\ 3}} + \frac{kq_1q_3\vec{r}_{31}}{r_{31}^{\ 3}} = \frac{-kq^2}{(d^2+\delta^2)^{3/2}}(d\hat{x}+\delta\hat{y}) - \frac{kq^2}{(d^2+\delta^2)^{3/2}}(-d\hat{x}+\delta\hat{y}) = \frac{-2kq^2\delta}{(d^2+\delta^2)^{3/2}}\hat{y}$

$$= -\frac{2kq^2\delta}{d^3\left(1+\frac{\delta^2}{d^2}\right)^{3/2}}\hat{y} = -\frac{2kq^2\delta}{d^3}\left(1+\frac{\delta^2}{d^2}\right)^{-3/2}\hat{y}$$

Note the binomial expansion of $\left(1+\delta^2/d^2\right)^{-3/2} \approx 1-(3/2)\left(\delta^2/d^2\right)$. Neglecting the term δ^2/d^2 (keeping only terms linear in δ), the net force is $\vec{F}_{\text{net}} \approx -2\left(kq^2\delta/d^3\right)\hat{y}$. Then from $F=-\omega^2 mx$, $\omega=\sqrt{-F/(mx)}$ with $x=\delta$, the angular frequency is $\omega=\sqrt{2kq^2\delta/\left(md^3\delta\right)}=\sqrt{2kq^2/\left(md^3\right)}=\frac{q}{d}\sqrt{\frac{2k}{md}}$ and the frequency is $f=\frac{q}{2\pi d}\sqrt{\frac{2k}{md}}=\frac{q}{\pi d}\sqrt{\frac{k}{2md}}$.

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: The frequency of oscillation should depend directly on the magnitude of the charges and inverse on the distance separating the charges. This lends support to the formulas found above.

21.59. The gravitational force between the Earth and Moon is given by $F_{\text{g}} = GM_{\text{Earth}}m_{\text{Moon}}/r_{\text{EM}}^{\ 2}$. The static electrical force between the Earth and the Moon is $F=kQ^2/r_{\text{EM}}^{\ 2}$, where Q is the magnitude of the charge on each the Earth and the Moon. If the static electrical force is 1.00% that of the force of gravity, then the charge Q would be:

$$F=0.01F_{\text{g}} \Rightarrow \frac{kQ^2}{r_{\text{EM}}^{\ 2}} = \frac{0.0100GM_{\text{Earth}}m_{\text{Moon}}}{r_{\text{EM}}^{\ 2}} \Rightarrow Q=\sqrt{\frac{0.0100GM_{\text{Earth}}m_{\text{Moon}}}{k}}.$$

This gives $Q=\sqrt{\frac{0.0100(6.67\cdot 10^{-11}\text{ N m}^2/\text{kg})(5.97\cdot 10^{24}\text{ kg})(7.36\cdot 10^{22}\text{ kg})}{\left(8.99\cdot 10^{9}\text{ N m}^2/\text{C}^2\right)}} = 5.71\cdot 10^{12}\text{ C}.$

21.60. The gravitational force between the Earth and Moon is given by $F_{\text{g}} = GM_{\text{Earth}}m_{\text{moon}}/r_{\text{EM}}^{\ 2}$. If this is due solely to static electrical force between the Earth and Moon, the magnitude of Q would be:

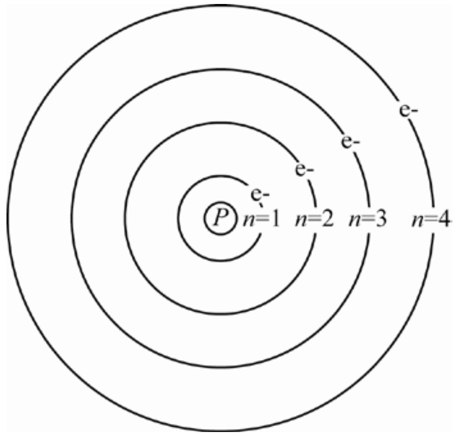
$$F_{\text{g}} = -G\frac{M_{\text{Earth}}m_{\text{Moon}}}{r_{\text{EM}}^{\ 2}} = -k\frac{Q^2}{r_{\text{EM}}^{\ 2}} \Rightarrow Q=\sqrt{\frac{GM_{\text{Earth}}m_{\text{Moon}}}{k}}.$$

So, $Q=\sqrt{\frac{(6.67\cdot 10^{-11}\text{ N m}^2/\text{kg})(5.97\cdot 10^{24}\text{ kg})(7.36\cdot 10^{22}\text{ kg})}{\left(8.99\cdot 10^{9}\text{ N m}^2/\text{C}^2\right)}} = 5.71\cdot 10^{13}\text{ C}.$

This is a large amount of charge, on the order of 10^{31} electrons worth of charge. This is equivalent to about 60 million moles of electrons.

21.61. **THINK:** The radii of the electron orbits are $r_{\text{n}} = n^2 a_{\text{B}}$, where n is an integer (not 0) and $a_{\text{B}} = 5.29\cdot 10^{-11}$ m. Calculate the electrostatic force between the electron (charge $-e$ and mass $m_{\text{e}} = 9.109\cdot 10^{-31}$ kg) and the proton (charge e and mass $m_{\text{p}} = 1.673\cdot 10^{-27}$ kg) for the first 4 orbits and compare them to the gravitational interaction between the two. Note $e = 1.602\cdot 10^{-19} C$.

SKETCH:



RESEARCH: The Coulomb force is $F = k|q_1||q_2|/r^2$, or $F_n = ke^2/r_n^{\ 2}$ in this case. The gravitational force is $F_g = Gm_1m_2/r^2$, or $F_{g,\,n} = Gm_em_p/r_n^{\ 2}$.

SIMPLIFY: $n=1$: $F_1 = \dfrac{ke^2}{a_B^{\ 2}}$; $F_{g,\,1} = \dfrac{Gm_em_p}{a_B^{\ 2}}$, $n=2$: $F_2 = \dfrac{ke^2}{(4a_B)^2}$; $F_{g,\,2} = \dfrac{Gm_em_p}{(4a_B)^2}$

$n=3$: $F_3 = \dfrac{ke^2}{(9a_B)^2}$; $F_{g,\,3} = \dfrac{Gm_em_p}{(9a_B)^2}$, $n=4$: $F_4 = \dfrac{ke^2}{(16a_B)^2}$; $F_{g,\,4} = \dfrac{Gm_em_p}{(16a_B)^2}$

CALCULATE: Note that: $\dfrac{ke^2}{a_B^{\ 2}} = \dfrac{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)(1.602\cdot 10^{-19}\text{ C})^2}{\left(5.29\cdot 10^{-11}\text{ m}\right)^2} = 8.2465\cdot 10^{-8}\text{ N}$ and

$$\frac{Gm_em_p}{a_B^{\ 2}} = \frac{(6.67\cdot 10^{-11}\text{ N m}^2/\text{kg})(9.109\cdot 10^{-31}\text{ kg})(1.673\cdot 10^{-27}\text{ kg})}{\left(5.29\cdot 10^{-11}\text{ m}\right)^2} = 3.632\cdot 10^{-47}\text{ N}.$$

Then for $n=1$: $F_1 = \dfrac{ke^2}{a_B^{\ 2}} = 8.2465\cdot 10^{-8}\text{ N}$; $F_{g,\,1} = \dfrac{Gm_em_p}{a_B^{\ 2}} = 3.6342\cdot 10^{-47}\text{ N}$

$n=2$: $F_2 = \dfrac{ke^2}{\left(4a_B\right)^2} = 5.1515\cdot 10^{-9}\text{ N}$; $F_{g,\,2} = \dfrac{Gm_em_p}{\left(4a_B\right)^2} = 2.2712\cdot 10^{-48}\text{ N}$

$n=3$: $F_3 = \dfrac{ke^2}{\left(9a_B\right)^2} = 1.1081\cdot 10^{-9}\text{ N}$; $F_{g,\,3} = \dfrac{Gm_em_p}{\left(9a_B\right)^2} = 4.4863\cdot 10^{-49}\text{ N}$

$n=4$: $F_4 = \dfrac{ke^2}{\left(16a_B\right)^2} = 3.2213\cdot 10^{-10}\text{ N}$; $F_{g,\,4} = \dfrac{Gm_em_p}{\left(16a_B\right)^2} = 1.4195\cdot 10^{-49}\text{ N}$

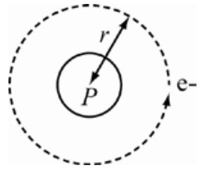
ROUND: Since a_B has three significant figures, $F_1 = 8.25\cdot 10^{-8}\text{ N}$, $F_{g,\,1} = 3.63\cdot 10^{-47}\text{ N}$, $F_2 = 5.15\cdot 10^{-9}\text{ N}$, $F_{g,\,2} = 2.27\cdot 10^{-48}\text{ N}$, $F_3 = 1.12\cdot 10^{-9}\text{ N}$, $F_{g,\,3} = 4.49\cdot 10^{-49}\text{ N}$, $F_4 = 3.22\cdot 10^{-10}\text{ N}$, and $F_{g,\,4} = 1.42\cdot 10^{-49}\text{ N}$. In every case the gravitational force between the proton and the electron is almost forty orders of magnitude smaller than the electrostatic force between them.

DOUBLE-CHECK: As *n* increases, the distance between the proton and the electron increases. Since each force follows an inverse-square law with respect to the distance, the forces decrease as *n* increases

21.62. **THINK:** The net force on the orbiting electron is the centripetal force, F_C. This is due to the electrostatic force between the electron and the proton, F. The radius of the hydrogen atom is $r = 5.29\cdot 10^{-11}$ m. The charge of an electron is $q_e = -e = -1.602\cdot 10^{-19}$ C, and the charge of a proton is $q_p = e = 1.602\cdot 10^{-19}$ C.

Find the velocity v and the kinetic energy K of the electron orbital. The mass of an electron is $m_e = 9.109 \cdot 10^{-31}$ kg.

SKETCH:



RESEARCH: The centripetal force is $F_C = m_e v^2 / r$. The electrostatic force is $F = k|q_1||q_2| / r^2$. The kinetic energy is $K = mv^2 / 2$.

SIMPLIFY: Solve for v^2:

$$F_C = F \Rightarrow \frac{m_e v^2}{r} = \frac{k|q_1||q_2|}{r^2} = \frac{ke^2}{r^2} \Rightarrow v^2 = \frac{ke^2}{rm_e}$$

Find K: $K = \frac{1}{2} m_e v^2 = \frac{1}{2} m_e \left(\frac{ke^2}{rm_e} \right) = \frac{ke^2}{2r}.$

CALCULATE: $K = \dfrac{\left(8.99 \cdot 10^9 \text{ N m}^2 / \text{C}^2\right)(1.602 \cdot 10^{-19} \text{ C})^2}{2(5.29 \cdot 10^{-11} \text{ m})} = 2.1807 \cdot 10^{-18} \text{ J} = 13.6077 \text{ eV}$

ROUND: $K = 13.6$ eV

DOUBLE-CHECK: Because the electron has very little mass, it is capable of approaching speeds on the order of $0.01c$ or $0.1c$ (where c is the speed of light). For the same reason, its kinetic energy is small (on the order of a few electron volts, in the case of the hydrogen atom).

21.63. For the atom described in the previous question, the ratio of the gravitational force between the electron and proton to the electrostatic force is:

$$F_g / F = \frac{\frac{Gm_e m_p}{r^2}}{\frac{k|q_1||q_2|}{r^2}} = \frac{Gm_e m_p}{ke^2}$$

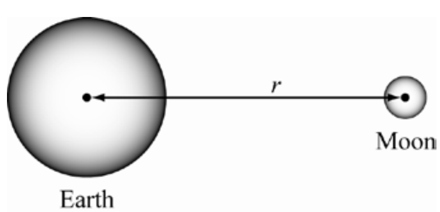
$$= \frac{\left(6.6742 \cdot 10^{-11} \text{m}^3 / (\text{kg s}^2)\right)(9.109 \cdot 10^{-31} \text{ kg})(1.673 \cdot 10^{-27} \text{ kg})}{\left(8.99 \cdot 10^9 \text{ N m}^2 / \text{C}^2\right)(1.602 \cdot 10^{-19} \text{ C})^2}$$

$$= 4.41 \cdot 10^{-40}$$

This value is independent of the radius; if this radius is doubled, the ratio does not change.

21.64. **THINK:** The Earth and the Moon each have a charge $q = -1.00 \cdot 10^6$ C. Their masses are $m_E = 5.97 \cdot 10^{24}$ kg and $m_M = 7.36 \cdot 10^{22}$ kg, respectively. The distance between them is $r = 384,403$ km, center-to-center. (a) Compare their electrostatic repulsion, F, with their gravitational attraction, F_g. (b) Discuss the effects of the electrostatic force on the size, shape and stability of the Moon's orbit around the Earth.

SKETCH:



RESEARCH: Treat each object as a point particle. The electrostatic force is $F = k|q_1||q_2|/r^2$, and the gravitational force is $F_g = GMm/r^2$.

SIMPLIFY:

(a) $F = kq^2/r^2$; $F_g = GM_E m_M / r^2$

(b) Not applicable.

CALCULATE:

(a) $$F = \frac{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)(-1.00\cdot 10^6 \text{ C})^2}{\left(3.84403\cdot 10^8 \text{ m}\right)^2} = 60839.6 \text{ N}$$

$$F_g = \frac{\left(6.6742\cdot 10^{-11} \text{ m}^3/\left(\text{kg s}^2\right)\right)\left(5.9742\cdot 10^{24} \text{ kg}\right)\left(7.36\cdot 10^{22} \text{ kg}\right)}{\left(3.84403\cdot 10^8 \text{ m}\right)^2} = 1.986\cdot 10^{20} \text{ N}$$

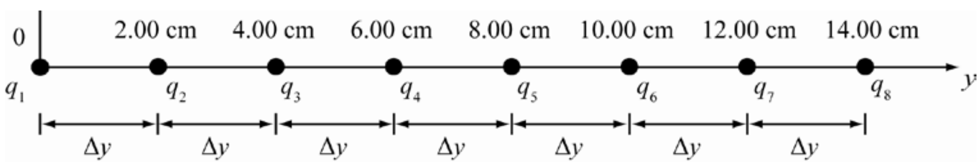
(b) The force of gravity is about 16 orders of magnitude greater than the electrostatic repulsion. The electrostatic force is an inverse-square central force. It therefore has no effect on the shape or stability of the Moon's orbit. It could only affect the size of the orbit, but given the orders of magnitude in difference between this and F_g, the effect is probably undetectable.

ROUND:

(a) $F = 6.08\cdot 10^4$ N and $F_g = 1.99\cdot 10^{20}$ N

DOUBLE-CHECK: F_g should greater than F, otherwise the Moon would not remain in the Earth's orbit.

21.65. Eight $+1.00$-μC charges are aligned on the y-axis with a distance $\Delta y = 2.00$ cm between each closest pair:



The force on the charge at $y = 4.00$ cm, q_3, is:

$$\vec{F}_{\text{tot},3} = \sum_{n=1,\, n\neq 3}^{8} \vec{F}_{n,3} = \vec{F}_{13} + \vec{F}_{23} + \vec{F}_{43} + \vec{F}_{53} + \vec{F}_{63} + \vec{F}_{73} + \vec{F}_{83} = (F_{13} + F_{23} - F_{43} - F_{53} - F_{63} - F_{73} - F_{83})\hat{y}$$

All terms have in common the factor $k|q_3|$. Then,

$$F_{\text{tot},3} = k|q_3|\left(\frac{|q_1|}{\left|\vec{y_1} - \vec{y_3}\right|^2} + \frac{|q_2|}{\left|\vec{y_2} - \vec{y_3}\right|^2} - \frac{|q_4|}{\left|\vec{y_4} - \vec{y_3}\right|^2} - \frac{|q_5|}{\left|\vec{y_5} - \vec{y_3}\right|^2} - \frac{|q_6|}{\left|\vec{y_6} - \vec{y_3}\right|^2} - \frac{|q_7|}{\left|\vec{y_7} - \vec{y_3}\right|^2} - \frac{|q_8|}{\left|\vec{y_8} - \vec{y_3}\right|^2}\right)$$

Since $q_1 = q_2 = \ldots = q_8 = q$,

$$F_{\text{tot},3} = kq^2\left(\frac{1}{(2\Delta y)^2} + \frac{1}{(\Delta y)^2} - \frac{1}{(\Delta y)^2} - \frac{1}{(2\Delta y)^2} - \frac{1}{(3\Delta y)^2} - \frac{1}{(4\Delta y)^2} - \frac{1}{(5\Delta y)^2}\right)$$

$$\vec{F}_{\text{tot},3} = \frac{kq^2}{(\Delta y)^2}\left(\frac{1}{2^2} + 1 - 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \frac{1}{5^2}\right)\hat{y}$$

$$= \frac{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(1.00\cdot 10^{-6} \text{ C}\right)^2}{\left(0.0200 \text{ m}\right)^2}\left(-\frac{769}{3600}\right)\hat{y}$$

$$= -4.80 \text{ N } \hat{y}$$

21.66. The distance between the electron (charge $q_e = -e$) and the proton (charge $q_p = e$) is $r = 5.29 \cdot 10^{-11}$ m. The net force on the electron is the centripetal force, $F_c = m_e a_c = m_e v^2 / r$. This is due to the Coulomb force, $F = k|q_1||q_2|/r^2$. That is, $F_c = F \Rightarrow m_e v^2 / r = k|q_1||q_2|/r^2$. The speed of the electron is:

$$m_e v^2 = \frac{ke^2}{r} \Rightarrow v = \sqrt{\frac{ke^2}{m_e r}} = \sqrt{\frac{\left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)(1.602 \cdot 10^{-19} \text{ C})^2}{(9.109 \cdot 10^{-31} \text{ kg})(5.29 \cdot 10^{-11} \text{ m})}} = 2.18816 \cdot 10^6 \text{ m/s} \approx 2.19 \cdot 10^6 \text{ m/s.}$$

21.67. The radius of the nucleus of ^{14}C is $r_0 = 1.505$ fm. The nucleus has charge $q_0 = +6e$.
(a) A proton (charge $q = e$) is placed $d = 3.00$ fm from the surface of the nucleus. Treating the nucleus as a point charge, the distance between the proton and the charge of the nucleus is $r = d + r_0$. The force is repulsive due to the like charges. The magnitude of this force is

$$F = \frac{k|q||q_0|}{r^2} = \frac{k6e^2}{(d+r_0)^2} = \frac{\left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)6(1.602 \cdot 10^{-19} \text{ C})^2}{\left(3.00 \cdot 10^{-15} \text{ m} + 1.505 \cdot 10^{-15} \text{ m}\right)^2} = 68.2097 \text{ N} \approx 68.2 \text{ N}$$

(b) The proton's acceleration is:

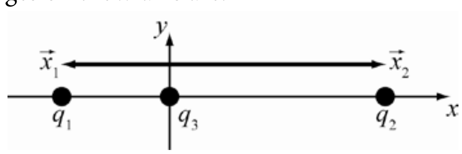
$$F = m_p a \Rightarrow a = \frac{F}{m_p} = \frac{68.210 \text{ N}}{1.673 \cdot 10^{-27} \text{ kg}} = 4.077 \cdot 10^{28} \text{ m/s}^2 \approx 4.08 \cdot 10^{28} \text{ m/s}^2$$

21.68. The original force is $F = k|q_1||q_2|/r^2 = 0.100$ N. Now q_1 becomes $(1/2)q_1$, while r becomes $2r$. The new force is:

$$F' = \frac{k\left|\frac{1}{2}q_1\right||q_2|}{(2r)^2} = \frac{1}{8}\frac{k|q_1||q_2|}{r^2} = \frac{1}{8}F = \frac{1}{8}(0.100 \text{ N}) = 0.0125 \text{ N}$$

21.69. The charge and position of three point charges on the x-axis are:

$q_1 = +19.0$ μC; $\vec{x}_1 = -10.0$ cm
$q_2 = -57.0$ μC; $\vec{x}_2 = +20.0$ cm
$q_3 = -3.80$ μC; $\vec{x}_3 = 0$



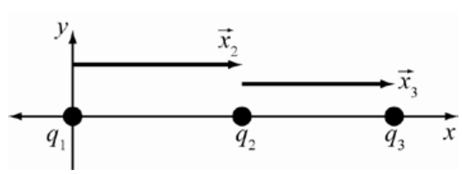
The magnitude of the total electrostatic force on q_3 is:

$$F_{\text{tot, 3}} = \left|\vec{F}_{13} + \vec{F}_{23}\right| = \left|-F_{13} - F_{23}\right| = \left(F_{13} + F_{23}\right) = \frac{k|q_3||q_1|}{\left|\vec{x}_1 - \vec{x}_3\right|^2} + \frac{k|q_3||q_2|}{\left|\vec{x}_2 - \vec{x}_3\right|^2} = k|q_3|\left(\frac{|q_1|}{(x_1)^2} + \frac{|q_2|}{(x_2)^2}\right)$$

$$= \left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(|-3.80 \text{ μC}|\right)\left(\frac{|19.0 \text{ μC}|}{(0.100 \text{ m})^2} + \frac{|-57.0 \text{ μC}|}{(0.200 \text{ m})^2}\right) \text{ 113.59 N} \quad 114 \text{ N} \approx$$

21.70. The charge and position of three point charges on the x-axis are:

$q_1 = +64.0$ μC; $\vec{x}_1 = 0.00$ cm
$q_2 = +80.0$ μC; $\vec{x}_2 = 25.0$ cm
$q_3 = -160.0$ μC; $\vec{x}_3 = 50.0$ cm



The magnitude of the total electrostatic force on q_1 is:

$$F_{\text{tot},1} = \left|\vec{F}_{21} + \vec{F}_{31}\right| = \left|-F_{21} + F_{31}\right| = \left|\frac{-k|q_2||q_1|}{\left|\vec{x}_2 - \vec{x}_1\right|^2} + \frac{k|q_3||q_1|}{\left|\vec{x}_3 - \vec{x}_1\right|^2}\right| = k|q_1|\left|\frac{|q_3|}{(x_3)^2} - \frac{|q_2|}{(x_2)^2}\right|$$

$$= \left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(|64.0 \text{ μC}|\right)\left(\left|\frac{160.0 \text{ μC}}{(0.500 \text{ m})^2} - \frac{80.0 \text{ μC}}{(0.250 \text{ m})^2}\right|\right) = 368.23 \text{ N} \approx 368 \text{ N}$$

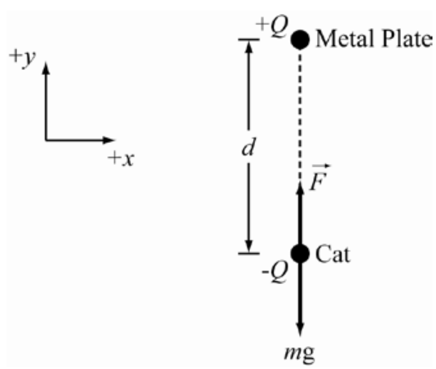
21.71. The charge of the Earth is $Q = -6.8\cdot 10^5$ C. The mass of the object is $m = 1.0$ g. For this object to be levitated near the Earth's surface ($r_\text{E} = 6378$ km), the Coulomb force and the force of gravity must be the same. The charge q of the object can be found from balancing these forces:

$$F_\text{g} = F_\text{Coulomb} \;\Rightarrow\; mg = \frac{k|Qq|}{r_\text{E}^{\,2}} \;\Rightarrow\; |q| = \frac{mgr_\text{E}^{\,2}}{k|Q|}$$

$$|q| = \frac{\left(0.0010 \text{ kg}\right)(9.81 \text{ m/s}^2)(6.378\cdot 10^6 \text{ m})^2}{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left|-6.8\cdot 10^5 \text{ C}\right|} = 6.5278\cdot 10^{-5} \text{ C} \approx 65 \text{ μC}$$

Since Q is negative, and the object is levitated by the repulsion of like charges, it must be that $q \approx -65$ μC .

21.72. The mass of the cat is 7.00 kg. The distance between the cat and the metal plate is 2.00 m. The cat is suspended due to attractive electric force between the cat and the metal plate.



The attractive force between the cat and the metal plate is $F = kQQ/d^2$. Since the cat is suspended in the air, this means that $F = mg$. Therefore $mg = kQ^2/d^2$. Solving for Q gives $Q = \sqrt{mgd^2/k} = d\sqrt{mg/k}$. Substituting $m = 7.00$ kg , $g = 9.81 \text{ m/s}^2$, $k = 8.99\cdot 10^9 \text{ N m}^2/C^2$ and $d = 2.00$ m yields

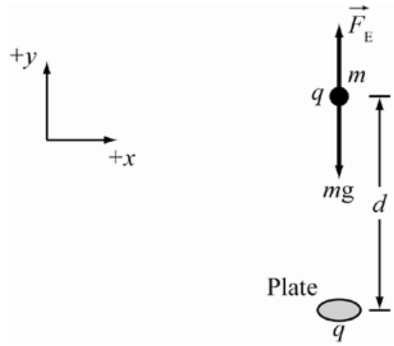
$$Q = 2.00 \text{ m}\sqrt{\frac{7.00 \text{ kg}\cdot 9.81 \text{ m/s}^2}{8.99\cdot 10^9 \text{ N m}^2/\text{C}^2}} = 1.748\cdot 10^{-4}\text{C}.$$

The number of electrons that must be extracted is

$$N = \frac{Q}{q_\text{e}} = \frac{1.748\cdot 10^{-4} \text{ C}}{1.602\cdot 10^{-19} \text{ C}} = 1.09\cdot 10^{15} \text{ electrons.}$$

21.73. **THINK:** A 10.0 g mass is suspended 5.00 cm above a non-conducting flat plate. The mass and the plate have the same charge q. The gravitational force on the mass is balanced by the electrostatic force.

SKETCH:



RESEARCH: The electrostatic force on the mass m is $F_E = kq^2/d^2$. This force is balanced by the gravitational force $F_g = mg$. Therefore, $F_E = F_g$ or $kq^2/d^2 = mg$.

SIMPLIFY: The charge on the mass m that satisfies the balanced condition is $q = d\sqrt{mg/k}$.

CALCULATE: Putting in the numerical values gives:

$$q = 0.0500 \text{ m}\sqrt{\frac{\left(10.0\cdot 10^{-3}\text{ kg}\right)\left(9.81\text{ m/s}^2\right)}{8.99\cdot 10^{9}\text{ N m}^2/\text{C}^2}} = 1.6517\cdot 10^{-7}\text{C}.$$

The number of electrons on the mass m is:

$$N = \frac{q}{e} = \frac{1.6517\cdot 10^{-7}\text{C}}{1.602\cdot 10^{-19}\text{C / electron}} = 1.0310\cdot 10^{12}\text{ electrons.}$$

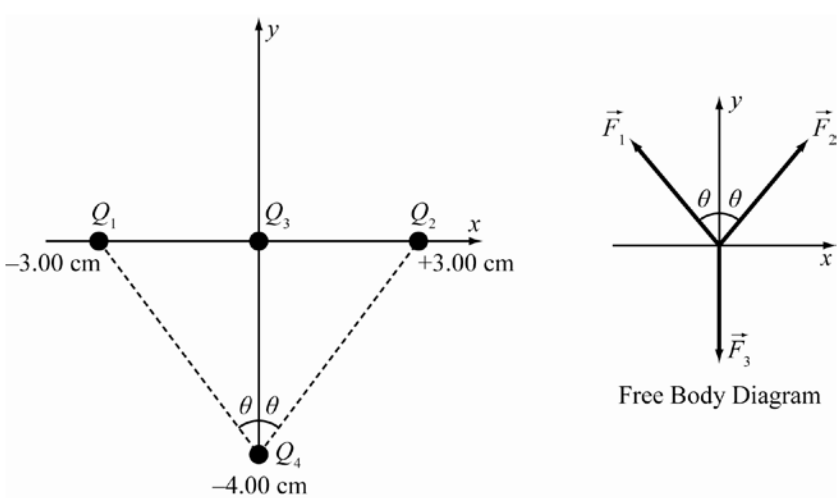
The additional mass of electrons is $\Delta m = \left(1.0310\cdot 10^{12}\right)\left(9.11\cdot 10^{-31}\text{ kg}\right) = 9.39263\cdot 10^{-19}$ kg.

ROUND: Rounding to three significant figures gives $q = 1.65\cdot 10^{-7} e$, and $\Delta m = 9.39\cdot 10^{-19}$ kg.

DOUBLE-CHECK: It is expected that Δm is negligible since the mass of electron is very small.

21.74. **THINK:** This problem involves superposition of forces. Since there are three forces on Q_4, the net force is the vector sum of three forces.

SKETCH:



RESEARCH: The magnitude of the forces between two charges, q_1 and q_2, is $F = kq_1q_2 / r^2$. The forces on Q_4 are

$$\vec{F}_1 = k\frac{Q_1Q_4}{r_{14}^2}\left(-\sin\theta\hat{x} + \cos\theta\hat{y}\right),\quad \vec{F}_2 = k\frac{Q_2Q_4}{r_{24}^2}\left(\sin\theta\hat{x} + \cos\theta\hat{y}\right),\quad \text{and}\quad \vec{F}_3 = k\frac{Q_3Q_4}{r_{34}^2}\left(-\hat{y}\right).$$

SIMPLIFY: By symmetry, the horizontal components of F_1 and F_2 cancel, and F_3 has no horizontal component. The net force is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = kQ_4\left[\left(\frac{Q_1}{r_{14}^2} + \frac{Q_2}{r_{24}^2}\right)\cos\theta - \frac{Q_3}{r_{34}^2}\right]\hat{y}.$$

Since $Q_1 = Q_2$ and $r_{14} = r_{24}$, the above equation simplifies to

$$\vec{F} = kQ_4\left[\frac{2Q_1\cos\theta}{r_{14}^2} - \frac{Q_3}{r_{34}^2}\right]\hat{y}.$$

CALCULATE: The distance r_{14} and r_{34} are $r_{14} = \sqrt{(3\text{ cm})^2 + (4\text{ cm})^2} = 5$ cm; $r_{34} = 4$ cm. Therefore $\cos\theta = 4/5$. Substituting the numerical values yields:

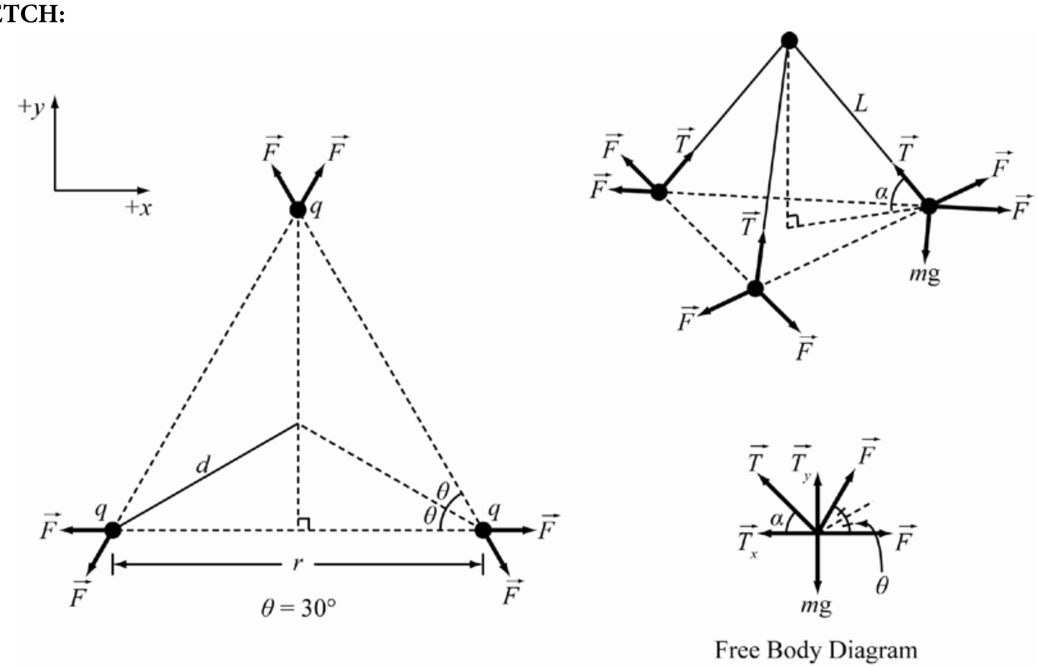
$$F = \left(8.99\cdot 10^9\text{ N m}^2/\text{C}^2\right)\left(2\cdot 10^{-3}\text{ C}\right)\left[\left(\frac{2\cdot 1\cdot 10^{-3}\text{ C}}{\left(5\cdot 10^{-2}\text{ m}\right)^2}\right)\left(\frac{4}{5}\right) - \frac{1.024\cdot 10^{-3}\text{ C}}{\left(4\cdot 10^{-2}\text{ m}\right)^2}\right] = 0\text{ N}.$$

ROUND: Not needed

DOUBLE-CHECK: It is clear from the symmetry of the problem that this is a reasonable outcome.

21.75. **THINK:** Three 5.00-g Styrofoam balls of radius 2.00 cm are tied to 1.00 m long threads and suspended freely from a common point. The charge of each ball is q and the balls form an equilateral triangle with sides of 25.0 cm.

SKETCH:



RESEARCH: The magnitude of the force between two charges, q_1 and q_2, is $F_{12} = kq_1q_2 / r^2$. The magnitude of F in the above figure is $F = kq^2 / r^2$. Using Newton's Second Law, it is found that $T_y = T\sin\alpha = mg$ and $T_x = T\cos\alpha = 2F\cos\theta$.

SIMPLIFY: Eliminating T in the above equations yields $\tan\alpha = mg/(2F\cos\theta)$. Rearranging gives, $F = mg/(2\tan\alpha\cos\theta) = kq^2/r^2$. Therefore, the charge q is

$$q = \sqrt{\frac{mgr^2}{2k\tan\alpha\cos\theta}}.$$

From the sketch, it is clear that the distance of the ball to the center of the triangle is $d = r/(2\cos\theta)$. Therefore $\tan\theta = \sqrt{L^2 - d^2}/d$.

CALCULATE: Substituting the numerical values, $r = 0.250$ m, $m = 5.00\cdot 10^{-3}$ kg, $g = 9.81$ m/s^2, $L = 1.00$ m and $\theta = 30°$ (exact) gives

$$d = \frac{0.250 \text{ m}}{2\cos(30°)} = 0.1443 \text{ m}$$

$$\tan\alpha = \frac{\sqrt{(1.00 \text{ m})^2 - (0.1443 \text{ m})^2}}{0.1443 \text{ m}} = 6.856$$

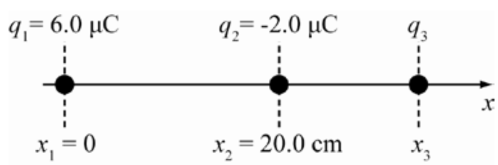
$$q = \sqrt{\frac{5.00\cdot 10^{-3} \text{ kg}(9.81 \text{ m/s}^2)(0.250 \text{ m})^2}{2\cdot(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2)6.856\cos(30°)}} = 1.69463\cdot 10^{-7} \text{ C}$$

ROUND: $q = 0.169$ μC

DOUBLE-CHECK: This charge is approximately 11 orders of magnitude larger than the elementary charge *e*. The charge required to deflect 5.00 g balls by a distance of 25.0 cm would need to be fairly large.

21.76. **THINK:** Two point charges lie on the *x*-axis. A third point charge needs to be placed on the *x*-axis such that it is in equilibrium. This means that the net force on the third charge due to the other charges is zero.

SKETCH:



RESEARCH: In order for the third charge to be in equilibrium, the force on it due to q_1, $\vec{F}_1$, must be equal in magnitude and opposite in direction to $\vec{F}_2$ the force due to q_2. Note that the sign of the third charge is irrelevant, so I can arbitrarily assume it is positive. Since $|q_1| > |q_2|$, the third charge must be closer to q_2 than to q_1. Also, since q_1 and q_2 are oppositely charged, the forces on a particle between them will be in the same direction and hence cannot cancel. The third charge must be in the region $x > 20.0$ cm. The net force on q_3 is $F_{\text{net}} = \dfrac{k|q_1||q_3|}{x_3^2} - \dfrac{k|q_2||q_3|}{(x_3 - x_2)^2}$.

SIMPLIFY: Solving $F_{\text{net}} = 0$ for x_3 yields $|q_2|x_3^2 = |q_1|(x_3 - x_2)^2$ or $\sqrt{|q_2|}(x_3 - x_2)$. Therefore the position of q_3 is $x_3 = \dfrac{\sqrt{q_1}x_2}{\sqrt{|q_1|} - \sqrt{|q_2|}}$.

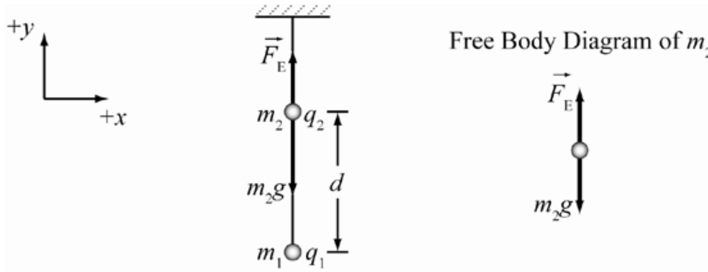
CALCULATE: Putting in the numerical values yields $x_3 = \dfrac{(\sqrt{6.0 \text{ μC}})(20.0 \text{ cm})}{\sqrt{6.0 \text{ μC}} - \sqrt{2.0 \text{ μC}}} = 47.32$ cm.

ROUND: Using only two significant digits, the position x_3 is $x_3 = 47$ cm

DOUBLE-CHECK: This is correct since $x_3 > x_2$.

21.77. **THINK:** In this problem, a gravitational force on an object is balanced by an electrostatic force on the object.

SKETCH:



RESEARCH: The electric force on q_2 is given by $F_E = kq_1q_2/d^2$. The gravitational force on m_2 is $F_g = m_2 g$.

SIMPLIFY: $\frac{kq_1q_2}{d^2} \rightarrow \frac{kq^2}{d^2} = m_2 g \Rightarrow m_2 = \frac{kq^2}{gd^2}.$

CALCULATE: Substituting the numerical values, $q_1 = q_2 = +2.67$ μe, $d = 0.360$ m produces

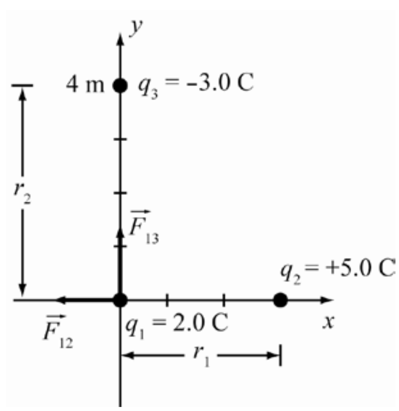
$$m_2 = \frac{\left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(2.67 \cdot 10^{-6} \text{ C}\right)^2}{\left(9.81 \text{ m/s}^2\right)\left(0.360 \text{ m}\right)^2} = 0.05041 \text{ kg}.$$

ROUND: Keeping only three significant digits gives $m_2 = 50.4$ g.

DOUBLE-CHECK: This makes sense since F_E is small.

21.78. **THINK:** Because this is a two-dimensional problem, the directions of forces are important for determining a net force.

SKETCH:



RESEARCH: The magnitude of the force between two charges is $F = k|q_1||q_2|/r^2$. The net force on q_1 is $\vec{F}_{\text{net}} = \vec{F}_{12} + \vec{F}_{13} = -\frac{k|q_1||q_2|}{r_1^2}\hat{x} - \frac{k|q_1||q_3|}{r_2^2}\hat{y}$. The direction of the net force is $\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right)$.

SIMPLIFY: Not needed

CALCULATE: $\vec{F}_{\text{net}} = \frac{-\left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)(2.00 \text{ C})(5.00 \text{ C})}{(3.00 \text{ m})^2}\hat{x} + \frac{\left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)(2.00 \text{ C})(3.00 \text{ C})}{(4.00 \text{ m})^2}\hat{y}$

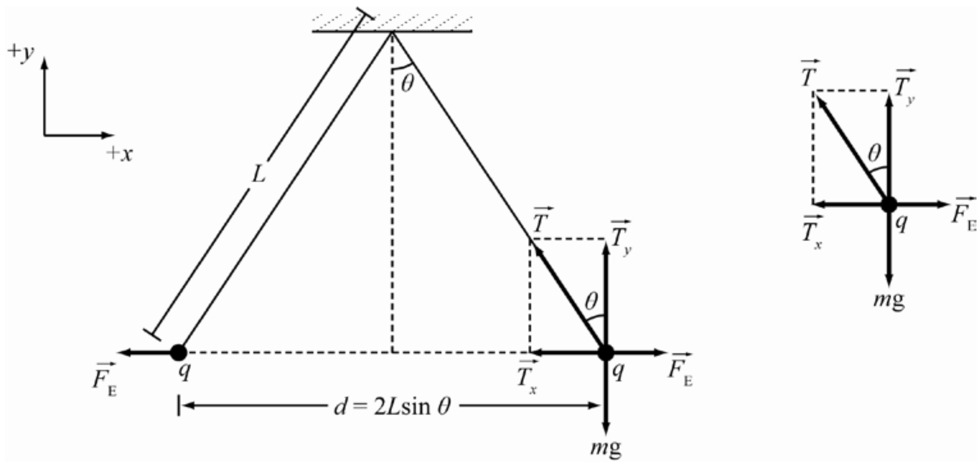
$$= \left(-9.998 \cdot 10^9 \text{ N}\right)\hat{x} + \left(3.371 \cdot 10^9 \text{ N}\right)\hat{y}$$

The magnitude of $\vec{F}_{\text{net}}$ is $\left|F_{\text{net}}\right| = \sqrt{9.99^2 + 3.37^2} \cdot 10^9 \text{ N} = 10.551 \cdot 10^9 \text{ N}$. The direction of $\vec{F}_{\text{net}}$ is $\theta = \tan^{-1}\left(\dfrac{3.37 \cdot 10^9 \text{ N}}{-9.99 \cdot 10^9 \text{ N}}\right) = 161.36°$ with respect to the positive x-axis, or 18.64° above the negative x-axis (the net force points up and to the left, in quadrant II).

ROUND: Keeping only three significant digits yields $F_{\text{net}} = \left(-1.00 \cdot 10^8 \text{ N}\right)\hat{x} + \left(3.4 \cdot 10^9 \text{ N}\right)\hat{y}$ and $\left|F_{\text{net}}\right| = 10.6 \cdot 10^9 \text{ N}$ at 18.6° above the negative x-axis.

21.79. **THINK:** To solve this problem, the force due to the charges and the tension in the string must balance the gravitational force on the spheres.

SKETCH:



RESEARCH: The force due to electrostatic repulsion of the two spheres is $F_{\text{E}} = kq_1q_2 / d^2 = kq^2 / d^2$. Applying Newton's Second Law yields (I) $T_x = T\sin\theta = F_{\text{E}}$ and (II) $T_y = T\cos\theta = mg$. $L = 0.45 \text{ m}$, $m = 2.33 \cdot 10^{-3} \text{ kg}$, $\theta = 10.0°$.

SIMPLIFY: Dividing (I) by (II) gives $\tan\theta = F_{\text{E}} / \left(mg\right) = kq^2 / \left(d^2 mg\right)$. After simple manipulation, it is found that the charge on each sphere is $q = \sqrt{d^2 mg \tan\theta / k} = 2L\sin\theta\sqrt{mg\tan\theta / k}$ using $d = 2L\sin\theta$.

CALCULATE: Substituting the numerical values, it is found that

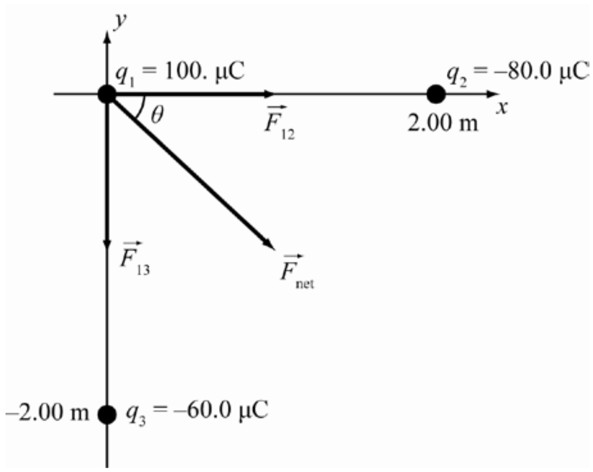
$$q = (2)(0.450 \text{ m})(\sin 10.0°)\sqrt{\frac{2.33 \cdot 10^{-3} \text{ kg}\left(9.81 \text{ m/s}^2\right)\tan\left(10.0°\right)}{8.99 \cdot 10^9 \text{ N m}^2 / \text{C}^2}} = 1.0464 \cdot 10^{-7} \text{ C}.$$

ROUND: Keeping only three significant digits gives $q = 0.105\ \mu\text{C}$.

DOUBLE-CHECK: This is reasonable. The relatively small spheres and small distance will mean the charge is small.

21.80. **THINK:** I want to find the magnitude and direction of the net force on a point charge q_1 due to point charges q_2 and q_3. The charges q_1, q_2, and q_3 are located at $(0,0)$, $(2.0,0.0)$, and $(0,-2.00)$, respectively.

SKETCH:



RESEARCH: The magnitude of the force between two charges is $F = k|q_1||q_2|/r^2$. The net force on q_1 is $\vec{F}_{\text{net}} = \vec{F}_{12} + \vec{F}_{13} = -\dfrac{k|q_1||q_2|}{r_1^2}\hat{x} - \dfrac{k|q_1||q_3|}{r_2^2}\hat{y}.$

SIMPLIFY: Not needed

CALCULATE: Putting in the numerical values yields

$$\vec{F}_{\text{net}} = \frac{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)(100.\cdot 10^{-9}\text{ C})(80.0\cdot 10^{-9}\text{ C})}{\left(2.00\text{ m}\right)^2}\hat{x} - \frac{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)(100.\cdot 10^{-9}\text{ C})(60.0\cdot 10^{-9}\text{ C})}{\left(2.00\text{ m}\right)^2}\hat{y}$$

$$= 1.798\cdot 10^{-5}\text{ N}\hat{x} - 1.348\cdot 10^{-5}\text{ N}\hat{y}$$

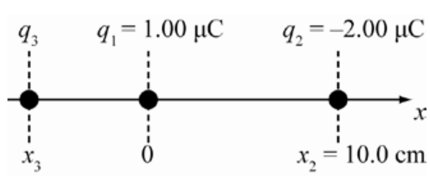
The magnitude of $\vec{F}_{\text{net}}$ is $\left|\vec{F}_{\text{net}}\right| = \sqrt{1.798^2 + 1.348^2}\cdot 10^{-5}\text{ N} = 2.247\cdot 10^{-5}\text{ N}$. The direction of $\vec{F}_{\text{net}}$ is $\theta = \tan^{-1}\left(\dfrac{1.348}{1.798}\right) = -36.860°.$

ROUND: Rounding to three significant digits, it is found that $\left|\vec{F}_{\text{net}}\right| = 2.25\cdot 10^{-5}\text{ N}$ and $\theta = 36.9°$ below the horizontal.

DOUBLE-CHECK: Since both forces acting on q_1 are attractive, it is expected that the direction of the net force would be between the two contributing force vectors.

21.81. **THINK:** If it is assumed that the third charge is positive, then the third charge experiences a repulsive force with q_1 and an attractive force with q_2.

SKETCH:



RESEARCH: Because $|q_1| > |q_2|$ and the force between q_1 and q_3 is attractive, the possible region where q_3 can experience zero net force is in the region $x < 0$. The net force on q_3 is $F_{\text{net}} = -\dfrac{k|q_1||q_3|}{\left(0 - x_3\right)^2} + \dfrac{k|q_2||q_3|}{\left(x_2 - x_3\right)^2}.$

SIMPLIFY: Solving $F_{\text{net}} = 0$ for x_3 yields $x_3^{\,2}|q_2| = |q_1|(x_2 - x_3)^2$ implies:

$$\text{(I)} \quad x_3\sqrt{|q_2|} = \sqrt{|q_1|}(x_2 - x_3) \ \text{ or } \ \text{(II)} \ -x_3\sqrt{|q_2|} = \sqrt{|q_1|}(x_2 - x_3)$$

Equation (I) gives $x_3 > 0$ and equation (II) gives $x_3 < 0$. Therefore the correct solution is the solution of Equation (II). Solving (II) yields $x_3 = \dfrac{-\sqrt{|q_1|}x_2}{\sqrt{|q_2|} - \sqrt{|q_1|}}$.

CALCULATE: Substituting $q_1 = 1.00\ \mu\text{C}$, $q_2 = -2.00\ \mu\text{C}$ and $x_2 = 10.0$ cm into above equation gives

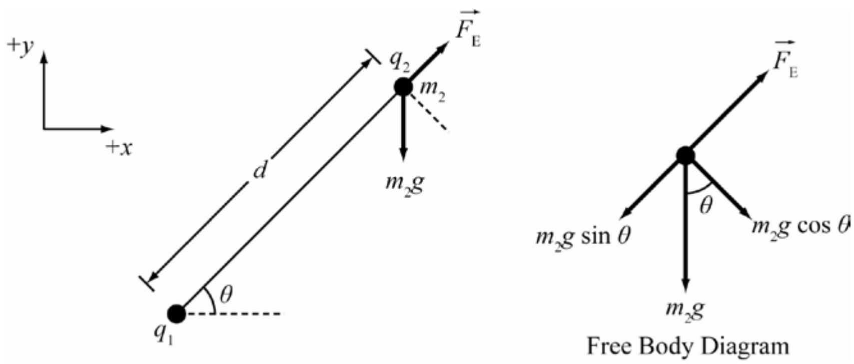
$$x_3 = \frac{-\sqrt{1.00\ \mu\text{C}} \cdot 10.0\ \text{cm}}{\sqrt{2.00\ \mu\text{C}} - \sqrt{1.00\ \mu\text{C}}} = -24.142\ \text{cm}$$

ROUND: To three significant figures: $x_3 = -24.1$ cm.

DOUBLE-CHECK: The negative value of x indicates that q_3 is located in the region $x < 0$, as expected.

21.82. **THINK:** The electrostatic force on a bead is balanced by its gravitational weight.

SKETCH:



RESEARCH: The repulsive force between two charged beads is $F_E = k\dfrac{q_1q_2}{d^2}$. Using Newton's Second Law,

$$F_E = k\frac{q_1q_2}{d^2} = m_2 g \sin\theta .$$

SIMPLIFY: Therefore the distance d is $d = \sqrt{\dfrac{kq_1q_2}{m_2 g \sin\theta}}$.

CALCULATE: Substituting the numerical values into the above equation gives

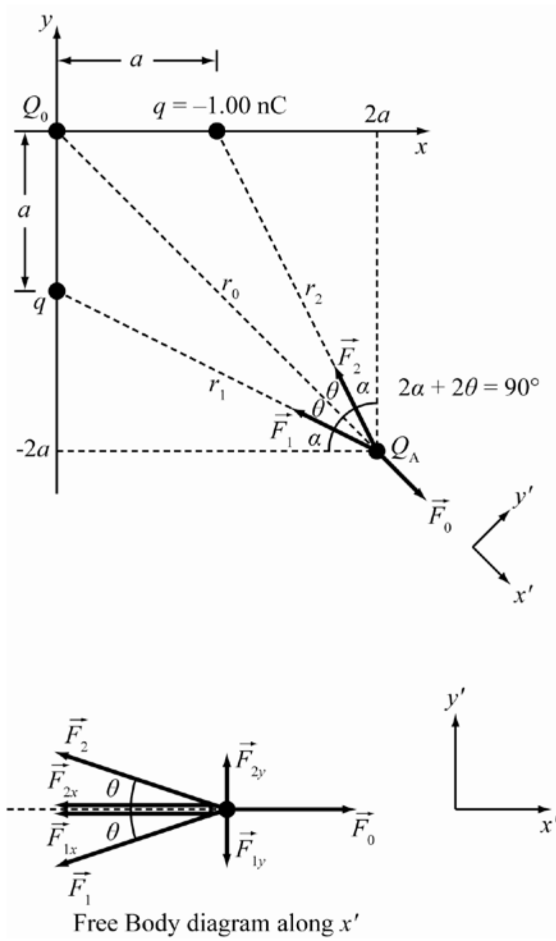
$$d = \sqrt{\frac{\left(8.99 \cdot 10^9\ \text{N m}^2/\text{C}^2\right)(1.27 \cdot 10^{-6}\ \text{C})(6.79 \cdot 10^{-6}\ \text{C})}{3.77 \cdot 10^{-3}\ \text{kg}\left(9.81\ \text{m/s}^2\right)\sin(51.3°)}} = 1.638\ \text{m}.$$

ROUND: Keeping only three significant digits gives $d = 1.64$ m.

DOUBLE-CHECK: The beads are very light, so a small charge is sufficient to cause a relatively large separation.

21.83. **THINK:** Since this is a two dimensional problem, electrostatic forces are added as vectors. It is assumed that Q_A is a positive charge.

SKETCH:



RESEARCH: To balance the forces F_1 and F_2, the charge on Q_0 must be positive. The electrostatic forces on Q_A are $F_1 = \frac{k|q|Q_A}{r_1^2}$, $F_2 = \frac{k|q|Q_A}{r_2^2}$, and $F_0 = \frac{kQ_0Q_A}{r_0^2}$. Applying Newton's Second Law, it is found that $F_0 = F_{1x} + F_{2x}$ or $kQ_0Q_A / r_0^2 = F_1\cos\theta + F_2\cos\theta$. Using $r_1 = r_2$ this becomes $\frac{kQ_0Q_A}{r_0^2} = \frac{k|q|Q_A}{r_1^2}2\cos\theta.$

SIMPLIFY: Solving the above equation for Q_0 gives the charge Q_0, $Q_0 = (r_0 / r_1)^2|q|2\cos\theta.$ From the above figure, it is noted that $r_0 = \sqrt{(2a)^2 + (2a)^2} = 2a\sqrt{2}$, $r_1 = \sqrt{(2a)^2 + a^2} = a\sqrt{5}$, and

$$\cos\theta = \cos(45° - \alpha) = \cos 45°\cos\alpha + \sin 45°\sin\alpha \Rightarrow \cos\theta = \frac{\sqrt{2}}{2}\frac{2a}{a\sqrt{5}} + \frac{\sqrt{2}}{2}\frac{a}{a\sqrt{5}} = \frac{3}{2}\sqrt{\frac{2}{5}} = \frac{3}{10}\sqrt{10}.$$

Therefore the magnitude of charge Q_0 is $|Q_0| = 2|q|\frac{8a^2}{5a^2}\frac{3}{10}\sqrt{10} = \frac{48}{50}\sqrt{10}|q|.$

CALCULATE: Substituting $q = -1.00$ nC yields $|Q_0| = \frac{48}{50}\sqrt{10}\cdot|-1.00\text{ nC}| = 3.036$ nC.

ROUND: Rounding to three significant figures gives $|Q_0| = 3.04$ nC.

DOUBLE-CHECK: Since r_0 is larger than r_1, it is expected that Q_0 is larger than $2|q| = 2$ nC.

Multi-Version Exercises

Exercises 21.84–21.86 The components of the forces in the x-direction give us $T\sin\theta = \dfrac{kq^2}{d^2}$.

The components of the forces in the y-direction give us $T\cos\theta = mg$.

We can divide these two equations to get

$$\tan\theta = \frac{kq^2}{mgd^2} \;\Rightarrow\; d = \sqrt{\frac{kq^2}{mg\tan\theta}}.$$

From the figure we can see that

$$\sin\theta = \frac{d/2}{\ell} \;\Rightarrow\; \ell = \frac{d}{2\sin\theta}.$$

Combining these two equations gives us

$$\ell = \frac{d}{2\sin\theta} = \frac{\sqrt{\dfrac{kq^2}{mg\tan\theta}}}{2\sin\theta} = \sqrt{\frac{kq^2}{4mg\sin^2\theta\tan\theta}}.$$

21.84. $$\ell = \sqrt{\frac{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(29.59\cdot 10^{-6}\text{ C}\right)^2}{4\left(0.9860 \text{ kg}\right)\left(9.81 \text{m/s}^2\right)\sin^2 29.79°\tan 29.79°}} = 1.211 \text{ m}.$$

21.85. $$\ell = \sqrt{\frac{kq^2}{4mg\sin^2\theta\tan\theta}}$$

$$\ell^2 = \frac{kq^2}{4mg\sin^2\theta\tan\theta}$$

$$m = \frac{kq^2}{4\ell^2 g\sin^2\theta\tan\theta} = \frac{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(15.71\cdot 10^{-6}\text{ C}\right)^2}{4\left(1.223 \text{ m}\right)^2\left(9.81 \text{m/s}^2\right)\sin^2 21.07°\tan 21.07°} = 0.7592 \text{ kg}.$$

21.86. $$\ell = \sqrt{\frac{kq^2}{4mg\sin^2\theta\tan\theta}}$$

$$\ell^2 = \frac{kq^2}{4mg\sin^2\theta\tan\theta}$$

$$q = \sqrt{\frac{4\ell^2 mg\sin^2\theta\tan\theta}{k}} = \sqrt{\frac{4\left(1.235 \text{ m}\right)^2\left(0.9935 \text{ kg}\right)\left(9.81 \text{m/s}^2\right)\sin^2 22.35°\tan 22.35°}{8.99\cdot 10^9 \text{ N m}^2/\text{C}^2}}$$

$$q = 1.989\cdot 10^{-5} \text{ C} = 19.89\ \mu\text{C}.$$

Exercises 21.87–21.89

$$F_{\text{net}} = F_{13} - F_{23} = \frac{kq_1q_3}{\left(x_3 - x_1\right)^2} - \frac{kq_2q_3}{\left(x_2 - x_3\right)^2} = 0$$

$$\left(x_3 - x_1\right)^2 q_2 = \left(x_2 - x_3\right)^2 q_1$$

$$\left(x_3 - x_1\right)\sqrt{q_2} = \pm\left(x_2 - x_3\right)\sqrt{q_1}.$$

We choose the + sign since we know that the force can only balance when $x_1 < x_3 < x_2$.

So we can write

$$x_3 = \frac{\sqrt{q_1}x_2 + \sqrt{q_2}x_1}{\sqrt{q_1} + \sqrt{q_2}}.$$

21.87. $$x_3 = \frac{\sqrt{3.979 \cdot 10^{-6}\text{ C}}(14.13\text{ m}) + \sqrt{8.669 \cdot 10^{-6}\text{ C}}(-5.689\text{ m})}{\sqrt{3.979 \cdot 10^{-6}\text{ C}} + \sqrt{8.669 \cdot 10^{-6}\text{ C}}} = 2.315\text{ m}$$

21.88. $$x_1 = \frac{x_3\left(\sqrt{q_1} + \sqrt{q_2}\right) - \sqrt{q_1}x_2}{\sqrt{q_2}}$$

$$x_1 = \frac{(2.358\text{ m})\left(\sqrt{4.325 \cdot 10^{-6}\text{ C}} + \sqrt{7.757 \cdot 10^{-6}\text{ C}}\right) - \sqrt{4.325 \cdot 10^{-6}\text{ C}}(14.33\text{ m})}{\sqrt{7.757 \cdot 10^{-6}\text{ C}}}$$
$$= -6.581\text{ m}$$

21.89. $$x_3 = \frac{\sqrt{q_1}x_2 + \sqrt{q_2}x_1}{\sqrt{q_1} + \sqrt{q_2}}$$

$$x_2 = \frac{x_3\left(\sqrt{q_1} + \sqrt{q_2}\right) - \sqrt{q_2}x_1}{\sqrt{q_1}}$$
$$= \frac{(4.625\text{ m})\left(\sqrt{4.671 \cdot 10^{-6}\text{ C}} + \sqrt{6.845 \cdot 10^{-6}\text{ C}}\right) - \sqrt{6.845 \cdot 10^{-6}\text{ C}}(-3.573\text{ m})}{\sqrt{4.671 \cdot 10^{-6}\text{ C}}}$$
$$= 14.55\text{ m}$$

Chapter 22: Electric Fields and Gauss's Law

Concept Checks

22.1. b **22.2.** b **22.3.** a **22.4.** c **22.5.** c **22.6.** e **22.7.** c **22.8.** c **22.9.** e **22.10.** e **22.11.** a **22.12.** a **22.13.** c **22.14.** d

Multiple-Choice Questions

22.1. e **22.2.** d **22.3.** a **22.4.** a **22.5.** d **22.6.** c **22.7.** c **22.8.** c **22.9.** a **22.10.** a & d **22.11.** a **22.12.** a, d and e

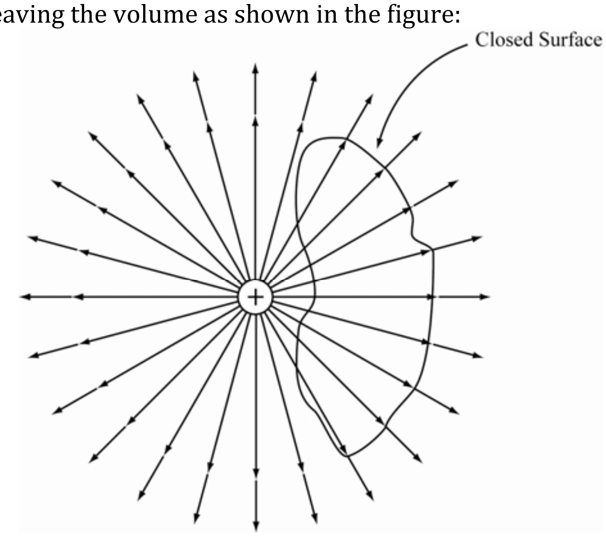
Conceptual Questions

22.13. The metal frame and sheet metal of the car form a Faraday cage, excluding the electric fields induced by the lightning. The current in the lightning strike flows around the outside of the car to ground. The passengers inside the car can be in contact with the inside of the car with no ill effects, but should not stick their hands out an open window.

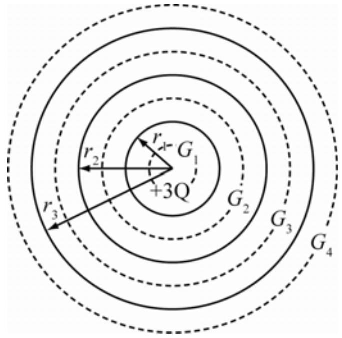
22.14. Since lightning can strike the tree and have the current flow through the wet tree, the current would jump to any object near the tree. To avoid lightning, go inside the house or a car. If I were outside, I would go to a low place and avoid trees or tall buildings. I should not lie down on the ground since the current can flow along the surface of the Earth.

22.15. If electric field lines crossed, there would be a charge at the crossing point. It is known that the electric field lines extend away from a positive change and the lines terminate at a negative charge. If in the vicinity of the crossing point there is no charge, then the lines cannot cross. Moreover, if we put a test charge on the crossing point, there would be two directions of the force; this is not possible; therefore the lines cannot cross.

22.16. The net flux through a closed surface is proportional to the net flux penetrating the surface, that is, the flux leaving the volume minus the flux entering the volume. This means that if there is a charge within a surface, the flux due to the charge will only exit through the surface creating a net flux no matter where the charge is located within the surface. If a charge moves just outside the surface, then the net flux crossing the surface would be zero since the flux entering the volume must be equal to the flux leaving the volume as shown in the figure:



22.17. Because of the spherical symmetry of this problem, Gauss's Law can be used to determine electric fields. The image below shows a cross-section of the nested spheres:



Gauss's Law is applied on four surfaces, G_1, G_2, G_3 and G_4 as shown in the figure.

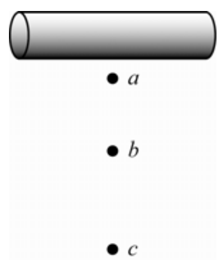
(a) In the region $r < r_1$, the electric field is zero because it is inside the conducting sphere.

(b) Applying Gauss's Law on the surface G_2 gives the electric field in the region $r_1 < r < r_2$, i.e., $E\left(4\pi r^2\right) = 3Q/\varepsilon_0$ or $E = 3Q/4\pi\varepsilon_0 r^2$.

(c) In the region $r_2 < r < r_3$, the electric field is zero since it is inside a conductor.

(d) In the region $r < r_3$, using Gauss's Law yields $E\left(4\pi r^2\right) = 3Q/\varepsilon_0$. Therefore, the electric field is $E = 3Q/4\pi\varepsilon_0 r^2$.

22.18.

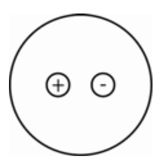


(a) If you are very close to the rod, the electric field can be approximated by the field produced by a very long rod. Then E is proportional to the linear charge density and to $1/r$.

(b) If you are a few centimeters away from the center, the rod's finite length becomes relevant and the rod can be treated as a line of charge with finite length, as in Example 22.3.

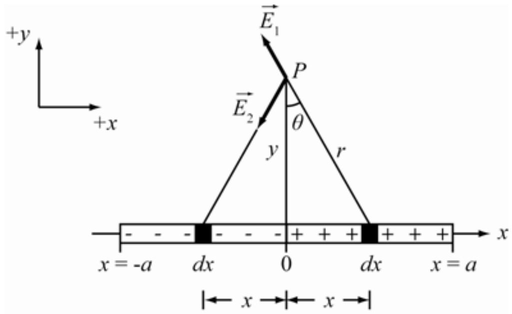
(c) If you are very far away, then the electric field behaves like that of a point charge. Therefore, the field is proportional to the total charge and to $1/r^2$.

22.19.



The total electric flux through a closed surface is equal to the net charge, q_{enc}, divided by the constant ε_0 or $\oint_{\text{net}} = q_{\text{enc}}/\varepsilon_0$. This is known as Gauss's Law. The strength of a dipole is $p = qd$. Because the dipole is completely enclosed by the spherical surface, the enclosed charge will be $q_{\text{enc}} = +q + (-q) = 0$. Thus the net flux through the closed surface will be zero.

22.20.



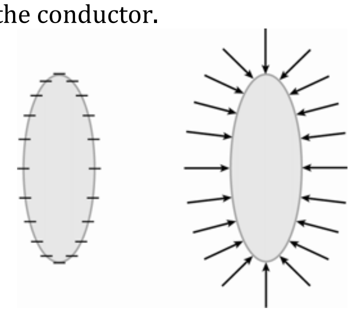
Consider two small elements dx at x and $-x$ as shown in the above figure. Due to the symmetry of the problem, it is found that the component of E_1 in the *y*-direction, E_{1y}, is equal in magnitude, but in the opposite direction, to the *y*-component of E_2. Therefore, only the *x*-components of electric fields contribute to the net field. Integrating over the length of wire yields $\vec{E} = \int_0^a \frac{2\sin\theta dq}{4\pi\varepsilon_0 r^2}(-\hat{x})$.

Using $dq = \lambda dx$, it simplifies to $\vec{E} = \left(\frac{-\hat{x}}{2\pi\varepsilon_0}\right)\int_0^a \frac{\sin\theta\lambda}{r^2}(dq)$. Substituting $r = \sqrt{x^2+y^2}$ and $\sin\theta = x/r$ yields $\vec{E} = \left(\frac{-\hat{x}\lambda}{2\pi\varepsilon_0}\right)\int_0^a \frac{xdx}{\left(x^2+y^2\right)^{3/2}}$. Using the substitution $z = x^2$ yields:

$$\vec{E} = \left(\frac{-\hat{x}\lambda}{2\pi\varepsilon_0}\right)\left(\frac{1}{2}\right)\int_0^{a^2}\frac{dz}{\left(z+y^2\right)} = \left(\frac{-\hat{x}\lambda}{2\pi\varepsilon_0}\right)\left[-2/\left(z+y^2\right)^{1/2}\right]_0^{a^2} = \left(\frac{-\hat{x}\lambda}{2\pi\varepsilon_0}\right)\left[-2/\left(z+y^2\right)^{1/2}\right]_0^{a^2}$$

$$= (-\hat{x})\left(\frac{\lambda}{2\pi\varepsilon_0}\right)\left[(1/y) - 1/\sqrt{a^2+y^2}\right].$$

22.21. Since the conductor has a negative charge, this means that the electric field lines are toward the conductor. Electrons inside the conductor can move freely and redistribute themselves such that the repulsion forces between electrons are minimized. As a consequence of this, the electrons are distributed on the surface of the conductor.



22.22. St. Elmo's Fire is a form of corona discharge; the same phenomenon whereby lightning rods bleed off accumulated ground charge to prevent lightening strokes. Lightning rods are not supposed to conduct a lightning strike to ground except as a last resort. In stormy weather, a ship or aircraft can become electrically charged by air friction. The charge will collect at the sharp edges or points on the structure of the ship or plane because the electric field is concentrated in areas of high curvature. Sufficiently large fields ionize the air at these areas, as the molecules of nitrogen and oxygen de-ionize they give off energy in the form of visible light. The ghostly glow known since the days of "wooden ships and iron men" is St. Elmo's Fire.

22.23. Consider the surface layer of charge to be divided into two component; a 'tile' in the vicinity of some point, and the 'rest' of the charge on the surface. Seen from close enough to the given point on the surface, the 'tile' appears as a flat plane of charge. Gauss's Law applied to the cylindrical surface pierced symmetrically by such a plane, implies that the 'tile' produces an electric field with the component $\sigma/2\varepsilon_0$ perpendicularly outward from the surface on the outside, inward on the inside. But Gauss's Law applied to a short cylinder ('pillbox') partially embedded in the conductor, implies that the entire charge layer produces an electric field with component σ/ε perpendicularly outward outside the surface, and zero inside. To yield this result, the 'rest' must produce electric field $\sigma/2\varepsilon_0$, outward, in the vicinity of the 'tile' inside and out. It is this electric field which exerts force on the 'tile', carries charge per unit area σ. Hence, every portion of the charge layer experiences outward force per unit area stress of magnitude $\Sigma = \sigma^2/2\varepsilon_0$. Note that the outward direction of the stress is independent of sign of σ.

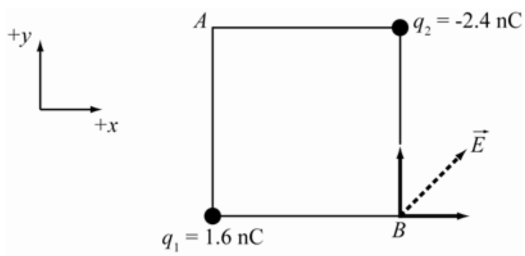
22.24. The net force on the dipole is zero, so there will be no translational motion of dipole. The net torque; however, is not zero, so the dipole will rotate. With the force on the positive charge to the right and the force on the negative charge to the left, the dipole will rotate counter-clockwise.

Exercises

22.25. The electric field produced by the charge is:

$$E = \frac{kq}{r^2} = \frac{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(4.00\cdot 10^{-9} \text{ C}\right)}{\left(0.250 \text{ m}\right)^2} = 575.36 \text{ N/C} \approx 575 \text{ N/C}.$$

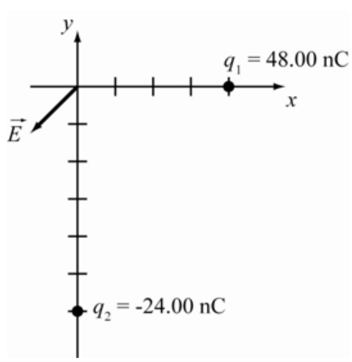
22.26.



The electric field vector will be $\vec{E} = \sum_i \vec{E}_i = \left(kq_1/r^2\right)\hat{x} + \left(kq_2/r^2\right)\hat{y} = k/r^2\left(q_1\hat{x} + q_2\hat{y}\right)$. The magnitude of the vector is:

$$\left|\vec{E}\right| = \sqrt{E_x^2 + E_y^2} = \frac{k}{r^2}\sqrt{q_x^2 + q_y^2} = \frac{8.99\cdot 10^9 \text{ N m}^2/\text{C}^2}{\left(1.0 \text{ m}\right)^2}\sqrt{\left(1.6\cdot 10^{-9} \text{ C}\right)^2 + \left(-2.410^{-9} \text{ C}\right)^2} = 25.931 \text{ N/C} = 26 \text{ N/C}.$$

22.27.



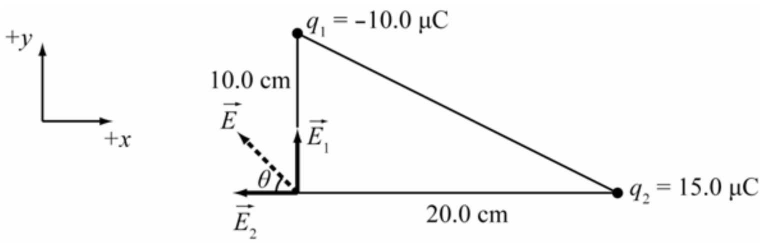
The electric field at the origin is $\vec{E} = \sum_i E_i = \left(k\left|q_1\right|/r_1^2\right)\left(\hat{x}\right) + \left(k\left|q_2\right|/r_2^2\right)\left(-\hat{y}\right)$. The direction is $\tan\theta = E_y/E_x$.

$$\theta_0 = \tan^{-1}\left(\frac{E_y}{E_x}\right) = \tan^{-1}\left[\frac{k|q_2|/r_2^2}{k|q_1|/r_1^2}\right] = \tan^{-1}\left[\frac{r_1^2|q_2|}{r_2^2|q_1|}\right] = \tan^{-1}\left[\frac{(4.000\text{ m})^2(24.00\text{ nC})}{(6.000\text{ m})^2(48.00\text{ nC})}\right] = 12.53°.$$

The electric field lies in the 3rd quadrant so $\theta = 180.00° + \theta_0 + 12.53° = 192.53°$. Rounding to four significant figures gives us $\theta = 192.5°$.

22.28. **THINK:** The electric field is the sum of the fields generated by the two charges of the corner triangle. The first charge is $q_1 = -1.0\cdot 10^{-5}$ C and is located at $\vec{r}_1 = (0.10\text{ m})\hat{y}$. The second charge is $q_2 = 1.5\cdot 10^{-5}$ C located at $\vec{r}_2 = (0.20\text{ m})\hat{x}$.

SKETCH:



RESEARCH: The electric field is given by the equation $\vec{E} = (kq/r^2)\hat{r}$.

SIMPLIFY: $\vec{E} = (kq_1/r_1^2)\hat{y} + (kq_2/r_2^2)\hat{x}$. The magnitude of the field is

$$|\vec{E}| = \sqrt{E_x^2 + E_y^2} = k\sqrt{\left(\frac{q_1}{r_1^2}\right)^2 + \left(\frac{q_2}{r_2^2}\right)^2},$$

and has a direction $\theta = \tan^{-1}\left(\frac{E_y}{E_x}\right) = \tan^{-1}\left(\left(\frac{q_1}{r_1^2}\right) \div \left(\frac{r_2^2}{q_2}\right)\right) = \tan^{-1}\left(\frac{r_2^2 q_1}{r_1^2 q_2}\right)$ where θ is in the second quadrant.

CALCULATE: $E = (8.99\cdot 10^9\text{ N m}^2/\text{C}^2)\sqrt{\left(\frac{-1.0\cdot 10^{-5}\text{ C}}{(0.100\text{ m})^2}\right)^2 + \left(\frac{1.5\cdot 10^{-5}\text{ C}}{(0.200\text{ m})^2}\right)^2} = 9.6013\cdot 10^6\text{ N/C}$

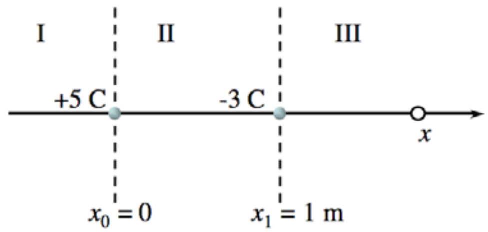
$$\theta = \tan^{-1}\frac{(0.200\text{ m})^2(1.0\cdot 10^{-5}\text{ C})}{(0.100\text{ m})^2(1.5\cdot 10^{-5}\text{ C})} = 69.444° \text{ or } \theta = 110.56°.$$

ROUND: The least precise value given in the question has two significant figures, so the answer should also be reported to two significant figures. The electric field produced at the corner is $E = 9.6\cdot 10^6$ N/C at 110° from the *x*-axis.

DOUBLE-CHECK: Dimensional analysis confirms the answer is in the correct units.

22.29. **THINK:** We want to find out where the combined electric field from two point charges can be zero. Since the electric field falls off as the inverse second power of the distance to the charge, and since both charges are on the *x*-axis, only points on the same line have any chance of canceling the electric field from these two charges, resulting in a net zero electric field. The first charge, $q_1 = 5.0$ C, is at the origin. The second charge, $q_2 = -3.0$ C, is at $x = 1.0$ m. Consider where along the *x*-axis it is possible to have zero electric field. On the sketch we have marked three regions (I, II, and III). If we place a positive charge anywhere in region II, the 5 C will repel it and the −3 C will attract it, so that the positive charge moves to the right. If we place a negative charge in the same region, it will move to the left. So we know that the electric field cannot be zero anywhere in region II. Region I is closer to the 5 C charge. Since this is also the charge with the larger magnitude, its electric field will dominate region I, and thus there is no place in region I where the electric field is 0. This leaves region III, where the two electric fields from the point charges can cancel.

SKETCH:



RESEARCH: The electric field due to the charge at the origin is $E_0 = kq_0/x^2$. The other charge produces a field of $E_1 = kq_1/(x-x_1)^2$.

SIMPLIFY: The combined electric field is $E = kq_0/x^2 + kq_1/(x-x_1)^2$. Setting the electric field to zero, solve for x:

$$\frac{kq_0}{x^2} + \frac{kq_1}{(x-x_1)^2} = 0 \Rightarrow \frac{kq_0}{x^2} = -\frac{kq_1}{(x-x_1)^2} \Rightarrow (x-x_1)^2 q_0 = -x^2 q_1 \Rightarrow (x-x_1)^2 |q_0| = x^2 |q_1|$$

We could now solve the resulting quadratic equation blindly and would obtain two solutions, each of which we would have to evaluate for validity. Instead, we can make use of the thinking we have done above. In the last step we used the fact that the charge at the origin is positive and the other is negative, replacing them with their absolute values. Now we can take the square root on both sides and choose the positive root, leaving us with

$$(x-x_1)\sqrt{|q_0|} = x\sqrt{|q_1|} \Rightarrow x = \frac{x_1\sqrt{|q_0|}}{\sqrt{|q_0|}-\sqrt{|q_1|}}$$

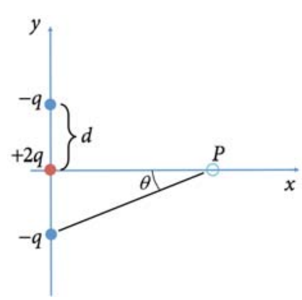
CALCULATE: $x = \dfrac{(1.00 \text{ m})\sqrt{5.00 \text{ C}}}{\sqrt{5.00 \text{ C}}-\sqrt{3.00 \text{ C}}} = 4.43649 \text{ m}$

ROUND: The positions are reported to three significant figures. The electric field is zero at $x = 4.44$ m.

DOUBLE-CHECK: This is a case where we can simply insert our result and verify that it does what it is supposed to: $E(x{=}4.4 \text{ m}) = k(5 \text{ C})/(4.4 \text{ m})^2 + k(-3 \text{ C})/(4.4 \text{ m} - 1 \text{ m})^2 = 0$.

22.30. **THINK**: Let's fix the coordinate notation first. The charges are located at points (0,*d*), (0,0), and (0,-*d*) on the *y*-axis, and the point *P* is *P* = (*x*,0). In order to specify the electric field at a point in space, we need to specify the magnitude and the direction. Lets first think about the direction. The distribution of the charges is symmetric with respect to the *x*-axis. Thus if we flip the charge distribution upside down, we see the same picture. This means also that we can do this for the electric field generated by these charges. Right away this means that the electric field anywhere on the *x*-axis cannot have a *y*-component and can only have an *x*-component.

SKETCH:



RESEARCH: The electric field strength is given by $E = kQ/r^2$, and the electric fields from different charges add as vectors. We need to add the *x*-components of the electric fields from all charges. They are (from top to bottom along the *y*-axis):

$$E_1 = \frac{-kq}{d^2+x^2}$$
$$E_{1,x} = E\cos\theta = \frac{-kq}{d^2+x^2}\frac{x}{\sqrt{d^2+x^2}} = \frac{-kqx}{\left(d^2+x^2\right)^{3/2}}$$
$$E_{2,x} = \frac{2kq}{x^2}$$
$$E_3 = E_1 = \frac{-kq}{d^2+x^2}$$
$$E_{3,x} = E_{1,x} = \frac{-kqx}{\left(d^2+x^2\right)^{3/2}}$$

SIMPLIFY: All we have to do is add the individual *x*-components to find our expression for the *x*-component of the electric field along the *x*-axis:

$$E_x(x,0) = E_{1,x} + E_{2,x} + E_{3,x} = \frac{2kq}{x^2} - \frac{2kqx}{\left(d^2+x^2\right)^{3/2}} = 2kq\left(\frac{1}{x^2} - \frac{x}{\left(d^2+x^2\right)^{3/2}}\right)$$

(This is the expression for $x>0$; for $x<0$ it has the opposite sign so that it always points away from the origin.)
CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: For $x \to 0$ we see that the first term diverges as we get very close to the positive charge at the origin, which is as expected.
For large distances, $x \to \infty$, $d/x \to 0$, we expect at most a very weak electric field because the net charge of our configuration is 0. We can factor out the $1/x^2$ term to get

$$E_x(x,0) = \frac{2kq}{x^2}\left(1 - \frac{x^3}{\left(d^2+x^2\right)^{3/2}}\right) = \frac{2kq}{x^2}\left(1 - \frac{1}{\left(\left(\frac{d}{x}\right)^2+1\right)^{3/2}}\right) = \frac{2kq}{x^2}\left(1 - \left(\left(\frac{d}{x}\right)^2+1\right)^{-3/2}\right).$$ For

$\left(d^2/x^2\right) \ll 1$, the binomial expansion gives us

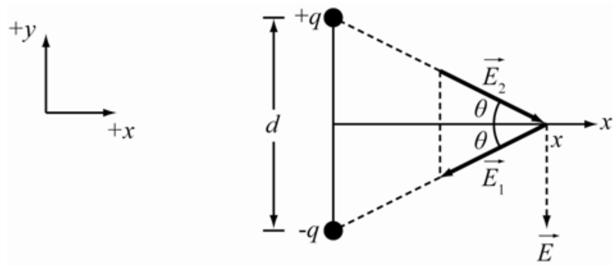
$$\left(\frac{d^2}{x^2}+1\right)^{-3/2} \approx 1 - \frac{3}{2}\frac{d^2}{x^2}.$$

The electric field then simplifies to

$$E_x(x \gg d,0) = \frac{2kq}{x^2}\left(1 - \left(1 - \frac{3}{2}\frac{d^2}{x^2}\right)\right) = \frac{2kq}{x^2}\frac{3}{2}\frac{d^2}{x^2} = \frac{3kqd^2}{x^4}.$$

Thus the electric field strength of this configuration, called a "quadrupole", falls with the inverse fourth power of the distance to the origin for large distances. (... as compared to the electric field from a dipole, which falls with the inverse third power).

22.31. The dipole is just two charges fixed together of opposite sign. The electric field at a point is the sum of the fields produced by each charge. The figure indicates that the electric field produced is created by the component of the field perpendicular to line *x*.



$$E = E_{1y} + E_{2y} = E_1 \sin\theta + E_2 \sin\theta = \frac{-kq}{\left(\left(\frac{d}{2}\right)^2 + x^2\right)}\sin\theta + \frac{-kq}{\left(\left(\frac{d}{2}\right)^2 + x^2\right)}\sin\theta = \frac{-2kq\sin\theta}{\left(\frac{d}{2}\right)^2 + x^2}$$

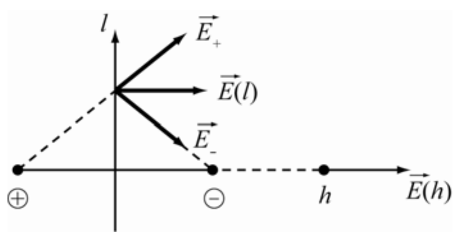
Note that $\sin\theta = \frac{d}{2\sqrt{(d/2)^2 + x^2}}$. This means the field is:

$$E = \frac{-2kqd}{2\left((d/2)^2 + x^2\right)^{3/2}} = \frac{-kqd}{\left((d/2)^2 + x^2\right)^{3/2}} = \frac{-kp}{\left((d/2)^2 + x^2\right)^{3/2}}.$$

If $x \gg d$ then $E = -kp/x^3$. The field along the axis of the dipole is $E = -2kp/x^3$, indicating that the field strength falls off more rapidly perpendicular to the dipole axis.

22.32. **THINK:** The field due to a dipole moment at a point *h* along the *x*-axis is $E(h) = k2qd/h^3$. I want to find the point perpendicular to the *x*-axis as measured from the origin (i.e., along the *y*-axis), where the electric field has this same value.

SKETCH:



RESEARCH: From the previous problem, the electric field along the *y*-axis is $E(l) = \frac{kqd}{\left(d^2/4 + l^2\right)^{3/2}}$.

Set $E(l) = E(h)$ and solve for l.

SIMPLIFY: $\frac{k(2qd)}{h^3} = \frac{kqd}{\left(\frac{d^2}{4} + l^2\right)^{3/2}} \Rightarrow \frac{2}{h^3} = \frac{1}{\left(\frac{d^2}{4} + l^2\right)^{3/2}} \Rightarrow 2\left(\frac{d^2}{4} + l^2\right)^{3/2} = h^3 \Rightarrow l = \sqrt{\left(\frac{h}{\sqrt[3]{2}}\right)^2 - \left(\frac{d}{2}\right)^2}.$

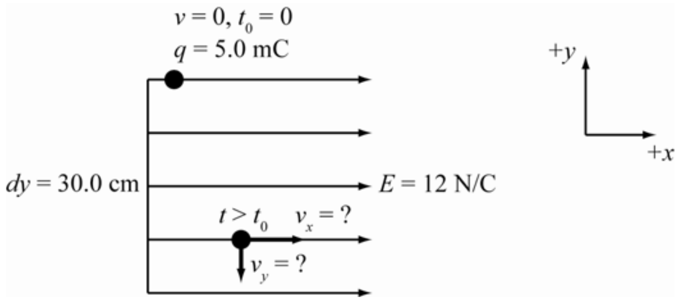
CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: According to this expression, *l* will always be less than *h*. This is consistent with the previous result that the electric field strength along a line perpendicular to the dipole axis falls off more rapidly than the field strength along the dipole axis.

22.33. **THINK:** As the $m = 4.0$ g ball falls the force of gravity acting on it will cause it to accelerate downwards. At the same time, the force due to the electric field acts on the ball causing it to accelerate towards the east. The forces act perpendicular to each other. The problem is solved by finding each component of the velocity. In order to find the velocity due to the electric field, the time required for the ball to travel 30.0 cm downwards is needed.

SKETCH:



RESEARCH: The velocity in the downward direction is found using $v_y^2 = v_{y0}^2 + 2gdy$. The time it takes to reach this velocity $t = v_y / g$. The acceleration eastward is calculated using $F = ma = qE$. The velocity is then $v_x = a_x t$.

SIMPLIFY: The *y*-component of the velocity is $v_y = \sqrt{2gdy}$ because the ball starts from rest. The time it takes for the ball to fall 30.0 cm is $t = \sqrt{2gdy} / g$. The acceleration eastward is $a = qE/m$. The velocity eastward is $v_x = a_x t \rightarrow v_x = \left(\frac{qE}{m}\right)\frac{\sqrt{2gdy}}{g} = \frac{qE}{m}\sqrt{\frac{2dy}{g}}$.

CALCULATE: $v_y = \sqrt{2(9.81 \text{ m/s}^2)(0.300 \text{ m})} = 2.4261 \text{ m/s}$ downward

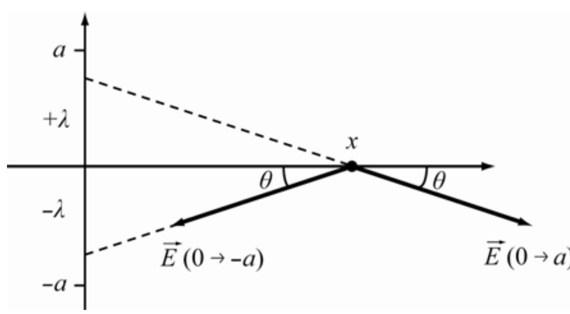
$$v_x = \left(5.0 \cdot 10^{-3} \text{ C}\frac{12 \text{ N/C}}{0.0040 \text{ kg}}\right)\sqrt{2\left(\frac{0.300 \text{ m}}{9.81 \text{ m/s}^2}\right)} = 3.7096 \text{ m/s} \text{ eastward}$$

ROUND: The velocity is report to three significant figures. The ball reaches a velocity of $(3.71 \text{ m/s})\hat{x} + (2.43 \text{ m/s})\hat{y}$.

DOUBLE-CHECK: This is a reasonable answer considering the size of the values given in the question.

22.34. **THINK:** A line of charge along the *y*-axis has linear charge density $+\lambda$ from $y = 0$ to $y = +a$, and $-\lambda$ from $y = 0$ to $y = -a$. I want to find an expression for the electric field at any point x along the *x*-axis. It is noted that the charge configuration is similar in structure to a dipole. By symmetry, the *x*-components of the field cancel out, and the net field is in the *y*-direction.

SKETCH:



RESEARCH: The electric field resulting from a charge distribution is the integral over the differential charge: $dE = kdq/r^2$. The *y*-component of the field is $dE_y = kdq\sin\theta / r^2$, where θ is the angle between the electric field produced by dq and the *y*-axis. Also, $r = \sqrt{x^2 + y^2}$, $\sin\theta = y/r$. From 0 to a, $dq = \lambda dy$, and from 0 to $-a$, $dq = -\lambda dy$.

SIMPLIFY: $dE_+ = dE_{y,+} = \frac{kdq}{r^2}\sin\theta = \left(\frac{k\lambda dy}{x^2 + y^2}\right)\left(\frac{y}{\sqrt{x^2 + y^2}}\right) = \frac{k\lambda y dy}{(x^2 + y^2)^{3/2}}$

$$dE_{-} = dE_{y,-} = \frac{kdq}{r^2}\sin(-\theta) = \left(\frac{-k\lambda dy}{x^2+y^2}\right)\left(-\frac{y}{\sqrt{x^2+y^2}}\right) = \frac{k\lambda ydy}{\left(x^2+y^2\right)^{3/2}}$$

The field due to the positive charge distribution is: $E_{+} = \int_0^a \frac{k\lambda ydy}{\left(x^2+y^2\right)^{3/2}} = k\lambda\int_0^a \frac{ydy}{\left(x^2+y^2\right)^{3/2}}$. Similarly, the field due to the negative charge distribution is: $E_{-} = \int_0^{-a} \frac{k\lambda ydy}{\left(x^2+y^2\right)^{3/2}} = k\lambda\int_0^{-a} \frac{ydy}{\left(x^2+y^2\right)^{3/2}}$.

CALCULATE: Let $u = x^2+y^2$ then $du = 2ydy$ then:

$$E_{+} = \frac{k\lambda}{2}\int_0^a \frac{du}{u^{3/2}} = \left(\frac{k\lambda}{2}\right)\left(\frac{-2}{u^{1/2}}\right)\Bigg|_0^a = -\frac{k\lambda}{u^{1/2}}\Bigg|_0^a = \left[\frac{-k\lambda}{\left(x^2+y^2\right)^{1/2}}\right]_0^a = -k\lambda\left[\frac{1}{x} - \frac{1}{\left(x^2+a^2\right)^{1/2}}\right], \text{ and}$$

$$E_{-} = \frac{k\lambda}{2}\int_0^{-a} \frac{du}{u^{3/2}} = \left(\frac{k\lambda}{2}\right)\left(\frac{-2}{u^{1/2}}\right)\Bigg|_0^{-a} = -\frac{k\lambda}{u^{1/2}} = \left[-\frac{k\lambda}{\left(x^2+y^2\right)^{1/2}}\right]_0^{-a} = -k\lambda\left[\frac{1}{x} - \frac{1}{\left(x^2+(-a)^2\right)^{1/2}}\right].$$

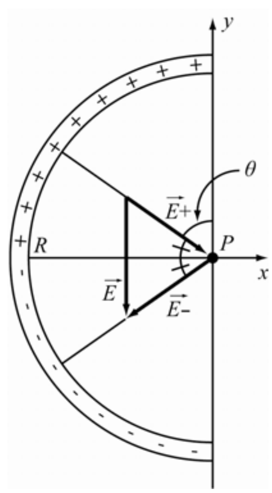
The total electric field at x is: $E = E_{+} + E_{-} = 2k\lambda\left[\frac{1}{\left(x^2+a^2\right)^{1/2}} - \frac{1}{x}\right]$.

ROUND: Not applicable.

DOUBLE CHECK: The electric field decreases inversely proportionally to the distance from the wire, as expected.

22.35. **THINK:** A semicircular rod carries a uniform charge of +Q along its upper half, and –Q along it's lower half. I want to determine the magnitude and direction of the electric field at the center of the semicircle. The rod has a length of $L = \pi R$. The charge density of the upper half of the rod is $\lambda = Q/L = Q/(1/2)\pi R = 2Q/\pi R$. Similarly, the lower half of the rod is $\lambda = -2Q/\pi R$.

SKETCH:



RESEARCH: From the symmetry of the semi-circle, the *x*-components of the field cancel, and the resulting electric field only has a *y*-component. The *y*-component of the electric field for the upper segment of the rod is given by

$$dE_{+y} = dE\cos\theta = \left(-kdq/R^2\right)\cos\theta = \left(-k\lambda dx/R^2\right)\cos\theta,$$

where $dx = Rd\theta$. Therefore, $dE_{+,y} = \left(-k\lambda Rd\theta/R^2\right)\cos\theta = -k\lambda\cos\theta d\theta/R$.

SIMPLIFY: Integrating both sides with respect to θ gives:

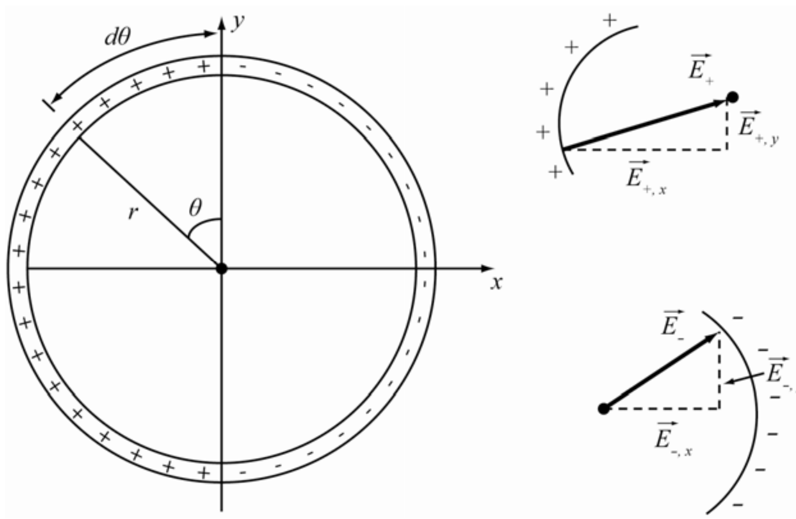
$$dE_{+y} = \left(-k\lambda/R\right)\int_0^{\pi/2}\cos\theta d\theta = \left(-k\lambda/R\right)\sin\theta\big|_0^{\pi/2} = \left(-k\lambda/R\right)\left(1-0\right) = \left(-k\lambda/R\right) = -k2Q/\pi R^2.$$

The lower half of the semicircle also contributes the same y-component. The total electric field at the origin is

$$\vec{E} = E_{+y\hat{y}} + E_{-y\hat{y}} = 2E_{y\hat{y}} = \left(\frac{-4kQ}{\pi R^2}\right)\hat{y} = \left(\frac{-4Q}{4\pi\varepsilon_0\cdot\pi R^2}\right)\hat{y} = \left(\frac{-Q}{\pi^2\varepsilon_0 R^2}\right)\hat{y}.$$

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE CHECK: The resulting field points in the direction from the positive charge to the negative charge, as required.

22.36. **THINK:** Two semicircular rods, with uniformly distributed charges of $+1.00\ \mu\text{C}$ and $-1.00\ \mu\text{C}$, respectively, form a circle of radius $r = 10.0$ cm. I want to determine the magnitude and direction on the electric field at the center of the circle.
SKETCH:



RESEARCH: The charge densities of the positively charged and negatively charged rods are $+\lambda = Q/\pi R$ and $-\lambda = -Q/\pi R$, respectively. The differential element of the electric field is given by $dE = kdq/R^2$, where the differential element of charge along the line is $dq = \lambda dx = \lambda R d\theta$. It is also necessary to consider the x- and y-components of the differential elements.

SIMPLIFY: $dE_{+,x} = \frac{k\lambda dx\sin\theta}{R^2} = \frac{k\lambda d\theta\sin\theta}{R^2} = \frac{kQRd\theta\sin\theta}{\pi R^3} = \frac{kQ\sin\theta d\theta}{\pi R^2}$. Similarly, $dE_{+,y} = \frac{kQ\cos\theta d\theta}{\pi R^2}$; $dE_{-,x} = \frac{-kQ\sin\theta d\theta}{\pi R^2}$; $dE_{-,y} = \frac{-kQ\cos\theta d\theta}{\pi R^2}$. Integrating both sides of each expression gives:

$$E_{+,x} = \frac{kQ}{\pi R^2}\int_0^{\pi}\sin\theta d\theta = \frac{kQ}{\pi R^2}\left(-\cos\theta\right)\Big|_0^{\pi} = \frac{2kQ}{\pi R^2}$$

$$E_{+,y} = \frac{kQ}{\pi R^2}\int_0^{\pi}\cos\theta d\theta = \frac{kQ}{\pi R^2}\left(\sin\theta\right)\Big|_0^{\pi} = 0$$

$$E_{-,x} = -\frac{kQ}{\pi R^2}\int_{\pi}^{2\pi}\sin\theta d\theta = -\frac{kQ}{\pi R^2}\left(-\cos\theta\right)\Big|_{\pi}^{2\pi} = \frac{2kQ}{\pi R^2}$$

$$E_{-,y} = -\frac{kQ}{\pi R^2}\int_{\pi}^{2\pi}\cos\theta d\theta = -\frac{kQ}{\pi R^2}\left(\sin\theta\right)\Big|_{\pi}^{2\pi} = 0$$

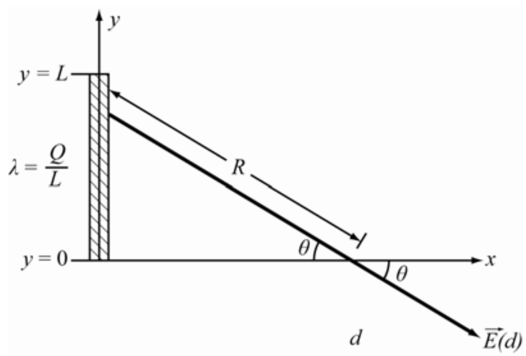
The total electric field at the center is given by: $E = E_{+,x} + E_{+,y} + E_{-,x} + E_{-,y} = \frac{2kQ}{\pi R^2} + 0 + \frac{2kQ}{\pi R^2} + 0 = \frac{4kQ}{\pi R^2}.$

CALCULATE: $E = \dfrac{4\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(1.00\cdot 10^{-6}\text{ C}\right)}{\pi\left(0.100\text{ m}\right)^2} = 1.1446\cdot 10^6 \text{ N/C}$

ROUND: The electric field is reported to three significant figures: $E = 1.14\cdot 10^6$ N/C. Because all of the *y*- components are zero, the resultant field is in the positive *x*-direction.
DOUBLE-CHECK: Given the symmetry of the charge configuration, this is a reasonable result.

22.37. **THINK:** The charge Q is uniformly distributed along the rod of length L. The rod has linear charge density $\lambda = Q/L$. The electric field at a position $x = d$ can be calculated by integrating over the differential electric field due to the differential charge on the rod. The electric field differential $dE = kdq/r^2$, where the differential is along the *y*-axis, and $R = \sqrt{d^2+y^2}$. The *x*- and *y*-components of the field must be considered individually. The *x*-component of the field differential is given by $dE_x = dE\cos\theta$, and the *y*-component is given by $dE_y = dE\sin\theta$.

SKETCH:



SIMPLIFY: $dE_x = \dfrac{kdQ}{R^2}\cos\theta = \dfrac{k\lambda dy}{R^2}\cos\theta = \dfrac{kQdy}{LR^2}\cos\theta = \dfrac{kQdy}{L\left(d^2+y^2\right)^2}\cos\theta$

$$\cos\theta = \frac{d}{R} = \frac{d}{\sqrt{d^2+y^2}} \;\Rightarrow\; dE_x = \left(\frac{kQdy}{L\left(d^2+y^2\right)^2}\right)\left(\frac{d}{\sqrt{d^2+y^2}}\right) = \frac{kdQdy}{L\left(d^2+y^2\right)^{3/2}}$$

$$dE_y = \frac{kQdy}{L\left(d^2+y^2\right)^2}\sin\theta;\;\; \sin\theta = \frac{y}{R} = \frac{y}{\sqrt{d^2+y^2}} \;\Rightarrow\; dE_y = \frac{ykQdy}{L\left(d^2+y^2\right)^{3/2}}$$

Integrate both expressions.

$$E_x = \int_0^L \frac{dkQdy}{L\left(d^2+y^2\right)^{3/2}} = \frac{dkQ}{L}\int_0^L \frac{1}{\left(d^2+y^2\right)^{3/2}}dy = \frac{kQd}{L}\left[\frac{y}{d^2\sqrt{d^2+y^2}}\right]_0^L$$

$$E_y = \int_0^L \frac{ykQdy}{L\left(d^2+y^2\right)^{3/2}} = \frac{kQ}{L}\int_0^L \frac{y}{\left(d^2+y^2\right)^{3/2}}dy = \frac{kQ}{L}\left[\frac{-1}{\sqrt{d^2+y^2}}\right]_0^L$$

$\vec{E}(d) = E_x\hat{x} - E_y\hat{y}$

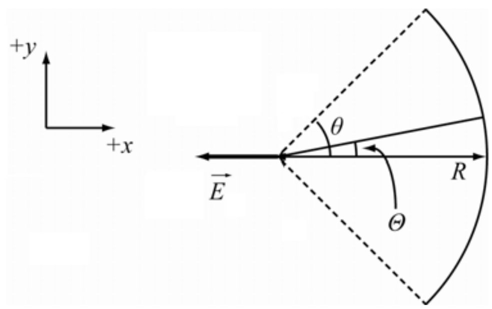
CALCULATE: $E_x = \dfrac{kQd}{L}\left[\dfrac{y}{d^2\sqrt{d^2+y^2}}\right]_0^L = \dfrac{kQd}{L}\left(\dfrac{L}{d^2\sqrt{d^2+L^2}} - 0\right) = \dfrac{kQ}{d\sqrt{d^2+L^2}}$

$$E_y = \frac{kQ}{L}\left[\frac{-1}{\sqrt{d^2+y^2}}\right]_0^L = \frac{kQ}{L}\left(\frac{-1}{\sqrt{d^2+L^2}} - \frac{-1}{d}\right) = \frac{kQ}{dL} - \frac{kQ}{L\sqrt{d^2+L^2}}$$

$$\vec{E}(d) = \left(\frac{kQ}{d\sqrt{d^2+L^2}}\right)\hat{x} - \left(\frac{kQ}{dL} - \frac{kQ}{L\sqrt{d^2+L^2}}\right)\hat{y}$$

ROUND: Not applicable.
DOUBLE CHECK: The magnitude of the electric field decreases as *d* increases, as expected.

22.38. **THINK:** A wire bent into an arc of radius *R* and carrying a uniformly distributed charge *Q* will have a linear charge density of $\lambda = Q/2\theta R$. By the symmetry of the charge distribution, the *y*-components cancel, and only the *x*-component of the charge contributes to the electric field.
SKETCH:



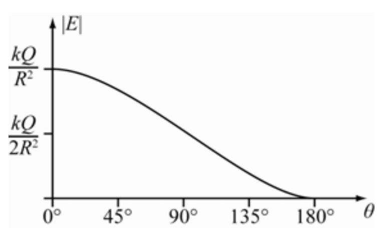
RESEARCH: An electric field produced by an infinitesimal segment of the arc is $dE = k\,dq/R^2 = k\lambda dx/R^2 = k\lambda R d\Theta/R^2 = k\lambda d\Theta/R.$ The total electric field can be calculated by integrating over the differential elements of the field. Since the *y*-component of the field is zero, $E = E_x = \int_{-\theta}^{\theta} \frac{k\lambda d\Theta}{R}\cos\Theta.$

SIMPLIFY:

$$E = \int_{-\theta}^{\theta} \frac{k\lambda}{R}\cos\Theta d\Theta = \frac{k\lambda}{R}\int_{-\theta}^{\theta}\cos\Theta d\Theta = \frac{kQ}{2\theta R^2}\int_{-\theta}^{\theta}\cos\Theta d\Theta = \left[\frac{kQ}{2\theta R^2}\sin\Theta\right]_{-\theta}^{\theta} = \frac{kQ}{2\theta R^2}\left(\sin\theta - \sin(-\theta)\right)$$

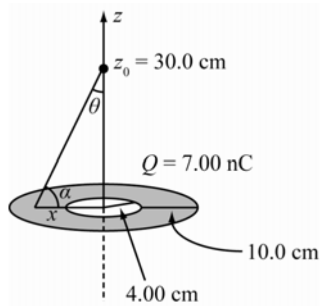
$$= \frac{kQ}{2\theta R^2}\left(\sin\theta + \sin(\theta)\right) = \frac{kQ\sin\theta}{\theta R^2}$$

CALCULATE:



ROUND: Not applicable.
DOUBLE CHECK: As $\theta \to 0$ the field is the same as that of a point charge, because $\lim_{\theta\to 0}\frac{kQ\sin\theta}{\theta R^2} = \frac{kQ}{R^2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{kQ}{R^2}$. The field becomes zero as the point is symmetrically enclosed by a ring of charge.

22.39. **THINK:** The washer will create an electric field that should be not to different from the electric field of the thin ring of charge we encountered in Solved Problem 22.1. The washer has a total charge $Q = 7.00$ nC, with inner and outer radius of the washer are $r_i = 2.00$ cm and $r_o = 5.0$ cm. The electric field at $z_o = 30.0$ cm away from the center of the washers is desired.
SKETCH:



RESEARCH: The surface density is $\sigma = Q/A$ where the area is $A = \pi\left(r_o^2 - r_i^2\right)$. The field will point in along the *z*-axis due to symmetry. The field due to a segment is $dE = kdq/R^2$. The distance from the segment of charges is $R = \sqrt{x^2 + z_o^2}$ and $\cos\theta = z_o / \sqrt{x^2 + z_o^2}$.

SIMPLIFY: $E = \int \frac{kdq}{R^2}\cos\theta = \int_0^{2\pi}\int_{r_i}^{r_o} \frac{k\sigma dA z_o}{R^3} = \int_0^{2\pi}\int_{r_i}^{r_o} \frac{kQz_o x\,dx\,d\theta}{R^3\pi\left(r_o^2 - r_i^2\right)} = \frac{2\pi kQz_o}{\pi\left(r_o^2 - r_i^2\right)}\int_{r_i}^{r_o} \frac{x\,dx}{\left(x^2 + z_o^2\right)^{3/2}}$. Evaluating the single integral gives:

$$E = \frac{2kQz_o}{\left(r_o^2 - r_i^2\right)}\left[\frac{-1}{\sqrt{x^2 + z_o^2}}\right]_{r_i}^{r_o} = \frac{-2kQz_o}{\left(r_o^2 - r_i^2\right)}\left[\frac{1}{\left(r_o^2 + z_o^2\right)^{1/2}} - \frac{1}{\left(r_i^2 + z_o^2\right)^{1/2}}\right] = \frac{2kQz_o}{\left(r_o^2 - r_i^2\right)}\left[\frac{1}{\sqrt{r_i^2 + z_o^2}} - \frac{1}{\sqrt{r_o^2 + z_o^2}}\right].$$

CALCULATE:

$$E = \frac{2\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(7.00\cdot 10^{-9} \text{ C}\right)\left(0.300 \text{ m}\right)}{\left(0.0500 \text{ m}\right)^2 - \left(0.0200 \text{ m}\right)^2}\left(\frac{1}{\sqrt{\left(0.0200\right)^2 + \left(0.300\right)^2}} - \frac{1}{\sqrt{\left(0.0500\right)^2 + \left(0.300\right)^2}}\right)\frac{1}{\text{m}}$$

$$= 682.715 \text{ N/C}$$

ROUND: The values are given to three significant figures. The electric field is $E = 6.83\cdot 10^2 \text{ N/C}$ pointing towards the positive *z*-axis.

DOUBLE-CHECK: In Solved Problem 22.1 we found for the thin ring: $E = kQz_o / \left(r^2 + z_0^2\right)^{3/2}$. Using the average of our outer and inner radius we then find from this formula:

$$E = \frac{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(7.00\cdot 10^{-9} \text{ C}\right)\left(0.300 \text{ m}\right)}{(\left(0.0350 \text{ m}\right)^2 + \left(0.300 \text{ m}\right)^2)^{3/2}} = 685.2 \text{ N/C}$$

Since this is fairly close to our result for a ring with finite thickness, we have added confidence in our result.

22.40. The force on the particle is $F = qE$. The charge is $q = -2e$ so the force is

$$F = qE = -2eE = -2\left(1.60\cdot 10^{-19} \text{ C}\right)\left(10.0\cdot 10^3 \text{ N/C}\right) = -3.20\cdot 10^{-15} \text{ N}.$$

22.41. The torque due to the field is

$$\left|\vec{\tau}\right| = \vec{p}\times\vec{E} = pE\sin\theta = qdE\sin\theta = \left(5.00\cdot 10^{-15} \text{ C}\right)\left(0.400\cdot 10^{-3} \text{ m}\right)\left(2.00\cdot 10^3 \text{ N/C}\right)\left(\sin 60.0^\circ\right)$$
$$= 3.46\cdot 10^{-15} \text{ N m}.$$

22.42. The maximum torque occurs when the dipole is perpendicular to the field. The electric field is $\left|\tau\right| = \left|\vec{p}\times\vec{E}\right| = pE\sin\theta = \left(1.05 \text{ D}\right)\left(3.34\cdot 10^{-30} \text{ C m/D}\right)\left(160.0 \text{ N/C}\right)\left(\sin 90^\circ\right) = 5.61\cdot 10^{-28} \text{ N m}.$

22.43. The force acting on the electron is $F = ma = qE$. The acceleration is then $a = qE/m$. Assuming the electron is moving in the same direction as the electric field, the acceleration will oppose the motion. The velocity is given by $v^2 = v_0^2 + 2ax = v_0^2 + 2\left(\frac{qE}{m}\right)x = v_0^2 - \frac{2eEx}{m}$. Solving this equation for *x*: $\left(\frac{2eE}{m}\right)x = v_0^2 - v^2$ and $v = 0$, therefore $x = \frac{mv_0^2}{2eE}$. The distance traveled is

$$x = \frac{\left(9.109\cdot 10^{-31}\text{ kg}\right)\left(27.5\cdot 10^{6}\text{ m/s}\right)^2}{2\left(1.602\cdot 10^{-19}\text{ C}\right)\left(11{,}400\text{ N/C}\right)} = 0.1885\text{ m}.$$

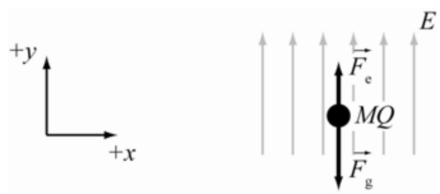
To three significant figures, the electron travels 0.189 m before it stops.

22.44. The dipole moment is $p = qd = ed = (1.602\cdot 10^{-19}\text{ C})(0.680\cdot 10^{-9}\text{ m}) = 1.089\cdot 10^{-28}\text{ C m} \approx 1.09\cdot 10^{-28}\text{ C m}$. The torque experienced by the dipole is

$$|\tau| = |\vec{p}\times\vec{\text{E}}| = p\text{E}\sin\theta = ed\text{E}\sin\theta = \left(1.089\cdot 10^{-28}\text{ C m}\right)\left(4.40\cdot 10^{3}\text{ N/C}\right)\left(\sin 45°\right) = 3.39\cdot 10^{-25}\text{ N m}.$$

22.45. **THINK:** The net force on falling object in an electric field is the sum of the force due to gravity and the force due to the electric field. If the falling object carries a positive charge, then the force on the object due to the electric field acts in the direction opposite to the force of gravity.

SKETCH:



RESEARCH: The net upward force acting on the object is $F = F_e - F_g = QE - Mg = Ma$. This corresponds to a downward acceleration of $a = g - \frac{QE}{M}$. Recall that the speed of an object in free fall is given by $v_f^2 = v_0^2 + 2a\Delta y \Rightarrow v = \sqrt{2ah}$.

SIMPLIFY:

(a) $v = \sqrt{2ah} \;\Rightarrow\; a = \frac{v^2}{2h} = g - \frac{QE}{M} \;\Rightarrow\; v = \sqrt{2h\left(g - QE/M\right)}$

(b) If the value $g - QE/M$ is less than zero, then the argument of the square root is negative. This means the value is non-real and the body does not fall.

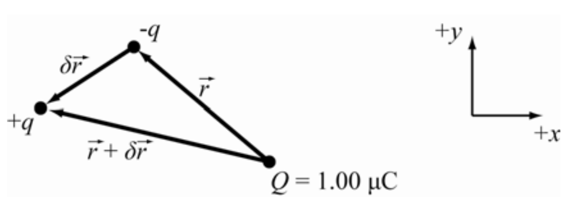
CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE CHECK: Dimensional analysis confirms that the units of the expression reduce to m/s, the correct units for velocity.

22.46. **THINK:** The force in between the charge and the dipole moment is equal to the force acting on each pole of the dipole. The dipole moment is $p = 6.20\cdot 10^{-30}$ C m and is $r = 1.00$ cm from the charge $Q = 1.00\ \mu\text{C}$.

SKETCH:



RESEARCH: The force due to an electric field is $\vec{F} = q\vec{E}(r)$, where the electric field is $E(r) = \left(kQ/r^2\right)\hat{r}$.

SIMPLIFY: The total force is $\vec{F} = q\vec{E}(r+\delta r) - q\vec{E}(r)$. From the fundamental theorem of calculus,

$$\vec{F} = q\delta r \frac{d}{dr}E(r) = p\frac{d}{dr}\frac{kQ}{r^2}\hat{r} = pkQ\left(\frac{-2}{r^3}\right)\hat{r} = \frac{-2kpQ}{r^3}\hat{r}$$

CALCULATE: $F = \dfrac{-2\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}\right)\left(6.2\cdot 10^{-30} \text{ C m}\right)\left(1.00\cdot 10^{-6} \text{ C}\right)}{\left(0.0100 \text{ m}\right)^3}\hat{r} = 1.11476\cdot 10^{-19} \text{ N}$

ROUND: The force is reported to 3 significant figures.

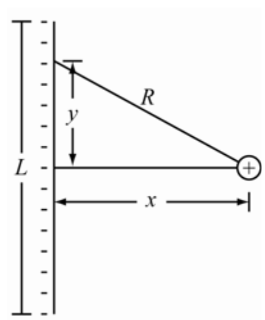
(a) The force between the dipole and the charge is $1.11\cdot 10^{-19}$ N.

(b) The molecule is attracted to the charge regardless of the sign of the charge. This occurs because the charge of opposite sign on the dipole will move closer to the charge creating an attractive force.

DOUBLE-CHECK: The mass of a water molecule is $3.01\cdot 10^{-26}$ kg, meaning the force is relatively large. To view a dipole attracted to a charge, place a charged rod or comb near running water from a faucet.

22.47. **THINK:** Assuming that the wire is made of a conducting material, the charges will be uniformly distributed over its length. The wire will produce an electric field. This field in turn produces a force on a proton, causing the proton to accelerate. The wire has a length of $L = 1.33$ m and a total charge of $Q = -3.05\cdot 10^6 e$. The proton is $x = 0.401$ m away from the center of the wire.

SKETCH:



RESEARCH: The linear density of the wire is $\lambda = Q/L$. Due to the symmetry around the center of the wire the field produced is only along the x-axis. The electric field due to a segment of charge is $dE = \left(kdq/R^2\right)\cos\theta$. The distance from the charge to the segment of the wire is $R = \sqrt{x^2+y^2}$. The force on the proton is $F = ma = qE(r)$.

SIMPLIFY: The electric field is:

$$|E| = \int \frac{k|dq|}{R^2}\cos\theta = \int_{-L/2}^{L/2} \frac{k|\lambda|dy}{R^2}\left(\frac{x}{R}\right) = k|\lambda|x\int_{-L/2}^{L/2}\frac{dy}{\left(x^2+y^2\right)^{3/2}} = k\lambda x\left[\frac{y}{x^2\sqrt{x^2+y^2}}\right]_{-L/2}^{L/2}$$

$$= \left(\frac{k|\lambda|}{x}\right)\left[\frac{L/2}{\left(x^2+L^2/4\right)^{1/2}} - \frac{-L/2}{\left(x^2+L^2/4\right)^{1/2}}\right] = \frac{k|\lambda|L}{x\left(x^2+L^2/4\right)^{1/2}} = \frac{k|Q|}{x\left(x^2+L^2/4\right)^{1/2}}$$

The acceleration of the proton is $a = \dfrac{|q||\text{E}|}{m} = \dfrac{k|q||Q|}{mx\left(x^2+L^2/4\right)^{1/2}}$.

CALCULATE: $|E| = \dfrac{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}\right)\left|\left(-3.05\cdot 10^6\right)\left(1.602\cdot 10^{-19} \text{ C}\right)\right|}{0.401 \text{ m}\left[\left(0.401 \text{ m}\right)^2 + \left(1.33\right)^2/4\right]^{1/2}} = 0.0141062 \text{ N/C}$

$$a = \frac{1.602\cdot 10^{-19} \text{ C}}{1.672\cdot 10^{-27} \text{ kg}}\left(0.0141062 \text{ N/C}\right) = 1{,}351{,}561 \text{ m/s}^2$$

ROUND: The values are reported to 3 significant figures.
(a) The electric field produced by the wire at 0.401 m from its center is 0.0141 N/C.
(b) The acceleration of the proton is $1.35 \cdot 10^6 \text{ m/s}^2$.
(c) The force is attractive since the wire is negatively charged and the proton is positively charged. The force points towards the wire.
DOUBLE-CHECK: These are reasonable answers with appropriate units.

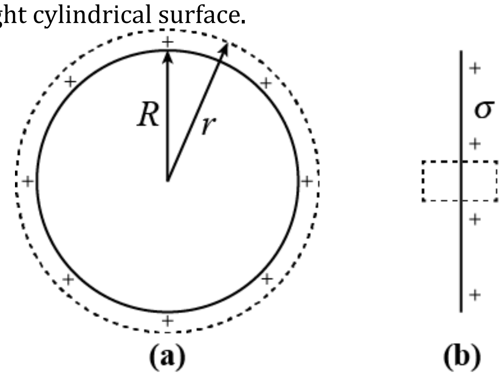
22.48. The flux through a Gaussian surface is the sum of the total charges within the surface divided by the permittivity of free space ε_0. $\Phi = \sum_{\varepsilon} Q_{\text{i}} = (3q) + (-q) + (2q) + (-7q)/\varepsilon_0 = -3q/\varepsilon_0$.

22.49. The sum of the flux through each surface is equal to the charge enclose divided by ε_0. $\sum_i \Phi_i = Q/\varepsilon_0$.

The charge is then

$$Q = \varepsilon_0 \sum_i \Phi_i = \left(8.85 \cdot 10^{-12} \text{ C}^2/\left(\text{N m}^2\right)\right)\left(-70.0 - 300.0 - 300.0 + 300.0 - 400.0 - 500.0\right) \text{ N m}^2$$
$$= -1.124 \cdot 10^{-8} \text{ C} \approx -1.12 \cdot 10^{-8} \text{ C}.$$

22.50. **THINK:** The first Gaussian surface is a sphere with radius $r = R +$ 0.00.0000010 m. This surface encloses all the charge on the sphere. The second Gaussian surface is a small, right cylinder, whose axis is perpendicular to the surface of the sphere and penetrates the surface. Taking the cylinder to be small compared to the sphere, we can consider the surface of the sphere to be locally flat. The charge density on the surface of the sphere will be the total charge divided by the surface area of the sphere. For this case, the electric field is constant outside the sphere and zero inside the sphere.
SKETCH: The sketch shows the two Gaussian surfaces.
(a) shows the spherical surface
(b) shows the small, right cylindrical surface.



RESEARCH: For the spherical Gaussian surface, the electric field just outside the surface of the sphere is the same as a point charge, so the electric field is radial and perpendicular to the Gaussian surface. So we have $\Phi = \oiint \vec{E} \bullet d\vec{A} = E\left(4\pi r^2\right) = \dfrac{q}{\varepsilon_0}$. We choose a very small right cylinder so that the surface of the sphere is locally flat as show in the sketch. In this case, the electric field is perpendicular to surface. The charge density is $\sigma = \dfrac{q}{4\pi R^2}$. The electric field is parallel to the sides of the cylinder and perpendicular to the ends of the cylinder. So we have $\Phi = \oiint \vec{E} \bullet d\vec{A} = E_{\text{inside}} A + E_{\text{outside}} A = \dfrac{q}{\varepsilon_0} = \dfrac{\sigma A}{\varepsilon_0}$. The electric field inside the sphere is zero.

SIMPLIFY: For the spherical surface, the electric field is $E = \dfrac{q}{\varepsilon_0\left(4\pi r^2\right)} = k\dfrac{q}{r^2}.$

For the cylindrical surface, the electric field is $E_{\text{outside}} = \dfrac{\sigma}{\varepsilon_0} = \dfrac{\dfrac{q}{4\pi R^2}}{\varepsilon_0} = k\dfrac{q}{R^2}.$

CALCULATE: In this case, is very close to , so the answer for both cases is $E = k\dfrac{q}{R^2} = \left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\dfrac{6.1\cdot 10^{-6}\text{ C}}{\left(0.15\text{ m}\right)^2} = 2.44373\cdot 10^6 \text{ N/C}.$ The charge is positive so the field points outward from the surface of the sphere.

ROUND: We round the magnitude of the electric field to two significant figures $E = 2.4\cdot 10^6 \text{ N/C}.$

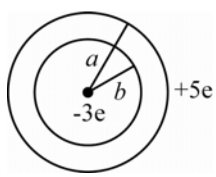
DOUBLE-CHECK: The units are correct for an electric field. The rather high magnitude results from the fact that field is calculated very close to the surface of the charged sphere. Our result for the small right cylindrical Gaussian surface in only correct very close to the surface of the sphere, so that the surface can considered locally flat.

22.51. The cube does not contain any charges, thus the total flux must be zero.
$A\left(E_A + E_B + E_C + E_D + E_E + E_F\right) = \sum_i \Phi_i = 0,$ and therefore,

$$\begin{aligned} E_F &= -\left(E_A + E_B + E_C + E_D + E_E\right) \\ &= -\left(-15.0\text{ N/C} + 20.0\text{ N/C} + 10.0\text{ N/C} + 25.0\text{ N/C} + 20.0\text{ N/C}\right) \\ &= -60\text{ N/C}. \end{aligned}$$

The field on the face F is 60.0 N/C into the face of the cube.

22.52.

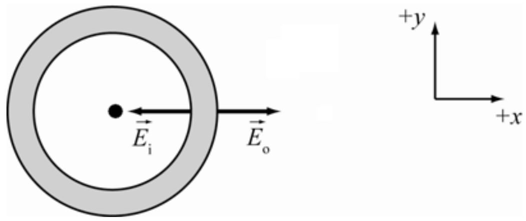


The charge inside the sphere induces a charge of $+3e$ on the inside surface of the sphere. The $+3e$ charge must come from somewhere. In this case the $+3e$ charge is removed from the outer surface charge. The outer surface charge is then $+2e$. The total charge within the material of the sphere is $+5e$.

22.53. Gauss's Law states that $\oiint E\cdot dA = \dfrac{q_{enc}}{\varepsilon_0}.$ The integral over the sphere gives $\oiint E\cdot dA = EA = E\left[4\pi R^2\right] = \dfrac{q_{\text{enc}}}{\varepsilon_0} \Rightarrow E = \dfrac{q_{enc}}{4\pi R^2\varepsilon_0}.$ The electric field outside a uniform distribution of charge is identical to the field created by a point charge of the same magnitude, located at the center of the distribution. Since the radius of the balloon never reaches R, the charge enclosed is constant and the electric field does not change.

22.54. **THINK:** The charges on the surface of the shell may be found using Gauss's Law. The inner and outer radii of the shell are $r_{\text{i}} = 8.00\text{ cm}$ and $r_{\text{o}} = 10.0\text{ cm}$ respectively. The electric field at the surface of the outer radius is 80.0 N/C pointing away from the center of the sphere. The electric field at the surface of the inner radius is 80.0 N/C and points towards the center of the sphere. Since the spherical shell does not produce any field in its interior, we can infer that there is a negative charge inside the hollow portion, equivalent to a point charge at the center.

SKETCH:



RESEARCH: Gauss's Law states that $\Phi_e = \frac{q_{enc}}{\varepsilon_0} = \oiint E \cdot dA.$

SIMPLIFY: For a spherically symmetric electric field, the charge enclosed within a Gaussian sphere of radius R is given by $\frac{q_{enc}}{\varepsilon_0} = \oiint E \cdot dA \Rightarrow q_{enc} = \varepsilon_0 E\left(4\pi R^2\right).$ This gives the (negative) charge at the center of the sphere. Since the field between the inner and outer surfaces of the shell is zero, this is also equal to the total (positive) surface charge at the inner radius of the conductor: $q_i = \varepsilon_o E_i\left(4\pi r_i^2\right).$ The Gaussian surface around the whole sphere contains the charge at the center and the charge of the shell. Since the charges at the center and on the inner surface are equal and opposite and therefore cancel, the field at the outer surface can be calculated as being due solely to the charge on the outer surface: $q_o = \varepsilon_o E_o\left(4\pi r_o^2\right).$

CALCULATE: $q_i = \left(8.854 \cdot 10^{-12} \text{ C}^2/\text{N m}^2\right)\left(80.0 \text{ N/C}\right)4\pi\left(0.0800 \text{ m}\right)^2 = 5.6966 \cdot 10^{-11} \text{ C}$

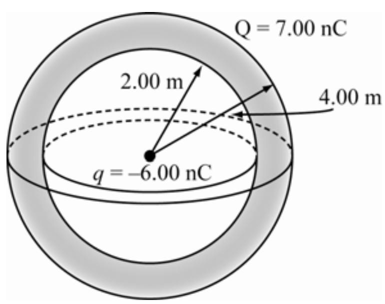
$$q_o = \left(8.854 \cdot 10^{-12} \text{ C}^2/\text{N m}^2\right)\left(80.0 \text{ N/C}\right)4\pi\left(0.100 \text{ m}\right)^2 = 8.9010 \cdot 10^{-11} \text{ C}$$

ROUND: Rounding to three significant figures, the inside and outside total charges over the surface of the sphere are $5.70 \cdot 10^{-11}$ C and $8.90 \cdot 10^{-11}$ C, respectively.

DOUBLE-CHECK: These are reasonable answers with appropriate units. As you would expect, given that the field strength is the same inside and out, the ratio of the charges is the ratio of the square of the radii: $8^2 : 10^2 = 5.70 : 8.90.$

22.55. **THINK:** The electric field at various points can be found using Gauss's Law. This law can also be used to find the charge on the outside surface of the conductor. There is a $q = -6.00$ nC charge at the center of the sphere. The shell has inner and outer radii of $r_i = 2.00$ m and $r_o = 4.00$ m respectively. The shell has a total charge of $Q = +7.00$ nC.

SKETCH:



RESEARCH: Gauss's Law states that $\oiint \vec{E} d\vec{A} = q_{enc}/\varepsilon_0.$

SIMPLIFY: The electric field of charges with spherical symmetry are given by Gauss' Law, where we take spherical Gaussian surfaces: $\oiint \vec{E} \cdot d\vec{A} = EA = E\left(4\pi r^2\right) = \frac{q_{enc}}{\varepsilon_0}$ or $\vec{E}(r) = \frac{q_{enc}}{4\pi\varepsilon_0 r^2}.$ The electric field at $r_1 < r_i$ is $E(r_1) = \dfrac{q_{enc}}{4\pi\varepsilon_0 r_1^2}$. The electric field inside any conductor is always zero: $E(r_2) = 0$ where $r_i < r_2 < r_o$. The electric field outside of the conductor is $r_3 > r_o$. $E(r_3) = \dfrac{q_{enc}}{4\pi\varepsilon_0 r_3^2} = \dfrac{Q+q}{4\pi\varepsilon_0 r_3^2}$. Because the field inside the conductor must be zero, Gauss's Law indicates that the charge at the center of the shell is equal and oppposite to the charge on the inside of the shell: $E(r_2) = 0 = \dfrac{q_{enc}}{4\pi\varepsilon_0 r_2^2} = \dfrac{q+q_i}{4\pi\varepsilon_0 r_2^2}$ or $q = -q_i$. The charge on the sphere is equal to the sum of charges on the inner and outer surfaces of the shell $q_i + q_o = Q$. Thus, the outer surface charge is $\sigma = q_o / 4\pi r_o^2 = (Q - q_i)/4\pi r_o^2 = (Q+q)/4\pi r_o^2$.

CALCULATE:

(a) The electric field at $r_1 = 1.00$ m is $E_1 = \dfrac{-6.00 \cdot 10^{-9} \text{ C}}{4\pi\left(8.85 \cdot 10^{-12} \text{ C}^2/\left(\text{N m}^2\right)\right)(1.00 \text{ m})^2} = -53.951$ N/C.

(b) The electric field at $r_2 = 3.00$ m is $E_2 = 0$ N/C.

(c) The electric field at $r_3 = 5.00$ m is $E = \dfrac{\left(7.00 \cdot 10^{-9} \text{ C} - 6.00 \cdot 10^{-9} \text{ C}\right)}{4\pi\left(8.85 \cdot 10^{-12} \text{ C}^2/\left(\text{N m}^2\right)\right)(5.00 \text{ m})^2} = 0.3597$ N/C.

(d) The surface charge on the outside part of the shell is

$$\sigma = \frac{\left(7.00 \cdot 10^{-9} \text{ C} - 6.00 \cdot 10^{-9} \text{ C}\right)}{4\pi(4.00 \text{ m})^2} = 4.974 \cdot 10^{-12} \text{ C/m}^2.$$

ROUND: All the values have an accuracy of three significant figures.

(a) The electric field at $r_1 = 1.00$ m is -54.0 N/C.

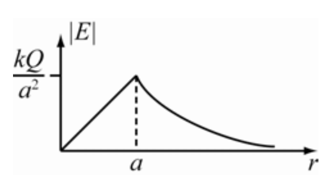
(b) The electric field at $r_2 = 3.00$ m is 0 N/C.

(c) The electric field at $r_3 = 5.00$ m is 0.360 N/C.

(d) The surface density on the outside surface is $4.97 \cdot 10^{-12}$ C/m^2.

DOUBLE-CHECK: These are reasonable results.

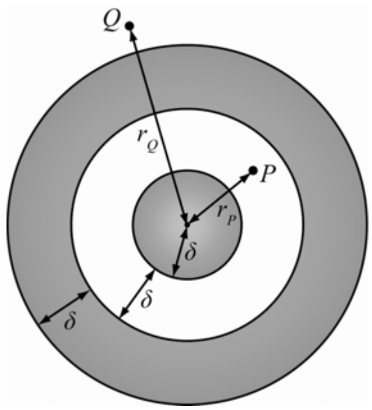
22.56. Inside the sphere of radius a, the charge density is $\rho = \dfrac{Q_{tot}}{V} = \dfrac{Q_{tot}}{(4/3)\pi a^3}$ and is zero anywhere else. Gauss's Law states $\oiint \vec{E}\, d\vec{A} = \dfrac{q_{enc}}{\varepsilon_0}$. The area of the Gaussian surface is always taken to be $A = 4\pi r^2$ and by spherical symmetry, the E-field points radially. Thus, $\oiint \vec{E} \cdot d\vec{A} = \dfrac{q_{enc}}{\varepsilon_0}$ gives $E = \left(\dfrac{q_{enc}}{A\varepsilon_0}\right)\hat{r} = \left(\dfrac{q_{enc}}{(4\pi\varepsilon_0)r^2}\right)\hat{r} = \left(\dfrac{kq_{enc}}{r^2}\right)\hat{r}$. If $r < a$, the enclosed charge is then $q_{enc} = \rho V = \dfrac{Q}{(4/3)\pi a^3}\left(\dfrac{4}{3}\pi r^3\right) = \dfrac{Qr^3}{a^3}$ and $E = \left(\dfrac{kq_{enc}}{r^2}\right) = \dfrac{kQr^3}{a^3 r^2} = \left(\dfrac{kQr}{a^3}\right)\hat{r}$. Otherwise, the surface encloses the whole charge Q. The electric field is then $E = \left(\dfrac{kQ}{r^2}\right)\hat{r}$ if $r > a$. Note that this behaves like a point charge, as would be expected once outside the radius. Below is a graph of $E(\vec{r})$.



22.57. Using Gauss's Law $\oiint E \cdot dA = EA = E\left(4\pi r_{\text{E}}^2\right) = \dfrac{q_{\text{Earth}}}{\varepsilon_0}$. Solving for the charge gives

$$q_{\text{Earth}} = \varepsilon_0 E\left(4\pi r_{\text{Earth}}^2\right) = \left(8.85\cdot 10^{-12}\text{ C}^2/\text{N m}^2\right)\left(-150.\text{ N/C}\right)4\pi\left(6371\cdot 10^3\text{ m}\right)^2 = -6.7711\cdot 10^5\text{ C}$$
$$\approx -6.77\cdot 10^5\text{ C}.$$

22.58. Let $\delta = 10.0$ cm be the radius of the solid sphere, the distance between the solid sphere and the inner part of the hollow sphere, and the thickness of the hollow sphere. Let $r_P = 15.0$ cm be the distance from the center to the point *P*, and let $r_Q = 35.0$ cm be the distance from the center to the point *Q*.



(a) The Gaussian surface at r_p encloses the charge on the inner sphere. $E_1\left(4\pi r_p^2\right) = \dfrac{q_{\text{enc}}}{\varepsilon_0}$. The charge on the inner sphere is

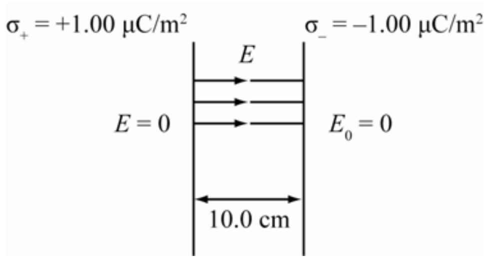
$$q = \varepsilon_0 4\pi E_1 r_p^2 = \left(8.85\cdot 10^{-12}\text{ C}^2/\text{N m}^2\right)4\pi\left(-10000\text{ N/C}\right)\left(0.150\text{ m}\right)^2 = -2.50\cdot 10^{-8}\text{ C} = -25.0\text{ nC}.$$

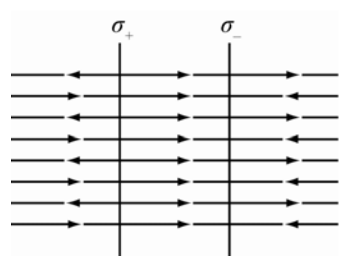
(b) For the electric field inside the shell to be zero, the charge on the inner surface of the shell must be equal to the negative of the charge on the inner sphere. $E = \dfrac{q_{\text{enc}}}{4\pi\varepsilon_0 r^2} = \dfrac{\left(q+q_{\text{i}}\right)}{4\pi\varepsilon_0 r^2} = 0$ or $q_{\text{i}} = -q$. The charge on the inner surface of the shell is then $q_{\text{i}} = -q = -\left(-25.0\text{ nC}\right) = 25.0\text{ nC}.$

(c) The Gaussian surface at $r_Q = 35.0$ cm from the center encloses the inner charge and the charge on the shell: $E_2\left(4\pi r_Q^2\right) = \dfrac{q_{\text{enc}}}{\varepsilon_0} = \dfrac{\left(q+q_{\text{shell}}\right)}{\varepsilon_0}$ or $q + q_{\text{shell}} = \varepsilon_0 4\pi E_2 r_Q^2$. The charge on the shell is the sum of the charge on the inner and outer surfaces of the shell: $q_{\text{shell}} = q_{\text{i}} + q_{\text{o}}$. The charge on the outer surface of the shell is

$$q_{\text{o}} = q_{\text{shell}} - q_{\text{i}} = q_{shell} - (-q) = q_{shell} + q = \varepsilon_0 4\pi E_2 r_Q^2$$
$$= \left(8.85\cdot 10^{-12}\text{ C}^2/\text{N m}^2\right)4\pi\left(1.00\cdot 10^4\text{ N/C}\right)\left(0.350\text{ m}\right)^2 = 1.36\cdot 10^{-7}\text{ C} = 0.136\,\mu\text{C}.$$

22.59.





The field due to either of the two sheets is found by taking a Gaussian cylinder with top-area A through either plane. Then $\oiint E \cdot dA = 2 \cdot EA = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{2\varepsilon_0}$. For the positively charged plate the field points normally away from it. The negatively charged plate has field lines pointing towards it. Adding these fields together gives zero on the outside of the two plates, and $E = 2E_0 = 2\left(\frac{\sigma}{2\varepsilon_0}\right) = \frac{\sigma}{\varepsilon_0}$ within the two plates, directed towards the negative plate. The field is

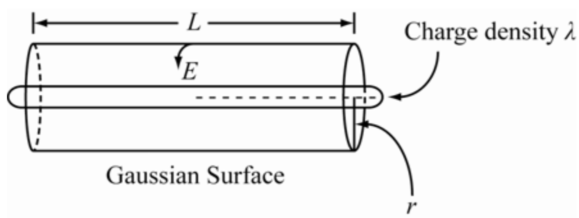
$$E = \frac{\left(1.00 \cdot 10^{-6} \text{ C}^2/\text{m}^2\right)}{\left(8.85 \cdot 10^{-12} \text{ C}^2/\left(\text{N m}^2\right)\right)} = 1.13 \cdot 10^5 \text{ N/C},$$

and points from the positive plate to the negative plate. Therefore, the force an electron will experience between the two plates is given by

$$F = qE = eE = \left(1.602 \cdot 10^{-19} \text{ C}\right)\frac{\left(1.00 \cdot 10^{-6} \text{ C}^2/\text{m}^2\right)}{\left(8.85 \cdot 10^{-12} \text{ C}^2/\left(\text{N m}^2\right)\right)} = 1.8107 \cdot 10^{-14} \text{ N} \approx 1.81 \cdot 10^{-14} \text{ N}$$

Since the E-field outside the plates is 0, the electron will experience no force outside of the two plates.

22.60. The magnitude of an electric field is $1.23 \cdot 10^3$ N/C at a distance 50.0 cm perpendicular to the wire. The direction of the electric field is pointing toward the wire.



Applying Gauss's Law on the surface shown above gives:

Noting that $\lambda = \frac{Q_{enc}}{L} \Rightarrow Q_{enc} = \lambda L$, $\oiint E \cdot dA = EA = E(2\pi rL) = \frac{\lambda L}{\varepsilon_0} \Rightarrow \lambda = 2\pi Er\varepsilon_0$. Observing the E-field's inward direction as negative, the charge density of the wire is

$$\lambda = 2\pi r\varepsilon_0 E = 2\pi\left(0.500 \text{ m}\right)\left(8.85 \cdot 10^{-12} \text{ C}^2/\left(\text{N m}^2\right)\right)\left(-1.23 \cdot 10^3 \text{ N/C}\right)$$
$$= -3.4204 \cdot 10^{-8} \text{ C/m} \approx -3.42 \cdot 10^{-8} \text{ C/m}.$$

The number of electrons per meter is

$$N = \frac{\left(-3.42 \cdot 10^{-8} \text{ C/m}\right)}{\left(-1.602 \cdot 10^{-19} \text{ C}\right)} = 2.135 \cdot 10^{11} \text{ electrons/m}$$
$$N \approx 2.14 \cdot 10^{11} \text{ electrons per meter.}$$

22.61. **THINK:** A solid sphere of radius R has a non-uniform charge density $\rho = Ar^2$. Integrate the sphere.

SKETCH: Not required.

RESEARCH: The total charge is given by $Q = \int_{\text{Sphere}} \rho dV$.

SIMPLIFY: Integrating in the spherical polar coordinate yields:

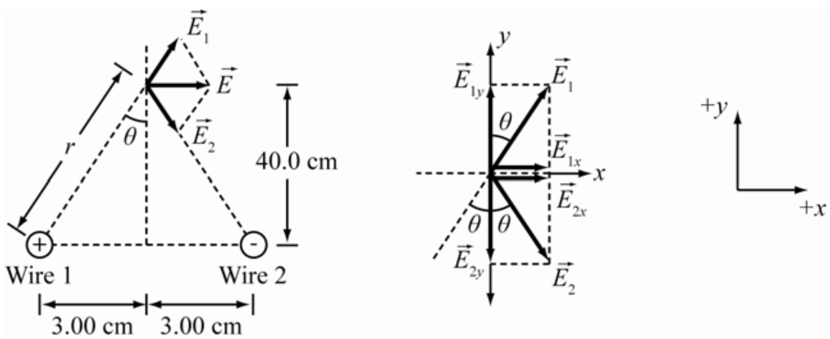
$$Q=\int_0^R\int_0^{2\pi}\int_0^{\pi}\rho(r)r^2\sin\theta d\theta d\phi dr=\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi\int_0^R\left(Ar^2\right)r^2dr=4\pi A\int_0^R r^4dr$$
$$=4\pi A\left[\frac{r^5}{5}\right]_{r=o}^{r=R}=4\pi A\frac{R^5}{5}=\frac{4}{5}\pi AR^5.$$

CALCULATE: Not required.
ROUND: Not required.
DOUBLE-CHECK: One can check the result by single-variable integration, using spherical shells:
$dV=A_{shell}dr=\left(4\pi r^2\right)dr$

$$Q=\int\rho dV=\int_0^R\left(Ar^2\right)\left(4\pi r^2\right)dr=4\pi A\int_0^R r^4dr=4\pi A\left[\frac{r}{5}\right]_0^R=\frac{4}{5}\pi AR^5$$

Which agrees with the previous answer.

22.62. **THINK:** This is a superposition of two electric fields.
SKETCH:



RESEARCH: The magnitude of the electric field of a charged wire at a distance r from the wire is, by simple application of Gauss' Law, $E=\lambda/2\pi\varepsilon_0 r$, where λ is the linear charge density of the wire. The net electric field at P is given by $\vec{E}_{\text{net}}=\frac{\lambda}{2\pi\varepsilon_0 r}\left(\sin\theta\hat{x}+\cos\theta\hat{y}\right)+\frac{\lambda}{2\pi\varepsilon_0 r}\left(\sin\theta\hat{x}-\cos\theta\hat{y}\right)$

SIMPLIFY: By symmetry, $\vec{E}_{\text{net}}=\vec{E}_x=\left(\frac{\lambda}{2\pi\varepsilon_0 r}\right)\sin\theta\hat{x}.$

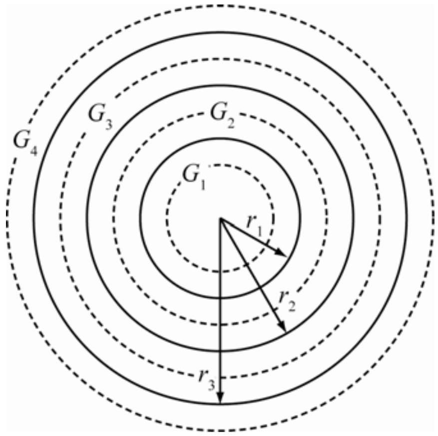
CALCULATE: $\lambda=1.00\ \mu\text{C/m}$, $r=\sqrt{3.00^2+40.0^2}\text{ cm}=40.11\text{ cm}$, $\sin\theta=\frac{3.00\text{ cm}}{40.11\text{ cm}}=0.07479$ and

$$\vec{E}_{\text{net}}=\frac{\left(1.00\cdot10^{-6}\text{ C/m}\right)\left(0.07479\right)}{2\pi\left(8.85\cdot10^{-12}\text{ C}^2/\left(\text{N m}^2\right)\right)\left(0.4011\text{ m}\right)}\hat{x}=\left(6707\text{ N/C}\right)\hat{x}.$$

ROUND: Keeping three significant figures yields $\vec{E}_{\text{net}}=\left(6.71\text{ kN/C}\right)\hat{x}.$

DOUBLE-CHECK: Since the vertical components cancel out, it makes sense that the answer is in the *x*-direction.

22.63. **THINK:** Since this problem has a spherical symmetry, it is possible to apply Gauss's Law.
SKETCH:



r_1 is the radius of a sphere with a charge density $\rho = 120 \text{ nC/cm}^3$. r_2 is the inner radius of a conducting shell. r_3 is the outer radius of the conducting shell. The shell has a net charge q_s.

RESEARCH: For this problem, four Gaussian surfaces, G_1 (within the sphere), G_2 (between the sphere and the shell), G_3 (within the shell), and G_4 (outside the shell) are used. By applying Gauss's Law on each surface, the electric field can be determined.

SIMPLIFY: For the Gaussian surface G_1, applying Gauss's Law gives

$$\oint \vec{E} d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0} \Rightarrow -E \oint dA = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\rho V_{\text{enc}}}{\varepsilon_0}.$$

(a) Using $V_{\text{enc}} = \frac{4}{3}\pi r_a^3$, the electric field is $E_1\left(4\pi r_a^2\right) = \frac{\rho\left(\frac{4}{3}\pi r_a^3\right)}{\varepsilon_0}$, $E = \frac{\rho r_a}{3\varepsilon_0}$.

(b) For the Gaussian surface G_2, applying Gauss's Law yields:

$$\oiint E \cdot dA = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\rho\left(\frac{4}{3}\pi r_1^3\right)}{\varepsilon_0} \rightarrow E\left(4\pi r_b^2\right) = \frac{4\rho\pi r_1^3}{3\varepsilon_0} \Rightarrow E = \frac{\rho r_1^3}{3\varepsilon_0 r_b^2}.$$

(c) For the Gaussian surface G_3, the electric field is zero since the surface is in a conductor.

(d) For the Gaussian surface G_4, applying Gauss's Law gives

$$\oiint E \cdot dA = \frac{q_{\text{enc}}}{\varepsilon_0} \Rightarrow E\left(4\pi r_d^2\right) = \frac{q_{sphere} + q_{shell}}{\varepsilon_0} = \frac{4\rho\pi r_1^3}{3\varepsilon_0} + \frac{q_{\text{shell}}}{\varepsilon_0} \Rightarrow E = \frac{\rho r_1^3}{3\varepsilon_0 r_d^2} + \frac{q_{shell}}{4\pi\varepsilon_0 r_d^2}$$

$$E = \frac{1}{r_d^2 \varepsilon_0}\left[\frac{\rho r_1^3}{3} + \frac{q_{shell}}{4\pi}\right].$$

CALCULATE: Substituting the numerical values, $\rho = 120 \text{ nC/cm}^3 = 0.12 \text{ C/m}^3$, $r_1 = 0.12$ m, $r_2 = 0.300$ m, $r_3 = 0.500$ m, $r_a = 0.100$ m, $r_b = 0.200$ m and $r_d = 0.800$ m yields the electric fields:

(a) $E = \frac{\rho r_a}{3\varepsilon_0} = \frac{\left(0.12 \text{ C/m}^3\right)\left(0.100 \text{ m}\right)}{3\left(8.85 \cdot 10^{-12} \text{ C}^2/\text{Nm}^2\right)} = 4.518 \cdot 10^8 \text{ N/C}$

(b) $E = \frac{\rho r_1^3}{3\varepsilon_0 r_b^2} = \frac{\left(0.12 \text{ C/m}^3\right)\left(0.12 \text{ m}\right)^3}{3\left(8.85 \cdot 10^{-12} \text{ C}^2/\text{Nm}^2\right)\left(0.200 \text{ m}\right)^2} = 1.953 \cdot 10^8 \text{ N/C}$

(c) $E = 0$ since it is in the conducting shell.

(d) $$E=\frac{1}{r_d^2\varepsilon_0}\left[\frac{\rho r_1^3}{3}+\frac{q_{shell}}{4\pi}\right]=\frac{1}{\left(8.85\cdot 10^{-12}\text{ C}^2/\text{Nm}^2\right)\left(0.800\text{ m}\right)^2}\left[\frac{\left(0.12\text{ C/m}^3\right)\left(0.12\text{ m}\right)^3}{3}+\frac{-2.00\cdot 10^{-3}\text{ C}}{4\pi}\right]$$
$$=-1.589\cdot 10^7\text{ N/C}$$

ROUND: Rounding to three significant figures:

(a) $E=4.52\cdot 10^8$ N/C

(b) $E=1.95\cdot 10^8$ N/C

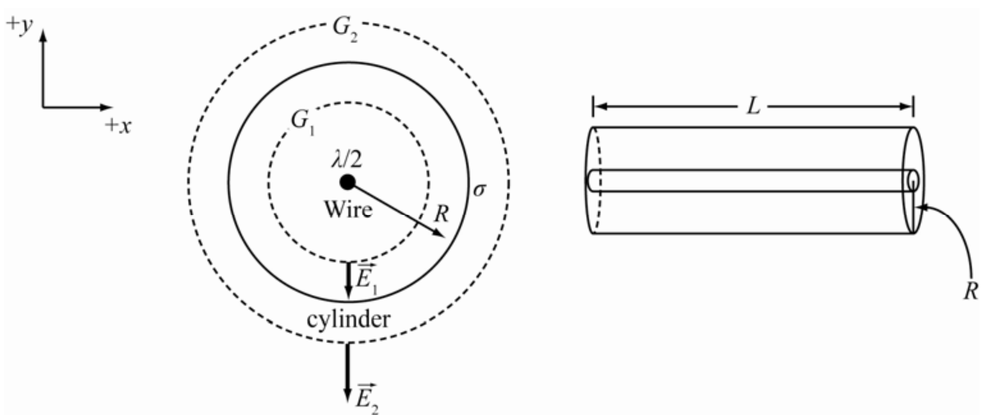
(c) $E=0$ since it is in the conducting shell.

(d) $E=-1.59\cdot 10^7$ N/C

DOUBLE-CHECK: The values of electric fields have the correct units and are of reasonable orders of magnitude.

22.64. **THINK:** Using the symmetry of a cylinder, Gauss's Law can be applied.

SKETCH:



Note that the Gaussian surfaces G_1 and G_2 are cylindrical surfaces with radii r_1 and r_2 and a length L.

RESEARCH: The electric field can be determined by applying Gauss's Law on the Gaussian surfaces G_1 and G_2.

SIMPLIFY: For the Gaussian surface G_1, applying Gauss's Law produces

$$\oint\vec{E}_1 d\vec{A}=\frac{q_{\text{enc}}}{\varepsilon_0}=\frac{\left(\lambda/2\right)L}{\varepsilon_0}\;\Rightarrow\;\vec{E}_1\left(2\pi r_1 L\right)=\frac{\left(\lambda/2\right)L}{\varepsilon_0}\hat{r}\;\Rightarrow\;E_1=\frac{\lambda}{4\pi\varepsilon_0 r_1}\hat{r}.$$

Similarly for the Gaussian surface G_2, using Gauss's Law gives

$$\oint\vec{E}_2 d\vec{A}=\frac{q_{\text{enc}}}{\varepsilon_0}=\frac{\left(\lambda/2\right)L+\sigma\left(2\pi RL\right)}{\varepsilon_0}\;\Rightarrow\;E_2\left(2\pi r_2 L\right)=\frac{\left(\lambda/2\right)L+\sigma\left(2\pi RL\right)}{\varepsilon_0}\hat{r}\;\Rightarrow\;\vec{E}_2=\frac{\lambda+4\pi R\sigma}{4\pi\varepsilon_0 r_2}.$$

Therefore, the expressions of the electric fields are:

(a) For $r\le R$, the electric field is $\vec{E}=\dfrac{\lambda}{4\pi\varepsilon_0 r}\hat{r}.$

(b) For $r\ge R$, the electric field is $\vec{E}=\dfrac{\lambda+4\pi R\sigma}{4\pi\varepsilon_0 r_2}\hat{r}.$

CALCULATE: Not required.

ROUND: Not required.

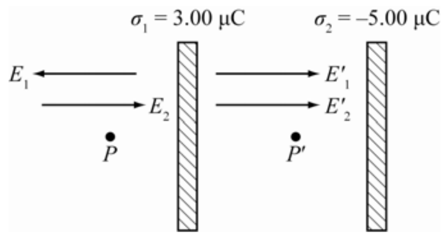
DOUBLE-CHECK: Since the metal cylinder is a conductor, all its charge resides on its outer surface. This means that the field inside the cylinder is not affected by the charge on the cylinder. Therefore, for $r\le R$, the electric field is only due to the wire. For $r\ge R$, the charge on the cylinder produces an electric field as if all its charge was concentrated in the center of the cylinder. Therefore, the electric field can be found by replacing $\lambda/2$ with $(\lambda/2)+2\pi R\sigma$ as the new linear density of a wire.

22.65. **THINK:** Use the values from the question: $\sigma_1 = 3.00\ \mu\text{C/m}^2$, and $\sigma_2 = -5.00\ \mu\text{C/m}^2$.

(a) The total field can be determined by superposition of the fields from both plates. The field contributions from the two charged sheets are opposing each other at point *P*, to the left of the first sheet.

(b) The situation is similar to a) except that the fields due to both charged sheets point in the same direction at point P'.

SKETCH:



RESEARCH:

(a) At point P, the field due to sheet #1 is given by $E_1 = -(\sigma_1/2\varepsilon_0)\hat{x}$, and the field due to sheet #2 is given by $E_2 = -(\sigma_2/2\varepsilon_0)\hat{x}$. Note that $E_{\text{total}} = E_1 + E_2$.

(b) At point P', the field due to sheet #1 is given by $E_1' = (\sigma_1/2\varepsilon_0)\hat{x}$, and the field due to sheet #2 is given by $E_2' = -(\sigma_2/2\varepsilon_0)\hat{x}$. Again, $E_{\text{total}}' = E_1' + E_2'$.

SIMPLIFY:

(a) $E = \left(\dfrac{-\sigma_1}{2\varepsilon_0}\right)\hat{x} + \left(\dfrac{-\sigma_2}{2\varepsilon_0}\right)\hat{x} = \dfrac{-(\sigma_1+\sigma_2)}{2\varepsilon_0}$

(b) $E' = \left(\dfrac{\sigma_1}{2\varepsilon_0}\right)\hat{x} + \left(\dfrac{-\sigma_2}{2\varepsilon_0}\right)\hat{x} = \dfrac{(\sigma_1-\sigma_2)}{2\varepsilon_0}$

CALCULATE:

(a) $E_{\text{total}} = \dfrac{-(3.00-5.00)\cdot 10^{-6}\ \text{C/m}^2}{2\left(8.85\cdot 10^{-12}\ \text{C}^2/\left(\text{N m}^2\right)\right)}\hat{x} = \left(1.130\cdot 10^5\ \text{N/C}\right)\hat{x}$

(b) $E_{\text{total}} = \dfrac{\left(3.00-(-5.00)\right)\cdot 10^{-6}\ \text{N/C}}{2\left(8.85\cdot 10^{-12}\ \text{C}^2/\left(\text{N m}^2\right)\right)}\hat{x} = \left(4.520\cdot 10^5\ \text{N/C}\right)\hat{x}$

ROUND:

(a) $E_{\text{total}} = 1.13\cdot 10^5$ N/C in the positive *x*-direction

(b) $E_{\text{total}} = 4.52\cdot 10^5$ N/C in the positive *x*-direction

DOUBLE-CHECK: The results are reasonable because the answer in (b) is four times larger than that found in (a) since in (a) the fields are opposing each other and in (b) the fields are in same direction.

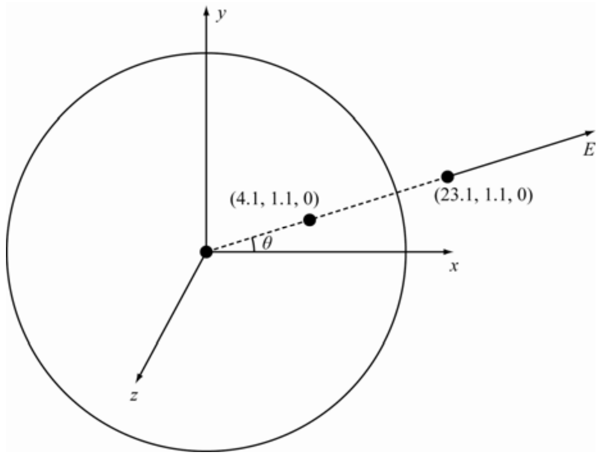
22.66. **THINK:**

(a) The field due to a charged sphere outside the radius of the sphere is equivalent to the field due to a point charge of equal magnitude at the center of the sphere.

(b) The electric field radiates outward, perpendicular to the surface of the sphere.

(c) The field inside a conductor is zero.

SKETCH:



RESEARCH:

(a) The field is given by: $q = 0.271\text{ nC}$ and $r^2 = (23.1\text{ cm})^2 + (1.10\text{ cm})^2 + (0\text{ cm})^2$.

(b) The angle is given by $\tan\theta = (1.10\text{ cm})/(23.1\text{ cm})$ or $\theta = \tan^{-1}(1.10\text{ cm})/(23.1\text{ cm})$.

(c) The field is zero inside a conductor.

SIMPLIFY: Not required.

CALCULATE:

(a) $$E = \frac{(0.271\text{ nC})(10^{-9}\text{ C/nC})}{4\pi(8.85\cdot 10^{-12}\text{ C}^2/\text{N m}^2)\left((0.231)^2 + (0.0110)^2\right)\text{ N/C}} = 45.56\text{ N/C}$$

(b) $$\theta = \tan^{-1}\left(\frac{1.10\text{ cm}}{23.1\text{ cm}}\right) = 2.7263^\circ$$

(c) 0 N/C

ROUND:

Rounding to three significant figures:

(a) $E \approx 45.6\text{ N/C}$

(b) $\theta \approx 2.73^\circ$

(c) 0 N/C

DOUBLE-CHECK:

(a) Not required.

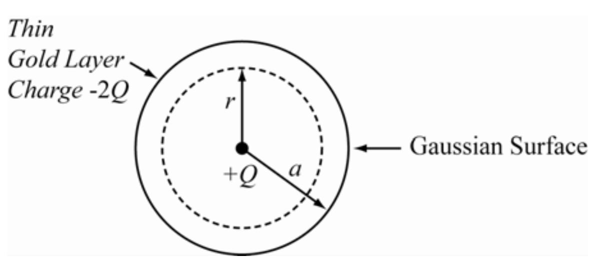
(b) Since the *y*-component is much less than the *x*-component I expected the angle to be small, which it is.

(c) Not required.

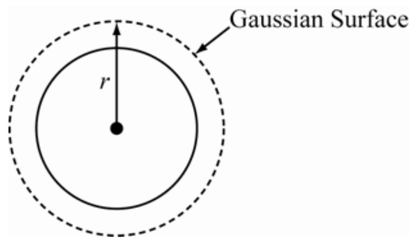
22.67. **THINK:** The spherical symmetry of the charged object allows the use of Gauss's Law to calculate the electric field. To do this, separate Gaussian surfaces must be considered for $r < a$ and $r > a$.

SKETCH:

(a)



(b)



RESEARCH:

(a) The total charge inside the Gaussian surface is given by $q=\int_0^r \rho_0 4\pi r^2 dr^1$. The charge density is $\rho_0 = Q/V_{\text{sphere}}$, and the volume is $V_{\text{sphere}} = (4/3)\pi a^3$.

(b) The total charge is simply the charge of the non-conducting layer and the gold layer: $q = \text{Total Charge} = Q - 2Q = -Q$.

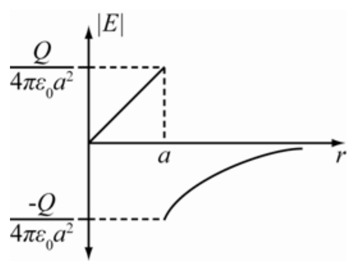
Gauss's Law states $\oint \vec{E}\cdot d\vec{A} = q_{\text{enc}}/\varepsilon_0$. Since the Gaussian surface in this case is a sphere, Gauss's Law simplifies to $E(4\pi r^2) = q/\varepsilon_0$.

SIMPLIFY:

(a) $q=\int_0^r \rho_0 4\pi r^2 dr^1 = \rho_0 \int_0^r 4\pi r^2 dr^1 = \rho_0 (4/3)\pi r^3 = \left(\frac{Q}{(4/3)\pi a^3}\right)\left(\frac{2}{3}\pi r^3\right)$. Substituting $\rho_0 = \frac{Q}{(4/3)\pi a^3}$, $q=\frac{Qr^3}{a^3}$. $E(4\pi r^2) = \frac{q_{enc}}{\varepsilon_0} = \frac{Qr^3}{a^3\varepsilon_0} \Rightarrow \vec{E}(\vec{r}) = \left(\frac{Qr}{4\pi a^3 \varepsilon_0}\right)\hat{r}$, $r<a$. The direction is radially outward.

(b) $E(4\pi r^2) = \frac{q_{enc}}{\varepsilon_0} = \frac{Q-2Q}{\varepsilon_0} = \frac{-Q}{\varepsilon_0} \Rightarrow E = -\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)$ for $r>a \Rightarrow \vec{E} = -\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)\hat{r}$. The direction is towards the center of the sphere.

(c)



The discontinuity at $r=a$ is due to the surface charge density of the gold. The charge on the gold layer causes a sudden spike in the total charge resulting in a discontinuity in the electric fields.

CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK:

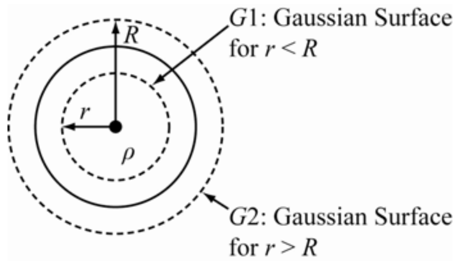
(a) The electric field increases r gets larger since the charge inside the Gaussian surface increases as a function of r^3 while the area increases as a function of r^2. Since the increase of the area decreases the field by a function of r^2 and the charge increases the field by r^3 it is reasonable that the field increases, as a function of r.

(b) The sphere acts like a point source is as expected.

(c) There is a discontinuity in the E v. r graph due to the presence of a surface charge density on the gold layer, which is expected.

22.68. THINK: By constructing Gaussian surfaces in both regions $r<R$ and $r>R$, the electric field can be calculated using Gauss's Law.

SKETCH:



RESEARCH: The total charge inside the Gaussian surface is given by $q=\int_0^r \rho_0 4\pi r^2 dr'$. The charge density is given by $\rho(r)=(\beta/r)\sin(\pi r/2R)$. For the Gaussian surface outside the sphere $(r>R)$, the total charge is given by $q=\int_0^r \rho_0 4\pi r^2 dr'$. The electric field can be calculate using Gauss's Law: $\oint \vec{E}\ d\vec{A}=q_{\text{enc}}/\varepsilon_0$, which for a spherical Gaussian surface is $E=(4\pi r^2)=q/\varepsilon_0$.

SIMPLIFY: For the case $r<R$, $q=\int_0^r\left(\frac{\beta}{r'}\right)\sin\left(\frac{\pi r'}{2R}\right)\left(4\pi r'^2\right)dr'=4\pi\beta\int_0^r r'\sin\left(\frac{\pi r'}{2R}\right)dr'$

$$=4\pi\beta\left[\left(\frac{-2Rr}{\pi}\right)\cos\left(\frac{\pi r}{2R}\right)_0^r+\left(\frac{2R}{\pi}\right)\int_0^r\cos\left(\frac{\pi r'}{2R}\right)dr'\right]$$

Integration by parts: $q=4\pi\beta\left[\left(\frac{-2Rr}{\pi}\right)\cos\left(\frac{\pi r}{2R}\right)+\left(\frac{2R}{\pi}\right)^2\sin\left(\frac{\pi r'}{2R}\right)\right]_0^r$

$$=\frac{-8\beta R}{\pi}\left[\pi r\cdot\cos\left(\frac{\pi r}{2R}\right)-2R\sin\left(\frac{\pi r}{2R}\right)\right].$$

For the case $r>R$, q is given by

$$q=\int_0^R\rho(r'')\left(4\pi r'^2\right)dr'=4\pi\beta\left[\left(\frac{-2R(R)}{\pi}\right)\cos\left(\frac{\pi R}{2R}\right)+\left(\frac{2R}{\pi}\right)^2\sin\left(\frac{\pi R}{2R}\right)\right]_0^R$$

$$=4\pi\beta\left[\left(\frac{-2R^2}{\pi}\right)(0)+\left(\frac{4R^2}{\pi^2}\right)\sin\left(\frac{\pi}{2}\right)\right]=4\pi\beta\left[\left(\frac{-2R^2}{\pi}\right)(0)+\left(\frac{4R^2}{\pi^2}\right)(1)\right]=\frac{16\beta R^2}{\pi}.$$

The electric field is given by $E\left(4\pi r^2\right)=\frac{q}{\varepsilon_0}\ \Rightarrow\ E=\frac{q}{4\pi r^2\varepsilon_0}$.

For the case $r<R$,

$$E=\left(\frac{1}{4\pi r^2\varepsilon_0}\right)\cdot\frac{-8\beta R}{\pi}\left[\pi r\cdot\cos\left(\frac{\pi r}{2R}\right)-2R\sin\left(\frac{\pi r}{2R}\right)\right]=\frac{-8\beta Rk}{\pi r^2}\left[\pi r\cdot\cos\left(\frac{\pi r}{2R}\right)-2R\sin\left(\frac{\pi r}{2R}\right)\right]\qquad(1)$$

For $r>R$,

$$q=\left(\frac{16\beta\rho^2}{\pi}\right)\left(\frac{1}{4\pi r^2\varepsilon_0}\right)=\left(\frac{4\beta R^2}{\pi^2 r^2\varepsilon_0}\right).\qquad(2)$$

For $r=R$,

$$(1)=\frac{-8\beta Rk}{\pi R^2}\left[\pi R\cdot\cos\left(\frac{\pi R}{2R}\right)-2R\sin\left(\frac{\pi R}{2R}\right)\right]=\frac{-8\beta k}{\pi}\left[\pi\cdot\cos\left(\frac{\pi}{2}\right)-2\sin\left(\frac{\pi}{2}\right)\right]=\frac{-8\beta k}{\pi}\left[0-2(1)\right]=\frac{16\beta k}{\pi}$$

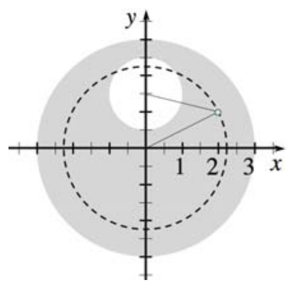
$$(2)=\left(\frac{4\beta R^2}{\pi^2\varepsilon_0 R}\right)=\left(\frac{4\beta}{\pi^2\varepsilon_0}\right)=\frac{16\beta k}{\pi}$$

$$\therefore(1)=(2)$$

The expressions are equal when $r=R$.

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: The two expressions are equal at $r = R,$ which should be the case since there are no surface charge densities present cause discontinuities for $r > R,$ the objects act like a point source, which is expected from a charged sphere.

22.69. **THINK:** The principle of superposition can be used to find the electric field at the specified point. The electric field at the point $(2.00,1.00)$ is modeled as the sum of a positively charged cylindrical rod with no hole and a negatively charged cylindrical rod whose size and location are identical to those of the cavity. Let's first think about the case of the positively charged cylindrical rod without a hole. Since the point of interest is inside the rod, the entire charge distribution of the rod cannot contribute. Instead we draw our Gaussian surface as a cylinder with our point of interest on its rim (see sketch below, where the dashed circle in the cross-sectional view represents the Gaussian cylinder).
SKETCH:



RESEARCH: In section 22.9 of the textbook it was shown that for cylindrical symmetry of the charge distribution the electric field outside the charge distribution can be written as $E = 2k\lambda / r$, where *r* is the distance to the central axis of the charge distribution and λ is the charge per unit length.
In the problem here the charge was initially uniformly distributed over the entire cross-sectional area, which means that the value of λ for the Gaussian surface and for the hole are proportional to their cross-sectional area: $\lambda_{\text{Gauss}} = \lambda_{\text{rod}}(r/R)^2$, and $\lambda_{\text{hole}} = -\lambda_{\text{rod}}(r_{\text{hole}}/R)^2$.
Now we have the tools to calculate the magnitudes of the individual electric fields of the rod and of the hole. What is left is to add the two, which is a vector addition. So we have to determine the *x*- and *y*-components of the fields individually and the combine them.
If E_1 is the field from the dashed cylinder and E_2 is that of the cavity then from considering the geometry the relation are given by: $E_{1x} = E_1 2/\left(2^2+1^2\right)^{1/2}$, $E_{2x} = E_2 2/\left(2^2+1^2\right)^{1/2}$, $E_{1y} = E_1/\left(2^2+1^2\right)^{1/2}$ and $E_{2y} = 0.5E_2/\left(2^2+0.5^2\right)^{1/2}$.
The net electric field is given by the following relations $E_x = E_{1x} + E_{2x}$ and $E_y = E_{1y} + E_{2y}$.
SIMPLIFY:
$E_1 = 2k\lambda_{\text{Gauss}}/r = 2k\lambda_{\text{rod}}(r/R)^2/r = 2k\lambda_{\text{rod}}r/R^2$
$E_2 = 2k\lambda_{\text{hole}}/r_2 = -2k\lambda_{\text{rod}}(r_{\text{hole}}/R)^2/r_2$
where r_2 is the distance between our point of interest and the center of the hole.
$$E_x = E_1\frac{2}{\left(2^2+1^2\right)^{1/2}} + E_2\frac{2}{\left(2^2+0.5^2\right)^{1/2}}, \text{ and } E_y = E_1\frac{1}{\left(2^2+1^2\right)} + E_2\frac{0.5}{\left(2^2+0.5^2\right)^{1/2}}.$$
CALCULATE: $r = \left((0.01\text{ m})^2 + (0.0200\text{ m})^2\right)^{1/2} = 0.02236\text{ m}$
$$r_2 = \left((0.00500\text{ m})^2 + (0.0200\text{ m})^2\right)^{1/2} = 0.02062\text{ m}$$

$$E_1 = \frac{2(8.99\cdot 10^9\text{ Nm}^2/\text{C}^2)(6.00\cdot 10^{-7}\text{ C/m})(0.02236\text{ m})}{(0.0300\text{ m})^2} = 2.680\cdot 10^5\text{ N/C}$$

$$E_{21} = -\frac{2(8.99\cdot 10^9\text{ Nm}^2/\text{C}^2)(6.00\cdot 10^{-7}\text{ C/m})}{(0.02062\text{ m})}\left(\frac{0.0100\text{ m}}{0.0300\text{ m}}\right)^2 = -0.581\cdot 10^5\text{ N/C}$$

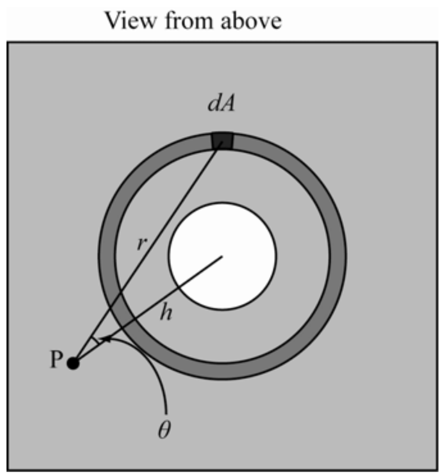
$E_x = 1.833\cdot 10^5$ N/C, $E_y = 1.339\cdot 10^5$ N/C

ROUND: $E_x = 183$ kN/C, $E_y = 134$ kN/C

DOUBLE-CHECK: We can calculate the magnitude and direction of the combined electric field and find: $E = \sqrt{E_x^2 + E_y^2} = 227$ kN/C, and $\theta = \tan^{-1}(E_y / E_x) = 36.1°$. If the hole would not have been drilled, the magnitude would have been the magnitude we calculated above for E_1, $E_1 = 268$ kN/C, and it would have pointed along the $\vec{r}$ vector with an angle of 26.6°. This means that our result states that the magnitude of the electric field is weakened due to the presence of the hole, and that it does not point radial outward any more, but further away from the *x*-axis. Both of these results are in accordance with expectations and add confidence to our result: the hole modifies the electric field somewhat, but does not do so radically.

22.70. **THINK:** Use the principle of superposition and model the problem as a positive infinite plane and a negative circular disc.

SKETCH:



RESEARCH: The electric field contributed by the plane is given by: $E_{\text{plane}} = \sigma / 2\varepsilon_0$. One can find the electric field of a disc by adding up the contributions from each small area. From the symmetry one many conclude that the field points vertically. The contribution of each small area to the field in the *y*-direction is given by: $dE = (-\sigma dA / 4\pi\varepsilon_0)(\cos\theta / r^2)$, $\cos\theta = h/r$, $r^2 = \rho^2 + h^2$, $E_{\text{disc}} = \int dE$. $E_{\text{total}} = E_{\text{plane}} + E_{\text{disc}}$.

$h = 0.200$ m, $R = 0.050$ m, $\sigma = 1.3$ C/m^2.

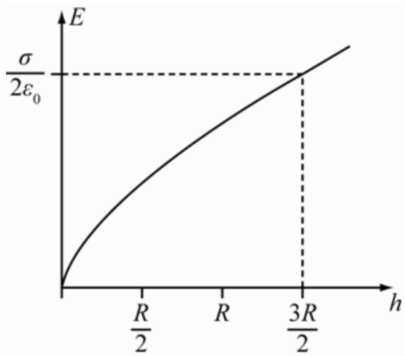
SIMPLIFY: $dE = -\frac{\sigma dA}{4\pi\varepsilon_0}\left(\frac{\cos\theta}{r^2}\right) = \frac{-\sigma(\rho d\varepsilon de)}{4\pi\varepsilon_0}\left(\frac{h}{(\rho^2 + h^2)^{3/2}}\right)$

$$E_{\text{disc}} = \left(-\frac{\sigma h}{4\pi\varepsilon_0}\right)\int_0^{2\pi} d\theta \int_0^R d\rho \frac{\rho}{(\rho^2 + h^2)^{3/2}} = \left(-\frac{\sigma h}{4\pi\varepsilon_0}\right)(2\pi)\left[(\rho^2 - h^2)^{-1/2}\right]_0^R = \frac{\sigma h}{2\varepsilon_0}\left[\frac{1}{(h^2 + R^2)^{1/2}} - \frac{1}{h}\right]$$

$$E_{\text{total}} = E_{\text{disc}} + E_{\text{plane}} = \frac{\sigma}{2\varepsilon_0} - \frac{\sigma h}{2\varepsilon_0}\left[\frac{1}{h} - \frac{1}{(h^2 + R^2)^{1/2}}\right] = \left(\frac{\sigma}{2\varepsilon_0}\right)\frac{h}{(h^2 + R^2)^{1/2}}$$

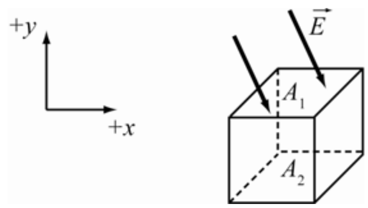
CALCULATE: $E_{\text{total}} = \dfrac{1.30\text{ C/m}^2}{2\left(8.85\cdot 10^{-12}\text{ C}^2/\text{ N m}^2\right)}\cdot\dfrac{0.200\text{ m}}{\left(\left(0.200\text{ m}\right)^2+\left(0.0500\text{ m}\right)^2\right)^{1/2}} = 7.125\cdot 10^{10}\text{ N/C}$

ROUND: $E_{\text{total}} \approx 7.13\cdot 10^{10}\text{ N/C}$



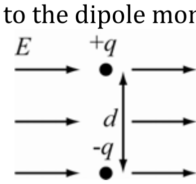
DOUBLE-CHECK: The plot shows that for large h the result is the same as that of an infinite plane without a hole as one would expect.

22.71. Regardless of what orientation the cube is in, we can always enclose it in a Gaussian surface that just covers the cube. Gauss's Law states that $\oiint \vec{E}d\vec{A} = q_{\text{enc}} / \varepsilon_0$.



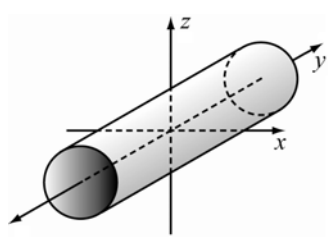
Now consider the flux through one particular face given by $\vec{E}\vec{A_1}$. There exists a flux through the opposite face given by $\vec{E}\vec{A_2}$ with the relation $\vec{E}\vec{A_1} = -\vec{E}\vec{A_2}$ since $\vec{A_1}$ and $\vec{A_2}$ point the opposite way. The sum of the flux contributed between the two opposite sides is $\vec{E}\vec{A_1} + \vec{E}\vec{A_2} = 0.$ If this calculation is done for each side then the total flux is 0 and hence the total charge must be 0 by Gauss's Law.

22.72. The dipole moment is given by $p = qd$ where d is the distance between the charges. The maximum torque is when the field is perpendicular to the dipole moment.



The torque is then $\tau = qEd = pE = \left(8.0\cdot 10^{-30}\text{ C m}\right)\left(500.0\text{ N/C}\right) = 4.0\cdot 10^{-27}\text{ N m}.$

22.73.

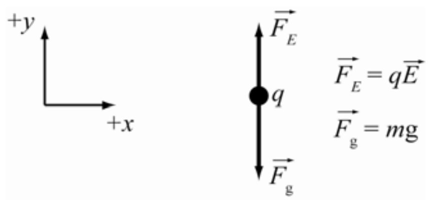


Consider a cylindrical Gaussian surface with a radius of 4.00 cm. By Gauss's Law, $\oiint \vec{E}d\vec{A} = q_{\text{enc}} / \varepsilon_0$. The charge inside the cylinder is $q = \rho\pi r^2 l$, so the field is given by

$$E(2\pi rl) = \frac{\rho\pi r^2 l}{\varepsilon_0} \Rightarrow E = \frac{\rho r}{2\varepsilon_0} = \frac{(6.40\cdot10^{-8}\text{ C/m}^3)(0.0400\text{ m})}{2(8.854\cdot10^{-12}\text{ m}^{-3}\text{ kg}^{-1}\text{ s}^4\text{ A}^2)} = 1.45\cdot10^2\text{ N/C}$$

away from the *y*-axis. The information concerning the radius of the cylinder is irrelevant.

22.74. The electric force and the gravitational force must balance.

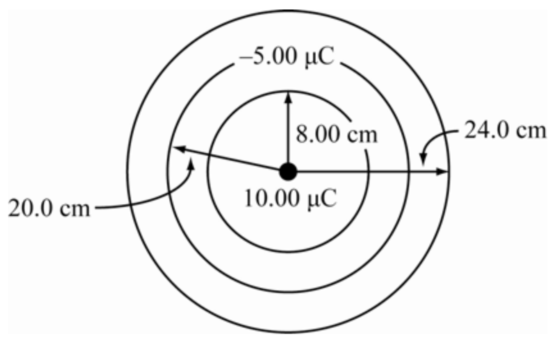


$qE - mg = 0 \Rightarrow E = mg/q, \quad g = 9.81\text{ m/s}^2$

(a) $m_{\text{electron}} = 9.109\cdot10^{-31}\text{ kg},\ q = -1.602\cdot10^{-19}\text{ C},\ E = \dfrac{(9.109\cdot10^{-31}\text{ kg})(9.81\text{ m/s}^2)}{-1.602\cdot10^{-19}\text{ C}} = -5.58\cdot10^{-11}\text{ N/C}$ with the field directed down.

(b) $m_{\text{proton}} = 1.672\cdot10^{-27}\text{ kg},\ q = 1.602\cdot10^{-19}\text{ C},\ E = \dfrac{(1.672\cdot10^{-27}\text{ kg})(9.81\text{ m/s}^2)}{1.602\cdot10^{-19}\text{ C}} = 1.02\cdot10^{-7}\text{ N/C}$ with the field directed up.

22.75.

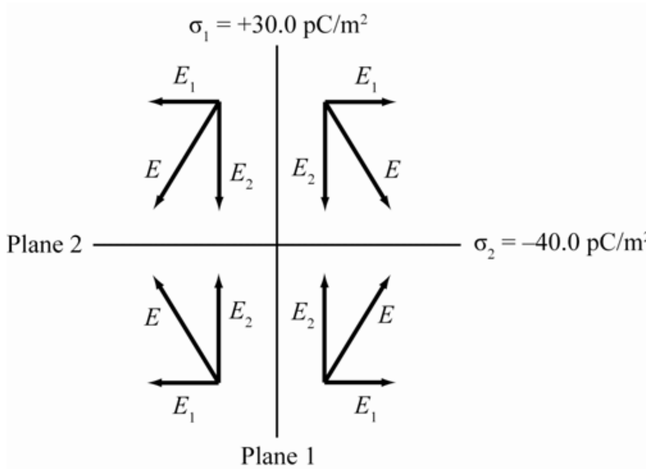


(a) Construct a Gaussian surface (spherical) with radius between 20.0 cm and 24.0 cm. Gauss's Law states that the total flux is equal to q/ε_0, since the electric field inside the last metallic shell is zero, the flux must be zero and hence the total charge must be zero. Since the total charge to be zero:

$$q_{\text{inside wall}} + 10.00\ \mu\text{C} - 5.00\ \mu\text{C} = 0 \Rightarrow q_{\text{inside wall}} = -5.00\ \mu\text{C}.$$

(b) Constructing a Gaussian sphere that contains all the shells, it can be determined that since the electric field is zero, outside the largest shell the flux is also zero and hence the total charge must be zero. $q_{\text{outside wall}} + q_{\text{inside wall}} + 10.00\ \mu\text{C} - 5.00\ \mu\text{C} = 0 \Rightarrow q_{\text{outside wall}} - 5.00\ \mu\text{C} - 5.00\ \mu\text{C} + 10.00\ \mu\text{C} = 0$, which then implies $q_{\text{outside wall}} = 0$.

22.76.



The fields from both plates are always perpendicular to each other. The field E_1 from plane 1 always points away from plane 1. The field E_2 from plane 2 always points toward plane 2. The combined field E points in different directions depending on where you measure it, but the magnitude of the field is the same everywhere.

$$E = \sqrt{E_1^2 + E_2^2} = \sqrt{\left(\frac{\sigma_1}{2\varepsilon_0}\right)^2 + \left(\frac{\sigma_2}{2\varepsilon_0}\right)^2}$$

$$E = \frac{\sqrt{\sigma_1^2 + \sigma_2^2}}{2\varepsilon_0} = \frac{\sqrt{\left(30.0 \text{ pC/m}^2\right)^2 + \left(-40.0 \text{ pC/m}^2\right)^2}}{2\left(8.85 \cdot 10^{-12} \text{ m}^{-3} \text{ kg}^{-1} \text{ s}^4 \text{ A}^2\right)} = 2.82 \text{ N/C}$$

22.77. The sum of the forces on the electron is given by $F_{\text{total}} = F_{\text{gravity}} + F_{\text{coulomb}} = -mg + qE.$ $E = -150. \text{ N/C},$ $q = -1.602 \cdot 10^{-19} \text{ C},$

$m = 9.11 \cdot 10^{-31}$ kg. Thus, $F_{net} = qE - mg = ma \Rightarrow a_e = \dfrac{eE}{m_e} - g.$

$$a_e = \frac{\left(1.602 \cdot 10^{-19} \text{ C}\right)\left(150. \text{ N/C}\right)}{\left(9.11 \cdot 10^{-31} \text{ kg}\right)} - \left(9.81 \text{ m/s}^2\right) = 2.64 \cdot 10^{13} \text{ m/s}^2.$$

22.78. This problem can be solved using Gauss's Law. $\text{Flux} = \dfrac{q_{\text{total}}}{\varepsilon_0} = \oiint \vec{E} \cdot d\vec{a} = \oiint \vec{E}_n \cdot d\vec{a} = 10 \text{ N m}^2/\text{C}.$ Since

$E da = E_n da,$ $q_{\text{total}} = \varepsilon_0 \left(10.0 \text{ N m}^2/\text{C}\right) = \left(8.85 \cdot 10^{-12} \text{ C}^2/\left(\text{N m}^2\right)\right)\left(10.0 \text{ N m}^2/\text{C}\right) = 8.85 \cdot 10^{-11} \text{ C}.$

22.79. This problem can be solved using Gauss's Law. $\text{Flux} = q_{\text{total}} / \varepsilon_0.$ The approximation can be made that the flux leaving the ends of the rod are negligible, so $\text{Flux} = q_{\text{total}} / \varepsilon_0 = \lambda l / \varepsilon_0$ where l is the length of the rod.

$$\lambda = \frac{\varepsilon_0 \Phi}{l} = \frac{\left(8.85 \cdot 10^{-12}\right)\left(1.46 \cdot 10^6 \text{ N m}^2/\text{C}\right)}{0.300 \text{ m}} = 4.31 \cdot 10^{-5} \text{ C/m}$$

22.80. **THINK:** I first need to find the relationship between the first wire and the second wire.
SKETCH: Not required.

RESEARCH: The field due to the first wire is given by: $E_1 = \dfrac{2k\lambda}{r} = 2.73 \text{ N/C}.$ The field due to the second wire is given by $E_2 = 2k\left(0.81\lambda\right)/\left(6.5r\right).$

SIMPLIFY: $E_2 = \frac{2k(0.81\lambda)}{6.5r} = \left(\frac{0.81}{6.5}\right)\frac{2k\lambda}{r} = \left(\frac{0.81}{6.5}\right)E_1$

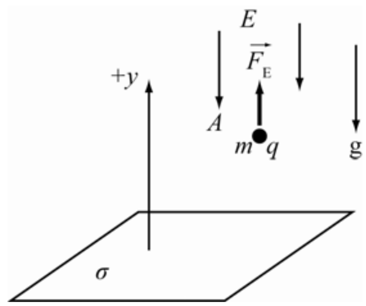
CALCULATE: $E_2 = \left(\frac{0.81}{6.5}\right)E_1 = \left(\frac{0.81}{6.5}\right)(2.73 \text{ N/C}) = 0.3402 \text{ N/C}$

ROUND: 0.340 N/C

DOUBLE-CHECK: The answer is comparable to the electric field of the original wire which makes it reasonable.

22.81. **THINK:** I want to find the charge, q_1 needed to balance out the force of gravity. After finding q, I can determine the number of electrons based on the charge of a single electron.

SKETCH:



RESEARCH: The net force on the object must equal zero in order for the object to remain motionless. $F_{\text{total}} = F_{\text{gravity}} + F_{\text{coulomb}} = 0$, $F_{\text{gravity}} = -mg$, $F_{\text{coulomb}} = Eq$, $E = \sigma/2\varepsilon_0$ for an infinite plane. The number of electrons is q/q_{electron}.

SIMPLIFY: $F_{\text{total}} = F_{\text{gravity}} + F_{\text{coulomb}} = 0 \Rightarrow F_{\text{total}} = -mg + Eq = 0,\ Eq = mg \Rightarrow \frac{\sigma}{2\varepsilon_0}q = mg \Rightarrow q = \frac{2mg\varepsilon_0}{\sigma}$.

Number of electrons $= \frac{2mg\varepsilon_0}{\sigma q_{\text{electron}}}$. $g = 9.81 \text{ m/s}^2$, $\sigma = -3.50 \cdot 10^{-5} \text{ C/m}^2$, $m = 1.00$ g.

CALCULATE:

Number of electrons $= \frac{2(1.00\cdot 10^{-3} \text{ kg})(9.81 \text{ m/s}^2)(8.85\cdot 10^{-12} \text{ m}^{-3} \text{ kg}^{-1} \text{ s}^4 \text{ A}^2)}{(-3.50\cdot 10^{-5} \text{ C/m}^2)(-1.602\cdot 10^{-19} \text{ C})} = 3.097\cdot 10^{10}$ electrons

ROUND: $3.10\cdot 10^{10}$ electrons

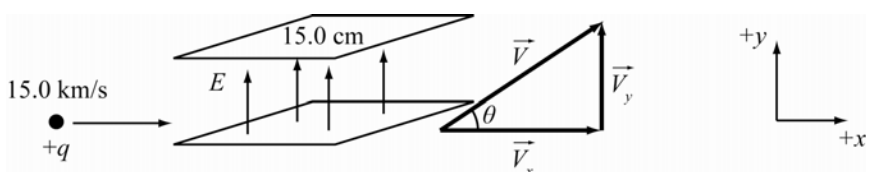
DOUBLE-CHECK: This number, though large, is reasonable since the amount of charge on each electron is tiny.

22.82. **THINK:**

(a) The necessary electric field strength can be determined by finding the acceleration required to achieve the desired deflection. The final speed of the proton can be found through the relation between the proton's initial velocity and its angle of deflection.

(b) The electric field strength required to give the protons a specific acceleration will impart a different acceleration to the kaons due to difference in mass.

SKETCH:



RESEARCH:
(a,b) Initially the velocity in the *y*-direction, v_y, is zero. The only part of the velocity affected by the electric field is v_y, v_x is the same before and after the deflection. $v_y = at$, t is the time the proton spends in between the plates. $F = m_0 a = Eq$, $\tan\theta = v_y / v_x$, $v^2 = v_x^2 + v_y^2$, where v is the new speed. $t = l / v_x$, l is the distance the proton has to traverse between the plates. $\theta = 1.50 \cdot 10^{-3}$ rad, $l = 15.0$ cm, $v_x = 15.0$ km/s.
(c) The mass of a proton $= 1.67 \cdot 10^{-27}$ kg. The mass of a kaon is $= 8.81 \cdot 10^{-28}$ kg. The speed of the kaon is given by setting the momentum of a kaon equal to the momentum of a proton:
$m_{\text{kaon}} v_{\text{kaon}} = m_{\text{proton}} v_{\text{proton}}$.
SIMPLIFY:
(a,b) $v_y = v_x \tan\theta = at \Rightarrow v_y / t = a$

$$Eq = ma = \frac{mv_y}{t} = \frac{mv_x \tan\theta}{t} = \frac{mv_x \tan\theta}{l / v_x} = \frac{mv_x^2 \tan\theta}{l} \Rightarrow E = \frac{mv_x^2 \tan\theta}{lq}$$

$$v^2 = v_x^2 + v_y^2 = v_x^2 + \left(v_x \tan\theta\right)^2 = v_x^2\left(1 + \tan^2\theta\right)$$

(c) Take the result from part (a) to find θ.

$$E = \frac{mv_x^2 \tan\theta}{lq} \Rightarrow \tan\theta = \frac{qEl}{mv_x^2} \Rightarrow \theta = \tan^{-1}\left(\frac{qEl}{mv_x^2}\right)$$

CALCULATE:

(a) $$E = \frac{\left(1.67 \cdot 10^{-27} \text{ kg}\right)\left(15.0 \cdot 10^3 \text{ m/s}\right)^2 \tan\left(1.50 \cdot 10^{-3} \text{ rad}\right)}{\left(0.150 \text{ m}\right)\left(1.602 \cdot 10^{-19} \text{ C}\right)} = 0.023455 \text{ N/C}$$

(b) $$v = \left(15.0 \cdot 10^3 \text{ m/s}\right)\left[1 + \tan^2\left(1.50 \cdot 10^{-3} \text{ rad}\right)\right]^{1/2} = 15.000017 \text{ km/s}$$

(c) $$v_{\text{kaon}} = \frac{1.67 \cdot 10^{-27} \text{ kg}}{8.81 \cdot 10^{-28} \text{ kg}}\left(15.0 \cdot 10^3 \text{ m/s}\right) = 28434 \text{ m/s}$$

With the results from part (a), the electric field is $E = mv_x^2 \tan\theta / (lq)$.

$$\theta = 1.50 \cdot 10^{-3} \text{rad}, \quad E = \frac{\left(1.67 \cdot 10^{-27} \text{ kg}\right)\left(15.0 \cdot 10^3 \text{ m/s}\right)^2 \tan\left(1.50 \cdot 10^{-3} \text{ rad}\right)}{\left(0.150 \text{ m}\right)\left(1.602 \cdot 10^{-19} \text{ C}\right)} = 0.02345507 \text{ N/C}$$

$$\theta = \tan^{-1}\left(\frac{qEl}{mv_x^2}\right) = \tan^{-1}\left[\frac{\left(1.602 \cdot 10^{-19} \text{C}\right)\left(0.02345507 \text{ N/C}\right)\left(0.150 \text{ m}\right)}{\left(8.81 \cdot 10^{-28} \text{kg}\right)\left(28434 \text{ m/s}\right)^2}\right] = 7.91295 \cdot 10^{-4} \text{rad}$$

ROUND:
(a) $E \approx 0.0235$ N/C
(b) $v \approx 1.50 \cdot 10^4$ km/s
(c) $\theta \approx 7.91 \cdot 10^{-4}$ rad
DOUBLE-CHECK:
The change in speed is small compared to the magnitude of the speed, which is expected since the deflection was also small. The deflection of the kaon is less that the deflection of a proton with the same momentum because the kaon has a higher speed.

22.83. **THINK:** Using the charge density, Gauss's Law can be used to find the electric field as a function of the radius.
SKETCH: Not required.

RESEARCH: The charge inside a spherical Gaussian surface is given by $q=\rho V_{\text{sphere}}$. $V_{\text{sphere}}=(4/3)\pi r^3$, $\rho=3.57\cdot 10^{-6}\text{ C/m}^3$ and $r=0.530$ m. Gauss's Law gives the field $\oint\!\!\oint \vec{E}d\vec{A}=E(4\pi r^2)=q/\varepsilon_0$.

SIMPLIFY: $E(4\pi r^2)=\dfrac{q}{\varepsilon_0}\Rightarrow E=\dfrac{1}{4\pi r^2}\left(\dfrac{q}{\varepsilon_0}\right)=\dfrac{1}{4\pi r^2}\left(\dfrac{\rho V}{\varepsilon_0}\right)=\dfrac{1}{4\pi r^2}\left(\dfrac{\rho(4/3)\pi r^3}{\varepsilon_0}\right)=\dfrac{\rho r}{3\varepsilon_0}$

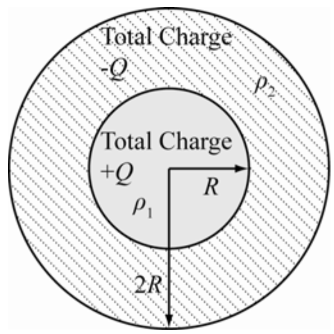
CALCULATE: $E=\dfrac{(3.57\cdot 10^{-6})(0.530)}{3(8.85\cdot 10^{-12})}\text{ N/C}=7.127\cdot 10^4\text{ N/C}$

ROUND: $E=7.13\cdot 10^4\text{ N/C}$

DOUBLE-CHECK: The result was independent of the actual radius of the sphere as it should be.

22.84. **THINK:** Gauss's Law can be used to determine the electric field as a function of radius for the three cases $r<R$, $R\le r\le 2R$ and $r>2R$.

SKETCH:



RESEARCH: The electric field through the surface of a sphere of radius r is given by Gauss's Law:

$$\oint\!\!\oint \vec{E}\cdot d\vec{A}=E(4\pi r^2)=q/\varepsilon_0.$$

For $r<R$, the enclosed charge is given by:

$$q_1=\int_0^r \rho_1(4\pi r'^2)dr',$$

where

$$\rho_1=\frac{Q}{(4/3)\pi R^3}.$$

For $R\le r\le 2R$, the enclosed charge is given by:

$$q_2=Q+\int_R^r \rho_2(4\pi r'^2)dr',$$

where

$$\rho_2=\frac{-Q}{(4/3)\pi\left((2R)^3-R^3\right)}.$$

For $r>2R$, the enclosed charge is $q_3=Q-Q=0$.

SIMPLIFY:

For $r<R$:

$$q_1=\int_0^r \frac{3Q}{4\pi R^3}(4\pi r'^2)dr'=\frac{3Q}{R^3}\int_0^r r'^2dr'=\frac{3Q}{R^3}\left(\frac{r^3}{3}\right)=\left(\frac{Q}{R^3}\right)r^3$$

$$E_{r<R}(4\pi r^2)=\frac{q_1}{\varepsilon_0}=\left(\frac{Q}{\varepsilon_0 R^3}\right)r^3\Rightarrow E_{r<R}=\frac{Qr}{4\pi\varepsilon_0 R^3}$$

For $R\le r\le 2R$,

$$q_2 = Q + \int_R^r \rho_2\left(4\pi r'^2\right)dr' = Q + \int_R^r\left(\frac{-3Q}{28\pi R^3}\right)\left(4\pi r'^2\right)dr' = Q - \frac{3Q}{7R^3}\int_R^r r'^2 dr' = Q - \frac{Q}{7R^3}\left(r^3 - R^3\right)$$

$$E_{R<r<2R} = \left(\frac{1}{4\pi\varepsilon_0 r^2}\right)\left(Q - \frac{Q}{7R^3}\left(r^3 - R^3\right)\right) = \left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)\left(1 - \left(\frac{r^3}{7R^3} - \frac{1}{7}\right)\right) = \left(\frac{Q}{28\pi\varepsilon_0}\right)\left(\frac{8}{r^2} - \frac{r}{R^3}\right)$$

For $r > 2R$: Since the total charge is zero, by Gauss's Law $E_{r>2R} = 0$.

CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: It is expected that the expression for $r < R$ and $R < r < 2R$ are equal at $r = R$ and the expressions for $r > 2R$ and $R \le r \le 2R$, are equal at $r = 2R$. For $r = R$:

$$E_{r<R} = \frac{Qr}{4\pi\varepsilon_0 R^3} = \frac{Q}{4\pi\varepsilon_0 R^2}$$

$$E_{R<r<2R} = \left(\frac{Q}{28\pi\varepsilon_0}\right)\left(\frac{8}{r^2} - \frac{r}{R^3}\right) = \left(\frac{Q}{28\pi\varepsilon_0}\right)\left(\frac{8}{R^2} - \frac{R}{R^3}\right) = \frac{Q}{4\pi\varepsilon_0 R^2} = E_{r<R}$$

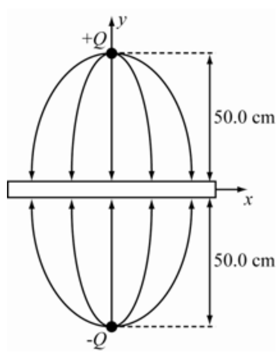
For $r = 2R$:

$$E_{R<r<2R} = \left(\frac{Q}{28\pi\varepsilon_0}\right)\left(\frac{8}{r^2} - \frac{r}{R^3}\right) = \left(\frac{Q}{28\pi\varepsilon_0}\right)\left(\frac{8}{4R^2} - \frac{2R}{R^3}\right) = \left(\frac{Q}{28\pi\varepsilon_0}\right)\left(\frac{2}{R^2} - \frac{2}{R^2}\right) = 0 = E_{r>2R}$$

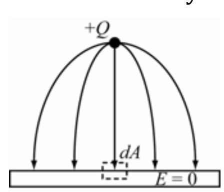
The expressions are equal, so the solution is reasonable.

22.85. **THINK:** The electric field due to the charge induces a charge distribution on the floor below it. As a result, the charge experiences a force directed toward the floor. Since the charge and its 'mirror image" describe a dipole, the electric field lines are perpendicular to the floor. I want to determine the force acting on the charge, the electric field just above the floor, the surface charge density and the total surface charge induced on the floor.

SKETCH:



A Gaussian pill box may be drawn along an infinitesimally small area as follows:



RESEARCH: The electric field due to the charge is given by $E = kq/r^2$, where q is the magnitude of the charge and r is the distance from the charge to the floor. The force experienced by the charge is given by Coulomb's law; $F = \left(1/4\pi\varepsilon_0\right)\left(q_1q_2/r^2\right)$. Since the electric field points in the negative y-direction, only the y-contribution from each charge need be found. The y-contribution is given by

$$E=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\right)\cos\theta,\ \cos\theta=\frac{a}{r}\text{, and } r=\left(a^2+\rho^2\right)^{1/2}.$$

Since the *y*-component from both charges is the same (i.e. since the charges are equal in magnitude), the total electric field is then: $E_{\text{total}}=\frac{2}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\right)\cos\theta.$ Using Gauss's Law on the pillbox,

$EdA=dq/\varepsilon_0 \;\Rightarrow\; dq=\sigma dA.$ The total charge is given by $q=\int_{\substack{\text{infinite}\\\text{plane}}}\sigma dA.$

SIMPLIFY:

(b) $F=\left(\frac{q_1q_2}{4\pi\varepsilon_0(2a)^2}\right)$

(c) $E_{\text{total}}=\frac{2}{4\pi\varepsilon_0}\left(\frac{Q}{r^2}\right)\cos\theta=\frac{2}{4\pi\varepsilon_0}\left(\frac{Qa}{r^3}\right)=\frac{1}{2\pi\varepsilon_0}\left(\frac{Qa}{\left(a^2+\rho^2\right)^{3/2}}\right)$

(d) $EdA=\frac{dq}{\varepsilon_0}=\frac{\sigma dA}{\varepsilon_0},\ E=\frac{\sigma}{\varepsilon_0}$ and $\sigma=E\varepsilon_0=\frac{1}{2\pi}\left(\frac{Qa}{\left(a^2+\rho^2\right)^{3/2}}\right)$

(e) $q=\int\sigma dA\int_0^\infty\left(\frac{\rho}{2\pi}\right)\left(\frac{Qa}{\left(a^2+\rho^2\right)^{3/2}}\right)dp=2\pi\int_0^\infty\left(\frac{\rho}{2\pi}\right)\left(\frac{Qa}{\left(a^2+\rho^2\right)^{3/2}}\right)d\rho=\frac{aQ}{2}(-2)\left[\left(a^2+\rho^2\right)^{-1/2}\right]_0^\infty=Q$

CALCULATE:

(b) $F=\frac{\left(1.00\cdot10^{-6}\text{ C}\right)\left(-1.00\cdot10^{-6}\text{ C}\right)}{4\pi\left(8.85\cdot10^{-12}\text{C}^2/\text{N m}^2\right)\left(1.00\text{ m}\right)^2}=-8.9918\cdot10^{-3}\text{ N}$

(c) Not applicable.
(d) Not applicable.
(e) Not applicable.

ROUND:

To three significant figures:

(b) $F=8.99\cdot10^{-3}$ N downward

DOUBLE-CHECK:

(a) The sketch is symmetric as it should be.

(b) The force is downward as it should be since the positive charge is attracted to the negative charge.

(c) The field gets weaker as ρ gets larger as expected since the source is farther away with increasing ρ.

(d) The surface charge density gets smaller as ρ gets larger since the source is farther away with increasing ρ.

(e) Since all the field lines coming from the charge go onto the top of the slab it is not unreasonable that the total charge induced is equal to the charge in magnitude.

Multi-Version Exercises

Exercises 22.86–22.88 The electric field a distance d from the wire is $E=\frac{2k\lambda}{d}$. The force is then $F=qE=\frac{e2k\lambda}{d}$. From Newton's Second Law we have $F=ma=\frac{e2k\lambda}{d}$. So the acceleration is $a=\frac{e2k\lambda}{md}$.

22.86. $$a=\frac{e2k\lambda}{md}=\frac{2\left(1.602\cdot 10^{-19}\text{ C}\right)\left(8.99\cdot 10^{9}\text{ N m}^2/\text{C}^2\right)\left(2.849\cdot 10^{-12}\text{ C/m}\right)}{\left(1.673\cdot 10^{-27}\text{ kg}\right)\left(0.6815\text{ m}\right)}=7.198\cdot 10^{6}\text{ m/s}^2$$

22.87. $$a=\frac{e2k\lambda}{md}$$

$$\lambda=\frac{amd}{2ek}=\frac{\left(1.111\cdot 10^{7}\text{ m/s}^2\right)\left(1.673\cdot 10^{-27}\text{ kg}\right)\left(0.6897\text{ m}\right)}{2\left(1.602\cdot 10^{-19}\text{ C}\right)\left(8.99\cdot 10^{9}\text{ N m}^2/\text{C}^2\right)}=4.451\cdot 10^{-12}\text{ C/m}$$

22.88. $$a=\frac{e2k\lambda}{md}$$

$$d=\frac{2ek\lambda}{ma}=\frac{2\left(1.602\cdot 10^{-19}\text{ C}\right)\left(8.99\cdot 10^{9}\text{ N m}^2/\text{C}^2\right)\left(6.055\cdot 10^{-12}\text{ C/m}\right)}{\left(1.673\cdot 10^{-27}\text{ kg}\right)\left(1.494\cdot 10^{7}\text{ m/s}^2\right)}=0.6978\text{ m}.$$

Exercises 22.89–22.91 The magnitude of the electric field at the center due to a differential element $d\ell$ is $dE=\frac{k\lambda d\ell}{R^2}$. The x-components add to zero, leaving only a field in the y-direction. The y-component is $dE_y=\frac{k\lambda d\ell}{R^2}\sin\theta$. Taking $d\ell=Rd\theta$ we have $dE_y=\frac{k\lambda R}{R^2}\sin\theta d\theta=\frac{k\lambda}{R}\sin\theta d\theta$. We integrate from 0 to π to get the magnitude of the electric field:

$$\int_0^{\pi}\frac{k\lambda}{R}\sin\theta d\theta=-\frac{k\lambda}{R}\left[\cos\theta\right]_0^{\pi}=2\frac{k\lambda}{R}=\frac{2\pi k\lambda}{L}.$$

So $E=\frac{2\pi k\lambda}{L}$.

22.89. $$E=\frac{2\pi k\lambda}{L}=\frac{2\pi\left(8.99\cdot 10^{9}\text{ N m}^2/\text{C}^2\right)\left(5.635\cdot 10^{-8}\text{ C/m}\right)}{\left(0.2213\text{ m}\right)}=1.438\cdot 10^{4}\text{ N/C}$$

22.90. $$E=\frac{2\pi k\lambda}{L}$$

$$\lambda=\frac{EL}{2\pi k}=\frac{\left(3.117\cdot 10^{4}\text{ N/C}\right)\left(0.1055\text{ m}\right)}{2\pi\left(8.99\cdot 10^{9}\text{ N m}^2/\text{C}^2\right)}=5.822\cdot 10^{-8}\text{ C/m}$$

22.91. $$E=\frac{2\pi k\lambda}{L}$$

$$L=\frac{2\pi k\lambda}{E}=\frac{2\pi\left(8.99\cdot 10^{9}\text{ N m}^2/\text{C}^2\right)\left(6.005\cdot 10^{-8}\text{ C/m}\right)}{2.425\cdot 10^{4}\text{ N/C}}=0.1399\text{ m}=13.99\text{ cm}$$

Chapter 23: Electric Potential

Concept Checks

23.1. e **23.2.** e **23.3.** a **23.4.** a **23.5.** b **23.6.** e **23.7.** d **23.8.** e **23.9.** b **23.10.** a **23.11** e

Multiple-Choice Questions

23.1. a **23.2** c **23.3.** c **23.4.** c **23.5.** a **23.6.** d **23.7.** a **23.8.** d **23.9.** a **23.10.** c **23.11.** c **23.12.** b **23.13.** a **23.14.** d

Conceptual Questions

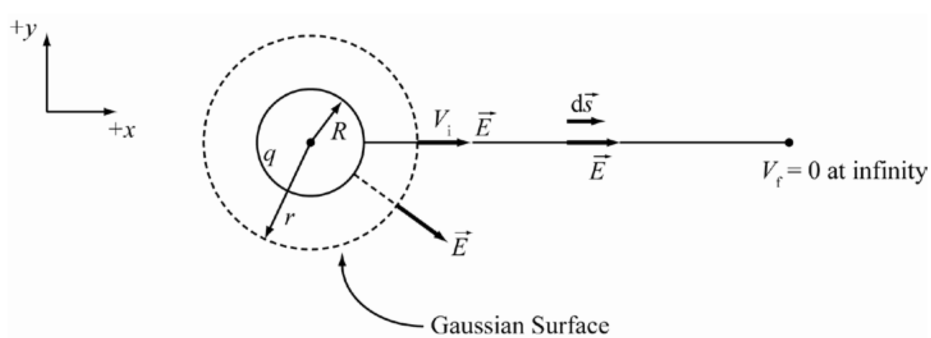
23.15. Birds are safe on a power line because there is no current flowing through the birds. A potential difference is needed in order for a current to flow. Since the potential in the bird is the same as the high voltage wire, the potential difference is zero. Therefore, the birds are safe resting on the wire.

23.16. It is unsafe to stand under a tree during an electrical storm because lightning is more likely to strike trees. This is due to the trees being made of materials making the tree a better conductor then air, which provide an easy path of least resistance for the electricity to the ground. After a strike trees have a high electric field in their vicinity, which helps initiate and guide lightning to the ground. The electricity can travel through the ground to inflict damage or even strike directly from the trees outer surface.

23.17. An equipotential line is defined as a line connecting points of the same potential. This means that if two equipotential lines were to cross, at the cross point, the potential would have two values at the same point. The equipotential lines are also always perpendicular to the electric field. If they were to cross, then there would have to be two different electric fields acting at the same point. If a point charge were put at this point where the electric fields crossed, there would be two separate forces acting from the two different electric fields. Both of these situations are not possible. Therefore two equipotential lines cannot cross.

23.18. In the vicinity of a pointy protrusion, the electric field can be very high. This can lead to a spark inside an electronic device which can make the device to stop functioning.

23.19.



Applying Gauss's law on a spherical Gaussian surfaces as shown above gives:

$$\oiint \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0}.$$

Since the spherical symmetry of the Gaussian surface, the above equation simplifies to:

$$E\oiint dA = \frac{q}{\epsilon_0} \;\Rightarrow\; E\left(4\pi r^2\right) = \frac{q}{\epsilon_0}.$$

Thus, the electric field is $E = \frac{q}{4\pi\epsilon_0 r^2}$. This is exactly the field of a point change with the same charge, q, located at the center of the spherical uniform charge. Using the relation between potential and field, $V_\text{f} - V_\text{i} = -\int_\text{i}^\text{f} \vec{E} \cdot d\vec{s}$, yields: $0 - V_i = -\int_r^\infty E dr = -\int_r^\infty \frac{q}{4\pi\epsilon_0 r^2} dr$. Thus,

$$V_i = \frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_r^\infty = \frac{q}{4\pi\epsilon_0}\left[\left(-\frac{1}{\infty}\right) - \left(-\frac{1}{r}\right)\right] = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r}\right) = \frac{q}{4\pi\epsilon_0 r}.$$

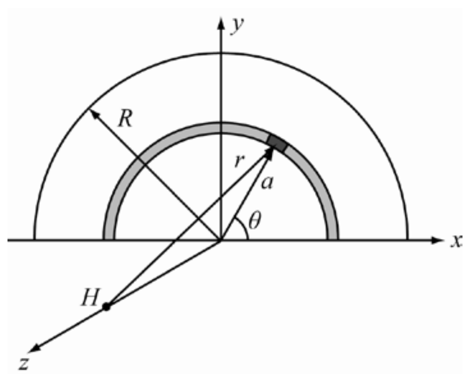
This potential is identical to the potential of a point change. Substituting $r = R$ gives the potential at R:

$$V = \frac{q}{4\pi\epsilon_0 R}.$$

Changing the charge distribution to non-uniform but spherical (radial) symmetry yields the same result as above. That is, its potential is the same as the potential produced by a point change in the center of spherical charge.

23.20. Since the distances of all points in the ring to the center of the ring are the same, the potential is $V = \frac{q}{4\pi\epsilon_0 R}$. The electric field is zero since electric field lines cannot cross where they would converge at the center of the ring.

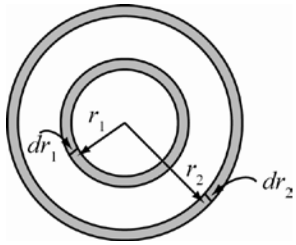
23.21.



The potential due to a small element dA is given by: $dV = \frac{\sigma dA}{4\pi\epsilon_0 r}$, where $dA = ad\theta da$. Integrating over the area of a half disk gives: $V = \int_0^R \int_0^\pi \frac{\sigma}{4\pi\epsilon_0 r} ad\theta da$. Substituting $r = \sqrt{H^2 + a^2}$ yields: $V = \int_0^R \int_0^\pi \frac{\sigma a}{4\pi\epsilon_0 \sqrt{H^2 + a^2}} d\theta da = \frac{\sigma}{4\epsilon_0}\int_0^R \frac{ada}{\sqrt{H^2 + a^2}}$. Now, for integration by substitution, let $z = H^2 + a^2$, which makes $dz = 2a\, da$. Then the previous integral becomes $\frac{\sigma}{4\epsilon_0}\int_{a=0}^{a=R} \frac{1}{2}\frac{dz}{\sqrt{z}} = \frac{\sigma}{4\epsilon_0}\sqrt{z}\Big|_{a=0}^{a=R} = \frac{\sigma}{4\epsilon_0}\sqrt{H^2 + a^2}\Big|_{a=0}^{a=R} = \frac{\sigma}{4\epsilon_0}\left(\sqrt{H^2 + R^2} - H\right)$.

23.22. As an electron moves away from a proton, it encounters a decreasing potential. Since the electron has a negative charge and the potential energy is defined as $U = qV$, then the potential energy increases as the electron moves away.

23.23.



The sphere is divided into many spherical shells. Consider two shells with radii r_1 and r_2. The potential energy of the two shells is: $U_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 r_2}$, where Q_1 and Q_2 are the total changes of shells of radii r_1 and r_2. The total charges are $Q_1 = 4\pi r_1^2 \rho(r_1) dr_1$ and $Q_2 = 4\pi r_2^2 \rho(r_2) dr_2$. Therefore,

$$U_{12} = \frac{1}{\epsilon_0} r_1^2 \rho(r_1) r_2 \rho(r_2) dr_1 dr_2.$$

The total potential energy is obtained by integrating over an interval $(0, \infty)$ for the variable r_2 and an interval $(0, r_2)$ for r_1. Thus,

$$U = \int_0^\infty \int_0^{r_2} U_{12} dr_1 dr_2 = \frac{1}{\epsilon_0} \int_0^\infty \int_0^{r_2} r_1^2 \rho(r_1) r_2 \rho(r_2) dr_1 dr_2.$$

Exercises

23.24. The work required to increase the distance between these ions is given by:

$$-W = \Delta U = U_f - U_i = \frac{1}{4\pi\epsilon_0}\left(\frac{|q_1 q_2|}{r_f}\right) - \frac{1}{4\pi\epsilon_0}\left(\frac{|q_1 q_2|}{r_i}\right) = \frac{1}{4\pi\epsilon_0}|q_1 q_2|\left(\frac{1}{r_f} - \frac{1}{r_i}\right).$$

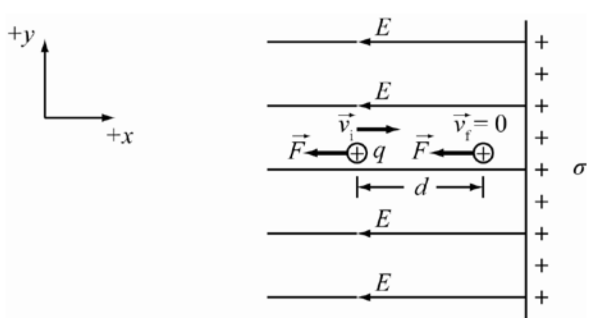
Substituting $q_1 = e = -1.602 \cdot 10^{-19}$ C, $q_2 = e = 1.602 \cdot 10^{-19}$ C, $r_f = 1.0 \cdot 10^{-2}$ m and $r_i = 0.24 \cdot 10^{-9}$ m yields:

$$-W = \left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right) \cdot \left|\left(-1.602 \cdot 10^{-19} \text{C}\right)\left(1.602 \cdot 10^{-19} \text{C}\right)\right| \cdot \left(\frac{1}{1.0 \cdot 10^{-2} \text{ m}} - \frac{1}{0.24 \cdot 10^{-9} \text{ m}}\right)$$

$$\Rightarrow W = 9.6 \cdot 10^{-19} \text{ J}.$$

23.25. **THINK:** As the positively charged ball approaches the positively charged plane, its potential energy will increase and its kinetic energy will decrease. At the point when the ball stops, all its initial kinetic energy will have been converted into potential energy. Work must be done on the ball to accomplish this change. The force necessary to do this work is supplied by the electric field created by the charged plane.

SKETCH:



RESEARCH: Since work is force times distance, the stopping distance can be calculated by exploiting the relationship between the work done on the ball, and the force exerted on the ball by the electric field. The electric field due to the charged plane is given by $E = \sigma / \epsilon_0$. The net force acting on the ball is given by

$F = qE = q\sigma / \epsilon_0$. The work done on the ball is equal to the change in kinetic energy, $\Delta K = K_\text{f} - K_\text{i}$. The work-energy relation states $W = -F \cdot d$. $W = \Delta K = K_\text{f} - K_\text{i} = -F \cdot d = -qE \cdot d = \dfrac{-q\sigma d}{\varepsilon_0}$.

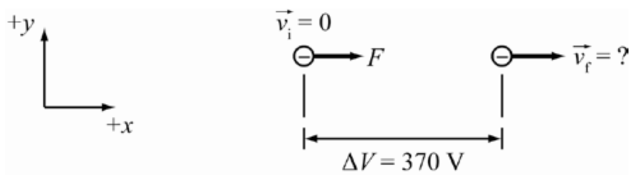
SIMPLIFY: $K_\text{f} - K_\text{i} = \dfrac{-q\sigma d}{\varepsilon_0} \Rightarrow d = -\dfrac{\varepsilon_0 (K_\text{f} - K_\text{i})}{q\sigma}$. Since $K_\text{f} = 0$, $d = \dfrac{K_\text{i}\epsilon_0}{q\sigma}$.

CALCULATE: $d = \dfrac{(6.00 \cdot 10^8 \text{ J})(8.85 \cdot 10^{-12} \text{ C}^2/\text{N m}^2)}{(5.00 \cdot 10^{-3} \text{ C})(4.00 \text{ C/m}^2)} = 0.2655 \text{ m}$; Therefore, the final distance from the plane is $x = 1 \text{ m} - 0.2655 \text{ m} = 0.7345 \text{ m}$.

ROUND: Keeping three significant figures gives $x = 0.734$ m.

DOUBLE-CHECK: This is a reasonable distance and less than the initial distance of 1 m.

23.26.



The change in potential energy, $\Delta U = \Delta Vq$. From the conservation of energy, $\Delta U = -\Delta K$. So $\Delta Vq = -\Delta K = -(K_\text{f} - K_\text{i}) = -\left(\frac{1}{2}mv_\text{f}^2 - \frac{1}{2}mv_\text{i}^2\right)$. Since $v_\text{i} = 0$,

$$\Delta Vq = -\frac{1}{2}mv_\text{f}^2 \Rightarrow v_\text{f} = \sqrt{\frac{-2\Delta Vq}{m}} = \sqrt{\frac{-2(370. \text{ V})(-1.602 \cdot 10^{-19} \text{ C})}{9.11 \cdot 10^{-31} \text{ kg}}} = 1.1407 \cdot 10^7 \text{ m/s} \approx 1.14 \cdot 10^7 \text{ m/s}.$$

23.27. The work done by the electric field on a proton is given by $W = -\Delta U = -q\Delta V = -q(V_\text{f} - V_\text{i})$. Substituting $V_\text{f} = -60.0$ V, $V_\text{i} = +180.$ V and $q = 1.602 \cdot 10^{-19}$ C yields:

$$W = (-1.602 \cdot 10^{-19} \text{ C})(-60.0 \text{ V} - 180. \text{ V}) = 3.84 \cdot 10^{-17} \text{ J}.$$

23.28. The work done by an electric field is given by $W = \Delta K = -q\Delta V$. This means that the potential difference is $\Delta V = -W / q$. Putting in $q = 2(1.602 \cdot 10^{-19} \text{ C})$ and $W = 200 \text{ keV} = 200 \cdot 10^3 (1.602 \cdot 10^{-19} \text{ J})$ yields:

$$\Delta V = -\frac{200 \cdot 10^3 (1.602 \cdot 10^{-19} \text{ J})}{2(1.602 \cdot 10^{-19} \text{ C})} = -100 \cdot 10^3 \text{ V or } -100 \text{ kV}.$$

23.29. Using the work-energy relation and $W = -q\Delta V$, it is found that: $W = \Delta K = -q\Delta V \Rightarrow \frac{1}{2}mv_\text{f}^2 - \frac{1}{2}mv_\text{i}^2 = -q\Delta V$. Since the proton is initially at rest, $v_\text{i} = 0$: $\frac{1}{2}mv_\text{f}^2 = -q\Delta V \Rightarrow v_\text{f} = \sqrt{\frac{-2q\Delta V}{m}}$. $q = 1.602 \cdot 10^{-19}$ C, $\Delta V = -500.$ V, and $m = 1.67 \cdot 10^{-27}$ kg, and this means that: $v_\text{f} = \sqrt{\dfrac{-2(1.602 \cdot 10^{-19} \text{ C})(-500. \text{ V})}{(1.67 \cdot 10^{-27} \text{ kg})}} = 3.10 \cdot 10^5 \text{ m/s}.$

23.30. The kinetic energy is calculated using the work-energy relation and $W = -q\Delta V$, that is:

$$W = \Delta K = -q\Delta V \Rightarrow K_\text{f} - K_\text{i} = -q\Delta V.$$

Since the initial kinetic energy is zero, the find kinetic energy is $K_\text{f} = -q\Delta V$.

(a) Inserting $q = -1.602 \cdot 10^{-19}$ C and $\Delta V = 10$ V yields the final kinetic energy:

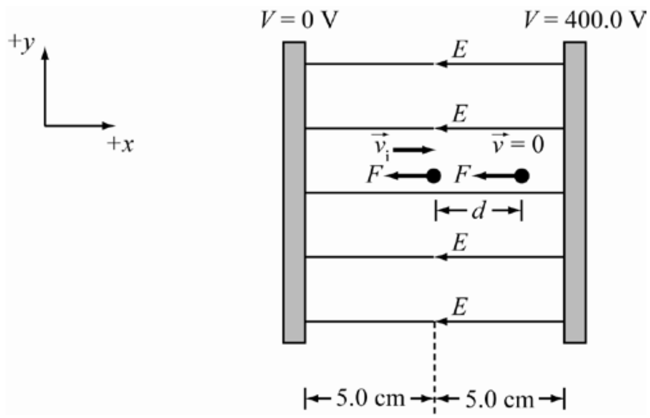
$$K_{\text{f}} = (1.602 \cdot 10^{-19}\text{ C})(10.0\text{ V}) = 1.60 \cdot 10^{-18}\text{ J}.$$

(b) The final velocity of the electron is:

$$\frac{1}{2}mv_{\text{f}}^2 = K_{\text{f}} \Rightarrow v_{\text{f}} = \sqrt{\frac{2K_{\text{f}}}{m}} = \sqrt{\frac{2\left(1.602 \cdot 10^{-18}\text{ J}\right)}{9.11 \cdot 10^{-31}\text{ kg}}} = 1.88 \cdot 10^6\text{ m/s}.$$

23.31. **THINK:** The force due to the electric field opposes the motion of the proton. Work must be done on the proton to move it toward the plate of higher potential. This work must come from the initial kinetic energy of the proton. If the proton has sufficient kinetic energy to provide the necessary work, the proton will reach plate *B*.

SKETCH:



RESEARCH: The work done is equal to the change in kinetic energy. The work is also equal to the scalar product of the force and the distance over which the force acts. So, $W = \Delta K = -F \cdot d$. The force on the proton due to the electric field is $F = qE$, and electric field due to the two plates, separarated a distance L, is $\Delta U = qEL$ and $\frac{\Delta U}{q} = \Delta V \Rightarrow E = \frac{\Delta V}{L}$. So $d = \frac{\Delta K}{-F} = \frac{\Delta K}{-qE} = -\frac{\Delta KL}{q\Delta V}$.

SIMPLIFY:

(a) $d = -\frac{(K_{\text{f}} - K_{\text{i}})L}{q(V_{\text{B}} - V_A)}$; Since $K_{\text{f}} = 0$ and $V_A = 0$, the distance is: $d = \frac{K_{\text{f}}L}{qV_{\text{B}}} = \frac{mv^2L}{2qV_{\text{B}}}$.

(b) If *d* is less than 5.0 cm, then the proton will turn around at *x* = 5.0 cm + *d*. If *d* is greater than 5.0 cm, the proton will reach plate B.

(c) The proton will reach the plate *A* with speed determined from the work-energy relation:

$$W = K_{\text{f}} - K_{\text{i}} = F\left(\frac{L}{2}\right) \Rightarrow \frac{1}{2}mv_{\text{f}}^2 - \frac{1}{2}mv_{\text{i}}^2 = \frac{q(V_{\text{B}} - V_{\text{A}})L}{2L}. \text{ Thus, } v_{\text{f}} = \sqrt{v_{\text{i}}^2 + q\left(\frac{V_{\text{B}} - V_{\text{A}}}{m}\right)}.$$

CALCULATE:

(a) This distance, *d*, is: $d = \frac{\left(1.67 \cdot 10^{-27}\text{ kg}\right)\left(150.0 \cdot 10^3\text{ m/s}\right)^2(0.100\text{)m}}{2\left(1.602 \cdot 10^{-19}\text{ C}\right)\left(400.0\text{ V} - 0\text{ V}\right)} = 0.029319$ m. Therefore the proton will not reach the plate B.

(b) The proton will turn around at a distance *x* = (10.0/2) cm + 2.9319 cm = 7.9319 cm from the plate *A*.

(c) The speed of the proton when it reaches the plate *A* is:

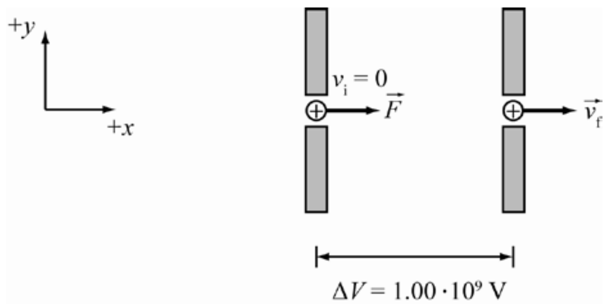
$$v_{\text{f}} = \sqrt{\left(150.0 \cdot 10^3\text{ m/s}\right)^2 + 1.602 \cdot 10^{-19}\text{ C}\left(\frac{400.0\text{ V} - 0\text{ V}}{1.67 \cdot 10^{-27}\text{ kg}}\right)} = 246721\text{ m/s}.$$

ROUND:
(a) no
(b) 7.93 cm
(c) $v_{\text{f}} = 247$ km/s

DOUBLE-CHECK: The work required to move the proton all the way to plate B is $W = qEd = 3.2 \cdot 10^{-17}$ J. The initial kinetic energy of the proton is $K = (1/2)mv^2 = 1.9 \cdot 10^{-17}$ J. This is not enough energy to provide the work required. The proton experiences constant acceleration in the electric field. The proton moves toward plate B, stops, and then moves back toward plate A, passing through its original position where it has the same speed as it had initially. The proton continues to accelerate until it strikes plate A. Thus it makes sense that the magnitude of its final velocity is greater than the initial velocity.

23.32. **THINK:** ^{32}S ions are accelerated from rest using a total voltage of $1.00 \cdot 10^9$ V. ^{32}S has 16 protons and 16 neutrons. The accelerator produces a beam of $6.61 \cdot 10^{12}$ ions/s. Note that the total power is the total energy absorbed per second.

SKETCH:



RESEARCH: The kinetic energy of each ion is determined from the work-energy relation, that is, $W = \Delta K = -q\Delta V$. Since $K_{\text{i}} = 0$, the final kinetic energy of the ion is $K_{\text{f}} = q\Delta V$.

SIMPLIFY: The total power is equal to $P = NK_{\text{f}} = Nq\Delta V$, where N is the number of ions per second.

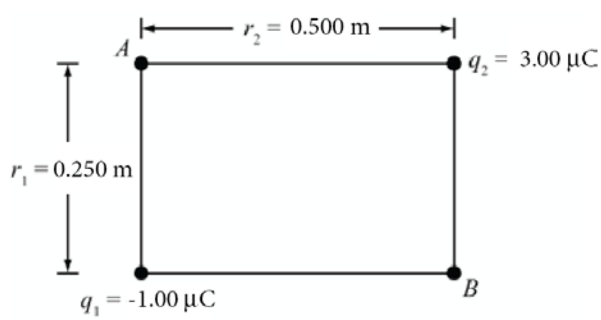
CALCULATE: Substituting the numerical values yields:

$$P = \left(6.61 \cdot 10^{12} \text{ ions/s}\right)(16)\left(1.602 \cdot 10^{-19} \text{ C}\right)\left(1.00 \cdot 10^{9} \text{ V}\right) = 16.94 \text{ kW}.$$

ROUND: $P = 16.9$ kW (keeping three significant digits).

DOUBLE-CHECK: Watts are an appropriate unit of power.

23.33.



(a) The potential at the point A is given by:

$$V_{\text{A}} = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} = k\left(\frac{q_1}{r_1} + \frac{q_2}{r_2}\right) = \left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(\frac{-1.00 \cdot 10^{-6} \text{ C}}{0.250 \text{ m}} + \frac{3.00 \cdot 10^{-6} \text{ C}}{0.500 \text{ m}}\right) = 1.798 \cdot 10^4 \text{ V} \approx 1.80 \text{ kV}.$$

(b) The potential difference between points *A* and *B* is:

$$V_{AB} = V_A - V_B = k\left(\frac{q_1}{r_1}+\frac{q_2}{r_2}\right) - k\left(\frac{q_1}{r_2}+\frac{q_2}{r_1}\right) = k(q_1 - q_2)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

$$= \left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(-1.00\cdot 10^{-6}\text{ C} - 3.00\cdot 10^{-6}\text{ C}\right)\left(\frac{1}{0.250\text{ m}}-\frac{1}{0.500\text{ m}}\right) = -7.192\cdot 10^4 \text{ V} \approx -7.19 \text{ kV}.$$

23.34. Each charge, q = 1.61 nC, is the same distance from the center of the rectangle, $r=\sqrt{(a/2)^2+(b/2)^2}$. Since there are four point charges, the total electric potential is sum of four individual electric potentials:

$$V = \sum_{i=1}^{4}\frac{kq}{r} = \frac{4kq}{(1/2)\sqrt{a^2+b^2}} = \frac{8\left(8.9875\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(1.61\cdot 10^{-9}\text{ C}\right)}{\sqrt{(3.00\text{ m})^2+(5.00\text{ m})^2}} = 19.9 \text{ V}.$$

23.35. For the van de Graff generator, all the excess charge is on the surface, so the electric potential is:

$$V = \frac{kQ}{r} = 1.00\cdot 10^5 \text{ V}, \text{ where } r = d/2 \text{ is the radius.}$$

The total charge, Q, is: $Q = \frac{Vr}{k} = \frac{Vd}{2k}$. The total charge is a result of the number of electrons, $Q = n|e|$, where n is given by:

$$n = \frac{Vd}{2k|e|} = \frac{\left(1.00\cdot 10^5 \text{ V}\right)(0.200\text{ m})}{2\left(8.9875\cdot 10^9 \text{ N m}^2/\text{c}^2\right)\left(1.602\cdot 10^{-19}\text{ C}\right)} = 6.95\cdot 10^{12} \text{ electrons.}$$

23.36. The electric potential of a charged uniform sphere is $V = kq/r$, where V = 100 V and r = 1.0 m, so the total charge is:

$$q = \frac{Vr}{k} = \frac{(100.\text{ V})(1.00\text{ m})}{8.9875\cdot 10^9 \text{ N m}^2/\text{C}^2} = 11.1 \text{ nC}.$$

23.37. All the charge, $Q = 5.60\ \mu\text{C}$, is equidistant from the center, R = 4.50 cm, so the electric potential at the center is:

$$V = \frac{kQ}{r} = \frac{\left(8.9875\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(5.60\cdot 10^{-6}\text{ C}\right)}{(0.0450\text{ m})} = 1118444 \text{ V} = 1.12 \text{ MV}.$$

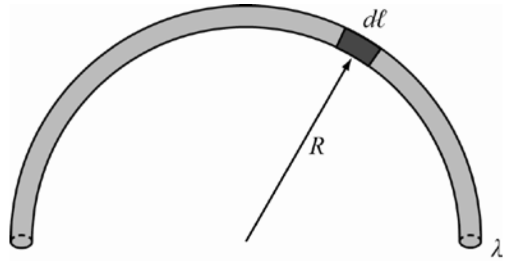
23.38. When considering a charged conducting sphere, the sphere can be considered to be a point charge Q = 8.0 nC for any distance $r > R$, where R is radius of sphere and R = 5.0 cm. Since all the charge is spread out evenly across the surface of the sphere, every point inside the sphere has the same electric potential.

(a) $V = \frac{kQ}{r_1} = \frac{\left(8.9875\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(8.00\cdot 10^{-9}\text{ C}\right)}{0.0800\text{ m}} = 8.987\cdot 10^2 \text{ V} \approx 8.99\cdot 10^2 \text{ V}$

(b) $V = \frac{kQ}{r_2} = \frac{kQ}{R} = \frac{\left(8.9875\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(8.00\cdot 10^{-9}\text{ C}\right)}{0.0500\text{ m}} = 1438 \text{ V} \approx 144 \text{ V}$

(c) $V = \frac{kQ}{r_3} = \frac{kQ}{R} = 1400 \text{ V}$

23.39.

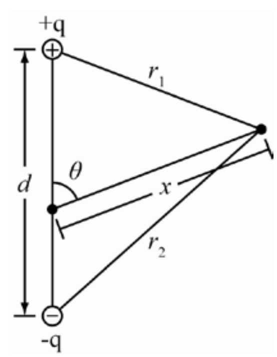


Since the wire is a half circle of radius, $R = 8.00$ cm, the length of the wire is $L = R\theta$. The total charge of the wire is $q = \lambda L$, where $\lambda = 3.00 \cdot 10^{-8}$ C/m. The charge for a small length of the wire is $dq = \lambda\ dL$. For constant R, $dq = \lambda R\ d\theta$. The electric potential is then:

$$V = k\int \frac{dq}{r} = k\int_0^{\pi} \frac{\lambda R}{R} d\theta = k\lambda \int_0^{\pi} d\theta = k\lambda\pi \ \Rightarrow\ V = \left(8.9875 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(3.00 \cdot 10^{-8} \text{ C/m}\right)\pi = 847 \text{ V}.$$

23.40. **THINK:** To find the electric potential, consider the dipole as a system of two point charges, $+q$ and $-q$. The two charges are a distance d away from each other. The potential as a function of θ and x can be found by summing the potentials due to each charge.

SKETCH:



RESEARCH: Using the law of cosines, the two distances can be determined:

$$r_1^2 = x^2 + \frac{d^2}{4} - 2\left(\frac{d}{2}\right)x\cos\theta \text{ and } r_2^2 = x^2 + \frac{d^2}{4} - 2\left(\frac{d}{2}\right)x\cos(180° - \theta).$$

The electric potential is: $V = \sum_{i=1}^{2} \frac{kq_i}{r_i}$.

SIMPLIFY: $V = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} = kq\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = kq\left(\frac{r_2 - r_1}{r_1 r_2}\right)$. Since $\cos(180° - \theta) = -\cos(\theta)$, the electric potential is

$$V = kq\frac{\sqrt{x^2 + \frac{d^2}{4} + xd\cos\theta} - \sqrt{x^2 + \frac{d^2}{4} - xd\cos\theta}}{\sqrt{x^2 + \frac{d^2}{4} - xd\cos\theta}\sqrt{x^2 + \frac{d^2}{4} + xd\cos\theta}}.$$

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: In the case when $x = 0$ it is seen that $V = 0$, which is expected for the point between two opposite charges. Then consider next the limit $x >> d$. For this limit, the denominator simplifies to:

$$\sqrt{x^2 - xd\cos\theta}\sqrt{x^2 + xd\cos\theta} = \left(x\sqrt{1 - \frac{d\cos\theta}{x}}\right)\left(x\sqrt{1 + \frac{d\cos\theta}{x}}\right) \approx x^2.$$

The numerator is simplified to (using series expansion):

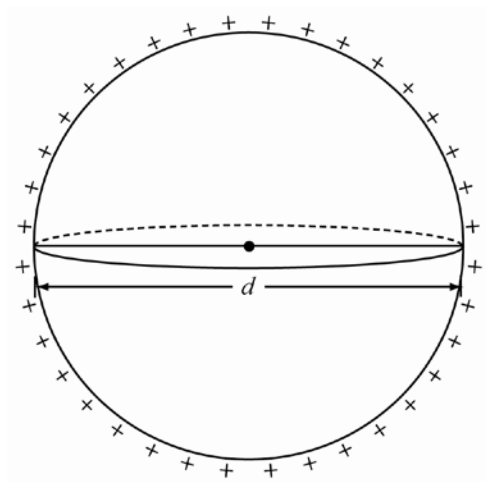
$$\sqrt{x^2+xd\cos\theta}-\sqrt{x^2-xd\cos\theta}=x\left(1+\frac{d\cos\theta}{x}\right)^{1/2}-x\left(1-\frac{d\cos\theta}{x}\right)^{1/2}$$

$$\approx x\left(1+\frac{1}{2}\frac{d\cos\theta}{x}\right)-x\left(1-\frac{1}{2}\frac{d\cos\theta}{x}\right)=d\cos\theta.$$

The potential simplifies to: $V=kq\left(\frac{d\cos\theta}{x^2}\right)=\frac{kp\cos\theta}{x^2}$, where $p=qd$ is the electric dipole moment of two point charges. These two cases confirm that our answer is correct.

23.41. **THINK:** The water droplet can be thought of as a solid insulating sphere of diameter $d=50.0\ \mu\text{m}$ and a total charge of $q=20.0$ pC. The potential is then found by integrating the electric field it produces from infinity to the center. The electric fields inside and outside the sphere are different.

SKETCH:



RESEARCH: The electric potential is found by: $V(r)-V(\infty)=-\int_\infty^r E(r)dr$. Since the water droplet is a non-conducting sphere, the electric field outside the sphere is $E_1=kq/r^2$, while inside the sphere is $E_2=kqr/R^3$ by Gauss' Law, where $R=d/2$ is the radius of the sphere.

SIMPLIFY:

(a) The potential on its surface, $r=R$, is: $V(R)-0=-\int_\infty^R \frac{kq}{r^2}dr=-kq\int_\infty^R\frac{dr}{r^2}=-kq\left[-\frac{1}{r}\right]_\infty^R=\frac{kq}{R}.$

(b) The potential inside the sphere at center, $r=0$, must be broken into 2 parts.

$$V(0)-0=-\int_\infty^R E_1dr-\int_R^0 E_2dr=-\int_\infty^R\frac{kq}{r^2}dr-\int_R^0\frac{kqr}{R^3}dr=\frac{kq}{R}-\frac{kq}{R^2}\int_R^0 rdr=\frac{kq}{R}-\frac{kq}{R^3}\left[\frac{1}{2}r^2\right]_R^0$$

$$=\frac{kq}{R}-\frac{kq}{R^3}\left(0-\frac{1}{2}R^2\right)=\frac{kq}{R}+\frac{kq}{2R}\Rightarrow V(0)=\frac{3}{2}\left(\frac{kq}{R}\right)=\frac{3}{2}V(R).$$

CALCULATE:

(a) $V(R)=\dfrac{\left(8.9875\cdot10^9\ \text{N m}^2/\text{C}^2\right)\left(20.0\ \text{pC}\right)}{50.0\ \mu\text{m}/2}=7190\ \text{V}$

(b) $V(0)=\frac{3}{2}\left(7190\ \text{V}\right)=10785\ \text{V}$

ROUND:

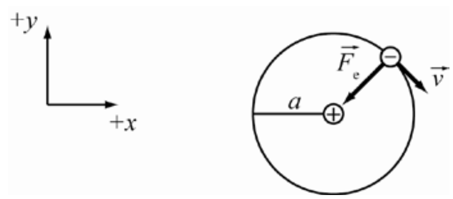
(a) $V(R)=7.19$ kV

(b) $V(0) = 10.8\text{ kV}$

DOUBLE-CHECK: Though these values seem large, the droplet has a charge density of 300 C/m^3, which is quite large for an object. Therefore, the values seem reasonable.

23.42. **THINK:** Both a proton and electron of a Hydrogen atom have a charge of $q = \pm e$. If the electron orbits the proton at a distance of $a = 0.529 \cdot 10^{-10}\text{ m}$, then the electric force is the same as the centripetal force. The escape speed of an object is the speed needed for its kinetic energy to equal its potential energy. The kinetic energy the electron needs to escape minus the potential energy it has in orbit is then the energy needed to remove the electron from orbit.

SKETCH:



RESEARCH: The electric force the electron feels is $F_e = ke^2/a^2$. The centripetal force to keep electron in orbit is $F_c = m_e v^2/a$. The potential energy of electron in orbit is $U = ke^2/a$. The kinetic energy it has for escape speed is $K_2 = m_e v_e^2/2$. The kinetic energy the electron has in orbit is $K_1 = m_e v^2/2$..

SIMPLIFY:

(a) Since the electric force is the only force acting on the electron:

$$F_e = F_c \Rightarrow \frac{ke^2}{a^2} = \frac{m_e v^2}{a} \Rightarrow v = \sqrt{\frac{ke^2}{m_e a}}.$$

(b) If electron escapes its orbit, it needs enough kinetic energy to counter its potential energy:

$$K_2 = U \Rightarrow \frac{1}{2} m_e v_e^2 = \frac{ke^2}{a} \Rightarrow v_e = \sqrt{\frac{2ke^2}{m_e a}} = \sqrt{2}v.$$

(c) The additional energy the electron needs to escape is equal in the change in kinetic energy:

$$E = \Delta K = K_2 - K_1 = \frac{1}{2} m_e v_e^2 - \frac{1}{2} m_e v^2 = \frac{1}{2} m_e \left(v_e^2 - v^2\right) = \frac{1}{2} m_e \left(\frac{2ke^2}{m_e a} - \frac{ke^2}{m_e a}\right) = \frac{1}{2}\frac{ke^2}{a}.$$

CALCULATE:

(a) $v = \sqrt{\dfrac{\left(8.9875 \cdot 10^9\text{ N m}^2/\text{C}^2\right)\left(1.602 \cdot 10^{-19}\text{ C}\right)^2}{\left(9.109 \cdot 10^{-31}\text{ kg}\right)\left(0.529 \cdot 10^{-10}\text{ m}\right)}} = 2.188 \cdot 10^6\text{ m/s}$

(b) $v_e = \sqrt{2}\left(2.188 \cdot 10^6\text{ m/s}\right) = 3.094 \cdot 10^6\text{ m/s}$

(c) $E = \dfrac{\left(8.9875 \cdot 10^9\text{ N m}^2/\text{C}^2\right)\left(1.602 \cdot 10^{-19}\text{ C}\right)^2}{2\left(0.529 \cdot 10^{-10}\text{ m}\right)} = 2.18 \cdot 10^{-18}\text{ J} = 13.6\text{ eV}$

ROUND:

(a) $v = 2.19 \cdot 10^6\text{ m/s}$

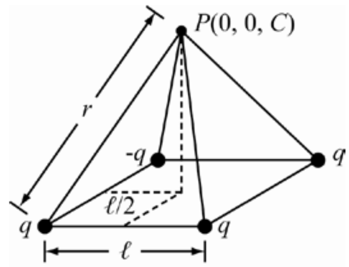
(b) $v_e = 3.09 \cdot 10^6\text{ m/s}$

(c) $E = 13.6\text{ eV}$

DOUBLE-CHECK: Both velocities are less than speed of light, so they make sense. Also, 13.6 eV is the experimentally found energy of an electron in a ground state of a hydrogen atom, so it makes sense too.

23.43. **THINK:** Each charge, three at $q = 1.50$ nC and one at $-q$, are placed the corners of a square of sides $l = 2a = 5.40$ cm. Since the point P in space is located above the very center of the square, each charge is the exact same distance from P. The point P is a distance $c = 4.10$ cm above the center of the square. The electric potential can be determined as the sum of the four individual point charges, taking the zero of electrical potential to be at an infinite distance.

SKETCH:



RESEARCH: The distance from each charge to point P is: $r = \sqrt{\left(\frac{l}{2}\right)^2 + \left(\frac{l}{2}\right)^2 + c^2} = \sqrt{\frac{l^2}{2} + c^2}$. The electric potential at this point is $V = \sum_{i=1}^{4} \frac{kq_i}{r_i}$.

SIMPLIFY: $V = k\left(\frac{q}{r} + \frac{q}{r} + \frac{q}{r} + \frac{-q}{r}\right) = \frac{2kq}{r} = \frac{2kq}{\sqrt{\frac{l^2}{2} + c^2}}$

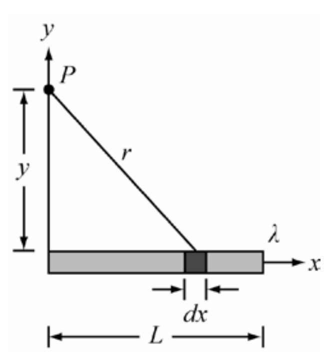
CALCULATE: $V = \dfrac{2\left(8.9875 \cdot 10^9 \text{ N m}^2/\text{c}^2\right)\left(1.50 \cdot 10^{-9} \text{ nC}\right)}{\sqrt{\dfrac{(0.0540 \text{ cm})^2}{2} + (0.0410 \text{ cm})^2}} = 481.2 \text{ V}$

ROUND: $V = 481$ V

DOUBLE-CHECK: Given the charges and distances involved, this value seems reasonable.

23.44. **THINK:** The electric potential at a point P, a distance y above the end of a rod, can be derived by simply integrating the charge over the length of the rod, L. The distance to the point P, from a point on the rod is found by using the Pythagorean theorem. The charge distribution of the road is $\lambda = cx$.

SKETCH:



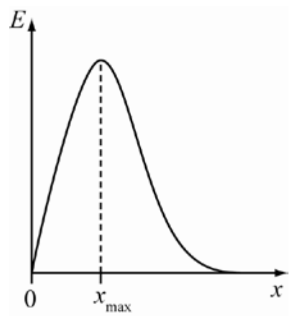
RESEARCH: The total charge of the rod is $q = \lambda L$, so a small element of length dx has a charge $dq = \lambda dx$. At any given point along the rod, the distance from it to P is $r = \sqrt{x^2 + y^2}$. The electric potential at point P is $V = \int_0^L kdq / r$.

SIMPLIFY:

$$V=\int_0^L \frac{kdq}{r}=k\int_0^L \frac{\lambda dx}{r}=k\int_0^L \frac{cxdx}{\sqrt{x^2+y^2}}=kc\int_0^L \frac{xdx}{\sqrt{x^2+y^2}}=kc\left[\sqrt{x^2+y^2}\right]_0^L=kc\left(\sqrt{L^2+y^2}-y\right)$$

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: From the expression, if $y >> L$ (far away point) then $V = 0$, which would be expected. Likewise, if $L >> y$, the potential becomes $V = K_e cL$, which is constant. This is expected for an infinite distance, so it makes sense.

23.45. **THINK:** The electric field, $\vec{E}=E_0xe^{-x}\hat{x}$, has a maximum when its derivative with respect to x is zero. The electric potential is found by integrating the electric field between the two points 0 and x_{max}.
SKETCH:



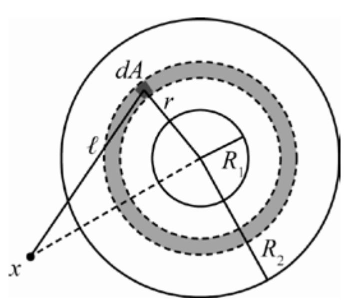
RESEARCH: Electric field is at maximum when $dE/dx=0$. The potential difference between 0 and x_{max} is $V=-\int_0^{x_{\text{max}}}\vec{E}\bullet dx.$
SIMPLIFY:

(a) $\dfrac{dE}{dx}=\dfrac{d}{dx}\left(E_0xe^{-x}\right)=E_0\left[\dfrac{d(x)}{dx}e^{-x}+x\dfrac{d\left(e^{-x}\right)}{dx}\right]=E_0\left(e^{-x}-xe^{-x}\right).$ If $\dfrac{dE}{dx}=0$: $e^{-x}=xe^{-x}\Rightarrow x_{\text{max}}=1$

(b) $V=-\int_0^1 E_0xe^{-x}dx=-E_0\int_0^1 xe^{-x}dx=-E_0\left[-(1+x)e^{-x}\right]_0^1=E_0\left[(1+x)e^{-x}\right]_0^1=E_0\left(2e^{-1}-1\right)$

CALCULATE: There is no need to calculate.
ROUND: There is no need to round.
DOUBLE-CHECK: The answer is reasonable.

23.46. **THINK:** The electric potential at a point a distance x from the center of a disk with inner radius R_1 and outer radius R_2 is found by integrating the charge over the radius of the disk and considering a ring of charge for a given radius. The distance to the point of interest and any point along a ring of given radius is found using the Pythagorean theorem.
SKETCH:



RESEARCH: Assuming the disk has a uniform charge distribution, the total charge is $q = \sigma A$, where A is the area of the disk and σ is area charge density. The area of a thin ring along disk is $dA = 2\pi r dr$. The distance from a point along the disk to a point x along the central axis of the disk is $l = \sqrt{r^2 + x^2}$. A small element of charge along the disk is written as $dq = \sigma dA = 2\pi\sigma r dr$. The potential then at a point along the x-axis is $V = k\int dq / l.$

SIMPLIFY:

$$V = k\int_{R_2}^{R_2} \frac{2\pi\sigma r dr}{\sqrt{r^2 + x^2}} = k2\pi\sigma\int_{R_2}^{R_2} \frac{r dr}{\sqrt{r^2 + x^2}} = 2\pi\sigma k\left[\sqrt{r^2 + x^2}\right]_{R_1}^{R_2} = 2\pi\sigma k\left(\sqrt{R_2^2 + x^2} - \sqrt{R_1^2 + x^2}\right)$$

If $R_1 = 0$, then $V = 2\pi\sigma k\left[\sqrt{r^2 + x^2}\right]_0^{R_2} = 2\pi\sigma k\left(\sqrt{R_2^2 + x^2} - x\right)$, as we would expect for the potential due to a charged disk (compare with Solved Problem 23.4).

CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: To determine if this value is reasonable, determine the electric field it produces:

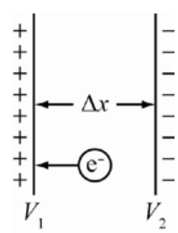
$$E_x = -\frac{dV}{dx} = \frac{\sigma}{2\epsilon_0}\left(\frac{X}{\sqrt{R_2^2 + x^2}} - \frac{X}{\sqrt{R_1^2 + x^2}}\right).$$

If $R_1 \to \infty$ and $R_2 \to 0$, the disk is an infinite plane and the electric field is $E_x = \sigma / 2\epsilon_0$, so it makes sense.

23.47. The electric field is related to the potential difference by $E = -\frac{\Delta V}{\Delta x}$. So, when $V = V_0 x^2$, where $V_0 = 270.\text{ V/m}^2$, the x-component of the electric field at x = 13.0 cm is then:

$$E_x = -\left[\frac{dV}{dx}\right]_{x=13.0\text{ cm}} = -2V_0\left[x\right]_{x=13.0\text{ cm}} = -2\left(270.\text{ V/m}^2\right)\left(0.130\text{ m}\right) = 70.2\text{ V/m}.$$

23.48.



(a) The left plate has a potential of $V_1 = +200.0\text{ V}$ and the right plate has a potential of $V_2 = -100.0\text{ V}$, so the potential difference across the plates is $\Delta V = V_1 - V_2 = 300.0\text{ V}$. The electric field from plate to plate is:

$$E = -\left|\frac{dV}{dx}\right| \approx \frac{\Delta V}{\Delta x}, \text{ where } \Delta x = 1.00\text{ cm. Therefore, } E = \frac{300.0\text{ V}}{0.0100\text{ m}} = 3.00\cdot 10^4\text{ V/m}.$$

(b) If the electron only travels $d = \Delta x / 2$, the change in electric potential is $\Delta V' = Ed = \frac{E\Delta x}{2}$. Since all its initial potential energy becomes kinetic energy:

$$K = U_i = e\Delta V' = \frac{1}{2}eE\Delta x = \frac{1}{2}\left(1.602\cdot 10^{-19}\text{ C}\right)\left(3.00\cdot 10^4\text{ V/m}\right)\left(0.0100\text{ m}\right) = 2.40\cdot 10^{-17}\text{ J}$$

23.49. The electric field from an electric potential, $V(x) = V_1 x^2 - V_2 x^3$, where $V_1 = 2.00\text{ V/m}^2$ and $V_2 = 3.00\text{ V/m}^3$ is found by:

$$E = -\frac{dV}{dx} = -\frac{d}{dx}\left(V_1 x^2 - V_2 x^3\right) = 3V_2 x^2 - 2V_1 x.$$

This field produces a force on a charge, $q = 1.00\ \mu\text{C}$, of $F = qE$. The acceleration of the charge is $a = \dfrac{F}{m} = \dfrac{qE}{m}$, where $m = 2.50$ mg. Therefore,

$$a = \frac{q\left(3V_2x^2 - 2V_1x\right)}{m} = \frac{\left(1.00\cdot 10^{-6}\ \text{C}\right)\left[3\left(3.00\ \text{V/m}^3\right)\left(2.00\ \text{m}\right)^2 - 2\left(2.00\ \text{V/m}^2\right)\left(2.00\ \text{m}\right)\right]}{\left(2.50\cdot 10^{-6}\ \text{kg}\right)} = 11.2\ \text{m/s}^2.$$

23.50. In three dimensions, the electric field is:

$$\vec{E}(x, y, z) = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right).$$

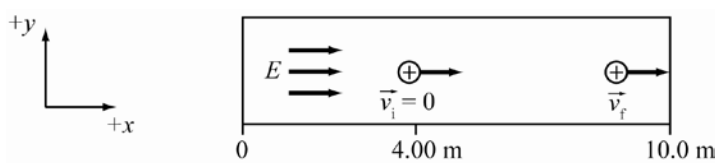
Therefore, if $V(x, y, z) = x^2 + xy^2 + yz$, $\dfrac{\partial V}{\partial x} = 2x + y^2$, $\dfrac{\partial V}{\partial y} = 2xy + z$ and $\dfrac{\partial V}{\partial z} = y$:

$$\vec{E}(x, y, z) = -\left(2x + y^2\right)\hat{x} - \left(2xy + z\right)\hat{y} - y\hat{z}$$

$$\vec{E}(3, 4, 5) = -\left(2(3) + (4)^2\right)\hat{x} - \left(2(3)(4) + (5)\right)\hat{y} - (4)\hat{z} = -22\hat{x} - 29\hat{y} - 4\hat{z}$$

23.51. **THINK:** The electric potential, $V = \left(3000 - 5x^2/\text{m}^2\right)$ V, is a function of x and thus acts only in one dimension. The electric field is found by differentiating the electric potential. The acceleration of a proton ($q = +e$, $x = 4$ m and $m_\text{p} = 1.673\cdot 10^{-27}$ kg) is then found by relating the electric field to the force on the proton. Since the electric field is not constant, kinematics cannot be used to determine the final speed. Conservation of energy must then be used to relate its final kinetic energy to the initial potential energy it has.

SKETCH:



RESEARCH: The electric field is determined by $E(x) = -dV(x)/dx$. The force on the proton is given by $F(x) = qE(x)$, and this force is also related to acceleration by $F(x) = m_\text{p}a(x)$. The change in electric potential from x_1 and x_2 is $\Delta V = V(x_2) - V(x_1)$, so the change in potential energy is $\Delta U = q\Delta V$. From conservation of energy: $\Delta U = -\Delta K$.

SIMPLIFY:

(a) $E(x) = -\dfrac{d}{dx}\left(3000 - \dfrac{5x^2}{\text{m}^2}\right)\text{V} = 10x\ \text{V/m}^2$

(b) $F(x) = m_\text{p}a(x) \Rightarrow a(x) = \dfrac{F(x)}{m_\text{p}} = \dfrac{qE(x)}{m_\text{p}} = \dfrac{10qx\ \text{V/m}^2}{m_\text{p}}$

(c) $\Delta V = V(x_2) - V(x_1) = -\dfrac{5\ \text{V}}{\text{m}^2}\left(x_2^2 - x_1^2\right)$

Therefore, $\Delta U = -5q\dfrac{\text{V}}{\text{m}^2}\left(x_2^2 - x_1^2\right)$. Use the equation: $\Delta K = \dfrac{1}{2}m_\text{p}\left(v_\text{f}^2 - v_\text{i}^2\right) = \dfrac{1}{2}m_\text{p}v_\text{f}^2$, when $\Delta K = -\Delta U \Rightarrow \dfrac{1}{2}m_\text{p}v_\text{f}^2 = 5q\ \text{V/m}^2\left(x_2^2 - x_1^2\right) \Rightarrow v_\text{f} = \sqrt{\dfrac{10q}{m_\text{p}}\left(x_2^2 - x_1^2\right)\ \text{V/m}^2}$.

CALCULATE:
(a) Not applicable.

(b) $a(4.00\text{ m}) = \dfrac{10(1.602\cdot10^{-19}\text{ C})(4.00\text{ m})\text{ V/m}^2}{1.673\cdot10^{-27}\text{ kg}} = 3.8302\cdot10^9\text{ m/s}^2$

(c) $v_\text{f} = \sqrt{\dfrac{10(1.602\cdot10^{-19}\text{ C})}{1.673\cdot10^{27}\text{ kg}}\left[(10.0\text{ m})^2-(4.00\text{ m})^2\right]\text{ V/m}^2} = 2.836\cdot10^5\text{ m/s}$

ROUND:
(a) $E(x) = 10x\text{ V/m}^2$

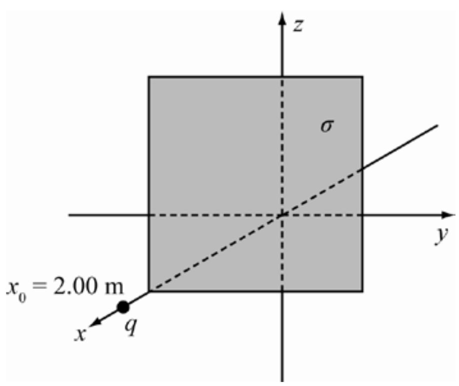
(b) $a = 3.83\cdot10^9\text{ m/s}^2$

(c) $v_\text{f} = 2.84\cdot10^5\text{ m/s}$

DOUBLE-CHECK: The units for the $E(x)$ expression are valid. The final velocity is lower than the speed of light, so it is reasonable. The acceleration is high; however, the purpose of the device is to accelerate particles to large speeds over short distances.

23.52. **THINK:** All points in space will be influenced by the infinite plane of charge with surface charge density, $\sigma = 4.00\text{ nC/m}^2$, and a point charge, q = 11.0 nC, located $x_0 = 2.00\text{ m}$ in a perpendicular direction from the plane. The plane produces a constant electric field. The overall potential at any point between the two will be the sum of the two individual potentials. The minimum is found by differentiating the potential in one dimension and setting it to zero. The derivative of the potential with respect to position is also the electric field. Therefore, when the potential is a minimum, the electric field is zero.

SKETCH:



RESEARCH: The electric field produced by the plane of charge is $E_\text{p} = \sigma / 2\varepsilon_0$. The electric potential from a constant electric field is $Ex = V$. The electric potential from a point charge is $V = kq/r$, where $r = x_0 - x$. The electric potential is at a minimum when $dV/dx = E = 0$.

SIMPLIFY:
(a) The electric potential from the plane along the x-axis is $V_1 = E_\text{p}x = \sigma x / 2\varepsilon_0 = 2\pi k\sigma x$. The electric potential from the charge, q, is $V_2 = kq/(x_0 - x)$. The total electric potential is:

$$V_\text{tot} = V_1 + V_2 = k\left(2\pi\sigma x + \frac{q}{x_0 - x}\right).$$

(b) $\dfrac{dV}{dx} = 2\pi k\sigma\dfrac{dx}{dx} + kq\dfrac{d}{dx}(x_0 - x)^{-1} = 2\pi k\sigma - \dfrac{kq}{(x_0 - x)^2} = 0$

$$\Rightarrow (x_0 - x)^2 = x_0^2 - 2x_0x + x^2 = \frac{q}{2\pi\sigma} \;\Rightarrow\; x^2 - 2x_0x + x_0^2 - \frac{q}{2\pi\sigma} = 0$$

$$\Rightarrow x = \frac{2x_0 \pm \sqrt{4x_0^2 - 4\left(x_0^2 - q/2\pi\sigma\right)}}{2} = x_0 \pm \sqrt{q/2\pi\sigma} = x_0 - \sqrt{q/2\pi\sigma} \quad \left(x < x_0\right).$$

(c) E is zero at the same position of minimum in V.

CALCULATE:

(a) Not applicable.

(b) $x = 2.00 \text{ m} - \sqrt{\dfrac{11.0 \text{ nC}}{2\pi\left(4.00 \text{ nC/m}^2\right)}} = 1.338 \text{ m}$

(c) $x = 1.338$ m

ROUND:

(a) Not applicable.

(b) $x = 1.34$ m

(c) $x = 1.34$ m

DOUBLE-CHECK: The minimum is located closer to the point charge than it is to the plane of charge, as it should be.

23.53. **THINK:** The position, r, must be defined for three-dimensional space so that each derivative has a non-zero answer. While the potential is a scalar, each derivative is actually a vector that points in that direction, i.e. E_x points in the x-direction.

SKETCH: Not applicable.

RESEARCH: The position in three-dimensional space is given by $r = \sqrt{x^2 + y^2 + z^2}$. The electric field in direction $\hat{\alpha}$ is $\vec{E}_\text{i} = -\delta V \hat{\alpha} / \delta\alpha$, where $\alpha = x, y, z$.

SIMPLIFY: $\vec{E}_x = -\dfrac{\delta V}{\delta x}\hat{x} = -kq\dfrac{\delta}{\delta x}\left(\sqrt{x^2 + y^2 + z^2}\right)\hat{x} = \dfrac{kq}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}\left(2x\right)\hat{x} = \dfrac{kq}{r^3}x\hat{x}$. Likewise, $\vec{E}_y = \dfrac{kq}{r^3}y\hat{y}$ and $\vec{E}_z = \dfrac{kq}{r^3}z\hat{z}$. Therefore, $\vec{E}\left(\vec{r}\right) = \dfrac{kq}{r^3}\left(x\hat{x} + y\hat{y} + z\hat{z}\right)$.

CALCULATE: Not applicable.

ROUND: Not applicable.

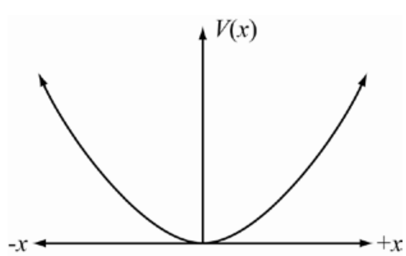
DOUBLE-CHECK: In vector notation, $x\hat{x} + y\hat{y} + z\hat{z}$ can be written as $\vec{r} = r\hat{r}$. This is evident if you let $r^2 = \vec{r} \bullet \vec{r}$. Therefore, the expression for electric field for a point charge can be written as:

$$\vec{E}\left(\vec{r}\right) = \frac{kq}{r^3}r\hat{r} = \frac{kq}{r^2}\hat{r}.$$

Since the potential was for a point charge, this makes sense.

23.54. **THINK:** Given the electric potential, $V\left(x\right) = Ax^2$, the potential energy can be determined. The force a particle feels is related to the derivative of the potential energy of a particle. If the particle is to behave like a harmonic oscillator, then the force needs to be related to a force resulting from a spring. This will yield a spring constant, k, which is then related to the period of the motion. The units of A are V/m^2. To avoid confusing the spring constant with the Coulomb constant, the spring constant will be denoted K here.

SKETCH:



RESEARCH: Given an electric potential, $V(x)$, the potential for an electron is $eV(x)$. The force such a potential causes is $F(x) = -dU(x)/dx$. If this force causes simple harmonic motion, it should resemble the force of a spring, $F = -Kx$. The period of an oscillating spring is given by $T = 2\pi\sqrt{m/K}$.

SIMPLIFY: The force of this potential is $F(x) = -\frac{d}{dx}\left(eAx^2\right) = -2Aex$. Relating this force to $F = -Kx$:

$$F = -2Aex = -Kx \Rightarrow K = 2Ae.$$

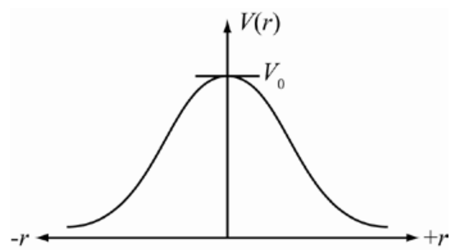
The period of this oscillation is then: $T = 2\pi\sqrt{\frac{m_e}{2Ae}}$.

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: Checking the units in the expression for the period:

$$[T] = \sqrt{\frac{\text{kg}}{\text{V/m}^2\ \text{C}}} = \sqrt{\frac{\text{kg}}{\text{N/C}\ (\text{C/m})}} = \sqrt{\frac{\text{kg}}{\text{N/m}}} = \sqrt{\frac{\text{kg}}{\text{kg m/s}^2/\text{m}}} = \sqrt{\frac{1}{1/\text{s}^2}} = \text{s}.$$

Units of time are necessary for the period.

23.55. **THINK:** The electric potential is given as $V(r) = V_0 e^{-r^2/a^2}$. The electric field and charge density are related to the first and second derivative of the electric potential. The total charge is the charge density integrated over all space. Rather than work in Cartesian coordinates, remain in r-space.
SKETCH:



RESEARCH: Given an electric potential $V(r)$, the electric field it produces is $E_r = -\frac{d}{dr}\left(V(r)\right)$. The charge distribution is given by an electric field, $E(r)$, as $\rho(r) = \varepsilon_0 \frac{d}{dr}E(r)$. The total charge is then:

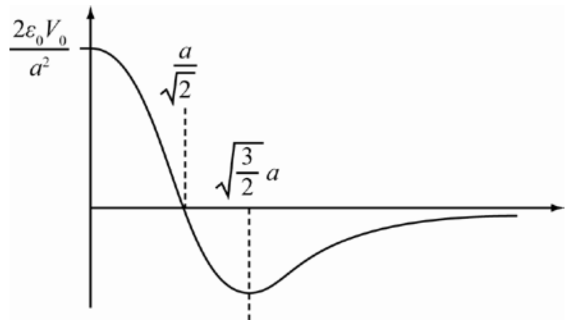
$$Q = \int_0^\infty p(r)dr.$$

SIMPLIFY:

(a) The electric field is: $E(r) = -\frac{dV(r)}{dr} = -V_0\frac{d}{dr}\left(e^{-r^2/a^2}\right) = -V_0\left(-\frac{2r}{a^2}\right)e^{-r^2/a^2} = \frac{2V_0 r}{a^2}e^{-r^2/a^2}$.

(b) The charge distribution is then: $p(r) = \varepsilon_0\frac{dE(r)}{dr} = \varepsilon_0\frac{d}{dr}\left(\frac{2V_0 r}{a^2}e^{-r^2/a^2}\right) = \frac{2\varepsilon_0 V_0}{a^2}\left[1-2\left(\frac{r^2}{a^2}\right)\right]e^{-r^2/a^2}$.

(c) The total charge is: $Q = \frac{2\varepsilon_0 V_0}{a^2}\int_0^\infty e^{-r^2/a^2}\left[1-2\left(\frac{r^2}{a^2}\right)\right]dr$. Let $A = \frac{2\varepsilon_0 V_0}{a^2}$, $x = \frac{r}{a} \Rightarrow dx = \frac{dr}{a}$ and the above equation becomes $Q = Aa\int_0^\infty\left(e^{-x^2} - 2x^2e^{-x^2}\right)dx$. Referring to a table of definite integrals: $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ and $2\int_0^\infty x^2e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$. Therefore, $Q = Aa\left(\sqrt{\pi}/2 - \sqrt{\pi}/2\right) = 0$. The total net charge is zero, i.e. there is equal negative charge and is positive charge. A plot of $p(r)$ vs. r is below.



Notice that the area above the r-axis (positive charge) is equal to the area below the r-axis (negative charge).

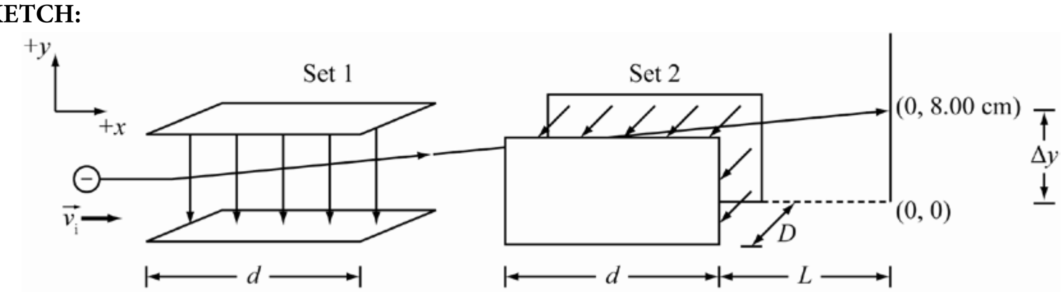
CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: Given that the electric potential is a Gaussian distribution, which shows symmetry, then symmetry in charge is expected. Such symmetry in charge would mean equal negative and positive charge, resulting in a zero net charge.

23.56. **THINK:** For this problem, assume the directions of the electric fields across the plates are appropriate to cause the necessary deflection, so only the magnitudes must be considered. The electron, $m_\text{e} = 9.109 \cdot 10^{-31}$ kg and $q = |e|$, is deflected from $(0,0)$ to $(0, 8.00 \text{ cm})$. Since the deflection is only in the y-direction, the second pair of plates $(d = 5 \text{ cm}, D = 4 \text{ cm})$ that cause horizontal deflection must have no potential across them, so only the first set of plates cause deflection. The voltage across the plates causes an electric field and then in turn a force that causes the electron to accelerate vertically in this area. Once out of this field, the electron is moving with constant velocity. Kinematics can then be used to determine what voltage is needed to cause the proper deflection. L = 40.0 cm and $v_\text{i} = 2.00 \cdot 10^7$ m/s.

SKETCH:



RESEARCH: The electric potential across the second set of plates is $V_\text{V} = 0,$ while across the first set the potential is $V_\text{H} = E_\text{H} D.$ The force that the horizontal plates cause is $F = qE_\text{H} = m_\text{e} a_y.$ During the whole trajectory, the horizontal velocity, v_i, is constant. The time it take to cross the first set of plates is $t_1 = d / v_\text{i}$, while its vertical displacement is $\Delta y_1 = a_y t_1^2 / 2.$ After the first plate, its vertical velocity remains constant as $v_0 = \sqrt{2a_y \Delta y_1}.$ The time after the first plate is $t_2 = (d + L) / v_\text{i}.$ Then the vertical displacement is $\Delta y_2 = v_0 t_2.$ The total y-displacement is then $\Delta y = \Delta y_1 + \Delta y_2.$

SIMPLIFY: The vertical acceleration is: $a_y = \dfrac{qE_\text{H}}{m_\text{e}} = \dfrac{|e| V_\text{H}}{m_\text{e} D}.$ The total y-displacement is then:

$$\Delta y = \Delta y_1 + \Delta y_2 = \frac{1}{2} a_y t_1^2 + v_0 t_2 = \frac{a_y d^2}{2v_1^2} + \sqrt{2a_y \Delta y}\left(\frac{d+L}{v_\text{i}}\right) = \frac{a_y d^2}{2v_1^2} + \sqrt{a_y^2 t_1^2}\left(\frac{d+L}{v_\text{i}}\right)$$

$$= a_y\left[\frac{d^2}{2v_\text{i}^2} + t_1\left(\frac{d+L}{v_\text{i}}\right)\right] = a_y\left[\frac{d^2}{2v_\text{i}^2} + \frac{2d(d+L)}{2v_\text{i}^2}\right] = \frac{|e| V_\text{H}}{m_\text{e} D}\left[\frac{d^2 + 2d(d+L)}{2v_\text{i}^2}\right].$$

The potential across the plates is then: $V_H = \frac{m_e D\Delta y}{|e|}\left[\frac{d^2+2d(d+L)}{2v_i^2}\right]^{-1}$.

CALCULATE:

$$V_H = \frac{(9.109\cdot 10^{-31}\text{ kg})(4.00\text{ cm})(8.00\text{ cm})}{1.602\cdot 10^{-19}\text{ C}}\left[\frac{(5.00\text{ cm})^2+2(5.00\text{ cm})((5.00\text{ cm})+(40.0\text{ cm}))}{2(2.00\cdot 10^7\text{ m/s})^2}\right]^{-1}$$

$= 306.45$ V

ROUND: $V_H = 306$ V

DOUBLE-CHECK: This is the same principle that a TV works by and 300 V is within the range of electric potentials that a TV can produce.

23.57. If the proton comes to a complete stop at $r = 1.00\cdot 10^{-15}$ m, then all of its initial kinetic energy is converted to potential energy:

$$U = \frac{k|e|^2}{r} = \frac{(8.9875\cdot 10^9\text{ N m}^2/\text{C}^2)(1.602\cdot 10^{-19}\text{ C})^2}{1.00\cdot 10^{-15}\text{ m}} = 2.31\cdot 10^{-13}\text{ J or } 1.44\text{ MeV}.$$

23.58. The barium nucleus has a charge of $q_1 = 56|e|$ and the krypton nucleus has a charge of $q_2 = 36|e|$. Their combined kinetic energy is $K_f = 200.$ MeV, which is equal to their initial potential energy, $U_i = kq_1q_2/r$. r is the separation of the two atoms, assumed to be the average size of the uranium atom, so:

$$U_i = \frac{kq_1q_2}{r} = K_f \Rightarrow r = \frac{k(36)(56)|e|^2}{K_f} = \frac{(8.9875\cdot 10^9\text{ N m}^2/\text{C}^2)(2016)(1.602\cdot 10^{-19}\text{ C})^2}{(200\cdot 10^6\text{ eV})(1.602\cdot 10^{-19}\text{ J}/(1\text{ eV}))} = 1.45\cdot 10^{-14}\text{ m}.$$

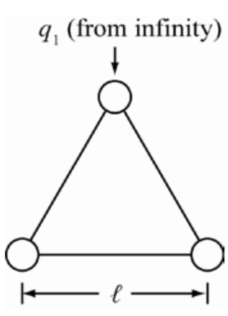
23.59. Assuming the first ion is brought in from an infinite distance, then the work needed to bring it a distance of $r = 10^{-14}$ m to the other ion is the potential energy of the two ions:

$$U = \frac{k|e|^2}{r} = \frac{(8.9875\cdot 10^9\text{ N m}^2/\text{C}^2)(1.602\cdot 10^{-19}\text{ C})^2}{10^{-14}\text{ m}}\left(\frac{1\text{ eV}}{1.602\cdot 10^{-19}\text{ J}}\right) = 143{,}980\text{ eV} \approx 144\text{ keV}.$$

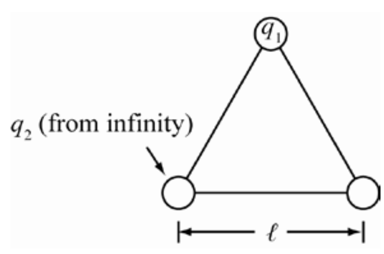
23.60. **THINK:** If each charge initially starts at an infinite distance, then the work done to move each charge to its final position is simply the potential energy of each charge in that position (the potential energy at infinity is zero). The charges are $q_1 = 1.0$ pC, $q_2 = 2.0$ pC and $q_3 = 3.0$ pC. Since they are on the corners of an equilateral triangle, each charge is the same distance, l = 1.2 m, from the others.

SKETCH:

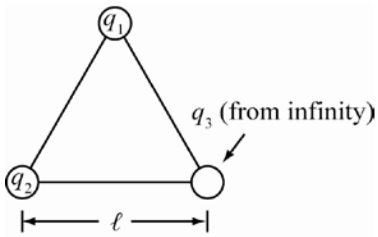
(a)



(b)



(d)



RESEARCH: In general, the potential for point charges is: $U = \sum_{i,j} \frac{kq_i q_j}{r_{ij}}$, where $r_{ij} = l$ for all *i* and *j*.

SIMPLIFY: The work done to bring all the charges together is $W = \sum_{i,j} \frac{kq_i q_j}{r_{ij}}$, where each pair of charges *i*, *j* is counted exactly once.

(a) Since there is no charge for q_1 to interact with, $U_1 = W_1 = 0$ J.

(b) Charge q_1 is present as q_2 is moved to its corner, so $U_2 = W_2 = \frac{kq_1 q_2}{l}$.

(c) Charges q_1 and q_2 are present as q_3 is moved to its corner, so

$$U_3 = W_3 = \frac{kq_1 q_3}{l} + \frac{kq_2 q_3}{l} = \frac{k}{l}(q_1 q_3 + q_2 q_3).$$

(d) The total energy is $U_{\text{tot}} = U_1 + U_2 + U_3$.

CALCULATE:

(a) $W_1 = 0$ J

(b) $W_2 = \frac{(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2)(1.0 \cdot 10^{-12} \text{ C})(2.0 \cdot 10^{-12} \text{ C})}{(1.20 \text{ m})} = 1.498 \cdot 10^{-14}$ J

(c) $W_3 = \frac{(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2)}{1.20 \text{ m}} \left[(1.0 \cdot 10^{-12} \text{ C})(3.0 \cdot 10^{-12} \text{ C}) + (2.0 \cdot 10^{-12} \text{ C})(3.0 \cdot 10^{-12} \text{ C}) \right]$
$= 6.743 \cdot 10^{-14}$ J

(d) $U_{\text{tot}} = (0 \text{ J}) + (1.498 \cdot 10^{-14} \text{ J}) + (6.743 \cdot 10^{-14} \text{ J}) = 8.241 \cdot 10^{-14}$ J

ROUND:

To three significant figures:

(a) $W_1 = 0$ J

(b) $W_2 = 1.50 \cdot 10^{-14}$ J

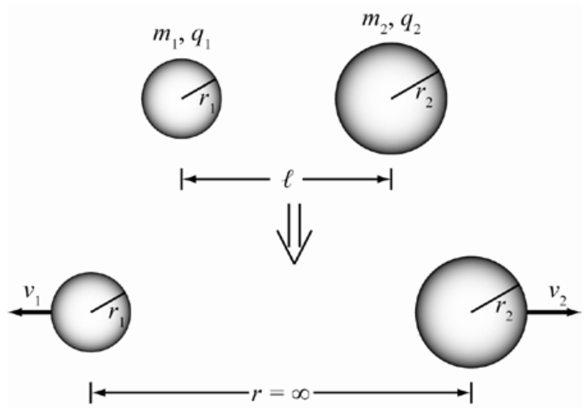
(c) $W_3 = 6.74 \cdot 10^{-14}$ J

(d) $U_{\text{tot}} = 8.24 \cdot 10^{-14}$ J

DOUBLE-CHECK: These small energy values are reasonable for such small amounts of charge.

23.61. **THINK:** Two balls have masses, $m_1 = 5.00$ g and $m_2 = 8.00$ g, and charges, $q_1 = 5.00$ nC and $q_2 = 8.00$ nC. Their center separation is l = 8.00 mm, and although the balls are not point charges, use the center separation to determine the potential energy stored in the two. Conservation of momentum and energy will allow the velocities of each to be determined. Since they are like charges, they repel and so the velocities will be in different directions.

SKETCH:



RESEARCH: The balls have no initial momentum, so by the conservation of momentum: $m_1 v_1 = m_2 v_2$. The initial potential energy of the two balls is given by $U_{\text{i}} = kq_1q_2 / l$. The final kinetic energy of the balls is given by $K_{\text{f}} = \left(m_1 v_1^2 / 2\right) + \left(m_2 v_2^2 / 2\right)$.

SIMPLIFY: From the conservation of momentum: $v_2 = m_1 v_1 / m_2$. Conservation of energy then requires:

$$U_{\text{i}} = K_{\text{f}} \Rightarrow \frac{kq_1q_2}{l} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}\left[m_1v_1^2 + m_2\left(\frac{m_1v_1}{m_2}\right)^2\right].$$

Therefore, $\dfrac{2kq_1q_2}{l} = \left[m_1 + \dfrac{m_1^2}{m_2}\right]v_1^2 \Rightarrow v_1 = \sqrt{\dfrac{2kq_1q_2}{l}\left(\dfrac{m_2}{m_1m_2 + m_1^2}\right)}.$

CALCULATE:

$$v_1 = \sqrt{\frac{2\left(8.9875\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(5.00 \text{ nC}\right)\left(8.00 \text{ nC}\right)}{0.008.00 \text{ m}}\left(\frac{0.00800 \text{ kg}}{0.00500 \text{ kg}\left(0.00800 \text{ kg}\right) + \left(0.00500 \text{ kg}\right)^2}\right)}$$

$$= 0.1052 \text{ m/s}$$

$$v_2 = \frac{5.00 \text{ g}\left(0.1052 \text{ m/s}\right)}{8.00 \text{ g}} = 0.06575 \text{ m/s}$$

ROUND: $v_1 = 0.105 \text{ m/s}$ and $v_2 = 0.0658 \text{ m/s}$

DOUBLE-CHECK: The charges are small and the masses relatively large, so the velocities obtained for the masses should be small.

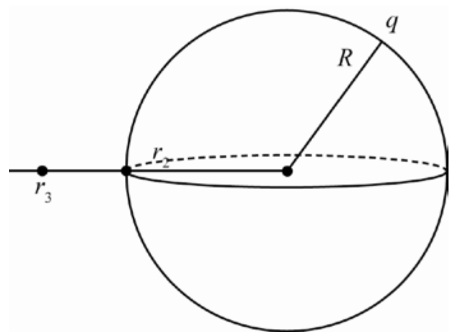
23.62. Conservation of energy can be considered to relate the change in kinetic energy to the change in potential energy by: $\Delta K = -\Delta U = K_{\text{f}} - K_{\text{i}} = U_{\text{i}} - U_{\text{f}} \Rightarrow K_{\text{f}} = U_{\text{i}} - U_{\text{f}}$. Each proton has the same mass, $m_{\text{p}} = 1.673\cdot 10^{-27}$ kg, and thus has the same kinetic energy, so the total kinetic energy is $K_{\text{f}} = \frac{1}{2}m_pv^2 + \frac{1}{2}m_pv^2 = m_{\text{p}}v^2$. Therefore,

$$K_{\text{f}} = U_{\text{i}} - U_{\text{f}} \Rightarrow m_{\text{p}}v^2 = k|e|^2\left(\frac{1}{r_{\text{i}}} - \frac{1}{r_{\text{f}}}\right) = k|e|^2\left(\frac{r_{\text{f}} - r_{\text{i}}}{r_{\text{i}}r_{\text{f}}}\right) \Rightarrow v = \sqrt{\frac{k|e|^2}{m_{\text{p}}}\left(\frac{r_{\text{f}} - r_{\text{i}}}{r_{\text{i}}r_{\text{f}}}\right)}$$

$$= \sqrt{\frac{\left(8.9875\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(1.602\cdot 10^{-19} \text{ C}\right)^2}{1.673\cdot 10^{-27} \text{ kg}}\left(\frac{10.0\cdot 10^{-3} \text{ m} - 1.00\cdot 10^{-3} \text{ m}}{10.0\cdot 10^{-3} \text{ m}\left(1.00\cdot 10^{-3} \text{ m}\right)}\right)} = 11.1 \text{ m/s}.$$

23.63. The battery places an electric potential of 12 V on the entire conducting surface of the hollow metal sphere. Inside the conducting shell, the electric field is zero and the electric potential remains at 12 V.

23.64.



Since the object is a solid conducting sphere, the electric potential is uniform throughout the sphere, so it is the same at $r_1 = 0$ m and $r_2 = 3$ m by $V = kq/R$, where $q = 4$ mC and $R = 3$ m. Outside the sphere, the distribution acts like a point charge, so the electric potential is $V = kq/r_3$.

(a) $V = \dfrac{kq}{R} = \dfrac{\left(8.9875\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(4.00\cdot 10^{-3} \text{ C}\right)}{3.00 \text{ m}} = 1.20\cdot 10^4 \text{ kV}$

(b) $V = \dfrac{kq}{R} = \dfrac{\left(8.9875\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(4.00\cdot 10^{-3} \text{ C}\right)}{3.00 \text{ m}} = 1.20\cdot 10^4 \text{ kV}$

(c) $V = \dfrac{kq}{r_3} = \dfrac{\left(8.9875\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(4.00\cdot 10^{-3} \text{ C}\right)}{5.00 \text{ m}} = 7.19\cdot 10^3 \text{ kV}$

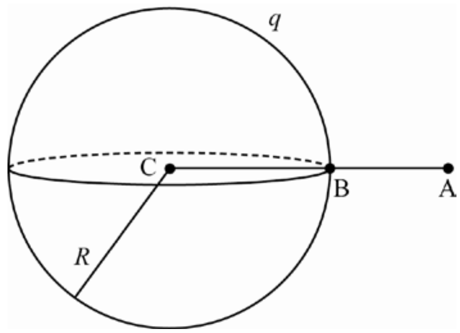
23.65. The infinite plate of surface charge density, $\sigma = 3.5\cdot 10^{-6} \text{ C/m}^2$, produces a constant electric field, $E = \sigma/2\varepsilon_0$. In going from point *A* to *B*, any movement perpendicular to the electric field results in no change in electric potential. Therefore, the only displacement of importance is $\Delta y = -1.0$ m. The change in potential is independent of the charge *Q*, and since the electric field is constant, it is the product of the electric field times the displacement:

$$\Delta V = -E\Delta y = -\frac{\sigma}{2\varepsilon_0}\Delta y = -\frac{\left(3.50\cdot 10^{-6} \text{ C/m}^2\right)}{2\left(8.854\cdot 10^{-12} \text{ C}^2/\left(\text{N m}^2\right)\right)}\left(-1.00 \text{ m}\right) = 1.98\cdot 10^5 \text{ V}.$$

23.66. Conservation of energy means the change in kinetic energy is equal to the magnitude of change in potential energy, $|\Delta K| = |\Delta U|$. The change in potential is $\Delta U = q\Delta V$, where $q = +e$ and $\Delta V = 21.9$ kV. The initial velocity is zero, so the change in kinetic energy is $\Delta K = m_e v_f^2/2$, where $m_e = 9.11\cdot 10^{-31}$ kg.

Therefore, $\Delta K = |\Delta U| = \dfrac{1}{2}m_e v_f^2 = e\Delta V \;\Rightarrow\; v_f = \sqrt{\dfrac{2e\Delta V}{m_e}} = \sqrt{\dfrac{2\left(1.602\cdot 10^{-19} \text{ C}\right)\left(21.9 \text{ kV}\right)}{9.11\cdot 10^{-31} \text{ kg}}} = 8.78\cdot 10^4 \text{ km/s}.$

23.67.



Since the object is a solid conducting sphere, the electric potential is distributed evenly through the sphere, so it is the same at points B and C and is given by $V = kq/R$, where $q = 6.10 \cdot 10^{-6}$ C and R = 18.0 cm. Therefore, $V_B = V_C = \frac{(8.9875 \cdot 10^9 \text{ N m}^2/\text{C}^2)(6.10 \cdot 10^{-6} \text{ C})}{0.180 \text{ m}} = 3.05 \cdot 10^5$ V. Outside the sphere, at $r_A = 24$ cm, the electric potential is that of a point charge, so:

$$V_A = \frac{kq}{r_A} = \frac{(8.9875 \cdot 10^9 \text{ N m}^2/\text{C}^2)(6.10 \cdot 10^{-6} \text{ C})}{0.240 \text{ m}} = 2.28 \cdot 10^5 \text{ V}.$$

23.68. The electric field of a spherical conductor is the same as that of a point charge at the center of the sphere with a charge equal to that of the spherical conductor. The potential outside the sphere is therefore also the same as a point charge:

$$V_{\text{surface}} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r}\right) = (8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2)\left(\frac{1.00 \cdot 10^{-6} \text{ C}}{1.00 \cdot 10^{-1} \text{ m}}\right) = 8.99 \cdot 10^4 \text{ V}.$$

23.69. First, determine the relationship between the electric field and the potential. The electric field is given by $E = kq/r^2$. The potential is given by $V = kq/r$. Therefore, the maximum voltage is

$$V_{\text{max}} = E_{\text{max}} r = (2.00 \cdot 10^6 \text{ V/m})(0.250 \text{ m}) = 5.00 \cdot 10^5 \text{ V}.$$

The maximum charge that it can hold is

$$q_{\text{max}} = \frac{r^2 E_{\text{max}}}{k} = \frac{(0.250 \text{ m})^2 (2.00 \cdot 10^6 \text{ V/m})}{(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2)} = 1.39 \cdot 10^{-5} \text{ C}.$$

23.70. Consider the conservation of energy to solve the problem. The potential energy is given by $U = qV$.

$$V = \frac{kq}{r} \Rightarrow U = \frac{kq^2}{r}$$

The moving proton will stop a distance r from the stationary proton, where the electric potential energy is equal to the initial kinetic energy:

$$K = U \Rightarrow \frac{1}{2} m_p v^2 = \frac{kq^2}{r}$$

$$r = \frac{2kq^2}{m_p v^2} = \frac{2(8.99 \cdot 10^9 \text{ Nm}^2/\text{C}^2)(1.602 \cdot 10^{-19} \text{ C})^2}{(1.673 \cdot 10^{-27} \text{ kg})(1.23 \cdot 10^4 \text{ m/s})^2} = 1.82 \cdot 10^{-9} \text{ m}.$$

23.71. (a) First an expression must be determined for the surface charge on each sphere. The surface area of a sphere is $4\pi r^2$. The surface charge density is given by:

$$\sigma_1 = \frac{q_1}{4\pi r_1^2} \text{ for the first sphere, and } \sigma_2 = \frac{q_2}{4\pi r_2^2} \text{ for the second sphere.}$$

$$\frac{\sigma_1}{\sigma_2} = \frac{q_1}{q_2}\left(\frac{r_2}{r_1}\right)^2 = \left(\frac{20.0 \text{ cm}}{10.0 \text{ cm}}\right)^2 = 4:1$$

(b) The charge flow stops when the potential is equal. If q_1 and q_2 are the final charge distributions after the potential of the two spheres are equal, then the following equations describe the potentials:

$$V_1 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1}\right), \; V_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_2}{r_2}\right).$$

$$V_1 = V_2 \;\Rightarrow\; \frac{q_1}{r_1} = \frac{q_2}{r_2} \;\Rightarrow\; q_1 = q_2\left(\frac{r_1}{r_2}\right) = q_2\left(\frac{10.0 \text{ cm}}{20.0 \text{ cm}}\right) = \frac{1}{2}q_2.$$

Also, $q_1 + q_2 = 200.\ \mu\text{C}$. Solving the two equations yields $q_1 = 200./3\ \mu\text{C} = 66.7\ \mu\text{C}$ and $q_2 = 400./3\ \mu\text{C} = 133.3\ \mu\text{C}$. The amount of charge that flows through the wire is then $|q_1 - q_2|/2 = |66.7\ \mu\text{C} - 133.3\ \mu\text{C}|/2 = 33.3\ \mu\text{C}$.

23.72. The potential of a sphere is given by $V(r) = q/4\pi\varepsilon_0 r$, where q is the total charge of the sphere. The total charge is given by $q = 4\pi r_s^2\sigma$, where r_s is the radius of the sphere. The potential difference between the surface of the sphere and the point, *P*, is then given by:

$$V(r_s) - V(r_p) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_s} - \frac{1}{r_p}\right] = \sigma\frac{4\pi r_s^2}{4\pi\varepsilon_0}\left[\frac{1}{r_s} - \frac{1}{r_p}\right] = \Delta V \;\Rightarrow\; \sigma = \frac{\varepsilon_0}{r_s^2}\left[\frac{1}{r_s} - \frac{1}{r_p}\right]^{-1}\Delta V$$

$$= \frac{8.85 \cdot 10^{-12} \text{ F/m}}{(0.200 \text{ m})^2 \text{ m}^3}\left[\frac{1}{0.200 \text{ m}} - \frac{1}{0.500 \text{ m}}\right]^{-1}(12.566 \text{ V}) = 9.27 \cdot 10^{-10} \text{ C/m}^2.$$

23.73. Consider the conservation of energy to determine the final kinetic energy:

$$\Delta K = -\Delta U \;\Rightarrow\; K_{\text{final}} - K_{\text{initial}} = U_{\text{initial}} - U_{\text{final}} \;\Rightarrow$$

$$K_{\text{final}} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{\text{initial}}}\right) - \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{\text{final}}}\right) = \frac{q_1 q_2}{4\pi\varepsilon_0}\left[\frac{1}{r_{\text{initial}}} - \frac{1}{r_{\text{final}}}\right].$$

Thus,

$$K_{\text{final}} = \left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(5.00 \cdot 10^{-6} \text{ C}\right)\left(9.00 \cdot 10^{-6} \text{ C}\right)\left[\frac{1}{0.100 \text{ m}} - \frac{1}{0.200 \text{ m}}\right]$$

$$K_{\text{final}} = 2.02275 \text{ J} \approx 2.02 \text{ J}.$$

23.74. The potential of a spherical object with a uniform charge distribution is the same as that of a point charge at the center of the sphere: $V = q/4\pi\varepsilon_0 r = \left(\frac{q}{r}\right)\frac{1}{4\pi\varepsilon_0} = \left(\frac{2.00 \cdot 10^{-6} \text{ C}}{2.00 \cdot 10^{-3} \text{ m}}\right)\left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right) = 8.99 \cdot 10^6 \text{ V}.$

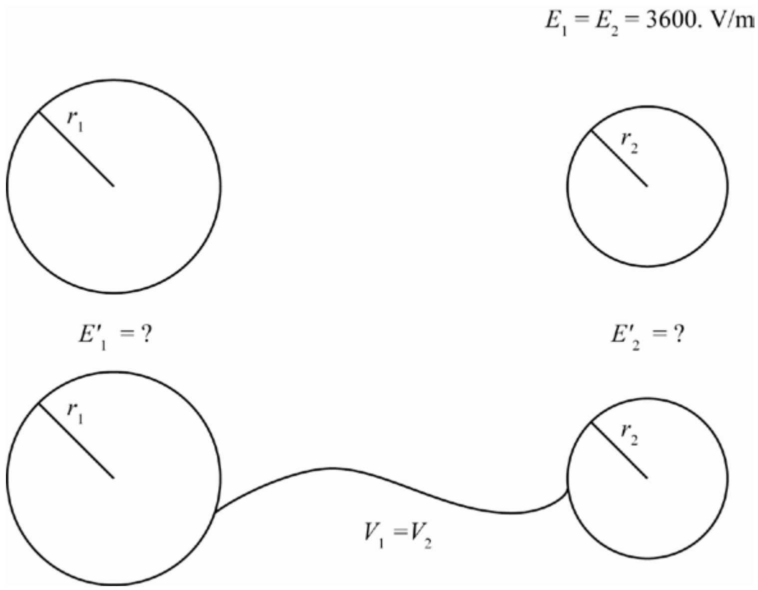
The potential difference has no angular dependence. If the potential in defined in terms of a charge distribution that depends on θ,

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\theta)}{r}dV,$$

the potential difference will have an angular dependence. Note that dV in the integral stands for differential volume.

23.75. **THINK:** First determine the total charge in each sphere based on the field. The charge from one sphere will flow into the other after they are connected until the potential of the two spheres are equal.

SKETCH:



RESEARCH: The electric fields of the two spheres are given by:

$$E_1 = \frac{kq_1}{r_1^2} \text{ and } E_2 = \frac{kq_2}{r_2^2}.$$

The potentials are given by:

$$V_1 = \frac{kq_1'}{r_1} \text{ and } V_2 = \frac{kq_2'}{r_2},$$

where q_1' and q_2' are the charges of the first and second sphere after they reach the same potential (when $V_1 = V_2$). Conservation of charges requires that, $q_1 + q_2 = q_1' + q_2'$. The final field strengths are given by:

$$E_1' = \frac{kq_1'}{r_1^2} \text{ and } E_2' = \frac{kq_2'}{r_2^2}.$$

The given values are $r_1 = 10$ cm, $r_2 = 5$ cm and $E_1 = E_2 \equiv E = 3600.$ V/m.

SIMPLIFY: The charge on each sphere before the two are connected is $q_1 = Er_1^2 / k$ and $q_2 = Er_2^2 / k$. Once the spheres are connected, their potentials are equal:

$$V_1 = V_2 \Rightarrow \frac{q_1'}{r_1} = \frac{q_2'}{r_2} \Rightarrow q_2' = \frac{r_2}{r_1} q_1'.$$

$$q_1' + q_2' = q_1 + q_2 \Rightarrow q_1' \left(1 + \frac{r_2}{r_1}\right) = \frac{E}{k}\left(r_1^2 + r_2^2\right) \Rightarrow q_1' = \frac{E\left(r_1^2 + r_2^2\right)}{k\left(1 + r_2 / r_1\right)}$$

$$\Rightarrow E_1' = k\left(\frac{E\left(r_1^2 + r_2^2\right)}{k\left(1 + r_2 / r_1\right)}\right)\frac{1}{r_1^2} = \frac{r_1^2 + r_2^2}{r_1^2 + r_1 r_2} E$$

Using $q_2' = \frac{r_2}{r_1} q_1'$ and $E_2' = kq_2' / r_2^2$ gives

$$E_2' = \frac{r_2}{r_1}\frac{kq_1'}{r_2^2} = \frac{r_1}{r_2}\frac{kq_1'}{r_1^2} = \frac{r_1}{r_2} E_1'.$$

CALCULATE: $E_1' = \dfrac{(10.0\text{ cm})^2+(5.00\text{ cm})^2}{(10.0\text{ cm})^2+(10.0\text{ cm})(5.00\text{ cm})}(3600.\text{ V/m}) = 3000.\text{ V/m}$

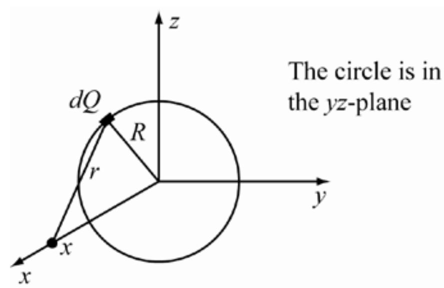
$E_2' = \dfrac{(10.0\text{ cm})}{(5.00\text{ cm})}(3000.\text{ V/m}) = 6000.\text{ V/m}$

ROUND: $E_1' = 3.00\cdot 10^3$ V/m and $E_2' = 6.00\cdot 10^3$ V/m

DOUBLE-CHECK: Since $r_1 > r_2$ it is expected that $E_2' > E_1'$ for the electric fields at the surfaces of the spheres.

23.76. **THINK:** First determine the potential for each infinitesimal part of the ring and then sum over the whole ring. Using the relationship between the potential field and the electric field, *E* can be determined.

SKETCH:



RESEARCH: The potential of each small dQ is given by $dQ = \lambda dL$ The total potential is then $V = \int \frac{kdQ}{r} = \int \frac{k\lambda dL}{r} = \int_0^{2\pi} \frac{k\lambda R d\theta}{\sqrt{x^2+R^2}} = \frac{2\pi Rk\lambda}{\sqrt{x^2+R^2}}$. Since $\lambda = \frac{Q}{L} = \frac{Q}{2\pi R}$, $V = \frac{kQ}{\sqrt{x^2+R^2}} = \frac{kQ}{r} = \frac{Q}{4\pi\varepsilon_0 r}$, where $r = \sqrt{x^2+R^2}$. From the symmetry, it can be inferred that $\vec{E}$ is pointing in the *x*-direction. The relation E_x and *x* is given by $E_x = -\partial V/\partial x$.

SIMPLIFY: $V = \dfrac{Q}{4\pi\varepsilon_0}\left(\dfrac{1}{r}\right) = \dfrac{Q}{4\pi\varepsilon_0}\left(\dfrac{1}{\sqrt{x^2+R^2}}\right)$

$$E_x = -\frac{\partial V}{\partial x} = -\frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{2}\right)\left[\frac{2x}{\left(x^2+R^2\right)^{3/2}}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{x}{\left(x^2+R^2\right)^{3/2}}\right]$$

CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: Check if *E* is truly zero in the *y*- and *z*-directions:

$$E_y = -\frac{\partial V}{\partial y} = 0 \text{ and } E_z = -\frac{\partial V}{\partial z} = 0, \text{ which confirms the result.}$$

23.77. **THINK:**

(a) First determine an expression for the total potential from both charges. After finding the expression, the potential can be determined.

(b) The derivative of the expression determined in part (a) can be used to determine the minimum point.

SKETCH: A sketch is not necessary.

RESEARCH:

(a) Let $q_1 = 0.681$ nC and $q_2 = 0.167$ nC be the two charges with positions $r_1 = 0$ and $r_2 = 10.9$ cm, respectively. The total potential is given by:

$$V_{\text{tot}} = \frac{q_1}{4\pi\varepsilon_0}\frac{1}{|r-r_1|} + \frac{q_2}{4\pi\varepsilon_0}\frac{1}{|r-r_2|}.$$

There are three cases, depending on the value of r:

$$V_{\text{tot}} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r-r_1}+\frac{q_2}{r-r_2}\right) \text{ for } r > r_1, r_2, \ V_{\text{tot}} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r-r_1}-\frac{q_2}{r-r_2}\right) \text{ for } r_1 < r < r_2 \text{ and}$$

$$V_{\text{tot}} = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q_1}{r-r_1}-\frac{q_2}{r-r_2}\right) \text{ for } r < r_1, r_2.$$

(b) The minima occur each time the derivative is equal to zero: $\partial V_{\text{tot}} / \partial r = 0.$

SIMPLIFY:

(a) There is nothing to simplify.

(b) Take the derivative for all three cases.

$r > r_1, r_2$: $\dfrac{\partial V_{\text{tot}}}{\partial r} = \dfrac{1}{4\pi\varepsilon_0}\left[-\dfrac{q_1}{(r-r_1)^2}-\dfrac{q_2}{(r_1-r_2)^2}\right]$. The expression is equal to zero at infinity.

$r_1 < r < r_2$: $\dfrac{\partial V_{\text{tot}}}{\partial r} = \dfrac{1}{4\pi\varepsilon_0}\left[-\dfrac{q_1}{(r-r_1)^2}+\dfrac{q_2}{(r-r_2)^2}\right]$. The expression is zero when:

$$\frac{q_1}{(r-r_1)^2} = \frac{q_2}{(r-r_2)^2} \Rightarrow \frac{q_1}{r^2} = \frac{q_2}{(r-r_2)^2} \quad (r_1 = 0 \text{ cm})$$

$$\Rightarrow \frac{(r-r_2)^2}{r^2} = \frac{q_2}{q_1} \Rightarrow \sqrt{\left(1-\frac{r_2}{r}\right)^2} = \sqrt{\frac{q_2}{q_1}} \Rightarrow \left|1-\frac{r_2}{r}\right| = \sqrt{\frac{q_2}{q_1}} \Rightarrow \frac{r_2}{r}-1 = \sqrt{\frac{q_2}{q_1}} \Rightarrow \frac{r_2}{\sqrt{q_2/q_1}+1} = r.$$

CALCULATE:

(a) $V_{\text{tot}} = \left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(\dfrac{0.681 \text{ nC}}{20.1 \text{ cm}-0}+\dfrac{0.167 \text{ nC}}{20.1 \text{ cm}-10.9 \text{ cm}}\right) = 46.78 \text{ V}$

(b) $r = \dfrac{10.9 \text{ cm}}{\sqrt{0.167 \text{ nC}/(0.681 \text{ nC})}+1} = 7.28997 \text{ cm}$

ROUND:

(a) 46.8 V

(b) 7.29 cm

DOUBLE-CHECK:

(a) The potential is positive and the potential from both charges is the sign that one would expect. This makes sense, since if a test charge was placed at 20.1 cm, it would move away from either one of the charges.

(b) An equilibrium point will exist between the two charges, where the force from one is balanced by the other. Note that $0 < 7.29 \text{ cm} < 10.9 \text{ cm}$.

23.78. THINK:

(a) The total potential of the origin can be determined using superposition.

(b) The expression for the potential determined in part (a) can be used to find the point where the potential is zero.

SKETCH: A sketch is not necessary.

RESEARCH:

(a) $V_{\text{tot}} = \dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{q_1}{r_1}+\dfrac{q_2}{r_2}\right)$, $r_1^2 = x_1^2 + y_1^2 = (2.5 \text{ m})^2 + (3.2 \text{ m})^2$, $r_2^2 = x_2^2 + y_2^2 = (-2.1 \text{ m})^2 + (1.0 \text{ m})^2$

$q_1 = 2.0$ μC and $q_2 = -3.1$ μC.

(b) Think of this as a one-dimensional problem with q_1 at the origin. The distance between q_1 and q_2 is given by $d = |\vec{r}_1 - \vec{r}_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. If r is the distance from q_1 to the point where $V_{\text{tot}} = 0$, then:

$$V_{\text{tot}} = k\left(\frac{q_1}{r} + \frac{q_2}{d-r}\right) = 0 \text{ for } r > d.$$

To determine the new point, simply switch to Cartesian coordinates:

$$\vec{r}_{\text{zero}} = \left(\frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}\right)r, \quad x_{\text{zero}} = x_1 + \frac{x_2 - x_1}{d}r \text{ and } y_{\text{zero}} = y_1 + \frac{y_2 - y_1}{d}r.$$

SIMPLIFY:

(a) Nothing to simplify.

(b) $V_{\text{tot}} = k\left(\frac{q_1}{r} + \frac{q_2}{d-r}\right) = 0 \Rightarrow \frac{q_1}{r} + \frac{q_2}{d-r} = 0 \Rightarrow \frac{q_1}{r} = -\frac{q_2}{d-r} \Rightarrow \frac{d-r}{r} = -\frac{q_2}{q_1} \Rightarrow \frac{d}{r} - 1 = -\frac{q_2}{q_1}$

$$\Rightarrow r = \frac{d}{1 - q_2 / q_1}$$

CALCULATE:

(a) $V_{\text{tot}} = \left(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(\frac{2.0\ \mu\text{C}}{\sqrt{(2.5\text{ m})^2 + (3.2\text{ m})^2}} + \frac{-3.1\ \mu\text{C}}{\sqrt{(-2.1\text{ m})^2 + (1.0\text{ m})^2}}\right) = -7.554 \cdot 10^3 \text{ V}$

(b) $d = \sqrt{(2.5\text{ m} + 2.1\text{ m})^2 + (3.2\text{ m} - 1.0\text{ m})^2} = 5.099\text{ m}, \quad r = \frac{5.099\text{ m}}{1 - (-3.1\ \mu\text{C}/2.0\ \mu\text{C})} = 2.000\text{ m}$

$x_{\text{zero}} = 2.5\text{ m} + \left(\frac{-2.1\text{ m} - 2.5\text{ m}}{5.099\text{ m}}\right)(2.000\text{ m}) = 0.6957\text{ m}, \quad y_{\text{zero}} = 3.2\text{ m} + \frac{1.0\text{ m} - 3.2\text{ m}}{5.099\text{ m}}(2.000\text{ m}) = 2.337\text{ m}$

ROUND:

(a) $-7.6 \cdot 10^3$ V

(b) $(0.70\text{ m}, 2.3\text{ m})$

DOUBLE-CHECK:

(a) The total voltage has appropriate units: volts.

(b) The point is between the two points, as one would expect because when going from a negative potential to a positive potential, the zero point is expected to be between the negative and positive charges.

23.79. **THINK:**

(a) Since the electric field of a conducting sphere is the same as that of a point charge its center, the expression for the potential is the same.

(b) The charge flow will stop when the potential of the two surfaces is equal.

SKETCH: A sketch is not necessary.

RESEARCH:

(a) $V_{\text{sphere}} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R}\right), \quad Q = 4.20 \cdot 10^{-6}\text{ C}, \ R = 0.400\text{ m}.$

(b) $V_1 = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{R_1}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_2}{R_2}\right) = V_2, \quad Q_1 + Q_2 = Q = 4.20 \cdot 10^{-6}\text{ C}, \ R_1 = 0.400\text{ m}, \ R_2 = 0.100\text{ m}$

SIMPLIFY:

(a) Nothing to simplify.

(b) $\frac{Q_1}{R_1} = \frac{Q_2}{R_2} \Rightarrow Q_1 = \frac{R_1}{R_2}Q_2$

Substitute this expression into $Q_1 + Q_2 = Q$ to get:

$$\frac{R_1}{R_2}Q_2 + Q_2 = Q \Rightarrow Q_2 = \frac{Q}{1+(R_1/R_2)}, \quad Q_1 = Q - Q_2.$$

Charge flow is Q_2. $E_1 = \frac{Q_1}{4\pi\varepsilon_0}\left(\frac{1}{R_1^2}\right)$ and $E_2 = \frac{Q_2}{4\pi\varepsilon_0}\left(\frac{1}{R_2^2}\right)$.

CALCULATE:

(a) $V_{\text{sphere}} = \left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(\frac{4.20\cdot 10^{-6} \text{ C}}{0.400 \text{ m}}\right) = 9.44\cdot 10^4 \text{ V}$

(b) $Q_2 = \frac{4.20\cdot 10^{-6} \text{ C}}{1+(0.400 \text{ m}/0.100 \text{ m})} = 0.840\cdot 10^{-6} \text{ C}, \; Q_1 = 4.20\cdot 10^{-6} \text{ C} - 0.840\cdot 10^{-6} \text{ C} = 3.36\cdot 10^{-6} \text{ C}$

$$\frac{E_2}{E_1} = \frac{Q_2}{R_2^2}\left(\frac{R_1^2}{Q_1}\right) = \frac{(0.840\cdot 10^{-6} \text{ C})(0.400 \text{ m})^2}{(3.36\cdot 10^{-6} \text{ C})(0.100 \text{ m})^2} = 4$$

The electric field on the surface of the second sphere is four times larger than the first sphere. This is the inverse of the ratio of their radii.
The electric field at the surface of sphere 1 is

$$E_1 = \frac{Q_1}{4\pi\varepsilon_0}\left(\frac{1}{R_1^2}\right) = \left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(\frac{3.36\cdot 10^{-6} \text{ C}}{(0.400 \text{ m})^2}\right) = 1.8879\cdot 10^5 \text{ V/m}.$$

$$E_2 = \frac{Q_2}{4\pi\varepsilon_0}\left(\frac{1}{R_2^2}\right) = \left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)\left(\frac{0.840\cdot 10^{-6} \text{ C}}{(0.100 \text{ m})^2}\right) = 7.5516\cdot 10^5 \text{ V/m}.$$

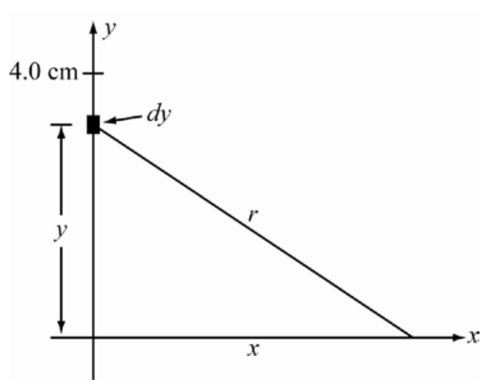
ROUND:

(a) $V_{\text{sphere}} = 9.44\cdot 10^4 \text{ V}$

(b) $Q_2 = 0.840\cdot 10^{-6}$ C, $E_1 = 1.89\cdot 10^5$ V/m, $E_2 = 7.55\cdot 10^5$ V/m.

DOUBLE-CHECK:
(a) The correct units of a voltage are volts.
(b) The charge flow is non-zero and comparable to the total charge, as one would expect.

23.80. **THINK:** Determine the potential of an infinitesimally small piece dy along the y-axis on the x-axis. Then integrate to determine the potential.
SKETCH:



RESEARCH: $dV = \frac{\lambda dy}{4\pi\varepsilon_0}\frac{1}{r}$, $V = \int_0^L dV$, $r = \sqrt{x^2+y^2}$, $\lambda = Ay$, $x = 3.06$ m, $L = 4.0$ cm,

$A = 8.0\cdot 10^{-7} \text{ C/m}^2$

SIMPLIFY: $V=\int dV=\int_0^L \frac{\lambda dy}{4\pi\varepsilon_0}\frac{1}{r}=\int_0^L \frac{dy}{4\pi\varepsilon_0}\frac{Ay}{\sqrt{x^2+y^2}}=\frac{A}{4\pi\varepsilon_0}\left[\sqrt{x^2+y^2}\right]_0^L=\frac{A}{4\pi\varepsilon_0}\left(\sqrt{x^2+L^2}-x\right)$

CALCULATE:

$V=\left(8.0\cdot 10^{-7}\text{ C/m}\right)\left(8.99\cdot 10^9\text{ N m}^2/\text{C}^2\right)\left[\sqrt{(0.03\text{ m})^2+(0.04\text{ m})^2}-0.03\text{ m}\right]=1.438\cdot 10^2\text{ V}$

ROUND: $V=1.4\cdot 10^2\text{ V}$

DOUBLE-CHECK: As x gets larger, $\sqrt{x^2+y^2}-x\approx 0,$ as expected.

23.81. (a) Let $q_1=-3.00\text{ mC}$ and $q_2=5.00\text{ mC}$ be located at $x_1=2.00\text{ m}$ and $x_2=-4.00\text{ m},$ respectively. There are three cases:

$$V(x)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{x-x_1}+\frac{q_2}{x-x_2}\right)\text{ for } x>x_1,\ x_2,\quad V(x)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{x_1-x}+\frac{q_2}{x-x_2}\right)\text{ for } x_1<x<x_2\text{ and}$$

$$V(x)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{x_1-x}+\frac{q_2}{x_2-x}\right)\text{ for } x<x_1,\ x_2.$$

The three cases stem from $|x-x_1|,\ |x-x_2|>0.$

(b) Case $x>x_1,x_2$: $V_{\text{tot}}=0\Rightarrow \frac{q_1}{x-x_1}=-\frac{q_2}{x-x_2}\Rightarrow x=\frac{q_1x_2+q_2x_1}{q_1+q_2}.$ Case $x_1<x<x_2$:

$\frac{q_1}{x_1-x}=-\frac{q_2}{x-x_2}\Rightarrow x=\frac{q_1x_2-q_2x_1}{q_1-q_2}.$ Case $x<x_1,x_2$: This case yields the same results as the first case.

Zeroes occur at the following points:

$$x=\frac{q_1x_2+q_2x_1}{q_1+q_2}=\frac{(-3.00\text{ mC})(-4.00\text{ m})+(5.00\text{ mC})(2.00\text{ m})}{-3.00\text{ mC}+5.00\text{ mC}}=11.0\text{ m},$$

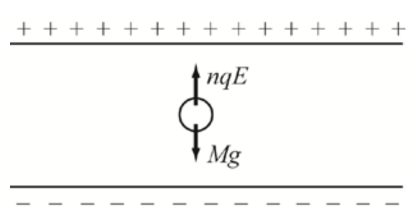
$$x=\frac{q_1x_2-q_2x_1}{q_1-q_2}=\frac{(-3.00\text{ mC})(-4.00\text{ m})-(5.00\text{ mC})(2.00\text{ m})}{-3.00\text{ mC}-5.00\text{ mC}}=-0.250\text{ m}.$$

(c) $E=-\frac{\partial V}{\partial x}$. $E=\frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{(x-x_1)^2}+\frac{q_2}{(x-x_2)^2}\right]$ for $x>x_1,\ x_2,$ $E=\frac{1}{4\pi\varepsilon_0}\left[-\frac{q_1}{(x_1-x)^2}+\frac{q_2}{(x-x_2)^2}\right]$ for $x_1<x<x_2$ and

$$E=\frac{1}{4\pi\varepsilon_0}\left[-\frac{q_1}{(x_1-x)^2}-\frac{q_2}{(x_2-x)^2}\right]\text{ for } x<x_1,\ x_2.$$

23.82. **THINK:** The forces acting on the charge are the coulomb and gravitational forces. For equilibrium, the total force must be zero.

SKETCH:



RESEARCH: $F_{\text{gravity}}=-Mg,\ F_{\text{coulomb}}=nqE,\ E=\frac{V}{d},\ F_{\text{total}}=F_{\text{gravity}}+F_{\text{coulomb}}$

SIMPLIFY: For equilibrium:

$$F_{\text{tot}} = 0 \;\Rightarrow\; Mg = nqE = \frac{nqV}{d} \;\Rightarrow\; V = \frac{Mgd}{nq}.$$

If the voltage is halved: $V = \frac{1}{2}\frac{Mgd}{nq}$. The total force is then:

$$Ma = F_{\text{tot}} = \frac{nqV}{d} - Mg = nq\left(\frac{1}{2}\frac{Mgd}{nq}\right)\frac{1}{d} - Mg = -\frac{Mg}{2} \;\Rightarrow\; a = -\frac{g}{2}.$$

If the voltage is doubled: $V = 2\frac{Mgd}{nq}$. The total force is then:

$$Ma = F_{\text{tot}} = \frac{nqV}{d} - Mg = nq\left(2\frac{Mgd}{nq}\right)\frac{1}{d} - Mg = Mg \;\Rightarrow\; a = g.$$

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: If the voltage is halved, the particle goes down. If the voltage is doubled, the particle goes up. In both cases, the result makes sense.

23.83. THINK:
(a) The potential is a function of distance but not direction. Since every segment, dq, is the same distance, R, from the origin, they have the same potential.
(b) Same as part (a).
(c) The charge distribution is symmetric with respect to reflection on the y-axis. This means that the E-field cannot have an x-component at the point O. In Chapter 22 we learned that the electric field points away from positive charges (and towards negative ones). Therefore we predict that the electric field at point O points in the negative x-direction. We can also make a prediction for the magnitude of the electric field. If the entire charge Q were all concentrated at the point $(0,R)$, then the electric field would be that of a point charge, $E = V/R$, where V is the answer obtained in parts a) and b). This means that we predict that $|E| < V/R$.
SKETCH: A sketch is not necessary.
RESEARCH:

(a) $V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R}\right)$

(b) $dV = \frac{dq}{4\pi\varepsilon_0}\left(\frac{1}{R}\right)$, $V = \int dV$

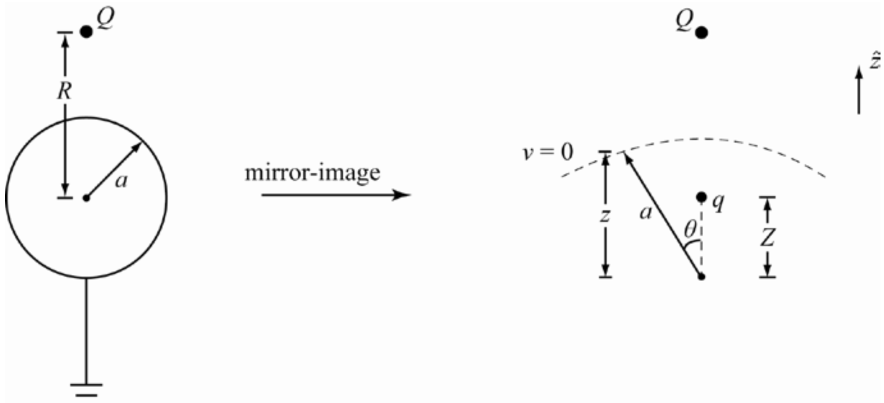
(c) Nothing to research.
SIMPLIFY:
(a) Nothing to simplify.

(b) $V = \int_0^q \frac{dq}{4\pi\varepsilon_0}\left(\frac{1}{R}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R}\right)$, which is the same result as part (a).

(c) $|E| < \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R^2}\right)$

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: The fact that the result from (b) matches the prediction made in (a) supports the prediction.

23.84. THINK:
(a) First determine the expression for the potential contribution.
(b,c) Can be determined after determining the expression for part (a).
SKETCH:



RESEARCH:
(a) Let the center of the sphere be at the origin of coordinates, with the exterior charge at $z-R$ on the positive z-axis. Let the image charge be at a coordinate z on that axis, with $|z|<a.$ The requirement that the surface of the charge be equipotential with potential zero takes the form:

$$0=\frac{Q}{\sqrt{x^2+y^2+(R-z)^2}}+\frac{q}{\sqrt{x^2+y^2+(z-Z)^2}}=\frac{Q}{\sqrt{R^2+a^2-2aR\cos\theta}}+\frac{q}{\sqrt{Z^2+a^2-2az\cos\theta}} \quad (1).$$

(x,y,z) is any point on the surface of the sphere (so $x^2+y^2+z^2=a^2$) with $z=a\cos\theta.$
(b) Since the electric field at the exterior charge is the same whether the sphere or the image charge is present, the force on the exterior charge toward the sphere is the same as the image charge would exert.

$$F=\frac{1}{4\pi\varepsilon_0}\left[\frac{qQ}{(R-z)^2}\right]e_z=-\frac{1}{4\pi\varepsilon_0}\left(\frac{Q^2aR}{R^2-a^2}\right)e_z$$

e_z is the unit vector in the positive z-direction, as defined above.
(c) The surface charge density, $\sigma(\theta)$, on the sphere is given by Gauss' law applied to a "pillbox" partially embedded at any point in the surface of the sphere: $\sigma(\theta)=\varepsilon_0 E_r,$ where E_r is the radial component of the net electric field at the surface of the sphere. This can be determined from the contributions of the exterior and image charges via Coulomb's law:

$$E(\theta)=\frac{1}{4\pi\varepsilon_0}\left\{\frac{Q\left[xe_x+ye_y+(z-R)e_z\right]}{\left(R^2+a^2-2aR\cos\theta\right)^{3/2}}+\frac{q\left[xe_x+ye_y+(z-Z)e_z\right]}{\left(Z^2+a^2-2aZ\cos\theta\right)^{3/2}}\right\}$$

$$=\frac{1}{4\pi\varepsilon_0}\left[\frac{Q\left[1-\left(R^2/a^2\right)\right]}{\left(R^2+a^2-2aR\cos\theta\right)^{3/2}}\right]\left(xe_x+ye_y+ze_z\right)=\frac{1}{4\pi\varepsilon_0}\frac{Qa\left[\left(R^2/a^2\right)-1\right]}{\left(R^2+a^2-2aR\cos\theta\right)^{3/2}}e_r.$$

SIMPLIFY:
(a) Rearranging yields:

$$\frac{q^2}{Q^2}\left(R^2+a^2\right)-\left(z^2+a^2\right)=2a\left[\left(\frac{q^2}{Q^2}R\right)-z\right]\cos\theta.$$

Since the right side of this equation depends on $\theta,$ while the left side does not, they are equal for all θ if and only if both are zero. This implies $q^2/Q^2=z/R.$ Therefore, $z\left(R^2+a^2\right)-R\left(z^2+a^2\right)=0.$ The quadratic formula gives two solutions for this $z = R$. Hence, $q = -Q$, which is trivial and $z = a^2 / R$ and $q = -Qa / R$, the desired solution. Equation (1) requires that q be opposite in sign to Q.

(c) Using the coordinate and result of part (a), and the radial unit vector, e_r, which is equal to $\left(xe_x + ye_y + ze_z\right)/a$ at the surface of the sphere. Note that, as expected, the net electric field is in the radial (normal) direction at the spherical surface. The surface charge density is therefore given by:

$$\sigma(\theta) = \frac{Qa\left[\left(R^2/a^2\right)-1\right]}{4\pi\left(R^2 + a^2 - 2aR\cos\theta\right)^{3/2}}.$$

The total induced charge on the sphere can be determined by integrating this over the surface.

CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: Gauss' law applied to a spherical "skin" around the conductor implies that the total surface charge is equal to the imagine charge, $-Qa/R$.

Multi-Version Exercises

Exercises 23.85–23.87 When a wire connects the two spheres, they have the same potential at the surface of both spheres:

$$\frac{kQ_1}{R_1} = \frac{kQ_2}{R_2} \quad \Rightarrow \quad \frac{Q_1}{R_1} = \frac{Q_2}{R_2}$$

The charge on the two spheres must sum to the original charge on the first sphere, $Q = Q_1 + Q_2$. We can write the charge on the first sphere as $Q_1 = Q - Q_2$. Now we can write

$$\frac{Q - Q_2}{R_1} = \frac{Q_2}{R_2}.$$

Solving for Q_2,

$$\frac{Q}{R_1} - \frac{Q_2}{R_1} = \frac{Q_2}{R_2}$$

$$\frac{Q_2}{R_1} + \frac{Q_2}{R_2} = Q_2\left(\frac{R_1 + R_2}{R_1 R_2}\right) = \frac{Q}{R_1}$$

$$Q_2\left(\frac{R_1 + R_2}{R_2}\right) = Q$$

$$Q_2 = Q\frac{R_2}{R_1 + R_2}.$$

23.85. $Q_2 = Q\dfrac{R_2}{R_1 + R_2} = \left(1.953 \cdot 10^{-6}\text{ C}\right)\dfrac{(0.6115\text{ m})}{1.206\text{ m} + 0.6115\text{ m}} = 6.571 \cdot 10^{-7}\text{ C}$

23.86. $Q_2 = Q\dfrac{R_2}{R_1 + R_2}$

$$Q = Q_2\frac{R_1 + R_2}{R_2} = \left(0.9356 \cdot 10^{-6}\text{ C}\right)\frac{1.435\text{ m} + 0.6177\text{ m}}{0.6177\text{ m}} = 3.109 \cdot 10^{-6}\text{ C}$$

23.87. $Q_2 = Q\dfrac{R_2}{R_1+R_2}$

$$R_1 + R_2 = R_2\frac{Q}{Q_2}$$

$$R_1 = R_2\left(\frac{Q}{Q_2}-1\right) = (0.6239\text{ m})\left(\frac{4.263\cdot 10^{-6}\text{ C}}{1.162\cdot 10^{-6}\text{ C}}-1\right) = 1.665\text{ m}.$$

Exercises 23.88–23.90 The electric field at the surface of the sphere is given by $E = \dfrac{kQ}{R^2}$. The potential a distance d from the surface is $V = \dfrac{kQ}{R+d}$. The charge on the sphere is $Q = \dfrac{ER^2}{k}$. So we can express the potential a distance d from the surface as

$$V = \frac{kQ}{R+d} = \frac{k\left(ER^2/k\right)}{R+d} = \frac{ER^2}{R+d}.$$

23.88. $V = \dfrac{ER^2}{R+d} = \dfrac{\left(3.165\cdot 10^5\text{ V/m}\right)\left(1.895\text{ m}\right)^2}{1.895\text{ m}+0.2981\text{ m}} = 5.182\cdot 10^5\text{ V}$

23.89. $V = \dfrac{ER^2}{R+d}$

$$V(R+d) = ER^2$$

$$VR + Vd = ER^2$$

$$ER^2 - VR - Vd = 0$$

$$R = \frac{V \pm \sqrt{V^2 + 4EVd}}{2E}$$

$$= \frac{2.843\cdot 10^5\text{ V} \pm \sqrt{\left(2.843\cdot 10^5\text{ V}\right)^2 + 4\left(3.269\cdot 10^5\text{ V/m}\right)\left(2.843\cdot 10^5\text{ V}\right)\left(0.3237\text{ m}\right)}}{2\left(3.269\cdot 10^5\text{ V/m}\right)}$$

$$= 1.121\text{ m}$$

(The other solution would have yielded a negative radius.)

23.90. $V = \dfrac{ER^2}{R+d}$

$$E = \frac{V(R+d)}{R^2} = \frac{\left(3.618\cdot 10^5\text{ V}\right)\left(1.351\text{ m}+0.3495\text{ m}\right)}{\left(1.351\text{ m}\right)^2} = 3.371\cdot 10^5\text{ V/m}$$

Chapter 24: Capacitors

Concept Checks

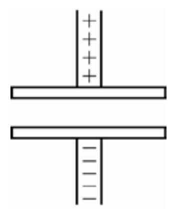
24.1. a **24.2.** a **24.3.** a **24.4.** d **24.5.** d **24.6.** c **24.7.** a **24.8.** b **24.9.** a **24.10.** b **24.11.** a **24.12.** a **24.13.** (a) True (b) False (c) True (d) False (e) False

Multiple-Choice Questions

24.1. b **24.2.** c **24.3.** c **24.4.** d **24.5.** c **24.6.** a **24.7.** d **24.8.** c **24.9.** a **24.10.** (a) F (b) F (c) T (d) F (e) T **24.11** b **24.12** a **24.13** b **24.14** a

Conceptual Questions

24.15.



If two insulators were used the charge would not be able to flow into the insulators and no charge would be stored; thus, conductors must be used.

24.16. Work has to be done to separate a positively charged plate from a negatively charged plate. When the battery is disconnected, the charge on the plates has nowhere to go and must remain the same. The electric field from a plane of charge depends only on the charge, not upon the distance from the plane (ignoring edge effects) so the electric field will remain the same. The voltage difference between the plates will just be the product of the electric field with the separation distance (since the electric field is constant), so as you pull the plates apart you'll be moving the same charge against an even voltage. When the battery remains connected, the voltage remains the same as the battery voltage. So as the plates are pulled apart, the electric field must decrease to make up for the increase in separation, which means the charge must flow off the plates (which it can do, because there's a path to the battery). Thus the force becomes less and less with greater separation; a smaller charge against a smaller field. The work done in increasing the separation is less. Therefore, the work done is greater when the capacitor is disconnected from the battery.

24.17. Since capacitors can store charge and are found in a lot of electrical equipment, grounding is done to ensure the excess charge can be discharged safely.

24.18. Imagine a conductor inserted into a parallel-plate capacitor, with ideally thin insulating sheets on each side to prevent charge transfer between the conductor and the capacitor plates. As current flows through the capacitor, charge will build up on the capacitor plates, but there will be an equal and opposite charge separation on the conductor between them, so that E between the plates will remain 0 and ΔV for the capacitor will remain zero. This corresponds to an infinitely large dielectric constant.

24.19. $U=\frac{C\Delta V^2}{2}=\frac{q^2}{2C}$, $U_{\text{old}}=\frac{q^2}{2C_{\text{old}}}$, $U_{\text{new}}=\frac{q^2}{2C_{\text{new}}}$, $C_{\text{old}}=\frac{\varepsilon_0 A}{d}$, $C_{\text{new}}=\frac{\varepsilon_0 A}{(d+d')}=\frac{C_{\text{old}}d}{(d+d')}$. The separation distance is increased by d'.

$$U_{\text{new}}=\frac{1}{2}\left(\frac{q^2}{C_{\text{new}}}\right)\left(\frac{d+d'}{d}\right)=U_{\text{old}}\left(1+\frac{d'}{d}\right)$$

The energy stored has increased from the work in pulling the charges apart.

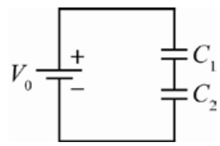
24.20. In order to increase the capacitance from $10.0\ \mu\text{F}$ to $18.0\ \mu\text{F}$ in a capacitor, you could add a dielectric in the capacitor with a dielectric constant of 1.80. The alternative would be to change the capacitor geometry by narrowing the plate separation or increasing the plate area.

24.21. For two capacitors $C_{\text{series}} = \left(\frac{1}{C_1}+\frac{1}{C_2}\right)^{-1} = \frac{C_1C_2}{\left(C_1+C_2\right)}$, then

$$C_{\text{series}} = \frac{C_1C_2}{C_1+C_2} = \frac{C_2}{1+\left(C_2/C_1\right)} < C_2, \text{ and } C_{\text{series}} = \frac{C_1C_2}{C_1+C_2} = \frac{C_1}{1+\left(C_1/C_2\right)} < C_1.$$

The resultant capacitance is always smaller than the smaller of the two values. In particular, if the difference between the two value is large (an order of magnitude or more), the resultant capacitance is less than but very close to the smaller of the two. For example, if we connect in series a capacitor $C_1 = 1\ \mu\text{F}$ with a capacitor $C_2 = 10\ \mu\text{F}$, we get a capacitance of $0.91\ \mu\text{F}$.

24.22. Two capacitors are connected in series. Assume the potential difference V_0 is due to a battery. The circuit is:



In series, the equivalent capacitance is $\frac{1}{C_{\text{eq}}} = \frac{1}{C_1}+\frac{1}{C_2}$. The potential difference supplied by the battery is $V_0 = \frac{q}{C_{\text{eq}}} = q\left(\frac{1}{C_1}+\frac{1}{C_2}\right) = \frac{q}{C_1}+\frac{q}{C_2}$, where $V_1 = \frac{q}{C_1}$ and $V_2 = \frac{q}{C_2}$. Solving for V_1 in the above yields: $V_0 = V_1 + \frac{q}{C_2} \Rightarrow V_1 = V_0 - \frac{q}{C_2}$. Note that since $V_0 = q\left(\frac{1}{C_1}+\frac{1}{C_2}\right)$,

$$q = \frac{V_0}{\left(1/C_1\right)+\left(1/C_2\right)} = \frac{V_0C_1C_2}{C_1+C_2}.$$

Then

$$V_1 = V_0 - \left(\frac{V_0C_1C_2}{C_1+C_2}\right)\left(\frac{1}{C_2}\right) = V_0 - \left(\frac{V_0C_1}{C_1+C_2}\right) = \frac{V_0\left(C_1+C_2\right)-V_0C_1}{C_1+C_2} = \frac{V_0C_2}{C_1+C_2}.$$

Similarly for V_2,

$$V_2 = V_0 - \left(\frac{V_0C_1C_2}{C_1+C_2}\right)\left(\frac{1}{C_1}\right) = V_0 - \left(\frac{V_0C_2}{C_1+C_2}\right) = \frac{V_0\left(C_1+C_2\right)-V_0C_2}{C_1+C_2} = \frac{V_0C_1}{C_1+C_2}.$$

24.23. (a) The limit is when the field reaches the dielectric strength of the material. The dielectric strength of air is given as $2.5\text{ kV/mm} = 2.5\cdot 10^6\text{ V/m}$. $E = kq/r^2$ for a sphere, so

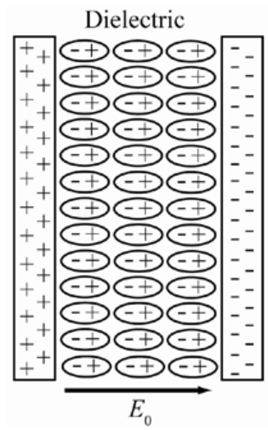
$$2.5\cdot 10^6\text{ V/m} = \frac{\left(8.99\cdot 10^9\text{ N m}^2/\text{C}^2\right)q}{\left(0.0500\text{ m}\right)^2} \Rightarrow q = 6.952\cdot 10^{-7}\text{ C} \approx 7.0\cdot 10^{-7}\text{ C}.$$

(b) When the charge in the sphere exceeds the limit specified in (a), the charge on the sphere will create a strong enough electric field to create an ionized conductive channel of air. The charge will spark though the air discharging the sphere slightly.

24.24. (a) The energy stored on the capacitor is $U = \dfrac{C_0V^2}{2}.$

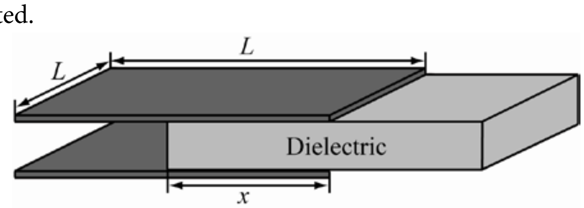
(b) Due to the power supply, the potential difference across the capacitors' plates remains constant as V when the dielectric material is inserted. To maintain this constant V, the power supply must supply additional charge to the plates. The capacitance becomes $C = \kappa C_0$, and the energy becomes $U = \dfrac{CV^2}{2} = \dfrac{\kappa C_0 V^2}{2}.$

(c) The dielectric is pulled into the space between the plates. There is an applied electric field $\vec{E}_0$ between the plates. When the dielectric is inserted, the molecules of the dielectric align with the field:



Then on each surface of the dielectric, there is an induced charge opposite to the charge on the adjacent plate. Since unlike charges attract, the dielectric is pulled into the space between the plates.

24.25. The capacitor is disconnected from the power supply; the charge Q on each plate remains constant while the dielectric is inserted, while the potential difference across the plates is reduced by a factor of κ. The force with which the slab is pulled into the capacitor is $F = -\left(\dfrac{d}{dx}\right)U = -\left(\dfrac{d}{dx}\right)\left(\dfrac{Q^2}{2C}\right)$, Q is constant. Consider two regions of the capacitor, one which is empty and one which contains dielectric material. These two "pieces" are in parallel, so $C = C_{\text{empty}} + C_{\text{dielectric}} = \dfrac{\varepsilon_0 L(L-x)}{d} + \dfrac{\kappa\varepsilon_0 L(x)}{d}$, where x is the depth that the dielectric is inserted.



Then $F = -\dfrac{d}{dx}\left(\dfrac{Q^2}{2C}\right) = -\left(\dfrac{Q^2}{2}\right)\left(-\dfrac{1}{C^2}\right)\dfrac{dC}{dx}$. Now

$$\frac{dC}{dx} = \frac{d}{dx}\left[\frac{\varepsilon_0 L(L-x)}{d} + \frac{\kappa\varepsilon_0 L(x)}{d}\right] = -\frac{\varepsilon_0 L}{d} + \frac{\kappa\varepsilon_0 L}{d} = \frac{\varepsilon_0 L}{d}(\kappa - 1).$$

$$F = \left(\frac{Q^2}{2C^2}\right)\left(\frac{\varepsilon_0 L}{d}\right)(\kappa-1) = \frac{V^2\varepsilon_0 L}{2d}(\kappa-1)$$

It turns out that the force is constant; it does not depend on *x*.

24.26. Assume the coaxial capacitor contains a dielectric material of dielectric material of dielectric constant κ (as opposed to air). The capacitance of a cylindrical capacitor is:

$$C=\frac{2\pi\kappa\varepsilon_0 L}{\ln\left(r_2/r_1\right)}=\frac{2\pi\kappa\varepsilon_0 L}{\ln\left(R/(R-d)\right)}=\frac{2\pi\kappa\varepsilon_0 L}{\ln R-\ln\left(R-d\right)}=\frac{2\pi\kappa\varepsilon_0 L}{\ln R-\ln\left(R(1-d/R)\right)}$$
$$=\frac{2\pi\kappa\varepsilon_0 L}{\ln R-\left(\ln R+\ln(1-d/R)\right)}=-\frac{2\pi\kappa\varepsilon_0 L}{\ln(1-d/R)}$$

Consider the series expansion for $\ln\left(1-x\right)$, where $|x|\le 1,\ x\neq 1$: $\ln\left(1-x\right)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\ldots$, which can be approximated as $\ln\left(1-x\right)\approx -x$ for x close to zero. This approximation is valid, since the question states that $d << R$, so d/R is close to zero. Then:

$$C=-\frac{2\pi\kappa\varepsilon_0 L}{\ln\left(1-d/\text{R}\right)}\approx\frac{2\pi\kappa\varepsilon_0 LR}{d}.$$

Now consider the surface area of the cylinder: $A=2\pi LR.$ In the limit of small distance d, this is the area of both cylinders. Then we can make the replacement: $C=\frac{\kappa\varepsilon_0 A}{\text{d}}$. This result is the formula for a parallel plate capacitor, and in the limit of d/R approaching zero, this makes a great deal of sense.

24.27. The charge on each plate will be same. The battery will move charge from one plate to the other, keeping the overall charge on the device neutral.

24.28. The parallel plate capacitor is connected to a battery. As the plates are pulled apart:
(a) The electrical potential on the plates does not change; therefore the charge on the plates would have to decrease.
(b) The capacitance for the parallel plate capacitor is $C=\varepsilon_0 A/d.$ As the distance between the plates increases, C must decrease. By definition $C=q/V$; since V remains the same, it must be that the charge on the plates decreases as they are pull apart.
(c) The electric field between the plates of a parallel capacitor is uniform, and is equal to $E=V/\text{d}.$ As the plates are pulled apart the electric field must decrease.

Exercises

24.29. Assume the supercapacitor is made from parallel plates. The capacitance is $C=\frac{\varepsilon_0 A}{d}$. Rearranging for A yields: $A=\frac{Cd}{\varepsilon_0}$. With $C=1.00$ F, $d=1.00\text{ mm}=1.00\cdot 10^{-3}$ m and $\varepsilon_0=8.85\cdot 10^{-12}$ F/m, the area is

$$A=\frac{\left(1.00\text{ F}\right)\left(1.00\cdot 10^{-3}\text{ m}\right)}{8.85\cdot 10^{-12}\text{ F/m}}=1.13\text{ km}^2.$$

24.30. The potential difference across the collinear cylinders is $V=100.$ V. The inner radius is $r_1=10.0\text{ cm}=0.100$ m. The outer radius is $r_2=15.0\text{ cm}=0.150$ m. The length of both cylinders is $L=40.0\text{ cm}=0.400$ m. By definition, $q=CV$. For a cylindrical capacitor, $C=2\pi\varepsilon_0 L/\ln\left(r_2/\text{r}_1\right)$. For this system,

$$C=\frac{2\pi\left(8.85\cdot 10^{-12}\text{ F/m}\right)\left(0.400\text{ m}\right)}{\ln\left(0.150/0.100\right)}=5.486\cdot 10^{-11}\text{ F}\approx 54.9\text{ pF}.$$

Then $q=\left(5.488\cdot 10^{-11}\text{ F}\right)\left(100.\text{ V}\right)=5.488\cdot 10^{-9}\text{ F}\approx 5.49\text{ nC}.$ The electric field between the plates of a cylindrical capacitor is $E=q/2\pi\varepsilon_0 rL.$

The magnitude of the electric field just outside the inner surface is:

$$E_1 = \frac{q}{2\pi\varepsilon_0 r_1 L} = \frac{5.488\cdot 10^{-9}\text{ F}}{2\pi\left(8.85\cdot 10^{-12}\text{ F/m}\right)\left(0.100\text{ m}\right)\left(0.400\text{ m}\right)} = 2466\text{ V/m} \approx 2470\text{ V/m}.$$

Its magnitude just inside the outer surface is:

$$E_2 = \frac{q}{2\pi\varepsilon_0 r_2 L} = \frac{5.488\cdot 10^{-9}\text{ F}}{2\pi\left(8.85\cdot 10^{-12}\text{ F/m}\right)\left(0.150\text{ m}\right)\left(0.400\text{ m}\right)} = 1644\text{ V/m} \approx 1640\text{ V/m}.$$

24.31. For a spherical conductor, the capacitance is $C = 4\pi\varepsilon_0 R.$ With $C = 1.00$ F, the radius must be

$$R = \frac{C}{4\pi\varepsilon_0} = \frac{1.00\text{ F}}{4\pi\left(8.85\cdot 10^{-12}\text{ F/m}\right)} = 8.988\cdot 10^{9}\text{ m} \approx 8.99\cdot 10^{9}\text{ m}.$$

24.32. The capacitance of a spherical capacitor made from two concentric conducting shells is: $C_\text{s} = 4\pi\varepsilon_0 r_1 r_2 / \left(r_2 - \text{r}_1\right).$ The capacitance of a parallel plate capacitor is: $C_\text{p} = \varepsilon_0 A/d.$ The area A of the parallel plate capacitor is equal to the area of the inner sphere in the spherical capacitor: $A = 4\pi \text{r}_1.$ With $d = \left(r_2 - r_1\right),$ the fractional difference in capacitance between the two geometries is

$$\frac{C_\text{s} - C_\text{p}}{C_\text{p}} = \left(\frac{\left(4\pi\varepsilon_0 r_1 r_2/\text{d}\right) - \left(4\pi\varepsilon_0 r_1^2/\text{d}\right)}{\left(4\pi\varepsilon_0 r_1^2/\text{d}\right)}\right) = \frac{r_2}{r_1} - 1.$$

The capacitance of the parallel plate capacitor is $\left(r_2 / r_1 - 1\right)100\%$ smaller than that of the spherical capacitor.

24.33. The capacitance of a spherical conductor is: $C = 4\pi\varepsilon_0 R.$ With the radius of the Earth being $R = 6371\text{ km} = 6.371\cdot 10^6\text{ m},$ the Earth's capacitance is:

$$C = 4\pi\left(8.85\cdot 10^{-12}\text{ F/m}\right)\left(6.371\cdot 10^{6}\text{ m}\right) = 7.0887\cdot 10^{-4}\text{ F} \approx 7.089\cdot 10^{-4}\text{ F}.$$

24.34. By definition, capacitance is $C = q/V.$ The capacitance of the spherical conductor is

$$C = 4\pi\varepsilon_0 \frac{r_1 r_2}{\left(r_2 - \text{r}_1\right)}.$$

The potential difference across the inner and outer spheres is $V = 900.$ V when a charge of $q = 6.726\cdot 10^{-8}$ C is applied to them. The radius of the outer sphere is $r_2 = 0.210$ m. The radius of the inner sphere is:

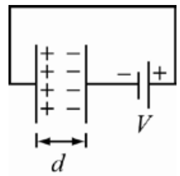
$$C = 4\pi\varepsilon_0 \frac{r_1 r_2}{r_2 - r_1} \;\Rightarrow\; C\left(r_2 - r_1\right) = 4\pi\varepsilon_0 r_1 r_2 \;\Rightarrow\; Cr_2 = r_1\left(C + 4\pi\varepsilon_0 r_2\right)$$

$$r_1 = \frac{Cr_2}{C + 4\pi\varepsilon_0 r_2} = \frac{r_2}{1 + \dfrac{4\pi\varepsilon_0 r_2 V}{q}}$$

$$r_1 = \frac{\left(0.210\text{ m}\right)}{1 + \dfrac{4\pi\left(8.85\cdot 10^{-12}\text{ F/m}\right)\left(0.210\text{ m}\right)\left(900.\text{ V}\right)}{\left(6.726\cdot 10^{-8}\text{ C}\right)}} \approx 0.160\text{ m}.$$

24.35. **THINK:** The capacitor is a parallel plate capacitor of variable separation distance. The material between the two plates is air. When the initial separation is $d_0 = 0.500\text{ cm} = 0.00500\text{ m}$, the initial capacitance is $C_0 = 32.0$ pF.

(a) A battery is connected with a potential difference $V = 9.00$ V. Find the charge density σ on the left plate. When the separation is changed to $d' = 0.250 \text{ cm} = 0.00250$ m, find the new capacitance C' and charge density, σ'.
(b) The separation between the plates is again $d_0 = 0.00500$ m and the battery is disconnected. The plates are moved to a separation of $d' = 0.00250$ m. Find the new potential difference V' between the plates.
SKETCH:



RESEARCH: By definition capacitance is $C = q/V$. Charge density is $\sigma = q/A$. Note that as long as the battery is connected, the potential difference across the plates is constant. When the battery is disconnected, the total charge in the system must remain constant.
SIMPLIFY:
(a) When the battery is first hooked up, the charge density is $\sigma = \frac{q}{A} = \frac{C_0 V}{A} = \frac{\varepsilon_0 V}{d_0}$ (for a parallel plate capacitor). When the separation is decreased from d_0 to d', the new capacitance is

$$C' = \frac{\varepsilon_0 A}{d'} \cdot \frac{d_0}{d_0} = \frac{\varepsilon_0 A}{d_0} \cdot \frac{d_0}{d'} = \frac{C_0 d_0}{d'}.$$

The new charge density is: $\sigma' = \frac{C'V}{A} = \frac{C_0 d_0}{d'} \cdot \frac{V}{A} = \frac{A\varepsilon_0}{d_0} \cdot \frac{d_0}{d'} \cdot \frac{V}{A} = \frac{\varepsilon_0 V}{d'}$ where V is the battery voltage.
(b) Now that the battery is disconnected, q remains constant, assuming ideal conditions, but the potential difference between the plates can change. When the plates are moved to a separation of d' the new capacitance is $\varepsilon_0 A/d'$, which is C' from above. The new potential difference is then $V' = \frac{q}{C'} = \frac{C_0 V}{C'}$.
CALCULATE:
(a) $\sigma = \frac{(8.85 \cdot 10^{-12} \text{ F/m})(9.00 \text{ V})}{0.00500 \text{ m}} = 1.593 \cdot 10^{-8} \text{ C/m}^2$, $\quad C' = \frac{(32.0 \text{ pF})(0.00500 \text{ m})}{0.00250 \text{ m}} = 64.0$ pF,

$\sigma' = \frac{(8.85 \cdot 10^{-12} \text{ F/m})(9.00 \text{ V})}{0.00250 \text{ m}} = 3.186 \cdot 10^{-8} \text{ C/m}^2$

(b) $V' = \frac{(32.0 \text{ pF})(9.00 \text{ V})}{64.0 \text{ pF}} = 4.50$ V

ROUND: For σ, σ' and V' the precision is limited to two significant figures from V. C' has three significant figures.
(a) $\sigma = 1.59 \cdot 10^{-8} \text{ C/m}^2$, $C' = 64.0$ pF, $\sigma' = 3.19 \cdot 10^{-8} \text{ C/m}^2$
(b) $V' = 4.50$ V
DOUBLE-CHECK: C is proportional to $1/d$ for parallel plates; as d decreases C must increase. When q is constant C is proportional to $1/V$, so as C increases, V decreases.

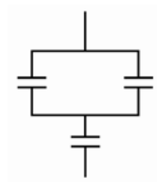
24.36. For equivalent capacitors of capacitance C in parallel, the equivalent capacitance is

$$C_{\text{eq}} = \sum_{i=1}^{n} C_i = nC.$$

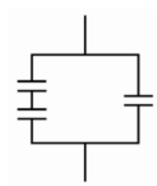
For a single capacitor, $C_{\text{eq}} = C.$ For two capacitors in parallel, $C_{\text{eq}} = 2C.$ For for three capacitors in parallel, $C_{\text{eq}} = 3C.$ For equivalent capacitors of capacitance C in series, the equivalent capacitance is

$$\frac{1}{C_{\text{eq}}} = \sum_{i=1}^{n} \frac{1}{C_i} = \frac{n}{C} \Rightarrow C_{\text{eq}} = \frac{C}{n}.$$

For two capacitors in series $C_{\text{eq}} = C/2.$ For three capacitors in series, $C_{\text{eq}} = C/3.$ Another combination is to have two capacitors in parallel and add the third in series with the first two, as shown:



In this case, the equivalent capacitance is $\frac{1}{C_{\text{eq}}} = \frac{1}{2C} + \frac{1}{C} = \frac{3}{2C} \Rightarrow C_{\text{eq}} = \frac{2}{3}C.$ Lastly, there can be two capacitors in series, with the third added in parallel with the first two, as shown:

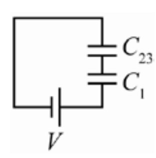


In this case, the equivalent capacitance is $C_{\text{eq}} = C + \frac{C}{2} = \frac{3}{2}C.$

24.37. The capacitor can be treated like two capacitors in parallel where each has an area equal to half that of the original. One capacitor has the original plate spacing $d_1 = 1.00 \text{ mm} = 0.00100 \text{ m},$ and the other has a plate spacing $d_2 = 0.500 \text{ mm} = 0.000500 \text{ m}.$ The original area was $A = 1.00 \text{ cm}^2 = 1.00 \cdot 10^{-4} \text{ m}^2.$ For capacitors in parallel, the equivalent capacitance is $C_{\text{eq}} = C_1 + C_2.$ For this system,

$$C_{\text{eq}} = \frac{\varepsilon_0 A/2}{d_1} + \frac{\varepsilon_0 A/2}{d_2} = \frac{\left(8.85 \cdot 10^{-12} \text{ F/m}\right)\left(1.00 \cdot 10^{-4} \text{ m}^2\right)}{2\left(0.00100 \text{ m}\right)} + \frac{\left(8.85 \cdot 10^{-12} \text{ F/m}\right)\left(1.00 \cdot 10^{-4} \text{ m}^2\right)}{2\left(0.000500 \text{ m}\right)}$$
$$= 1.3281 \cdot 10^{-12} \text{ F} \approx 1.33 \text{ pF}.$$

24.38. The capacitors have values $C_1 = 3.1 \text{ nF},$ $C_2 = 1.3 \text{ nF}$ and $C_3 = 3.7 \text{ nF}.$ The battery provides a voltage of $V = 14.9 \text{ V}.$ The circuit can be reduced to:



where $C_{23} = C_2 + C_3$ since C_2 and C_3 are in parallel. Now C_1 and C_{23} are in series, and the equivalent capacitance is:

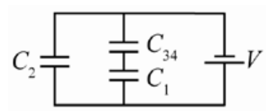
$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_{23}} = \frac{1}{C_1} + \frac{1}{C_2 + C_3} = \frac{C_1 + C_2 + C_3}{C_1\left(C_2 + C_3\right)} \Rightarrow C_{\text{eq}} = \frac{C_1\left(C_2 + C_3\right)}{C_1 + C_2 + C_3}.$$

For capacitors in series, the charge on each capacitor is the total charge, $q = C_{\text{eq}}V$. So, the charge on C_{23} is q. For C_2 and C_3 in parallel, this means that $q_2 + q_3 = q$. Also, the potential difference across C_2 and C_3 is the same, so $V_2 = V_3$. Then

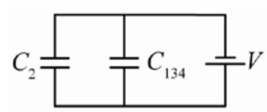
$$q = q_2 + q_3 = C_2V_2 + C_3V_3 = C_2V_2 + C_3V_2 = V_2\left(C_2 + C_3\right) \Rightarrow V_2 = \frac{q}{C_2 + C_3} = \frac{C_{\text{eq}}V}{C_2 + C_3} = \frac{C_1V}{C_1 + C_2 + C_3}$$

$$V_2 = \frac{(3.1\text{ nF})(14.9\text{ V})}{(3.1\text{ nF}) + (1.3\text{ nF}) + (3.7\text{ nF})} = 5.702\text{ V} \approx 5.70\text{ V}.$$

24.39. The capacitors have values $C_1 = 3.50$ nF, $C_2 = 2.10$ nF, $C_3 = 1.30$ nF and $C_4 = 4.90$ nF. The battery has voltage $V = 10.3$ V. First, C_3 and C_4 are in parallel. Then $C_{34} = C_3 + C_4$ and the circuit becomes



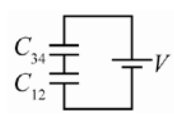
Next, C_1 and C_{34} are in series. Then $\frac{1}{C_{134}} = \frac{1}{C_1} + \frac{1}{C_{34}} = \frac{1}{C_1} + \frac{1}{C_3 + C_4}$. Then $C_{134} = \frac{C_1\left(C_3 + C_4\right)}{C_1 + C_3 + C_4}$ and the circuit becomes



Now C_2 and C_{134} are in parallel. The equivalent capacitance of the circuit is:

$$C_{\text{eq}} = C_2 + C_{134} = C_2 + \frac{C_1\left(C_3 + C_4\right)}{C_1 + C_3 + C_4} = (2.10\text{ nF}) + \frac{(3.50\text{ nF})(1.30\text{ nF} + 4.90\text{ nF})}{3.50\text{ nF} + 1.30\text{ nF} + 4.90\text{ nF}} = 4.337\text{ nF} \approx 4.34\text{ nF}.$$

24.40. The capacitors have values $C_1 = 18.0\ \mu\text{F}$, $C_2 = 11.3\ \mu\text{F}$, $C_3 = 33.0\ \mu\text{F}$ and $C_4 = 44.0\ \mu\text{F}$. The potential difference is $V = 10.0$ V. Capacitors C_1 and C_2 are in parallel, as are C_3 and C_4. Write $C_{12} = C_1 + C_2$ and $C_{34} = C_3 + C_4$. The circuit becomes



with C_{34} and C_{12} in series, the equivalent capacitance is

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_{12}} + \frac{1}{C_{34}} \Rightarrow C_{\text{eq}} = \frac{C_{12}C_{34}}{C_{12} + C_{34}} = \frac{\left(C_1 + C_2\right)\left(C_3 + C_4\right)}{C_1 + C_2 + C_3 + C_4}.$$

The total charge required to charge the capacitors in the circuit is

$$q = C_{\text{eq}}V = \frac{\left(C_1 + C_2\right)\left(C_3 + C_4\right)V}{C_1 + C_2 + C_3 + C_4} = \frac{(18.0\ \mu\text{F} + 11.3\ \mu\text{F})(33.0\ \mu\text{F} + 44.0\ \mu\text{F})(10.0\text{ V})}{18.0\ \mu\text{F} + 11.3\ \mu\text{F} + 33.0\ \mu\text{F} + 44.0\ \mu\text{F}} = 2.12 \cdot 10^{-4}\text{ C}.$$

24.41. **THINK:** Six capacitors are arranged as shown in the question.

(a) The capacitance of capacitor 3 is $C_3 = 2.300$ nF. The equivalent capacitance of the combination of capacitors 2 and 3 is $C_{23} = 5.000$ nF. Find the capacitance of capacitor 2, C_2. C_3 and C_2 are in parallel. Use the formula for parallel capacitance.

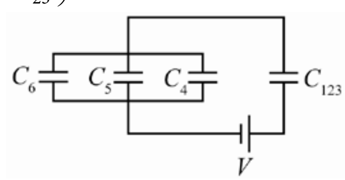
(b) The equivalent capacitance of the combination of capacitors 1, 2, and 3 is $C_{123} = 1.914$ nF. Find the capacitance of capacitor 1, C_1. C_1 and C_{23} are in series. Use the formula for series capacitance.

(c) The remaining capacitances are $C_4 = 1.300$ nF, $C_5 = 1.700$ nF, and $C_6 = 4.700$ nF. Find the equivalent capacitance of the whole system, C_{eq}.

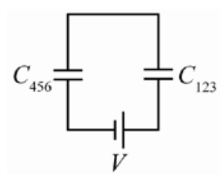
(d) A battery with a potential difference of $V = 11.70$ V is connected as shown. Find the total charge q deposited on this system of capacitors.
(e) Find the potential drop across capacitor 5 in this case.
SKETCH: Consider the sketch in the question. Sketches of the simplified system are provided in the simplify step.

RESEARCH: For capacitors in series, the equivalent capacitance is $1/C_{\text{eq}} = \sum_{i=1}^{n} 1/C_i$, and the charge on each is the same. For capacitors in parallel, the equivalent capacitance is $C_{eq} = \sum_{i=1}^{n} C_i$, and the potential drop across each is the same. By definition, capacitance is $C = |q/\Delta V|$.
SIMPLIFY:
(a) $C_{23} = C_2 + C_3 \Rightarrow C_2 = C_{23} - C_3$

(b) $\dfrac{1}{C_{123}} = \dfrac{1}{C_1} + \dfrac{1}{C_{23}} \Rightarrow C_1 = \left(\dfrac{1}{C_{123}} - \dfrac{1}{C_{23}}\right)^{-1}$



(c) C_4, C_5, and C_6 are in parallel. The equivalent capacitance between these capacitors is $C_{456} = C_4 + C_5 + C_6$.



Now, C_{123} and C_{456} are in series. The equivalent capacitance of the entire circuit is

$$C_{\text{eq}} = \left(\frac{1}{C_{123}} + \frac{1}{C_{456}}\right)^{-1}.$$

(d) The total charge required to charge the capacitors is $q = C_{\text{eq}}\Delta V$.

(e) C_4, C_5, and C_6 are in parallel. The voltage drop across each capacitor is the same: $V_4 = V_5 = V_6$. The equivalent capacitors C_{123} and C_{456} are in series. The charge on each C_{123} and C_{456} is the same, and they are equal to the total charge in the system: $q = q_{123} = q_{456}$. Since C_4, C_5, and C_6 are in parallel, $q = q_{456} = q_4 + q_5 + q_6$. Then,

$$q = C_4V_4 + C_5V_5 + C_6V_6 = V_5\left(C_4 + C_5 + C_6\right),\ V_5 = \frac{q}{\left(C_4 + C_5 + C_6\right)}.$$

CALCULATE:
(a) $C_2 = 5.000 \text{ nF} - 2.300 \text{ nF} = 2.700 \text{ nF}$

(b) $C_1 = \left(\dfrac{1}{1.914 \text{ nF}} - \dfrac{1}{5.000 \text{ nF}}\right)^{-1} = 3.1011 \text{ nF}$

(c) $C_{456} = 1.300 \text{ nF} + 1.700 \text{ nF} + 4.700 \text{ nF} = 7.700 \text{ nF},\quad C_{\text{eq}} = \left(\dfrac{1}{1.914 \text{ nF}} + \dfrac{1}{7.700 \text{ nF}}\right)^{-1} = 1.533 \text{ nF}$

(d) $q = (1.533 \text{ nF})(11.70 \text{ V}) = 17.94 \text{ nC}$

(e) $V_5 = \dfrac{(17.94\text{ nC})}{(1.300\text{ nF}+1.700\text{ nF}+4.700\text{ nF})} = 2.32987\text{ V}$

ROUND:

(a) To 4 significant figures, $C_2 = 2.700$ nF.

(b) To 4 significant figures, $C_1 = 3.101$ nF.

(c) To 4 significant figures, $C_{\text{eq}} = 1.533$ nF.

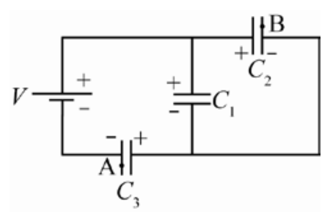
(d) To 4 significant figures, $q = 17.94$ nC.

(e) To 4 significant figures, $V_5 = 2.330$ V.

DOUBLE-CHECK: Note for equivalent capacitance C_{23}, where C_2 and C_3 are in parallel, $C_{23} > C_2, C_3$. Similarly, for equivalent capacitance C_{456}, where C_4, C_5, and C_6 are in parallel, $C_{456} > C_4, C_5, C_6$. Finally, the equivalent capacitance of the entire circuit $C_{\text{eq}} < C_{123}, C_{456}$, where the equivalent capacitances C_{123} and C_{456} are in series.

24.42. **THINK:** The capacitance values are $C_1 = 15.0$ nF, $C_2 = 7.00$ nF and $C_3 = 20.0$ nF. The potential difference provided by the battery is $V = 80.0$ V. Note that C_1 and C_2 are in parallel with each other, while C_3 is in series with the equivalent capacitor of C_1 and C_2. Find the magnitude and the sign of the charge $q_{3\text{l}}$ on the left plate of C_3, the electric potential V_3 across C_3, and the magnitude and sign of the charge $q_{2\text{r}}$ on the right plate of C_2.

SKETCH:



RESEARCH: Capacitors C_1 and C_2 are in parallel; their equivalent capacitance is $C_{12} = C_1 + C_2$. Capacitor C_3 is in series with C_{12}; the equivalent capacitance of the circuit is; therefore,

$$C_{\text{eq}} = \left(\frac{1}{C_3} + \frac{1}{C_{12}}\right)^{-1} = \frac{C_3 C_{12}}{C_3 + C_{12}}.$$

Since C_1 and C_2 are in parallel the potential drop across them is equal, so $V_1 = V_2$. The potential drop across C_3 and the equivalent capacitor C_{12} must sum to the potential drop across battery, $V_3 + V_{12} = V$, since they are in series. Finally, the total charge in the circuit is $q = C_{\text{eq}} V$, while the charge on a specific capacitor C_i is $q_\text{i} = C_\text{i} V_\text{i}$. Since C_3 and C_{12} are in series, the charges on these capacitors are equal to each other, and equal to the total charge in the circuit. The charges on C_1 and C_2 must sum to the total charge on their equivalent capacitor C_{12}, which is equal to the total charge in circuit q.

SIMPLIFY: As explained above,

$$q_3 = q = C_{\text{eq}} V = \frac{C_3 C_{12}}{C_3 + C_{12}} V = \frac{C_3 (C_1 + C_2)}{C_1 + C_2 + C_3} V.$$

At point A, q_3 is negative: the battery sets up an electric field in the wires. The field drives electrons from the negative end of the battery to the left of C_3. Then,

$$q_{3\text{l}} = -\frac{C_3 (C_1 + C_2)}{C_1 + C_2 + C_3} V.$$

V_3 is then $V_3 = q_3 / C_3$ and q_2 is $q_2 = C_2V_2$. V_2 is found from $V_3 + V_{12} = V$, where $V_{12} = V_1 = V_2$. Then $V_2 = V - V_3$, and $q_2 = C_2\left(V - V_3\right)$. At point B, q_2 is negative. The electric field pulls electrons from the left plate of C_2 to the positive end of the battery so the net charge on the left plate of C_2 is positive. The right plate must, therefore, be negatively charged. Then, $q_{2,b} = -C_2\left(V - V_3\right)$.

CALCULATE: $q_{31} = -\dfrac{(20.0 \text{ nF})(15.0 \text{ nF} + 7.00 \text{ nF})}{15.0 \text{ nF} + 7.00 \text{ nF} + 20.0 \text{ nF}}(80.0 \text{ V}) = -8.381 \cdot 10^{-7}\text{C}$

$V_3 = \left|\dfrac{-8.381 \cdot 10^{-7} \text{ C}}{20.0 \text{ nF}}\right| = 41.905 \text{ V}$

$q_{2\text{r}} = -(7.00 \text{ nF})(80.0 \text{ V} - 41.905 \text{ V}) = -2.6667 \cdot 10^{-7} \text{ C}$

ROUND: To three significant figures, $q_{31} = -8.38 \cdot 10^{-7}\text{C}$, $V_3 = 41.9 \text{ V}$, and $q_{2\text{r}} = -2.67 \cdot 10^{-7} \text{ C}$.

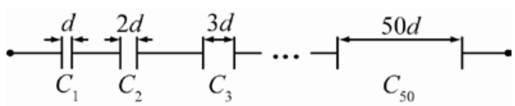
DOUBLE-CHECK: Note that $q_1 + q_2 = q_3$. q_1 is found from $q_1 = C_1V_1 = C_1V_2 = C_1\left(V - V_3\right)$, so

$$q_1 = (15.0 \text{ nF})(80.0 \text{ V} - 41.905 \text{ V}) = 5.7143 \cdot 10^{-7} \text{ C}.$$

Then $q_1 + q_2 = \left(5.7143 \cdot 10^{-7} \text{ C}\right) + \left(2.6667 \cdot 10^{-7} \text{ C}\right) = 8.381 \cdot 10^{-7} \text{ C}$, which is the magnitude of q_3 that was found above.

24.43. **THINK:** Fifty parallel plate capacitors are connected in series. The distance between the plates of the first capacitor is d, between the plates of the second capacitor $2d$, the third capacitor $3d$, and so on. The area of the plates remains the same for all capacitors. Find the equivalent capacitance C_{eq} in terms of C_1 (the capacitance of the first capacitor).

SKETCH:



RESEARCH: For capacitors in series, the equivalent capacitance is

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots = \sum_{\text{i}=1}^{n} \frac{1}{C_{\text{i}}}.$$

The capacitance of a single parallel plate capacitor is: $C = \varepsilon_0 A / d$.

SIMPLIFY: The equivalent capacitance is:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots + \frac{1}{C_{50}} = \frac{d_1}{\varepsilon_0 A} + \frac{d_2}{\varepsilon_0 A} + \frac{d_3}{\varepsilon_0 A} + \ldots \frac{d_{50}}{\varepsilon_0 A} = \frac{d}{\varepsilon_0 A}(1 + 2 + 3 + \ldots + 50).$$

Since $d_1 = d,\ d_2 = 2d,\ \ldots,\ d_n = nd$, it follows that: $\dfrac{1}{C_{\text{eq}}} = \dfrac{d}{\varepsilon_0 A} = \displaystyle\sum_{\text{i}=1}^{50} \text{i}$. Note $C_1 = \dfrac{\varepsilon_0 A}{d}$ and $\displaystyle\sum_{\text{i}=1}^{50} \text{i} = \dfrac{n(n+1)}{2}$,

and therefore $\dfrac{1}{C_{\text{eq}}} = \left(\dfrac{1}{C_1}\right)\left(\dfrac{n(n+1)}{2}\right) \Rightarrow C_{\text{eq}} = \dfrac{2C_1}{n(n+1)}$.

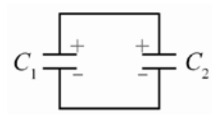
CALCULATE: With $n = 50$, $C_{\text{eq}} = \dfrac{2C_1}{50(50+1)} = \dfrac{C_1}{1275}$.

ROUND: The answer is precise. No rounding is required.

DOUBLE-CHECK: For capacitors in series, the equivalent capacitance must be less than the value of the largest capacitor, in this case C_1.

24.44. **THINK:** A $C_1 = 5.00 \text{ nF}$ capacitor initially charged to $V_1 = 60.0 \text{ V}$ and a $C_2 = 7.00 \text{ nF}$ capacitor charged to $V_2 = 40.0 \text{ V}$ are connected to each other with the negative plate of C_1 connected to the negative plate C_2. Find the final charge on C_2.

SKETCH:



RESEARCH: For capacitors connected in parallel, the potential drop across each capacitor is the same, that is $V_1 = V_2$. In addition, charge must be conserved, that is the initial charge q_i before the capacitors are connected must equal the final charge q_f when the capacitors are connected. Note $V_0 = q_0 / C_0$. In general, the charge q on a capacitor is $q = CV$.

SIMPLIFY: $q_\text{i} = q_{1\text{i}} + q_{2\text{i}} = C_1V_1 + C_2V_2$, $\quad q_\text{f} = C_1V_\text{f} + C_2V_\text{f} = C_2V_\text{f}\left(\dfrac{C_1V_\text{f}}{C_2V_\text{f}} + 1\right) = q_{2,\text{f}}\left(\dfrac{C_1}{C_2} + 1\right)$. Then,

$$q_\text{f} = q_\text{i},\ q_{2,\text{f}}\left(\frac{C_1}{C_2} + 1\right) = \left(C_1V_1 + C_2V_2\right) \Rightarrow q_{2,\text{f}} = \left(\frac{C_1V_1 + C_2V_2}{C_1/C_2 + 1}\right).$$

CALCULATE: $q_{2,\text{f}} = \left(\dfrac{(5.00\text{ nF})(60.0\text{ V}) + (7.00\text{ nF})(40.0\text{ V})}{(5.00\text{ nF})/(7.00\text{ nF}) + 1}\right) = 3.3833 \cdot 10^{-7}\text{ C}.$

ROUND: Rounding to three significant figures, $q_{2,\text{f}} = 3.38 \cdot 10^{-7}\text{ C} = 0.338\ \mu\text{C}.$

DOUBLE-CHECK: The solution is reasonable given the magnitudes of the capacitors.

24.45. The charge on each plate has a magnitude of $q = 60.0\ \mu\text{C}$ and potential difference of $V = 12.0\text{ V}$. The capacitance is; therefore, $C = q/V = 60.0\ \mu\text{C}/12.0\text{ V} = 5.00 \cdot 10^{-6}\text{ F}.$ When the potential difference is $V' = 120.\text{ V}$, the potential energy stored in the capacitor is

$$U = \frac{1}{2}C\left(V'\right)^2 = \frac{1}{2}\left(5.00 \cdot 10^{-6}\text{ F}\right)\left(120.\text{ V}\right)^2 = 0.0360\text{ J}.$$

24.46. The potential difference across the defibrillator is $V = 7500\text{ V}$. It stores $U = 2400\text{ J}$. Generally $U = (1/2)CV^2$. Solving for C yields $C = \dfrac{2U}{V^2} = \dfrac{2(2400\text{ J})}{(7500\text{ V})^2} = 8.533 \cdot 10^{-5}\text{ F} \approx 85.3\ \mu\text{F}.$

24.47. The Earth has an electric field $E = 150\text{ N/C} = 150\text{ V/m}$. The electric energy density is $U = (1/2)\varepsilon_0E^2 = (1/2)\left(8.85 \cdot 10^{-12}\text{ F/m}\right)\left(150.\text{ V/m}\right)^2 = 9.956 \cdot 10^{-8}\text{ J/m}^3$
$U \approx 9.96 \cdot 10^{-8}\text{ J/m}^3.$

24.48. **THINK:** The battery potential difference across two capacitors in series is $V = 120.\text{ V}$. The capacitances are $C_1 = 1.00 \cdot 10^3\ \mu\text{F}$ and $C_2 = 1.50 \cdot 10^3\ \mu\text{F}$. Find

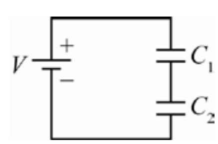
(a) The total capacitance C_eq of this circuit.

(b) The charge on each capacitor, q_1 and q_2.

(c) The potential difference across each capacitor, V_1 and V_2.

(d) The total energy stored in the circuit, U.

SKETCH:



RESEARCH:

(a) For a circuit of two capacitors connected in series, the equivalent capacitance is $1/C_\text{eq} = 1/C_1 + 1/C_2$.

(b) For a circuit of two capacitors connected in series, the charge on each capacitor is the same, and equal to the charge on a capacitor plate in the equivalent, one-capacitor circuit, that is $q_1 = q_2 = q.$ The total charge is found from $q = C_{\text{eq}} V.$

(c) For a circuit of two capacitors connected in series, the sum of the potential differences across each capacitor must be equal to the potential difference across the leads of the battery, that is $V_1 + V_2 = V.$ In general, for each capacitor in series, $q_i = C_i V_i.$

(d) The total energy stored in the circuit will be the sum of the energy stored in each capacitor, $U = U_1 + U_2.$ For each capacitor C_i, $U_i = q_i^2/2C_i.$

SIMPLIFY:

(a) $\dfrac{1}{C_{\text{eq}}} = \dfrac{1}{C_1} + \dfrac{1}{C_2} \Rightarrow C_{\text{eq}} = \dfrac{C_1 C_2}{C_1 + C_2}$

(b) $q_1 = q_2 = q = C_{\text{eq}} V$

(c) $V_1 = \dfrac{q_1}{C_1} = \dfrac{q}{C_1}; \quad V_2 = \dfrac{q_2}{C_2} = \dfrac{q}{C_2}$

(d) $U = U_1 + U_2 = \dfrac{1}{2}\dfrac{q_1^2}{C_1} + \dfrac{1}{2}\dfrac{q_2^2}{C_2} = \dfrac{q^2}{2}\left(\dfrac{1}{C_1} + \dfrac{1}{C_2}\right)$

CALCULATE:

(a) $C_{\text{eq}} = \dfrac{\left(1.00 \cdot 10^3 \text{ μF}\right)\left(1.50 \cdot 10^3 \text{ μF}\right)}{1.00 \cdot 10^3 \text{ μF} + 1.50 \cdot 10^3 \text{ μF}} = 600. \text{ μF} = 6.00 \cdot 10^{-4} \text{ F} = 0.600 \text{ mF}$

(b) $q_1 = q_2 = q = (0.600 \text{ mF})(120. \text{ V}) = 0.0720 \text{ C}$

(c) $V_1 = \dfrac{0.0720 \text{ C}}{1.00 \cdot 10^3 \text{ μF}} = 72.0 \text{ V}; \quad V_2 = \dfrac{0.0720 \text{ C}}{1.50 \cdot 10^3 \text{ μF}} = 48.0 \text{ V}$

(d) $U = \dfrac{(0.0720 \text{ C})^2}{2}\left(\dfrac{1}{1.00 \cdot 10^3 \text{ μF}} + \dfrac{1}{1.50 \cdot 10^3 \text{ μF}}\right) = 4.32 \text{ J}$

ROUND: Since C_1 and C_2 are given to three significant figures,

(a) $C_{\text{eq}} = 0.600 \text{ mF}$

(b) $q_1 = q_2 = 0.0720 \text{ C}$

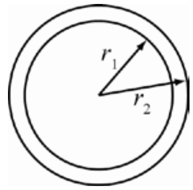
(c) $V_1 = 72.0 \text{ V}; V_2 = 48.0 \text{ V}$

(d) $U = 4.32 \text{ J}$

DOUBLE-CHECK: It should be the case that $V_1 + V_2 = (72.0 \text{ V}) + (48.0 \text{ V}) = 120. \text{ V}.$ This is correct. In addition, the potential energy in the circuit be equal to $(1/2)C_{\text{eq}}V^2 : U = (1/2)(0.600 \text{ mF})(120. \text{ V})^2 = 4.32 \text{ J}.$

24.49. **THINK:** Treat the neutron star as a spherical capacitor. The inner radius of the capacitor is the radius of the neutron star, $r_1 = 10.0 \text{ km} = 1.00 \cdot 10^4 \text{ m}.$ The outer radius is the radius of the neutron star and the 1.00 cm dipole layer. The charge density is $\sigma = \left(1.00 \text{ } \mu\text{C/cm}^2\right)(100. \text{ cm/m})^2 = 0.0100 \text{ C/m}^2.$ Find both the capacitance C of the star and electrical energy U stored in the star's dipole layer.

SKETCH:



RESEARCH: The capacitance of a spherical capacitor is $C = 4\pi\varepsilon_0 \dfrac{r_1 r_2}{r_2 - r_1}$. The total charge on the dipole layer is $q = \sigma A = \sigma 4\pi r_1^2$. Note since $r_1 \approx r_2$, assume the areas of the inner and outer shells are the same. The potential energy of a capacitor is $U = (1/2)q^2 / C$.

SIMPLIFY: $C = 4\pi\varepsilon_0 \dfrac{r_1 r_2}{r_2 - r_1}$, $U = \dfrac{1}{2}\dfrac{q^2}{C} = \dfrac{\left(\sigma 4\pi r_1^2\right)^2}{2C}$.

CALCULATE: $C = \dfrac{4\pi\left(8.854 \cdot 10^{-12} \text{ F/m}\right)\left(1.00 \cdot 10^4 \text{ m}\right)\left(10{,}000.01 \text{ m}\right)}{0.0100 \text{ m}} = 1.11263 \text{ F}$

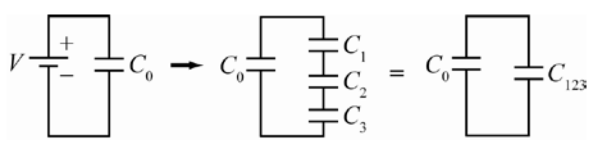
$$U = \frac{\left(\left(0.0100 \text{ C/m}^2\right)4\pi\left(1.00 \cdot 10^4 \text{ m}\right)^2\right)^2}{2\left(1.11263 \text{ F}\right)} = 7.096 \cdot 10^{13} \text{ J}.$$

ROUND: To 3 significant figures due to the thickness of the dipole layer, $C = 1.11$ F and $U = 7.10 \cdot 10^{13}$ J.

DOUBLE-CHECK: There is an enormous amount of charge on the dipole layer, $q \approx 1.30 \cdot 10^7$ C. Since both C and U are proportional to q, they should also be large (especially U, where $U \propto q^2$).

24.50. **THINK:** A $C_0 = 4.00 \cdot 10^3$ nF parallel plate capacitor is connected to a $V = 12.0$ V battery and charged. Use what you know about capacitors in series.
(a) Find the charge Q on the positive plate of the capacitor.
(b) Find the electrical energy U stored in the capacitor. The capacitor is then disconnected from the 12.0 V battery and used to charge 3 uncharged capacitors, a $C_1 = 100.$ nF capacitor, a $C_2 = 200.$ nF capacitor, and a $C_3 = 300.$ nF capacitor, arranged in series.
(c) Find the potential difference across each of the 4 capacitors, V_0, V_1, V_2, and V_3. The capacitors are in series.
(d) Determine the amount of electrical energy stored in the C_0 capacitor that was transferred to the other 3 capacitors.

SKETCH:



RESEARCH: In general, $q = CV$, and for capacitors in series, $\dfrac{1}{C_{\text{eq}}} = \sum_{i=1}^{n} \dfrac{1}{C_i}$.

(a) From above, the charge Q is $Q = C_0 V$.

(b) The electrical energy U is $U = Q^2 / 2C_0$.

(c) For capacitors in series, the charge on each plate is the same, that is $q_1 = q_2 = q_3$. The three capacitors can be replaced with an equivalent capacitor C_{123}, where C_{123} has a charge $q_{123} = q_1 = q_2 = q_3$. Note the first capacitor C_0 provides the voltage to charge the capacitors C_1, C_2, and C_3 or rather C_{123}. Some of

the charge Q originally on C_0 flows to the equivalent capacitor C_{123}. By conservation of charge, $Q = q_0 + q_{123}$. Once q_0 and q_{123} are determined, V_0, V_1, V_2, and V_3 can be determined from $q = CV$.
(d) The energy transferred to the other 3 capacitors is the sum of the energy on each capacitor. Use $U = q^2 / 2C$ to determine the energy on each capacitor.

SIMPLIFY:

(a) $Q = C_0 V$

(b) $U = Q^2 / 2C_0$

(c) From $V_0 = V_{123}$,

$$\frac{q_0}{C_0} = \frac{q_{123}}{C_{123}} \Rightarrow q_{123} = \left(\frac{C_{123}}{C_0}\right) q_0 \text{ where } \frac{1}{C_{123}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

From $Q = q_0 + q_{123}$,

$$Q = q_0 + \frac{C_{123}}{C_0} q_0 \Rightarrow q_0 = \frac{Q}{1 + \left(C_{123} / C_0\right)} \text{ with } q_{123} = q_1 = q_2 = q_3,$$

$V_0 = \dfrac{q_0}{C_0}$, $V_1 = \dfrac{q_{123}}{C_1}$, $V_2 = \dfrac{q_{123}}{C_2}$, and $V_3 = \dfrac{q_{123}}{C_3}$.

(d) $\Delta U = U_1 + U_2 + U_3 = \dfrac{{q_{123}}^2}{2}\left(\dfrac{1}{C_1} + \dfrac{1}{C_2} + \dfrac{1}{C_3}\right) = \dfrac{{q_{123}}^2}{2C_{123}}$

CALCULATE:

(a) $Q = C_0 V = \left(4.00 \cdot 10^3 \text{ nF}\right)\left(12.0 \text{ V}\right) = 4.80 \cdot 10^{-5} \text{ C}$

(b) $U = \dfrac{\left(4.80 \cdot 10^{-5} \text{ C}\right)^2}{2\left(4.00 \cdot 10^3 \text{ nF}\right)} = 2.88 \cdot 10^{-4} \text{ J}$

(c) $C_{123} = \left(\dfrac{1}{100. \text{ nF}} + \dfrac{1}{200. \text{ nF}} + \dfrac{1}{300. \text{ nF}}\right)^{-1} = 54.5 \text{ nF};\ q_0 = \dfrac{4.88 \cdot 10^{-5} \text{ C}}{1 + \left(\left(54.5 \text{ nF}\right) / \left(4.00 \cdot 10^3 \text{ nF}\right)\right)} = 4.735 \cdot 10^{-5} \text{ C}$

$q_{123} = q_1 = q_2 = q_3 = \left(\dfrac{54.5 \text{ nF}}{4.00 \cdot 10^3 \text{ nF}}\right)\left(4.735 \cdot 10^{-5} \text{ C}\right) = 6.46 \cdot 10^{-7} \text{ C}$

Then $V_0 = \dfrac{4.735 \cdot 10^{-5} \text{ C}}{4.00 \cdot 10^3 \text{ nF}} = 11.84 \text{ V}$, $V_1 = \dfrac{6.46 \cdot 10^{-7} \text{ C}}{100. \text{ nF}} = 6.46 \text{ V}$, $V_2 = \dfrac{6.46 \cdot 10^{-7} \text{ C}}{200. \text{ nF}} = 3.23 \text{ V}$ and $V_3 = \dfrac{6.46 \cdot 10^{-7} \text{ C}}{300. \text{ nF}} = 2.15 \text{ V}$.

(d) The transferred energy is $\Delta U = \dfrac{\left(6.46 \cdot 10^{-7} \text{ C}\right)^2}{2\left(54.5 \text{ nF}\right)} = 3.822 \cdot 10^{-6} \text{J}$.

ROUND:

(a) $Q = 4.80 \cdot 10^{-5} \text{ C}$

(b) $U = 2.88 \cdot 10^{-4} \text{ J} = 0.288 \text{ mJ}$

(c) $V_0 = 11.8 \text{ V}$, $V_1 = 6.46 \text{ V}$, $V_2 = 3.23 \text{ V}$, $V_3 = 2.15 \text{ V}$

(d) The energy transferred is $3.822 \cdot 10^{-6} \text{ J} = 3.82\ \mu\text{J}$

DOUBLE-CHECK: Because C_0 acts like a battery for C_1, C_2 and C_3 (is series), $V_0 = V_1 + V_2 + V_3$: $V_1 + V_2 + V_3 = 6.46 \text{ V} + 3.23 \text{ V} + 2.15 \text{ V} = 11.84 \text{ V} = V_0$.

24.51. THINK: The circuit has $V = 12.0$ V, $C_1 = 500.$ pF and $C_2 = 500.$ pF.

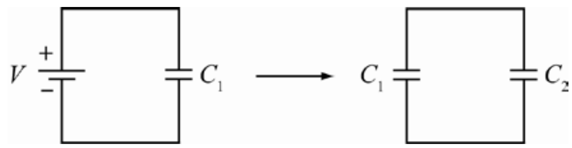
(a) Find the energy U_0 delivered by the battery while the switch is closed to A and the capacitor C_1 is fully charged.

(b) Find the energy stored on C_1 while the switch is closed to A and the capacitor C_1 is fully charged. The potential difference across C_1 is equal to the potential difference of the battery, V.

(c) Find the total energy stored at C_1 and C_2 when the switch is thrown to B. The first capacitor C_1 provides the voltage to charge the capacitor C_2, i.e. C_1 acts as a battery.

(d) Explain the energy loss, if there is any. Energy lost is the difference between the initial and final energies of the system.

SKETCH:



RESEARCH: In general $q = CV$

(a) The energy provided by the battery is given by $U = QV$.

(b) The energy stored in a capacitor is $U = CV^2/2$.

(c) Then the new potential difference V_1 across C_1 must be equal to V_2, the potential difference across C_2, or $V_1 = V_2$. This implies that $q_1/C_1 = q_2/C_2$ (since $V = q/C$). By conservation of charge, the original charge q_0 on C_1 (before the switch was thrown to point B) is $q_0 = q_1 + q_2$. Each q_1 and q_2 can be determined. Then the energy on each capacitor is $U = q^2/(2C)$.

(d) $\Delta E = E_f - E_i$

SIMPLIFY:

(a) $U = QV = C_1V^2$ as $Q = q_0 = C_1V$.

(b) $U_0 = \dfrac{C_1V^2}{2}$

(c) From $\dfrac{q_1}{C_1} = \dfrac{q_2}{C_2} \Rightarrow q_2 = \dfrac{C_2}{C_1}q_1$. Then $q_0 = q_1 + q_2 = q_1\left(1 + \dfrac{C_2}{C_1}\right)$. Also $q_0 = C_1V$. Then $q_1 = \dfrac{C_1V}{1 + C_2/C_1}$.

$U_1 = \dfrac{q_1^2}{2C_1}$, $U_2 = \dfrac{q_2^2}{2C_2}$

(d) $\Delta E = \left|E_f - E_i\right| = \left|U_1 + U_2 - U_0\right|$

CALCULATE:

(a) $U = (500.\text{ pF})(12.0\text{ V})^2 = 7.20 \cdot 10^{-8}$ J

(b) $U_0 = (500.\text{ pF})(12.0\text{ V})^2/2 = 3.60 \cdot 10^{-8}$ J

(c) $q_1 = \dfrac{(500.\text{ pF})(12.0\text{ V})}{1 + \left((500.\text{ pF})/(500.\text{ pF})\right)} = 3.00 \cdot 10^{-9}$ C, $q_2 = \left(\dfrac{C_2}{C_1}\right)q_1 = q_1 = 3.00 \cdot 10^{-9}$ C

Then $U_1 = U_2 = \dfrac{q_2^2}{2C_2}$ (since $C_1 = C_2$ and $q_1 = q_2$), $U_1 = \dfrac{\left(3.00 \cdot 10^{-9}\text{ C}\right)^2}{2(500.\text{ pF})} = 9.00 \cdot 10^{-9}$ J.

(d) $\Delta E = \left|2U_2 - U_0 = 2\left(9.00 \cdot 10^{-9}\text{ J}\right) - 3.60 \cdot 10^{-8}\text{ J}\right| = 1.80 \cdot 10^{-8}$ J

Even though the battery supplies $7.20 \cdot 10^{-8}$ J, half of this is lost to heat in the system. Again, when C_1 is connected to C_2, half of the energy is lost to heat.

ROUND: The capacitors have 3 significant figures. The answers should be rounded to 3 significant figures as well.

(a) $U = 7.20 \cdot 10^{-8}\text{ J} = 72.0\text{ nJ}$

(b) $U_0 = 36.0\text{ nJ}$

(c) $U_1 = U_2 = 9.00\text{ nJ}$

(d) $\Delta E = 18.0\text{ nJ}$

DOUBLE-CHECK: These answers are reasonable considering the initial values given.

24.52. THINK:

(a) The energy density, which is related to the electric field, can be integrated to determine the total electrostatic potential energy.

(b) The given formula can be integrated over the volume of the sphere to determine the gravitational potential energy.

(c) The magnitudes of the values found in parts (a) and (b) can be used to determine what impact the electrostatic forces have on the structure of the Earth.

SKETCH: Not required.

RESEARCH:

(a) The energy density is given by $U = (1/2)\varepsilon_0 E^2$. For the Earth of radius *R*, the electric field as a function of the radial position *r* is given by:

$$E = \frac{R^2 E_{\text{surface}}}{r^2}.$$

The Earth can be treated as a conductor, so all excess charge resides at the surface of the Earth. Therefore, by Gauss's Law $E = 0$ for $r < R$.

(b) As given in the question, the differential gravitational potential energy is $dU_g = -(Gm/r)dm$.

SIMPLIFY:

(a) The electrostatic potential energy of the Earth is given by:

$$U_e = \frac{1}{2}\varepsilon_0 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_R^{\infty} r^2 \left(\frac{E_{\text{surface}} R^2}{r^2}\right)^2 dr = \left(\frac{1}{2}\varepsilon_0\right) 4\pi \int_R^{\infty} \left(\frac{E_{\text{surface}}^2 R^4}{r^2}\right) dr$$

$$= 2\pi\varepsilon_0 E_{\text{surface}}^2 R^4 \int_R^{\infty} r^{-2} dr = 2\pi\varepsilon_0 E_{\text{surface}}^2 R^4 \left[-\frac{1}{r}\right]_R^{\infty} = 2\pi\varepsilon_0 E_{\text{surface}}^2 R^3.$$

(b) The gravitational potential energy of the Earth is given by:

$$U_g = \int dU = \int -\frac{Gm}{r} dm$$

The mass is a function of the radius,

$$m = \frac{4}{3}\pi\rho r^3,$$

and the differential mass element can be written as,

$$dm = \rho dV = \rho d\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi\rho d\left(r^3\right) = 4\pi\rho r^2 dr.$$

Therefore,

$$U_g = \int_0^R -\frac{G}{r}\left(\frac{4}{3}\pi\rho r^3\right)\left(4\pi\rho r^2\right)dr = -\frac{G(4\pi)^2\rho^2}{3}\int_0^R r^4 dr = -\frac{G(4\pi)^2\rho^2}{3}\left[\frac{1}{5}r^5\right]_0^R = -\frac{16}{15}\pi^2 G\rho^2 R^5.$$

CALCULATE:

(a) $U_e = 2\pi\left(8.854 \cdot 10^{-12}\text{ F/m}\right)\left(-150.\text{ V/m}\right)^2\left(6.371 \cdot 10^6\text{ m}\right)^3 = 3.237 \cdot 10^{14}\text{ J}$

(b) $U_g = -\frac{16}{15}\pi^2\left(6.6742 \cdot 10^{-11}\text{ m}^3\text{/kg s}^2\right)\left(5.515 \cdot 10^3\text{ kg/m}^3\right)^2\left(6.371 \cdot 10^6\text{ m}\right)^5 = -2.243 \cdot 10^{32}\text{ J}$

(c) $\left|U_{\text{g}}/U_{\text{e}}\right| = \left|\dfrac{\left(-2.243\cdot 10^{32}\text{ J}\right)}{\left(3.237\cdot 10^{14}\text{ J}\right)}\right| = 6.930\cdot 10^{17}$

The effects of the electrostatic forces on the Earth's structure are insignificant compared to the effects of gravity.

ROUND:

(a) To three significant figures, $U_{\text{e}} = 3.24\cdot 10^{14}$ J.

(b) To four significant figures, $U_{\text{g}} = -2.243\cdot 10^{32}$ J.

(c) To three significant figures, $\left|U_{\text{g}}/U_{\text{e}}\right| = 6.93\cdot 10^{17}$.

DOUBLE-CHECK: The gravitational potential energy greatly exceeds the electrostatic potential energy, which is reasonable since the gravitational potential energy is proportional to R^5, while the electrostatic potential energy is proportional to R^3.

24.53. In general, the energy stored in a capacitor is $U = (1/2)CV^2$. In terms of its dielectric strength V_{max}/mm, which yields $U = \dfrac{1}{2}\left(\dfrac{\kappa\varepsilon_0 A}{d}\right)V^2 = \dfrac{\kappa\varepsilon_0 A}{2d}\left(\left(V_{\text{max}}/\text{mm}\right)d\right)^2 = \dfrac{\kappa\varepsilon_0 Ad}{2}\left(V_{\text{max}}/\text{mm}\right)^2$. The ratio of U_{Mylar} to U_{air} is $\dfrac{U_{\text{Mylar}}}{U_{\text{air}}} = \dfrac{\kappa_{\text{Mylar}}\left(V_{\text{max}}/\text{mm}\right)^2_{\text{Mylar}}}{\kappa_{\text{air}}\left(V_{\text{max}}/\text{mm}\right)^2_{\text{air}}}$. From Table 24.1, this becomes:

$$\frac{U_{\text{Mylar}}}{U_{\text{air}}} = \frac{(3.1)(280\text{ kV/mm})^2}{(1)(2.5\text{ kV/mm})^2} = 3.89\cdot 10^4.$$

24.54. This set-up can be treated as two capacitors in parallel, one with the dielectric material and one with air. The total capacitance is $C = C_{\text{dielectric}} + C_{\text{air}}$. With $\kappa_{\text{air}} = 1$, this becomes

$$C = \frac{\kappa\varepsilon_0 L(L/2)}{s} + \frac{\varepsilon_0 L(L/2)}{s} = \frac{\varepsilon_0 L^2}{2s}(\kappa+1).$$

24.55. The dielectric constant of air is $\kappa = 1.00059$. Its dielectric strength is $V/d = 2.5\text{ kV/mm}$. Treating the surface as having area A, and assuming it is a plane, the charge is $q = CV = \left(\dfrac{\varepsilon_0\kappa_{\text{air}}A}{d}\right)V$. The charge density of the surface is $\sigma = \dfrac{q}{A} = \dfrac{\varepsilon_0\kappa_{\text{air}}V}{d}$. The maximum charge density is

$$\sigma_{\text{max}} = \varepsilon_0\kappa_{\text{air}}\left(\frac{V}{d}\right)_{\text{max}} = \left(8.85\cdot 10^{-12}\text{ F/m}\right)(1.00059)(2.5\text{ kV/mm}) = 2.2149\cdot 10^{-8}\text{ C/mm} \approx 2.2\cdot 10^{-5}\text{ C/m}.$$

24.56. The Thermocoax cable can be modeled as a cylindrical capacitor with $r_1 = 0.085$ mm and $r_2 = 0.175$ mm. With $k = 9.7$,

$$\frac{C}{L} = \frac{2\pi\varepsilon_0\kappa}{\ln(r_2/r_1)} = \frac{2\pi\left(8.85\cdot 10^{-12}\text{ F/m}\right)(9.7)}{\ln\left(0.175\text{ mm}/(0.085\text{ mm})\right)} = 7.473\cdot 10^{-10}\text{ F/m} = 750\text{ pF/m}.$$

24.57. This system is treated as two capacitors in parallel, one with dimensions $L\cdot L/5$ and dielectric κ_1, the other with dimensions $L\cdot 4L/5$ and dielectric κ_2. Then

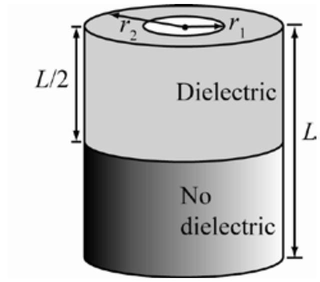
$$C_{\text{eq}}=C_1+C_2=\frac{\varepsilon_0\kappa_1A_1}{d}+\frac{\varepsilon_0\kappa_2A_2}{d}=\frac{\varepsilon_0\kappa_1\left(L^2/5\right)}{d}+\frac{\varepsilon_0\kappa_2\left(4L^2/5\right)}{d}=\frac{\varepsilon_0L^2}{d}\left(\frac{\kappa_1}{5}+\frac{4\kappa_2}{5}\right)$$
$$=\frac{\left(8.85\cdot 10^{-12}\text{ F/m}\right)\left(0.100\text{ m}\right)^2}{0.0100\text{ m}}\left(\frac{20.0}{5}+\frac{4\left(5.00\right)}{5}\right)=7.08310^{-11}\text{ F}\approx 70.8\text{ pF}.$$

24.58. **THINK:** The capacitor has capacitance $C=4.0$ nF and contains a sheet of Mylar with dielectric constant $\kappa=3.1$. The capacitor is charged to $V=120$ V and the power supply is then disconnected.
(a) Determine the work W required to completely remove the sheet of Mylar from the space between the two plates.
(b) Determine the potential difference between the plates of the capacitor once the Mylar is completely removed.
SKETCH:

RESEARCH:
(a) The work done is the change in potential energy: $W=\Delta U$, where $U=q^2/2C$. Note that initially the capacitance is $C_{\text{i}}=\kappa C_{\text{air}}$, while the final capacitance is $C_{\text{f}}=C_{\text{air}}$.
(b) The final potential is $V_{\text{f}}=q/C_{\text{f}}$. Because the power supply is disconnected, the charge remains constant and is $q=C_{\text{i}}V_{\text{i}}$.
SIMPLIFY:
(a) $W=U_{\text{f}}-U_{\text{i}}=\dfrac{q^2}{2C_{\text{f}}}-\dfrac{q^2}{2C_{\text{i}}}=\dfrac{q^2}{2C_{\text{air}}}\left(1-\dfrac{1}{\kappa}\right)$. Since q is conserved with the power supply being removed, $q=C_{\text{i}}V_{\text{i}}$, and $W=\dfrac{C_{\text{i}}^2V_{\text{i}}^2}{2C_{\text{air}}}\left(1-\dfrac{1}{\kappa}\right)$. Since $C_{\text{air}}=C_{\text{i}}/\kappa$, $W=\dfrac{C_{\text{i}}^2V_{\text{i}}^2}{2C_{\text{i}}/\kappa}\left(1-\dfrac{1}{\kappa}\right)=\left(\dfrac{C_{\text{i}}V_{\text{i}}^2}{2}\right)\left(\kappa-1\right)$.
(b) $V_{\text{f}}=\dfrac{q}{C_{\text{f}}}=\dfrac{C_{\text{i}}V_{\text{i}}}{C_{\text{f}}}=\dfrac{\kappa C_{\text{air}}V_{\text{i}}}{C_{\text{air}}}=\kappa V_{\text{i}}$.
CALCULATE:
(a) $W=\dfrac{\left(4.0\text{ nF}\right)\left(120\text{ V}\right)^2}{2}\left(3.1-1\right)=6.048\cdot 10^{-5}\text{ J}$
(b) $V_{\text{f}}=\left(3.1\right)\left(120\text{ V}\right)=372\text{ V}$
ROUND: To 2 significant figures, $W=6.0\cdot 10^{-5}$ J and $V_{\text{f}}=370$ V.
DOUBLE-CHECK: When the dielectric material is removed and the power supply is disconnected, the capacitance must decrease while the charge stays constant. Since $C\propto 1/V$, the potential must increase.

24.59. **THINK:** A cylindrical capacitor is half-filled with a dielectric of constant κ. This can be treated as two cylindrical capacitors in parallel. It is connected with a battery of potential difference V across its two electrodes. Find the charge q deposited on the capacitor, and find the ratio of this charge to the charge q_0 deposited on a completely empty capacitor connected in the same way across the same potential drop.

SKETCH:

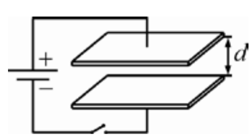


RESEARCH: The capacitance of a cylindrical capacitor is $C = 2\pi\varepsilon_0 L/\ln(r_2/r_1)$. For capacitors in parallel, $C_{\text{eq}} = C_1 + C_2$. In general, for a given potential, $q = CV$.

SIMPLIFY: When the capacitor is half-full, $C_{\text{eq}} = \dfrac{2\pi\kappa\varepsilon_0 L/2}{\ln(r_2/r_1)} + \dfrac{2\pi\varepsilon_0 L/2}{\ln(r_2/r_1)} = \dfrac{\pi\varepsilon_0 L}{\ln(r_2/r_1)}(\kappa+1)$. Then, $q = \dfrac{\pi\varepsilon_0 L/2}{\ln(r_2/r_1)}(\kappa+1)V$. In the absence of a dielectric, $C = (2\pi\varepsilon_0 L)/\ln(r_2/r_1)$. This gives the result: $q_0 = (2\pi\varepsilon_0 LV)/\ln(r_2/r_1)$. The ratio is $\dfrac{q}{q_0} = \left(\dfrac{\pi\varepsilon_0 LV(\kappa+1)/\ln(r_2/r_1)}{2\pi\varepsilon_0 LV/\ln(r_2/r_1)}\right) = \dfrac{\kappa+1}{2}$.

CALCULATE: Not required.
ROUND: Not required.
DOUBLE-CHECK: Since the power source supplies a constant V, the charge is not constant in the capacitor. It should be greater with the dielectric since $q \propto C$ and dielectrics increase capacitance.

24.60. **THINK:** The dielectric slab has thickness *d*. The parallel plate capacitor with area $A = 100.\text{ cm}^2 = 0.0100\text{ m}^2$ is charged by a battery of $V = 110.\text{ V}$. The plate separation distance is $d = 250.\text{ cm} = 0.0250\text{ m}$. The dielectric constant is $\kappa = 2.31$.
(a) Find the capacitance C, the potential difference V, the electric field E, the total charge stored on the plate Q, and electric energy stored U in the capacitor *before* inserting the dielectric material.
(b) Find the above physical quantities, when the dielectric slab is inserted while the battery is kept connected.
(c) Find the above physical quantities, when the dielectric slab is inserted *after* the battery was disconnected.
SKETCH:



RESEARCH:
(a) For a parallel plate capacitor, $C = \varepsilon_0 A/d$. While connected to a battery, V across the capacitor is equal to that of the battery. The electric field is $E = V/d$. The total charge Q can be found from $Q = CV$. The electrical energy of a capacitor is $U = (1/2)QV$.
(b) When the dielectric inserted the capacitance becomes $C' = \kappa\varepsilon_0 A/d = \kappa C$. With the battery still connected, the potential V stays constant but the charge changes; $Q' = C'V$. The electric field is $E = V/d$; it does not change. The electrical energy becomes $U' = (1/2)Q'V$.
(c) When the dielectric slab is inserted *after* the battery is disconnected the capacitance is still the same as in part (b). Now the potential does not remain constant, but the charge does; the charge is the same Q as

in part (a), and the potential becomes $V'' = Q/V'$. The electric field is $E'' = V''/d$, and the electrical energy is $U'' = (1/2)QV''$.

SIMPLIFY:

(a) $C = \varepsilon_0 A/d$, $V = V_{\text{batt}}$, $E = V/d$, $Q = CV_{\text{batt}}$ and $U = (1/2)QV$.

(b) $C' = \kappa C$, $V = V_{\text{batt}}$, $E = V/d$, $Q' = C'V_{\text{batt}}$ and $U = (1/2)Q'V$.

(c) $C' = \kappa C$, $V'' = Q/C' = CV_{\text{batt}}/C' = V_{\text{batt}}/\kappa$, $E'' = V''/d$, $Q = CV_{\text{batt}}$ and $U = (1/2)QV''$.

CALCULATE:

(a) $C = \dfrac{(8.8542 \cdot 10^{-12} \text{ F/m})(0.0100 \text{ m}^2)}{0.0250 \text{ m}} = 3.5417 \cdot 10^{-12} \text{ F}$, $V = 110. \text{ V}$,

$E = \dfrac{110. \text{ V}}{0.0250 \text{ m}} = 4.40 \cdot 10^3 \text{ V/m}$, $Q = (3.5417 \cdot 10^{-12} \text{ F})(110. \text{ V}) = 3.896 \cdot 10^{-10} \text{ F}$ and

$U = (1/2)(3.896 \cdot 10^{-10} \text{ J})(110. \text{ V}) = 2.143 \cdot 10^{-8} \text{ J}$.

(b) $C' = (2.31)(3.5417 \cdot 10^{-12} \text{ F}) = 8.181 \cdot 10^{-12} \text{ F}$, $V = 110. \text{ V}$, $E = 4.40 \cdot 10^3 \text{ V/m}$,

$Q' = (8.181 \cdot 10^{-12} \text{ F})(110. \text{ V}) = 8.999 \cdot 10^{-10} \text{ C}$ and $U = (1/2)(8.999 \cdot 10^{-10} \text{ C})(110. \text{ V}) = 4.949 \cdot 10^{-8} \text{ J}$.

(c) $C' = 8.181 \cdot 10^{-12} \text{ F}$, $V'' = \dfrac{110. \text{ V}}{2.31} = 47.62 \text{ V}$, $E'' = \dfrac{47.62 \text{ V}}{0.0250 \text{ m}} = 1905 \text{ V/m}$, $Q = 3.896 \cdot 10^{-10} \text{ C}$ and

$U = (1/2)(3.896 \cdot 10^{-10} \text{ C})(47.62 \text{ V}) = 9.276 \cdot 10^{-9} \text{ J}$.

ROUND: Rounding to three significant figures:

(a) $C = 3.54 \text{ pF}$, $V = 110. \text{ V}$, $E = 4.40 \cdot 10^3 \text{ V/m}$, $Q = 390. \text{ pF}$ and $U = 21.4 \text{ nJ}$.

(b) $C' = 8.18 \text{ pF}$, $V = 110. \text{ V}$, $E = 4.40 \cdot 10^3 \text{ V/m}$, $Q' = 900. \text{ pC}$ and $U = 49.5 \text{ nJ}$.

(c) $C' = 8.18 \text{ pF}$, $V'' = 47.6 \text{ V}$, $E'' = 1.90 \cdot 10^3 \text{ V/m}$, $Q = 390. \text{ pC}$ and $U = 9.28 \text{ nJ}$.

DOUBLE-CHECK: Capacitance increases with a dielectric material, when the battery stays connected, q must increase. When the battery is disconnected before the dielectric is inserted, V must decrease.

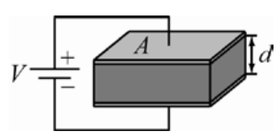
24.61. **THINK:** A parallel plate capacitor has a capacitance of $C = 120. \text{ pF}$ and plate area of $A = 100. \text{ cm}^2 = 0.0100 \text{ m}^2$. The space between the plates is filled with mica of dielectric constant $\kappa = 5.40$. The plates of the capacitor are kept at $V = 50.0 \text{ V}$. I want to find:

(a) The strength of the electric field mica, E.

(b) The amount of free charge on the plates, Q.

(c) The amount of induced charge on mica, Q_{ind}.

SKETCH:



RESEARCH: For a parallel plate capacitor, $C = \varepsilon_0 A\kappa/d$. While connected to a battery, V across the capacitor is equal to that of the battery. The field in the mica is; therefore, just the field between the plates, $E = V/d$. The charge Q is $Q = CV$. The induced charge in the mica is found by considering $E_{\text{net}} = E_0 - E_{\text{induced}}$.

SIMPLIFY:

(a) $C = \dfrac{\varepsilon_0 A\kappa}{d} \Rightarrow d = \dfrac{\varepsilon_0 A\kappa}{C}$ and $E = \dfrac{V}{d} = \dfrac{VC}{\varepsilon_0 A\kappa}$.

(b) $Q = CV$

(c) $E_{\text{net}} = E - E_{\text{ind}} \Rightarrow E_{\text{ind}} = E - E_{\text{net}}$, $\dfrac{Q_{\text{ind}}}{\varepsilon_0 A} = \dfrac{Q}{\varepsilon_0 A} - \dfrac{Q_{\text{net}}}{\varepsilon_0 A} = \dfrac{Q}{\varepsilon_0 A} - \dfrac{Q}{\kappa\varepsilon_0 A}$. Then $Q_{\text{ind}} = Q\left(1 - \dfrac{1}{\kappa}\right)$.

CALCULATE:

(a) $E = \dfrac{(50.0 \text{ V})(120. \text{ pF})}{(8.85 \cdot 10^{-12} \text{ F/m})(0.0100 \text{ m}^2)(5.40)} = 12554.9 \text{ V/m}$

(b) $Q = (120. \text{ pF})(50.0 \text{ V}) = 6.00 \text{ nC}$

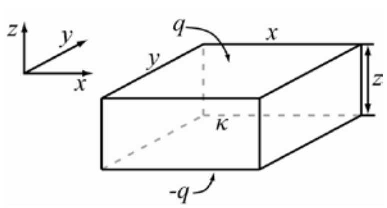
(c) $Q_{\text{ind}} = (6.00 \text{ nC})\left(1 - \dfrac{1}{5.40}\right) = 4.8888 \text{ nC}$

ROUND: To three significant figures, $E = 12.6 \text{ kV/m}$, $Q = 6.00 \text{ nC}$ and $Q_{\text{ind}} = 4.89 \text{ nC}$.

DOUBLE-CHECK: The charge induced in the dielectric should be less than the charge on the capacitor plates.

24.62. **THINK:** Given a material with dielectric constant $\kappa = 3.40$ and dielectric strength $V_{\text{max}} / d = 3.00 \cdot 10^7$ V/m, I want to design a capacitor with capacitance $C = 47.0$ pF, which can hold a charge of $q = 7.50$ nC. Let x and y be the dimensions of the parallel plates, and let z be the plate separation.

SKETCH:



RESEARCH: $C = 47.0 \text{ pF} = \kappa\varepsilon_0 A / z$, $A = xy$, $q = 7.50 \text{ nC} = C\Delta V$, $E = \Delta V / z = 3.00 \cdot 10^7$ V/m, and $\kappa = 3.40$.

SIMPLIFY: $z = \dfrac{\Delta V}{E} = \dfrac{q}{EC}$. The dimensions are minimized when $x = y$,

$$z = \kappa\frac{\varepsilon_0 A}{C} = \kappa\frac{\varepsilon_0 x^2}{C} \Rightarrow \left(\frac{zC}{\kappa\varepsilon_0}\right)^{1/2} = x.$$

CALCULATE: $z = \dfrac{7.50 \text{ nC}}{(3.00 \cdot 10^7 \text{ V/m})(47.0 \text{ pF})} = 5.3191 \cdot 10^{-6}$ m,

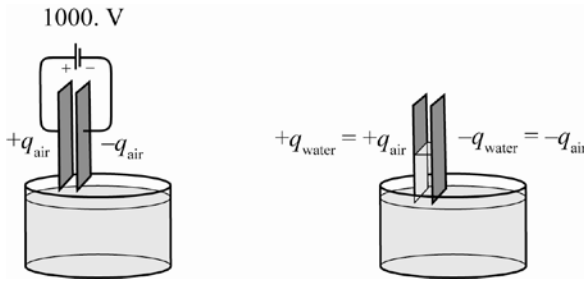
$$x = \left(\frac{(5.32 \cdot 10^{-6} \text{ m})(47.0 \text{ pF})}{(3.40)(8.85 \cdot 10^{-12} \text{ m}^{-3} \text{ kg}^{-1} \text{ s}^4 \text{ A}^2)}\right)^{1/2} = 2.8827 \cdot 10^{-3} \text{ m}$$

ROUND: $z = 5.32 \cdot 10^{-6}$ m and $x = y = 2.88 \cdot 10^{-3}$ m.

DOUBLE-CHECK: Since the charge stored and the capacitance was very small, it is not surprising, though perhaps unrealistic, to have such small dimensions.

24.63. **THINK:** The plates on the parallel plate capacitor have a width $W = 1.00 \text{ cm} = 0.0100 \text{ m}$ and a length $L = 10.0 \text{ cm} = 0.100 \text{ m}$, and therefore an area of $A = 10.0 \text{ cm}^2 = 0.00100 \text{ m}^2$. The separation between the plates is $d = 0.100 \text{ mm} = 0.000100 \text{ m}$. It is charged by a power supply at a potential difference of $V = 1.00 \cdot 10^3$ V. The power supply is then removed, and without being discharged, the capacitor is placed in a vertical position above a container holding de-ionized water, such that the short sides of the plates are in contact with the water. Demonstrate that the water will rise between the plates, and determine the system of equations that would allow one to calculate the height to which the water rises between the plates.

SKETCH:



RESEARCH: The capacitance of a parallel plate capacitor with air between the plates is $C_{air} = \varepsilon_0 A / d$. The capacitance of a parallel plate capacitor with de-ionized water between the plates is $C_{water} = \kappa\varepsilon_0 A / d = \kappa C_{air}$, where $\kappa = 80.4$. The charge on a capacitor is $q = CV$. The energy stored in a capacitor is $U = q^2 / (2C)$. The potential energy of a column of water with height *h* inside the parallel plate capacitor is $U_g = mgh_{CM}$, where h_{CM} is the height of the center of mass of the column of water. The mass of the water is $m = \rho V$, where $\rho = 1.00 \cdot 10^3$ kg/m^3.

SIMPLIFY: With air, the capacitor has a capacitance of $C_{air} = \dfrac{\varepsilon_0 A}{d}$. The charge on the capacitor is $q_{air} = C_{air}V$. The energy stored in the capacitor is $U_{air} = \dfrac{q_{air}^2}{2C}$. Once the capacitor is charged and the battery is removed, the charge on the capacitor stays the same. Since de-ionized water does not conduct electricity, the charge on each plate will stay the same even after the capacitor touches the water. By bringing the edge of the capacitor in contact with the free surface of the water in the tank, an upward force will act on the water. This can be proven from energy considerations. Assume that the water is indeed pulled upward between the plates until it completely fills the space between the plates (the water column height equals the length of the plates, *L*). The new capacitor with water as dielectric has a capacitance $C_{water} = \dfrac{\kappa\varepsilon_0 A}{d} = \kappa C_{air}$. The energy stored by this capacitor would be: $U_{water} = \dfrac{q_{water}^2}{2C_{water}} = \dfrac{q_{air}^2}{2\kappa C_{air}} = \dfrac{1}{\kappa}U_{air}$. The energy stored by the capacitor with water as a dielectric is less than the energy stored by the capacitor with air: $\Delta U = U_{water} - U_{air} = \dfrac{1}{\kappa}U_{air} - U_{air} = \dfrac{1-\kappa}{\kappa}U_{air}$. The change (final minus initial) in the energy of the system (capacitor) is negative, which means the system is doing work. This work is done against the force of gravity to pull the water upward between the plates. The potential energy of a column of water with height equal to the length of the capacitor plate would be $U_g = mgh_{CM} = \dfrac{1}{2}\rho VgL = \dfrac{1}{2}\rho gWL^2 d$. The work done by the electric field in the capacitor is not enough to pull the water all the way up to a height *L*, but is enough to pull the water to some height *h* between the plates. The new capacitor is, in effect, a parallel combination of two capacitors: one with air, another with a dielectric (water). Their respective capacitances are: $C_1 = \dfrac{\kappa\varepsilon_0 A_1}{d} = \dfrac{\kappa\varepsilon_0 Wh}{d}$ and $C_2 = \dfrac{\varepsilon_0 A_2}{d} = \dfrac{\varepsilon_0 W(L-h)}{d}$. The charge will no longer be uniformly distributed on the plates. Rather, it will be redistributed such that the voltages on the parallel capacitors are the same: $V_1 = V_2 \Rightarrow \dfrac{q_1}{C_1} = \dfrac{q_2}{C_2}$. In addition, the total charge remains $q_1 + q_2 = q_{air}$. The energies stored in the two capacitors are $U_1 = \dfrac{q_1^2}{2C_1}$ and $U_2 = \dfrac{q_2^2}{2C_2}$. By conservation of energy, $U_i = U_f$, or more specifically, $U_{\text{air}} = U_1 + U_2 + mg\dfrac{h}{2},$ where $mg\dfrac{h}{2}$ represents the gravitational energy of the column of water that is sucked upward between the plates of the capacitor. For a height *h* of the water between the plates, the mass of water is $m = \rho V = \rho dWh.$

CALCULATE: $C_{\text{air}} = \dfrac{\left(8.85\cdot 10^{-12}\text{ F/m}\right)\left(1.00\cdot 10^{-2}\text{ m}\right)\left(10.0\cdot 10^{-2}\text{ m}\right)}{1.00\cdot 10^{-4}\text{ m}} = 8.85\cdot 10^{-11}\text{ F},$

$q_{\text{air}} = \left(8.85\cdot 10^{-11}\text{ F}\right)\left(1.00\cdot 10^{3}\text{ V}\right) = 8.85\cdot 10^{-8}\text{ C},\ U_{\text{air}} = \dfrac{\left(8.85\cdot 10^{-8}\text{ C}\right)^2}{2\left(8.85\cdot 10^{-11}\text{ F}\right)} = 4.425\cdot 10^{-5}\text{ J},$

$U_{\text{water}} = \dfrac{4.425\cdot 10^{-5}\text{ J}}{80.4} = 5.51\cdot 10^{-7}\text{ J}$

Recall that $\Delta U = U_{\text{water}} - U_{\text{air}}$: $\Delta U = \left(\dfrac{1-80.4}{80.4}\right)\left(4.425\cdot 10^{-5}\text{ J}\right) - 4.372\cdot 10^{-5}\text{ J}.$ So $U_{\text{water}} < U_{\text{air}}$, and the capacitor is doing work against gravity to "suck" water up between the plates. So it is true that the water is drawn up to some height *h* between the plates of the capacitor. Note that if the water rose to height $h = L$, the gravitational potential energy stored in the system would be:

$$U_{\text{g}} = \frac{\left(1.00\cdot 10^{3}\text{ kg/m}^3\right)\left(9.81\text{ m/s}^2\right)\left(0.0100\text{ m}\right)\left(0.100\text{ m}\right)^2\left(0.000100\text{ m}\right)}{2} = 4.905\cdot 10^{-5}\text{ J}.$$

So the work done by the capacitor's electric field is not enough to suck up the water to the full height of *L*, since $U_{\text{g}} > U_{\text{water}}$. The parallel combination of the two "new" capacitors with water and air, respectively, has capacitances

$$C_1 = \frac{(80.4)\left(8.85\cdot 10^{-12}\text{ F/m}\right)\left(1.00\cdot 10^{-2}\text{ m}\right)h}{1.00\cdot 10^{-4}\text{ m}} = 7.12\cdot 10^{-8}h\text{ F}$$

$$C_2 = \frac{\left(8.85\cdot 10^{-12}\text{ F/m}\right)\left(1.00\cdot 10^{-2}\text{ m}\right)(L-h)}{1.00\cdot 10^{-4}\text{ m}} = \left(8.85\cdot 10^{-10}\right)\left(0.100\text{ m}-h\right)\text{ F},$$

where *h* is measured in meters. By considering the new charge distribution and equivalent voltages on each C_1 and C_2, there are two equations:

$$\frac{q_1}{7.08\cdot 10^{-8}h} = \frac{q_2}{8.85\cdot 10^{-10}\left(0.100\text{ m}-h\right)} \Rightarrow 8.85\cdot 10^{-10}q_1\left(0.100\text{ m}-h\right) = \left(7.08\cdot 10^{-8}\right)\left(q_2 h\right)$$

$$q_1 + q_2 = 8.85\cdot 10^{-8}\text{ C}$$

In addition, the energies stored in each capacitor is $U_1 = \dfrac{q_1^2}{2C_1}$ and $U_2 = \dfrac{q_2^2}{2C_2}$, or $U_1 = \dfrac{q_1^2}{1.424\cdot 10^{-7}h}$ and $U_2 = \dfrac{q_2^2}{1.771\cdot 10^{-9}\left(0.100\text{ m}-h\right)}$. The mass of the water between the parallel plates is $m = \rho dWh = \left(1.00\cdot 10^{3}\text{ kg/m}^3\right)\left(0.000100\text{ m}\right)\left(0.0100\text{ m}\right)h = \left(0.00100\text{ kg}\right)h.$ The gravitational potential energy of this water is: $U_{\text{g}} = mgh/2 = \dfrac{\left(0.00100\text{ kg}\right)\left(9.81\text{ m/s}^2\right)}{2}h^2 = \left(4.91\cdot 10^{-3}\text{ J}\right)h^2.$ By conservation of energy,

$$U_{\text{air}} = U_1 + U_2 + mg\frac{h}{2}$$

$$4.425\cdot 10^{-5} = \frac{q_1^2}{1.424\cdot 10^{-7}h} + \frac{q_2^2}{1.771\cdot 10^{-9}\left(0.1\text{ m}-h\right)} + 4.91\cdot 10^{-3}h^2$$

where the energies above are each measured in Joules. This provides the third equation needed to determine the height h. The system of equations contains the three unknowns, q_1, q_2, and h, where the charges are in Coulombs and the height in meters:

$$8.85\cdot 10^{-2} q_1 (0.100\text{ m} - h) = (7.08)(q_2 h)$$
$$q_1 + q_2 = 8.85\cdot 10^{-8}$$
$$4.425\cdot 10^{-5} = \frac{q_1^2}{1.424\cdot 10^{-7} h} + \frac{q_2^2}{1.771\cdot 10^{-9}(0.100\text{ m} - h)} + 4.91\cdot 10^{-3} h^2$$

ROUND: A computer algebra system can be used to solve the system. There are four solutions:

$$\left(h = -0.0956\text{ m},\ q_1 = 9.08\cdot 10^{-8}\text{ C},\ q_2 = -2.32\cdot 10^{-9}\text{ C}\right),$$
$$\left(h = -7.11\cdot 10^{-7}\text{ m},\ q_1 = -5.04\cdot 10^{-11}\text{ C},\ q_2 = 8.86\cdot 10^{-8}\text{ C}\right),$$
$$\left(h = -1.27\cdot 10^{-3}\text{ m},\ q_1 = 1.76\cdot 10^{-5}\text{ C},\ q_2 = -1.75\cdot 10^{-5}\text{ C}\right),$$
$$\left(h = 9.43\cdot 10^{-2}\text{ m},\ q_1 = 8.84\cdot 10^{-8}\text{ C},\ q_2 = 6.67\cdot 10^{-11}\text{ C}\right).$$

DOUBLE-CHECK: Note that if the height of the water column h were to equal the length of the capacitor plate, L (where $L = 0.100$ m), it would defy the first equation in the system of equations above. This is consistent with an earlier finding, where it was shown that the final height of the water cannot be $h = L$ since there is not enough energy stored in the capacitor's electric field (with $h = L$) to balance the gravitational potential energy in the water column of height $h = L$.

24.64. This is like two parallel plate capacitors in parallel, each with an area $A = (1/2)\pi r^2$. In parallel,

$$C_{\text{eq}} = C_1 + C_2 = \frac{\varepsilon_0 A}{d} + \frac{\varepsilon_0 \kappa A}{d} = \frac{\varepsilon_0 A}{d}(1+\kappa) = \frac{\varepsilon_0 \pi r^2}{2d}(1+\kappa)$$
$$= \frac{(8.85\cdot 10^{-12}\text{ F/m})\pi(0.610\text{ m})^2}{2(0.00210\text{ m})}(1+11.1) = 2.980\cdot 10^{-8}\text{ F} \approx 30.0\text{ nF}.$$

24.65. The largest potential difference that can be sustained without breakdown is about $(2.5\text{ kV/mm})(15\text{ mm}) = 37.5\text{ kV}$. Next, consider the relationship between charge deposited and change in potential:

$$Q = CV = \frac{\varepsilon_0 A}{d} V = \left(8.85\cdot 10^{-12}\text{ C}^2\text{ N}^{-1}\text{ m}^{-2}\right)\left(\frac{0.0025\text{ m}^2}{0.015\text{ m}}\right)\left(3.75\cdot 10^4\text{ V}\right) = 5.5\cdot 10^{-8}\text{ C}.$$

24.66. The capacitances are given as $C_1 = 2.0$ nF and $C_2 = C_3 = 4.0$ nF. The potential difference applied is $V = 1.5$ V. The potential difference on C_1 is equal to that of the battery. Then $q_1 = C_1 V_1 = C_1 V = (2.0\text{ nF})(1.5\text{ V}) = 3.0$ nC. Note C_2 and C_3 are in series. Their equivalent capacitance C_{23} is

$$C_{23} = \frac{C_2 C_3}{C_2 + C_3} = \frac{(4.0\text{ nF})(4.0\text{ nF})}{8.0\text{ nF}} = 2.0\text{ nF}.$$

With C_1 and C_{23} in parallel, $V_1 = V_{23} = V$. The charges for capacitors in series are equal; then $q_2 = q_3 = q_{23} = C_{23} V = (2.0\text{ nF})(1.5\text{ V}) = 3.0$ nC.

24.67. The energy stored in a capacitor is $U = (1/2)CV^2$. Putting a Mylar insulator between the plates of a vacuum gap capacitor will increase the capacitance C by a factor of $\kappa = 3.1$, which is Mylar's dielectric constant. Therefore, the percentage increase is

$$\frac{(U_{\text{f}} - U_{\text{i}})}{U_{\text{i}}}\cdot 100\% = \frac{\left((1/2)\kappa CV^2 - (1/2)CV^2\right)}{(1/2)CV^2}\cdot 100\% = (\kappa - 1)\cdot 100\% = \left((3.1) - 1\right)\cdot 100\% = 210\%$$

24.68. The capacitance of a parallel plate capacitor in air is given by $C = \varepsilon_0 A / d$, where A is the area of the plates and d is the separation between the plates. Increasing the distance between the plates reduces the capacitance and therefore, at constant voltage, reduces the charge on the plates. The lost charge backs up into the battery, and therefore a calculation of total energy must account for the battery energy, as well.

Model the battery as a very large capacitance C_b which is in parallel with the capacitor being manipulated (because the connection is positive-to-positive and negative-to-negative), which has capacitances before and after of C_i and C_f, respectively. The total charge (capacitor and battery combined) is q, the charge on the battery is q_b, and since $C_b \gg C_\text{i}, C_\text{f}$ it follows that the $q \approx q_b$ and the voltage, $V = q_b / C_b$, is essentially constant. The energy stored in a capacitor is in general $\frac{1}{2}\left(\frac{q^2}{C}\right)$, so here the change in total potential energy is

$$\begin{aligned}
\Delta U &= U_\text{f} - U_\text{i} \\
&= \frac{1}{2}\left(\frac{q^2}{C_b + C_\text{f}}\right) - \frac{1}{2}\left(\frac{q^2}{C_b + C_\text{i}}\right) \\
&= \frac{q^2}{2}\left(\frac{1}{C_b + C_\text{f}} - \frac{1}{C_b + C_\text{i}}\right) \\
&= \frac{q^2}{2}\left(\frac{C_\text{i} - C_\text{f}}{(C_b + C_\text{f})(C_b + C_\text{i})}\right) \\
&= \frac{q^2}{2}\left(\frac{C_\text{i} - C_\text{f}}{C_b^2 + C_b C_\text{f} + C_b C_\text{i} + C_\text{f} C_\text{i}}\right) \\
&\approx \frac{q^2}{2C_b^2}(C_\text{i} - C_\text{f}) \qquad \text{because } C_b \gg C_\text{i}, C_\text{f} \\
&\approx \frac{1}{2}V^2(C_\text{i} - C_\text{f}) \qquad \text{because } q \approx q_b
\end{aligned}$$

Note that the capacitor being manipulated *loses* energy, since for the capacitor $\Delta U = \frac{1}{2}V^2(C_\text{f} - C_\text{i}) < 0$.

Overall, however, the system *gains* energy, $\Delta U = \frac{1}{2}V^2(C_\text{i} - C_\text{f}) > 0$, which means the battery gains twice as much energy as the capacitor loses. Since work W done on the system equals ΔU for the system, the work done is

$$\begin{aligned}
W &= \frac{1}{2}V^2(C_\text{i} - C_\text{f}) \\
&= \frac{1}{2}V^2\left(\frac{\varepsilon_0 A}{d_\text{i}} - \frac{\varepsilon_0 A}{d_\text{f}}\right) \\
&= \frac{\varepsilon_0 A V^2}{2}\left(\frac{1}{d_\text{i}} - \frac{1}{d_\text{f}}\right) \\
&= \frac{(8.85 \cdot 10^{-12} \text{ F/m})(12.0 \cdot 10^{-4} \text{ m}^2)(9.00 \text{ V})^2}{2}\left(\frac{1}{(1.50 \cdot 10^{-3} \text{ m})} - \frac{1}{(2.75 \cdot 10^{-3} \text{ m})}\right) \\
&= 1.30 \cdot 10^{-10} \text{ J}
\end{aligned}$$

24.69. The capacitance is $C = 1.00\ \text{F}$ for a square, parallel-plate capacitor. The separation is $d = 0.100\ \text{mm} = 0.000100\ \text{m}$, and is filled with paper of $\kappa = 5.00$. For a parallel plate capacitor $C = \dfrac{\kappa\varepsilon_0 A}{d} = \dfrac{\kappa\varepsilon_0 L^2}{d}$. Then

$$L = \sqrt{\frac{dC}{\kappa\varepsilon_0}} = \sqrt{\frac{(0.000100\ \text{m})(1.00\ \text{F})}{(5.00)(8.85\cdot 10^{-12}\ \text{F/m})}} = 1503\ \text{m} \approx 1.50\ \text{km}.$$

24.70. The capacitance is $C = 4.00\ \text{pF}$ and the potential difference is $V = 10.0\ \text{V}$. The plate separation is $d = 3.00\ \text{mm}$.

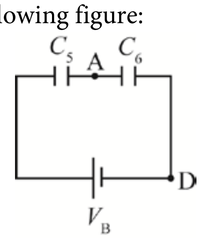
(a) The charge is $Q = CV = (4.00\ \text{pF})(10.0\ \text{V}) = 40.0\ \text{pC}$.

(b) The energy stored is $U = (1/2)CV^2 = (1/2)(4.00\ \text{pF})(10.0\ \text{V})^2 = 200.\ \text{pJ}$.

(c) The area is $A = Cd/\varepsilon_0 = (4.00\ \text{pF})(0.00300\ \text{m})/(8.85\cdot 10^{-12}\ \text{F/m}) = 1.36\cdot 10^{-3}\ \text{m}^2$.

(d) The dielectric constant of polystyrene is $\kappa = 2.6$. Then $C = \kappa C_{\text{air}} = 2.6(4.00\ \text{pF}) = 10.\ \text{pF}$.

24.71. First, simplify the circuit diagram to the following figure:



with $C_5 = C_1 + C_2$ and $C_6 = C_3 + C_4$. The total capacitance is given by $C_{\text{tot}} = (1/C_5 + 1/C_6)^{-1}$. Since the capacitors are in series, the charge on each capacitor is $q = C_{\text{tot}}V_B$. The potential at point A is then given by $V_A = V_D + V_B - q/C_5$. Also, $C_5 = (1.00\ \text{mF}) + (2.00\ \text{mF}) = 3.00\ \text{mF}$, $C_6 = (3.00\ \text{mF}) + (4.00\ \text{mF}) = 7.00\ \text{mF}$, $C_{\text{tot}} = 2.10\ \text{mF}$, and $q = (2.10\ \text{mF})(1.00\ \text{V}) = 2.10\ \text{mC}$. Therefore, the potential at point A is $V_A = 0.00\ \text{V} + (1.00\ \text{V}) - (2.10\ \text{mC})/(3.00\ \text{mF}) = 0.300\ \text{V}$.

24.72. The energy is given by $U = q^2/2C$, with $C = \kappa\varepsilon_0 A/d$ for a parallel-plate capacitor. The energy stored is:

$$U = \frac{q^2 d}{2\kappa\varepsilon_0 A} = \frac{(4.20\cdot 10^{-4}\ \text{C})^2(1.30\cdot 10^{-3}\ \text{m})}{2(7.0)(8.85\cdot 10^{-12}\ \text{F/m})(6.40\cdot 10^{-3}\ \text{m}^2)} = 289.19\ \text{J} \approx 289\ \text{J}.$$

24.73. The capacitance is given by $C = \dfrac{\kappa\varepsilon_0 A}{d} = \dfrac{(9.10)(8.85\cdot 10^{-12}\ \text{F/m})(1.00\cdot 10^{-10}\ \text{m}^2)}{(2.00\cdot 10^{-8}\ \text{m})} = 4.03\cdot 10^{-13}\ \text{F}$.

24.74. (a) Since the capacitors C_1 and C_2 have the same potential,

$$V = \frac{q}{C} \Rightarrow \frac{Q_1}{C_1} = \frac{Q_2}{C_2} \Rightarrow Q_1 = \left(\frac{C_1}{C_2}\right)Q_2 = \left(\frac{6.00\ \mu\text{F}}{3.00\ \mu\text{F}}\right)(40.0\ \mu\text{C}) = 80.0\ \mu\text{C}.$$

(b) Since the total charge on C_1 and C_2 is equal to that on C_3, it is required that: $Q_3 = Q_1 + Q_2 = 120.\ \mu\text{C}$.

(c) The total voltage applied is:

$$\frac{Q_1}{C_1} + \frac{Q_3}{C_3} = \frac{80.0\ \mu\text{C}}{6.00\ \mu\text{F}} + \frac{120.\ \mu\text{C}}{5.00\ \mu\text{F}} = 37.3\ \text{V}.$$

24.75. The capacitance of a cylindrical capacitor is given by:

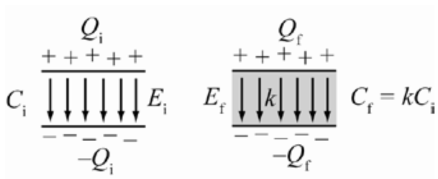
$$C=\frac{\kappa 2\pi\varepsilon_0 L}{\ln\left(r_2/r_1\right)}=\left(63\right)\frac{2\pi\left(8.85\cdot 10^{-12}\text{ m}^{-3}\text{ kg}^{-1}\text{ s}^{4}\text{ A}^{2}\right)\left(0.0724\text{ m}\right)}{\ln\left(4.16\text{ cm}/3.02\text{ cm}\right)}=0.792\text{ nF}.$$

24.76. (a) The capacitance is given by: $C=4\pi\varepsilon_0 R=4\pi\left(8.85\cdot 10^{-12}\text{ F/m}\right)\left(6.328\cdot 10^{6}\text{ m}\right)=704\ \mu\text{F}$

(b) $U=\dfrac{Q^2}{2C}=\dfrac{\left(-7.8\cdot 10^{5}\right)^2}{2\left(704\ \mu\text{F}\right)}=4.32\cdot 10^{14}\text{ J}.$

24.77. **THINK:** A parallel plate capacitor with an air gap is connected to a 6.00 V battery. The initial energy of the capacitor is $U_\text{i}=72.0\text{ nJ}$. After a dielectric material is inserted, the capacitor has an additional energy of 317 nJ. The final energy stored in the capacitor is $U_\text{f}=72.0\text{ nJ}+317\text{ nJ}=389\text{ nJ}$.

SKETCH:



RESEARCH: The energy stored in a capacitor is given by $U=\left(1/2\right)CV^2$. The initial and final energy are $U_\text{i}=\left(1/2\right)C_\text{i}V^2$ and $U_\text{f}=\left(1/2\right)C_\text{f}V^2$.

SIMPLIFY:

(a) Taking a ratio of U_f and U_i yields $U_\text{f}/U_\text{i}=C_\text{f}/C_\text{i}$. Using $C_\text{f}=\kappa C_\text{i}$, the dielectric constant is found to be $\kappa=U_\text{f}/U_\text{i}$.

(b) The charge in the capacitor is given by $Q=CV$. Using $U=\left(1/2\right)CV^2$, it is found that the charge is $Q=\left(2U/V^2\right)V=2U/V$.

(c) The electric field inside a parallel plate capacitor is $E=\dfrac{Q}{A\kappa\varepsilon_0}=\dfrac{2U}{\kappa\varepsilon_0 AV}$.

(d) The electric field inside the capacitor after the dielectric material is inserted is $E_\text{f}=\dfrac{2U_\text{f}}{\kappa\varepsilon_0 AV}=\dfrac{2\left(U_\text{f}/\kappa\right)}{\varepsilon_0 AV}$. Using the result in (a) $U_\text{i}=U_\text{f}/\kappa$ yields $E_\text{f}=2U_\text{i}/\left(\varepsilon_0 AV\right)=E_\text{i}$. This means that the field does not change.

CALCULATE:

(a) The dielectric constant is $\kappa=\dfrac{389\text{ nJ}}{72.0\text{ nJ}}=5.403.$

(b) The charge in the capacitor after the dielectric material has been inserted is

$$Q_\text{f}=\frac{2U_\text{f}}{V}=\frac{2\left(389\cdot 10^{-9}\text{ J}\right)}{6.00\text{ V}}=0.129\ \mu\text{C}.$$

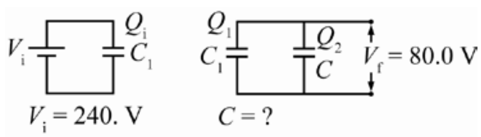
(c) The electric field inside the capacitor before the dielectric material is inserted is

$$E_\text{i}=\frac{2U_\text{i}}{\varepsilon_0 AV}=\frac{2\left(72.0\text{ nJ}\right)}{\left(8.85\cdot 10^{-12}\text{ C}^2/\text{N m}^2\right)\left(5.00\cdot 10^{-3}\text{ m}^2\right)\left(6.00\text{ V}\right)}=5.424\cdot 10^{5}\text{ N/C}.$$

(d) $E_\text{f}=E_\text{i}=5.424\cdot 10^{5}\text{ N/C}$

ROUND:
(a) $\kappa = 5.40$.
(b) $Q_{\text{f}} = 0.130\ \mu\text{C}$.
(c) $E_{\text{i}} = 542$ kV/m.
(d) $E_{\text{f}} = 542$ kV/m.
DOUBLE-CHECK: The numerical results are reasonable.

24.78. **THINK:** Since charge is conserved, the initial charge on the capacitor charged by the battery is equal to the total charge on the capacitors that are connected.
SKETCH:



RESEARCH: Since the charge is conserved,

$$Q_{\text{i}} = Q_1 + Q_2 \ \Rightarrow\ C_1V_{\text{i}} = C_1V_{\text{f}} + CV_{\text{f}}.$$

The energy stored in a capacitor is $U = (1/2)CV^2$.

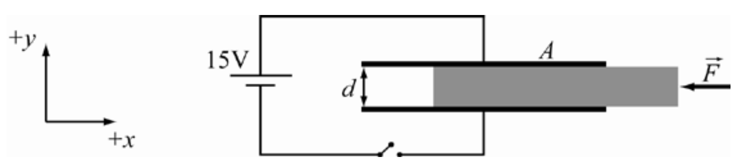
SIMPLIFY: Solving for C in the above equation yields $C = \dfrac{C_1(V_{\text{i}} - V_{\text{f}})}{V_{\text{f}}}$. The energy stored in the second capacitor is $U = (1/2)CV_{\text{f}}^2$.

CALCULATE: $C = \dfrac{(8.00\ \mu\text{F})(240.\ \text{V} - 80.0\ \text{V})}{(80.0\ \text{V})} = 16.0\ \mu\text{F}$

$U = \dfrac{1}{2}(16.0\ \mu\text{F})(80.0\ \text{V})^2 = 51.2$ mJ

ROUND: Keeping only three significant figures gives $C = 16.0\ \mu\text{F}$ and $U = 51.2$ mJ.
DOUBLE-CHECK: The equivalent capacitance of the two capacitors is $C_{\text{eq}} = C_1 + C = 24.0\mu\text{F}$. The charge on the equivalent capacitor is $Q = C_{\text{eq}}V_{\text{f}} = (24.0\mu\text{F})(80.0\ \text{V}) = 1.92$ mC. The initial charge on the first capacitor is $Q_{\text{i}} = C_1V_{\text{i}} = (8.00\mu\text{F})(240.\ \text{V}) = 1.92$ mC, so charge is conserved, as expected.

24.79. **THINK:** In this problem, the work-energy relation is used. The dielectric constant of nylon is $\kappa = 3.50$. Use the work-energy relation, and think about the initial and final energies of the capacitor.
SKETCH:



RESEARCH: From work-energy relation, it is found that $W = -\Delta U = U_{\text{i}} - U_{\text{f}}$. The initial and final energies of the capacitor are $U_{\text{i}} = \dfrac{1}{2}CV_{\text{i}}^2$ and

$$U_2 = \frac{1}{2}\left(\frac{Q_{\text{i}}^2}{C_{\text{f}}}\right) = \frac{1}{2}\left(\frac{(C_{\text{i}}V_{\text{i}})^2}{\kappa C_{\text{i}}}\right) = \frac{1}{2}\left(\frac{C_{\text{i}}V_{\text{i}}^2}{\kappa}\right).$$

SIMPLIFY: Therefore, $W = \frac{1}{2}CV_i^2\left(1-\frac{1}{\kappa}\right)$. The work done by the electric field is $W = FL$, where L is the length of the square plate. Using the capacitances of a parallel plate, $C = \frac{\varepsilon_0 A}{d}$, it is found that

$$FL = \frac{1}{2}\left(\frac{\varepsilon_0 A}{d}\right)V_i^2\left(1-\frac{1}{\kappa}\right) \Rightarrow F = \frac{1}{2}\left(\frac{\varepsilon_0 L^2}{Ld}\right)\left(V_i^2\right)\left(1-\frac{1}{\kappa}\right) = \frac{1}{2}\cdot\frac{V_i^2\varepsilon_0 L}{d}\left(1-\frac{1}{\kappa}\right).$$

CALCULATE: Thus the net force done by the electric field is

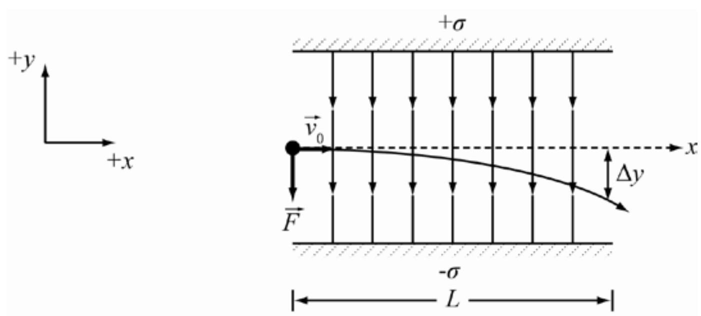
$$F = \frac{1}{2}\left(\frac{\left(8.85\cdot 10^{-12}\text{ C}^2/\text{N m}^2\right)\left(0.0200\text{ m}\right)}{1.00\cdot 10^{-3}\text{ m}}\right)\left(15.0\text{ V}\right)^2\left(1-\frac{1}{3.50}\right) = 1.4223\cdot 10^{-8}\text{ N}.$$

ROUND: Keeping three significant figures gives $F = 1.42\cdot 10^{-8}$ N. The positive sign of the force means that the direction of the force is the same direction as the motion of the dielectric material.

DOUBLE-CHECK: The magnitude and direction of the force make sense, considering the scale. Note also that although here the force was treated as an *average* force, the solution to Conceptual Question 24.25 shows that the force is fact *constant.*

24.80. **THINK:** This problem is similar to the motion of a projectile under a gravitational force. In this case the gravitational force is replaced by an electrostatic force. Let L be the length of the plates.

SKETCH:



RESEARCH: The force acting on a proton is given by $F = qE$. For a parallel plate, the electric field is $E = \sigma/\varepsilon_0$. Thus, $F = q\sigma/\varepsilon_0$. The time required to reach the far edge of the capacitor is $t = L/V_0$. Therefore, the deflection distance ΔY is

$$\Delta Y = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{F}{m}\right)\left(\frac{L}{V_0}\right)^2 = \frac{1}{2}\left(\frac{q\sigma}{m\varepsilon_0}\right)\left(\frac{L^2}{V_0^2}\right).$$

SIMPLIFY: Not applicable.

CALCULATE: Putting in the numerical values gives:

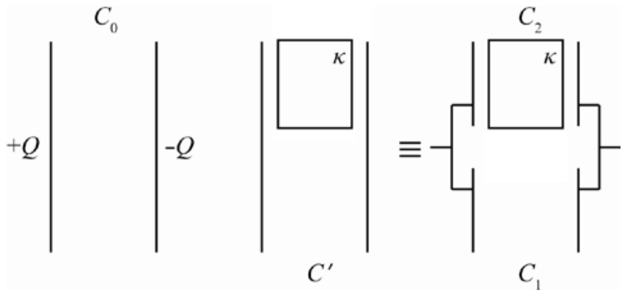
$$\Delta Y = \frac{1}{2}\left(\frac{\left(1.602\cdot 10^{-19}\text{ C}\right)\left(1.0\cdot 10^{-6}\text{ C/m}^2\right)}{\left(1.67\cdot 10^{-27}\text{ kg}\right)\left(8.85\cdot 10^{-12}\text{ C}^2/\left(\text{N m}^2\right)\right)}\right)\left(\frac{\left(2.0\cdot 10^{-2}\text{ m}\right)^2}{\left(1.0\cdot 10^6\text{ m/s}\right)^2}\right) = 0.002168\text{ m}.$$

ROUND: $\Delta Y \approx 2.2$ mm.

DOUBLE-CHECK: This is reasonable.

24.81. **THINK:** A parallel plate capacitor with a squared area of side $L = 10.0$ cm and separation distance $d = 2.50$ mm is charged to a potential difference of $V_0 = 75.0$ V, and then disconnected from the battery. I want to determine the capacitor's capacitance, C_0, and the energy, U_0, stored in it at this point. A dielectric with constant $\kappa = 3.40$ is then inserted into the capacitor such that it fills 2/3 of the volume between the plates. I want to determine the new capacitance, new potential difference between the plates and energy of the capacitor, C', V' and U'. I want to determine how much work, if any, is required to insert the dielectric into the capacitor.

SKETCH:



RESEARCH:

(a) The capacitance of a parallel plate capacitor is given by $C_0 = \varepsilon_0 A/d$. The energy stored in the capacitor is $U_0 = (1/2)C_0V_0^2$.

(b) After a dielectric has been inserted, the capacitor can be treates as two capacitors in parallel, one with a dielectric, and one without. The new capacitance is obtained by adding the contributions of the two parts of the capacitor, i.e., $C' = C_1 + C_2$. Since the charge on the capacitor is unchanged, and the potential is the same across both parts of the new capacitor, $C_0V_0 = C'V' \Rightarrow V' = (C_0/C')V_0$. The new energy stored on the capacitor is $U' = (1/2)C'V'^2$.

SIMPLIFY:

(a) $C_0 = \varepsilon_0 A/d;\quad U_0 = (1/2)C_0V_0^2$

(b) The new capacitance is $C' = C_1 + C_2 = (1/3)C_0 + (2/3)\kappa C_0 = \left(\frac{1+2\kappa}{3}\right)C_0$. The new potential between the plates is $V' = (C_0/C')V_0$. The new energy stored in the capacitor is

$$U' = (1/2)C'V'^2 = \frac{\left(\frac{1+2\kappa}{3}\right)(C_0)\left(\frac{C_0V_0}{\left(\frac{1+2\kappa}{3}\right)(C_0)}\right)^2}{2} = \frac{1}{2}\left(\frac{3C_0V_0^2}{1+2\kappa}\right) = U_0\left(\frac{3}{1+2\kappa}\right).$$

(c) By using the work energy relation, it is found that the applied work is

$$W = \Delta U = U' - U_0 = \left(\frac{3}{1+2\kappa} - 1\right)U_0 = \frac{2(1-\kappa)}{1+2\kappa}U_0.$$

Since κ is larger than 1, this means that the applied work is negative. Therefore, the external agent does not need to do work to insert the dielectric slab.

CALCULATE: Substituting the numerical values yields,

(a) $C_0 = \frac{\left(8.85\cdot 10^{-12}\text{ C}^2/\text{N m}^2\right)(0.100\text{ m})^2}{0.0025\text{ m}} = 35.42\text{ pF},\quad U_0 = \frac{1}{2}\left(35.42\cdot 10^{-12}\text{ F}\right)(75.0\text{ V})^2 = 9.961\cdot 10^{-8}\text{ J}$

(b) $C' = \frac{1+2(3.4)}{3}(35.42\text{ pF}) = 92.08\text{ pF},\quad V' = \frac{35.42\text{ pF}}{92.08\text{ pF}}(75.0\text{ V}) = 28.85\text{ V},$

$U' = \frac{3}{1+2(3.4)}\left(9.961\cdot 10^{-8}\text{ J}\right) = 3.83\cdot 10^{-8}\text{ J}$

(c) Not required.

ROUND: Rounding all results to three significant figures gives:

(a) $C_0 = 35.4$ pF and $U_0 = 0.996\cdot 10^{-8}$ J.

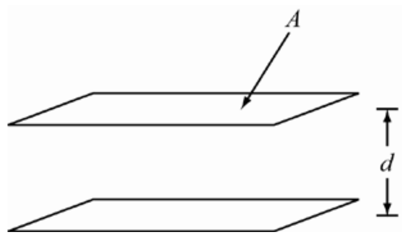
(b) $C' = 92.1$ pF, $V' = 28.9$ V and $U = 3.83\cdot 10^{-8}$ J.

(c) Not required.

DOUBLE-CHECK: The answers are of reasonable magnitudes and their respective units make sense.

24.82. **THINK:** What would be the area of a parallel plate capacitor, with plate separation $d = 1.0$ mm, capable of storing the same amount of energy as a AAA battery, or 3400 J ? The potential difference of a AAA battery is $V = 1.5$ V. What would be the area of such a capacitor be if the potential difference is to be the maximum that can be applied without dielectric breakdown of the air between the plates?

SKETCH:



RESEARCH: The capacitance of the parallel plate capacitor is given by $C = \varepsilon_0 A / d$. The energy that can be stored in the capacitor is $U = (1/2)CV^2$. The dielectric strength of air is 2.5 kV/mm.

SIMPLIFY: Substituting $C = \varepsilon_0 A / d$ into the expression for U and solving for A:

$$U = \frac{1}{2}\frac{\varepsilon_0 A}{d}V^2 \;\Rightarrow\; A = \frac{2dU}{\varepsilon_0 V^2}.$$

CALCULATE:

(a) Putting in the numerical values into the above expression gives

$$A = \frac{2\left(1.0 \cdot 10^{-3}\text{ m}\right)\left(3400\text{ J}\right)}{\left(8.85 \cdot 10^{-12}\text{ C}^2/\left(\text{N m}^2\right)\right)\left(1.5\text{ V}\right)^2} = 3.413 \cdot 10^{11}\text{ m}^2.$$

(b) Replacing V with the maximum voltage before dielectric breakdown occurs yields

$$A = \frac{2\left(1.0 \cdot 10^{-3}\text{ m}\right)\left(3400\text{ J}\right)}{\left(8.85 \cdot 10^{-12}\text{ C}^2/\left(\text{N m}^2\right)\right)\left(\left(2.5 \cdot 10^{3}\text{ V/mm}\right)\left(1.0\text{ mm}\right)\right)^2} = 1.229 \cdot 10^{5}\text{ m}^2.$$

Note that $V_{\text{max}} = (2.5\text{ kV/mm})(1.0\text{ mm}) = 2.5$ kV.

ROUND: Rounding the results to three significant figures yields

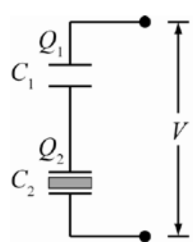
(a) $A = 3.41 \cdot 10^{11}\text{ m}^2$ and (b) $A = 1.23 \cdot 10^{5}\text{ m}^2$.

(c) These areas are in the order of many square km, making such a capacitor highly impractical.

DOUBLE-CHECK: This makes sense. Capacitors are excellent devices for allowing access to small amounts of energy extremely quickly; they are not feasible for slow release of large amounts of energy. It makes sense that only a horrendously large capacitor would have the same physical capabilities as a battery.

24.83. **THINK:** Recall the equivalent capacitance of capacitors connected in series. The areas of the two capacitors in this question are the same. The charges are the same for both capacitors.

SKETCH:



RESEARCH: An equivalent capacitance of two capacitors connected in series is $\frac{1}{C_{eq}}=\frac{1}{C_1}+\frac{1}{C_2}$. The capacitance of a parallel plate capacitor is given by $C=\frac{\varepsilon_0 A}{d}$. Since C_1 and C_2 have the same area and the same plate separation, the capacitance of C_2 is equal to $C_2=\kappa C_1$, where κ is the dielectric constant of material in C_2. The equivalent capacitance, $C_{eq}=\left(\frac{1}{C_1}+\frac{1}{C_2}\right)^{-1}=\left(\frac{1}{C_1}+\frac{1}{\kappa C_1}\right)^{-1}=\frac{\kappa C_1}{1+\kappa}$.

(a) The charge on the capacitors is $Q=CV$.

(b) The total energy stored in the capacitors is $U=(1/2)CV^2=(1/2)Q^2/C$.

(c) The electric field inside C_2 is given by $E_2=(Q_2/A)/(\kappa\varepsilon_0)$, and $Q_2=Q_1=Q$

SIMPLIFY:

(a) $Q=CV=\frac{\kappa C_1}{1+\kappa}V=\frac{\kappa\varepsilon_0 A}{d(1+\kappa)}V$

(b) $$U_1=\frac{Q_1^2}{2C_1}=\frac{\left(\frac{\kappa C_1}{\kappa+1}\right)^2V^2}{2C_1}=\frac{\kappa^2C_1V^2}{2(\kappa+1)^2}=\frac{\kappa^2\varepsilon_0AV^2}{2d(\kappa+1)^2}$$

$$U_2=\frac{Q_2^2}{2C_2}=\frac{\left(\frac{\kappa C_1}{\kappa+1}\right)^2V^2}{2\kappa C_1}=\frac{\kappa C_1V^2}{2(\kappa+1)^2}=\frac{\kappa\varepsilon_0AV^2}{2d(\kappa+1)^2}$$

(c) $E_2=\frac{Q}{\kappa\varepsilon_0 A}=\frac{\kappa\varepsilon_0 AV}{d\kappa\varepsilon_0A(\kappa+1)}=\frac{V}{d(\kappa+1)}$

CALCULATE: Substituting the numerical values into the above equations yields

(a) $$Q_1=Q_2=\frac{7.00(8.854\cdot 10^{-12}\text{ F/m})(1.00\cdot 10^{-4}\text{ m})(96.0\text{ V})}{(0.100\cdot 10^{-3}\text{ m})(7.00+1)}=7.44\cdot 10^{-10}\text{ C}$$

(b) $$U_1=\frac{(7.00)^2\left(\frac{(8.854\cdot 10^{-12}\text{ F/m})(1.00\cdot 10^{-4}\text{ m})}{(0.100\cdot 10^{-3}\text{ m})}\right)(96.0\text{ V})^2}{2(7.00+1)^2}=31.24\cdot 10^{-9}\text{ J}.$$

$$U_2=\frac{7.00\left(\frac{(8.854\cdot 10^{-12}\text{ F/m})(1.00\cdot 10^{-4}\text{ m})}{(0.100\cdot 10^{-3}\text{ m})}\right)(96.0\text{ V})^2}{2(7.00+1)^2}=4.46\cdot 10^{-9}\text{ J}$$

The total energy is $U=U_1+U_2=31.24\text{ nJ}+4.456\text{ nJ}=35.70\text{ nJ}$.

(c) $$E_2=\frac{96.0\text{ V}}{(0.100\cdot 10^{-3}\text{ m})(7.00+1)}=120{,}000\text{ V/m}$$

ROUND: Rounding all results to three significant figures gives the following answers.

(a) $Q=0.744$ nC

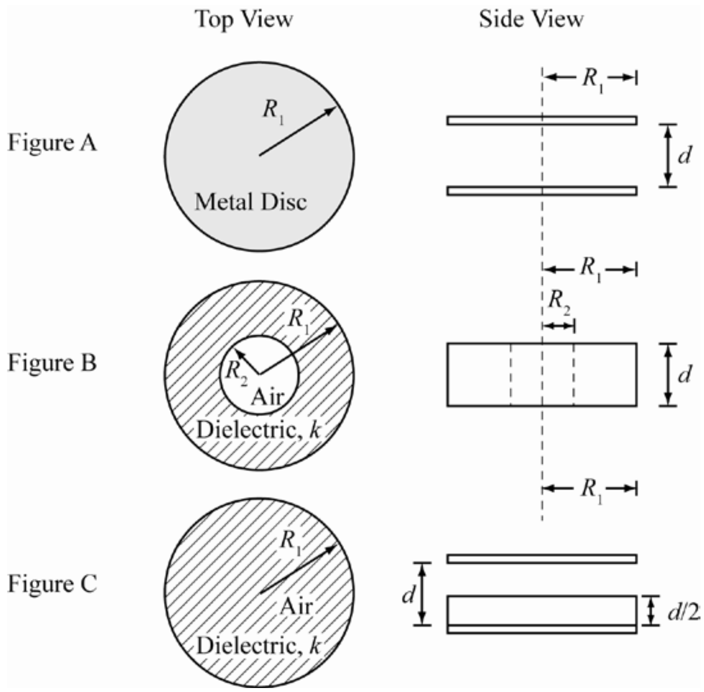
(b) $U=35.7$ nJ

(c) $E_2=1.20\cdot 10^5$ V/m

DOUBLE-CHECK: Dimensional analysis confirms that all results are in the correct units.

24.84. **THINK:** The plates of the parallel plates capacitor consist of two metal discs of radius $R_1 = 4.00 \text{ cm} = 0.0400 \text{ m}$. The separation between the plates is $d = 2.00 \text{ mm} = 0.00200 \text{ m}$. In part a) the space between the plates is filled with air. In part b) a dielectric of out radius R_1, inner radius $R_2 = 2.00 \text{ cm} = 0.0200 \text{ m}$, thickness $d = 2.00 \text{ mm} = 0.00200 \text{ m}$ and dielectric constant $\kappa = 2.00$ is placed between the plates. In part B) the situation can be modeled as two capacitors in parallel. In part c) a solid disc of radius R_1, made of dielectric material having the dielectric constant $\kappa = 2.00$ is placed between the plates. This situation can be modeled as two capacitors in series. The thickness of the dielectric layer is $d/2$.

SKETCH:



RESEARCH:

(a) The capacitance of the parallel plate capacitor in air is $C_1 = \varepsilon_0 A/d$, while $A = \pi R_1^2$.

(b) The equivalent capacitance of the capacitors in parallel is given by $C_2 = C_1' + C_2''$, where $C_2' = \kappa\varepsilon_0 A'/d = \kappa\varepsilon_0\pi\left(R_1^2 - R_2^2\right)/d$ and $C_2'' = \varepsilon_0 A''/d = \varepsilon_0 \pi R_2^2/d$.

(c) The equivalent capacitance C_3 of the capacitors in series is $C_3 = \left(1/C_3' + 1/C_3''\right)^{-1}$, where $C_3' = \kappa\varepsilon_0\pi R_1^2/\left(d/2\right)$ and $C_3'' = \varepsilon_0\pi R_1^2/\left(d/2\right)$.

SIMPLIFY:

(a) $C_1 = \varepsilon_0 \pi \text{R}_1^2/d$.

(b) $C_2 = \kappa\varepsilon_0\pi\dfrac{R_1^2 - R_2^2}{d} + \dfrac{\varepsilon_0\pi R_2^2}{d} = \dfrac{\varepsilon_0\pi}{d}\left(\kappa\left(R_1^2 - R_2^2\right) + R_2^2\right)$

(c) Not necessary.

CALCULATE:

(a) $C_1 = \dfrac{\left(8.85\cdot 10^{-12} \text{ C}^2/\left(\text{N m}^2\right)\right)\pi\left(0.0400 \text{ m}\right)^2}{0.00200 \text{ m}} = 2.224\cdot 10^{-11} \text{ F}$

(b) $C_2 = \dfrac{\left(8.85\cdot 10^{-12} \text{ C}^2/\left(\text{N m}^2\right)\right)\pi}{0.00200 \text{ m}}\left(2.00\left(\left(0.0400 \text{ m}\right)^2 - \left(0.0200 \text{ m}\right)^2\right) + \left(0.0200 \text{ m}\right)^2\right)$

$= 3.892\cdot 10^{-11}\text{ F}$

(c) $C_3' = \dfrac{2.00\left(8.85\cdot 10^{-12}\text{ C}^2/\left(\text{N m}^2\right)\right)\pi\left(0.0400\text{ m}\right)^2}{0.00100\text{ m}} = 8.897\cdot 10^{-11}\text{ F}$

$C_3'' = \dfrac{\left(8.85\cdot 10^{-12}\text{ C}^2/\left(\text{N m}^2\right)\right)\pi\left(0.0400\text{ m}\right)^2}{0.00100\text{ m}} = 4.4485\cdot 10^{-11}\text{ F}$

$C_3 = \left(\dfrac{1}{8.897\cdot 10^{-11}\text{ F}} + \dfrac{1}{4.4485\cdot 10^{-11}\text{ F}}\right)^{-1} = 2.9657\cdot 10^{-11}$

ROUND: To 3 significant figures,

(a) $C_1 = 2.22\cdot 10^{-11}\text{ F}$

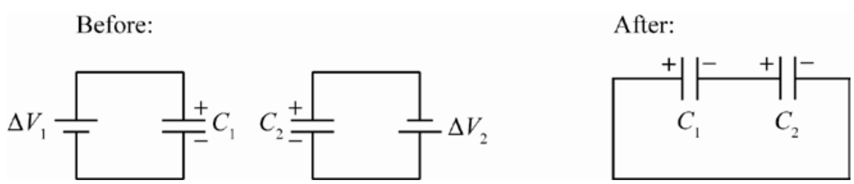
(b) $C_2 = 3.89\cdot 10^{-11}\text{ F}$

(c) $C_3 = 2.97\cdot 10^{-11}\text{ F}$

DOUBLE-CHECK: The value of a capacitor with a dielectric medium is greater than the value of the capacitor without a dielectric medium.

24.85. **THINK:** Capacitor $C_1 = 1.00\ \mu\text{F}$ has an electric potential of $\Delta V_1 = 50.0\text{ V}$. Capacitor $C_2 = 2.00\ \mu\text{F}$ has an electric potential of $\Delta V_2 = 20.0\text{ V}$. The two capacitors are connected positive plate to negative plate. Calculate the final charge, $Q_{1,\text{ f}}$ on capacitor C_1 after the two capacitors have come to equilibrium.

SKETCH:



RESEARCH: Because the capacitors are connected in such a way that the positive plate of each is connected to the negative plate of the other, they must be in series. Therefore the final charges on C_1 and C_2 must be equal. Because charge is conserved, the total initial charge must equal the total final charge. The initial charge of the system is $Q_\text{i} = Q_{1,\text{ i}} + Q_{2,\text{ i}}$ where $Q_{1,\text{ i}} = C_1\Delta V_1$ and $Q_{2,\text{ i}} = C_2\Delta V_2$.

SIMPLIFY: $Q_\text{i} = Q_{\text{i}1} + Q_{\text{i}2} = C_1V_1 + C_2V_2$, $Q_\text{i} = Q_\text{f}$, $Q_{\text{f}1} = Q_{\text{f}2}$, and $Q_{\text{f}1} = \dfrac{Q_\text{f}}{2} = \dfrac{Q_\text{i}}{2} = \dfrac{C_1V_1 + C_2V_2}{2}$.

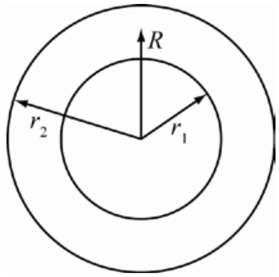
CALCULATE: $Q_{\text{f}1} = \dfrac{\left(1.00\cdot 10^{-6}\text{ F}\right)\left(50.0\text{ V}\right) + \left(2.00\cdot 10^{-6}\text{ F}\right)\left(20.0\text{ V}\right)}{2} = 4.50\cdot 10^{-5}\text{ C}$

ROUND: There were three significant figures provided in the question so the answer should be written as $Q_{1,\text{ f}} = 4.50\cdot 10^{-5}\text{ C}$ or $45.0\ \mu\text{C}$.

DOUBLE-CHECK: It is reasonable that there is less charge stored on capacitor C_1 after it was connected to C_2 because the potential across capacitor C_1 would have to decrease in order for it to come into equilibrium with C_2.

24.86. **THINK:** The spherical capacitor consists of two concentric conducting spheres of radius r_1 and r_2, where $r_2 > r_1$. The space between the spheres is filled with a dielectric material of electric permeability $\varepsilon = 10\varepsilon_0$. The dielectric material starts at r_1 and extends to radius $R\left(r_1 < R < r_2\right)$. The problem can be modeled as two spherical capacitors connected in series.

SKETCH:



RESEARCH: The equivalent capacitance for capacitors in series is $1/C_{\text{eq}} = 1/C_1 + 1/C_2$. The equation for the first spherical capacitor is $C_1 = 4\pi\varepsilon\left(r_1 R/\left(R - r_1\right)\right)$.

SIMPLIFY: $1/C_{\text{eq}} = 1/C_1 + 1/C_2 = \left(C_2 + C_1\right)/C_1C_2$, substituting the values for C_1 and C_2 into the equation gives:

$$\frac{1}{C_{\text{eq}}} = \frac{R - r_1}{4\pi\varepsilon r_1 R} + \frac{r_2 - R}{4\pi\varepsilon r_2 R}.$$

Recall that $\varepsilon = 10\varepsilon_0$, substituting this value into the equation gives:

$$\frac{1}{C_{\text{eq}}} = \frac{R - r_1}{4\pi 10\varepsilon_0 r_1 R} + \frac{r_2 - R}{4\pi\varepsilon_0 r_2 R} = \frac{1}{4\pi\varepsilon_0}\left(\frac{R - r}{10 r_1 R} + \frac{r_2 - R}{r_2 R}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{R^2 r_2 - r_1 r_2 R + 10 r_1 r_2 R - 10 r_1 R^2}{10 r_1 r_2 R^2}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{R r_2 + 9 r_1 r_2 - 10 r_1 R}{10 r_1 r_2 R}\right) \Rightarrow C_{\text{eq}} = 4\pi\varepsilon_0\left(\frac{10 r_1 r_2 R}{R r_2 + 9 r_1 r_2 - 10 r_1 R}\right).$$

In the limit of $R = r_1$, $C_{\text{eq}} = 4\pi\varepsilon_0\left[10 r_1^2 r_2 / \left(10 r_1 r_2 - 10 r_1^2\right)\right] = 4\pi\varepsilon_0\left(r_1 r_2 / \left(r_2 - r_1\right)\right)$. In the limit of $R = r_2$, $C_{\text{eq}} = 4\pi\varepsilon_0\left[10 r_1 r_2 / \left(r_2 - r_1\right)\right]$.

CALCULATE: Not required.

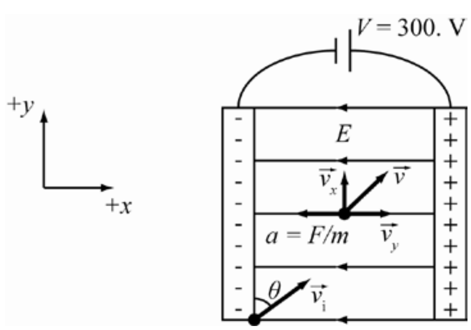
ROUND: Not required.

DOUBLE-CHECK: It is reasonable that the equivalent capacitance is greater in the limit $R = r_2$, because there would be more dielectric material present in the spherical capacitor.

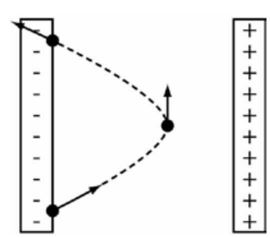
24.87. **THINK:** This problem is similar to finding the maximum height in a projectile motion. However, in this case, the acceleration is due to the electric field of a parallel plate capacitor. Use the conservation of energy.

SKETCH:

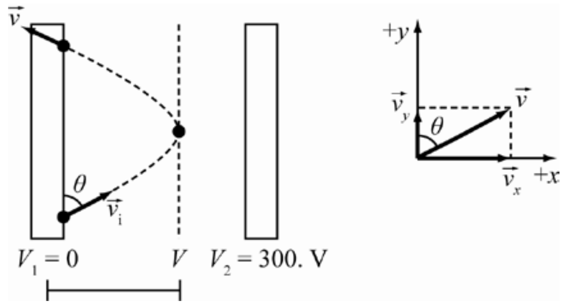
(a)



(b)



(c)



RESEARCH:

(a) The energy required to reach the positive plate is given by $\Delta U = q\Delta V$. The energy of a proton is $K_i = (1/2)mv^2$. In order for proton to reach the positive plate, the kinetic energy must be larger than ΔU, that is, $K_i > \Delta U$.

(b) Not needed.

(c) From conservation of energy, it is found that $K_i + U_i = K_f + U_f$. Since at the maximum *x* displacement, the kinetic energy is zero, $K_f = 0$.

(d) Using conservation of energy, it is found that $K_i + U_i = K_f + U_f$, but in this case $U_i = U_f$.

SIMPLIFY:

(c) Using $K_i = (1/2)mv_x^2$ and $U = qV$, the above equation becomes $(1/2)mv_x^2 + 0 = 0 + qV$. Using $v_x = v_i \sin\theta$, thus, the potential is $V = mv_i^2 \sin^2\theta / (2q)$.

(d) It follows that $K_i = K_f$. Therefore, the speed of the proton as it reaches the negative plate is the same as the initial speed, $2.00 \cdot 10^5$ m/s.

CALCULATE:

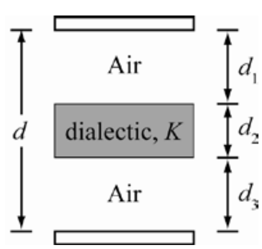
(a) The kinetic energy of the proton is $K_i = (1/2)(1.67 \cdot 10^{-27} \text{ kg})(2.00 \cdot 10^5 \text{ m/s})^2 = 3.34 \cdot 10^{-17}$ J. The energy required to reach the positive plate is $\Delta U = (1.602 \cdot 10^{-19})(300. \text{ V}) = 4.806 \cdot 10^{-17}$ J. Thus K_i is less than ΔU. This means that the proton cannot reach the positive plate regardless of the angle θ.

ROUND: Not needed.

DOUBLE-CHECK: It is reasonable that the proton cannot reach the plate. Conservation of energy was used to balance the potential and kinetic energies, which is a key method of computing velocities.

24.88. **THINK:** The parallel plate capacitor has a dielectric, κ that is positioned between the plates. There is an air gap separating the dielectric material and the plates. The thickness of the dielectric material is d_2. The thickness of the air gaps above and below the dielectric material are d_1 and d_3 respectively. The overall distance between the plates is d.

SKETCH:



RESEARCH: The parallel plate capacitor can be modeled as three parallel plate capacitors in series. The equivalent capacitance of the capacitors in series is $1/C_{eq} = 1/C_1 + 1/C_2 + 1/C_3$. Two of the capacitors are air filled and one has the material of dielectric, κ. The capacitance, C of each of the capacitors can be determined using the equations $C_1 = \varepsilon_0 A/d_1$, $C_2 = \varepsilon_0 \kappa A/d_2$ and $C_3 = \varepsilon_0 A/d_3$. All three capacitors have the same plate area A. The total distance between the plates is $d = d_1 + d_2 + d_3$.

SIMPLIFY: $\dfrac{1}{C_{\text{eq}}} = \dfrac{1}{C_1} + \dfrac{1}{C_2} + \dfrac{1}{C_3} = \dfrac{C_3C_2 + C_1C_3 + C_1C_2}{C_1C_2C_3}$. Substituting in the values for C_1, C_2 and C_3 gives

$$\frac{1}{C_{\text{eq}}} = \frac{(\varepsilon_0 A)^2\left(\dfrac{\kappa}{d_2d_3} + \dfrac{1}{d_1d_3} + \dfrac{\kappa}{d_1d_2}\right)}{(\varepsilon_0 A)^3\left(\dfrac{\kappa}{d_1d_2d_3}\right)} = \frac{d_1d_2d_3}{\varepsilon_0\kappa A}\left(\frac{\kappa}{d_2d_3} + \frac{1}{d_1d_3} + \frac{\kappa}{d_1d_2}\right) = \frac{1}{\varepsilon_0\kappa A}\left(d_1\kappa + d_2 + d_3\kappa\right),$$

but $d_1 + d_3 = d - d_2$, substituting this into the equation gives:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{\varepsilon_0\kappa A}\left[(d - d_2)\kappa + d_2\right] \Rightarrow C_{\text{eq}} = \frac{\varepsilon_0\kappa A}{(d - d_2)\kappa + d_2}.$$

CALCULATE: Not required.
ROUND: Not required.
DOUBLE-CHECK: It seems reasonable that the overall capacitance depends on the distance between the plates and the thickness of the dielectric material and not on the position of the material between the plates.

Multi-Version Exercises

Exercises 24.89–24.91 The energy stored in one supercapacitor is $U_1 = \frac{1}{2}C(\Delta V)^2$. The number of required supercapacitors is then

$$n = \frac{U}{U_1} = \frac{U}{\frac{1}{2}C(\Delta V)^2} = \frac{2U}{C(\Delta V)^2}.$$

24.89. $$n = \frac{2U}{C(\Delta V)^2} = \frac{2(53.63 \cdot 10^6 \text{ J})}{(3.361 \cdot 10^3 \text{ F})(2.121 \text{ V})^2} = 7094.$$

24.90. $$n = \frac{2U}{C(\Delta V)^2}$$

$$\Delta V = \sqrt{\frac{2U}{nC}} = \sqrt{\frac{2(60.51 \cdot 10^6 \text{ J})}{(6990)(3.423 \cdot 10^3 \text{ F})}} = 2.249 \text{ V}$$

24.91. $$n = \frac{2U}{C(\Delta V)^2}$$

$$C = \frac{2U}{n(\Delta V)^2} = \frac{2(67.39 \cdot 10^6 \text{ J})}{(6845)(2.377 \text{ V})^2} = 3485 \text{ F} = 3.485 \text{ kF}$$

Exercises 24.92–24.94 The energy stored in the capacitor with dielectric inserted is $U_{\text{in}} = \frac{1}{2}\kappa C(\Delta V)^2$.

The energy stored in the capacitor with the dielectric removed is $U_{\text{in}} = \frac{1}{2}C(\Delta V)^2$. The work required is equal to the change in the energy stored in the capacitor:

$$W = \Delta U = \frac{1}{2}\kappa C(\Delta V)^2 - \frac{1}{2}C(\Delta V)^2$$
$$W = \frac{1}{2}C(\Delta V)^2(\kappa - 1).$$

24.92. $W = \frac{1}{2}C(\Delta V)^2(\kappa - 1) = \frac{1}{2}(3.547 \cdot 10^{-6} \text{ F})(10.03 \text{ V})^2(4.617 - 1) = 6.453 \cdot 10^{-4} \text{ J}.$

24.93. $W = \frac{1}{2}C(\Delta V)^2(\kappa - 1)$

$$\kappa = \frac{2W}{C(\Delta V)^2} + 1 = \frac{2(4.804 \cdot 10^{-4} \text{ J})}{(3.607 \cdot 10^{-6} \text{ F})(11.33 \text{ V})^2} + 1 = 3.075$$

24.94. $W = \frac{1}{2}C(\Delta V)^2(\kappa - 1)$

$$\Delta V = \sqrt{\frac{2W}{C(\kappa - 1)}} = \sqrt{\frac{2(7.389 \cdot 10^{-4} \text{ J})}{(3.669 \cdot 10^{-6} \text{ F})(3.533 - 1)}} = 12.61 \text{ V}$$

Chapter 25: Current and Resistance

Concept Checks

25.1. e **25.2.** a **25.3.** a **25.4.** a **25.5.** a **25.6.** e **25.7.** c **25.8.** b

Multiple-Choice Questions

25.1. d **25.2.** d **25.3.** c **25.4.** c **25.5.** c **25.6.** a **25.7.** c **25.8.** a **25.9.** c **25.10.** a **25.11.** c **25.12.** d **25.13.** d **25.14.** a

Conceptual Questions

25.15. Subject to the applied potential and electric field E, the electrons will accelerate indefinitely due to the electric force $F = qE = ma$. The drift velocity and current will increase indefinitely until some other effect takes over.

25.16. The voltage across the light bulb is constant. The resistance of a piece of metal (the filament in the bulb) is lower at low temperatures compared to higher temperatures. Since $V = iR$ and V is constant, and the resistance is low, the current i must be large. When the light bulb is first turned on, the filament is cold, so the current is large. A large current increases the likelihood of the light bulb burning out.

25.17. They will be brighter if they are connected in parallel. In parallel, the light bulbs will pull twice the current from the battery, which is twice the power. In series, the circuit has twice the resistance, as it draws only half the current.

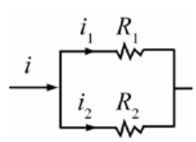
25.18. Resistors in parallel

$$R_{\text{parallel}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 R_2}{R_1 + R_2} = \frac{R_2}{1 + R_2 / R_1} < R_2 \text{ and } R_{\text{parallel}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 R_2}{R_1 + R_2} = \frac{R_1}{1 + R_1 / R_2} < R_1.$$

The resultant resistance is always smaller than the smaller of the two values. In particular, if the difference between the two values is large (an order of magnitude or more), the resultant resistance is less than but very close to the smaller of the two. Example: if you connect in parallel a resistor $R_1 = 1\ \Omega$ and $R_2 = 10\ \Omega$, you get a resistance of $0.91\ \Omega$. If $R_1 = 1$ Ohm and $R_2 = 1\ \text{k}\Omega$, you get a resistance of $0.999\ \Omega$.

25.19. In calculating power, we can use any of the following three equivalent formulas: $P = iV = Ri^2 = \dfrac{V^2}{R}$. For resistors in series, the current is the same through all the resistors, so it makes sense to use $P = Ri^2$, and it is thus apparent that the higher the resistance, R of a resistor, the higher the power dissipated on that resistor. For resistors in parallel, the voltage across all the resistors is the same, so it makes sense to use $P = \dfrac{V^2}{R}$, and it is thus apparent that the resistor with the lowest resistance will dissipate most power.

25.20. Consider the following diagram.



$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow R_{\text{eq}} = \frac{R_1 R_2}{R_1 + R_2}, \text{ so}$$

$$V_1 = V_2 = V = iR_{\text{eq}} \Rightarrow i_1 R_1 = i_2 R_2 = iR_{\text{eq}} \Rightarrow i_1 R_1 = i\frac{R_1 R_2}{R_1 + R_2} \Rightarrow i_1 = \frac{R_2}{R_1 + R_2} i.$$

25.21. Since the resistors are in parallel, $\frac{1}{R_T}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+...=\frac{1}{R_T}=\frac{1}{R}+\frac{1}{R^2}+\frac{1}{R^3}+...,$ and $R=10\ \Omega$. Let $x=\frac{1}{R}=\frac{1}{10\ \Omega}$. The series can be rewritten as: $\frac{1}{R_T}=x+x^2+x^3+... \Rightarrow 1+\frac{1}{R_T}=1+x+x^2+x^3+...,$ but $1+x+x^2+x^3+...=\frac{1}{(1-x)}$ for $|x|<1$, and $\frac{1}{10}<1,$ so that means that $1+\frac{1}{R_T}=\frac{1}{1-x} \Rightarrow \frac{1}{R_T}=\frac{1}{1-x}-1=\frac{1-1+x}{1-x} \Rightarrow \frac{1}{R_T}=\frac{x}{1-x} \Rightarrow R_T=\frac{1-x}{x}=\frac{1}{x}-1,$ which then gives the formula $\frac{1}{x}=R \Rightarrow R_T=R-1=10\ \Omega-1\ \Omega=9\ \Omega.$

25.22. The black wire, having lower resistance, will draw more power than the red wire since $P=iV=i^2R=\frac{\Delta V^2}{R}$, where ΔV is the battery voltage. Since the wire converts this electrical energy to thermal energy, the black wire will get hotter. Note that if the battery has significant internal resistance that will affect the temperature of the wires, but the black wire will still be hotter than the red wire.

25.23. No, ordinary incandescent light bulbs are not actually Ohm resistors. They can be operated over a wide enough range of currents, hence temperatures, that the temperature dependence of the filament resistance is significant. The resistance of an ordinary light bulb measured with an Ohmmeter at room temperature is substantially lower, that its resistance at an operating temperature of order 2000 K. By connecting light bulbs in series it is possible to operate them at a range of voltages, hence currents, wide enough to display this variation in resistance. The experiment is quite pretty as the light bulbs can be made to glow colors ranging from red through orange and yellow to white. A plot of V versus i for the light bulbs is not the straight line of an Ohm resistor it steepens noticeably the resistance increases as i increases.

25.24. (a) No assumptions can be made about the geometry and this is certainly not a steady state of equilibrium situation. However, if we consider a surface S surrounding the injection region as a Gaussian surface then the charge $Q(t)$ is given by Gauss' Law $Q(t)=\varepsilon\oint_S \vec{E}\cdot d\vec{A},$ where the permittivity incorporates the dielectric properties of the material. The material is ohmic so the electric field E drives current density $J=\sigma E$. Hence, the above equation can be written $Q(t)=(e/\sigma)\oint_S \vec{J}\cdot d\vec{A}$. By the definition of J, the integral here is the net rate of charge transport out of the volume surrounded by S. Charge conservation requires that this be equal to the rate of decrease of the charge within that volume (no charge can be gained or lost): $\oint_S \vec{J}\cdot d\vec{A}=-dQ/dt$. This is the *certainty equation* for electric charge: it is similar to the continuity equation of field mechanics, which expresses the conservation of field mass for particle number. It has the advantage that it applies in every situation. Here it implies $\frac{dQ}{dt}=-(\sigma/e)Q$ is the desired differential equation.

(b) Students at this level should recognize immediately that the solution of a differential equation of this form is an exponential function. Explicitly, the equation implies

$$\int_{Q_0}^{Q(t)}\frac{dQ'}{Q'}=-\frac{\sigma}{e}\int_0^t dt', \quad \text{or} \quad \ln\left(\frac{Q(t)}{Q_0}\right)=-\frac{\sigma t}{e}, \quad \text{i.e.} \quad Q(t)=Q_0\exp\left(-\frac{\sigma t}{e}\right),$$

for all $t\geq 0$. The charge in the injection region decays exponentially rapidly for a good conductor slowly for a poor one and the injected charge moves to the outer surface of the conductor.

(c) The proceeding result implies that the time required for the charge in the injection region to decrease by half is $t_{1/2} = \varepsilon \ln\left(\frac{2}{\sigma}\right)$. For copper, $\sigma = \left(1.678 \cdot 10^{-8}\ \Omega\ \text{m}\right)^{-1}$ at 20 °C and $\varepsilon - \varepsilon_0$ by assumption yielding

$$t_{1/2}^{\text{Cu}} = \frac{\left(8.85 \cdot 10^{-12}\ \text{C}^2/\text{N m}^2\right)\ln 2}{5.959 \cdot 10^7\ (\Omega\ \text{m})^{-1}} = 1.03 \cdot 10^{-19}\ \text{s}.$$

This is less than the crossing time of light over a single atom so this calculation particularly the assumptions of Ohmic behavior and unit "dielectric constant" may not be very accurate in this case. It does indicate; however, that the evacuation of free charge from the interior of a good conductor is very rapid. For quartz the data is somewhat varied. *The Handbook of Chemistry and Physics* gives $\sigma = \left(1 \cdot 10^3\ \Omega\ \text{m}\right)^{-1}$ at 20 °C for SiO_2 and $\varepsilon = (3.75 - 4.1)\varepsilon_0$ for fused quartz. A typical value for $t_{1/2}$ would be

$$t_{1/2}^{\text{SiO}_2} = \frac{3.9\left(8.85 \cdot 10^{-12}\ \text{C}^2/\left(\text{N m}^2\right)\right)\ln 2}{1 \cdot 10^{-13}\ (\Omega\ \text{m})^{-1}} = 200\ \text{s}$$

over three minutes, some twenty one orders of magnitude longer. *Reitz and Milford* gives $\sigma = \left(7.5 \cdot 10^{17}\ \Omega\ \text{m}\right)^{-1}$ for fused quartz.

This implies a value $t_{1/2}^{\text{SiO}_2} = \dfrac{3.9\left(8.854 \cdot 10^{-12}\ \text{C}^2/\left(\text{N m}^2\right)\right)\ln 2}{1.3 \cdot 10^{-18}\ (\Omega\ \text{m})^{-1}} = 1.8 \cdot 10^7\ \text{s}$, or about 210 days.

25.25. You can write the drift speed of electrons in a wire as $v = \dfrac{i}{(nqA)}$. For a wire connected across a potential difference V, you can find the current i in the wire by determining the resistance of the gold wire, which is just $R = \rho_{\text{resisitivity, Au}} \Delta x / A$, where Δx is its length. Therefore,

$$v = \frac{i}{nqA} = \frac{V}{nqAR} = \frac{V}{nqA\rho_{\text{resisitivity, Au}}(\Delta x / A)} = \frac{V}{nq\rho_{\text{resisitivity, Au}}\Delta x}.$$

Thus, since none of the quantities in the equation above depend on A, it has been shown that the speed of electrons does not depend on the cross-sectional area of the wire.

25.26. The brightness of a light bulb is proportional to its current, so to rank the brightness of the bulbs, you will need to find and rank the currents. The currents can be found by calculating the equivalent resistance for the different circuit elements. Bulbs 1 and 2 are in series, so $i_1 = i_2$. The equivalent resistance for the 2 bulbs is $2R$. The current through bulbs 1 and 2 is $i_1 = i_2 = V/(2R)$. Bulbs 5 and 6 are in series, so $i_5 = i_6$. The equivalent resistance for the 2 bulbs is $2R$. The equivalent resistance for bulbs 4, 5, and 6 is $R_{456} = \left[1/R + 1/(2R)\right]^{-1} = (2/3)R$. Adding bulb 3 in series gives: $R_{3456} = (5/3)R$ and the current in bulb 3 is: $i_3 = 3V/(5R)$. The voltage across bulbs 4, 5 and 6 is then $V - (3/5)V = (2/5)V$. This makes the currents in bulbs 4, 5 and 6: $i_4 = 2V/(5R)$ and $i_5 = i_6 = V/(5R)$. Ranking the bulbs from dimmest to brightest: $(i_5 = i_6) < i_4 < (i_1 = i_2) < i_3$.

25.27. Conductor 1: length $= L$, Radius $= R$, Area $= A$, Resistance $= R$ and Voltage $= V$. Conductor 2: length $= L$, Radius $= R$, Area $= A$, Resistance $= 2R$ and Voltage $= V$. Power delivered is given as $P = \dfrac{V^2}{R}$, $P_1 = \dfrac{V^2}{R}$, $P_2 = \dfrac{V^2}{2R} = \dfrac{P_1}{2}$. Therefore, the power delivered to the first would be twice that delivered to the second.

Exercises

25.28. The total charge in the Tevatron is $Q = i\Delta t$. Now, $\Delta t = L / v$, where L is the beam circumference and v is the speed of the protons. $i = 11\cdot 10^{-3}$ A, $L = 6.3\cdot 10^{3}$ m and $v = c = 3.00\cdot 10^{8}$ m/s. This charge is made up of n protons:

$$Q = ne = i\Delta t = \frac{iL}{v} \Rightarrow n = \frac{iL}{e\cdot c} = \frac{\left(1.10\cdot 10^{-2}\text{ A}\right)\left(6.30\cdot 10^{3}\text{ m}\right)}{\left(1.602\cdot 10^{-19}\text{ C}\right)\left(3.00\cdot 10^{8}\text{ m/s}\right)} = 1.4\cdot 10^{12}.$$

25.29. The area A is $A = \pi r^2 = 3.14\cdot 10^{-6}\text{ m}^2$, so the current density is

$$J = \frac{i}{A} = \frac{1.00\cdot 10^{-3}\ A}{3.14\cdot 10^{-6}\text{ m}^2} = 318.3\text{ A/m}^2 \approx 318\text{ A/m}^2.$$

The density of electrons is

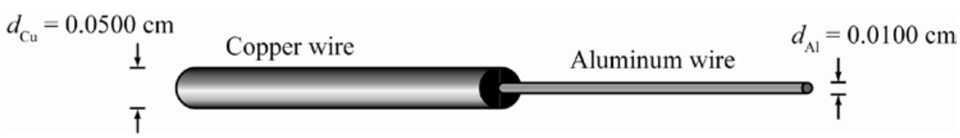
$$n = \left(\frac{1\text{ electron}}{\text{atom}}\right)\left(\frac{6.02\cdot 10^{23}\text{ atoms}}{26.98\text{ g}}\right)\left(\frac{2.700\cdot 10^{6}\text{ g}}{\text{m}^3}\right) = 6.02\cdot 10^{28}\text{ electrons / m}^3,$$

and the drift speed is

$$v_d = \frac{J}{ne} = \frac{318.3\text{ A/m}^2}{\left(6.02\cdot 10^{28}\text{ electrons/m}^3\right)\left(1.602\cdot 10^{-19}\text{ A s}\right)} = 3.30\cdot 10^{-8}\text{ m / s}.$$

25.30. **THINK:** The current is the same in both wires due to conservation of charge. This can be used to compute the ratio of current densities. The ratio of the drift velocities can then be computed by expressing the drift velocities in terms of the current density. The densities of charge carriers are charges per electron: $n_{\text{Cu}} = 8.50\cdot 10^{28}\text{ m}^{-3}$ and $n_{\text{Al}} = 6.02\cdot 10^{28}\text{ m}^{-3}$. The other values given in the question that will be needed are $d_{\text{Cu}} = 5.00\cdot 10^{-4}$ m, and $d_{\text{Al}} = 1.00\cdot 10^{-4}$ m. The lengths of the wires and the amount of current are not necessary to solve the question.

SKETCH:



RESEARCH: $J = i / A$, $A =$ cross-sectional area, $J = nev_{\text{d}}$ and $A = \pi r^2$.

SIMPLIFY:

(a) $$\frac{J_{\text{Cu}}}{J_{\text{Al}}} = \frac{i / A_{\text{Cu}}}{i / A_{\text{Al}}} = \frac{A_{\text{Al}}}{A_{\text{Cu}}} = \frac{\pi\left(d_{\text{Al}} / 2\right)^2}{\pi\left(d_{\text{Cu}} / 2\right)^2} = \frac{d_{\text{Al}}^2}{d_{\text{Cu}}^2}$$

(b) $$v_{\text{d}} = \frac{J}{ne} \Rightarrow \frac{v_{\text{d,Cu}}}{v_{\text{d,Al}}} = \frac{J_{\text{Cu}} / \left(n_{\text{Cu}}e\right)}{J_{\text{Al}} / \left(n_{\text{Al}}e\right)} = \left(\frac{J_{\text{Cu}}}{J_{\text{Al}}}\right)\left(\frac{n_{\text{Al}}}{n_{\text{Cu}}}\right)$$

CALCULATE:

(a) $$\frac{J_{\text{Cu}}}{J_{\text{Al}}} = \frac{\left(1.00\cdot 10^{-4}\text{ m}\right)^2}{\left(5.00\cdot 10^{-4}\text{ m}\right)^2} = 0.040000$$

(b) $$\frac{v_{\text{d-Cu}}}{v_{\text{d-Al}}} = \left(\frac{J_{\text{Cu}}}{J_{\text{Al}}}\right)\left(\frac{n_{\text{Al}}}{n_{\text{Cu}}}\right) = \left(0.0400\right)\left(\frac{6.02\cdot 10^{28}\text{ m}^{-3}}{8.50\cdot 10^{28}\text{ m}^{-3}}\right) = 0.02833$$

ROUND:

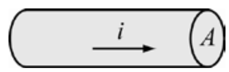
(a) $\frac{J_{\text{Cu}}}{J_{\text{Al}}}=0.0400$

(b) $\frac{v_{\text{d-Cu}}}{v_{\text{d-Al}}}=0.0283$

DOUBLE-CHECK: The answers are dimensionless since they are ratios.

25.31. **THINK:** From the atomic weight and density of silver, the conduction electron density can be computed. Since both the current and the cross-sectional area of the wire are given, the current density can be computed and, using the calculated quantities, the drift speed of the electrons can be computed. Use the data: $A=0.923\text{ mm}^2$, $i=0.123\text{ mA}$, $M=107.9\text{ g/mol}$, $\rho_{\text{Ag}}=10.49\text{ g/cm}^3$, $N_{\text{A}}=6.02\cdot 10^{23}\text{ mol}^{-1}$, $N=1$ electron/atom and $e=1.602\cdot 10^{-9}\text{ C}$.

SKETCH:



RESEARCH:

(a) $n=\frac{N\rho_{\text{Ag}}N_{\text{A}}}{M}$

(b) $J=\frac{i}{A}$

(c) $J=nev_{\text{d}}$

SIMPLIFY:

(a) Not required.

(b) Not required.

(c) $v_{\text{d}}=J/ne$

CALCULATE:

(a) $n=\frac{1\left(10.49\text{ g/cm}^3\right)\left(6.02\cdot 10^{23}\text{ mol}^{-1}\right)}{107.9\text{ g/mol}}=5.853\cdot 10^{22}\text{ cm}^{-3}$

(b) $J=\frac{0.123\cdot 10^{-3}\text{ A}}{\left(0.923\text{ mm}^2\right)\left(10^{-3}\text{m/mm}\right)^2}=133.3\text{ A/m}^2$

(c) $v_{\text{d}}=\frac{133.3\text{ A/m}^2}{\left(5.853\cdot 10^{22}\text{ cm}^{-3}\right)\left(10^2\text{ cm/m}\right)^3\left(1.602\cdot 10^{-19}\text{ C}\right)}=1.421\cdot 10^{-8}\text{ m/s}$

ROUND: Three significant figures:

(a) $n=5.85\cdot 10^{22}\text{ cm}^{-3}$

(b) $J=133\text{ A/m}^2$

(c) $v_{\text{d}}=1.42\cdot 10^{-8}\text{ m/s}$

DOUBLE-CHECK: These are reasonable values. Note that for part (a), only the composition and not the dimensions of the wire are relevant.

25.32. $R=\rho_{\text{Cu}}\dfrac{L}{A}=\left(1.72\cdot 10^{-8}\ \Omega\text{m}\right)\dfrac{10.9\text{ m}}{\pi\left(1.3\cdot 10^{-3}\text{ m}/2\right)^2}=0.141\ \Omega$

25.33. The resistances will be the same when their cross-sectional areas are the same. $\pi R_B^2=\pi\left(\dfrac{d_o}{2}\right)^2-\pi\left(\dfrac{d_i}{2}\right)^2\Rightarrow R_B=\dfrac{1}{2}\sqrt{d_o^2-d_i^2}$

$\Rightarrow R_{\text{B}}=\dfrac{1}{2}\sqrt{\left(3.00\text{ mm}\right)^2-\left(2.00\text{ mm}\right)^2}=1.12\text{ mm}$

25.34. The copper coil's resistance increases linearly with temperature. At $T_0=20.\ ^\circ\text{C}=293.15\text{ K},$ it has resistance $R_0=0.10\ \Omega.$ The temperature coefficient of copper is $\alpha=3.9\cdot 10^{-3}\text{ K}^{-1}.$ At $T=-100.\ ^\circ\text{C}=173.15\text{ K},$

$R=R_0\left(1+\alpha\left(T-T_0\right)\right)=\left(0.10\ \Omega\right)\left(1+\left(3.9\cdot 10^{-3}\text{ K}^{-1}\right)\left(173.15\text{ K}-293.15\text{ K}\right)\right)=0.053\ \Omega.$

25.35. The area of 12 gauge copper wire is $A_{\text{Cu}}=3.308\text{ mm}^2.$ The resistivity of copper and aluminum are, $\rho_{\text{Cu}}=1.72\cdot 10^{-8}\ \Omega\text{ m}$ and $\rho_{\text{Al}}=2.82\cdot 10^{-8}\ \Omega\text{ m}.$ In general the equation for resistance is $R=\rho L/A,$ meaning if the two wires have equal resistance per length $(L),$ then

$$\frac{\rho_{\text{Cu}}}{A_{\text{Cu}}}=\frac{\rho_{\text{Al}}}{A_{\text{Al}}}\ \Rightarrow\ A_{\text{Al}}=\frac{\rho_{\text{Al}}A_{\text{Cu}}}{\rho_{\text{Cu}}}=\frac{\left(2.82\cdot 10^{-8}\ \Omega\text{ m}\right)\left(3.308\text{ mm}^2\right)}{1.72\cdot 10^{-8}\ \Omega\text{ m}}=5.42\text{ mm}^2.$$

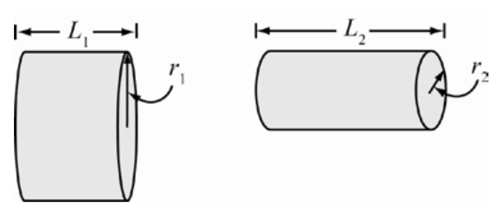
This value corresponds to between 9 and 10 gauge wire.

25.36. Since the resistance is given as $R=\rho L/A,$ then the setup that maximizes L and minimizes A will give the largest resistance. This corresponds to choosing $L=3.00\text{ cm}$ and $A=\left(2.00\text{ cm}\right)\left(0.010\text{ cm}\right)=0.020\text{ cm}^2.$ The resistivity is $\rho=2300\ \Omega\text{ m}.$ Therefore, the maximum resistance is

$$R_{\max}=\frac{\rho L}{A}=\frac{\left(2300\ \Omega\text{ m}\right)\left(3.00\text{ cm}\right)}{\left(0.020\text{ cm}^2\right)}\cdot\left(\frac{100\text{ cm}}{1\text{ m}}\right)=3.5\cdot 10^7\ \Omega=35\text{ M}\Omega.$$

25.37. **THINK:** The copper wire, $L_1=1\text{ m}$ and $r_1=0.5\text{ mm},$ has an area of $A_1.$ The wire is then stretched to $L_2=2\text{ m}.$ Since the overall volume $\left(V=AL\right)$ of the wire remains constant, if the wire doubles in length, the area must be halved.

SKETCH:



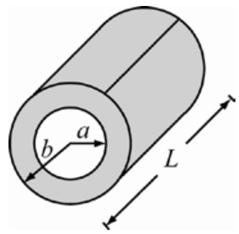
RESEARCH: The resistance of the wire is $R_i=\rho L_i/A_i.$ From the conservation of volume, it follows that $V=A_1L_1=A_2L_2.$ The fractional change in resistance is $\Delta R/R=\left(R_2-R_1\right)/R_1.$

SIMPLIFY: Since $L_2=2L_1,$ then $A_2=\left(1/2\right)A_1.$ The change in resistance is then $\dfrac{\Delta R}{R}=\dfrac{\left(R_2-R_1\right)}{R_1}=\dfrac{\rho\left(L_2/A_2-L_1/A_1\right)}{\rho\left(L_1/A_1\right)}=\dfrac{2L_1/\left(1/2\right)A_1-L_1/A_1}{L_1/A_1}=\dfrac{4L_1/A_1-L_1/A_1}{L_1/A_1}=3.$ It is the same for aluminum, independent of $\rho.$

CALCULATE: Not required.

ROUND: Not required.
DOUBLE-CHECK: It would seem to make sense that the fractional change in resistance is the same for all materials, so having the equation independent of ρ makes sense.

25.38. **THINK:** The actual cross-sectional area of the resistor, $L = 60$ cm and $R_0 = 1$ Ohm, is the difference in area from outer radius, $b = 2.5$ cm, and inner radius, $a = 1.5$ cm. The resistance of the resistor should vary linearly with temperature from $T = 300$ °C to $T_0 = 20$ °C with $\alpha = 2.14 \cdot 10^{-3}$ K^{-1}.
SKETCH:



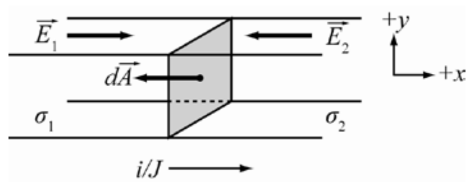
RESEARCH: The resistivity of the wire is given by $\rho = RA/L$, where the area of interest is $A = \pi\left(b^2 - a^2\right)$. The resistance varies with the temperature: $R = R_0\left(1 + \alpha\left(T - T_0\right)\right)$. The percentage change of resistance is found by $\left(\Delta R / R\right)\left(100\%\right)$.
SIMPLIFY:
(a) The resistivity is $\rho = \dfrac{RA}{L} = \dfrac{R_0\pi\left(b^2 - a^2\right)}{L}$.
(b) The fractional change in resistance is:

$$\%\frac{\Delta R}{R} = \frac{\left(R - R_0\right)}{R_0}\left(100\%\right) = \frac{R_0\left(1 + \alpha\left(T - T_0\right)\right) - R_0}{R_0}\left(100\%\right) = \alpha\left(T - T_0\right)\left(100\%\right).$$

CALCULATE:
(a) $\rho = \dfrac{\left(1.00\ \Omega\right)\pi\left(\left(2.50\text{ cm}\right)^2 - \left(1.50\text{ cm}\right)^2\right)}{60.0\text{ cm}} = 0.20944\ \Omega\text{ cm}$
(b) $\%\dfrac{\Delta R}{R} = \left(2.14 \cdot 10^{-3}\text{ K}^{-1}\right)\left(300.\text{ °C} - 20.\text{ °C}\right)\left(100\%\right) = 59.92\%$
ROUND:
(a) $\rho = 0.209\ \Omega$ cm, which is much higher than $\rho = 1.7 \cdot 10^{-6}\ \Omega$ cm for copper.
(b) $\%\dfrac{\Delta R}{R} = 59.9\%$
DOUBLE-CHECK: It makes sense that a resistor would be made of material that has a much higher resistivity than the wiring it connects to but a much lower resistivity than insulating materials that block current altogether (such as glass, $\rho > 10^7\ \Omega$ cm).

25.39. **THINK:** Since the current density across the junction is constant, J, and they share the same cross-sectional area, they have the same current, i. At the junction, a positive charge will build up, and this means both electric fields, E_1 and E_2, are pointing towards the junction. Gauss's Law can then be used to determine the total charge built up on the interface. The electric fields are also related to the conductivities, σ_1 and σ_2.

SKETCH:



RESEARCH: The conductivity is the inverse of resistivity, $\sigma = 1/\rho$. The resistivity is related to electric field by $\rho = E/J$. At the junction, Gauss's law states $\oint\!\!\oint \vec{E} \bullet d\vec{A} = q/E_0$. The current density J, is $J = i/A$.

SIMPLIFY: From the conductivity and resistivity, the current density is $\sigma = 1/\rho = 1/(E/J) = J/E \Rightarrow J = \sigma E$; therefore, $\sigma_1 E_1 = \sigma_2 E_2$ or $E_1 = (\sigma_2/\sigma_1)E_2$. Since $\vec{E}_2$ is parallel and $\vec{E}_1$ is antiparallel to $d\vec{A}$, $\oint\!\!\oint \vec{E} \bullet d\vec{A} = (E_2 - E_1)A = q/\varepsilon_0$. Solving this expression for q yields the following.

$$q = \varepsilon_0 (E_2 - E_1)A = \varepsilon_0 \left(E_2 - \left(\frac{\sigma_2}{\sigma_1} E_2 \right) \right) A = \varepsilon_0 E_2 \left(1 - \left(\frac{\sigma_2}{\sigma_1} \right) \right) \left(\frac{i}{J} \right) = \varepsilon_0 E_2 \left(1 - \frac{\sigma_2}{\sigma_1} \right) \left(\frac{i}{\sigma_2 E_2} \right) = \varepsilon_0 i \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1} \right)$$

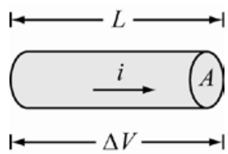
This is what was required to be shown.

CALCULATE: Not required.

ROUND: Not required.

DOUBLE-CHECK: The equation was verified, so it makes sense.

25.40. (a)



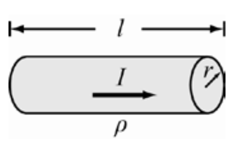
Since the potential across wire is $\Delta V = 12.0 \text{ V}$ and the current is $i = 3.20 \cdot 10^{-3} \text{ A}$. Ohm's Law states $\Delta V = iR \Rightarrow R = \dfrac{\Delta V}{i} = \dfrac{12.0 \text{ V}}{3.20 \cdot 10^{-3} \text{ A}} = 3750 \ \Omega.$

(b) Since the wire is $L = 1000. \text{ km}$ long and has area of $A = 4.50 \text{ mm}^2$, the resistivity of it is $R = \rho \dfrac{L}{A} \Rightarrow \rho = \dfrac{RA}{L} = \dfrac{(3750 \ \Omega)(4.50 \cdot 10^{-6} \text{ m}^2)}{(1000 \cdot 10^3 \text{ m})} = 1.69 \cdot 10^{-8} \ \Omega \text{ m}$, therefore, the wire is most likely copper $(\rho_C = 1.72 \cdot 10^{-6} \ \Omega \text{ m})$.

25.41. The current is $i = 600. \text{ A}$ and the potential difference is $\Delta V = 12.0 \text{ V}$. Therefore, Ohm's Law states $\Delta V = iR \Rightarrow R = \Delta V / i = 12.0 \text{ V}/(600. \text{ A}) = 0.0200 \ \Omega.$

25.42. **THINK:** The resistance of the wire of radius $r = 0.0250 \text{ cm}$ and length $L = 3.00 \text{ m}$ is found by using its resistivity $\rho = 1.72 \cdot 10^{-8} \ \Omega \text{ m}$. The potential drop is found using the current, $i = 0.400 \text{ A}$, and Ohm's Law. Assuming the electric field is constant, it is simply found through the potential drop over the length.

SKETCH:



RESEARCH: The resistance is $R=\rho L/A$. The area of the wire is $A=\pi r^2$. The potential difference across the wire is $\Delta V=iR$. The electric field across the wire is $E=\Delta V/L$.

SIMPLIFY:

(a) The resistance is $R=\frac{\rho L}{A}=\frac{\rho L}{\pi r^2}$.

(b) The potential difference across the wire is $\Delta V=iR$.

(c) The electric field across the wire is $E=\frac{\Delta V}{L}$.

CALCULATE:

(a) $R=\frac{\left(1.72\cdot 10^{-8}\ \Omega\text{ m}\right)\left(3.00\text{ m}\right)}{\pi\left(0.0250\text{ cm}\right)^2}=0.2628\ \Omega$

(b) $\Delta V=\left(0.400\text{ A}\right)\left(0.2628\ \Omega\right)=0.10512\text{ V}$

(c) $E=\frac{0.10512\text{ V}}{3.00\text{ m}}=0.03504\text{ V/m}=0.0350\text{ V/m}$

ROUND: Rounding to three significant figures;

(a) $R=0.263\ \Omega$

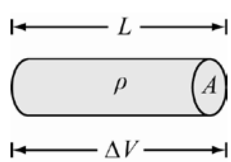
(b) $\Delta V=0.105\text{ V}$

(c) $E=0.0350\text{ V/m}$

DOUBLE-CHECK: The wire loses very little potential over a long length. This means that this wire used in a standard circuit which would only be a few centimeters in length would only lose about 1 or 2 mV, making it a suitable material for a circuit.

25.43. **THINK:** The length and resistivity of the wire are $L=1.0\text{ m}$ and $\rho=1.72\cdot 10^{-8}\ \Omega\text{ m}$. The area of the wire is $A=0.0201\text{ mm}^2$. Since the resistance changes linearly with the temperature, $T_0=20.\ ^\circ\text{C}$ and $T=-196\ ^\circ\text{C}$, the percentage change in resistance is proportional, by $\alpha=3.9\cdot 10^{-3}\text{ K}^{-1}$, to the temperature difference. Since the current is directly related to the resistance, the percentage change in resistance is related to the resistances themselves. Using the molar mass and density of copper, $m=0.06354\text{ kg}$ and $\rho_{\text{Cu}}=8.92\cdot 10^3\text{ kg/m}^3$, the carrier density, n, can be determined, which in turn allows velocity to be found. Use the value $\Delta V=0.1\text{ V}$.

SKETCH:



RESEARCH: The resistance of the wire is $R=\rho L/A$. From Ohm's Law, the potential drop across the wire is $\Delta V=IR$. The resistance changes linearly with temperature by $R=R_0\left(1+\alpha\left(T-T_0\right)\right)$. For a given quantity, x, the percentage change in it is therefore, $\Delta x/x=\left(\left(x-x_0\right)/x_0\right)\left(100\%\right)$. Current density is $J=nev_2=i/A$.

SIMPLIFY:

(a) The original resistance is $R_0=\rho L/A$. The cooled resistance is $R=R_0\left(1+\alpha\left(T-T_0\right)\right)$; therefore,

$$\%\frac{\Delta R}{R}=\frac{R-R_0}{R_0}=\frac{R_0\left(1+\alpha\left(T-T_0\right)\right)-R_0}{R_0}=-\alpha\left(T-T_0\right)\left(100\%\right).$$

(b) The percentage change in current is found using the following equations.

$$\frac{\Delta i}{i}=\frac{i-i_0}{i_0}=\frac{\Delta V/R-\Delta V/R_0}{\Delta V/R_0}=\frac{1/R-1/R_0}{1/R_0}=\frac{R_0-R}{RR_0}R_0=\frac{R_0-R}{R}=\frac{R_0-R_0\left(1+\alpha\left(T-T_0\right)\right)}{R_0\left(1+\alpha\left(T-T_0\right)\right)}$$
$$=\frac{-\alpha\left(T-T_0\right)}{\left(1+\alpha\left(T-T_0\right)\right)}\left(100\%\right)$$

(c) Assume each copper atom contributes just $1\,e^-$. The molar volume of copper is $V_{Cu}=\rho_{Cu}/m$; therefore, $n=N_A\rho_C/m$. The drift velocity is then, in general,

$$J=\frac{i}{A}=nev_d$$
$$\Rightarrow v_d=\frac{i}{neA}=\frac{\Delta V}{neAR}=\frac{\Delta V}{neAR_0\left(1+\alpha\left(T-T_0\right)\right)}$$
$$\Rightarrow v_d=\frac{\Delta V}{neA\left(\rho L/A\right)\left(1+\alpha\left(T-T_0\right)\right)}=\frac{\Delta V}{ne\rho L\left(1+\alpha\left(T-T_0\right)\right)}.$$

At $T=T_0$ and $V_d=\Delta V/ne\rho L$.

CALCULATE:

(a) $\%\Delta R/R=-\left(3.9\cdot 10^{-3}\text{ K}^{-1}\right)\left(-196\text{ °C}-20.\text{ °C}\right)\left(100\%\right)=-84.24\%$

(b) $\%\Delta i/i=\dfrac{-\left(3.9\cdot 10^{-3}\text{ K}^{-1}\right)\left(-196\text{ °C}-20.\text{ °C}\right)}{1+\left(3.9\cdot 10^{-3}\text{ K}^{-1}\right)\left(-196\text{ °C}-20.\text{ °C}\right)}\left(100\%\right)=534.5\%$

(c) $n=\dfrac{\left(6.022\cdot 10^{23}\text{ e}^-\right)\left(8.92\cdot 10^{3}\text{ kg/m}^3\right)}{0.06354\text{ kg}}=8.4539\cdot 10^{28}\text{ e}^-/\text{m}^3$

At room temperature, $V_d=\dfrac{0.10\text{ V}}{\left(8.4535\cdot 10^{28}\text{ e}^-/\text{m}^3\right)\left(1.602\cdot 10^{-19}\text{ C}\right)\left(1.72\cdot 10^{-8}\ \Omega\text{m}\right)\left(1.0\text{ m}\right)}=0.4293\text{ mm/s}.$

At temperature, $T=-196\text{ °C}$, $V_d=\dfrac{0.4293\text{ mm/s}}{1+\left(3.9\cdot 10^{-3}\text{ K}^{-1}\right)\left(-196\text{ °C}-20.\text{ °C}\right)}=2.724\text{ mm/s}.$

ROUND:

(a) $\%\Delta R/R=-84\%$ (decrease in resistance)

(b) $\%\Delta I/I=530\%$ (increase in current)

(c) At room temperature, $V_d=0.43$ mm/s. At 77 K the speed is $V_d=2.7$ mm/s.

DOUBLE-CHECK: Supercooling a resistor should greatly reduce the resistance and increase the current and drift velocity, which it does, so it makes sense.

25.44. If the current, $i=11$ A, went entirely through the known resistor, $R_0=35\ \Omega$, the potential drop across it would be $\Delta V=iR_0=\left(11\text{ A}\right)\left(35\ \Omega\right)=385$ V, which is too large, so the other resistor must be parallel to R_0. Therefore, by Ohm's Law, $\Delta V=i\left(\dfrac{1}{R}+\dfrac{1}{R_0}\right)^{-1}\Rightarrow R=\left(\dfrac{i}{\Delta V}-\dfrac{1}{R_0}\right)^{-1}=\left(\dfrac{11\text{ A}}{120\text{ V}}-\dfrac{1}{35\ \Omega}\right)^{-1}=15.849\ \Omega.$ Hence, $\Delta V=15.8\ \Omega$.

25.45. When the external resistor, $R=17.91\ \Omega$ is connected, the potential drop across it is $\Delta V=12.68$ V, so the current through the circuit is, by Ohm's Law, $i=\Delta V/R=12.68\text{ V}/17.91\ \Omega=0.70798\text{ A}=0.7080\text{ A}$. This is the same current running through the internal resistor, R_i, which is in series with R, so since the battery has a total emf of $\Delta V_{emf}=14.50$ V, the internal resistance is found using the following calculation. $\Delta V_{emf}=i\left(R+R_i\right)\Rightarrow R_i=\dfrac{\Delta V_{emf}}{i}-R=\dfrac{14.50\text{ V}}{0.7080\text{ A}}-17.91\ \Omega=2.5702\ \Omega=2.570\ \Omega$

25.46. The two resistors, $R_1 = 100.\ \Omega$ and $R_2 = 400.\ \Omega$, cause currents, $i_1 = 4.00$ A and $i_2 = 1.01$ A, respectively. The currents they cause are the same through the internal resistor, R_i, and in both cases the emf of the battery, ΔV, is the same. Since R_i is in series with each of the other resistors, Ohm's Laws says:

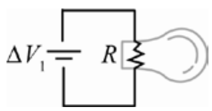
$$\Delta V = i_1(R_1 + R_i) = i_2(R_2 + R_i) \Rightarrow i_2R_2 - i_1R_1 = R_i(i_1 - i_2) \Rightarrow R_i = \frac{i_2R_2 - i_1R_1}{i_1 - i_2}$$

$$\Rightarrow R_i = \frac{(1.01\text{ A})(400.\ \Omega) - (4.00\text{ A})(100.\ \Omega)}{(4.00\text{ A} - 1.01\text{ A})} = 1.3378\ \Omega \approx 1.34\ \Omega.$$

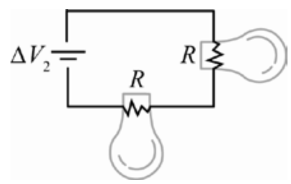
25.47. **THINK:** The resistance in each bulb is directly calculated by Ohm's Law. Consider temperature effects on resistance to explain discrepancy with parts (a) and (b). Use the values: $\Delta V_1 = 6.20$ V, $i_1 = 4.1$ A, $i_2 = 2.9$ A and $\Delta V_2 = 6.29$ V.

SKETCH:

(a)



(b)



RESEARCH: By Ohm's Law, $\Delta V_1 = i_1R$ for one light bulb and $\Delta V_2 = i_1(R + R)$ for both light bulbs.

SIMPLIFY:

(a) $\Delta V_1 = i_1R \Rightarrow R = \dfrac{\Delta V_1}{i_1}$

(b) $\Delta V_2 = i_2(R + R) \Rightarrow R = \dfrac{\Delta V_2}{2i_2}$

CALCULATE:

(a) $R = \dfrac{6.20\text{ V}_1}{4.1\text{ A}} = 1.5122\ \Omega$

(b) $R = \dfrac{6.29\text{ V}}{2(2.9\text{ A})} = 1.084\ \Omega$

ROUND:

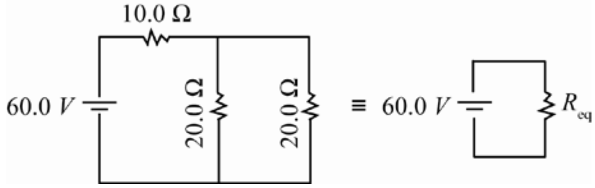
(a) $R = 1.51\ \Omega$

(b) $R = 1.08\ \Omega$.

(c) When two bulbs are put in series, it is expected that they glow dimmer than only one bulb. This would mean the one bulb would be hotter and thus have a larger resistance.

DOUBLE-CHECK: Answer to part (c) helps to verify that the answers to parts (a) and (b) are reasonable.

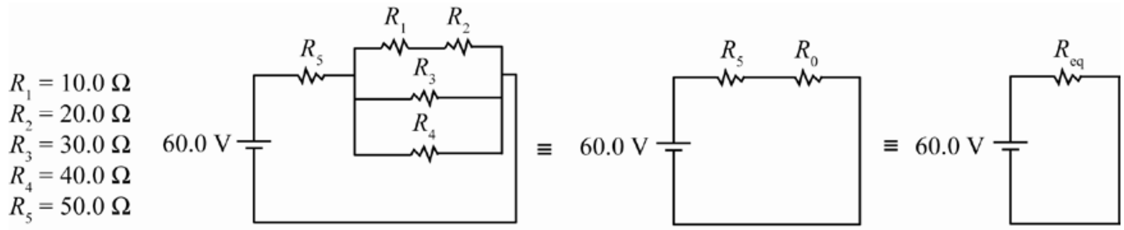
25.48. Simplifying the circuit gives



$R_{eq} = 10.0\ \Omega + \left[1/(20.0\ \Omega) + 1/(20.0\ \Omega)\right]^{-1} = 20.0\ \Omega$. By Ohm's Law, the current through R_{eq} is $i_{eq} = 60.0\text{ V}/(20.0\ \Omega) = 3.00$ A. From the circuit setup, the current through R_{eq} is the same as that through the $10.0\ \Omega$ resistor, which is 3.00 A.

25.49. **THINK:** The circuit can be redrawn to have the $10.0\ \Omega$ and $20.0\ \Omega$ resistors in series, both of which are parallel to the $30.0\ \Omega$ resistor, and then parallel again with the $40.0\ \Omega$ resistor. These resistors are then put in series with the $50.0\ \Omega$ resistor and the 60.0 V battery.

SKETCH:



RESEARCH: Resistors in series combine as $R_{eq} = \sum_{i=1}^{n} R_i$. Resistors in parallel combine as $\frac{1}{R_{eq}} = \sum_{i=1}^{n} \frac{1}{R_i}$.

SIMPLIFY: The combined resistors in parallel become R_P where

$$R_P^{-1} = \left(R_1 + R_2\right)^{-1} + R_3^{-1} + R_4^{-1} \Rightarrow \mathrm{R_P} = \left(\frac{1}{R_1 + R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)^{-1}.$$

The equivalent resistance is $R_{eq} = R_5 + R_P$.

CALCULATE: $\mathrm{R_P} = \left(\frac{1}{10.0\ \Omega + 20.0\ \Omega} + \frac{1}{30.0\ \Omega} + \frac{1}{40.0\ \Omega}\right)^{-1} = 10.909091\ \Omega$

$R_{eq} = 50.0\ \Omega + 10.909091\ \Omega = 60.909091\ \Omega$

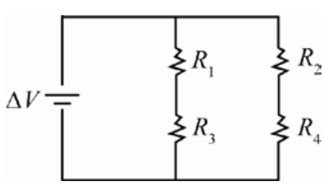
ROUND: The result should be rounded to three significant figures: $R_{eq} = 60.9\ \Omega$.

DOUBLE-CHECK: If you add N equal resistors, R, in parallel, the equivalent resistance is R/N. Since the resistors in parallel are all about $30\ \Omega$ in each branch, the equivalent resistance should be about $10\ \Omega$, which is close to the calculated answer of $11\ \Omega$.. Therefore, the values of R_P and R_{eq} are reasonable.

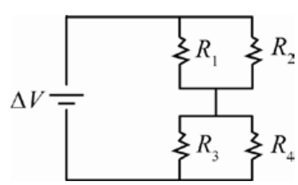
25.50. **THINK:** When the switch is open, the current clearly breaks up into paths and the two left resistors, $R_1 = R_3 = 3.00\ \Omega$, are in parallel with the two resistors, $R_2 = 5.00\ \Omega$ and $R_4 = 1.00\ \Omega$. When the switch is closed, it is not as obvious. Consider the potential drop from before the current splits to the switch arm. It should be the same regardless of which path is taken, likewise with the potential drop from the switch arm to after the current recombines. This means that the pairs of resistors R_1 and R_2, and R_3 and R_4, are connected in parallel. The pairs are subsequently connected in series with each other.

SKETCH:

(a)



(b)



RESEARCH: Resistors in series combine as $R_{eq} = \sum_{i=1}^{n} R_i$. Resistors in parallel combine as $\frac{1}{R_{eq}} = \sum_{i=1}^{n} \frac{1}{R_i}$.

The current is given by Ohm's Law $i = \Delta V / R_{eq}$.

SIMPLIFY:

(a) Equivalent resistance is $R_{eq} = \left(1/\left(R_1 + R_3\right) + 1/\left(R_2 + R_4\right)\right)^{-1} \Rightarrow i = \Delta V / R_{eq}$.

(b) Equivalent resistance is $R_{eq} = \left(1/\left(R_1 + R_2\right)\right)^{-1} + \left(1/\left(R_3 + R_4\right)\right)^{-1} \Rightarrow i = \Delta V / R_{eq}$.

CALCULATE:

(a) $R_{eq} = \left(\dfrac{1}{3.00\ \Omega + 3.00\ \Omega} + \dfrac{1}{5.00\ \Omega + 1.00\ \Omega} \right)^{-1} = 3.00\ \Omega$; therefore, $i = \dfrac{24.0\text{ V}}{3.00\ \Omega} = 8.00\text{ A}.$

(b) $R_{eq} = \left(\dfrac{1}{3.00\ \Omega} + \dfrac{1}{5.00\ \Omega} \right)^{-1} + \left(\dfrac{1}{3.00\ \Omega} + \dfrac{1}{1.00\ \Omega} \right)^{-1} = 2.625\ \Omega$; therefore, $i = \dfrac{24\text{ V}}{2.625\ \Omega} = 9.1429\text{ A}.$

ROUND:

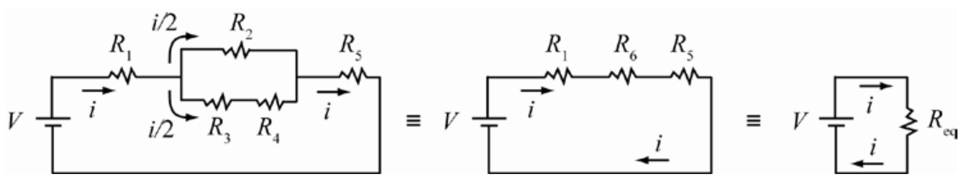
(a) $i = 8.00\text{ A}$

(b) $i = 9.14\text{ A}$

DOUBLE-CHECK: Typically, when resistors are in parallel, they have a lower equivalent resistance than when in series and thus would yield a larger current, so it makes sense.

25.51. **THINK:** The circuit can be redrawn to have $R_3 = 2.00\ \Omega$ and $R_4 = 4.00\ \Omega$ in series, which are then parallel to $R_2 = 6.00\ \Omega$. These are then in series with $R_1 = 6.00\ \Omega$, $R_5 = 3.00\ \Omega$, and the battery $V = 12.0\text{ V}$. Since R_5 is in series with the equivalent resistance, the current through it is the same as the current through the whole circuit. Since $R_2 = R_3 + R_4$, the current through each branch is equal, and half of the total current.

SKETCH:



RESEARCH: Resistors in series $R_{eq} = R_1 + R_2$. Resistors in parallel $R_{eq} = \left(1/R_1 + 1/R_2\right)^{-1}$. The current is given by Ohm's Law $i = \Delta V / R_{eq}$.

SIMPLIFY:

(a) The resistors in parallel combine as R_6, where $R_6 = \left(\dfrac{1}{R_2} + \dfrac{1}{R_3 + R_4} \right)^{-1}.$

The total equivalent resistance is $R_{eq} = R_1 + R_6 + R_5$.

(b) The current through R_5 is $i = V / R_{eq}$.

(c) The current through each branch is $i/2$, so potential across R_3 is $\Delta V_3 = \dfrac{1}{2} i R_3$.

CALCULATE:

(a) $R_6 = \left[\dfrac{1}{6.00\ \Omega} + \dfrac{1}{(2.00 + 4.00)\ \Omega} \right]^{-1} = 3.00\ \Omega.$ $R_{eq} = 6.00\ \Omega + 3.00\ \Omega + 3.00\ \Omega = 12.00\ \Omega$

(b) $i = (12.0\text{ V})/(12.0\ \Omega) = 1.00\text{ A}$

(c) $\Delta V_3 = \Delta V = \dfrac{(1.00\text{ A})(2.00\ \Omega)}{2} = 1.00\text{ V}$

ROUND:

(a) $R_{eq} = 12.00\ \Omega$

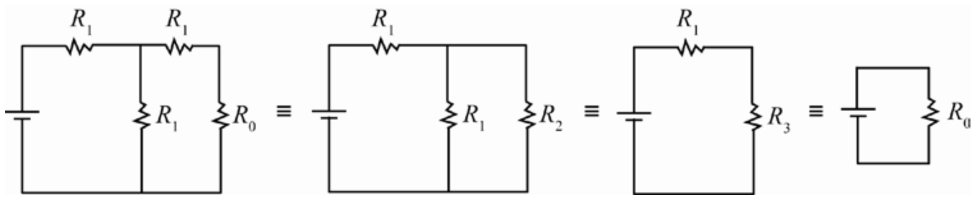
(b) $i = 1.00\text{ A}$

(c) $\Delta V_3 = 1.00\text{ V}$

DOUBLE-CHECK: By considering the potential drop across $R_1, R_2,$ and R_5, the values are: $\Delta V_1 = 6\text{ V}$, $\Delta V_2 = 3\text{ V}$, and $\Delta V_5 = 3\text{ V}$. Hence, $\Delta V_1 + \Delta V_2 + \Delta V_5 = 12\text{ V}$. This value matches the total voltage provided by the battery. Using $\Delta V_3 + \Delta V_4$ instead of ΔV_2 also gives 12 V, as expected.

25.52. **THINK:** The circuit can be redrawn to have R_0 and R_1 in series, which are then parallel to R_1. These are then in series with R_1.

SKETCH:



RESEARCH: Resistors in series combine as $R_{eq} = \sum_{i=1}^{n} R_i$. Resistors in parallel combine as $\frac{1}{R_{eq}} = \sum_{i=1}^{n} \frac{1}{R_i}$.

SIMPLIFY: Resistance R_2 is $R_2 = R_1 + R_0$. Resistance R_3 is $R_3 = \left(\frac{1}{R_1} + \frac{1}{R_1 + R_0}\right)^{-1}$. Equivalent resistance R_0 is $R_0 = R_1 + R_3 = R_1 + \left(\frac{1}{R_1} + \frac{1}{R_1 + R_0}\right)^{-1}$. Thus, $R_0 - R_1 = \frac{1}{R_1} + \frac{1}{R_0 + R_1} = \left(\frac{R_1 + R_0 + R_1}{R_1(R_1 + R_0)}\right)^{-1}$.

$$R_0 - R_1 = \frac{R_1^2 + R_1R_0}{2R_1 + R_0} \Rightarrow (R_0 - R_1)(2R_1 + R_0) = R_1^2 + R_1R_0 \Rightarrow R_0^2 + R_1R_0 - 2R_1^2 = R_1^2 + R_1R_0$$

$$R_0^2 = 3R_1^2 \Rightarrow R_1 = \frac{R_0}{\sqrt{3}}.$$

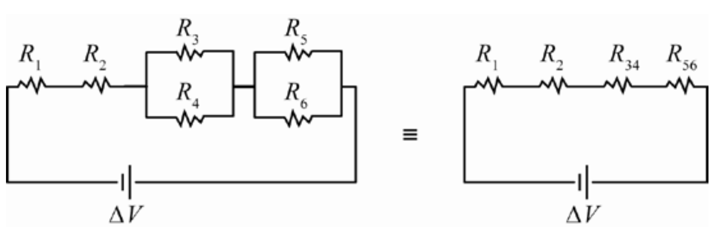
CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: If $R_1 = R_0$, the total equivalent résistance would be $5R_0/3$. If $R_1 = R_0/2$, the total equivalent resistance would be $7R_0/8$. Therefore, for $R_{eq} = R_0$, $R_0/2 < R_1 < R_0$. The answer satisfies this condition.

25.53. **THINK:** From the circuit, it is clear that resistors $R_1 = 5.00\ \Omega$ and $R_2 = 10.00\ \Omega$ are in series. Resistors $R_3 = R_4 = 5.00\ \Omega$ are in parallel, with equivalent resistance R_{34}. This is also true of resistors $R_5 = R_6 = 2.00\ \Omega$, whose equivalent resistance is R_{56}. This second pair of resistors are in turn connected in series with resistors R_1 and R_2. Ohm's Law can be used to determine the current through the whole circuit which is the same as each resistor in series.

SKETCH:



RESEARCH: Equivalent resistances if resistors are in parallel are $R_{34} = (1/R_3 - 1/R_4)^{-1}$. Total current through 4 resistors in series is $i = \Delta V/(R_1 + R_2 + R_{34} + R_{56})$. The potential drop across a resistor is $\Delta V_i = iR_i$.

SIMPLIFY:

(a) The total current is $i = \Delta V/(R_1 + R_2 + R_{34} + R_{56})$, $\Delta V_1 = iR_1$, $\Delta V_2 = iR_2$, $\Delta V_3 = \Delta V_4 = iR_{34}$ and $\Delta V_5 = \Delta V_6 = iR_{56}$.

(b) Current through R_1 and R_2 is i. Since $R_3 = R_4$ and $R_5 = R_6$, the current splits evenly among them so $i' = i/2$ through each of them.

CALCULATE:

(a) $R_{34} = \left(1/(5.00\ \Omega) - 1/(5.00\ \Omega)\right)^{-1} = 2.50\ \Omega$ $R_{56} = \left(1/(2.00\ \Omega) - 1/(2.00\ \Omega)\right)^{-1} = 1.00\ \Omega$,

$i = 20.0\text{ V}/(5.00\ \Omega + 10.00\ \Omega + 2.50\ \Omega + 1.00\ \Omega) = 1.08108\text{ A}$, $\Delta V_1 = (1.08108\text{ A})(5.00\ \Omega) = 5.405\text{ V}$,

$\Delta V_2 = (1.08108\text{ A})(10.0\ \Omega) = 10.81\text{ V}$, $\Delta V_3 = \Delta V_4 = (1.08108\text{ A})(2.50\ \Omega) = 2.702\text{ V}$ and

$\Delta V_5 = \Delta V_6 = (1.08108\text{ A})(1.00\ \Omega) = 1.081\text{ V}$

(b) $i_1 = i_2 = 1.08108\text{ A}$, $i_3 = i_4 = i_5 = i_6 = 1.08108/2 = 0.5405\text{ A}$

ROUND:

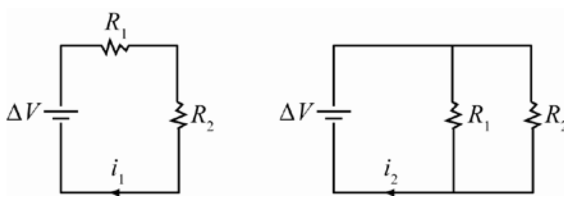
(a) $\Delta V_1 = 5.41\text{ V}$, $\Delta V_2 = 10.8\text{ V}$, $\Delta V_3 = \Delta V_4 = 2.70\text{ V}$ and $\Delta V_5 = \Delta V_6 = 1.08\text{ V}$.

(b) $i_1 = i_2 = 1.08\text{ A}$ and $i_3 = i_4 = i_5 = i_6 = 0.541\text{ A}$.

DOUBLE-CHECK: The sum of the four potential drops equals 20 V, so energy is conserved, so the answers make sense.

25.54. **THINK:** Ohm's law can be used to relate the potential drop, $\Delta V = 40.0\text{ V}$, to current $i_1 = 10.0\text{ A}$ when resistors R_1 and R_2 are in series and to current $i_2 = 50.0\text{ A}$ when the resistors are in parallel. This means there are two equations (the series and parallel configurations) and two unknowns (the resistors). Let R_1 be the one that is larger, since the choice is arbitrary, and solve for it.

SKETCH:



RESEARCH: For the series setup, $\Delta V = i_1(R_1 + R_2)$. For the parallel setup, $\Delta V = i_2(1/R_1 + 1/R_2)^{-1}$.

SIMPLIFY: The second resistor is, from series setup, $\Delta V = i_1(R_1 + R_2) \Rightarrow R_2 = (\Delta V/i_1) - R_1$ and $\dfrac{\Delta V}{i} = R_1 + R_2$. From the parallel setup, $\Delta V = i_2\left(\dfrac{R_1 R_2}{R_1 + R_2}\right) = \dfrac{\Delta V}{i_2}(R_1 + R_2) = R_1\left(\dfrac{\Delta V}{i_2} - R_1\right)$. Therefore,

$$\frac{\Delta V^2}{i_1 i_2} = \frac{\Delta V}{i_1} R_1 - R_1^2 \Rightarrow P = SR_1 - R_1^2.$$

With $P = \dfrac{\Delta V^2}{i_1 i_2}$, and $S = \dfrac{\Delta V}{i_1}$, $R_1^2 - SR_1 + P = 0 \Rightarrow R_1 = \dfrac{S \pm \sqrt{S^2 - 4P}}{2}$. R_1 must be positive to get the largest possible value, so $R_1 = \dfrac{S + \sqrt{S^2 - 4P}}{2}$.

CALCULATE: $P = \dfrac{(40.0\text{ V})^2}{(10.0\text{ A})(50.0\text{ A})} = 3.2\ \Omega^2$, and $S = \dfrac{40.0\text{ V}}{10.0\text{ A}} = 4\ \Omega$. Therefore,

$$R_1 = \frac{(4\ \Omega) + \sqrt{(4\ \Omega)^2 - 4(3.2\ \Omega^2)}}{2} = 2.8944\ \Omega.$$

ROUND: $R_1 = 2.89\ \Omega$

DOUBLE-CHECK: The other value for R_1, the negative in the quadratic, gives $R_1 = 1.11\ \Omega$, so R_{eq} in series and parallel is $4\ \Omega$ and $0.8\ \Omega$, respectively. $(4\ \Omega)(10.0\text{ A}) = 40\text{ V}$ and $(0.8\ \Omega)(50.0\text{ A}) = 40\text{ V}$, which is consistent with emf voltage.

25.55. The voltage changes, from $\Delta V_0 = 110.\text{ V}$ to $\Delta V_1 = 150.\text{ V}$, and the initial power is $P_0 = 100.\text{ W}$. Since the resistance does not change, the power is, generally, $P = \Delta V^2 / R$, so the fractional change in power is $\%P = \dfrac{P_1 - P_0}{P_0} = \dfrac{\Delta V_1^2 / R - \Delta V_0^2 / R}{\Delta V_0^2 / R} = \dfrac{\Delta V_1^2}{\Delta V_0^2} - 1 = \dfrac{(150\text{ V})^2}{(110\text{ V})^2} - 1 = 0.8595.$ Therefore, $\%P = 86.0\%$ (brighter).

25.56. (a) The average current, i, is simply the change in charge, $\Delta Q = 5.00\text{ C}$, over change in time, $\Delta t = 0.100\text{ ms}$. $i = \dfrac{\Delta Q}{\Delta t} = \dfrac{(5.00\text{ C})}{(0.100 \cdot 10^{-3}\text{ s})} = 50.0\text{ kA}.$

(b) If over the lightning bolt there is a $V = 70.0\text{ MV}$ potential, the power is $P = iV = (50.0\text{ kA})(70.0\text{ MV}) = 3.50 \cdot 10^{12}\text{ W}$

(c) The energy is the power times the change in time, $E = P\Delta t = (3.50 \cdot 10^{12}\text{ W})(0.100\text{ ms}) = 3.50 \cdot 10^{8}\text{ J}.$

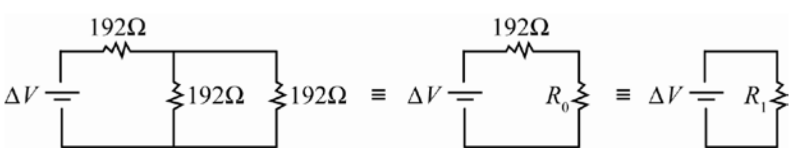
(d) Assuming the lightning obeys Ohm's Law, the resistance is $R = \Delta V / i = \dfrac{70.0\text{ MV}}{50.0\text{ kA}} = 1.40 \cdot 10^{3}\ \Omega.$

25.57. (a) If the hair dryer has power $P = 1600.\text{ W}$ and requires a potential of $V = 110.\text{ V}$, the current supplied is then $i = P / V = 1600.\text{ W} / 110.\text{ V} = 14.545\text{ A} = 14.5\text{ A}$. i does not exceed 15.0 A, so it will not trip the circuit.

(b) Assuming the hair dryer obeys Ohm's Law, its effective resistance is given by $R = \Delta V / i = (110.\text{ V}) / (14.545\text{ A}) = 7.56\ \Omega.$

25.58. For a year of use, the time it is active is $\Delta t = (1\text{ year})(365\text{ days / year})(24\text{ hours / day}) = 8760\text{ h}$. The power of a regular light bulb is $P = 100.00\text{ W} = 0.10000\text{ kW}$. The power of the fluorescent bulb is $P_F = 26.000\text{ W} = 0.026000\text{ kW}$. Since it costs \$0.12 / kWh to have each on, the cost of running each is: $C = (\$0.12/\text{kWh})(0.10000\text{ kW})(8760\text{ h}) = \$105.12,\ C_F = (\$0.12/\text{kWh})(0.026\text{ kW})(8760\text{ h}) = \$27.33.$

25.59. To find the current through each, reduce circuit to



(a) The two 192 Ω resistors in parallel have an equivalent resistance given by

$$\frac{1}{R_0} = \frac{1}{192\ \Omega} + \frac{1}{192\ \Omega} = \frac{2}{192\ \Omega} \quad \Rightarrow \quad R_0 = 96.0\ \Omega.$$

The resistance R_0 is is series with another 192 Ω resistor. This system has the equivalent resistance given by

$$R_1 = R_0 + 192\ \Omega = 96.0\ \Omega + 192\ \Omega = 288\ \Omega.$$

The total power in the circuit is given by

$$P = \frac{(\Delta V)^2}{R_1} = \frac{(120.\text{ V})^2}{288\ \Omega} = 50.0\text{W}.$$

(b) The current supplied by the emf source is given by

$$\Delta V = iR_1 \quad \Rightarrow \quad i = \frac{\Delta V}{R_1} = \frac{120.\text{ V}}{288\ \Omega} = 0.4167\text{ A}.$$

This current flows through the first resistor. So the potential drop across the first resistor is

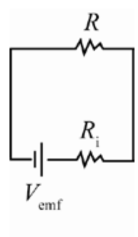
$$\Delta V_1 = (0.4167\text{ A})(192\ \Omega) = 80.0\text{ V}.$$

The remaining two resistors are in parallel, so the potential drop across each of these two resistors must sum to the potential difference of the source of emf
$\Delta V_2 = \Delta V_3 = 120.\text{ V} - 80.0\text{ V} = 40.0\text{ V}.$

25.60. The current through the light bulb is $i = P/V$. The charge through the bulb is $q = i\Delta t$, so

$$q = i\Delta t = \frac{P\Delta t}{\Delta V} \Rightarrow \Delta t = \frac{q\Delta V}{P}\frac{(625\text{ mAh})\left(\frac{1\text{ A}}{1000\text{ mA}}\right)\left(\frac{60\text{ min}}{\text{h}}\right)(1.5\text{ V})}{5.0\text{ W}} = 11\text{ min}.$$

25.61. **THINK:** The overall current through the resistor, R (which takes the values $1.00\ \Omega$, $2.00\ \Omega$ and $3.00\ \Omega$), is found using Ohm's Law for when the load resistance is in series with the internal resistance, $R_\text{i} = 2.00\ \Omega$, and the external emf, $V_\text{emf} = 12.0\text{ V}$. I will determine an expression for the power across the load resistor and differentiate with respect to *R*, and solve this derivative equal to zero, in order to find a maximum in power.
SKETCH:



RESEARCH: With R and R_i in series, current through circuit is $i = V_\text{emf}/(R_\text{i} + R)$. Power through load resistor is $P = i^2 R$. The power is maximized when $dP/dR = 0$ and $d^2P/dR^2 < 0$.

SIMPLIFY: Power is $P = \left(\frac{V_\text{emf}}{R_\text{i} + R}\right)^2 R = \frac{V_\text{emf}^2}{(R_\text{i} + R)^2} R.$ Therefore,

$$\frac{dP}{dR} = V_\text{emf}^2 \frac{d}{dR}\left[R(R_\text{i} + R)^{-2}\right] = V_\text{emf}^2\left((R_\text{i} + R)^{-2} - 2(R)(R_\text{i} + R)^{-3}\right)$$

$$= V_\text{emf}^2\left(\frac{R_\text{i} + R}{(R_\text{i} + R)^3} - \frac{2R}{(R_\text{i} + R)^3}\right) = V_\text{emf}^2\left(\frac{R_\text{i} - R}{(R_\text{i} + R)^3}\right) = 0,$$

and hence $R = R_\text{i}$ is a critical point of *P*. The double-check step will verify that $R = R_\text{i}$ leads to a maximum.

CALCULATE: $P_1 = \frac{(12.0\text{ V})^2(1.00\ \Omega)}{(1.00\ \Omega + 2.00\ \Omega)^2} = 16.0\text{ W},$ $P_2 = \frac{(12.0\text{ V})^2(2.00\ \Omega)}{(2.00\ \Omega + 2.00\ \Omega)^2} = 18.0\text{ W}$ and

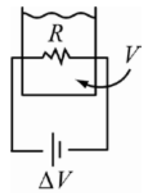
$$P_3 = \frac{(12.0\text{ V})^2(3.00\ \Omega)}{(3.00\ \Omega + 2.00\ \Omega)^2} = 17.28\text{ W}.$$

ROUND: The values should be rounded to three significant figures each: $P_1 = 16.0\text{ W}$, $P_2 = 18.0\text{ W}$ and $P_3 = 17.3\text{ W}$.

DOUBLE-CHECK: The second derivative of *P*, $\frac{d^2P}{dR^2} = 2\frac{V_\text{emf}{}^2(R - 2R_\text{i})}{(R_i + R)^4}$, is clearly negative when $R = R_\text{i}$, which verifies that $R = R_\text{i}$ yields a maximum for *P*.

25.62. **THINK:** Using the density and volume of water, $\rho = 1000\text{ kg/m}^3$ and $V = 250\text{ mL}$, the mass of the water can be determined. Along with the specific heat of water, $c = 4.186\text{ kJ/(kg K)}$, and the fact that the water goes from $T_\text{i} = 20\ ^\circ\text{C} = 293\text{ K}$ to $T_\text{f} = 100\ ^\circ\text{C} = 373\text{ K}$ the energy gained by the water can be determined.

The change in energy over the time $\Delta t = 45$ s is equal to the power that the coil, $\Delta V = 15$ V, dissipates.
SKETCH:



RESEARCH: Power dissipated by coil is $P = \Delta V^2 / R$. The energy gained by heating water, $\Delta Q = mc\Delta T$. The rate of energy gained by water is $P = \Delta Q / \Delta t$, mass of water is $m = \rho V$.
SIMPLIFY: Equating power dissipated by coil to energy rate gained by water gives the equation:

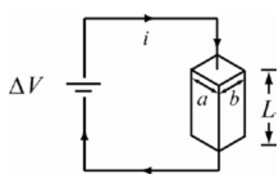
$$P = \frac{\Delta V^2}{R} = \frac{\Delta Q}{\Delta t} = \frac{mc\Delta T}{\Delta t} = \frac{\rho V\mathrm{c}\left(T_\mathrm{f} - T_\mathrm{i}\right)}{\Delta t}.$$

Therefore, $R = \dfrac{\Delta V^2 \Delta t}{\rho Vc\left(T_\mathrm{f} - T_\mathrm{i}\right)}$.

CALCULATE: $R = \dfrac{(15 \text{ V})^2 (45 \text{ s})}{\left(1000 \text{ kg/m}^3\right)(0.25 \text{ L})(4186 \text{ J/kg K})(373 \text{ K} - 293 \text{ K})\left(1 \text{ m}^3/1000 \text{ L}\right)} = 0.1209\ \Omega$

ROUND: The values in the question have two significant figures, so round the answer to $R = 1.2$ m Ω.
DOUBLE-CHECK: Since power is inversely proportional to resistance, the optimal way to heat using it would be to make the resistance as small as possible, so it makes sense.

25.63. **THINK:** Both the copper wire length ($l = 75.0$ cm, diameter $d = 0.500$ mm and resistivity $\rho_\mathrm{C} = 1.69 \cdot 10^{-8}\ \Omega$ m) and silicon block (length $L = 15.0$ cm, width $a = 2.00$ mm, thickness b, resistivity $\rho_\mathrm{S} = 8.70 \cdot 10^{-4}\ \Omega$ m and resistance $R_\mathrm{S} = 50.0\ \Omega$) can be thought of as resistors in series with a total potential drop of $\Delta V = 0.500$ V and a density of charge carriers of $1.23 \cdot 10^{23}$ m^{-3}. The current density, J, can be used to determine the velocity of the carriers through the silicon.
SKETCH:



RESEARCH: The resistance of the material is in general $R = \rho L / A$. The current through circuit is found by Ohm's Law with copper wire and silicon block in series, $\Delta V = i\left(R_\mathrm{C} - R_\mathrm{S}\right)$. Area of the silicon block is $A = ab$. The current density is $J = i / A = nev_\mathrm{d}$. The time it takes to pass through silicon block is $\Delta t = L / v_\mathrm{d}$. Power dissipated by silicon block is $P_\mathrm{S} = i^2 R_\mathrm{S}$.
SIMPLIFY:

(a) Resistance of wire is $R_\mathrm{C} = \dfrac{\rho_\mathrm{C} l}{(1/4)\pi d^2}$.

(b) Current through circuit is $i = \dfrac{\Delta V}{R_\mathrm{C} + R_\mathrm{S}}$.

(c) Thickness of silicon block $R_\mathrm{S} = \dfrac{\rho_\mathrm{S} L}{ab} \Rightarrow b = \dfrac{\rho_\mathrm{S} L}{a R_\mathrm{S}}$.

(d) Drift velocity of electrons $J = \frac{i}{A} = nev_d \;\Rightarrow\; \text{v}_d = \frac{i}{abne}$. So the time to cross the block is

$$\Delta t = \frac{L}{v_\text{d}} = \frac{Labne}{i}.$$

(e) $P_\text{S} = i^2 R_\text{S}$

(f) Electric power is lost via heat.

CALCULATE:

(a) $R_\text{C} = \dfrac{(1.69 \cdot 10^{-8}\ \Omega\ \text{m})(75.0\ \text{cm})}{(1/4)\pi(0.500\ \text{mm})^2} = 0.064553\ \Omega = 64.553\ \text{m}\ \Omega$

(b) $i = \dfrac{0.500\ \text{V}}{50.0\ \Omega + 0.064533\ \Omega} = 0.009987\ \text{A} = 9.987\ \text{mA}$

(c) $b = \dfrac{(8.70 \cdot 10^{-4}\ \Omega\text{m})(15.0\ \text{cm})}{(2.00\ \text{mm})(50.0\ \Omega)} = 0.001305\ \text{m} = 1.305\ \text{mm}$

(d) $\Delta t = \dfrac{(15.0\ \text{cm})(2.00\ \text{mm})(1.305\ \text{mm})(1.23 \cdot 10^{23}\ \text{m}^{-3})(1.602 \cdot 10^{-19}\ \text{C})}{9.987\ \text{mA}} = 0.77243\ \text{s}$

(e) $P_\text{S} = (9.987\ \text{mA})^2 (50.0\ \Omega) = 0.004987\ \text{W} = 4.987\ \text{mW}$

ROUND:

(a) $R_\text{C} = 64.6\ \text{m}\ \Omega$
(b) $i = 9.99\ \text{mA}$
(c) $b = 1.31\ \text{mm}$
(d) $\Delta t = 0.772\ \text{s}$
(e) $P_\text{S} = 4.99\ \text{mW}$

DOUBLE-CHECK: Drift velocity is $L / \Delta t \approx 20\ \text{cm/s}$ which is reasonable for such a small resistance. Also, the power lost is small, which is desirable in silicon, which is used in many electronic devices, so it makes sense.

25.64. Electrical power is defined as $P = i^2 R$ or $P = \Delta V^2 / R$ or $P = i\Delta V$. In the normal operation, the radio has a resistance of $r = \Delta V^2 / P = (10.0\ \text{V})^2 / (30.0\ \text{W}) = 3.33\ \Omega$ and the current flowing through the radio is $i = \frac{P}{\Delta V} = \frac{30.0\ \text{W}}{10.0\ \text{V}} = 3.00\ \text{A}$. Now, if a $25.0\ \text{kV}$ power supply is used, the required total resistance, such that the current flowing through the radio is the same, is $R_\text{T} = \frac{\Delta V}{i} = \frac{25.0\ \text{kV}}{3.0\ \text{A}} = 8333.33\ \Omega$. Thus, the external resistance required is $R = R_\text{T} - r$. The closest number of resistors is

$$N = \frac{R}{R_1} = \frac{R_\text{T} - r}{R_1} = \frac{8333.33\ \Omega - 3.33\ \Omega}{25\ \Omega} = 333.2 \approx 333\ \text{resistors}.$$

All resistors are connected in series, since the potential drop across the resistors makes up for the rest of the enormous potential drop provided by the power supply.

25.65. It is given $\Delta V = 120\ \text{V}$, $\Delta t = 2.0\ \text{min} = 120\ \text{s}$ and $U_1 = 48\ \text{kJ}$. The power needed to cook one hot dog is $P_1 = U_1 / \Delta t = 4.8 \cdot 10^4\ \text{J} / 120\ \text{s} = 4.0 \cdot 10^2\ \text{W}$. The current to produces this power is $i_1 = P_1 / \Delta V = 4.0 \cdot 10^2\ \text{W} / 120\ \text{V} = 3.3\ \text{A}$. The current to cook three hot dogs is $i = 3i_1 = 3(3.3\ \text{A}) = 10.\ \text{A}$.

25.66. The aluminum wire and the copper wire dissipate the same power. Since the voltages across the wires are the same, this means that the resistances of the wires are the same. That is $R_{\text{Al}} = R_{\text{Cu}}$, $\frac{\rho_{\text{Al}} L_{\text{Al}}}{A_{\text{Al}}} = \frac{\rho_{\text{Cu}} L_{\text{Cu}}}{A_{\text{Cu}}}$. Using the area of a circle $A = \pi r^2$, it is found that $\frac{r_{\text{Al}}^2}{r_{\text{Cu}}^2} = \frac{\rho_{\text{Al}} L_{\text{Al}}}{\rho_{\text{Cu}} L_{\text{Cu}}}$. Thus, $r_{\text{Al}} = r_{\text{Cu}} \sqrt{\frac{\rho_{\text{Al}} L_{\text{Al}}}{\rho_{\text{Cu}} L_{\text{Cu}}}}$. Substituting the numerical values:

$$r_{\text{Al}} = (1.00 \text{ mm}) \sqrt{\frac{(2.82 \cdot 10^{-8} \ \Omega \text{ m})(5.00 \text{ m})}{(1.72 \cdot 10^{-8} \ \Omega \text{ m})(10.0 \text{ m})}} = 0.905 \text{ mm}.$$

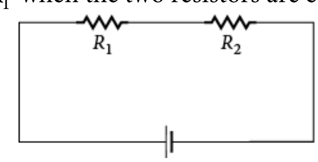
25.67. The resistance of a cylindrical wire is $R = \rho L / A$. The length of the resistor is $L = RA / \rho$. Substituting the numerical values yields $L = (10.0 \ \Omega)(1.00 \cdot 10^{-6} \text{ m}^2)/(1.00 \cdot 10^{-5} \ \Omega \text{ m}) = 1.00$ m.

25.68. Two resistive cylindrical wires of identical length are made of copper and aluminum. They carry the same current and have the same potential difference across their length. This means that they have the same resistance, that is, $R_{\text{Cu}} = R_{\text{Al}}$, $\frac{\rho_{\text{Al}} L_{\text{Al}}}{A_{\text{Al}}} = \frac{\rho_{\text{Cu}} L_{\text{Cu}}}{v}$. Since $L_{\text{Al}} = L_{\text{Cu}}$ and $A_{\text{Al}} = \pi r_{\text{Al}}^2$ and $A_{\text{Cu}} = \pi r_{\text{Cu}}^2$, it becomes $\frac{r_{\text{Cu}}^2}{r_{\text{Al}}^2} = \frac{\rho_{\text{Cu}}}{\rho_{\text{Al}}}$ or $\frac{r_{\text{Cu}}}{r_{\text{Al}}} = \sqrt{\frac{\rho_{\text{Cu}}}{\rho_{\text{Al}}}}$. Therefore, the ratio of their radii is

$$\frac{r_{\text{Cu}}}{r_{\text{Al}}} = \sqrt{\frac{1.72 \cdot 10^{-8} \ \Omega \text{ m}}{2.82 \cdot 10^{-8} \ \Omega \text{ m}}} = 0.781.$$

25.69. Consider a circuit with $R_1 = 200. \ \Omega$ and $R_2 = 400. \ \Omega$.

(a) What is the power dissipated in R_1 when the two resistors are connected in series?



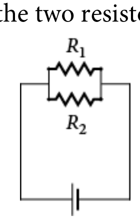
The current in the circuit is given by

$$\Delta V = iR_{\text{eq}} \quad \Rightarrow \quad i = \frac{\Delta V}{R_{\text{eq}}} = \frac{\Delta V}{R_1 + R_2}.$$

The power dissipated in R_1 is then

$$P = i^2 R_1 = \left(\frac{\Delta V}{R_1 + R_2} \right)^2 R_1 = \frac{R_1 (\Delta V)^2}{(R_1 + R_2)^2} = \frac{(200. \ \Omega)(9.00 \text{ V})^2}{(200. \ \Omega + 400. \ \Omega)^2} = 0.0450 \text{ W}.$$

(b) What is the power dissipated in R_1 when the two resistors are connected in parallel?



The potential difference across R_1 is 9.00 V so the power dissipated in this case is

$$P = \frac{(\Delta V)^2}{R_1} = \frac{(9.00 \text{ V})^2}{200. \ \Omega} = 0.405 \text{ W}.$$

The ratio of the power delivered to the $200.\ \Omega$ resistor by the 9.00 V battery when the resistors are connected in parallel to the power delivered when connected in series is

$$\frac{P_{\text{parallel}}}{P_{\text{series}}}=\frac{\dfrac{(\Delta V)^2}{R_1}}{\dfrac{R_1(\Delta V)^2}{(R_1+R_2)^2}}=\frac{(R_1+R_2)^2}{R_1^2}=\left(\frac{R_1+R_2}{R_1}\right)^2=\left(\frac{200.\ \Omega+400.\ \Omega}{200.\ \Omega}\right)^2=9.00.$$

25.70. The conductance of a wire is given by $G=\dfrac{1}{R}=\dfrac{A}{\rho L}.$

(a) From the electrical power $P=\dfrac{\Delta V^2}{R}$, the conductance of the element is

$$G=\frac{P}{\Delta V^2}=\frac{1500.\ \text{W}}{(110.\ \text{V})^2}=0.124\ \Omega^{-1}.$$

(b) Using $A=\pi r^2$, the radius of the wire is $r^2=\dfrac{\rho LG}{\pi}$ or $r=\sqrt{\dfrac{\rho LG}{\pi}}$. Substituting $\rho=9.7\cdot 10^{-8}\ \Omega$ m, $L=3.5$ m and $G=0.124\ \Omega^{-1}$ gives

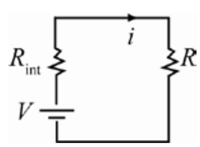
$$r=\sqrt{\frac{(9.7\cdot 10^{-8}\ \Omega\ \text{m})(3.50\ \text{m})(0.124\ \Omega^{-1})}{\pi}}=0.116\ \text{mm}.$$

25.71. The resistance of the light bulb is $R=\Delta V_1^2/P_1$. Consider each value in the problem to have three significant figures. The power consumed by the bulb in a US household is

$$P_2=\frac{\Delta V_2^2}{R}=\left(\frac{\Delta V_2}{\Delta V_1}\right)^2 P_1=\left(\frac{120.\ \text{V}}{240.\ \text{V}}\right)^2(100.\ \text{W})=25.0\ \text{W}.$$

25.72. (a) The minimum overall resistance is $R=\Delta V/i=115\ \text{V}/200.\ \text{A}=0.575\ \Omega.$
(b) The maximum electrical power is $P=i\Delta V=(200.\ \text{A})(115\ \text{V})=23.0$ kW.

25.73. **THINK:** A battery with emf 12.0 V and internal resistance $R_i=4.00\ \Omega$ is attached across an external resistor of resistance R. The maximum power that can be delivered to the resistor R is required.
SKETCH:



RESEARCH: The power delivered to the resistor R is given by $P=i^2R$. The current flowing through the circuit is $i=\Delta V/(R+R_i)$. Therefore, the power is $P=\Delta V^2R/(R+R_i)^2$. The maximum power delivered to the resistor R is given when R satisfies $dP/dR=0$. That is

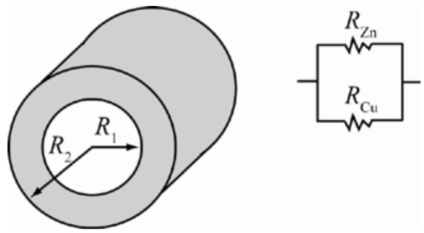
$$\frac{dP}{dR}=\frac{\Delta V^2}{(R+R_i)^2}+\frac{\Delta V^2R(-2)}{(R+R_i)^3}=0.$$

SIMPLIFY: Solving the above equation for R yields $R+R_i-2R=0$ or $R=R_i$. Thus, the maximum power delivered to R is $P=\dfrac{\Delta V^2R_i}{(2R_i)^2}=\dfrac{\Delta V^2}{4R_i}.$

CALCULATE: Substituting the numerical values gives $P=\dfrac{(12.0\ \text{V})^2}{4(4.00\ \Omega)}=9.00$ W.

ROUND: $P = 9.00 \text{ W}$
DOUBLE-CHECK: This is a reasonable amount of power for a 12 V battery to supply.

25.74. **THINK:** Calculate the resistance of a 10.0 m length of multiclad wire consisting of a zinc core of radius 1.00 mm surrounded by a copper sheath of thickness 1.00 mm. The wire can be treated as two resistors in parallel.
SKETCH:



RESEARCH: The resistivity of zinc is $\rho_{\text{Zn}} = 5.964 \cdot 10^{-8} \; \Omega \text{ m}$, and the resistivity of copper is $\rho_{\text{Cu}} = 1.72 \cdot 10^{-8} \; \Omega \text{ m}$. Resistance is given by $R = \rho \dfrac{L}{A}$. The resistance of the zinc wire is therefore $R_{\text{Zn}} = \dfrac{\rho_{\text{Zn}} L}{\pi R_1^2}$, and the resistance of the hollow copper wire is $R_{\text{Cu}} = \dfrac{\rho_{\text{Cu}} L}{\pi R_2^2 - \pi R_1^2}$. The net resistance is $1/R = 1/R_{\text{Zn}} + 1/R_{\text{Cu}}$.

SIMPLIFY: $$R_{\text{eq}} = \left(\frac{1}{R_{\text{Zn}}} + \frac{1}{R_{\text{Cu}}} \right)^{-1} = \left(\frac{R_{\text{Cu}} + R_{\text{Zn}}}{R_{\text{Zn}} R_{\text{Cu}}} \right)^{-1} = \frac{R_{\text{Cu}} R_{\text{Zn}}}{R_{\text{Cu}} + R_{\text{Zn}}} = \frac{\dfrac{\rho_{\text{Cu}} L}{A_{\text{Cu}}} \cdot \dfrac{\rho_{\text{Zn}} L}{A_{\text{Zn}}}}{\dfrac{\rho_{\text{Cu}} L}{A_{\text{Cu}}} + \dfrac{\rho_{\text{Zn}} L}{A_{\text{Zn}}}}$$

$$R_{\text{eq}} = \frac{\dfrac{\rho_{\text{Zn}} L}{\pi r_1^2} \cdot \dfrac{\rho_{\text{Cu}} L}{\pi \left(R_2^2 - R_1^2 \right)}}{\dfrac{\rho_{\text{Cu}} L}{\pi \left(R_2^2 - R_1^2 \right)} + \dfrac{\rho_{\text{Zn}} L}{\pi R_1^2}} = \frac{L \left(\rho_{\text{Zn}} \rho_{\text{Cu}} \right)}{\pi \left(\rho_{\text{Cu}} r_1^2 + \rho_{\text{Zn}} \left(R_2^2 - R_1^2 \right) \right)}$$

CALCULATE:
$$R_{\text{eq}} = \frac{10.0 \text{ m} \left(1.72 \cdot 10^{-8} \; \Omega \text{ m} \right) \left(5.964 \cdot 10^{-8} \; \Omega \text{ m} \right)}{\pi \left(\left(1.72 \cdot 10^{-8} \text{ m} \right) \left(1.00 \cdot 10^{-3} \text{ m} \right)^2 + \left(5.964 \cdot 10^{-8} \; \Omega \text{ m} \right) \left(\left(2.00 \cdot 10^{-3} \text{ m} \right)^2 - \left(1.00 \cdot 10^{-3} \text{ m} \right)^2 \right) \right)}$$
$$= 0.01664925 \; \Omega$$
ROUND: Keeping only three significant digits gives $R = 0.0166 \; \Omega$.
DOUBLE-CHECK: The combined resistance of the components of the wire is less than the resistance of either material alone, as expected for resistances in parallel.

25.75. **THINK:** The Stanford Linear Accelerator accelerated a beam of $2.0 \cdot 10^{14}$ electrons per second through a potential difference of $2.0 \cdot 10^{10}$ V.
SKETCH: Not required.
RESEARCH: Electrical current is defined by $i = q/t$, i.e. the amount of charging passing per unit of time. The power in the beam is calculated by $P = i\Delta V$ and the effective ohmic resistance is $R = \Delta V / i$.
SIMPLIFY:
(a) The electrical current in the beam is $i = q/t = |e|(n/t)$.
(b) The power in the beam is $P = i\Delta V$.
(c) The effective resistance is $R = \Delta V / i$.

CALCULATE:

(a) $i = (1.602 \cdot 10^{-19}\text{ C})(2.0 \cdot 10^{14}\text{ electrons/second}) = 3.204 \cdot 10^{-5}\text{ A}.$

(b) $P = (3.204 \cdot 10^{-5}\text{ A})(2.0 \cdot 10^{10}\text{ V}) = 640.8\text{ kW}.$

(c) $R = \dfrac{2.0 \cdot 10^{10}\text{ V}}{3.204 \cdot 10^{-5}\text{ A}} = 6.2422 \cdot 10^{14}\ \Omega.$

ROUND: Keeping only two significant digits, the results become,

(a) $i = 3.2 \cdot 10^{-5}\text{ A}$

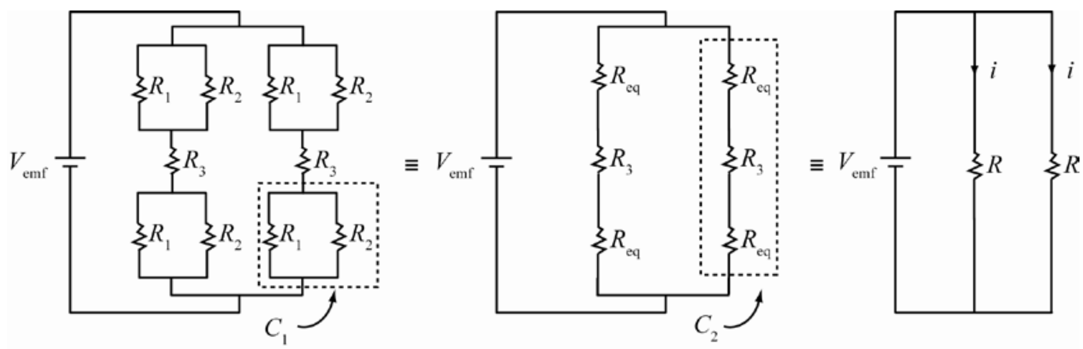
(b) $P = 640\text{ kW}$

(c) $R = 6.2 \cdot 10^{14}\ \Omega$

DOUBLE-CHECK: Such large values are reasonable for a device meant to accelerate particles to relativistic speeds.

25.76. **THINK:** To solve this problem, the circuit needs to be simplified by finding equivalent resistances. Use the relationships of parallel and series resistors.

SKETCH:



RESEARCH: If two resistors in series, the equivalent resistance is $R_{eq} = R_A + R_B$, and if two resistors in parallel, the equivalent resistances is $1/R_{eq} = 1/R_A + 1/R_B$ or $R_{eq} = R_A R_B / (R_A + R_B)$.

SIMPLIFY: The equivalent resistance of two resistors in the circuit C_1 is $R_{eq} = R_1 R_2 / (R_1 + R_2)$. The equivalent resistance of three resistors in the circuit C_2 is $R = R_{eq} + R_3 + R_{eq} = 2R_{eq} + R_3$ $= 2R_1 R_2 / (R_1 + R_2) + R_3$. Thus, the effective resistance is given by

$$\frac{1}{R_{eff}} = \frac{1}{R} + \frac{1}{R} \Rightarrow R_{eff} = \frac{1}{2}R = \frac{R_1 R_2}{R_1 + R_2} + \frac{1}{2}R_3.$$

The current flowing through R_3 is the same as current through R. Therefore, $i = \dfrac{V_{emf}}{R} = \dfrac{V_{emf}}{2R_{eff}}.$

CALCULATE:

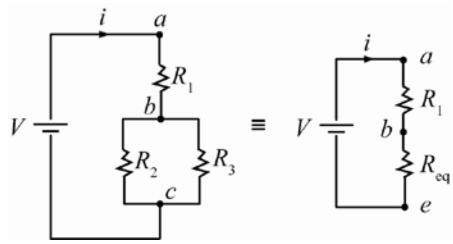
(a) $R_{eff} = \dfrac{(3.00\ \Omega)(6.00\ \Omega)}{3.00\ \Omega + 6.00\ \Omega} + \dfrac{20.0\ \Omega}{2} = 10.0\ \Omega$

(b) $i = \dfrac{12.0\text{ V}}{2(12.0\ \Omega)} = 0.500\text{ A}$

ROUND: Not required.

DOUBLE-CHECK: Both of the calculated values have appropriate units for what the values represent.

25.77. **THINK:** For resistors connected in parallel, the potential differences across the resistors are the same.
SKETCH:



RESEARCH: The equivalent resistance of two resistors in parallel $(R_2 \text{ and } R_3)$ is $R_{\text{eq}} = (R_2)(R_3)/(R_2 + R_3)$.

SIMPLIFY:

(a) The potential difference across R_3 is $V_{\text{bc}} = V_{\text{ac}} - V_{\text{ab}} = V - iR_1 = V\left(\dfrac{R_1}{R_1 + R_{\text{eq}}}\right)$.

(b) Since R_1 and R_{eq} are in series, the current flowing through R_1 and R_{eq} is $i = \dfrac{V}{R} = \dfrac{V}{R_1 + R_{\text{eq}}}$.

(c) The rate thermal energy dissipated from R_2 is $P = \dfrac{V_{\text{bc}}^2}{R_2}$.

CALCULATE: $R_{\text{eq}} = \dfrac{(3.00\ \Omega)(6.00\ \Omega)}{3.00\ \Omega + 6.00\ \Omega} = 2.00\ \Omega$

(a) The potential difference across R_3 is $V_{\text{bc}} = 110.\text{ V}\left(\dfrac{2.00\ \Omega}{2.00\ \Omega + 2.00\ \Omega}\right) = 55.0\text{ V}$.

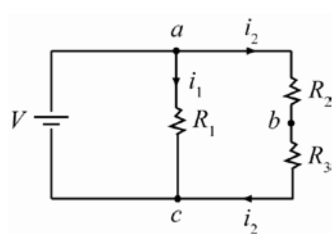
(b) The current through R_1 is $i = \dfrac{110.\text{ V}}{2.00\ \Omega + 2.00\ \Omega} = 27.5\text{ A}$.

(c) The thermal energy dissipated from R_2 is $P = \dfrac{(55.0\text{ V})^2}{3.00\ \Omega} = 1.008\text{ kW}$.

ROUND: Keeping three significant digits gives:
(a) $V_{\text{bc}} = 55.0\text{ V}$
(b) $i = 27.5\text{ A}$
(c) $P = 1.01\text{ kW}$

DOUBLE-CHECK: Each value has appropriate units for what is being measured.

25.78. **THINK:** When a potential difference V is applied across resistors connected in series, the resistors have identical currents. The potential difference across R_1 is $V_{\text{ac}} = V$.
SKETCH:



RESEARCH:

$V_{ac} = V_{ab} + V_{bc}$; The potential differences across R_1, R_2 and R_3 are $V_{ab} = i_2 R_2 = \dfrac{VR_2}{R_2 + R_3}$ and $V_{bc} = i_2 R_3 = \dfrac{VR_3}{R_2 + R_3}$, respectively. The currents are: $i_1 = \dfrac{V}{R_1}$ and $i_2 = i_3 = \dfrac{V}{R_2 + R_3}$.

SIMPLIFY: Not required.

CALCULATE: Substituting the values of the resistors and the potential difference across the battery yields

(a) $V_{ac} = 1.500 \text{ V}$, $V_{ab} = \dfrac{(1.500 \text{ V})(4.00\ \Omega)}{4.00\ \Omega + 6.00\ \Omega} = 0.600 \text{ V}$, and

$V_{bc} = V_{ac} - V_{ab} = 1.500 \text{ V} - 0.600 \text{ V} = 0.900 \text{ V}.$

(b) $i_1 = \dfrac{1.500 \text{ V}}{2.00\ \Omega} = 0.750 \text{ A}$ and $i_2 = i_3 = \dfrac{1.500 \text{ V}}{4.00\ \Omega + 6.00\ \Omega} = 0.150 \text{ A}.$

ROUND: Keeping three significant figures gives:

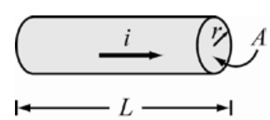
(a) $V_{ac} = 1.500 \text{ V}$, $V_{ab} = 0.600 \text{ V}$ and $V_{bc} = 0.900 \text{ V}$.

(b) $i_1 = 0.750 \text{ A}$ and $i_2 = i_3 = 0.150 \text{ A}$.

DOUBLE-CHECK: The resistance through the right path is five times larger than the resistance through the center path. Therefore the current should be five times smaller in the right path, which the calculation shows before rounding for significant figures.

25.79. **THINK:** In order for a copper cable to start melting, its temperature must be increased to a melting point temperature, which for copper is 1359 K (Table 18.2). Copper has a specific heat of 386 J/kg K (Table 18.1) and a mass density of 8960 kg/m^3. The cable is insulated. This means the energy dissipated by the cable is used to increase its temperature. Use $L = 2.5$ m, and $V = 12$ V.

SKETCH:



RESEARCH: The resistance of the copper cable is $R = \rho \dfrac{L}{A}$. The energy dissipated by the cable is $E = P \cdot \Delta t = \left(\dfrac{V^2}{R}\right)\Delta t = \left(\dfrac{V^2 A}{\rho L}\right)\Delta t$. This energy must be equal to the amount of heat required to increase the temperature of the cable from the room temperature to the melting point temperature, which is $Q = cm(T_M - T_R) = cm\Delta T$, where c is the specific heat of copper.

SIMPLIFY: Using the mass of the copper cable given by $m = \rho_D AL$, the time required to start melting the cable is $\left(\dfrac{V^2 A}{\rho L}\right)\Delta t = c\rho_D AL\Delta T \Rightarrow \Delta t = \dfrac{c\rho_D \rho L^2 \Delta T}{V^2}$.

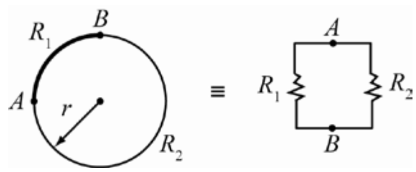
CALCULATE: $\Delta t = \dfrac{(386 \text{ J/kg K})(8960 \text{ kg/m}^3)(1.72 \cdot 10^{-8}\ \Omega \text{ m})(2.5 \text{ m})^2 (1359 \text{ K} - 300 \text{ K})}{(12.0 \text{ V})^2} = 2.734 \text{ s}$

ROUND: Rounding Δt to three significant digits produces $\Delta t = 2.73$ s.

DOUBLE-CHECK: This is a short time interval, but the 12 V battery supplies a large voltage, and the insulation does not allow the heat of the wire to dissipate.

25.80. **THINK:** A piece of copper wire is used to form a circular loop of radius 10 cm. The cross-sectional area of the wire is 10 mm^2. The resistance between two points on the wire is needed.

SKETCH:



RESEARCH: The resistance of a wire is given by $R = \rho L / A$. Thus, the resistances of each segment of the wire are $R_1 = \rho L_1 / A$ and $R_2 = \rho L_2 / A$.

SIMPLIFY: Since $L_2 = 3L_1$, the resistance R_2 is $R_2 = \rho 3L_1 / A = 3R_1$. Because R_1 and R_2 are in parallel, the effective resistance is $R = \dfrac{R_1R_2}{R_1 + R_2} = \dfrac{3R_1^2}{R_1 + 3R_1} = \dfrac{3}{4}R_1 = \dfrac{3}{4}\dfrac{\rho L_1}{A}$. Putting in $L_1 = \dfrac{1}{4} \cdot 2\pi r$ gives $R = \dfrac{3}{8} \cdot \dfrac{\rho \pi r}{A}$.

CALCULATE: $R = \dfrac{3}{8}\left(1.72 \cdot 10^{-8} \ \Omega \text{ m}\right)\dfrac{\pi\left(0.100 \text{ m}\right)}{1.00 \cdot 10^{-5} \text{ m}^2} = 2.03 \cdot 10^{-4} \ \Omega$

ROUND: $R = 2.03 \cdot 10^{-4} \ \Omega$

DOUBLE-CHECK: The resistance of L_1 is

$$R_1 = \rho L_1 / A = \left(1.72 \cdot 10^{-8} \ \Omega \text{ m}\right)\frac{2\pi\left(10 \cdot 10^{-2} \text{ m}\right)}{4\left(10 \cdot 10^{-6} \text{ m}^2\right)} = 2.702 \cdot 10^{-4} \ \Omega.$$

The resistance of L_2 is $R_2 = 3R_1 = 8.105 \cdot 10^{-4} \ \Omega$. Our result is less than R_1 and R_2, because the two resistances are in parallel. So our answer seems reasonable.

25.81. **THINK:** Two conducting wires have identical length and identical radii of circular cross-sections. I want to calculate the ratio of the power dissipated by the two resistors (copper and steel).

SKETCH: Not required.

RESEARCH: The resistance of a wire is given by $R = \rho L / A$. The power dissipated by the wire is $P = \dfrac{V^2}{R} = \dfrac{V^2 A}{\rho L}$.

SIMPLIFY: Therefore, the ratio of powers of two wires is

$$\frac{P_{\text{copper}}}{P_{\text{steel}}} = \left(\frac{V^2 A_{\text{copper}}}{\rho_{\text{copper}} L_{\text{copper}}}\right) \div \left(\frac{V^2 A_{\text{steel}}}{\rho_{\text{steel}} L_{\text{steel}}}\right).$$

Since $L_{\text{copper}} = L_{\text{steel}}$ and $A_{\text{copper}} = A_{\text{steel}}$, the ratio becomes $\dfrac{P_{\text{copper}}}{P_{\text{steel}}} = \dfrac{\rho_{\text{steel}}}{\rho_{\text{copper}}}$.

CALCULATE: $\dfrac{P_{\text{copper}}}{P_{\text{steel}}} = \dfrac{40.0 \cdot 10^{-8} \ \Omega \text{ m}}{1.68 \cdot 10^{-8} \ \Omega \text{ m}} = 23.8095$

ROUND: Rounding the result to three significant digits yields a ratio of 23.8:1. This is because copper is a better conducting material than steel. Moreover, the specific heat of copper is less than steel. This means that copper is less susceptible to heat than steel.

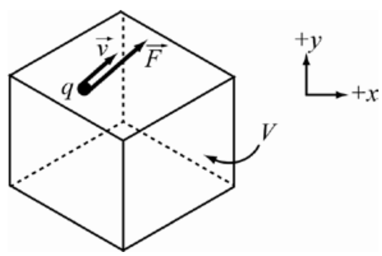
DOUBLE-CHECK: Since the two wires have identical dimensions, and the power dissipated is inversely proportional to the resistivity of the wires, it is reasonable that the material with the higher resistivity dissipates the larger amount of power.

25.82. **THINK:** The resistance of a wire increases or decreases linearly as a function of temperature.
SKETCH: Not required.
RESEARCH: The resistance of the wire at temperature T is given by $R = R_0\left(1+\alpha\left(T-T_0\right)\right)$. The resistance at temperature T is $R = V^2 / P$.
SIMPLIFY: The resistance at the temperature T_0 becomes $R_0 = \left(V^2/P\right)\left[1+\alpha\left(T-T_0\right)\right]^{-1}$.
CALCULATE: Substituting $T = 20$ °C, $T_0 = 1800$ °C, $\alpha = -5\cdot 10^{-4}$ °C^{-1}, $V = 110$ V and $P = 40.0$ W gives

$$R_0 = \frac{(110\text{ V})^2}{40\text{ W}}\left(1-\left(5.0\cdot 10^{-4}\text{ °C}^{-1}\right)\left(1800\text{ °C}-20\text{ °C}\right)\right)^{-1} = 2750\ \Omega.$$

ROUND: Rounding to two significant figures yields $R_0 = 2800\ \Omega = 2.8$ kΩ.
DOUBLE-CHECK: Ohms are appropriate units for resistance. The calculation can be checked by rounding the values to the nearest power of 10. $T \approx 0$ °C, $T_0 \approx 1000$ °C, $\alpha \approx -10^{-3}$ °C^{-1}, $V \approx 100$ V, and $P \approx 100$ W (note that P was rounded to 100 since $\log 40 > 1.5$). Then,
$R_0 = (100\text{ V}/100\text{ W})\left[1-\left(10^{-3}\text{ °C}^{-1}\right)\left(1000\text{ °C}-0\text{ °C}\right)\right] \approx 1000\ \Omega$. The approximated value is the same order of magnitude as the calculated value. This lends support to the calculation.

25.83. **THINK:** The energy dissipated in a resistor is equal to the energy required to move electrons along the direction of a current. The material may or may not be ohmic. The rate of energy dissipation in a resistor is equal to the amount of power required to push electrons.
SKETCH:



RESEARCH: If the force on the electrons is $\vec{F}$ and the average velocity of the electrons is $\vec{v}$, the required power is $P = Fv$. Since $F = qE$, the power becomes $P = qEv$.
SIMPLIFY: Using the charge $q = neV$, where V is a small finite volume, the required power is $P = nevVE$ or $P = EJV$.
(a) Therefore, the power dissipated per unit volume is $P/V = EJ$.
(b) For an ohmic material the current density $\vec{J}$ is related to $\vec{E}$ by $\vec{J} = \sigma\vec{E}$. This equation yields the power dissipated per unit volume of $P/V = E(\sigma E) = \sigma E^2$ or $P/V = (J)(J)/\sigma = (1/\sigma)J^2 = \rho J^2$.
CALCULATE: Not required.
ROUND: Not required.
DOUBLE-CHECK: Examine the two sides of the equation $P/V = EJ$. The units of P/V are watts per cubic meter. The units of the product $E\ J$ are $(\text{V/m})(\text{A/m}^2) = \text{V A/m}^3 = \text{W/m}^3$. In (b), the units of P/V are still watts per cubic meter. Since $\vec{J} = \sigma\vec{E}$, the units of σ are $\text{A}^2/\text{ mW}$. Therefore, the units of $(1/\sigma)J^2 = \rho J^2$ are $(\text{m W/A}^2)(\text{A/m}^2)^2 = (\text{m W/A}^2)(\text{A}^2/\text{m}^4) = \text{W/m}^3$. Thus, by dimensional analysis, the computed equations are sensible.

Multi-Version Exercises

Exercises 25.84–25.86 Following Solved Problem 25.4, we find $f=\dfrac{4P\rho_{\text{Cu}}L}{\pi(\Delta V)^2d^2}$.

25.84. $$f=\frac{4P\rho_{\text{Cu}}L}{\pi(\Delta V)^2d^2}=\frac{4\left(7935\cdot 10^6\text{ W}\right)\left(1.72\cdot 10^{-8}\ \Omega\text{m}\right)\left(643.1\cdot 10^3\text{ m}\right)}{\pi\left(1.177\cdot 10^6\text{ V}\right)^2\left(0.02353\text{ m}\right)^2}=0.1457=14.6\%.$$

25.85. $$f=\frac{4P\rho_{\text{Cu}}L}{\pi(\Delta V)^2d^2}$$

$$\Delta V=\sqrt{\frac{4P\rho_{\text{Cu}}L}{\pi fd^2}}=\sqrt{\frac{4\left(5319\cdot 10^6\text{ W}\right)\left(1.72\cdot 10^{-8}\ \Omega\text{m}\right)\left(411.7\cdot 10^3\text{ m}\right)}{\pi\left(0.07538\right)\left(0.02125\text{ m}\right)^2}}=1.187\cdot 10^6\text{ V}=1.19\text{ MV}.$$

25.86. $$f=\frac{4P\rho_{\text{Cu}}L}{\pi(\Delta V)^2d^2}$$

$$L=\frac{\pi f(\Delta V)^2d^2}{4P\rho_{\text{Cu}}}=\frac{\pi\left(0.1166\right)\left(1.197\cdot 10^6\text{ V}\right)^2\left(0.01895\text{ m}\right)^2}{4\left(5703\cdot 10^6\text{ W}\right)\left(1.72\cdot 10^{-8}\ \Omega\text{m}\right)}=4.804\cdot 10^5\text{ m}=480.\text{ km}.$$

Exercises 25.87–25.88 The energy stored in the battery is equal to the power output of the battery multiplied by the time the battery delivers that power. The power delivered by the battery is $P=i\Delta V$. The energy stored in the battery is then $U=Pt=i\Delta Vt$.

25.87. The time is $t=110.0\text{ min}\dfrac{60\text{ s}}{\text{min}}=6600\text{ s}.$

$U=i\Delta Vt=\left(25.0\text{ A}\right)\left(10.5\text{ V}\right)\left(6600\text{ s}\right)=1732500\text{ J}=1.73\text{ MJ}.$

25.88. $U=i\Delta Vt$

$$t=\frac{U}{i\Delta V}=\frac{1.843\cdot 10^6\text{ J}}{\left(25.0\text{ A}\right)\left(10.5\text{ V}\right)}=7020\text{s}=117\text{ min}$$

RC = 117

Exercises 25.89–25.91 The temperature dependence of resistance is $R-R_0=R_0\alpha\left(T-T_0\right)$. The resistance at operating temperature is given by $\Delta V=iR\ \Rightarrow\ R=\dfrac{\Delta V}{i}$. Combining these equations gives us

$$\frac{\Delta V}{i}-R_0=R_0\alpha\left(T-T_0\right)$$

$$T=T_0+\frac{\dfrac{\Delta V}{i}-R_0}{R_0\alpha}.$$

25.89. $$T=20.00\ ^\circ\text{C}+\frac{\dfrac{3.907\text{ V}}{0.3743\text{ A}}-1.347\ \Omega}{\left(1.347\ \Omega\right)\left(4.5\cdot 10^{-3}\ ^\circ\text{C}^{-1}\right)}=1520\ ^\circ\text{C}$$

25.90.
$$\frac{\Delta V}{i} - R_0 = R_0\alpha\left(T - T_0\right)$$
$$\frac{\Delta V}{i} = R_0 + R_0\alpha\left(T - T_0\right)$$
$$R_0 = \frac{\Delta V / i}{1+\alpha\left(T - T_0\right)} = \frac{\Delta V}{i\left[1+\alpha\left(T-T_0\right)\right]} = \frac{3.949\text{ V}}{\left(0.4201\text{ A}\right)\left[1+\left(4.5\cdot 10^{-3}\text{ °C}^{-1}\right)\left(1291\text{ °C} - 20.00\text{ °C}\right)\right]} = 1.399\ \Omega$$

25.91.
$$\frac{\Delta V}{i} - R_0 = R_0\alpha\left(T - T_0\right)$$
$$\frac{\Delta V}{i} = R_0 + R_0\alpha\left(T - T_0\right)$$
$$i = \frac{\Delta V}{R_0\left[1+\alpha\left(T-T_0\right)\right]} = \frac{3.991\text{ V}}{\left(1.451\ \Omega\right)\left[1+\left(4.5\cdot 10^{-3}\text{ °C}^{-1}\right)\left(1.110\cdot 10^{3}\text{ °C} - 20.00\text{ °C}\right)\right]} = 0.4658\text{ A} = 465.8\text{ mA}$$

Chapter 26: Direct Current Circuits

Concept Checks

26.1. b **26.2.** c **26.3.** c **26.4.** e **26.5.** d **26.6.** b **26.7.** d

Multiple-Choice Questions

26.1. a **26.2.** b **26.3.** b **26.4.** d **26.5.** a **26.6.** d **26.7.** c **26.8.** b **26.9.** d, e & f **26.10.** c **26.11.** e **26.12.** e **26.13.** c

Conceptual Questions

26.14. In the first diagram, the voltmeter does not measure the voltage across the load resistor R_{Load}, but it measures the voltage across the series $R_{\text{Load}} + R_{\text{Ammeter}}$. As long as the internal resistance of the Ammeter is much less than R_{Load}, the effect can be neglected. Similarly, the Ammeter does not measure only the current through the load resistor R_{Load}, but also the current through the R_{Ammeter}. This means that the current measurement is affected by the value of R_{Ammeter}. As long as R_{Ammeter} is much less than R_{Load} the effect can be neglected.

In the second diagram, the voltmeter measures the voltage across the load resistor. However, the current flowing through R_{Load} is affected by the internal resistance of the voltmeter. .As a result the measured voltage is altered from the original value. As long as the load resistance of the voltmeter $R_{\text{Voltmeter}}$ is much larger than the load resistance R_{Load}, the effect can be neglected. The ammeter measures the net current flowing through the resistors R_{Ammeter}, R_{Load} and $R_{\text{Voltmeter}}$. The effects of the internal resistors of the ammeter and the voltmeter are negligible if $R_{\text{Load}} \ll R_{\text{Voltmeter}}$ and $R_{\text{Ammeter}} \ll R_{\text{Load}}$.

26.15. The capacitive time constant is given by $\tau_0 = RC$. Since the equivalent of two identical capacitors connected in series is $C_{\text{eq}} = \left(\frac{1}{C} + \frac{1}{C}\right)^{-1} = \frac{1}{2}C$, the time constant is $\tau = RC_{\text{eq}} = (1/2)RC = (1/2)\tau_0$. Therefore, the time constant decreases by a factor of 2.

26.16. The resistance in the two-point probe measurement is given by the resistance of the device and the wires since they are in series. The four-point measurement is designed such that the resistance of the wires is no longer a part of the measurement so the real potential drop measured is that of the device. The four-point measurement, therefore, gives a better measurement of the real resistance.

26.17. For a capacitor, the rate of which it discharges is based on the current. If the resistance is high and the current is low then the discharge rate is slower. If the capacitance is large, then for a given voltage the amount of charge is higher, because $C = \frac{q}{\Delta V}$, and it will take a longer time for the capacitor to discharge. Hence, increasing R or C can increase the time constant.

26.18. The charge builds up on the capacitor. Thus, the emf of the capacitor balances the emf of the batteries. Summing around the circuit balances the emf, which must be zero regardless of the resistor. The current that satisfies this condition is zero.

26.19. An appropriate resistor R may be connected in series with the bulb and the battery, the value of the resistor can be solved by applying Kirchoff's loop rule for the circuit containing ε (the voltage of the car battery), R and the bulb. $\varepsilon - iR - iR_{\text{bulb}} = 0 \Rightarrow R = \varepsilon / i - R_{\text{bulb}}$ where $i = P/V$ and $R_{\text{bulb}} = V^2/P$.

Therefore,

$$R = \frac{\varepsilon V}{P} - \frac{V^2}{P} = \frac{V}{P}(\varepsilon - V) = \frac{1.5\text{ V}}{1.0\text{ W}}(12.0\text{ V} - 1.5\text{ V}) = 16\ \Omega.$$

26.20. If the emf's are doubled then the currents will also double as Kirchoff's junction rule will still be satisfied as long as all the currents are doubled. Kirchoff's Loop rule implies that if the potential drop across all the resistors is equal to the emf, then doubling the emf means that the currents must also double to account for the needed increases in the potential drop.

26.21. With the capacitors uncharged at $t = 0$, the potential difference across each capacitor at that instant is zero, just like the potential difference of a connecting wire. After a long time, the capacitors will be fully charged, and the potential across the two points will be such that the charge cannot flow between the two points across the capacitor. This has the same effect as open segments in the circuit.

26.22. When using a voltmeter, I want to measure the potential difference across a device. To do this, I set it up parallel to the component, because the potential drop is the same for any two circuit elements in parallel. Ammeters are instruments with very low resistance designed to measure the current. Think of this as a device submerged into a running stream. In order for the instrument to measure the flow, it has to be "in the stream". Similarly, ammeters are in series with the components they wish to measure.

26.23. This question provides an example of meter loading. In connecting an ordinary voltmeter and ammeter simultaneously to some component of a circuit, only two possible orientations are available: one can connect the ammeter in series with the parallel combination of the voltmeter and the component, or one can connect the voltmeter in parallel with the series combination of the ammeter and the component. In the first case, the ammeter measures not the current through the component but the current through the component and the voltmeter, which is slightly greater for any voltmeter with non-infinite resistance. In the second case, the voltmeter measures not the potential difference across the component, but the potential difference across the component and the ammeter, which will be slightly greater for any ammeter with non zero resistance. Simultaneous exact measurements of the current and voltage for the components alone are not possible with ordinary meters. The restriction "with ordinary meters" is reiterated here because it is possible to measure the voltage across a component, for example, without drawing current. This can be done via a "null measurement" such as is done with a potentiometer.

26.24. $P = V^2/R$ implies $R = V^2/P$. A larger power rating implies a smaller filament resistance. Therefore, the answer is a 100 W bulb.

26.25. Since the constant is given by RC, the ratio of the times is equal to the ratio of the capacitances. The capacitances in each case are: series: $C_{\text{series}} = C_1C_2/(C_1 + C_2)$, parallel: $C_{\text{parallel}} = C_1 + C_2$. The ratio is then:

$$\frac{C_{\text{parallel}}}{C_{\text{series}}} = \frac{C_1 + C_2}{\left(\dfrac{C_1C_2}{C_1 + C_2}\right)} = \frac{(C_1 + C_2)^2}{C_1C_2} = \frac{C_1^2 + 2C_1C_2 + C_2^2}{C_1C_2} = \frac{C_1}{C_2} + \frac{C_2}{C_1} + 2.$$

The time to charge the capacitors in parallel is larger by a factor of $\dfrac{C_1}{C_2} + \dfrac{C_2}{C_1} + 2$ (which is at least 2).

26.26. (a) The current at any time t is given by: $i = i_{\text{initial}}\mathrm{e}^{-t/\tau}$ where $i_{\text{initial}} = V/R$ and $\tau = RC$.

(b) The power of the battery is $P = Vi$. Integrated over all time, the power gives us the energy

$$\int_0^\infty P\mathrm{d}t = \int_0^\infty Vi\mathrm{d}t = \int_0^\infty \left(V^2/R\right)\mathrm{e}^{-t/(RC)}dt = CV^2.$$

(c) The power dissipation from R is $P = \mathrm{i}R$. $\int_0^\infty iR\mathrm{d}t = \int_0^\infty \left(V^2/R\right)\mathrm{e}^{-2t/(RC)}dt = (1/2)CV^2.$

(d) Note that the energy provided by the battery less the energy dissipated by the resistor is the energy stored in the capacitor satisfying the law of conservation of energy.

Exercises

26.27. The total resistance of the circuit is given by: $R_{\text{total}} = R_1 + R_2$. The current is then $I = \Delta V / R_{\text{total}}$. The potential drop across each resistor is then:

$$V_1 = \left(\frac{R_1}{R_1 + R_2}\right)\Delta V, \quad V_2 = \left(\frac{R_2}{R_1 + R_2}\right)\Delta V.$$

The resistors in series construct a voltage divider. The voltage ΔV is divided between the two resistors with potential drop proportional to their respective resistances.

26.28. The current must be such that $P = I^2 R = 10.0$ W.

$$I = \frac{V_{\text{emf}}}{r+R} \Rightarrow \left(\frac{V_{\text{emf}}}{r+R}\right)^2 R = P \Rightarrow \left(V_{\text{emf}}\right)^2 R = r^2 P + 2rRP + R^2 P$$

$$\Rightarrow 12.0^2 R = 1.0^2(10.0) + 2(1.00)(10.0)R + 10.0R^2 \Rightarrow 0 = R^2 - 12.4R + 1.00.$$

Solving this quadratic equation yields: $R = \dfrac{12.4 \pm \sqrt{12.4^2 - 4.00}}{2.00}\ \Omega = 0.0812\ \Omega$ or $12.3\ \Omega$. Either of those two resistances will work.

26.29. Kirchhoff's Loop Rule around the upper loop and large loop yields

$$V_{\text{emf}} - (2.00\text{ A})R - (2.00\text{ A})(20.0\ \Omega) = 0 \text{ and } V_{\text{emf}} - (3.00\text{ A})R = 0 \Rightarrow V_{\text{emf}} = (3.00\text{ A})R,$$

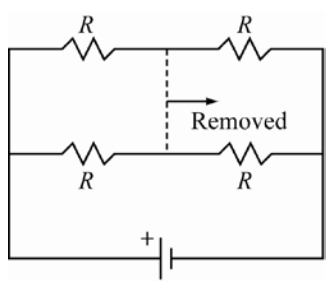
respectively. Therefore,

$$(3.00\text{ A})R - (2.00\text{ A})R - (2.00\text{ A})(20.0\ \Omega) = 0 \Rightarrow (1.00\text{ A})R = (2.00\text{ A})(20.0\ \Omega) \Rightarrow R = 40.0\ \Omega$$

$$V_{\text{emf}} = (3.00\text{ A})(40.0\ \Omega) = 120.\text{ V}.$$

26.30. **THINK:** Close inspection of the diagram shows that there is no current flowing across the middle resistor is zero. This is because there is nothing different between the point above and below the middle resistor. That resistor can therefore be removed while changing nothing.

SKETCH: The new diagram is then:



RESEARCH: The equivalent resistances of the top two resistors and the bottom two resistors are given by $R_{\text{top}} = R + R = 2R$, and $R_{\text{bottom}} = R + R = 2R$, respectively. The system's total equivalent resistance is given by $R_{\text{eq}} = \left(\dfrac{1}{2R} + \dfrac{1}{2R}\right)^{-1}$.

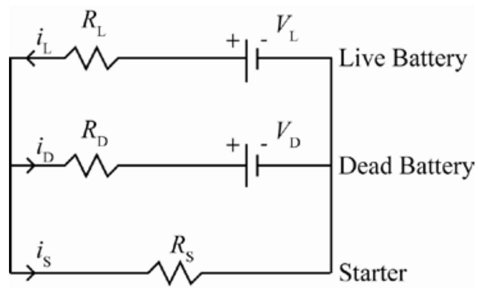
SIMPLIFY: $R_{\text{eq}} = \left(\dfrac{1}{2R} + \dfrac{1}{2R}\right)^{-1} = \left(\dfrac{2}{2R}\right)^{-1} = R$

CALCULATE: Not required.

ROUND: Not required.

DOUBLE-CHECK: The total resistance is comparable to each of the individual resistors as one would expect.

26.31. THINK:
(a) The dead battery is parallel to the starter and the live battery.
(b) Kirchoff's Laws can be used to find the currents. Use the data:
$V_L = 12.00$ V, $V_D = 9.950$ V, $R_L = 0.0100\ \Omega$, $R_D = 1.100\ \Omega$, $R_S = 0.0700\ \Omega$.
SKETCH:



RESEARCH: Kirchoff's Laws give:

$$i_L = i_D + i_S \quad (1)$$

$$i_D = i_L - i_S \quad (1.1)$$

$$V_L - i_L R_L - i_S R_S = 0 \quad (2)$$

$$V_D + i_D R_D - i_S R_S = 0 \quad (3)$$

SIMPLIFY: Substitute (1) into (2) and solve for i_D. $V_L - (i_D + i_S) R_L - i_S R_S = 0$ implies $V_L - i_D R_L - i_S (R_L + R_S) = 0$, which in turn implies

$$i_D = \frac{V_L - i_S (R_L + R_S)}{R_L} \quad (4)$$

Substitute (4) into (3) and solve for i_S. $V_D + \left(\frac{V_L - i_s (R_L + R_S)}{R_L} \right) R_D - i_S R_S = 0$ implies

$$i_S = \frac{V_D R_L + V_L R_D}{R_L R_D + R_S R_D + R_S R_L} \quad (5)$$

Substitute (1.1) into (3) and solve for i_S. $V_D + (i_L - i_S) R_D - i_S R_S = 0$ implies

$$i_S = \frac{V_D + i_L R_D}{R_D + R_S} \quad (6)$$

Substitute (6) into (2) and solve for i_L. $V_L - i_L R_L - \left(\frac{V_D + i_L R_D}{R_D + R_S} \right) R_S = 0$ implies

$$i_L = \frac{V_L R_D + V_L R_S - V_D R_S}{R_L R_D + R_L R_S + R_D R_S} \quad (7)$$

Substitute (5) and (7) into (1) and solve for i_L.

CALCULATE: $$i_S = \frac{(9.950\text{ V})(0.0100\ \Omega) + (12.00\text{ V})(1.100\ \Omega)}{(0.0100\ \Omega)(1.100\ \Omega) + (0.0700\ \Omega)(1.100\ \Omega) + (0.0700\ \Omega)(0.0100\ \Omega)} = 149.938\text{ A}$$

$$i_L = \frac{(12.00\text{ V})(1.100\ \Omega) + (12.00\text{ V})(0.0700\ \Omega) - (9.950\text{ V})(0.0700\ \Omega)}{(0.0100\ \Omega)(1.100\ \Omega) + (0.0100\ \Omega)(0.0700\ \Omega) + (1.100\ \Omega)(0.0700\ \Omega)} = 150.434\text{ A}$$

$150.434\text{ A} = i_D + 149.938\text{ A} \Rightarrow i_D = 150.434\text{ A} - 149.938\text{ A} = 0.496\text{ A}$

ROUND: Three significant figures: $i_S = 150.$ A, $i_L = 150.$ A, $i_D = 0.496$ A.

DOUBLE-CHECK: Inserting the calculated values back into the original Kirchoff's equations;

$$V_L - i_L R_L - i_S R_S = (12\text{ V}) - (150\text{ A})(0.01\ \Omega) - (150\text{ A})(0.07\ \Omega) = 0,$$

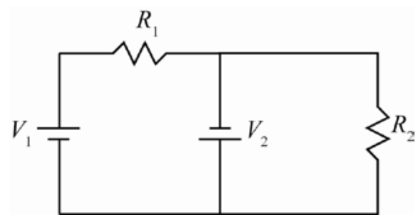
and

$$V_D + i_D R_D - i_S R_S = (9.95\text{ V}) + (0.496\text{ A})(1.1\ \Omega) - (150\text{ A})(0.07\ \Omega) = 0,$$

as required.

26.32. **THINK:** There is only one unknown, so one equation is sufficient to solve the problem. Use Kirchoff's Loop Law to obtain the answer.

SKETCH:



RESEARCH: Kirchoff's Loop Law gives $V_1 - i_1 R_1 + V_2 = 0$ for the first loop.

SIMPLIFY: $i_1 = \dfrac{V_2 + V_1}{R_1}$

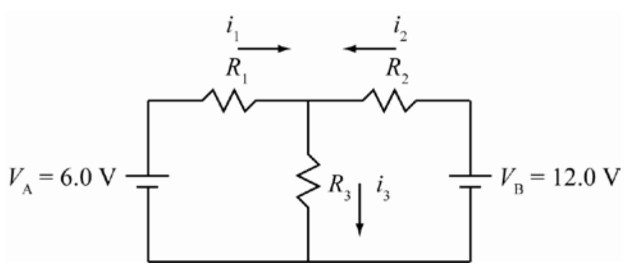
CALCULATE: $i_1 = \dfrac{2.5\text{ V} + 1.5\text{ V}}{4.0\ \Omega} = 1.0\text{ A}$

ROUND: $i_1 = 1.0\text{ A}$

DOUBLE-CHECK: Consider the outside loop: $V_1 - i_1 R_1 - i_2 R_2 = 0.$ From the second loop, $-V_2 - i_2 R_2 = 0 \Rightarrow i_2 = V_2 / R_2.$ So, $0 = V_1 - i_1 R_1 - i_2 R_2 = V_1 - i_1 R_1 - (V_2 / R_2) R_2 \Rightarrow i_1 = 1.0\text{ A},$ as before.

26.33. **THINK:** Kirchoff's Laws can be used to determine the currents. Use the values: $V_A = 6.0\text{ V},$ $V_B = 12.0\text{ V},$ $R_1 = 10.0\ \Omega,$ $R_2 = 40.0\ \Omega,$ and $R_3 = 10.0\ \Omega.$

SKETCH:



RESEARCH: $i_3 = i_1 + i_2,$ $V_A = i_1 R_1 + i_3 R_3,$ $V_B = i_2 R_2 + i_3 R_3,$ $V_A - i_1 R_1 + i_2 R_2 - V_B = 0,$ and $P = iV.$

SIMPLIFY: $V_A = i_1 R_1 + (i_1 + i_2) R_3 \Rightarrow i_1 = \dfrac{V_A - i_2 R_3}{R_1 + R_3}$

$$V_B = i_2 R_2 + (i_1 + i_2) R_3 \Rightarrow V_B = i_2 R_2 + \left[\frac{V_A - i_2 R_3}{R_1 + R_3} + i_2\right] R_3 \Rightarrow V_B = i_2 R_2 + \frac{V_A R_3}{R_1 + R_3} - \frac{i_2 R_3^2}{R_1 + R_3} + i_2 R_3$$

$$i_2 = \frac{\left(V_B - \dfrac{V_A R_3}{R_1 + R_3}\right)}{\left(R_2 - \dfrac{R_3^2}{R_1 + R_3} + R_3\right)}, \quad i_1 = \frac{V_A - i_2 R_3}{R_1 + R_3}, \quad i_3 = i_1 + i_2$$

$P_A = i_1 V_A,$ $P_B = i_2 V_B$

CALCULATE: $i_2 = \left(12.0\text{ V} - \dfrac{(6.0\text{ V})(10.0\ \Omega)}{10.0\ \Omega + 10.0\ \Omega}\right) \Big/ \left(40.0\ \Omega - \dfrac{(10.0\ \Omega)^2}{10.0\ \Omega + 10.0\ \Omega} + 10.0\ \Omega\right) = 0.20\text{ A}$

$i_1 = \dfrac{6.0\text{ V} - (0.20\text{ A})(10.0\ \Omega)}{10.0\ \Omega + 10.0\ \Omega} = 0.20\text{ A},\ \ i_3 = 0.20\text{ A} + 0.20\text{ A} = 0.40\text{ A}$

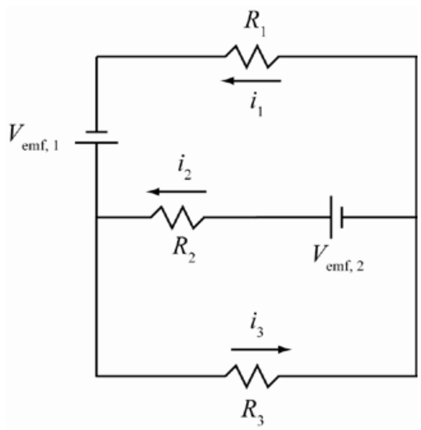
$P_\text{A} = (0.20\text{ A})(6.0\text{ V}) = 1.2\text{ W},\ P_\text{B} = (0.20\text{ A})(12.0\text{ V}) = 2.4\text{ W}$

ROUND: $i_1 = 0.20\text{A},\ i_2 = 0.20\text{A},\ i_3 = 0.40\text{A}, P_\text{A} = 1.2\text{ W},$ and $P_\text{B} = 2.4\text{ W}.$

DOUBLE-CHECK: The direction of i_1 and i_2 makes sense since they are in the direction of the driving force of the battery.

26.34. **THINK:** The circuit has three branches. Kirchoff's Loop and junction laws can be used to find at least three linearly independent equations. Use the values: $R_1 = 5.00\ \Omega,\ R_2 = 10.0\ \Omega,\ R_3 = 15.0\ \Omega,$ $V_{\text{emf},1} = 10.0\text{ V},$ and $V_{\text{emf},2} = 15.0\text{ V}.$

SKETCH:



RESEARCH:

$$V_{\text{emf},1} - V_{\text{emf},2} + i_2R_2 - i_1R_1 = 0 \quad (1)$$

$$V_{\text{emf},1} - i_3R_3 - i_1R_1 = 0 \quad (2)$$

$$i_1 + i_2 = i_3 \quad (3)$$

SIMPLIFY: Substitute (3) into (2) and solve for i_1: $V_{\text{emf},1} - (i_1 + i_2)R_3 - i_1R_1 = 0$ implies

$$i_1 = \frac{V_{\text{emf},1} - i_2R_3}{R_1 + R_3} \quad (4)$$

Substitute 4 into 1 and solve for i_2:

$$V_{\text{emf},1} - V_{\text{emf},2} + i_2R_2 - \left(\frac{V_{\text{emf},1} - i_2R_3}{R_1 + R_3}\right)R_1 = 0 \ \Rightarrow\ i_2 = \frac{V_{\text{emf},2} - V_{\text{emf},1} + \left(V_{\text{emf},1}R_1 / (R_1 + R_3)\right)}{\left(R_1R_3 / (R_1 + R_3)\right) + R_2} \quad (5)$$

CALCULATE: $i_2 = \dfrac{15.0\text{V} - 10.0\text{V} + (10.0\text{V})(5.00\ \Omega)/(5.00\ \Omega + 15.0\ \Omega)}{(5.00\ \Omega)(15.0\ \Omega)/(5.00\ \Omega + 15.0\ \Omega) + 10.0\ \Omega} = 0.54545\text{ A}$

$i_1 = \dfrac{10.0\text{ V} - i_2(15.0\ \Omega)}{5.00\ \Omega + 15.0\ \Omega} = 0.09091\text{ A}$

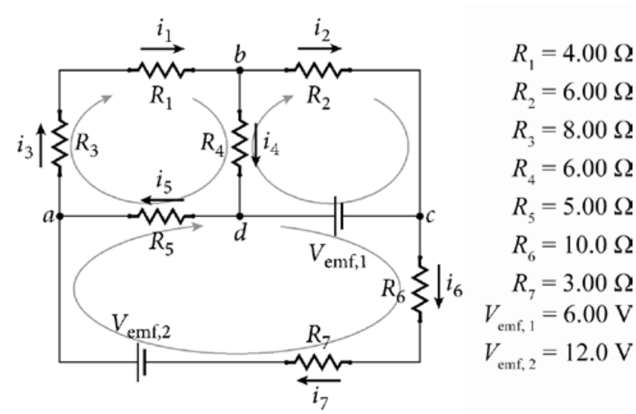
$i_3 = i_1 + i_2 = 0.636363\text{ A}$

ROUND: To three significant figures: $i_1 = 0.0909\text{ A},\ \ i_2 = 0.545\text{ A},\ \ i_3 = 0.636\text{ A}$

DOUBLE-CHECK: The calculated values for the currents are all positive, which is consistent with the direction specified in the problem.

26.35. **THINK:** Kirchhoff's Laws can be applied to this circuit. We must identify the junctions and the loops. We note that the currents through resistors 1 and 3 are the same and the currents through resistors 6 and 7 are the same. We have five unknowns, i_1, i_2, i_4, i_5, and i_6. We need five equations for the solution.

SKETCH:



RESEARCH: We have $i_1 = i_3$ and $i_6 = i_7$. We take the directions of the currents as shown in the sketch. There are four junctions giving the following equations

$$a: \ i_5 + i_6 = i_1$$
$$b: \ i_1 = i_2 + i_4.$$

There are three loops that can be analyzed using Kirchoff's loop rule. Analyzing each loop in the clockwise direction:

$$\text{Starting at } a: -i_1R_3 - i_1R_1 - i_4R_4 - i_5R_5 = 0$$
$$\text{Starting at } d: \ -V_{\text{emf},1} - i_6R_6 - i_6R_7 + V_{\text{emf},2} + i_5R_5 = 0$$
$$\text{Starting at } c: \ V_{\text{emf},1} + i_4R_4 - i_2R_2 = 0.$$

The power supplied by each battery is given by $P = Vi$.

SIMPLIFY: Cramer's rule is the most efficient method for solving a system of five equations and five unknowns. Rearranging the equations:

$$-i_1 + i_5 + i_6 = 0$$
$$i_1 - i_2 - i_4 = 0$$
$$-i_1\left(R_1 + R_3\right) - i_4R_4 - i_5R_5 = 0$$
$$i_5R_5 - i_6\left(R_6 + R_7\right) = V_{\text{emf},1} - V_{\text{emf},2}$$
$$-i_2R_2 + i_4R_4 = -V_{\text{emf},1}.$$

Taking the coefficients of the currents, we can write the matrix equation as:

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ -\left(R_1 + R_3\right) & 0 & -R_4 & -R_5 & 0 \\ 0 & 0 & 0 & R_5 & -\left(R_6 + R_7\right) \\ 0 & -R_2 & R_4 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V_{\text{emf},1} - V_{\text{emf},2} \\ -V_{\text{emf},1} \end{bmatrix}$$

CALCULATE: We can use Cramer's rule to solve this system of five equations and five unknowns

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ -12.00 & 0 & -6.00 & -5.00 & 0 \\ 0 & 0 & 0 & 5.00 & -13.00 \\ 0 & -6.00 & 6.00 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -6.00 \\ -6.00 \end{bmatrix}$$

The solution can be calculated by hand using Cramer's rule or using a computer algebra system. The matrix, when evaluated by such a program into reduced row echelon form, gives the numeric solution as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0.250746 \\ 0.625373 \\ -0.374627 \\ -0.152239 \\ 0.402985 \end{bmatrix} \Rightarrow \begin{matrix} i_1 = 0.250746 \text{ A} \\ i_2 = 0.625373 \text{ A} \\ i_4 = -0.374627 \text{ A} \\ i_5 = -0.152239 \text{ A} \\ i_6 = 0.402985 \text{ A} \end{matrix}$$

The current through resistors R_1 and R_3 is $i_1 = i_3 = 0.250746$ A in the assumed direction. The current through resistor R_2 is $i_2 = 0.625373$ A in the assumed direction. The current through resistor R_4 is $i_4 = 0.374627$ A in a direction opposite to the assumed direction. The current through resistor R_5 is $i_5 = 0.152239$ A in a direction opposite to the assumed direction. The current through resistors R_6 and R_7 is $i_6 = i_7 = 0.402985$ A in the assumed direction.

The current flowing through $V_{\text{emf},1}$ is given by

$i_2 + i_4 = 0.625373 \text{ A} - 0.374627 \text{ A} = 0.250746 \text{ A}.$

$P(V_{\text{emf},1}) = (6.00 \text{ V})(0.250746 \text{ A}) = 1.504476 \text{ W}.$

The current flowing through $V_{\text{emf},2}$ is given by

$i_1 + i_2 + i_3 + i_6 + i_7 = 0.250746 \text{ A} + 0.625373 \text{ A} + 0.250746 \text{ A} + 0.402985 \text{ A} + 0.402985 \text{ A} = 1.932835 \text{ A}.$

$P(V_{\text{emf},2}) = (12.0 \text{ V})(1.932835 \text{ A}) = 23.19402 \text{ W}.$

ROUND: Rounding to three significant digits and assigning the directions we have:

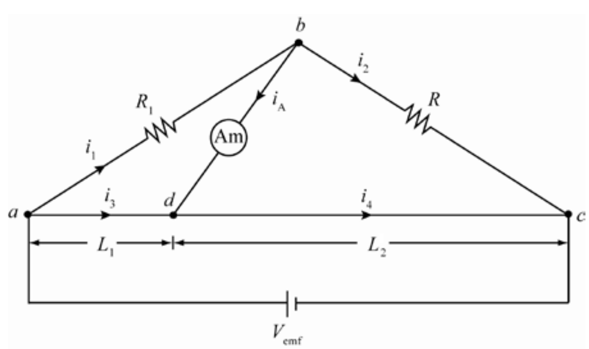
	Magnitude	Direction
i_1	0.251 A	to the right
i_2	0.625 A	to the right
i_3	0.251 A	upward
i_4	0.375 A	upward
i_5	0.152 A	to the right
i_6	0.403 A	downward
i_7	0.403 A	to the left

$P(V_{\text{emf},1}) = 1.50 \text{ W}, \; P(V_{\text{emf},2}) = 23.2 \text{ W}.$

DOUBLE-CHECK: We can substitute out results for the five currents back into our five equations and show that they are satisfied.

26.36. **THINK:** When the potential difference between *a* and *b* is zero, no current will flow. The potential difference will be zero when the ratio of the resistances above the ammeter is equal to the ratio of the resistances below the ammeter. Use $L_1 = 25.0$ cm and $L_2 = 75.0$ cm.

SKETCH:



RESEARCH: The current is zero when $\frac{R_1}{R_x}=\frac{R_{L_1}}{R_{L_2}} \Rightarrow R_1R_{L_2}=R_xR_{L_1}$, $R_1=100.\ \Omega$. $R_{L_1}=\rho\frac{L_1}{A}$ and $R_{L_2}=\rho\frac{L_2}{A}$.

SIMPLIFY: $R_1\rho\left(\frac{L_2}{A}\right)=R_x\rho\left(\frac{L_1}{A}\right)$, $R_x=R_1\frac{L_2}{L_1}$

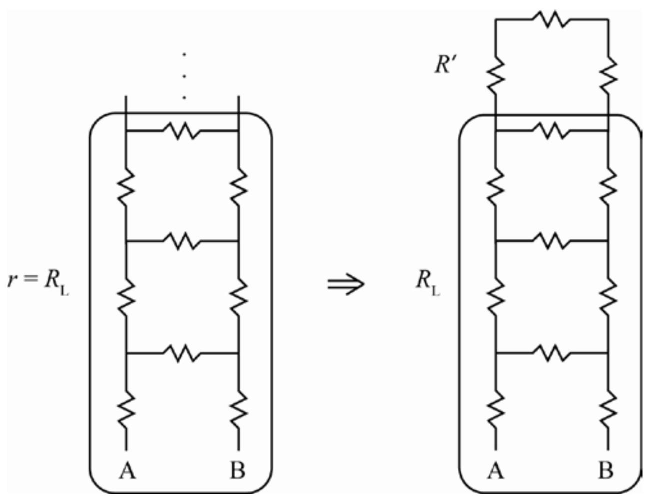
CALCULATE: $R_x=(100.\ \Omega)\left(\frac{75.0\text{ cm}}{25.0\text{ cm}}\right)=300.\ \Omega$

ROUND: $R_x=300.\ \Omega$

DOUBLE-CHECK: R_x is comparable to R_1 as one would expect.

26.37. **THINK:** Suppose the total equivalent resistance of the ladder up to some arbitrary point is given by R_L. Since the ladder is infinite, it does not matter what point on the ladder is chosen for the analysis, and adding one more segment to the end will not change the equivalent resistance of the network.

SKETCH:



RESEARCH: The ladder consists of the array with resistance R_L plus another segment with resistance R'. R' contributes one resistor of resistance R, in parallel with the array, and two resistors of resistance, R, in series with the array. The total resistance is now $R_L'=2R+R\,||\,R_L=R_L$.

SIMPLIFY: $R_L=2R+R\,||\,R_L=2R+\left(\frac{1}{R}+\frac{1}{R_L}\right)^{-1}=2R+\frac{RR_L}{R+R_L}$

$$R_L=2R+\frac{RR_L}{R+R_L} \Rightarrow R_L\left(R+R_L\right)=2R\left(R+R_L\right)+RR_L \Rightarrow R_L^2-2RR_L-2R^2=0$$

CALCULATE: Solving the quadratic equation for R_L gives $R_L=\left(1+\sqrt{3}\right)R$.

ROUND: Since no values are given in the question, it is best to leave the answer in its precise form, $R_L=\left(1+\sqrt{3}\right)R$.

DOUBLE-CHECK: Consider the first rung of three resistors in series. The equivalent resistance is $R_{eq}=3R$. Now, add another rung of three resistors. One resistor is in parallel with the first rung, and two resistors are in series with the first rung. The equivalent resistance is now

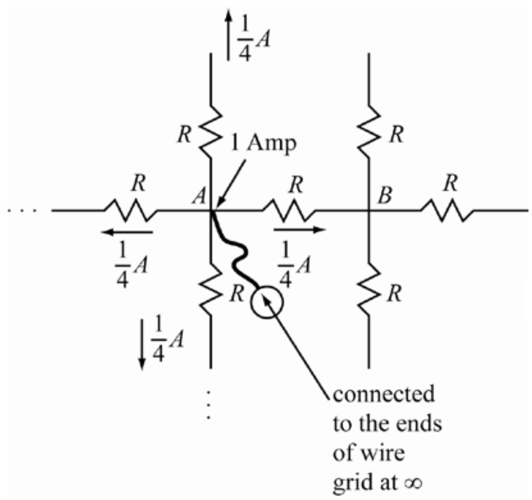
$$R_{eq,1}=\left(\frac{1}{3R}+\frac{1}{R}\right)^{-1}+2R=\frac{11}{4}R=2.75R.$$

Adding another rung gives $R_{\text{eq},2}=\left(\frac{4}{11R}+\frac{1}{R}\right)^{-1}+2R=\frac{41}{15}R=2.7333R.$ Repeating the process, $R_{\text{eq},3}=\frac{153}{56}R=2.7324R,\ R_{\text{eq},4}=\frac{571}{209}R=2.73206R,\ \ldots,\ R_n\to\left(1+\sqrt{3}\right)R.$ This verifies the value found in the solution.

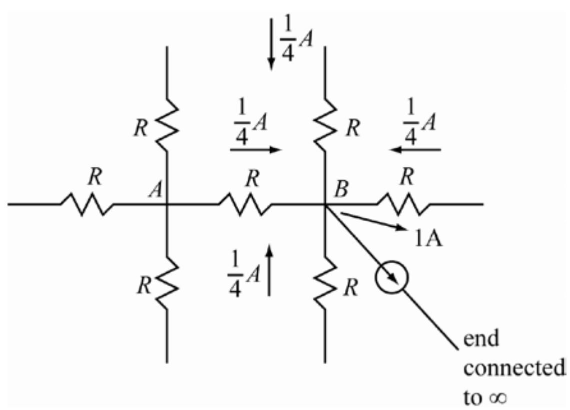
26.38. **THINK:** This is a very famous and very tricky problem. Superposition can be used to find the answer. I will inject 1 Amp into A as specified below and extract 1 Amp from B as specified below.

SKETCH:

1 Amp injected into A:



1 Amp extracted from B:



RESEARCH: The superposition of the two cases has 1 Amp entering A and leaving B. The superposition of the currents indicates that $(1/2)$A passes through the resistor R_{AB}, showing the effective resistance between the two points is $R/2$.

SIMPLIFY: Not required.

CALCULATE: Not required.

ROUND: Not required.

DOUBLE-CHECK: It makes sense that the effective resistance is less than R since there are other pathways for the current to flow.

26.39. Let i_{A} be the maximum current (i.e. full scale value) the ammeter can measure without the shunt. If the shunt is to extend the full scale value by a factor $N = i_{\mathrm{tot}} / i_{\mathrm{A}}$, then

$$i_{\mathrm{A}} + i_{\mathrm{shunt}} = Ni_{\mathrm{A}} \Rightarrow \frac{i_{\mathrm{shunt}}}{i_{\mathrm{A}}} = N - 1.$$

Since the ammeter and shunt have the same voltage across them,

$$R_{\mathrm{i,A}} i_{\mathrm{A}} = R_{\mathrm{shunt}} i_{\mathrm{shunt}} \Rightarrow R_{\mathrm{shunt}} = \frac{i_{\mathrm{A}}}{i_{\mathrm{shunt}}} R_{\mathrm{i,A}} = \frac{R_{\mathrm{i,A}}}{N-1}.$$

To allow a current of 100 A, the resistance of the shunt resistor must be

$$R_{\mathrm{shunt}} = \frac{(1.00\ \Omega)}{(100.\ \mathrm{A}/1.00\ \mathrm{A}) - 1} = \frac{1.00\ \Omega}{99.0} = 10.1\ \mathrm{m}\Omega.$$

The fraction of the total current flowing through the ammeter is

$$\frac{i_{\mathrm{A}}}{i_{\mathrm{tot}}} = \frac{(1.00\ \mathrm{A})}{(100.\ \mathrm{A})} = 0.0100.$$

The fraction of the total current flowing through the shunt is

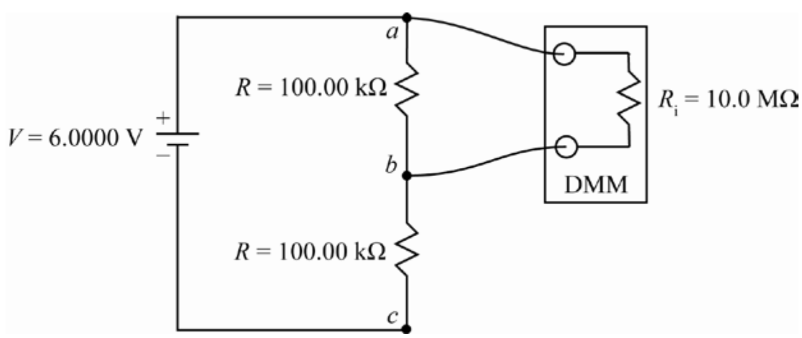
$$\frac{i_{\mathrm{shunt}}}{i_{\mathrm{tot}}} = 1 - \frac{(1.00\ \mathrm{A})}{(100.\ \mathrm{A})} = 0.990.$$

26.40. The voltage across the device must be smaller than the voltage across the device and the resistor by a factor of N. $N(V_{1,\mathrm{V}}) = V_{1,\mathrm{V}} + V_{\mathrm{series}}$. Since $i_{1,\mathrm{V}} = i_{\mathrm{series}}$:

$$\frac{V_{1,\mathrm{V}}}{R_{1,\mathrm{V}}} = \frac{V_{\mathrm{series}}}{R_{\mathrm{series}}} \Rightarrow N(V_{1,\mathrm{V}}) = V_{1,\mathrm{V}} + \left(\frac{R_{\mathrm{series}}}{R_{1,\mathrm{V}}}\right) V_{1,\mathrm{V}} \Rightarrow N = 1 + \frac{R_{\mathrm{series}}}{R_{1,\mathrm{V}}} \Rightarrow R_{\mathrm{series}} = (N-1) R_{1,\mathrm{V}}$$

Numerical Application: $R_{\mathrm{series}} = (100. - 1)1.00 \cdot 10^6\ \Omega = 99.0 \cdot 10^6\ \Omega = 99.0\ \mathrm{M}\Omega.$ The 1.00 V potential drop across the voltmeter is 1.00% of the total power. The other 99.0 V potential drop occurs across the added series resistor and is 99.0% of the total.

26.41. The sketch illustrates the case of measuring V_{ab}.



The total resistance is $\dfrac{RR_{\mathrm{i}}}{R+R_{\mathrm{i}}} + R.$ The total current is

$$i = \frac{V}{\dfrac{RR_{\mathrm{i}}}{R+R_{\mathrm{i}}} + R} = \frac{6.0000\ \mathrm{V}}{\dfrac{(1.0000 \cdot 10^5\ \Omega)(1.00 \cdot 10^7\ \Omega)}{1.0000 \cdot 10^5\ \Omega + 1.00 \cdot 10^7\ \Omega} + 1.0000 \cdot 10^5\ \Omega} = 3.0149 \cdot 10^{-5}\ \mathrm{A}.$$

The potential across the voltmeter is

$$V_{ab} = i\frac{RR_i}{R+R_i} = \left(3.0149\cdot 10^{-5}\text{ A}\right)\frac{\left(1.0000\cdot 10^{5}\ \Omega\right)\left(1.00\cdot 10^{7}\ \Omega\right)}{1.0000\cdot 10^{5}\ \Omega + 1.00\cdot 10^{7}\ \Omega} = 2.985\text{ V} = 2.99\text{ V},$$

Increasing R_i will reduce the error since the voltmeter will draw less current.

26.42. (a) The current is to be 10.0 mA for a voltage of 9.00 V. $R = V_{emf}/i = 9.00\text{ V}/\left(10.0\text{ mA}\right) = 900.\ \Omega$

(b) The current is 2.50 mA. The resistance is $R_{variable} + R$. The current is given by

$$i = \frac{V_{emf}}{R_{variable}+R} \Rightarrow iR_{variable} + iR = V_{emf} \Rightarrow R = \frac{V_{emf} - iR_{variable}}{i} = \frac{9.00\text{ V} - \left(2.50\text{ mA}\right)\left(900.\ \Omega\right)}{2.50\text{ mA}} = 2.70\text{ k}\Omega.$$

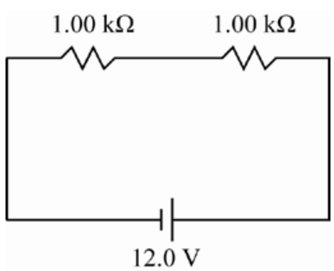
26.43. **THINK:**

(a) The total resistance must first be determined in order to find the current. Since the resistors are in series, the same current flows through both of them.

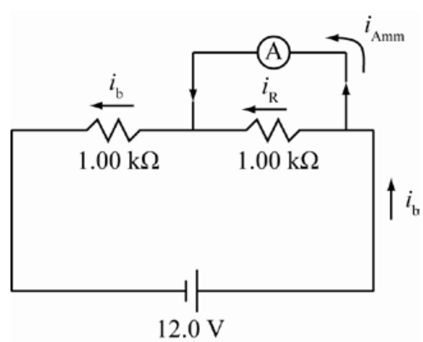
(b) The current that flows through the circuit is the result of the equivalent resistance including the ammeter. The same current flows through the 1.00 kΩ resistor and the parallel combination of resistor and ammeter. Of the current flowing through this combination, the majority will flow through the lower resistance, i.e., the ammeter. The fraction of the current that goes through the Ammeter can be calculated using the resistances.

SKETCH:

(a)



(b)



RESEARCH:

(a) $R_{eq} = 2R$, $R = 1.00$ kΩ, $i_a = V/R_{eq}$, $V = 12.0$ V

(b) The current that flows through the circuit is $i_b = \dfrac{V}{R_{eq}}$, where $R_{eq} = R + \left(\dfrac{RR_A}{R+R_A}\right)$, $R_A = 1.0\ \Omega$. The current flowing through the resistor/ammeter combination is split into two parts. $i_R = \Delta V_1/R$, and $i_{Amm} = \Delta V_2/R_A$.

SIMPLIFY:

(a) $i_a = \dfrac{V}{2R}$

(b) $$i_{Amm} = \frac{\Delta V_2}{R_{Amm}} = \frac{i_b RR_{Amm}}{R_{Amm}\left(R+R_{Amm}\right)} = \frac{V}{R+\left(\dfrac{RR_{Amm}}{R+R_{Amm}}\right)}\cdot\left(\frac{RR_{Amm}}{R_{Amm}\left(R+R_{Amm}\right)}\right) = \frac{V}{R+2R_{Amm}}$$

CALCULATE:

(a) $i_a = \dfrac{12.0\text{ V}}{2\left(1.00\cdot 10^{3}\ \Omega\right)} = 6.00$ mA

(b) $\dfrac{12.0 \text{ V}}{\left(1.00 \text{ k}\Omega + 2\cdot(1.0\ \Omega)\right)} = 0.01198 \text{ A}$

ROUND:

(a) $i_a = 6.00 \text{ mA}$

(b) $i_{\text{Amm}} = 0.012 \text{ A}$

DOUBLE-CHECK: The ammeter measures the current across the other resistor acting like a short across the first resistor, as would be expected.

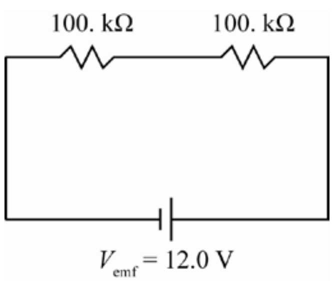
26.44. **THINK:**

(a) I need to find the total resistance and then find the potential drop in each resistor.

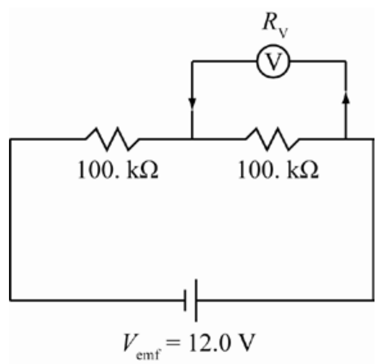
(b) When a voltmeter is connected across one of the resistors, the combination of the resistor and the voltmeter will have an equivalent resistance slightly different from that of the resistor alone. This will cause a change in the potential drop across the resistor/voltmeter combination. I need to calculate the new potential drop.

SKETCH:

(a)



(b)



RESEARCH:

(a) Since they are identical and in series, the resistors have the same potential drop of $V/2$.

(b) The total resistance is now given by $R_{\text{total}} = R + \left(\dfrac{R_{\text{voltmeter}}R}{R_{\text{voltmeter}} + R}\right)$. The potential drop across the voltmeter is then $V_{\text{voltmeter}} = iR = \left(\dfrac{V}{R_{\text{total}}}\right)\left(\dfrac{R_{\text{voltmeter}}R}{R_{\text{voltmeter}} + R}\right)$.

SIMPLIFY: Not required.

CALCULATE:

(a) $\dfrac{12.0\text{V}}{2} = 6.00\text{V}$

(b) $R_{\text{total}} = 100.\text{ k}\Omega + \dfrac{(10.0 \text{ M}\Omega)(100.\text{ k}\Omega)}{(10.0 \text{ M}\Omega + 100.\text{ k}\Omega)} = 199.009901 \text{ k}\Omega$

$$V_{\text{voltmeter}} = \frac{12.0 \text{ V}}{199.009 \text{ k}\Omega}\left[\frac{(10.0 \text{ M}\Omega)(100.\text{ k}\Omega)}{(10.0 \text{ M}\ \Omega + 100.\text{ k}\ \Omega)}\right] = 5.97 \text{ V}$$

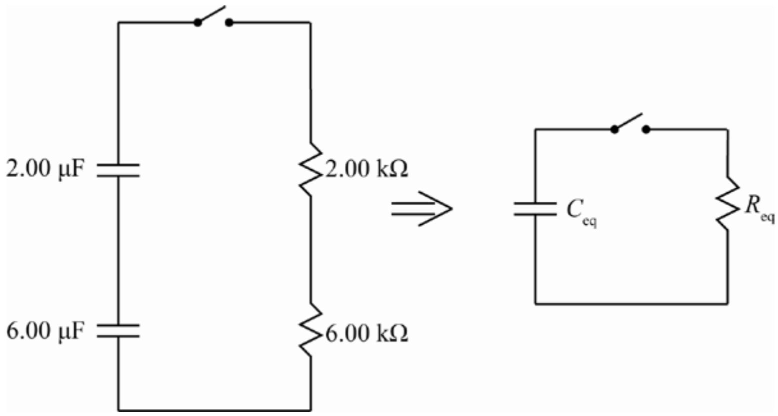
The percentage change is $\dfrac{6.00 \text{ V} - 5.97 \text{ V}}{6.00 \text{ V}} = 0.500\%$.

ROUND: The percentage change is 0.500%.

DOUBLE-CHECK: It make sense that the voltmeter will reduce the voltage since any voltmeter (with non-infinite resistance) will draw a small amount of current.

26.45. The equation for the charge of a capacitor in an RC circuit over time is $Q(t)=Q_{initial}e^{-t/\tau}$. Use the equations: $\tau = RC$, $R = 100.\ \Omega + 200.\ \Omega = 300.\ \Omega$, $C = 10.0$ mF, $\ln\left(\frac{Q(t)}{Q_{\text{initial}}}\right) = -t/\tau$, $t = -\tau \ln\left(\frac{Q(t)}{Q_{\text{initial}}}\right)$, and $t = -(300.\ \Omega)(10.0\text{ mF})\ln\left(\frac{5.00\text{ mC}}{100.\text{ mC}}\right) = 8.99$ s.

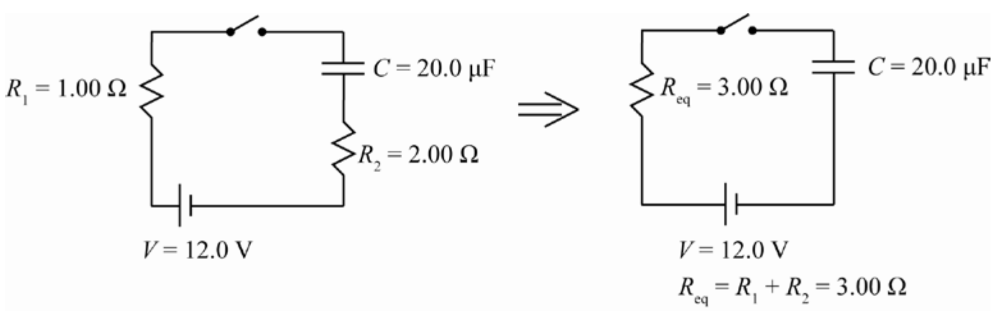
26.46. The circuit can be easily simplified to



where $R_{\text{eq}} = 2.00\text{ k}\Omega + 6.00\text{ k}\Omega = 8.00\text{ k}\Omega$ and $C_{\text{eq}} = \left[(1/2.00\,\mu\text{F}) + (1/6.00\,\mu\text{F})\right]^{-1} = 1.50\ \mu$F. The time constant is then $\tau = R_{\text{eq}}C_{\text{eq}} = (8.00\text{ k}\Omega)(1.50\ \mu\text{F}) = 12$ ms. The initial charge of the $2.00\ \mu$F capacitor, with initial potential $V = 10.0$ V, is $q_0 = CV = (2.00\ \mu\text{F})(10.0\text{ V}) = 2.00\cdot 10^{-5}$ C. The charge decays as $q(t) = q_0 e^{-t/\tau}$. When $t = \frac{\tau}{2}$, the charge left is

$$q\left(\frac{\tau}{2}\right) = q_0 e^{-1/2} = 0.6065 q_0 = 0.6065\cdot\left(2\cdot 10^{-5}\text{ C}\right) = 1.213\cdot 10^{-5}\text{ C}.$$

26.47. Since the position of the resistor with respect to the capacitor is irrelevant, the circuit is simplified to:



The maximum charge of the capacitor is $q_0 = C\Delta V = (20.0\ \mu\text{F})(12.0\text{ V}) = 2.40\cdot 10^{-4}$ C. In general, the capacitor charges as $q(t) = q_0\left(1-e^{-\frac{t}{RC}}\right)$. When $q(t) = (1/2)q_0$:

$$\frac{1}{2}q_0 = q_0\left(1-e^{-\frac{t}{RC}}\right) \Rightarrow e^{-\frac{t}{RC}} = \frac{1}{2} \Rightarrow t = -RC\ln\left(\frac{1}{2}\right) = RC\ln(2).$$

Therefore, $t = (3.00\ \Omega)(20.0\ \mu\text{F})\ln(2) = 41.6\ \mu$s.

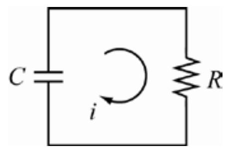
26.48. By Ohm's law, the power, $P = 1.21$ GW, is related to potential, $V = 12.0$ V, and resistance, R, by

$$P = \frac{V^2}{R} \Rightarrow R = \frac{V^2}{P} = \frac{(12.0\text{ V})^2}{1.21\text{ GW}} = 119\text{ n}\Omega.$$

The time to charge the capacitor, $C = 1.00\ F$, to 90.0% is $q(t) = q_0\left(1 - e^{-\frac{t}{RC}}\right) = 0.900 q_0 \Rightarrow e^{-\frac{t}{RC}} = 0.100.$

Therefore, $t = -RC\ln(0.100) = -(119\text{ n}\Omega)(1.00\text{ F})\ln(0.100) = 274\cdot 10^{-9}\text{ s} = 274\text{ ns}.$

26.49. **THINK:** The charge on the capacitor, $C = 90.0\ \mu\text{F}$, decays exponentially through the resistor, $R = 60.0\ \Omega$. The energy on the capacitor is proportional to the square of the charge, so the energy also decays exponentially. If 80.0% of the energy is lost, then 20.0% is left on the capacitor.
SKETCH:



RESEARCH: The charge on the capacitor is given by $q(t) = q_0 e^{-t/RC}$. The energy on the capacitor is given by $E(t) = \frac{1}{2}\frac{q(t)^2}{C}$. To determine the time when there is 20.0% energy remaining, consider the equation: $E(t) = 0.200E(0)$.
SIMPLIFY: Determine time, t:

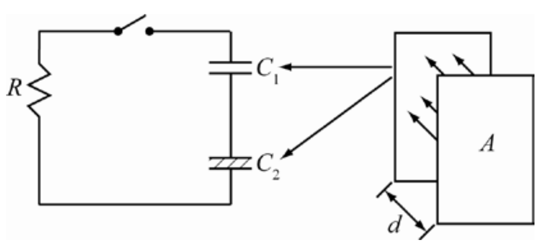
$$E(t) = \frac{q(t)^2}{2C} = \frac{q_0^2 e^{-2t/RC}}{2C} = 0.200E(0) = 0.200\frac{q_0^2}{2C}$$

$$\Rightarrow e^{-2t/RC} = 0.200 \Rightarrow \frac{-2t}{RC} = \ln(0.200) \Rightarrow t = -\frac{RC}{2}\ln(0.200).$$

CALCULATE: $t = -\frac{60.0\ \Omega(90.0\ \mu\text{F})}{2}\ln(0.200) = 4.3455\cdot 10^{-3}\text{ s}$
ROUND: To three significant figures, $t = 4.35$ ms

DOUBLE-CHECK: After $t = 4.35$ ms, the charge on the capacitor is 0.451 of the maximum charge. This value squared gives 0.203, which is 20.0% with rounding error considered.

26.50. **THINK:** After sufficient time, the potential on both plates (area $A = 2.00\text{ cm}^2$ and separation $d = 0.100$ mm) will be $\Delta V = 60.0$ V. Since the capacitors are in series, the total charge on each will be the same. The potential drop across a capacitor is needed to find its electric field. The second capacitor has dielectric constant $\kappa = 7.00$ and dielectric strength $S = 5.70$ kV/mm.
SKETCH:



RESEARCH: The capacitance of the air filled capacitor is $C_1 = \dfrac{\varepsilon_0 \text{A}}{d}$, an that with the dielectric is $C_2 = \dfrac{\kappa\varepsilon_0 \text{A}}{d}$. The charge on a capacitor is $Q = C\Delta V$. The energy stored in a capacitor is $U = \dfrac{Q^2}{2C}$. The electric field inside a capacitor is $E = \dfrac{V}{d}$.

SIMPLIFY:

(a) Equivalent capacitance is

$$C_{\text{eq}} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} = \left(\frac{d}{\varepsilon_0 \text{A}} + \frac{d}{\kappa\varepsilon_0 \text{A}}\right)^{-1} = \frac{\varepsilon_0 \text{A}}{d}\left(1 + \frac{1}{\kappa}\right)^{-1} = \frac{\varepsilon_0 \text{A}}{d}\left(\frac{\kappa}{\kappa + 1}\right).$$

Charge on the first capacitor is $Q = Q_1 = C_{\text{eq}}\Delta V$.

(b) Charge on the second capacitor is $Q = Q_2 = C_{\text{eq}}\Delta V$.

(c) The total energy on both plates is $U = \dfrac{Q^2}{2C_{\text{eq}}} = \dfrac{C_{\text{eq}}^2 \Delta V^2}{2C_{\text{eq}}} = \dfrac{1}{2} C_{\text{eq}} \Delta V^2$.

(d) The potential drop across the second capacitor is $\Delta V_2 = \dfrac{Q_2}{C_2} = \dfrac{Qd}{\kappa\varepsilon_0 \text{A}}$. The electric field across it is then $E_2 = \dfrac{\Delta V_2}{d} = \dfrac{Q}{\kappa\varepsilon_0 \text{A}}$.

CALCULATE:

(a) $C_{\text{eq}} = \dfrac{7.00}{7.00+1}\left[\dfrac{\left(8.854 \cdot 10^{-12} \text{ C}^2/\left(\text{N m}^2\right)\right)\left(2.00 \cdot 10^{-4} \text{ m}^2\right)}{1.00 \cdot 10^{-4} \text{ m}}\right] = 1.54945 \cdot 10^{-11} \text{ F}$

$Q_1 = \left(1.54945 \cdot 10^{-11} \text{ F}\right)\left(60.0 \text{ V}\right) = 9.2967 \cdot 10^{-10} \text{ C}$

(b) $Q_2 = 9.2967 \cdot 10^{-10} \text{ C}$

(c) $U = \dfrac{1}{2}\left(1.54945 \cdot 10^{-11} \text{ F}\right)\left(60.0 \text{ V}\right)^2 = 2.789 \cdot 10^{-8} \text{ J}$

(d) $E_2 = \dfrac{9.2967 \cdot 10^{-10} \text{ C}}{7.00\left(8.854 \cdot 10^{-12} \text{ C}^2/\left(\text{N m}^2\right)\right)\left(2.00 \cdot 10^{-4} \text{ m}^2\right)} = 75{,}000 \text{ V/m}$

ROUND:

(a) $Q_1 = 9.30 \cdot 10^{-10} \text{ C}$

(b) $Q_2 = 9.30 \cdot 10^{-10} \text{ C}$

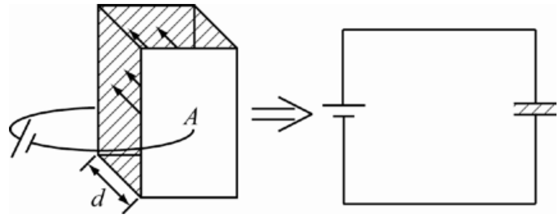
(c) $U = 2.79 \cdot 10^{-8} \text{ J}$

(d) $E_2 = 75.0 \text{ kV/m}$

DOUBLE-CHECK: Numerically, $\Delta V_2 = 7.5 \text{ V}$ and $\Delta V_1 = Qd/\varepsilon_0 A = 52.5 \text{ V}$, so $\Delta V_1 + \Delta V_2 = 60\text{V} = \Delta V$, which means energy was conserved. Also, since $E_2 < S$ (dielectric strength), this capacitor is clearly viable, so it makes sense.

26.51. **THINK:** Since the dielectric material ($\kappa = 2.5$, $d = 50.0 \text{ μm}$ and $\rho = 4.0 \cdot 10^{12} \text{ Ω m}$) acts as the resistor and it shares the same cross sectional area as the capacitor, $C = 0.050 \text{ μF}$, a time constant, τ, should be independent of the actual capacitance and resistance, and only depend on the material.

SKETCH:



RESEARCH: The capacitance is $C = \kappa\varepsilon_0 A / d$. The resistance is $R = \rho d / A$. The time constant is $\tau = RC$.

SIMPLIFY: The time constant is $\tau = RC = \left(\frac{\rho d}{A}\right)\left(\frac{\kappa\varepsilon_0 A}{d}\right) = \kappa\rho\varepsilon_0$.

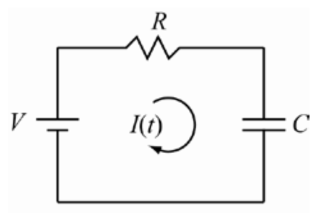
CALCULATE: $\tau = 2.5\left(4.0\cdot 10^{12}\ \Omega\ \text{m}\right)\left(8.85\cdot 10^{-12}\ \text{C}^2/\left(\text{N m}^2\right)\right) = 88.5\ \text{s}$

ROUND: $\tau = 89\ \text{s}$

DOUBLE-CHECK: While this value seems relatively high, it is nonetheless perfectly reasonable. The high resistivity and greater than 1 dielectric material, both imply bigger R and C, so a high τ is reasonable.

26.52. **THINK:** Since the current varies with time due to the charging of the capacitor, $C = 2.00\ \text{mF}$, the energy lost due to heat from the resistor, $R = 100.\ \Omega$, is found by integrating the power dissipation of the resistor over time. When the capacitor is fully charged it has the same potential as the battery, $\Delta V = 12.0\ \text{V}$.

SKETCH:



RESEARCH: When the capacitor is fully charged, the energy stored in it is $U = (1/2)C\Delta V^2$. The power across the resistor is $P = I^2 R$. The current decreases exponentially by $i(t) = i_0 e^{-t/\tau}$, where $\tau = RC$ and $i_0 = V/R$.

SIMPLIFY: Energy across the capacitor: $U_C = (1/2)CV^2$. Energy dissipated through the resistor:

$$U_R = \int_0^\infty P(t)dt = \int_0^\infty \left(I(t)\right)^2 R dt = R\int_0^\infty \left(I_o\right)^2 e^{-2t/\tau} dt = R\int_0^\infty \left(\frac{V}{R}\right)^2 e^{-2t/\tau} dt = \frac{V^2}{R}\int_0^\infty e^{-2t/\tau} dt.$$

Therefore,

$$U_R = \frac{V^2}{R}\left[-\frac{\tau}{2}e^{-2t/\tau}\right]_0^\infty = \frac{V^2}{R}\left(0 - \frac{-\tau}{2}e^0\right) = \frac{V^2\tau}{2R}.$$

Therefore, $\tau = RC$, $U_R = \frac{V^2(RC)}{2R} = \frac{1}{2}CV^2$ and $U_C = U_R$.

CALCULATE: $U_C = U_R = (1/2)(2.00\ \text{mF})(12.0\ \text{V})^2 = 0.144\ \text{J}$

ROUND: $U_C = U_R = 0.144\ \text{J}$, the same energy for both.

DOUBLE-CHECK: The energy stored in capacitor is same as energy lost to heat by the resistor. This makes sense if I consider that the total internal energy should stay the same. Therefore, energy lost by resistor is energy gained by capacitor, so energy is conserved.

26.53. **THINK:** Normally, to be fully discharged the time needs to go to infinity. After $\Delta t = 2.0$ ms, the capacitor should be as close to fully discharged as possible. A good standard of discharge is when the final charge is less than 0.01% which roughly corresponds to a time of 10τ, where τ is the time constant of the circuit. From τ, the capacitance, C, can be determined and using $E = 5.0$ J the potential difference on the plates is found. $R = 10.0\text{ k}\Omega$.

SKETCH: Not required.

RESEARCH: The time constant is approximated as $\tau = (1/10)\Delta t$, an is also $\tau = RC$. The energy stored in the capacitor is $E = (1/2)C\Delta V^2$.

SIMPLIFY: The capacitance is $C = \dfrac{\tau}{R} = \dfrac{\Delta t}{10R}$ the potential difference is then $\Delta V = \sqrt{\dfrac{2E}{C}}$.

CALCULATE: $C = \dfrac{2.0\text{ ms}}{10\cdot 10.0\text{ k}\Omega} = 2.0\cdot 10^{-8}\text{ F} = 0.020\ \mu\text{F}$ $\qquad \Delta V = \sqrt{\dfrac{2(5.0\text{ J})}{0.020\ \mu\text{F}}} = 22361\text{ V}$

ROUND: $C = 0.0200\ \mu\text{F}$, $\Delta V = 22.4$ kV

DOUBLE-CHECK: If instead I chose the capacitor to be only 99% discharged, corresponding to only $\Delta t = 5\tau$, the potential across the capacitor would be about 16 kV, which is also high, so our choice is reasonable.

26.54. **THINK:**

(a) When switch S_1 is closed, the current flows solely through resistors $R_1 = 100.\ \Omega$ and $R_3 = 300.\ \Omega$ which are in series with a battery $V_{\text{emf}} = 6.00$ V.

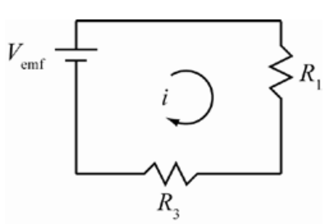
(b) When switch S_2 is closed, the current splits between $R_1 = 100.\ \Omega$ in one branch and $R_2 = 200.\ \Omega$ with a capacitor $C = 4.00$ mF in the other branch. Initially there is no charge on the capacitor so there is no potential drop across it, meaning it does not initially contribute to the current. These branches are then in series with resistor $R_3 = 300.\ \Omega$ and battery $V_{\text{emf}} = 6.00$ V.

(c) The capacitor, $C = 4.00$ mF, will charge but only through resistor $R_2 = 200.\ \Omega$, so as to give a time constant τ. As it charges over $t = 10.0\text{ min} = 600.\text{ s}$, the current through that branch will decrease exponentially.

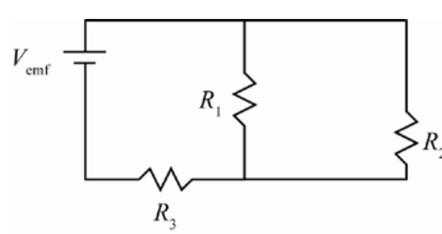
(d) When the capacitor, $C = 4.00$ mF, is fully charged, no current flows through that branch. This means that initially, the battery $V_{\text{emf}} = 6.00$ V, is in series with resistors $R_1 = 100.\ \Omega$ and $R_3 = 300.\ \Omega$. The initial potential in the capacitor must still equal the potential drop across resistor R_1. When switch S_1 is opened, the capacitor begins to discharge though resistors R_1 and $R_2 = 200.\ \Omega$. As capacitor discharges, the current will decrease exponentially to $i_{\text{f}} = 1.00$ mA .

SKETCH:

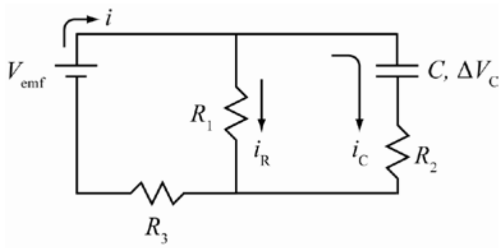
(a)



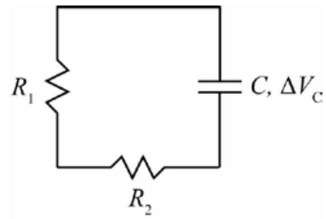
(b)



(c)



(d)



RESEARCH:

(a) The equivalent resistance is $R_{\text{eq}} = R_1 + R_3$. By Ohm's Law, the current through circuit is $i_1 = V_{\text{emf}} / R_{\text{eq}}$.

(b) The equivalent resistance of resistors 1 and 2 is $R_{12} = \left(1/\text{R}_1 + 1/R_2\right)^{-1}$. The total equivalent resistance is then $R_{\text{eq}} = R_3 + R_{12}$. By Ohm's law, the current through circuit is $I_2 = V_{\text{emf}}/R_{\text{eq}}$.

(c) As the capacitor charges, the current through it decrease as $i_C\left(t\right) = i_0 e^{-t/\tau}$ where, $\tau = R_2 C$. The current through resistor R_1 is i_R and the total current out of the battery is $i = i_R + i_C$.

(d) Potential drop across R_1 initially is $i_1 R_1 = \Delta V_1 = \Delta V_C$. Current decays exponentially as $i\left(t\right) = i_0 e^{-t/\tau}$, where $\tau = \left(R_1 + R_2\right)C$ and by Ohm's law, $i_0 = \Delta V_C / \left(R_1 + R_2\right)$.

SIMPLIFY:

(a) $i_1 = \dfrac{V_{\text{emf}}}{R_{\text{eq}}} = \dfrac{V_{\text{emf}}}{R_1 + R_3}$

(b) $R_{12} = \left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)^{-1} = \dfrac{R_1 R_2}{R_1 + R_2},\ i_2 = \dfrac{V_{\text{emf}}}{R_{\text{eq}}} = \dfrac{V_{\text{emf}}}{R_3 + R_{12}}.$

(c) Calculate $i_C\left(t\right)$ and infer i from it.

(d) Initial current is $i_0 = \dfrac{\Delta V_C}{R_1 + R_2} = \dfrac{R_1 i_1}{R_1 + R_2}$ when $i\left(t\right) = i_f$. Therefore,

$$i_f = \frac{R_1 i_1}{R_1 + R_2} e^{-t/\tau} \Rightarrow \frac{i_f\left(R_1 + R_2\right)}{R_1 i_1} = e^{-t/\tau} \Rightarrow t = -\tau \ln\left(\frac{i_f\left(R_1 + R_2\right)}{R_1 i_1}\right).$$

CALCULATE:

(a) $i_1 = \dfrac{6.00 \text{ V}}{100.\ \Omega + 300.\ \Omega} = 0.0150 \text{ A} = 15.0 \text{ mA}$

(b) $R_{12} = \dfrac{\left(100.\ \Omega\right)\left(200.\ \Omega\right)}{100.\ \Omega + 200.\ \Omega} = 66.67\ \Omega,\quad i_2 = \dfrac{6.00 \text{ V}}{300.\ \Omega + 66.67\ \Omega} = 0.01636 \text{ A} = 16.36 \text{ mA}$

(c) $\tau = \left(200.\ \Omega\right)\left(4.00 \text{ mF}\right) = 0.800 \text{ s}.\quad i_C\left(t\right) = i_0 e^{\frac{-600.\text{ s}}{0.8\text{ s}}} = i_0 e^{-750.} \approx 0 \text{ A}.$ Regardless of what i_0 is after 10.0 min, the current through that branch is effectively 0.0 A. Therefore, $i = i_R = 15.0$ mA. Since there is no current through the capacitor, the circuit is equivalent to having switch S_2 open, as in part (a) so current through battery is then the same as in part (a).

(d) $\tau = \left(100.\ \Omega + 200.\ \Omega\right)\left(4.0 \text{ mF}\right) = 1.20 \text{ s}$ and $t = -\left(1.20 \text{ s}\right)\ln\left(\dfrac{\left(1.00 \text{ mA}\right)\left(100.\ \Omega + 200.\ \Omega\right)}{\left(100.\ \Omega\right)\left(15.0 \text{ mA}\right)}\right) = 1.9313 \text{ s}$

ROUND:

(a) $i_1 = 15.0$ mA

(b) $i_2 = 16.4$ mA

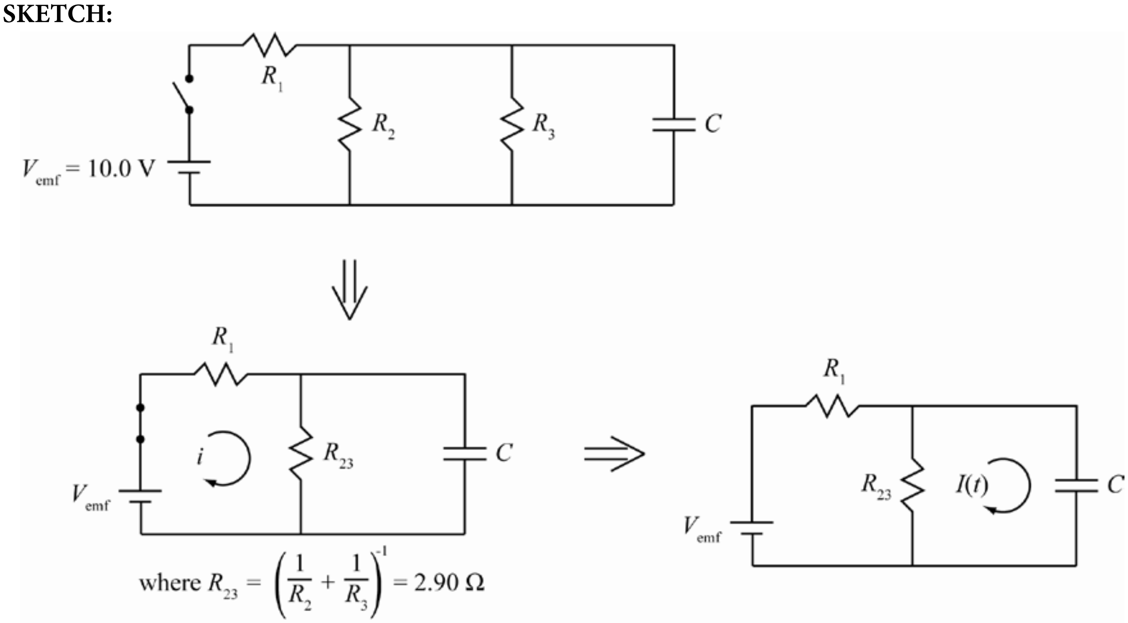
(c) $i = 15.0$ mA

(d) $t = 1.93$ s

DOUBLE-CHECK:
(a) This is a reasonable value for current.
(b) Since I added a resistor in parallel, the overall resistance is expected to decrease, and hence the current increase, so it makes sense.
(c) From part (b), I saw the overall current was greater than in part (a). If a piece of the current found in part (b) dies off, the final current should be smaller, so it makes sense.
(d) Since I know the current should reduce to zero in $t = 600.$ s, then reducing by 80.0% in only 1.93 s is very reasonable so it makes sense.

26.55. **THINK:** The capacitor, $C = 2.00\ \mu\text{F}$, charges via the battery, $\Delta V = 10.0$ V, through resistor, $R_1 = 10.0\ \Omega$, so the resistors, $R_2 = 4.00\ \Omega$ and $R_3 = 10.0\ \Omega$, can be simplified to be in parallel. After a long time, the capacitor becomes fully charged and no current goes through it. The potential drop across it is then the same as the drop across R_2 and R_3. The energy of the capacitor is proportional to the square of the potential drop across it. The total energy lost across R_3 is determined by integrating the power across it over time.

SKETCH:



RESEARCH: The current through the circuit after a long time is $i = V_{\text{emf}} / (R_1 + R_{23})$. Resistors in parallel add as $R_{23} = \left(R_2^{-1} + R_3^{-1}\right)^{-1}$. The potential drop across the capacitor is $\Delta V_C = iR_{23}$. The energy in the capacitor is given by $E = C(\Delta V_C)^2 / 2$. When the switch is open, the current through R_3 is $i_3 = \Delta V_C / R_2$. The current across R_3 varies as $i_3(t) = i_3 e^{-t/R_{23}C}$. The power across R_3 is given by $P_3 = i_3^2(t) R_3$. The energy across R_3 is given by $E_3 = \int_0^\infty P_3(t)\,dt$.

SIMPLIFY:

(a) The potential drop across the capacitor is given by: $\Delta V_C = iR_{23} = \dfrac{V_{\text{emf}} R_{23}}{R_1 + R_{23}}$.

(b) The energy in the capacitor is given by $E = \dfrac{1}{2} C (\Delta V_C)^2$.

(c) The energy across R_3 is given by: $E_3 = \int_0^\infty P_3(t)dt = \int_0^\infty R_3 i_3^2 e^{-2t/R_{23}C} dt = \frac{\Delta V_C^2}{R_3}\int_0^\infty e^{-2t/R_{23}C} dt$

$$= \frac{\Delta V_C^2}{R_3}\left[-\frac{R_{23}C}{2}e^{-2t/R_{23}C}\right]_{t=0}^{t=\infty} = \frac{\Delta V_C^2}{R_3}\left[0+\frac{R_{23}C}{2}\right] = \frac{\Delta V_C^2 R_{23}C}{2R_3}.$$

CALCULATE:

(a) $R_{23} = \left(\left(10.0\ \Omega\right)^{-1} + \left(4.00\ \Omega\right)^{-1}\right)^{-1} = 2.86\ \Omega,\ \Delta V_C = \frac{(10.0\ \text{V})(2.86\ \Omega)}{10.0\ \Omega + 2.86\ \Omega} = 2.22\ \text{V}$

(b) $E = \frac{1}{2}(2.00\ \mu\text{F})(2.22\ \text{V})^2 = 4.938\cdot 10^{-6}\ \text{J}$

(c) $E_3 = \frac{(2.22\ \text{V})^2(2.86\ \Omega)(2.00\ \mu\text{F})}{2(10.0\ \Omega)} = 1.411\cdot 10^{-6}\ \text{J}$

ROUND:

(a) $\Delta V_C = 2.22\ \text{V}$

(b) $E = 4.94\ \mu\text{J}$

(c) $E_3 = 1.41\ \mu\text{J}$

DOUBLE-CHECK: The energy across R_2 is $E_2 = (\Delta V_C)^2 R_{23}C/2R_2 = 3.527\ \mu\text{J}$ The result makes sense because energy is conserved: $E_2 + E_3 = E$.

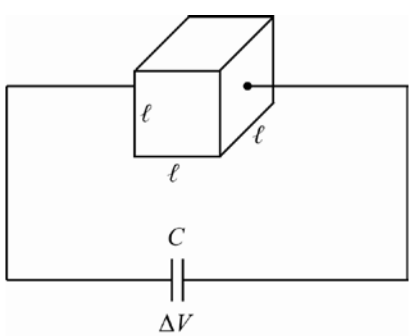
26.56. **THINK:**

(a) The capacitor, $C = 15\mu\text{F}$ and $\Delta V_C = 100.0\ \text{V}$, is fully discharged when the charge is less than 0.01%, which roughly corresponds to a time of 10τ, where τ is the time constant of the circuit. The resistor in question is a cube of gold of sides $l = 2.5\ \text{mm}$ and resistivity $\rho_R = 2.44\cdot 10^{-8}\ \Omega\ \text{m}$.

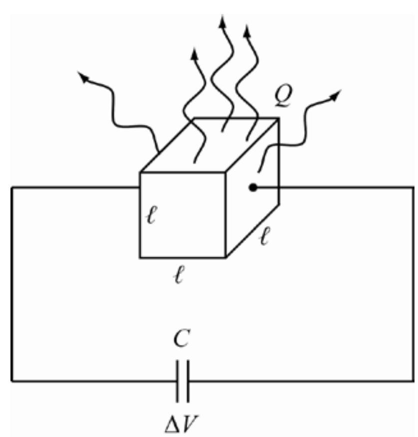
(b) The capacitor, $C = 15\ \mu\text{F}$ and $\Delta V_C = 100.0\ \text{V}$, is fully discharged so that all the initial stored energy has gone to heating the resistor. The resistor in question is a cube of gold of size $l = 2.5\ \text{mm}$, density $\rho_D = 19.3\cdot 10^3\ \text{kg/m}^3$ and specific heat $c = 129\ \text{J/kg °C}$. Assume the cube is initially at room temperature, $T_i = 20.0\ °\text{C}$.

SKETCH:

(a)



(b)



RESEARCH:

(a) The resistance of the cube is $R = \rho_R L/\text{A}$. The time constant is $t = 10\tau$.

(b) The energy of the capacitor is $U_C = (1/2)C\Delta V_C^2$. The energy gained by the gold block increases its temperature as $Q = mc\Delta T$. Mass of gold is $m = \rho_D V$. The energy the cube gains is same energy the capacitor dissipates, $U_C = Q$.

SIMPLIFY:

(a) The time for discharge is $t = 10\tau = 10RC = 10\left(\rho_R \frac{L}{\text{A}}\right)C = 10\left(\frac{\rho_R l^2}{l}\right)C = \frac{10\rho_R C}{l}$.

(b) To find the final temperature: $Q = U_C \Rightarrow mc\Delta T = (1/2)C(\Delta V)^2$. Therefore,

$$\rho_D Vc(T_\text{f} - T_\text{i}) = \frac{1}{2}C\Delta V_c^2 \quad\Rightarrow\quad T_\text{f} = \frac{C\Delta V_c^2}{2\rho_D l^3 c} + T_\text{i}.$$

CALCULATE:

(a) $t = \dfrac{10(2.44\cdot 10^{-8}\ \Omega\ \text{m})(15\ \mu\text{F})}{(2.5\ \text{mm})} = 1.464\cdot 10^{-9}\ \text{s}$

(b) $T_\text{f} = \dfrac{(15\ \mu\text{F})(100.0\ \text{V})^2}{2(1.93\cdot 10^4\ \text{kg/m}^3)(2.5\ \text{mm})^3(129\ \text{J/(kg °C)})} + 20.0\ \text{°C} = 21.928\ \text{°C}$

ROUND:

(a) $t = 1.46$ ns

(b) $T_\text{f} = 21.9$ °C

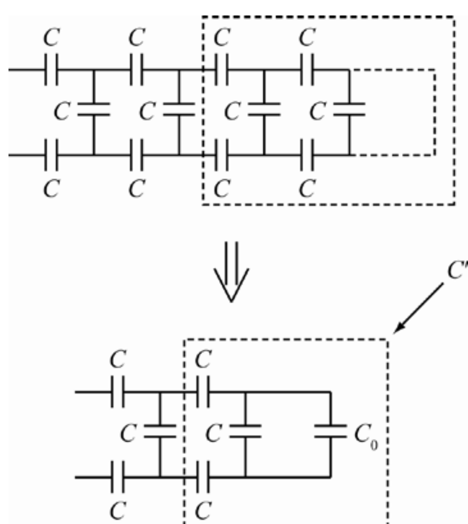
DOUBLE-CHECK:

(a) The calculated value has appropriate units for time, and the magnitude of the value is reasonable for a discharge time.

(b) The temperature of the gold cube does not change appreciable, which would be desirable for real circuits, so it makes sense.

26.57. **THINK:** Consider any given rung on the ladder to have a total equivalent capacitance of C_0. Next, determine the equivalent capacitance, C_1, of C_0 with the next rung and two legs. Since the ladder is infinite, it should not matter where on the ladder the analysis is performed. If C_0 is the equivalent capacitance of all the capacitors beyond some point, then adding another set of capacitors to the mix should not affect anything and C_1 should equal C_0, giving a recursive relation in *C* and thus the total equivalent capacitance, in terms of *C*, can be determined.

SKETCH:



RESEARCH: Capacitors add in series as $C_{eq}^{-1} = C_1^{-1} + C_2^{-1}$. Capacitors in parallel add as $C_{eq} = C_1 + C_2$.

SIMPLIFY: C_0 is parallel to *C*, which gives $C' = C_0 + C$. C' is in series with 2 *C*'s, which gives:

$$\frac{1}{C_1} = \frac{2}{C} + \frac{1}{C+C_0} = \frac{1}{C_0}.$$

Therefore, $2C_0 + \frac{CC_0}{C+C_0} = C \Rightarrow 2C_0\left(C+C_0\right) + CC_0 = C\left(C+C_0\right) \Rightarrow 2C_0C + 2C_0^2 + CC_0 = C^2 + CC_0$

$\Rightarrow 2C_0^2 + 2CC_0 - C^2 = 0.$

CALCULATE: Using the quadratic equation: $C_0 = \frac{-2C \pm \sqrt{4C^2 + 8C^2}}{4} = -\frac{C}{2} \pm \frac{\sqrt{12}}{4}C = C\left(\frac{-1 \pm \sqrt{3}}{2}\right).$

$C_0 = \left(\frac{\sqrt{3}-1}{2}\right)C,$ since C_0 must be positive.

ROUND: Not necessary.

DOUBLE-CHECK: Consider the first rung of three capacitors in series. The equivalent of these is *C*/3. Adding another rung of three capacitors puts one capacitor in parallel with *C*/3 and then two capacitors in series with this to get:

$$C + \frac{C}{3} = \frac{4C}{3} \text{ and then } \left(\frac{2}{C} + \frac{3}{4C}\right)^{-1} = \frac{4}{11}C.$$

Adding another rung performs the same operation as before to get:

$$C + \frac{4}{11}C = \frac{15}{11}C \text{ and then } \left(\frac{2}{C} + \frac{41}{56C}\right)^{-1} = \frac{56}{153}C.$$

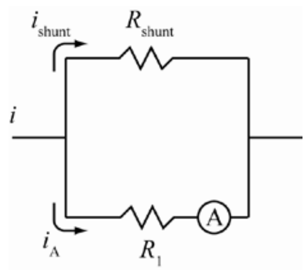
Continuing on gives: $C + \frac{15}{41}C = \frac{56}{41}C$ and then $\left(\frac{2}{C} + \frac{153}{209C}\right)^{-1} = \frac{209}{571}C$, $C + \frac{209}{571}C = \frac{780}{571}C$ and then $\left(\frac{2}{C} + \frac{571}{780C}\right)^{-1} = 0.36602534C$, $C + 0.36602534C = 1.36602534C$ and therefore $\left(\frac{2}{C} + \frac{1}{1.36602534C}\right)^{-1} = 0.366025399C$. The series converges around $C_0 = 0.366025C$. The solution is $C_0 = \left(\sqrt{3} - 1/2\right)C = 0.366025C$, which is the same as the above result. Therefore, by continuously adding rungs to the ladder, it converges to the previous result.

26.58. (a) If the switch is closed for a long time, the capacitor is fully charged and there is no current through that branch. Therefore, the current through The $4.0\ \Omega$ resistor is $i = 0$ A.

(b) With no current through R_2, the potential drop across it is $\Delta V_2 = 0$ V. The two resistors, $R_1 = 6.0\ \Omega$ and $R_3 = 8.0\ \Omega$, are in series with each other, so the current through them is $i = \Delta V / \left(R_1 + R_3\right) = (10.0 \text{ V})/(14.0\ \Omega) = 0.714$ A. The potential drop across the $6.0\ \Omega$ resistor is $\Delta V_1 = iR_1 = \left(0.714 \text{ A}\right)\left(6.0\ \Omega\right) = 4.286$ V, and across the $8.0\ \Omega$ resistor is $\Delta V_3 = iR_3 = \left(0.714 \text{ A}\right)\left(8.0\ \Omega\right) = 5.714$ V. Therefore, to three significant figures, $\Delta V_1 = 4.29$ V, $\Delta V_2 = 0.00$ V and $\Delta V_3 = 5.71$ V.

(c) The potential on the capacitor is the same as the potential drop across the $8.0\ \Omega$ resistor since they are parallel, so $\Delta V_C = \Delta V_3 = 5.71$ V.

26.59. (a) The maximum current through the ammeter is $i_A = 1.5$ mA. The ammeter has resistance $R_1 = 75\ \Omega$. The current through a resistor is given by $i = V / R$, where V is the potential difference across the resistor. Since current flows through the path of least resistance, when a shunt resistor of small resistance R_{shunt} is connected in parallel with the ammeter, most of the current flows through the shunt resistor. The shunt resistor carries most of the load so that the ammeter is not damaged.



From Kirchoff's rules $i = i_{\text{shunt}} + i_A$ and $i_{\text{shunt}} R_{\text{shunt}} = i_A R_1$. Therefore,

$$i = \frac{i_A R_1}{R_{\text{shunt}}} + i_A = i_A\left(\frac{R_1}{R_{\text{shunt}}} + 1\right).$$

For known current i_A (measured by ammeter) and known resistances R_1 and R_{shunt}, the new maximum current i can be calculated. Note that $i > i_A$. A shunt resistor is added in parallel with an ammeter so the current can be increased without damaging the ammeter.

(b) From Kirchoff's rules shown above,

$$R_{\text{shunt}} = \frac{i_A R_1}{i_{\text{shunt}}} = \frac{i_A R_1}{i - i_A} = \frac{(1.50 \text{ mA})(75.0\ \Omega)}{15.0 \text{ A} - 1.50 \text{ mA}} = 7.501 \cdot 10^{-3}\ \Omega = 7.50 \text{ m}\Omega.$$

26.60. The potential on the capacitor, $C = 150.\ \mu\text{F}$, when it is fully charged is $\Delta V = 200.$ V. The potential decreases exponentially as it discharges through $R = 1.00\ \text{M}\Omega$, by $\Delta V(t) = Ve^{-t/RC}$. When $\Delta V(t) = 50.0$ V, $\Delta V(t) = 50.0 \text{ V} = (200. \text{ V})e^{-\frac{t}{RC}} \Rightarrow e^{-\frac{t}{RC}} = \frac{1}{4}$. Therefore, the result is $t = RC\ln(4.00) = 207.94 \text{ s} = 208 \text{ s}$ or 3.47 min.

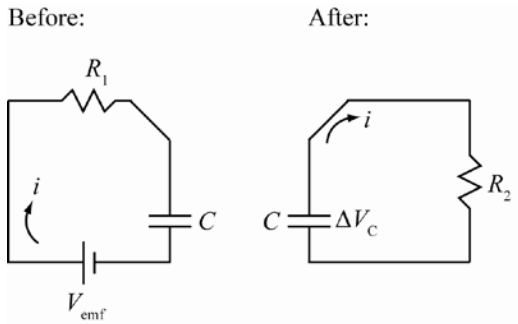
26.61. The capacitor, C, discharges through the bulb, $R_f = 2.5\ \text{k}\Omega$, in $\Delta t_D = 0.20$ ms. The charging time is $\Delta t_C = 0.80$ ms. For simplicity assume the charging and discharging time are the time constants of the circuits. Therefore, $\Delta t_d = \tau_d = R_f C \Rightarrow C = \frac{\Delta t_d}{R_f} = \frac{0.20 \text{ ms}}{2.5\ \text{k}\Omega} = 8.0 \cdot 10^{-8} \text{ F} = 80. \text{ nF}$, and

$$\Delta t_C = \tau_C = RC \Rightarrow R = \frac{\Delta t_C}{\text{C}} = \frac{0.80 \text{ ms}}{80. \text{ nF}} = 10.\ \text{k}\Omega.$$

26.62. The potential, V_{emf}, of the battery is the same with ammeter, $R_0 = 53\ \Omega$, as without. The external resistance $R = 1130\ \Omega$, has a current of $I = 5.25$ mA with ammeter, so by Ohm's law $V_{\text{emf}} = i(R_0 + R) = i'R \Rightarrow i' = \frac{i(R_0 + R)}{R} = \frac{(5.25 \text{ mA})(53\ \Omega + 1130\ \Omega)}{1130\ \Omega} = 5.4962 \text{ mA} = 5.50 \text{ mA}.$

26.63. **THINK:** When the switch is set to X for a long time, the capacitor, $C = 10.0\ \mu\text{F}$, charges fully so that it has the same potential as the battery, $\Delta V_C = V_{\text{emf}} = 9.00$ V. After placing the switch on Y, the capacitor discharges through resistor $R_2 = 40.0\ \Omega$ and decreases exponentially for both immediately $(t = 0 \text{ s})$ and $t = 1.00$ ms after the switch.

SKETCH:



RESEARCH: By Ohm's law, the current initially is $i_0 = \Delta V_C / R_2$, where $\Delta V_C = V_{emf}$. Current decays exponentially as $i(t) = i_0 e^{-t/\tau}$, where $\tau = RC$.

SIMPLIFY:

(a) Initial current is $i_0 = \Delta V_C / R_2 = V_{emf} / R_2$.

(b) After $t = 1$ ms, current is $i(t) = I_0 e^{-t/\tau} = i_0 e^{\frac{-t}{RC}}$.

CALCULATE:

(a) $i_0 = \dfrac{9.00 \text{ V}}{40.0 \ \Omega} = 0.225 \text{ A}$

(b) $i(1.00 \text{ ms}) = (0.225 \text{ A}) e^{\frac{-1.00 \text{ ms}}{(40.0 \ \Omega)(10.0 \ \mu\text{F})}} = 0.01847 \text{ A}$

ROUND:

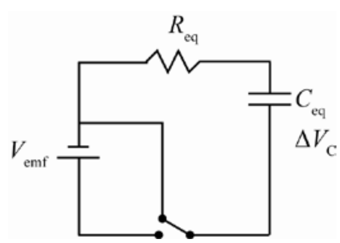
(a) $i_0 = 225$ mA

(b) $i(1.00 \text{ ms}) = 18.5$ mA

DOUBLE-CHECK: After only 1.00 ms, the current decreases by almost 90.0%, which would make this a desirable circuit, so it makes sense.

26.64. **THINK:** Since the two resistors, $R = 2.2 \text{ k}\Omega$, the two capacitors, $C = 3.8 \ \mu\text{F}$, and the battery, $V_{emf} = 12.0$ V, are all in series, the order doesn't matter, so equivalent resistance and capacitance are used to determine the time constant, τ. The current then decreases exponentially from it's initial current to $i_f = 1.50$ mA in time t.

SKETCH:



RESEARCH: The equivalent resistance is $R_{eq} = R + R = 2R$, and the equivalent capacitance is $C_{eq} = (1/C + 1/C)^{-1} = C/2$. Initial potential in capacitor, $\Delta V_C = V_{emf}$. By Ohm's law, the initial current is $i_0 = \Delta V_C / R_{eq}$.

SIMPLIFY: The current at time t is $i(t)=i_0e^{-t/\tau}=\dfrac{\Delta V_\text{C}}{R_{eq}}e^{\frac{-t}{R_{eq}C_{eq}}}=\dfrac{\Delta V_\text{C}}{2R}e^{\frac{-t}{(2R)(C/2)}}=\dfrac{\Delta V_\text{C}}{2R}e^{\frac{-t}{RC}}.$

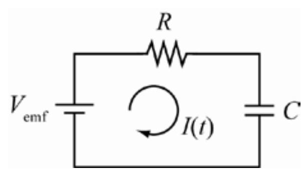
$$i(t)=i_\text{f}\ \Rightarrow\ i_\text{f}=\frac{\Delta V_\text{C}}{2R}e^{\frac{-t}{RC}}\ \Rightarrow\ \frac{2Ri_\text{f}}{\Delta V_\text{C}}=e^{\frac{-t}{RC}}\ \Rightarrow\ t=-RC\ln\left(\frac{2Ri_\text{f}}{\Delta V_\text{C}}\right)$$

CALCULATE: $t=-(2.20\text{ k}\Omega)(3.80\ \mu\text{F})\ln\left(\dfrac{2(2.20\text{ k}\Omega)(1.50\text{ mA})}{12.0\text{ V}}\right)=0.004998\text{ s}$

ROUND: $t=5.00$ ms

DOUBLE-CHECK: The initial value of the current was about 2.7 mA. The circuit decays to about half its original current in roughly 5 ms, which makes this a desirable circuit, so it makes sense.

26.65. The charge on the capacitor increases exponentially with a time constant, $\tau=3.1$ s. Since the amount of energy in the capacitor is proportional to the square of the charge, the energy also increases exponentially.



The charge on the capacitor is given by $q(t)=q_0\left(1-e^{-t/\tau}\right)$. The energy on the capacitor is given by $E(t)=q^2(t)/2C$. The time to get to half of the maximum energy is given by $E(t)=E_\text{max}/2$, where $E_\text{max}=q_0^2/2C$. This gives:

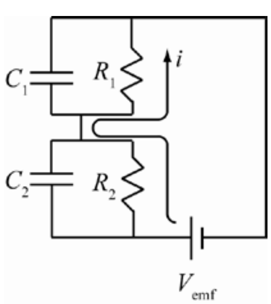
$$E(t)=\frac{1}{2}E_\text{max}=\frac{q_0^2}{4C}=\frac{q^2(t)}{2C}=\frac{q_0^2\left(1-e^{-t/\tau}\right)^2}{2C}\ \Rightarrow\ 1-e^{-t/\tau}=\frac{1}{\sqrt{2}}\ \Rightarrow$$

$$t=-\tau\ln\left(1-\frac{1}{\sqrt{2}}\right)=-(3.1\text{ s})\ln\left(1-\frac{1}{\sqrt{2}}\right)=3.8\text{ s}.$$

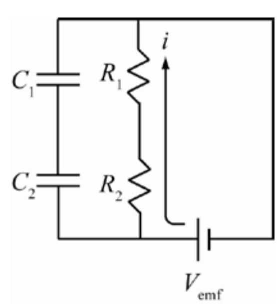
26.66. **THINK:** When the switch is closed for a long time, the capacitors, $C_1=1.00\ \mu\text{F}$ and $C_2=2.00\ \mu\text{F}$, are fully charged so no current flows through them, and thus the current only flows through the two resistors, $R_1=1.00\text{ k}\Omega$ and $R_2=2.00\text{ k}\Omega$, driven by a battery $V_\text{emf}=10.0$ V. At this point, the potential drop across each resistor is equal to the potential on its complementary capacitor. Since the capacitors are in series, they have the same charge on them.

SKETCH:

(a)



(b)



RESEARCH: The current, by Ohm's law is found in both cases as $i=V_\text{emf}/(R_1+R_2)$. When the switch is closed, the potential drop across capacitor C_j is $\Delta V_j=Q_j/C_j=IR_j$ (for $j=1,2$). When switch is open, charge on each plate is $Q=C_{eq}V_\text{emf}$.

SIMPLIFY:

(a) The charges on the capacitor are given by: $Q_j/C_j = iR_j \Rightarrow Q_j = iR_jC_j = \Delta V_{\text{emf}}R_jC_j/(R_1+R_2)$.

$Q_1 = V_{\text{emf}}R_1C_1/(R_1+R_2)$ and $Q_2 = V_{\text{emf}}R_2C_2/(R_1+R_2)$.

(b) The charge on each capacitor is $Q = C_{eq}V_{\text{emf}} = (1/C_1+1/C_2)^{-1}V_{\text{emf}} = C_1C_2/(C_1+C_2)V_{\text{emf}}$.

CALCULATE:

(a) $Q_1 = \dfrac{(10.0\text{ V})(1.00\text{ k}\Omega)(1.00\ \mu\text{F})}{1.00\text{ k}\Omega+2.00\text{ k}\Omega} = 3.33\cdot 10^{-6}\text{ C}$ and $Q_2 = \dfrac{(10.0\text{ V})(2.00\text{ k}\Omega)(2.00\ \mu\text{F})}{1.00\text{ k}\Omega+2.00\text{ k}\Omega} = 1.33\cdot 10^{-5}\text{ C}$

(b) $Q = \dfrac{(10.0\text{ V})(1.00\ \mu\text{F})(2.00\ \mu\text{F})}{1.00\ \mu\text{F}+2.00\ \mu\text{F}} = 6.67\cdot 10^{-6}\text{ C}$

ROUND:

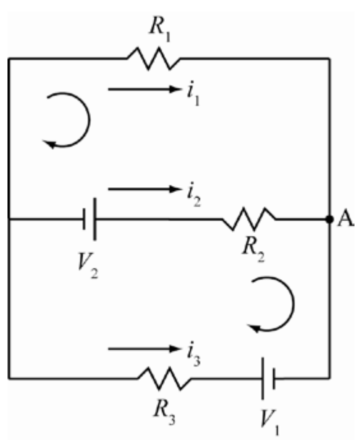
(a) $Q_1 = 3.33\ \mu\text{C}$, and $Q_2 = 13.3\ \mu\text{C}$

(b) $Q = 6.67\ \mu\text{C}$

DOUBLE-CHECK: More charge builds up when the capacitors have their own resistor than when they are paired together. This is because even though the potential drop across them is the same in both cases, when the switch is open, the overall capacitance of the circuit is less than the sum of two. So a smaller C gives a smaller Q, so it makes sense.

26.67. **THINK:** From Kirchoff's rules, an equation can be obtained for the sum of the three currents, i_1, i_2 and i_3, and two equations for the two inner loops of the circuit. This will yield 3 equations for 3 unknowns (the currents) and can be solved by simple substitution. Once the currents are known, the power over each resistor is found via Ohm's law. $R_1 = 10.0\ \Omega$, $R_2 = 20.0\ \Omega$, $R_3 = 30.0\ \Omega$, $V_1 = 15.0\text{ V}$ and $V_2 = 9.00\text{ V}$.

SKETCH:



RESEARCH: Looking at point A, the three currents all flow into it, so $i_1+i_2+i_3=0$. Going clockwise in each loop (upper and lower) yields two more equations: $-i_1R_1+i_2R_2-V_2=0$ and $V_2-i_2R_2+V_1+i_3R_3=0$. The power across a resistor is $P=i^2R$.

SIMPLIFY: Since all resistances and all voltages are known, the first three equations can be solved for the three separate currents:

$i_1+i_2+i_3=0 \Rightarrow i_3=-i_1-i_2$, and $-i_1R_1+i_2R_2-V_2=0$, then

$-i_1R_1+i_2R_2-V_2=0 \Rightarrow i_1=\dfrac{i_2R_2-V_2}{R_1}$.

$V_2 - i_2R_2 + V_1 + i_3R_3 = 0 \Rightarrow V_2 - i_2R_2 + V_1 + (-i_1 - i_2)R_3 = 0$

$\Rightarrow V_2 - i_2R_2 + V_1 + \left(-\left(\frac{i_2R_2 - V_2}{R_1}\right) - i_2\right)R_3 = 0 \Rightarrow i_2 = \frac{R_1V_1 + V_2(R_1 + R_3)}{R_1R_2 + R_1R_3 + R_2R_3}.$

$i_3 = -i_1 - i_2.$

The power across each is then $P_1 = i_1^2R_1$, $P_2 = i_2^2R_2$ and $P_3 = i_3^2R_3$.

CALCULATE:

$$i_2 = \frac{R_1V_1 + V_2(R_1 + R_3)}{R_1R_2 + R_1R_3 + R_2R_3} = \frac{(10.0\ \Omega)(15.0\text{ V}) + (9.00\text{ V})(10.0\ \Omega + 30.0\ \Omega)}{(10.0\ \Omega)(20.0\ \Omega) + (10.0\ \Omega)(30.0\ \Omega) + (20.0\ \Omega)(30.0\ \Omega)} = 0.4636\text{ A}$$

$$i_1 = \frac{i_2R_2 - V_2}{R_1} = \frac{(0.4636\text{ A})(20.0\ \Omega) - (9.00\text{ V})}{10.0\ \Omega} = 0.02727\text{ A}$$

$i_3 = -i_1 - i_2 = -(0.4636\text{ A}) - (0.02727\text{ A}) = -0.49087\text{ A}$, or $i_3 = 0.49087\text{ A}$ to the left.

$P_1 = (0.02727\text{ A})^2(10.0\ \Omega) = 0.00744\text{ W}$, $P_2 = (0.4636\text{ A})^2(20.0\ \Omega) = 4.299\text{ W}$, and

$P_3 = (0.49087\text{ A})^2(30.0\ \Omega) = 7.230\text{ W}.$

ROUND: $P_1 = 7.44\text{ mW}$, $P_2 = 4.30\text{ W}$ and $P_3 = 7.23\text{ W}$.

DOUBLE-CHECK: Looking back at the values for current, it is found that

$$i_1 + i_2 + i_3 = 0.4636\text{ A} + 0.02727\text{ A} - 0.49087\text{ A} = 0,$$

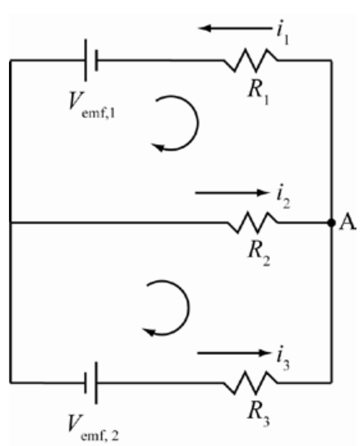
which is what would be expected. Going from left to right on each branch gives

$$-i_1R_1 = -\frac{3.00}{11.0}\text{ V},\ V_2 - i_2R_2 = -\frac{3.00}{11.0}\text{ V and } -i_3R_3 - V_1 = -\frac{3.00}{11.0}\text{ V}.$$

So the potential drop across each branch in parallel is the same, so the answers make sense.

26.68. **THINK:** From Kirchhoff's rules, an equation can be obtained for the sum of the three currents, i_1, i_2 and i_3, and two equations can be obtained for the two inner loops of the circuit. This will yield 3 equations for 3 unknowns (the currents) and can be solved by substitution. Once the currents are known, the voltage drop over resistor 2 is found via Ohm's law. $R_1 = 30.0\ \Omega$, $R_2 = 40.0\ \Omega$, $R_3 = 20.0\ \Omega$, $V_{\text{enf},1} = 12.0\text{ V}$ and $V_{\text{emf},2} = 16.0\text{ V}$.

SKETCH:



RESEARCH: By the choice of directions of currents, at point A, the currents sum as $i_1 - i_2 - i_3 = 0$. Going clockwise in the upper and lower loops gives 2 equations: $-V_1 + i_1R_1 + i_2R_2 = 0$ and $-i_2R_2 + i_3R_3 - V_2 = 0$. Potential drop across resistor 2 is $\Delta V = i_2R_2$.

SIMPLIFY:

$i_1 - i_2 - i_3 = 0 \Rightarrow i_3 = i_1 - i_2.$

$-V_1 + i_1R_1 + i_2R_2 = 0 \Rightarrow i_1 = \dfrac{V_1 - i_2R_2}{R_1}$

$-i_2R_2 + i_3R_3 - V_2 = 0 \Rightarrow -i_2R_2 + (i_1 - i_2)R_3 - V_2 = 0$

$\Rightarrow -i_2R_2 + \left(\left(\dfrac{V_1 - i_2R_2}{R_1}\right) - i_2\right)R_3 - V_2 = 0 \Rightarrow i_2 = \dfrac{R_3V_1}{R_1R_2 + R_1R_3 + R_2R_3}$

The potential drop across it is $\Delta V = |i_2|R_2$.

CALCULATE:

$$i_2 = \frac{R_3V_1}{R_1R_2 + R_1R_3 + R_2R_3} = \frac{(20.0\ \Omega)(12.0\text{ V})}{(30.0\ \Omega)(40.0\ \Omega) + (30.0\ \Omega)(20.0\ \Omega) + (40.0\ \Omega)(20.0\ \Omega)} = 0.092307\text{ A}$$

$\Delta V = (0.092307\text{ A})(40.0\ \Omega) = 3.6923\text{ V}$

ROUND: $\Delta V = 3.69\text{ V}$

DOUBLE-CHECK: Going back to equation for i_1 and i_3, I get $i_1 = 0.2769\text{ A}$ and $i_3 = -0.18461\text{ A}$, where $i_2 = 0.093207\text{ A}$, so the currents are consistent. Calculating the potential drop across the upper and lower branches gives $V_1 - i_1R_1 = 3.6923\text{ V}$ and $V_2 + i_3R_3 = 3.6923\text{ V}$, so each branch has the same potential drop across it, so it makes sense.

26.69. (a) From equation 24.10 of the textbook, the capacitance of a spherical capacitor is $C = \dfrac{4\pi\varepsilon_0 ab}{b-a}$, where $b = 1.10\text{ cm}$ and $a = 1.00\text{ cm}$. Since it is connected in series with resistor $R = 10.0\text{ M}\Omega$ and emf voltage, $V_{\text{emf}} = 10.0\text{ V}$, the time constant is

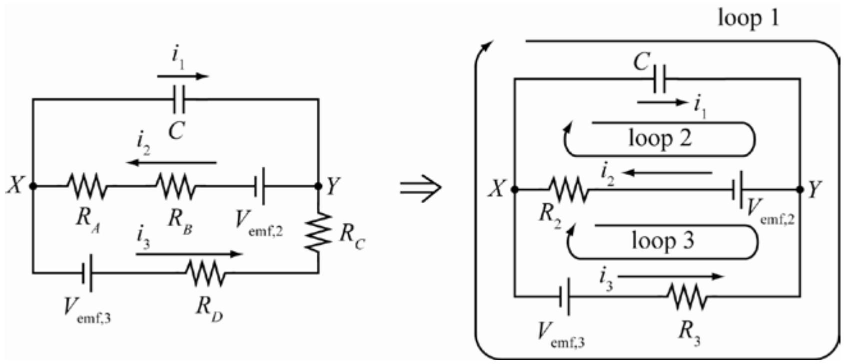
$$\tau = RC = \frac{4\pi\varepsilon_0 Rba}{(b-a)} = \frac{4\pi\left(8.8542\cdot 10^{-12}\text{ N m}^2/\text{C}^2\right)(10.0\text{ M}\Omega)(1.10\text{ cm})(1.00\text{ cm})}{(1.10\text{ cm} - 1.00\text{ cm})} = 1.22\cdot 10^{-4}\text{ s}.$$

(b) The charge on the capacitor still grows: $q(t) = q_0\left(1 - e^{-t/\tau}\right)$, where $q_0 = CV_{\text{emf}}$. Therefore,

$$\begin{aligned} q(0.100\text{ ms}) &= \frac{4\pi\varepsilon_0 V_{\text{emf}} ba}{(b-a)}\left(1 - e^{-\frac{(0.1\text{ ms})}{\tau}}\right) \\ &= \frac{4\pi\left(8.8542\cdot 10^{-12}\text{ N m}^2/\text{C}^2\right)(10.0\text{ V})(1.10\text{ cm})(1.00\text{ cm})}{(1.10\text{ cm} - 1.00\text{ cm})}\left(1 - e^{-\frac{(0.100\text{ ms})}{(1.22\cdot 10^{-4}\text{ s})}}\right) \\ &= 68.5\text{ pC} \end{aligned}$$

26.70. **THINK:** Since the circuit has three branches, four equations (one for each branch and the equation for the current at a junction) can be written down simply by inspection. However, a deeper analysis is required to fully understand the evolution of the circuit. For example, as the capacitor, $C = 30.0\ \mu\text{F}$, charges, the current through it, $i_1(t)$, starts at some maximum and decays to zero. When this happens, the other currents, $i_2(t)$ and $i_3(t)$ must become equal. Even though there are two branches with batteries, $V_{\text{emf},2} = 80.0\text{ V}$ and $V_{\text{emf},3} = 80.0\text{ V}$, and resistors, $R_\text{A} = 40.0\ \Omega$ and $R_\text{B} = 1.0\ \Omega$, and $R_\text{C} = 20.0\ \Omega$ and $R_\text{D} = 1.0\ \Omega$, respectively, the capacitor effectively sees two resistors in parallel to charge through. All three currents and the potential across each branch are time dependent.

SKETCH:



RESEARCH: Starting at junction X and going clockwise around the three loops gives three equations for the potential along them. Also, junction X gives an equation relating the currents.

$$-\Delta V_{\text{C}}(t)+R_3 i_3(t)-V_{\text{emf},3}=0 \quad (1)$$

$$-\Delta V_{\text{C}}(t)+V_{\text{emf},2}-R_2 i_2(t)=0 \quad (2)$$

$$R_2 i_2(t)-V_{\text{emf},2}+R_3 i_3(t)-V_{\text{emf},3}=0 \quad (3)$$

$$i_2(t)=i_1(t)+i_3(t) \quad (4)$$

where $R_2=R_{\text{A}}+R_{\text{B}}$ and $R_3=R_{\text{C}}+R_{\text{D}}$. The charge on the capacitor is $Q(t)=Q_{\max}\left(1-e^{-t/\tau}\right)$, where $\tau=R_{eq}C$. The equivalent resistor is R_2 and R_3 in parallel. The current through the capacitor decays as $i_1(t)=i_1(0)e^{-t/\tau}$, where $i_1(0)$ is the initial current through the capacitor.

SIMPLIFY: The time constant is given by: $\tau=R_{eq}C=C\left(\frac{1}{R_2}+\frac{1}{R_3}\right)^{-1}=\frac{CR_2R_3}{R_2+R_3}$. Consider the voltage drop from junction X to Y. Along each branch it must be equal, so for the bottom two branches: $R_2 i_2(t)-V_{\text{emf},2}=V_{\text{emf},3}-R_3 i_3(t) \Rightarrow V_{\text{emf},3}+V_{\text{emf},2}=R_2 i_2(t)+R_3 i_3(t)$. Using equation (4), and substituting in for $i_2(t)$ and $i_3(t)$ gives:

$$V_{\text{emf},2}+V_{\text{emf},3}=R_2\left[i_1(t)+i_3(t)\right]+R_3 i_3(t)=R_2 i_1(t)+\left(R_2+R_3\right)i_3(t) \Rightarrow i_3(t)=\frac{V_{\text{emf},2}+V_{\text{emf},3}-R_2 i_1(t)}{R_2+R_3},$$

$$V_{\text{emf},2}+V_{\text{emf},3}=R_2 i_2(t)+R_3\left[i_2(t)-i_1(t)\right]=-R_3 i_1(t)+\left(R_2+R_3\right)i_2(t) \Rightarrow i_2(t)=\frac{V_{\text{emf},2}+V_{\text{emf},3}+R_3 i_1(t)}{R_2+R_3}.$$

When $t\to\infty$, $i_1(t)\to 0$, so the steady state current is given by: $\lim_{t\to\infty} i_2(t)=\lim_{t\to\infty} i_3(t)=i_{\text{s}}=\frac{V_{\text{emf},2}+V_{\text{emf},3}}{R_2+R_3}$.

At all times, the voltage drop from X to Y is the same along any branch. Now that the steady state current is reached, the voltage drop across the capacitor can be determined and thus the maximum charge and initial current on the capacitor. Compare $\Delta V_{\text{C}}(t=\infty)$ to both branches:

$$\Delta V_{\text{C}}(t=\infty)=\Delta V_{\text{C,max}}=R_3 i_3(t=\infty)-V_{\text{emf},3}=R_3 i_{\text{s}}-V_{\text{emf},3} \text{ or}$$

$$\Delta V_{\text{C,max}}=V_{\text{emf},2}-R_2 i_2(t=\infty)=V_{\text{emf},2}-R_2 i_{\text{s}}.$$

$$\Rightarrow \ \Delta V_{\text{C,max}} = R_3\left(\frac{V_{\text{emf,2}}+V_{\text{emf,3}}}{R_2+R_3}\right) - V_{\text{emf,3}} = V_{\text{emf,2}} - R_2\left(\frac{V_{\text{emf,2}}+V_{\text{emf,3}}}{R_2+R_3}\right)$$

$$\Rightarrow \ R_3\left(\frac{V_{\text{emf,2}}+V_{\text{emf,3}}}{R_2+R_3}\right) - V_{\text{emf,3}}\left(\frac{R_2+R_3}{R_2+R_3}\right) = V_{\text{emf,2}}\left(\frac{R_2+R_3}{R_2+R_3}\right) - R_2\left(\frac{V_{\text{emf,2}}+V_{\text{emf,3}}}{R_2+R_3}\right)$$

$$\Rightarrow \ \frac{R_3V_{\text{emf,2}} - R_2V_{\text{emf,3}}}{R_2+R_3} = \frac{R_3V_{\text{emf,2}} - R_2V_{\text{emf,3}}}{R_2+R_3}.$$

The maximum charge is given by: $Q_{\text{max}} = C\Delta V_{\text{C,max}} = \dfrac{C\left(R_3V_{\text{emf,2}} - R_2V_{\text{emf,3}}\right)}{R_2+R_3}$. In general, $Q(t)$ is related to $i_1(t)$ by: $i_1(t) = \dfrac{dQ(t)}{dt} = \dfrac{d}{dt}Q_{\text{max}}\left(1-e^{-t/\tau}\right) = Q_{\text{max}}\left(0 - -\dfrac{1}{\tau}e^{-t/\tau}\right) = \dfrac{Q_{\text{max}}}{\tau}e^{-t/\tau}$.

$$i_1(0) = \frac{Q_{\text{max}}}{\tau} = \frac{C\left(R_3V_{\text{emf,2}} - R_2V_{\text{emf,3}}\right)}{R_2+R_3}\left(\frac{R_2+R_3}{CR_2R_3}\right) = \frac{V_{\text{emf,2}}}{R_2} - \frac{V_{\text{emf,3}}}{R_3}$$

Now that $i_1(0)$ is determined, $i_2(t)$ and $i_3(t)$ can be expressed in simpler terms:

$$i_2(t) = \frac{V_{\text{emf,2}}+V_{\text{emf,3}}+R_3i_1(t)}{R_2+R_3} = \frac{V_{\text{emf,2}}+V_{\text{emf,3}}}{R_2+R_3} + \frac{R_3\left(V_{\text{emf,2}}/R_2 - V_{\text{emf,3}}/R_3\right)e^{-t/\tau}}{R_2+R_3}$$

$$= i_{\text{s}} + \left[\frac{\left(R_3/R_2\right)V_{\text{emf,2}} - V_{\text{emf,3}}}{R_2+R_3}\right]e^{-t/\tau}.$$

$$i_3(t) = \frac{V_{\text{emf,2}}+V_{\text{emf,3}}-R_2i_1(t)}{R_2+R_3} = \frac{V_{\text{emf,2}}+V_{\text{emf,3}}}{R_2+R_3} - \frac{R_2\left(V_{\text{emf,2}}/R_2 - V_{\text{emf,3}}/R_3\right)e^{-t/\tau}}{R_2+R_3}$$

$$= i_{\text{s}} + \left[\frac{\left(R_2/R_3\right)V_{\text{emf,3}} - V_{\text{emf,2}}}{R_2+R_3}\right]e^{-t/\tau}$$

The potential across the capacitor for any given times is then given by:

$$\Delta V_{\text{C}}(t) = \frac{Q(t)}{C} = \frac{Q_{\text{max}}}{C}\left(1-e^{-t/\tau}\right) = \frac{R_3V_{\text{emf,2}} - R_2V_{\text{emf,3}}}{R_2+R_3}\left(1-e^{-t/\tau}\right).$$

Therefore, in addition to the previous four equations, there are six additional ones.

$$\tau = \frac{CR_2R_3}{R_2+R_3} \tag{5}$$

$$i_{\text{s}} = \frac{V_{\text{emf,2}}+V_{\text{emf,3}}}{R_2+R_3} \tag{6}$$

$$i_1(t) = \left(\frac{V_{\text{emf,2}}}{R_2} - \frac{V_{\text{emf,3}}}{R_3}\right)e^{-t/\tau} \tag{7}$$

$$i_2(t) = i_{\text{s}} + \left[\frac{\left(R_3/R_2\right)V_{\text{emf,2}} - V_{\text{emf,3}}}{R_2+R_3}\right]e^{-t/\tau} \tag{8}$$

$$i_3(t) = i_{\text{s}} + \left[\frac{\left(R_2/R_3\right)V_{\text{emf,3}} - V_{\text{emf,2}}}{R_2+R_3}\right]e^{-t/\tau} \tag{9}$$

$$\Delta V_{\text{C}}(t) = \frac{R_3V_{\text{emf,2}} - R_2V_{\text{emf,3}}}{R_2+R_3}\left(1-e^{-t/\tau}\right) \tag{10}$$

CALCULATE: $R_1 = 40.0\ \Omega + 1.0\ \Omega = 41.0\ \Omega,\ \ R_2 = 20.0\ \Omega + 1.0\ \Omega = 21.0\ \Omega$

$$\tau^{-1} = \left[\frac{(30.0\ \mu\text{F})(41.0\ \Omega)(21.0\ \Omega)}{41.0\ \Omega + 21.0\ \Omega}\right]^{-1} = \left(4.16613\cdot 10^{-4}\ \text{s}\right)^{-1} = 2400.31\ \text{s}^{-1}$$

$$i_s = \frac{80.0\ \text{V} + 80.0\ \text{V}}{41.0\ \Omega + 21.0\ \Omega} = 2.580645161\ \text{A},\quad \frac{V_{\text{emf},2}}{R_2} - \frac{V_{\text{emf},3}}{R_3} = \frac{80.0\ \text{V}}{41.0\ \Omega} - \frac{80.0\ \text{V}}{21.0\ \Omega} = -1.858304297\ \text{A}$$

$$\frac{(R_3/R_2)V_{\text{emf},2} - V_{\text{emf},3}}{R_2 + R_3} = \frac{(21.0\ \Omega / 41.0\ \Omega)80.0\ \text{V} - 80.0\ \text{V}}{21.0\ \Omega + 41.0\ \Omega} = -0.6294256\ \text{A}$$

$$\frac{(R_2/R_3)V_{\text{emf},3} - V_{\text{emf},2}}{R_2 + R_3} = \frac{(41.0\ \Omega / 21.0\ \Omega)80.0\ \text{V} - 80.0\ \text{V}}{21.0\ \Omega + 41.0\ \Omega} = 1.228878648\ \text{A}$$

$$\frac{R_3 V_{\text{emf},2} - R_2 V_{\text{emf},3}}{R_2 + R_3} = \frac{21.0\ \Omega(80.0\ \text{V}) - 41.0\ \Omega(80.0\ \text{V})}{21.0\ \Omega + 41.0\ \Omega} = -25.8069516\ \text{V}$$

ROUND: The initial four equations, rounded to three significant figures are:

(1) $79.98\ \text{V} + (0.03\ \text{V})e^{-(2400\ \text{s}^{-1})t} - 80.0\ \text{V} = 0,$ (2) $80.0\ \text{V} - \left[79.98\ \text{V} + (0.011\ \text{V})e^{-(2400\ \text{s}^{-1})t}\right] = 0,$

(3) $159.96\ \text{V} + (0.041\ \text{V})e^{-(2400\ \text{s}^{-1})t} - 160.0\ \text{V} = 0$ and

$(-0.629\ \text{A})e^{-(2400\ \text{s}^{-1})t} = (-1.86\ \text{A})e^{-(2400\ \text{s}^{-1})t} + (1.23\ \text{A})e^{-(2400\ \text{s}^{-1})t}.$

DOUBLE-CHECK: The initial four equations within rounding are still valid, so the values of the coefficients are correct. Checking $i_2(0)$ and $i_3(0)$ using equations (8) and (9) gives:

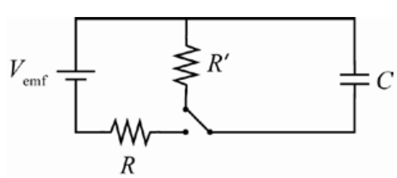
$$i_2(0) = \frac{V_{\text{emf},2} + V_{\text{emf},3} + R_3\left(V_{\text{emf},2}/R_2 - V_{\text{emf},3}/R_3\right)}{R_2 + R_3} = \frac{V_{\text{emf},2}\left(1 + R_3/R_2\right)}{R_2\left(1 + R_3/R_2\right)} = \frac{V_{\text{emf},2}}{R_2}\ \text{and}$$

$$i_3(0) = \frac{V_{\text{emf},2} + \Delta V_3 - R_2\left(V_{\text{emf},2}/R_2 - V_{\text{emf},3}/R_3\right)}{R_2 + R_3} = \frac{V_{\text{emf},3}\left(1 + R_2/R_3\right)}{R_3\left(1 + R_2/R_3\right)} = \frac{V_{\text{emf},3}}{R_3}.$$

These results satisfy $i_1(0) = i_2(0) - i_3(0) = \left(V_{\text{emf},2}/R_2\right) - \left(V_{\text{emf},3}/R_3\right)$. Also, consider that initially the $V_{\text{emf},2}$ battery "sees" only R_2 first (likewise for battery $V_{\text{emf},3}$ and R_3), so the initial current is simply $V_{\text{emf},2}/R_2$ (or $V_{\text{emf},3}/R_3$), so the equations for the currents make sense.

26.71. **THINK:** The capacitor of capacitance is $C = 10.0\ \mu\text{F},$ is charged through a resistor of resistance $R = 10.0\ \Omega,$ with a battery, $V_{\text{emf}} = 10.0\ \text{V}.$ It is discharged through a resistor, $R' = 1.00\ \Omega.$ For either charging or discharging, it takes the same number of time constants to get to half of the maximum value. The energy on the capacitor is proportional to the square of the charge.

SKETCH:



RESEARCH: The capacitor's charge is given by $q(t) = q_0\left(1 - e^{-t/\tau}\right).$ In general, the energy on the capacitor is given by $E(t) = q^2(t)/2C.$ The time constant is either $\tau = RC$ or $\tau' = R'C.$

SIMPLIFY:

(a) When $q(t) = q_0/2,$ then: $q(t) = \frac{1}{2}q_0 = q_0\left(1 - e^{-t/\tau}\right) \Rightarrow \frac{1}{2} = 1 - e^{-t/\tau} \Rightarrow t = -\tau\ln\left(\frac{1}{2}\right) = \tau\ln 2.$

(b) If $q(t)=\frac{1}{2}q_0$, the energy is: $E(t)=\frac{q^2(t)}{2C}=\frac{(q_0/2)^2}{2C}=\frac{1}{4}\left(\frac{q_0^2}{2C}\right)=\frac{1}{4}E_{\text{max}}$.

(c) The time constant for discharging is $\tau'=R'C$.

(d) The capacitor discharges to half the original charge in $t=\tau'\ln(2)$.

CALCULATE:

(a) $t=\tau\ln 2.00$, or $(0.693)\tau$

(b) 1.00:4.00.

(c) $\tau'=(1.00\ \Omega)(10.0\ \mu\text{F})=10.0\ \mu\text{s}$

(d) $t=(10.0\ \mu\text{s})\ln(2.00)=6.93\ \mu\text{s}$

ROUND:

(a) 0.693τ

(b) 1.00:4.00.

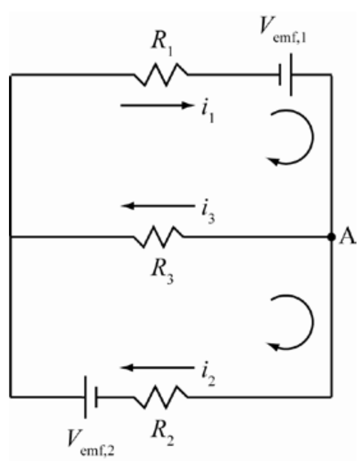
(c) $\tau'=10.0\ \mu\text{s}$

(d) $t=6.93\ \mu\text{s}$

DOUBLE-CHECK: In general, charge decreases exponentially as $q(t)=q_0e^{-t/\tau}$. For $\tau'=10\ \mu\text{s}$ and $t=6.93\ \mu\text{s}$, the charge is $q(6.93\ \mu\text{s})=q_0e^{-6.93/10}=0.497q_0$, which is about half the original charge.

26.72. **THINK:** From Kirchhoff's rules, an equation can be obtained for the sum of the three currents, i_1, i_2 and i_3, and two equations can be obtained for the two inner loops of the circuit. This will yield 3 equations for 3 unknowns (the currents) and can be solved by simple substitution. Once the currents are known, the voltage drop over resistor 2 is found via Ohm's law. $R_1=3.00\ \Omega$, $R_2=2.00\ \Omega$, $R_3=5.00\ \Omega$, $V_{\text{emf,1}}=10.0\text{ V}$ and $V_{\text{emf,2}}=6.00\text{V}$.

SKETCH:



RESEARCH: By the choice of directions of currents, at point A, the currents sum as $i_1-i_2-i_3=0$. Going clockwise in the upper and lower loops gives 2 equations: $-i_1R_1+V_{\text{emf,1}}-i_3R_3=0$ and $i_3R_3-i_2R_2+V_{\text{emf,2}}=0$. Potential drop across resistor 2 is $\Delta V=i_2R_2$. The power across the third resistor is $P=i_3^2R_3$.

SIMPLIFY: From equation of currents: $i_1=i_2+i_3 \Rightarrow i_3=i_1-i_2$. From the upper loop: $V_{\text{emf,1}}-i_3R_3=i_1R_1 \Rightarrow i_1=(V_{\text{emf,1}}/R_1)-(i_3R_3/R_1)$. From the lower loop: $V_{\text{emf,2}}+i_3R_3=i_2R_2 \Rightarrow i_2=(V_{\text{emf,2}}/R_2)+(i_3R_3/R_2)$. Therefore,

$$i_3 = i_1 - i_2 = \frac{V_{\text{emf},1}}{R_1} - i_3\frac{R_3}{R_1} - \frac{V_{\text{emf},2}}{R_2} - i_3\frac{R_3}{R_2} \Rightarrow \frac{R_3}{R_3}i_3 = \frac{V_{\text{emf},1}}{R_1} - \frac{V_{\text{emf},2}}{R_2} - i_3\left(\frac{R_3}{R_1} + \frac{R_3}{R_2}\right)$$

$$\Rightarrow i_3\left(\frac{R_3}{R_3} + \frac{R_3}{R_1} + \frac{R_3}{R_2}\right) = \frac{V_{\text{emf},1}}{R_1} - \frac{V_{\text{emf},2}}{R_2} \Rightarrow i_3 = \left(\frac{V_{\text{emf},1}}{R_1} - \frac{V_{\text{emf},2}}{R_2}\right)\left(\frac{R_3}{R_3} + \frac{R_3}{R_1} + \frac{R_3}{R_2}\right)^{-1}$$

$$\Rightarrow i_3 = \left[\left(\frac{V_{\text{emf},1}}{R_1 R_3}\right) - \left(\frac{V_{\text{emf},2}}{R_2 R_3}\right)\right]\left(\frac{1}{R_3} + \frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$$

CALCULATE:

(a) $i_3 = \left[\frac{10.0 \text{ V}}{(3.00\ \Omega)(5.00\ \Omega)} - \frac{6.00 \text{ V}}{(2.00\ \Omega)(5.00\ \Omega)}\right]\left(\frac{1}{5.00\ \Omega} + \frac{1}{3.00\ \Omega} + \frac{1}{2.00\ \Omega}\right)^{-1} = 0.06452 \text{ A}$

(b) $P = (0.06452 \text{ A})^2(5.00\ \Omega) = 0.02081 \text{ W}$

ROUND:

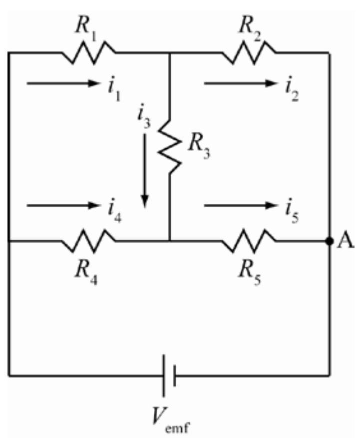
(a) $i = 64.5 \text{ mA}$

(b) $P = 20.8 \text{ mW}$

DOUBLE-CHECK: Going back to equation for i_1 and i_2, the currents can be calculated as $i_1 = 3.2258$ A and $i_2 = 3.1613$ A. Their difference is $i_1 - i_2 = 0.0045$ A, which is also i_3; therefore, the current is correct and it makes sense.

26.73. **THINK:** From Kirchhoff's rules, an equation can be obtained for the sum of the five currents, i_1, i_2, i_3, i_4 and i_5, and two equations can be obtained for the two inner loops of the circuit. Since the voltage drop across resistor 3 is zero, the current through that branch is also zero. Ohm's law allows an equation for the ratios of the resistors. Once R_2 is known, the current through it is obtained by equation the potential drop across both R_1 and R_2 is equal to the emf voltage. $R_1 = 8.00\ \Omega$, $R_4 = 2.00\ \Omega$, $R_5 = 6.00\ \Omega$ and $V_{\text{emf}} = 15.0$ V.

SKETCH:



RESEARCH: By the choice of directions of currents, two equations arise $i_1 = i_2 + i_3$ and $i_5 = i_3 + i_4$. Since the current through R_3 is zero, $\Delta V_3 = i_3 R_3 = 0$. Then, two sets of potential drops are equal: $i_1 R_1 = i_4 R_4$ and $i_2 R_2 = i_5 R_5$. The potential across R_1 and R_2 is $V_{\text{emf}} = i_1 R_1 + i_2 R_2$.

SIMPLIFY: Since i_3 is zero, the current becomes $i_1 = i_2$ and $i_4 = i_5$. Dividing the potential drops across each resistors yields

$$\frac{i_1 R_1}{i_2 R_2} = \frac{i_4 R_4}{i_5 R_5} \Rightarrow \frac{R_1}{R_2} = \frac{R_4}{R_5} \Rightarrow R_2 = \frac{R_1 R_5}{R_4}.$$

The current through it is $i_2 = i_1 = i$; therefore, $V_{\text{emf}} = i_1(R_1 + R_2) \Rightarrow i_1 = V_{\text{emf}} / (R_1 + R_2)$.

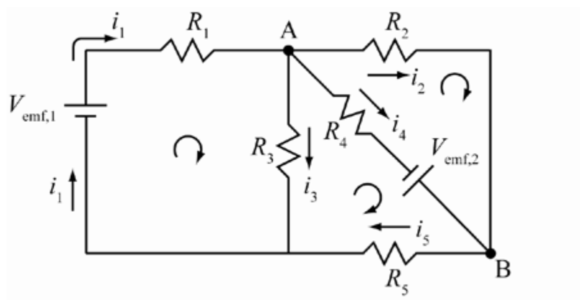
CALCULATE: $R_2 = \frac{(8.00\ \Omega)(6.00\ \Omega)}{2.00\ \Omega} = 24.0\ \Omega,\ i_2 = i_1 = \frac{15.0\ \text{V}}{8.00\ \Omega + 240\ \Omega} = 0.46875\ \text{A}$

ROUND: $R_2 = 24.0\ \Omega,\quad i_2 = 469\ \text{mA}$

DOUBLE-CHECK: Solving for $i_4 = i_5 = V_{\text{emp}} / (R_u + R_s) = 1.875\ \text{A}$ for a total current of $i_T = 2.34375\ \text{A}$ coming out of the battery. Since there is no current in R_3, the circuit is just R_1 and R_2 in parallel with R_4 and R_5 to gives $R_{\text{eq}} = \left[1/(R_1 + R_2) + 1/(R_4 + R_5)\right]^{-1}$, and produces a current of $i = \Delta V_{\text{emp}} / R_{\text{eq}} = 2.34375 = i_T$, so it makes sense.

26.74. **THINK:** From Kirchhoff's rules, equations can be obtained for the sum of the five currents, i_1, i_2, i_3, i_4 and i_5, and three equations for the three inner loops of the circuit. This will yield 5 equations and 4 unknowns (the currents) and can be solved by simple substitution. $R_1 = 1.00\ \Omega$, $R_2 = 2.00\ \Omega$, $R_3 = 3.00\ \Omega$, $R_4 = 4.00\ \Omega$, $R_5 = 5.00\ \Omega$, $V_{\text{emf,1}} = 12.0\ \text{V}$ and $V_{\text{emf,2}} = 6.00\ \text{V}$.

SKETCH:



RESEARCH: By the choice of directions of currents, at point A, $i_1 - i_2 - i_3 - i_4 = 0$, and at point B, $i_2 + i_4 - i_5 = 0$. By going clockwise in each loop yields 3 equations: $V_{\text{emf,1}} - i_1R_1 - i_3R_3 = 0$, $-i_2R_2 - V_{\text{emf,2}} - i_4R_4 = 0$ and $-i_4R_4 + V_{\text{emf,2}} - i_5R_5 + i_3R_3 = 0$. Potential drop across resistor 2 is $\Delta V = i_2R_2$. The power across the third resistor is $P = i_3^2R_3$.

SIMPLIFY:

(a) Using the equation $i_5 = i_2 + i_4$, the other 4 can be simplified to: 1) $i_1 - i_2 - i_3 - i_4 = 0$; 2) $R_1i_1 + R_3i_3 = V_{\text{emf,1}}$; 3) $-R_2i_2 + R_4i_4 = V_{\text{emf,2}}$; 4) $R_5i_2 - R_3i_3 + (R_4 + R_5)i_4 = V_{\text{emf,2}}$.

(b) Since the resistance are in Ω and all voltages are in V, the equations can be rewritten for simplicity with only the magnitude of the values, knowing that the final currents are in A, which yields: 1) $i_1 - i_2 - i_3 - i_4 = 0$; 2) $i_1 + 3i_3 = 12.0$; 3) $-2i_2 + 4i_4 = 6.00$; 4) $5i_2 - 3i_3 + 9i_4 = 6.00$.

$$(2)-(1): i_2 + 4i_3 + i_4 = 12.0 \equiv (2A)$$
$$2(2A)+(3)\ :\ 8i_3 + 6i_4 = 30.0 \equiv (3A)$$
$$-5(2A)+(4)\ :\ -23i_3 + 4i_4 = -54.0 \equiv (4A)$$
$$(23/8)(2A)+(4A)\ :\ (85.0/4)i_4 = 32.25 \equiv (4B).$$

Therefore, $(4B) \Rightarrow i_4 = 129/85.0$, $(3A) \Rightarrow i_3 = (30.0 - 6I_4)/8 = 222/85.0$,
$(2A) \Rightarrow i_2 = 12.0 - 4i_3 - i_4 = 3.00/85.0$, and $(1) \Rightarrow i_1 = i_2 + i_3 + i_4 = 354/85.0$.

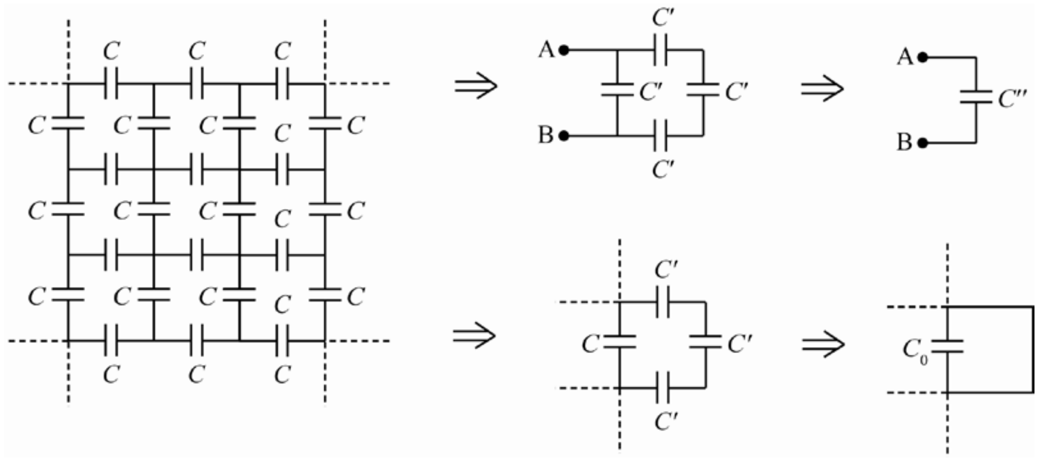
CALCULATE: $i_4 = \frac{129}{85.0}\ \text{A} = 1.5176\ \text{A}$

ROUND: $i_4 = 1.52\ \text{A}$

DOUBLE-CHECK: If these currents are used to calculate the potential drop from $A \to B$, you get $-i_3R_3 + i_5R_5 = 0.071\ \text{V}$, $-i_4R_4 + \Delta V_2 = 0.071\ \text{V}$, and $-i_2R_2 = 0.071\ \text{V}$, so the potential drops are all the same, so it makes sense.

26.75. **THINK:** Consider any square on the grid to have been reduced so that every side has a capacitance, C', which is the equivalent of all the capacitors above, below and along each side. Since the grid is infinite, then no side has more capacitors than any other, so all four are reduced to the same capacitance. Next, consider the same analysis except for only three sides, so that one side is still of capacitance, C, while the others are C'. When those four sides are reduced to one equivalent capacitance, the result should be equal to the original value of C'. This is because the grid is infinite and adding an extra square to the already reduced side should affect nothing, resulting in the same capacitance, giving a recursive relation in C, and thus the total equivalent capacitance, in terms of C, can be determined.

SKETCH:



RESEARCH: Capacitors in series add as $C_{\text{eq}}^{-1} = C_1^{-1} + C_2^{-1}$. Capacitors in parallel add as $C_{\text{eq}} = C_1 + C_2$.

SIMPLIFY: When all four sides are reduced to C', the equivalent capacitance (across A to B) is:

$$C'' = C' + \left(\frac{1}{C'} + \frac{1}{C'} + \frac{1}{C'}\right)^{-1} = \frac{4C'}{3}.$$

Looking at when one side is reduced using the other three reduced gives C_0 as:

$$C_0 = C + \left(\frac{1}{C'} + \frac{1}{C'} + \frac{1}{C'}\right)^{-1} = C + \frac{C'}{3}.$$

Since $C_0 = C'$: $C' = C + \frac{C'}{3} \Rightarrow C = \frac{2}{3}C'$ and $C' = \frac{3}{2}C$. Therefore, the total equivalent capacitance is:

$$C'' = \frac{4C'}{3} = \frac{4}{3}\left(\frac{3}{2}C\right) = \frac{4}{2}C = 2C.$$

CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: Consider an intersection on the grid. If a voltage was applied to this point, it would see equal capacitance (since it is infinite) in all four directions, meaning it would contribute an equal charge, q, to each direction. If the same voltage with opposite polarity was applied to any adjacent intersection, it would see a $-q$ along each direction. This means the capacitor that joins the two intersections is actually double the charge on one, meaning the potential sees an effective capacitance twice the size of any one capacitor, so an equivalent capacitance of $2C$ is correct. $C_0 = C + C'/3 \Rightarrow C' = (3/2)C$. Therefore the total equivalent capacitance is $C'' = (4/3)C' = (4/3)(3/2)C = 2C$.

Multi-Version Exercises

Exercises 26.76–26.78 Using Kirchhoff's Loop Rule we get $-iR_\text{i} - V_\text{t} + V_\text{e} = 0$. So the required battery charger emf is $V_\text{e} = iR_\text{i} + V_\text{t}$.

26.76. $V_\text{e} = iR_\text{i} + V_\text{t} = (9.759\text{ A})(0.1373\ \Omega) + 11.45\text{ V} = 12.79\text{ V}$

26.77. The potential difference across the terminals during charging is equal to the charger emf, 14.51 V. To find the open-circuit potential difference across the terminals, with the charger removed and no voltage drop due to internal resistance, use $V_\text{e} = iR_\text{i} + V_\text{t}$.

$$V_\text{t} = V_\text{e} - iR_\text{i} = 14.51\text{ V} - (5.399\text{ A})(0.1415\ \Omega) = 13.75\text{ V}$$

26.78. $V_\text{e} = iR_\text{i} + V_\text{t}$

$$R_\text{i} = \frac{V_\text{e} - V_\text{t}}{i} = \frac{16.93\text{ V} - 16.05\text{ V}}{6.041\text{ A}} = 0.15\ \Omega$$

Note that by the subtraction rule, the difference of the two voltages has only two significant figures.

Exercises 26.79–26.81 Kirchhoff's Loop Rule gives us

$V_{\text{emf},1} - \Delta V_1 - \Delta V_2 - V_{\text{emf},2} = V_{\text{emf},1} - iR_1 - iR_2 - V_{\text{emf},2} = 0.$

We can rearrange this equation to get

$$V_{\text{emf},1} - i(R_1 + R_2) - V_{\text{emf},2} = 0$$

$$i = \frac{V_{\text{emf},1} - V_{\text{emf},2}}{R_1 + R_2}.$$

26.79. $$i = \frac{V_{\text{emf},1} - V_{\text{emf},2}}{R_1 + R_2} = \frac{21.01\text{ V} - 10.75\text{ V}}{23.37\ \Omega + 11.61\ \Omega} = 0.2933\text{ A}$$

26.80. $$i = \frac{V_{\text{emf},1} - V_{\text{emf},2}}{R_1 + R_2}$$

$$R_2 = \frac{V_{\text{emf},1} - V_{\text{emf},2}}{i} - R_1 = \frac{16.37\text{ V} - 10.81\text{ V}}{0.1600\text{ A}} - 24.65\ \Omega = 10.10\ \Omega.$$

26.81. $$i = \frac{V_{\text{emf},1} - V_{\text{emf},2}}{R_1 + R_2}$$

$$i(R_1 + R_2) = V_{\text{emf},1} - V_{\text{emf},2}$$

$$V_{\text{emf},2} = V_{\text{emf},1} - i(R_1 + R_2) = 17.75\text{ V} - (0.1740\text{ A})(25.95\ \Omega + 13.59\ \Omega) = 10.87\text{ V}$$

Exercises 26.82–26.84 When the resistor is connected to the charged capacitor, the initial current i_0 will be given by $V_\text{emf} = i_0 R \;\Rightarrow\; i_0 = \frac{V_\text{emf}}{R}$. The time constant is $\tau = RC$. The current after time t is

$$i = i_0 e^{-t/\tau} = \frac{V_\text{emf}}{R} e^{-t/(RC)}.$$

26.82. $$i = \frac{V_\text{emf}}{R} e^{-t/(RC)} = \frac{131.1\text{ V}}{616.5\ \Omega} e^{-(3.871\text{ s})/((616.5\ \Omega)(15.19\cdot 10^{-3}\text{ F}))} = 0.1407\text{ A}$$

26.83. $i = \frac{V_{\text{emf}}}{R} e^{-t/(RC)}$

$$\frac{iR}{V_{\text{emf}}} = e^{-t/(RC)}$$

$$\ln\left(\frac{iR}{V_{\text{emf}}}\right) = -t/(RC)$$

$$C = -\frac{t}{R\ln\left(\frac{iR}{V_{\text{emf}}}\right)} = -\frac{1.743 \text{ s}}{(655.1\ \Omega)\ln\left(\frac{(0.1745 \text{ A})(655.1\ \Omega)}{133.1 \text{ V}}\right)} = 0.01749 \text{ F} = 17.49 \text{ mF}$$

26.84. $i = \frac{V_{\text{emf}}}{R} e^{-t/(RC)}$

$$V_{\text{emf}} = iRe^{t/(RC)} = (0.1203 \text{ A})(693.5\ \Omega)e^{(6.615 \text{ s})/((693.5\ \Omega)(19.79\cdot 10^{-3} \text{ F}))} = 135.1 \text{ V}$$

Chapter 27: Magnetism

Concept Checks

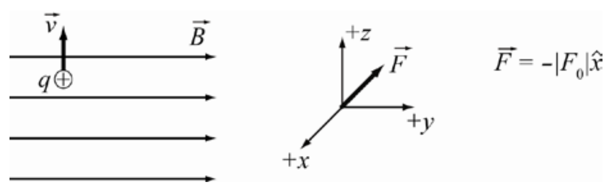
27.1. a **27.2.** a **27.3.** c **27.4.** a **27.5.** a

Multiple-Choice Questions

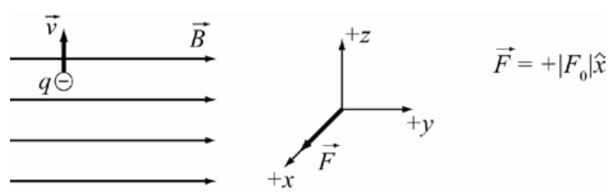
27.1. b **27.2.** c **27.3.** e **27.4.** b **27.5.** a **27.6.** a **27.7.** a,c,d,e are true; b is false **27.8.** b **27.9.** e **27.10.** d **27.11.** d **27.12.** d

Conceptual Questions

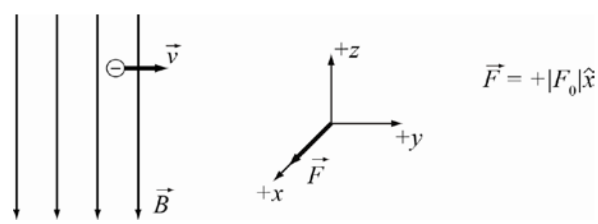
27.13. (a)



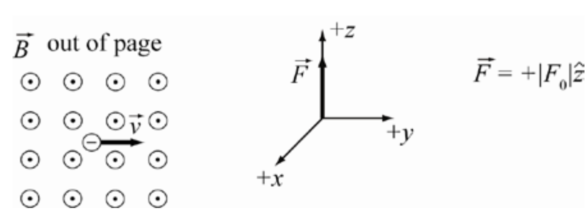
(b)



(c)



(d)



27.14. Zero. The force acting on a charged particle in a magnetic field is $\vec{F} = q\vec{v} \times \vec{B}$. By definition of the cross-product (and confirmed by experiment), this force is always perpendicular to the velocity of the particle at any point in the magnetic field. Thus, the work done by the magnetic field on the charged particle is zero. The effect of this force on the particle is that it changes the direction of the particle's velocity, but not its magnitude. Hence, the uniform circular motion the particle has in the magnetic field (the cyclotron motion).

27.15. A = Parabolic (electric field). B = Circular (magnetic field). The forces acting on a charged particle under either an electric field or a magnetic field is $\vec{F} = q\vec{E}$ or $\vec{F} = q\vec{v} \times \vec{B}$, respectively.

27.16.

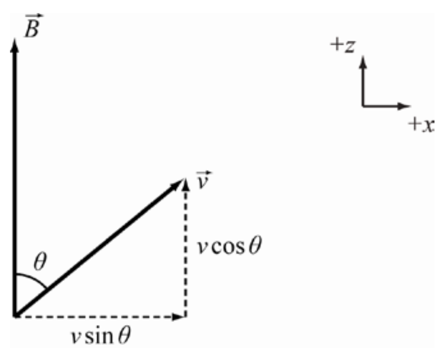
(a) The direction of the force acting on a charge moving in a magnetic field is given by the right-hand rule. If the fingers point in the direction of $\vec{v}$, then to produce a force in the negative x-direction, the magnetic field has to act out of the page, in positive z-direction.
(b) Yes, it does change. For the negatively charged electron, the field must point into the page, in negative z-direction. The direction of the force depends on the charge.

27.17. A magnetic potential is used to represent magnetic fields in regions of zero current density in some applications, but the construction is not as useful as its electrical counterpart. This is because the electric potential represents a potential energy (per unit charge), which is part of a conserved total energy. It keeps track of the work done by the electric field on a charge moving in that field, and can thus be used to analyze the dynamics of charged particles. But the magnetic field never does any work on a charged particle, as the magnetic force is perpendicular to the particle's velocity. There is no work for a magnetic scalar potential energy to track. It represents no contribution to a conserved total energy, and hence, does not enter into any dynamics. It is more useful in advanced treatments of electromagnetic theory to represent the magnetic field as the curl of a vector potential: $\vec{B} = \nabla \times \vec{A}$, for a suitable vector field, $\vec{A}$.

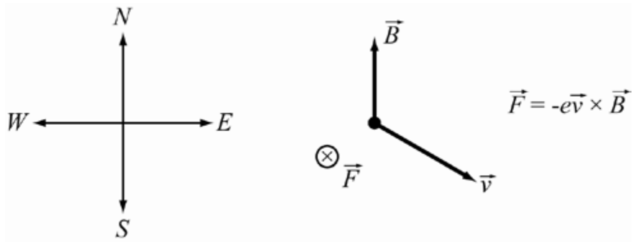
27.18. This is possible if the direction of the current is parallel or anti-parallel to that of the magnetic field. In such a case, $dF = i\,dL \times B = 0$.

27.19. Yes, it is possible. In order for this to work, the force due to the electric field, $\vec{F} = q\vec{E}$, has to be perpendicular to the velocity vector at all times. One way to achieve this is to have the electric field from a point particle, say a proton, for which the electric field points in radial outwards direction. An electron with suitable initial velocity can then make circular orbits around the proton. For these the speed does not change. If the electric field is replaced with a uniform magnetic field, the speed of a charged particle never changes. Note that in both cases described here, only the speed is constant, but the *direction* of the velocity vector changes. (In the case that the initial velocity vector is parallel or anti-parallel to the magnetic field even the direction stays constant.)

27.20. The charged particle will move in a helix around the magnetic field lines. Its motion in the z-direction is unaffected by the magnetic field, and therefore the time required involves determining the component of the initial velocity in the z-direction, which is simply v multiplied by the cosine of the angle. Thus, the time required is $\Delta t = \dfrac{\Delta z}{v_z} = \dfrac{\Delta z}{v\cos\theta}$, where Δz is the extent of the region along the z-direction.



27.21.

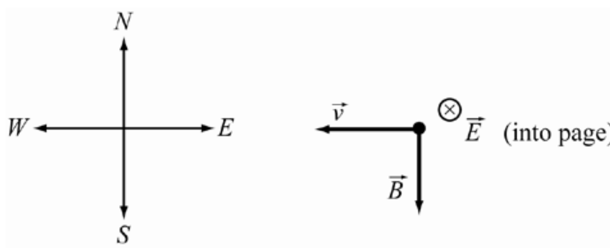


The magnetic force acts in a direction perpendicular to both the velocity and magnetic fields. Since these are both in the horizontal plane, the force acts into or out of the page. The right-hand rule shows that a positive charge experiences a net force outwards. Thus, for the negatively charged electron, the force is directed inwards.

27.22. Recall that a velocity selector works with perpendicular magnetic and electric fields. At the Earth's surface, there is an approximately perpendicular relation between the electric and magnetic fields. Thus, on a line perpendicular to *E* and *B*, charged particles will travel without deflection if they have the correct velocity. This velocity has a magnitude *E*/*B*. It is known that the Earth's magnetic field is approximately 0.3 gauss or $3\cdot 10^{-5}$ T. Therefore, the value of *E*/*B* at the Earth's surface is of the order:

$$\frac{150 \text{ N}}{3\cdot 10^{-5} \text{ T}} = 5\cdot 10^{6} \text{ m/s}.$$

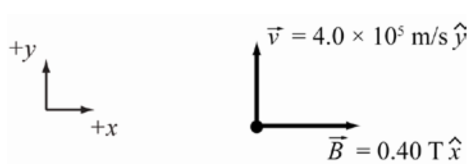
When pointed west, magnetically speaking the beam would be un-deflected. Once you are facing West, North is on your right.



27.23. A cyclotron has both electric and magnetic fields. It is the alternating electric field which does the work to increase the particle's kinetic energy. Although the magnetic field does not do any work (it does not change the particle's kinetic energy), it nevertheless plays an important role in keeping the particle in a circular orbit. As the electric field accelerates the particle, the radius of the circular orbit increases so that the particle follows a spiral trajectory. The alternating electric field and the static uniform magnetic are crucial for the operation of the cyclotron as a particle accelerator.

Exercises

27.24.



$\vec{F}_\text{B} = q\vec{v}\times\vec{B}$, $F_\text{B} = |q|vB\sin\theta$, $\theta = 90°$ and $q = e$, so, $F_\text{B} = evB$. Inserting the values gives:

$$F_\text{B} = (1.602\cdot 10^{-19} \text{ C})(4.00\cdot 10^{5} \text{ m/s})(0.400 \text{ T}) = 2.563\cdot 10^{-14} \text{ N} \approx 2.56\cdot 10^{-14} \text{ N}.$$

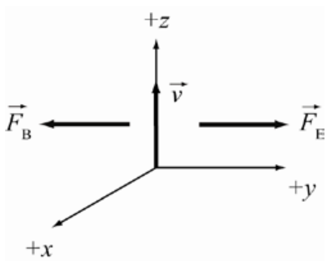
27.25. $\vec{F}_B = q\vec{v}\times\vec{B},\ F_B = |q|vB\sin\theta,\ \theta = 90°,\ q = -2e\ \Rightarrow\ \left|\vec{F}_B\right| = F_B = +2evB \Rightarrow$

$$B = \frac{F_B}{2ev} = \frac{3.00\cdot 10^{-18}\ \text{N}}{2\left(1.602\cdot 10^{-19}\ \text{C}\right)1.00\cdot 10^{5}\ \text{m/s}} = 9.363\cdot 10^{-5}\ \text{T} \approx 9.36\cdot 10^{-5}\ \text{T}.$$

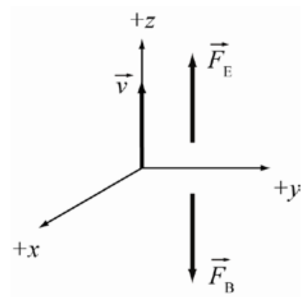
27.26. **THINK:** The particle moves in a straight line at constant speed. Thus, the net force must be zero. The electric force is the negative of the magnetic force. Using the right-hand rule, the direction of the magnetic field can be determined. For the magnitude, set the magnitude of the net force to zero. $q = 10.0\ \mu\text{C}$, $v =$ 300. m/s and $E =$ 100. V/m.

SKETCH:

(a)



(b)



RESEARCH: $\vec{F} = q\vec{E} + q\left(\vec{v}\times\vec{B}\right) = 0$

SIMPLIFY: $q\left(\vec{E} + \vec{v}\times\vec{B}\right) = 0\ \Rightarrow\ \vec{v}\times\vec{B} = -\vec{E}$

(a) $\vec{v}\times\vec{B} = -E\hat{y}\ \Rightarrow\ \left(v\hat{z}\times\vec{B}\right) = -E\hat{y}\ \Rightarrow\ \vec{B} = -\dfrac{E}{v}\hat{x}$ (by the right-hand rule)

(b) $\vec{v}\times\vec{B} = -E\hat{z}\ \Rightarrow\ \left(v\hat{z}\times\vec{B}\right) = -E\hat{z}$

There is no solution. $\hat{z}\times\vec{B}$ is either zero, or a vector in the xy-plane.

CALCULATE:

(a) $\left|\vec{B}\right| = -\dfrac{100.\ \text{V/m}}{300.\ \text{m/s}} = -\dfrac{1}{3}\ \text{T} = -0.3333\ \text{T}$; so $\vec{B} = -0.333\hat{x}$ T.

(b) No solution. No magnetic field will keep the particle moving at a constant speed in a straight line.

ROUND:

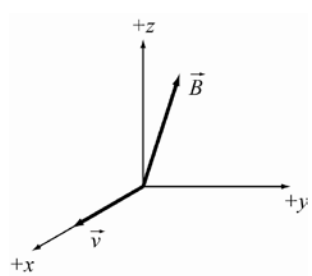
(a) $\vec{B} = -0.333\hat{x}$ T

(b) Not applicable.

DOUBLE-CHECK: No Lorenz force can counteract an electric force in z-direction, if the particle is also traveling in z-direction, because the Lorenz force is always perpendicular to the velocity vector.

27.27. **THINK:** First determine the components of the force. Once the components are determined, the magnitude and the direction of the force can be found. $q = 20.0\ \mu\text{C}$, $v =$ 50.0 m/s, $B_z = 0.700$ T and $B_y = 0.300$ T.

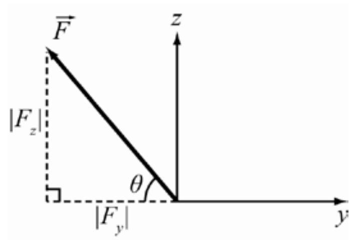
SKETCH:



RESEARCH: $\vec{F} = q\vec{v}\times\vec{B},\ \ \hat{x}\times\hat{y}=\hat{z},\ \ \hat{y}\times\hat{z}=\hat{x},\ \ \hat{z}\times\hat{x}=\hat{y},\ \ \vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$

SIMPLIFY: $\vec{F} = qv\hat{x}\times\left(B_z\hat{z}+B_y\hat{y}\right) = qv\left(B_z\hat{x}\times\hat{z}+B_y\hat{x}\times\hat{y}\right) = qv\left(-B_z\hat{y}+B_y\hat{z}\right) = F_y\hat{y}+F_z\hat{z}$

$\left|\vec{F}\right| = \sqrt{F_y^{\ 2}+F_z^{\ 2}} = qv\sqrt{B_z^{\ 2}+B_y^{\ 2}}$



$$\theta = \tan^{-1}\left(\frac{\left|F_z\right|}{\left|F_y\right|}\right) = \tan^{-1}\left(\frac{\left|B_y\right|}{\left|B_z\right|}\right) = \tan^{-1}\left(\frac{B_y}{B_z}\right)$$

CALCULATE: $\left|\vec{F}\right| = \left(20.0\cdot 10^{-6}\text{ C}\right)\left(50.0\text{ m/s}\right)\sqrt{\left(0.700\text{ T}\right)^2+\left(0.300\text{ T}\right)^2} = 7.616\cdot 10^{-4}\text{ N}$

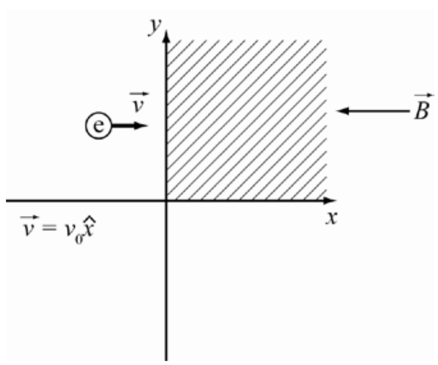
$$\theta = \tan^{-1}\left(\frac{0.300\text{ T}}{0.700\text{ T}}\right) = 23.20°$$

ROUND: $\left|\vec{F}\right| = 7.62\cdot 10^{-4}\text{ N}$ and the direction of the force is in the *yz*-plane, $\theta = 23.2°$ above the negative *y*-axis.

DOUBLE-CHECK: These results are reasonable. The Right Hand Rule dictates that the direction of the magnetic force be in the -*y*, +*z*-plane.

27.28. **THINK:** The only force acting on the particle is the magnetic force. The components of this force can be determined, and then the points where all the components vanish can be determined

SKETCH:



RESEARCH: $\vec{F} = q\vec{v}\times\vec{B} = 0,\ \hat{x}\times\hat{y}=\hat{z},\ \hat{y}\times\hat{z}=\hat{x},\ \hat{z}\times\hat{x}=\hat{y},\ \vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$

SIMPLIFY:

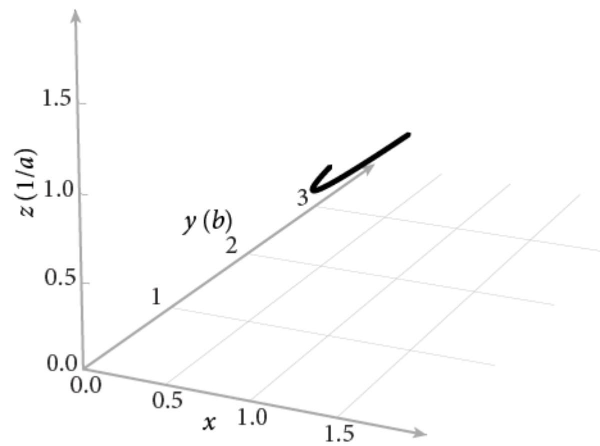
$$\vec{F} = 0 = qv_0\hat{x}\times\left[\left(x-az\right)\hat{y}\times\left(xy-b\right)\hat{z}\right] = qv_0\left[\left(x-az\right)\left(\hat{x}\times\hat{y}\right)+\left(xy-b\right)\left(\hat{x}\times\hat{z}\right)\right]$$

$$= qv_0\left[\left(x-az\right)\hat{z}+\left(b-xy\right)\hat{y}\right] = 0$$

$\Rightarrow\ x-az=0\ \Rightarrow\ x=az.$

$\Rightarrow\ b-xy=0\ \Rightarrow\ xy=b.$

The magnetic field will exert no force on the electron at all points satisfying the two equations, $x=az$ and $xy=b.$ The locus of these points in three dimensions is represented by the thick black line in the following figure.



CALCULATE: Not necessary.
ROUND: Not necessary.
DOUBLE-CHECK: For the given field, the results are correct. We can check to see if the magnetic field in this problem is physical. The fundamental equations, which govern the behavior of all electric and magnetic fields are called Maxwell's equations, named after James Clerk Maxwell, who first unified them. One of these equations states that:

$$\vec{\nabla}\cdot\vec{B}=\frac{\partial B_x}{\partial x}\hat{x}+\frac{\partial B_y}{\partial y}\hat{y}+\frac{\partial B_z}{\partial z}\hat{z}=0.$$

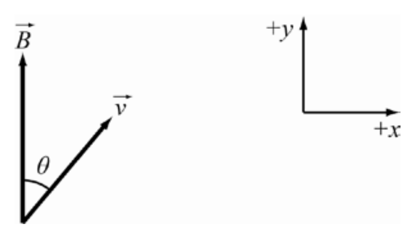
This equation is satisfied by the given $\vec{B}$ field so the specified field can exist. Interestingly, $\vec{\nabla}\cdot\vec{B}=0$ exists because as far as is now known, magnetic monopoles do not exist, only magnetic dipoles. This is in contrast to electric fields where monopoles (isolated positive and negative charges) do exist.

27.29. $\Delta K=\Delta U$

$$K=eV \;\Rightarrow\; \frac{1}{2}mv^2=eV \;\Rightarrow\; v=\sqrt{\frac{2eV}{m}}$$

$$B=\frac{mv}{er}=\frac{m}{er}\sqrt{\frac{2eV}{m}}=\frac{1}{r}\sqrt{\frac{2mV}{e}}=\frac{1}{0.200\text{ m}}\sqrt{\frac{2\left(1.67\cdot10^{-27}\text{ kg}\right)\left(400.\text{ V}\right)}{1.602\cdot10^{-19}\text{ C}}}=1.44\cdot10^{-2}\text{ T}.$$

27.30.



$\theta=35.0°$, $\left|\vec{B}\right|=0.0400\text{ T}=B$, $\left|\vec{v}\right|=4.00\cdot10^5\text{ m/s}=v$, $v_x=v\sin\theta=v_\perp$ (perpendicular to $\vec{B}$), $v_y=v\cos\theta=v_\parallel$ (parallel to $\vec{B}$).

(a) $r=\dfrac{mv_\perp}{|q|B}=\dfrac{\left(9.11\cdot10^{-31}\text{ kg}\right)\left(4.00\cdot10^5\text{ m/s}\right)\left(\sin35.0°\right)}{\left(1.602\cdot10^{-19}\text{ C}\right)\left(0.0400\text{ T}\right)}=3.262\cdot10^{-5}\text{ m}$

(b) The time it takes to travel 2π radians around the circle is:

$$t=\frac{2\pi r}{v_\perp}=\frac{2\pi}{v_\perp}\left(\frac{mv_\perp}{|q|B}\right)=\frac{2\pi m}{|q|B}.$$

During this time it is moving forward with speed $v_{\parallel}$, and will move a distance, *d*, given by:

$$d = v_{\parallel}t = \frac{v_{\parallel}2\pi m}{|q|B} = \frac{2\pi mv\cos\theta}{|q|B} = \frac{2\pi\left(9.11\cdot 10^{-31}\text{ kg}\right)\left(4.00\cdot 10^{5}\text{ m/s}\right)\cos 35.0^{\circ}}{\left(1.602\cdot 10^{-19}\text{ C}\right)\left(0.0400\text{ T}\right)} = 2.926\cdot 10^{-4}\text{ m}.$$

ROUND:

a) $r = 3.26\cdot 10^{-5}$ m.

b) $d = 2.93\cdot 10^{-4}$ m.

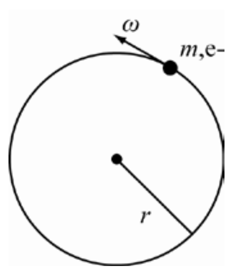
27.31. By Newton's second law, the quantity $d\vec{p}/dt$ is equal to the net force on the particle, exerted by the electric and magnetic fields. By the Work Energy Theorem, the quantity *dK*/*dt* is the rate at which work is done on the particle by the electric field. The magnetic force is always perpendicular to the particle's velocity and does no work. Hence, these quantities can be written:

$$\frac{d\vec{p}}{dt} = -q\left(\vec{E} + \vec{v}\times\vec{B}\right),\quad \frac{dK}{dt} = -q\vec{E}\bullet\vec{v}.$$

Slightly modified, these relationships can be put into a form that transforms simply from one reference frame to another according to Einstein's Special Theory of Relativity (see Chapter 35). They can be used to show that in a world governed by Einsteinian dynamics, the simplest force law that can be written is the combined electromagnetic force law above.

27.32. $$r = \frac{mv}{|q|B} = \frac{1.88\cdot 10^{-28}\text{ kg}\left(3.00\cdot 10^{6}\text{ m/s}\right)}{1.602\cdot 10^{-19}\text{ C}\left(0.500\text{ T}\right)} = 7.04\cdot 10^{-3}\text{ m} = 7.04\text{ mm}$$

27.33.



The net force is directed toward the center of the circle. From the right-hand rule, a positive charge requires the magnetic field to be oriented into the plane, in the negative *z*-direction. Since an electron is negatively charged, it can be concluded that the field points out of the page, along the positive z-direction. The magnitude is given by:

$$\omega = \frac{|q|B}{m} \Rightarrow B = \frac{m\omega}{|q|} = \frac{\left(9.11\cdot 10^{-31}\text{ kg}\right)\left(1.20\cdot 10^{12}\text{ s}^{-1}\right)}{1.602\cdot 10^{-19}\text{ C}} = 6.824\text{ T} \Rightarrow \vec{B} = 6.82\hat{z}\text{ T}.$$

27.34. $$r = \frac{mv}{|q|B},\quad K = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2K}{m}} \Rightarrow r = \frac{m}{|q|B}\left(\sqrt{\frac{2K}{m}}\right) = \frac{\sqrt{2Km}}{|q|B}$$

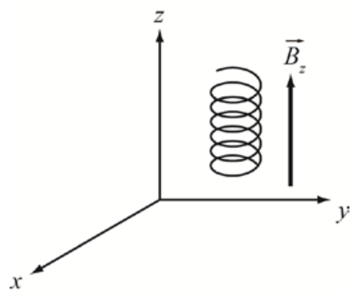
The mass, charge and fields are the same for the two particles.

$$\frac{r_1}{r_2} = \frac{\sqrt{2K_1m}}{|q|B}\left(\frac{|q|B}{\sqrt{2K_2m}}\right) = \sqrt{\frac{K_1}{K_2}} = \sqrt{\frac{4.00\cdot 10^{2}\text{ eV}}{2.00\cdot 10^{2}\text{ eV}}} = 1.41$$

So, the 400 eV particle travels in an orbit of radius 1.41 times that of the radius of the 200 eV particle.

27.35. **THINK:** The proton moves through a magnetic field. The component of the velocity parallel to the field is unchanged. The component perpendicular, however, will create a circular motion. The velocity of the proton is $v = \left(1.00\hat{x} + 2.00\hat{y} + 3.00\hat{z}\right)10^{5}$ m/s and the field is $B = 0.500\hat{z}$ T.

SKETCH:



RESEARCH: The radius of the circular motion in a magnetic field is:

$$r = \frac{mv}{|q|B}.$$

SIMPLIFY: The speed in the *xy*-plane is $v_{xy} = \sqrt{v_x^2 + v_y^2}$. The radius of the circle is: $r = \frac{m}{|q|B}\sqrt{v_x^2 + v_y^2}$.

CALCULATE: $r = \frac{1.6726 \cdot 10^{-27} \text{ kg}}{(1.6022 \cdot 10^{-19} \text{ C})(0.50 \text{ T})}\sqrt{(1.00 \cdot 10^5 \text{ m/s})^2 + (2.00 \cdot 10^5 \text{ m/s})^2} = 4.6688 \cdot 10^{-3} \text{ m}$

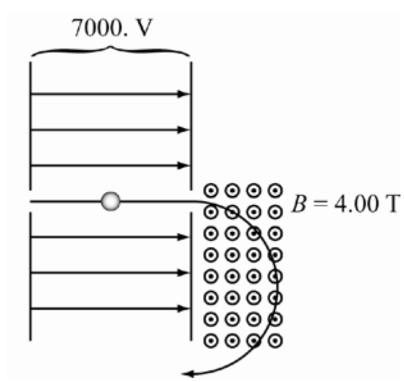
$v_{xy} = \sqrt{(1.00 \cdot 10^5 \text{ m/s})^2 + (2.00 \cdot 10^5 \text{ m/s})^2} = 2.23607 \cdot 10^5 \text{ m/s}$

ROUND: The values are given to two significant figures. The proton will follow a helical path with a velocity of $3.00 \cdot 10^5$ m/s along the *z*-axis, with the circular motion in the *xy*-plane having a speed of $2.24 \cdot 10^5$ m/s and a radius of 4.67 mm.

DOUBLE-CHECK: The angular velocity is on the same order of magnitude as the original velocity. Dimensional analysis confirms the units are correct.

27.36. **THINK:** The copper sphere accelerates in the region of the electric field, and gains an amount of kinetic energy equal to the potential difference times the charge on the sphere. At this speed, the sphere enters the magnetic field, which curves its path. The sphere has a mass of $m = 3.00 \cdot 10^{-6}$ kg and a charge of $5.00 \cdot 10^{-4}$ C. The potential difference is V = 7000. V, and the magnetic field is B = 4.00 T, perpendicular to the direction of the particle's initial velocity.

SKETCH:



RESEARCH: The kinetic energy will be $KE = mv^2/2 = PE = qV$. The radius of the path in the magnetic field is $r = mv/qB$.

SIMPLIFY: The velocity is $v^2 = 2qV/m$ or $v = \sqrt{2qV/m}$. The radius is then:

$$r = \frac{mv}{qB} = \frac{m}{qB}\sqrt{\frac{2qV}{m}} = \frac{1}{B}\sqrt{\frac{2mV}{q}}.$$

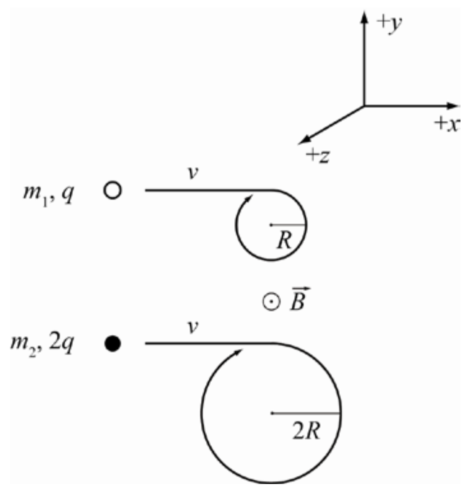
CALCULATE: $r = \frac{1}{4.00\text{ T}}\sqrt{\frac{2\left(3.00\cdot 10^{-6}\text{ kg}\right)\left(7000.\text{ V}\right)}{5.00\cdot 10^{-4}\text{ C}}} = 2.29129\text{ m}$

ROUND: The least precise values have three significant figures, so the radius of the copper's path in the magnetic field is 2.29 m.

DOUBLE-CHECK: The very large potential difference accelerates the sphere to a high speed, therefore a large radius of curvature is reasonable.

27.37. **THINK:** The particles will move in circular, clockwise paths (in the direction of $\vec{v}\times\vec{B}$) within the magnetic field. The radius of curvature of the path is proportional to the mass of the particle, and inversely proportional to the charge of the particle. Both particles move at the same speed within the same magnetic field. The radii, charges and masses of the particles can then be compared.

SKETCH:



RESEARCH: The radius is related to the mass and charge of the particles by $r = mv/|q|B.$ At the instant that the particles enter the magnetic field, the magnetic force acting on them is $\vec{F}_B = q\vec{v}\times\vec{B}.$ For the particles to travel in a straight line, the force on the particles due to the electric field must oppose the force due to the magnetic field: $\vec{F}_E = -\vec{F}_B \Rightarrow q\vec{E} = -q\vec{v}\times\vec{B}.$

SIMPLIFY: Since the velocity and the magnetic field is the same for both particles, $\frac{v}{B} = \frac{r_1 q_1}{m_1} = \frac{r_2 q_2}{m_2}.$ The ratio of the masses is:

$$\frac{m_1}{m_2} = \frac{r_1 q_1}{r_2 q_2} = \frac{Rq}{2R(2q)} = \frac{1}{4}.$$

For the particles to travel in a straight line,

$$q\vec{E} = -q\vec{v}\times\vec{B} = -qvB\left(\hat{x}\times\hat{z}\right) = -qvB\left(-\hat{y}\right)$$
$$\vec{E} = vB\hat{y}.$$

Therefore, the electric field must have magnitude $E = vB$ and point in the positive y-direction in order for the particles to move in a straight line.

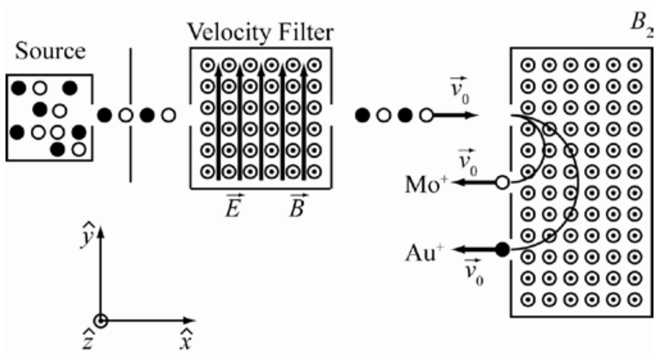
CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE CHECK: Since the mass increases with radius and charge, it makes sense that the particle with the smaller charge and radius has the smaller mass.

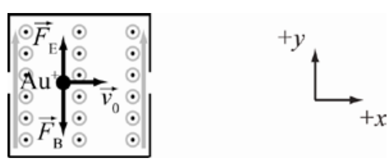
27.38. **THINK:** The question goes through the elements of a mass spectrometer. A source of gold and molybdenum emits singly ionized atoms at various velocities toward the velocity filter. The velocity filter described in the diagram uses a magnetic field, $\vec{B}_1$, and an electric field $\vec{E}$ to select the velocity of the exiting particles. It can be shown that the velocity of the ions must be $v = B_1 / E$. If the velocities are smaller, the force due to the electric field will dominate pushing the particles off the path through the filter. If the velocity is greater than B_1 / E then the force due to the magnetic field dominates and will also push the ion off its path. The particles exit the filter with the selected velocity and enter the mass spectrometer. The magnetic field inside the mass spectrometer curves the path of the entering particles based on their mass and charge. Thus, particles of different charges and masses can be separated. The gold and molybdenum have beams in the mass spectrometer with diameters of $d_2 = 40.00$ cm and $d_1 = 19.81$ cm, respectively. The mass of the gold ion is $m_{\text{gold}} = 3.271 \cdot 10^{-25}$ kg. Both ion types have charges of $q = e = 1.602 \cdot 10^{-19}$ C. The electric and magnetic field within the velocity filter are $\vec{E} = 1.789 \cdot 10^4 \hat{y}$ V/m and $\vec{B}_1 = 1.000\hat{z}$ T, respectively.

SKETCH:



RESEARCH:

(a)



(b) For the ions to pass through the velocity filter, the force due to the electric field must cancel the force due to the magnetic field: $F_e = qE = qv_0B$. The forces due to an electric and magnetic field are given by $\vec{F}_e = q\vec{E}$ and $\vec{F}_B = q\vec{v}_0 \times \vec{B}$, respectively.

(c) The radius of the ion's path in a magnetic field is given by $R = mv / (|q|B)$.

(d) The mass of the molybdenum can be determined by setting the velocity, charge and magnetic field equal to each other for each ion and comparing.

SIMPLIFY:

(b) The velocity of the exiting particle is then $v_0 = E / B_1$. This velocity does not depend on any parameters of the ions.

(c) The radius of the circular path is $r = mv_0 / (|q|B_2)$.

(d) $\dfrac{v_0}{|q|B_2} = \dfrac{r_{\text{Au+}}}{m_{\text{Au+}}} = \dfrac{r_{\text{Mo+}}}{m_{\text{Mo+}}}$. Solving for the mass of the molybdenum gives: $m_{\text{Mo+}} = \dfrac{r_{\text{Mo+}}}{r_{\text{Au+}}} m_{\text{Au+}}$.

CALCULATE:

(b) $v_0 = \dfrac{1.789 \cdot 10^4 \text{ V/m}}{1.000 \text{ T}} = 1.789 \cdot 10^4 \text{ m/s}$

(d) $m_{\text{Mo+}} = \dfrac{19.81 \text{ cm}/2}{40.00 \text{ cm}/2}\left(3.271 \cdot 10^{-25} \text{ kg}\right) = 1.6199 \cdot 10^{-25} \text{ kg}$

ROUND: The values are reported to four significant figures.

(b) The velocity filter allows particles traveling $1.789\cdot 10^4$ m/s to exit the filter. This value does not depend on the type of ion. It is based on the fact that the particle is charged and only depends on the fields.

(c) The equation for the radius of the semi-circular path is $r = mv/|q|B.$

(d) The mass of the molybdenum ion is $1.620\cdot 10^{-25}$ kg.

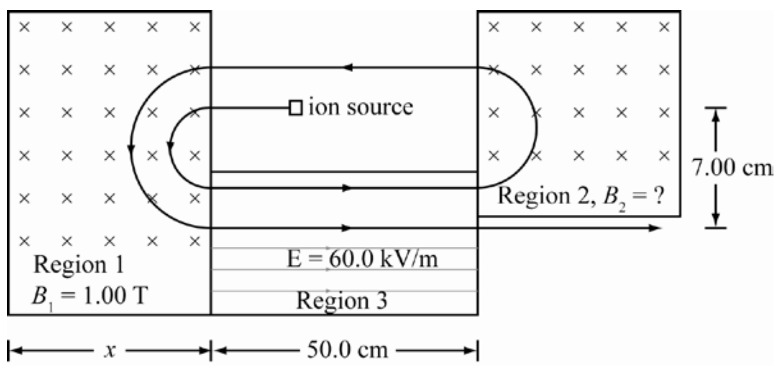
DOUBLE-CHECK: The calculated velocity has appropriate units, and the actual mass of molybdenum is about $1.64\cdot 10^{-25}$ kg. These facts help to support the answers as reasonable.

27.39. **THINK:** This question explores the function of an accelerator. The $^3\text{He}^+$ ion source ejects particles into region 1. The magnetic field in this region bends the path of the $^3\text{He}^+$ particle into a semicircular path. Particles then enter the 3rd region containing an electric field that accelerates the particles toward region 2. In region 2, the path of the particles is bent into a semicircle again. This time the particles do not pass through an electric field. The particle then enters region 1 again. The path is bent again and the particle accelerates once more before it exits the accelerator. The $^3\text{He}^+$ ions have a mass of $m = 5.02\cdot 10^{-27}$ kg and a charge of $q = e = 1.60\cdot 10^{-19}$ C. The ions start with a kinetic energy of:

$$K = 4.00 \text{ keV} = \frac{4.00\cdot 10^3 \text{ eV}\left(1.60\cdot 10^{-19} \text{ J}\right)}{1.00 \text{ eV}} = 6.40\cdot 10^{-16} \text{ J}.$$

The magnetic field in region 1 is $B_1 = 1.00$ T. In region 2, the magnetic field, B_2, is unknown. The 3rd region contains an electric field of E = 60.0 kV/m and has a length of l = 50.0 cm = 0.500 m. The distance between the source and the aperture is d = 7.00 cm = 0.0700 m.

SKETCH:



RESEARCH: The kinetic energy is equal to $KE = mv^2/2.$ The radius of the path of a charged particle in a magnetic field is $r = mv/|q|B.$ The force on the particle in region 3 is $F_e = qE,$ which must equal $F = ma$. With the acceleration of this region, the velocity at which it exits region 3 can be determined from:

$$v_f^2 = v_0^2 + 2ad.$$

SIMPLIFY: Let v_0, v_1 and v_2 be the velocity of the ion after it is ejected from the source, region 3 and region 3 the second time, respectively. The velocity after the source is given by $v_0 = \sqrt{2KE/m}.$ The acceleration of region 3 is $ma = qE$ or $a = qE/m$. The velocity, v_1, is then $v_1 = \sqrt{v_0^2 + 2al} = \sqrt{v_0^2 + 2qEl/m}.$ Similarly, the velocity, v_2, is given by:

$$v_2 = \sqrt{v_1^2 + 2al} = \sqrt{v_1^2 + 2qEl/m} = \sqrt{v_0^2 + 2qEl/m + 2qEl/m} = \sqrt{v_0^2 + 4qEl/m}.$$

The radius of the path, the first time the ion enters region 1 is $R_1 = mv_0/qB_1.$ The radius of the path in region 2 is $R_2 = mv_1/qB_2.$ The radius of the path the second time it goes through region 1 is $R_3 = mv_1/qB_1.$ For the particle to exit the aperture, a distance d = 7 cm from the ion source:

$$2R_1 - 2R_2 + 2R_3 = 2\left(\frac{mv_0}{qB_1} - \frac{mv_1}{qB_2} + \frac{mv_1}{qB_1}\right) = 2\frac{m}{q}\left(\frac{v_0 + v_1}{B_1} - \frac{v_1}{B_2}\right) = d.$$

Solve for B_2 to determine the magnetic field required in region 2:

$$\frac{v_0 + v_1}{B_1} - \frac{v_1}{B_2} = \frac{qd}{2m} \Rightarrow B_2\left(\frac{v_0 + v_1}{B_1}\right) - v_1 = \left(\frac{qd}{2m}\right)B_2$$

$$\Rightarrow B_2\left[\frac{v_0 + v_1}{B_1} - \frac{qd}{2m}\right] = v_1 \Rightarrow B_2 = \frac{v_1}{(v_0 + v_1)/B_1 - (qd/2m)} = \frac{2mB_1v_1}{2m(v_0 + v_1) - qdB_1}.$$

Region 1 must have dimensions larger than $R_3 = mv_1/qB_1$. The velocity of the ions as they exit the accelerator is $v_2 = \sqrt{v_0^2 + (4qEl/m)}$.

CALCULATE: $v_0 = \sqrt{\dfrac{2(6.40 \cdot 10^{-16}\ \text{J})}{5.02 \cdot 10^{-27}\ \text{kg}}} = 504995$ m/s

$$v_1 = \sqrt{\frac{2(6.40 \cdot 10^{-16}\ \text{J})}{5.02 \cdot 10^{-27}\ \text{kg}} + \frac{2(1.60 \cdot 10^{-19}\ \text{C})(60.0 \cdot 10^3\ \text{V/m})(0.500\ \text{m})}{5.02 \cdot 10^{-27}\ \text{kg}}} = 1472186\ \text{m/s}$$

$$v_2 = \sqrt{\frac{2(6.40 \cdot 10^{-16}\ \text{J})}{5.02 \cdot 10^{-27}\ \text{kg}} + \frac{4(1.60 \cdot 10^{-19}\ \text{C})(60.0 \cdot 10^3\ \text{V/m})(0.500\ \text{m})}{5.02 \cdot 10^{-27}\ \text{kg}}} = 2019822\ \text{m/s}$$

$$B_2 = \frac{2(5.02 \cdot 10^{-27}\ \text{kg})(1.00\ \text{T})(1472186\ \text{m/s})}{2(5.02 \cdot 10^{-27}\ \text{kg})(504955\ \text{m/s} + 1472186\ \text{m/s}) - (1.60 \cdot 10^{-19}\ \text{C})(0.0700\ \text{m})(1.00\ \text{T})} = 1.70866\ \text{T}$$

$$x = R_3 = \frac{mv_1}{qB_1} = \frac{(5.02 \cdot 10^{-27}\ \text{kg})(1472186\ \text{m/s})}{(1.60 \cdot 10^{-19}\ \text{C})(1.00\ \text{T})} = 0.046190\ \text{m}$$

ROUND: The values are reported to three significant figures.

(a) The magnetic field of region 2 is 1.71 T.

(b) The region must have dimensions greater than 4.62 cm.

(c) The velocity the ions leave the accelerator is $2.02 \cdot 10^6$ m/s.

DOUBLE-CHECK: Dimensional analysis confirms all the answers are in the correct units. These results are reasonable.

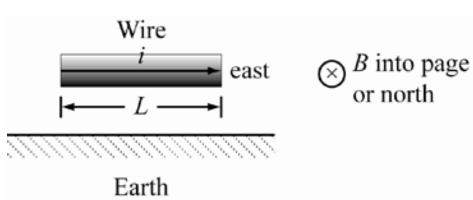
27.40. The force on a wire of length l and current i in a magnetic field B is given by $\vec{F} = li \times \vec{B} = liB\sin\theta$. The magnitude of the magnetic field is:

$$B = \frac{F}{li\sin\theta} = \frac{0.500\ \text{N}}{(2.00\ \text{m})(24.0\ \text{A})\sin 30.0°} = 0.02083\ \text{T} \approx 20.8\ \text{mT}.$$

27.41. The force on the wire is $\vec{F}_{\text{net}} = m\vec{a} = \vec{F}_{\text{B}} + \vec{F}_{\text{g}} = i\vec{L} \times \vec{B} + mg\hat{y} = iLB(-\hat{x} \times -\hat{z}) + mg\hat{y} = -iLB\hat{y} + mg\hat{y}$. For the conductor to stay at rest, $a = 0$ or $mg = iLB$. The suspended mass is then:

$$m = \frac{iLB}{g} = \frac{(20.0\ \text{A})(0.200\ \text{m})(1.00\ \text{T})}{(9.81\ \text{m/s}^2)} = 0.408\ \text{kg}.$$

27.42.



For the wire to levitate, the force of the magnetic field must equal the force of gravity on the wire:

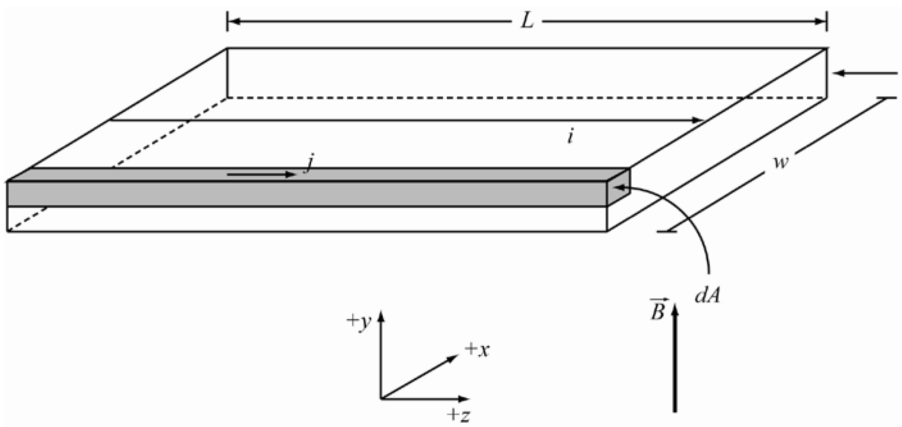
$$iLB = mg = \rho Vg = \rho\pi r^2 Lg.$$

The current required to levitate the wire is:

$$i = \frac{\rho\pi r^2 g}{B} = \frac{\left(8940\text{ kg/m}^3\right)\pi\left(0.000500\text{ m}\right)^2\left(9.81\text{ m/s}^2\right)}{\left(0.500\text{ G}\right)\left(0.0001\text{ T/G}\right)} = 1.38\cdot 10^3\text{ A}.$$

27.43. **THINK:** First, the relationship between the current in the sheet and the magnetic field must be established. The force on the sheet can then be determined. The sheet has length, L = 1.00 m, width, w = 0.500 m, and thickness, t = 1.00 mm = 0.00100 m. The magnetic field, B = 5.00 T, is perpendicular to the sheet and the current flowing through it. The current is $i = 3.00$ A.

SKETCH:



RESEARCH: The force on a wire carrying current in a magnetic field is $\vec{F} = i\vec{L}\times\vec{B}$.

SIMPLIFY: Imagine that the sheet is constructed of many wires of length, L, carrying a charge dq. The infinitesimal force on the sheet due to the wire is $d\vec{F} = dq\vec{L}\times\vec{B}$. Since L is perpendicular to B, $dF = dqLB$. The infinitesimal current is equal to the current density times the differential area, $dq = jdA$. The current density is equal to the total current divided by the cross sectional area, $j = i/A = i/wt$. The infinitesimal force is then $dF = dqLB = jdALB = iLBdA/wt$. Integrating over the area gives:

$$F = \frac{i}{wt}LB\int_0^w dx\int_0^t dy = \frac{i}{wt}LBwt = iLB.$$

This is the same result as that for a wire.

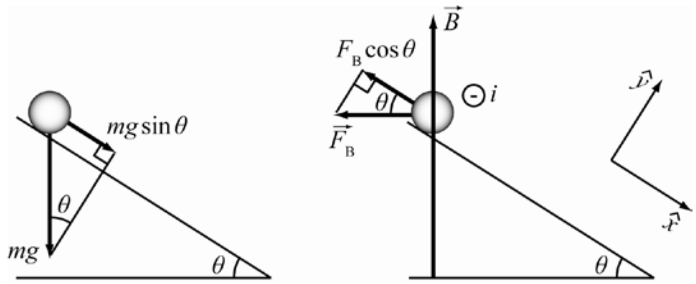
CALCULATE: $F = (3.00\text{ A})(1.00\text{ m})(5.00\text{ T}) = 15.0\text{ N}$

ROUND: The result is reported to two significant figures. The force on the sheet is 15.0 N. This is the same as the force on a wire of the same length with the same current and magnetic field.

DOUBLE-CHECK: The force on the sheet is the same as the force on a wire. This is expected since only the magnitude of the current matters in a wire (the size of the wire is not relevant).

27.44. **THINK:** For the rod to remain stationary, the forces of the magnetic field and gravity along the plane of the incline must cancel.

SKETCH:



RESEARCH: The force due to the magnetic field is $\left|\vec{F}_B\right| = i\vec{L}\times\vec{B} = iLB.$ Along the plane of the incline, the force is $F_{Bx} = F_B\cos\theta = iLB\cos\theta.$ The force due to gravity along the surface of the incline is $F_{gx} = mg\sin\theta.$

SIMPLIFY: Equating these forces gives the current:

$$iLB\cos\theta = mg\sin\theta \text{ or } i = \frac{mg}{LB}\tan\theta.$$

The current must go out of the page in the side view of the system, by the right-hand rule.

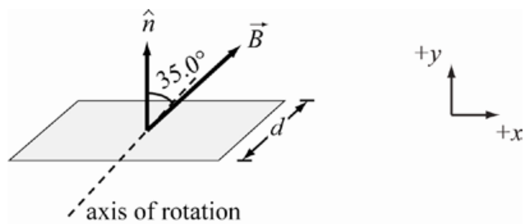
CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE CHECK: According to the derived expression, the strength of the current required to hold the wire stationary decreases as the magnetic field increases, which is logical.

27.45. **THINK:** The loop will experience a torque in the presence of a magnetic field as discussed in the chapter. The torque is also equal to the moment of inertia of the loop times its angular acceleration. The loop is square with sides of length, d = 8.00 cm and current, i = 0.150 A. The wire has a diameter of 0.500 mm (which corresponds to a radius of r = 0.250 mm) and a density of $\rho = 8960 \text{ kg/m}^3$. The magnetic field is B = 1.00 T and points $35.0°$ away from the normal of the loop.

SKETCH:



RESEARCH: The torque on the loop is $\tau = iAB\sin\theta.$ The moment of inertia for a rod about its center is $I = Md^2/12.$ The moment of a rod about an axis along its length is $I = \rho\pi r^4 L/4.$ The parallel axis theorem states $I_{\text{off center}} = I_{\text{cm}} + Md^2.$

SIMPLIFY: The moment of inertia of the loop is:

$$I = \frac{1}{12}Md^2 + \frac{1}{12}Md^2 + \left[\frac{1}{4}\rho\pi r^4 d + M\left(\frac{d}{2}\right)^2\right] + \left[\frac{1}{4}\rho\pi r^4 d + M\left(\frac{d}{2}\right)^2\right]$$

$$= \frac{1}{6}Md^2 + \frac{1}{2}\rho\pi r^4 d + \frac{1}{2}Md^2 = \left(\frac{1}{6}+\frac{1}{2}\right)Md^2 + \frac{1}{2}\rho\pi r^4 d = \frac{4}{6}\left(\rho\pi r^2 d\right)d^2 + \frac{1}{2}\rho\pi r^4 d = \frac{2}{3}\rho\pi r^2 d^3 + \frac{1}{2}\rho\pi r^4 d.$$

The angular acceleration is $\tau = I\alpha = iAB\sin\theta$ or $\alpha = iAB\sin\theta / I.$

$$\Rightarrow \alpha = \frac{id^2 B\sin\theta}{\rho\pi r^2 d^2\left(2d/3 + r^2/2d\right)} = \frac{iB\sin\theta}{\rho\pi r^2\left(2d/3 + r^2/2d\right)}$$

CALCULATE:

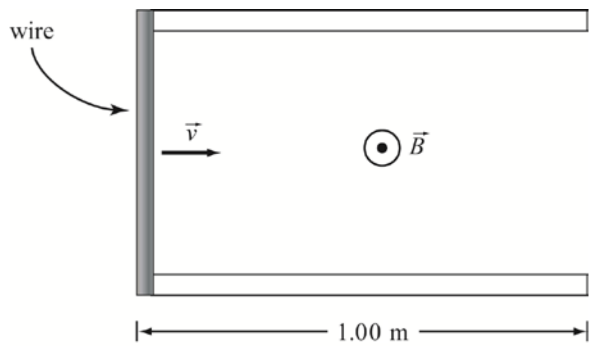
$$\alpha = \frac{(0.150\text{ A})(1.00\text{ T})(\sin 35.0°)}{(8960\text{ kg/m}^3)\pi(0.000250\text{ m})^2\left\{\left[2(0.0800\text{ m})/3\right]+\left[(0.000250\text{ m})^2/2(0.0800\text{ m})\right]\right\}} = 916.94\text{ rad/s}^2$$

ROUND: The values are given to two significant figures, thus the loop experiences an initial angular acceleration of $\alpha = 917\text{ rad/s}^2$.

DOUBLE-CHECK: One Tesla represents a magnetic field of large magnitude, resulting in a correspondingly large acceleration. This result is reasonable.

27.46. **THINK:** The rail-gun uses a magnetic force to accelerate a current carrying wire. The wire has a radius of $r = 5.10\cdot 10^{-4}\text{ m}$ and a density of $\rho = 8960\text{ kg/m}^3$. The current through the wire is $1.00\cdot 10^4\text{ A}$ and the magnetic field is B = 2.00 T. The wire travels a distance of L = 1.00 m before being ejected.

SKETCH:



RESEARCH: The force on the wire is $F = iLB$. The velocity of the wire is given by $v^2 = v_0^2 + 2ad$.

SIMPLIFY: The wire accelerates at a rate of:

$$a = \frac{F}{m} = \frac{iLB}{m} = \frac{iLB}{\rho\pi r^2 L} = \frac{iB}{\rho\pi r^2}.$$

The ejected velocity is:

$$v^2 = 2aL = \frac{2iBL}{\rho\pi r^2} \text{ or } v = \sqrt{\frac{2iBL}{\rho\pi r^2}}.$$

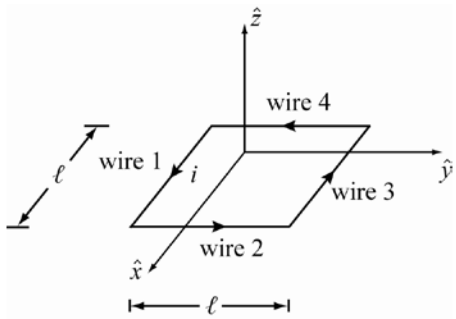
CALCULATE: $v = \sqrt{\dfrac{2(1.00\cdot 10^4\text{ A})(2.00\text{ T})(1.00\text{ m})}{(8960\text{ kg/m}^3)\pi(5.10\cdot 10^{-4}\text{ m})^2}} = 2337\text{ m/s}$

ROUND: The velocity is reported to two significant figures, like the given values. The wire exits the rail-gun at a speed of 2.34 km/s.

DOUBLE-CHECK: This result is very fast, about 7 times the speed of sound. This Navy has used this technology to accelerate 7 lb objects to this speed. As a comparison, this is double the speed of a bullet from a conventional rifle. A bullet weighs 55 g.

27.47. **THINK:** Using integration, the force on each segment of the loop can be found. From this, the net force can be found.

SKETCH:



RESEARCH: In the differential limit, the magnetic force on current carrying wire in a magnetic field is:

$$d\vec{F}_B = i\, d\vec{L}\times\vec{B}.$$

SIMPLIFY: The force on wire 1 is:

$$\vec{F}_{B,1} = i\int_{-l/2}^{l/2} d\vec{x}\times\vec{B} = \frac{iB_0}{a}\int_{-l/2}^{l/2} d\hat{x}\times\left(z\hat{x}+x\hat{z}\right) = -\frac{iB_0}{a}\int_{-l/2}^{l/2} x\,dx\,\hat{y} = \frac{iB_0}{2a}\left[x^2\right]_{-l/2}^{l/2} = 0.$$

The force on wire 2 is:

$$\vec{F}_{B,2} = \frac{iB_0}{a}\int_{-l/2}^{l/2} d\hat{y}\times\left(z\hat{x}+x\hat{z}\right) = \frac{iB_0}{a}\int_{-l/2}^{l/2}\left(-z\hat{z}+x\hat{x}\right)dy = \frac{iB_0}{a}\left(-z\hat{z}+x\hat{x}\right)\left[y\right]_{-l/2}^{l/2} = \frac{iB_0 l}{a}\left(-z\hat{z}+x\hat{x}\right).$$

Along wire 2, $x = l/2$ and $z = 0$, so

$$\vec{F}_{B,2} = \frac{iB_0 l^2}{2a}\hat{x}.$$

Wire 3 is similar to wire 1, so $F_{B,3} = F_{B,1} = 0$. The force on wire 4 is:

$$\vec{F}_{B,4} = \frac{iB_0}{a}\int_{l/2}^{-l/2} -d\hat{y}\times\left(z\hat{x}+x\hat{z}\right) = \frac{iB_0}{a}\left(z\hat{z}-x\hat{x}\right)\left[y\right]_{l/2}^{-l/2} = \frac{iB_0 l}{a}\left(-z\hat{z}+x\hat{x}\right) = \frac{iB_0 l}{a}\left[-(0)\hat{z}+(l/2)\hat{x}\right] = \frac{iB_0 l^2}{2a}\hat{x}.$$

The net force is:

$$\vec{F}_{\text{net}} = \vec{F}_{B,2}+\vec{F}_{B,4} = \frac{iB_0 l^2}{a}\hat{x}.$$

CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE CHECK: It is reasonable that the force is directly proportional to the magnetic field strength B_0, the side length *l* of the loop, and the current *i*.

27.48. The loop experiences a torque of:

$$\tau = NiAB\sin\theta = 20.0\left(2.00\cdot 10^{-3}\text{ A}\right)\left(0.0800\text{ m}\right)\left(0.0600\text{ m}\right)50.0\cdot 10^{-6}\text{ T} = 9.60\cdot 10^{-9}\text{ N m}.$$

By the right-hand rule, the torque is in the positive *y*-direction. To hold the loop steady, a torque of the same magnitude must be applied in the negative *y*-direction.

27.49. The torque on the loop due to the magnetic field is $\tau = \text{N}iAB\sin\theta = NiAB$. This is equal to the applied torque, $\tau = rF$. Equating the torques gives the magnetic field:

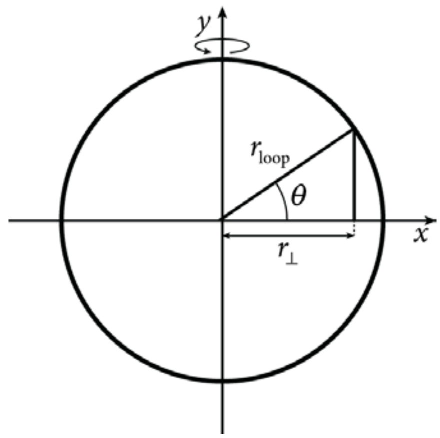
$$B = \frac{rF}{NiA} = \frac{rF}{Ni\pi r^2} = \frac{F}{Ni\pi r} = \frac{1.2\text{ N}}{120.\left(0.490\text{ A}\right)\pi\left(0.0480\text{ m}\right)} = 0.1353358\text{ T} = 0.135\text{ T}.$$

27.50. The torque on the pencil is:

$$\tau = NiAB\sin\theta = Ni\pi\left(\frac{d}{2}\right)^2 B\sin\theta = 20\left(3.00\text{ A}\right)\pi\left(\frac{0.00600\text{ m}}{2}\right)^2\left(5.00\text{ T}\right)\sin 60.0^\circ = 7.35\cdot 10^{-3}\text{ N m}.$$

27.51. **THINK:** The loop feels a torque if it carries a current in the presence of a uniform magnetic field. The maximum torque occurs when the magnetic moment of the loop is perpendicular to the magnetic field. The loop has a radius of r_{loop} = 0.500 m, density of ρ = 8960 kg/m^3, and the wire has a cross-sectional area of $A = 1.00 \cdot 10^{-5}$ m^2. A potential difference of $\Delta V = 0.0120$ V is applied to the wire. The loop is in a magnetic field of $B = 0.250$ T.

SKETCH:



RESEARCH: The resistivity of the wire is given by $\rho_{\text{R}} = 16.78 \cdot 10^{-9}\ \Omega$ m. The current is found using $\Delta V = iR$. The magnetic moment of the loop is given by $\vec{\mu} = iA\vec{n}$. The torque is equal to $\tau = I\alpha$, where I is the moment of inertia of the loop about its diameter. The moment of inertia is given by $I = \int r^2{}_\perp dm$. The torque due to the magnetic field is given by $\vec{\tau} = \vec{\mu} \times \vec{B}$.

SIMPLIFY: The mass of the loop is given by

$$m = \int dm = \int_0^{2\pi} \left(\rho A r_{\text{loop}} d\theta\right) = 2\pi\rho A r_{\text{loop}}.$$

The moment of inertia of one half of the loop is:

$$I_{1/2} = \int r_\perp{}^2 dm = \int_{-\pi/2}^{\pi/2} \left(r_{\text{loop}} \cos\theta\right)^2 \left(\rho A r_{\text{loop}} d\theta\right) = \rho A r_{\text{loop}}^3 \int_{-\pi/2}^{\pi/2} \left(\cos\theta\right)^2 d\theta = \rho A r_{\text{loop}}^3 \left(\frac{\pi}{2}\right) = \frac{1}{2}\pi\rho A r_{\text{loop}}^3.$$

The total inertial moment is twice this magnitude:

$$I = \pi\rho A r_{\text{loop}}^3 = \frac{1}{2} m r_{\text{loop}}^2.$$

The torque and thus the angular acceleration is maximized when the magnetic moment is perpendicular to the magnetic field. The torque is then given by

$$\tau_{\text{max}} = \left|\vec{\mu} \times \vec{B}\right| = \mu B \sin\theta = \mu B = iAB = i\pi r_{\text{loop}}^2 B.$$

The angular acceleration is:

$$\alpha_{\text{max}} = \frac{\tau_{\text{max}}}{I} = \frac{i\pi r_{\text{loop}}^2 B}{\frac{1}{2} m r_{\text{loop}}^2} = \frac{2\pi i B}{m}.$$

The current is:

$$i = \frac{\Delta V}{R}.$$

The resistance of the wire making up the loop is

$$R = \frac{2\pi r_{\text{loop}} \rho_{\text{R}}}{A}.$$

CALCULATE: For this problem, it is instructive to calculate the various quantities separately and then combine the intermediate results to get the maximum angular acceleration of the loop. The mass of the loop is

$$m = 2\pi\rho A r_{\text{loop}} = 2\pi\left(8960\text{ kg/m}^3\right)\left(1.00\cdot 10^{-5}\text{ m}^2\right)\left(0.500\text{ m}\right) = 0.2814867\text{ kg}.$$

The resistance of the wire making up the loop is

$$R = \frac{2\pi r_{\text{loop}}\rho_{\text{R}}}{A} = \frac{2\pi\left(0.500\text{ m}\right)\left(16.78\cdot 10^{-9}\ \Omega\text{ m}\right)}{1.00\cdot 10^{-5}\text{ m}^2} = 0.00527159\ \Omega.$$

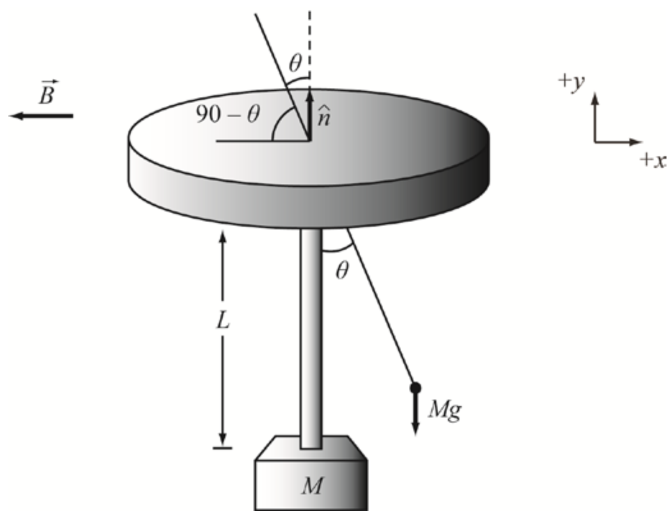
The current in the loop is $i = \frac{\Delta V}{R} = \frac{0.0120\text{ V}}{0.00527159\ \Omega} = 2.27635\text{ A}.$

The maximum angular acceleration is $\alpha_{\text{max}} = \frac{2\pi i B}{m} = \frac{2\pi\left(2.27635\text{ A}\right)\left(0.250\text{ T}\right)}{0.2814867\text{ kg}} = 12.702857\ \text{rad/s}^{-2}.$

ROUND: The result is reported to three significant figures. The maximum angular acceleration is $\alpha_{\text{max}} = 12.7\text{ rad/s}^2.$

DOUBLE-CHECK: The mass of the loop, the current in the loop, and the resistance of the loop are reasonable and all have the correct units.

27.52.



The torque on the coil as a function of θ is $\tau = \left|\vec{\mu}\times\vec{B}\right| = \mu B\sin\left(90° - \theta\right) = \mu B\cos\theta.$ The magnetic moment of the coil is $\mu = NiA.$ Assume the coil contributes little to the inertial moment of the galvanometer. Assume the mass is distributed evenly through the rod. The torque on the rod due to gravity is:

$$\tau = \left|\vec{r}\times\vec{F}\right| = LF\sin\theta = LMg\sin\theta,$$

where r is the distance to the center of mass of the rod. Equating the two torques gives:

$$\mu B\cos\theta = LMg\sin\theta \ \Rightarrow\ \tan\theta = \frac{\mu B}{LMg} = \frac{NiAB}{LMg}\ \Rightarrow\ \theta = \tan^{-1}\left(\frac{NIAB}{LMg}\right).$$

27.53. Assume the electron orbits the hydrogen with speed, v. The current of the electron going around its orbit is:

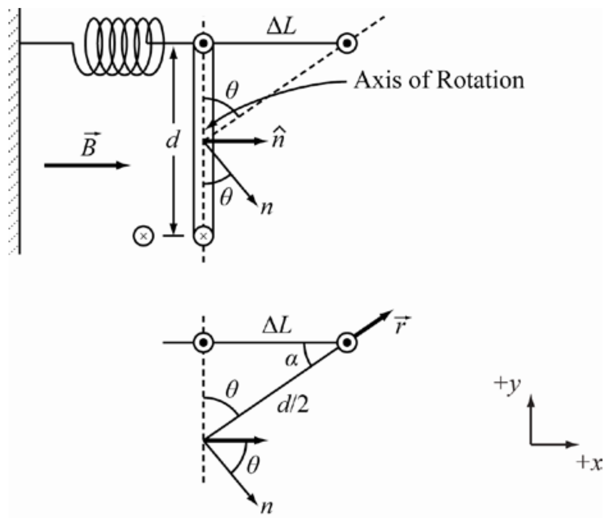
$$i = \frac{q}{t} = -\frac{e}{d/v} = -\frac{ev}{d} = -\frac{ev}{2\pi r}.$$

The magnetic moment of the orbit is: $\mu = iA = i\pi r^2 = -\frac{ev}{2\pi r}\left(\pi r^2\right) = -\frac{1}{2}evr.$ Angular momentum is given by $L = rp = rmv$. Using the angular momentum, the moment is:

$$\vec{\mu} = -\frac{1}{2}er\vec{v} = -\frac{1}{2}e\left(\frac{m}{m}\right)r\vec{v} = -\frac{erm\vec{v}/2}{m} = -\frac{e\vec{L}}{2m}.$$

27.54. **THINK:** The magnetic field produces a torque on the coil. This stretches the spring until it creates a torque equal but opposite to the torque due to the magnetic field. The ring has a diameter of d = 0.0800 m and carries a current of $i = 1.00$ A. The spring constant is k = 100. N/m and the magnetic field is B = 2.00 T.

SKETCH:



RESEARCH: The torque due to the spring is $\tau = \vec{r} \times \vec{F}_s$. The torque due to the magnetic field is $\tau = iAB\sin\theta$.

SIMPLIFY: The angle θ is given by: $\sin\theta = \frac{\Delta L}{d/2} = \frac{2\Delta L}{d}$. The torque due to the spring is $\tau_s = \vec{r} \times \vec{F}_s = dk\Delta L\sin\theta/2$. Equating this to the torque due to the magnetic field gives:

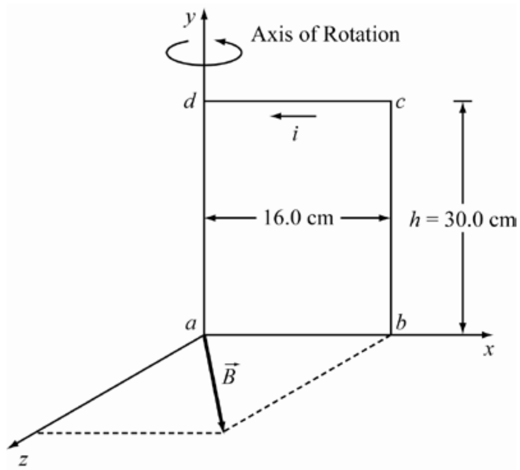
$$dk\Delta L\cos\theta/2 = iAB\sin\theta \Rightarrow \Delta L = \frac{2iAB}{dk} = \frac{2i\left(\pi d^2/4\right)B}{dk} = \frac{i\pi dB}{2k}.$$

CALCULATE: $\Delta L = \frac{(1.00 \text{ A})\pi(0.0800 \text{ m})(2.00 \text{ T})}{2(100. \text{ N/m})} = 0.002513274 \text{ m}$

ROUND: The values are given to three significant figures, thus the extension is $\Delta L = 2.51$ mm.
DOUBLE-CHECK: This result is reasonable.

27.55. **THINK:** The coil experiences a torque due to the magnetic field. The coil, however, is hinged along one of its lengths. The torque then can be determined in the normal way. The force on each segment is calculated to determine the torque on the coil. The coil has N = 40, a width of w = 16.0 cm, a height of h = 30.0 cm and carries a current of 0.200 A. The magnetic field is $\vec{B} = \left(0.0650\hat{x} + 0.250\hat{z}\right)$ T.

SKETCH:



RESEARCH: The force on a length of wire carrying current in a magnetic field is $\vec{F} = Ni\vec{l} \times \vec{B}$. The torque is given by $\tau = \vec{r} \times \vec{F}$.

SIMPLIFY: The force on the segment *a*-*b* is $\vec{F}_{\text{ab}} = Nil_{\text{ab}}\hat{x} \times \left[B_x\hat{x} + B_z\hat{z}\right] = NiwB_z\left(\hat{x} \times \hat{z}\right) = -NiwB_z\hat{y}$. The force on the segment *b*-*c* is:

$$\vec{F}_{\text{bc}} = Nil_{\text{bc}}\hat{y} \times \left[B_x\hat{x} + B_z\hat{z}\right] = Nih\left[B_x\left(\hat{y} \times \hat{x}\right) + B_z\left(\hat{y} \times \hat{z}\right)\right] = Nih\left[B_x\left(-\hat{z}\right) + B_z\left(\hat{x}\right)\right] = Nih\left[-B_x\hat{z} + B_z\hat{x}\right].$$

The net force on the coil is zero since the magnetic field does no work. The coil is free to rotate about the *y*-axis. The only force that can contribute is F_{bc}: $\tau = \vec{r} \times \vec{F} = w\hat{x} \times \vec{F}_{\text{bc}} = w\hat{x} \times Nih\left[B_z\hat{x} - B_x\hat{z}\right] = wNihB_x\hat{y}$. The door rotates counterclockwise when looking from the top.

CALCULATE: $\vec{F}_{\text{ab}} = -(40)(0.200 \text{ A})(0.160 \text{ m})(0.250 \text{ T})\hat{y} = -0.320\hat{y} \text{ N}$

$\vec{F}_{\text{bc}} = (40)(0.200 \text{ A})(0.300 \text{ m})\left[(-0.0650 \text{ T})\hat{z} + (0.250 \text{ T})\hat{x}\right] = (-0.156\hat{z} + 0.600\hat{x}) \text{ N} = (0.600\hat{x} - 0.156\hat{z}) \text{ N}$

$\tau = (40)(0.200 \text{ A})(0.160 \text{ m})(0.300 \text{ m})(0.0650 \text{ T})\hat{y} = 0.02496\hat{y} \text{ N m}$

ROUND:

(a) The force on segment *a*-*b* is $\vec{F}_{\text{ab}} = -0.320\hat{y}$ N

(b) The force on segment *b*-*c* is $\vec{F}_{\text{bc}} = (0.600\hat{x} - 0.156\hat{z})$ N or $\left|F_{\text{bc}}\right| = 0.620$ N directed 14.6° from the *x*-axis toward the negative *z*-axis.

(c) The total force is $F_{\text{net}} = 0$.

(d) The torque on the coil is $\left|\tau\right| = 0.0250$ N m and rotates along the *y*-axis in counterclockwise fashion.

(e) The coil rotates in a counterclockwise fashion as seen from above.

DOUBLE-CHECK: This result is reasonable.

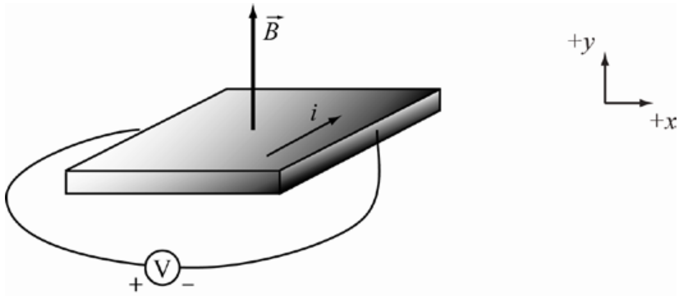
$$\tau = NiAB\sin\theta = 40(0.200 \text{ A})(0.160 \text{ m})(0.300 \text{ m}), \quad \sqrt{(0.0650 \text{ T})^2 + (0.250 \text{ T})^2}\sin(14.6°) = 0.0250 \text{ N m}$$

27.56. The Hall voltage is given by: $\Delta V_{\|} = \dfrac{iB}{neh}$. The carrier density of the electron sheet is:

$$n = \frac{iB}{eh\Delta V_{\|}} = \frac{10.0 \cdot 10^{-6}(1.00 \text{ T})}{(1.60 \cdot 10^{-19} \text{ C})(10.0 \cdot 10^{-9} \text{ m})(0.680 \cdot 10^{-3} \text{ V})} = 9.19 \cdot 10^{24} \text{ e/m}^3.$$

27.57. **THINK:** The question asks for the carrier density of the thin film, and the nature of the carriers. The film has a thickness of $h = 1.50$ μm. The current is i = 12.3 mA and the voltage reads V = -20.1 mV. The magnetic field is B = 0.900 T.

SKETCH:



RESEARCH:
(a) Due to the magnetic force, the charge carriers are accumulated on the visible edge of the sample. Since the polarity of the Hall potential is negative, the charge carriers are holes.
(b) The Hall voltage magnitude is given by $\Delta V_{\text{H}} = \dfrac{iB}{neh}$.

SIMPLIFY:
(b) The charge carrier density is $n = \dfrac{iB}{he\Delta V_{\text{H}}}$.

CALCULATE:

(b) $n = \dfrac{12.3 \cdot 10^{-3}(0.900 \text{ T})}{(1.50 \cdot 10^{-6} \text{ m})(1.602 \cdot 10^{-19} \text{ C})(20.1 \cdot 10^{-3} \text{ V})} = 2.2919 \cdot 10^{24} \text{ holes/m}^3$

ROUND:
(b) The values are given to three significant figures, so the carrier density of the film is $n = 2.29 \cdot 10^{24} \text{ holes/m}^3$.
DOUBLE-CHECK: This is a reasonable value for a carrier density.

27.58. The radius of the proton's path is: $r = \dfrac{mv}{qB} = \dfrac{m}{qB}\left(\dfrac{2\pi r}{T}\right) = \dfrac{m}{qB}\left(\dfrac{2\pi r}{1/f}\right) = \dfrac{2\pi mrf}{qB}$. The radius of the path and its frequency are: $r = \dfrac{mv}{qB}$ and $f = \dfrac{qB}{2\pi m}$, respectively. In the cyclotron:

$$r = \frac{(1.67 \cdot 10^{-27} \text{ kg})(2.998 \cdot 10^{8} \text{ m/s})/2}{(1.602 \cdot 10^{-19} \text{ C})(9.00 \text{ T})} = 0.1736 \text{ m} \approx 0.174 \text{ m},$$

$$f = \frac{(1.602 \cdot 10^{-19} \text{ C})(9.00 \text{ T})}{2\pi(1.67 \cdot 10^{-27} \text{ kg})} = 1.374 \text{ MHz} \approx 1.37 \text{ MHz}.$$

In the Earth's magnetic field:

$$r = \frac{(1.67 \cdot 10^{-27} \text{ kg})(2.998 \cdot 10^{8} \text{ m/s})/2}{(1.602 \cdot 10^{-19} \text{ C})(0.500 \text{ G})(0.0001 \text{ T/G})} = 31.25 \text{ km} \approx 31.3 \text{ km},$$

$$f = \frac{(1.602 \cdot 10^{-19} \text{ C})(0.500 \cdot 10^{-4} \text{ T})}{2\pi(1.67 \cdot 10^{-27} \text{ kg})} = 0.7634 \text{ kHz} \approx 0.763 \text{ kHz}.$$

27.59. The force on the wire is:

$$F = i\vec{L} \times \vec{B} = iLB\sin\theta = (3.41 \text{ A})(0.100 \text{ m})(0.220 \text{ T})\sin(90.0° - 10.0°) = 0.07388 \text{ N} \approx 7.39 \cdot 10^{-2} \text{ N}.$$

27.60. The radius of a charged particle's path in a magnetic field is $r = mv/|q|B$. For this electron, the radius of its path is:

$$r = \frac{\left(9.109 \cdot 10^{-31} \text{ kg}\right)\left(6.00 \cdot 10^{7} \text{ m/s}\right)}{\left(1.602 \cdot 10^{-19} \text{ C}\right)\left(0.500 \cdot 10^{-4} \text{ T}\right)} = 6.82 \text{ m}.$$

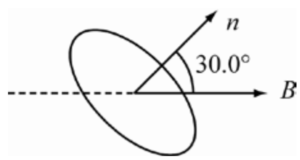
27.61. The force on a current carrying wire in a magnetic field is $F = ilB\sin\theta$. To determine the minimum current, set $\theta = 90°$:

$$i = \frac{F}{lB} = \frac{1.00 \text{ N}}{0.100 \text{ m}\left(0.430 \cdot 10^{-4} \text{ T}\right)} = 232{,}558 \text{ A} \approx 2.33 \cdot 10^{5} \text{ A}.$$

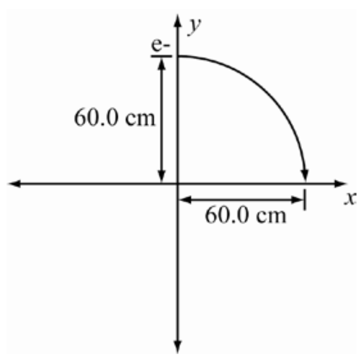
The minimum current required for the wire to experience a force of 1.0 N is $i = 2.33 \cdot 10^{5}$ A.

27.62. The torque on a current carrying coil in a magnetic field is:

$$\begin{aligned}\tau &= NiAB\sin\theta \\ &= (100)\left(100. \cdot 10^{-3} \text{ A}\right)\pi\left(0.100 \text{ m}\right)^2\left(0.0100 \text{ T}\right)\left(\sin 30.0°\right) \\ &= 0.0015707 \text{ N m} \approx 1.57 \cdot 10^{-3} \text{ N m}.\end{aligned}$$



27.63. (a) The electron must travel in a circular path with a radius of 60.0 cm, as shown in the figure below.



By the right-hand rule, B must be in the negative z-direction.

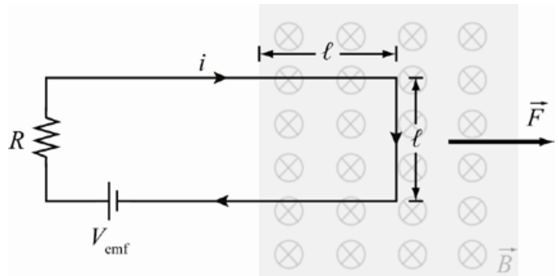
$$B = \frac{mv}{q_e r} = \frac{\left(9.11 \cdot 10^{-31} \text{ kg}\right)\left(2.00 \cdot 10^{5} \text{ m/s}\right)}{\left(1.60 \cdot 10^{-19} \text{ C}\right)\left(0.600 \text{ m}\right)} = 1.89 \cdot 10^{-6} \text{ T}$$

(b) The magnetic force is perpendicular to the motion and does no work.

(c) Since the speed of an electron does not change, the time the electron takes to travel a quarter-circle is given by:

$$t = \frac{\pi r}{2v} = \frac{3.14159\left(0.600 \text{ m}\right)}{2\left(2.00 \cdot 10^{5} \text{ m/s}\right)} = 4.71 \cdot 10^{-6} \text{ s}.$$

27.64. **THINK:** First, determine the current using Ohm's law and then determine the force on the wire.
SKETCH:

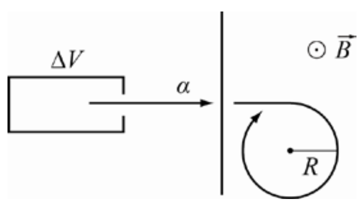


RESEARCH: Ohm's law is $V = iR$. Use the values: V = 12.0 V, B = 5.00 T, and $R = 3.00\ \Omega$. $\vec{F} = i\vec{L}\times\vec{B}$. In this case, since the top and bottom part of the loop have currents traveling in opposite directions, their forces will cancel. Only the right side of the loop will contribute to the force.
SIMPLIFY: $F = ilB$, to the right (from the right-hand rule), $V = iR \Rightarrow i = V/R$. Substitute the expression for I into the expression for F to get: $F = VlB/R$.

CALCULATE: $F = \dfrac{12.0\text{ V}}{3.00\ \Omega}(1.00\text{ m})(5.00\text{ T}) = 20.0\text{ N}$

ROUND: F = 20.0 N to the right.
DOUBLE-CHECK: The final expression makes sense since it is expected that if a larger voltage is applied, a larger force is attained.

27.65. **THINK:** As the alpha particle enters the region of the magnetic field, its motion will be deflected into a curved path. The radius of curvature is determined by the mass, charge, and initial velocity, and by the strength of the field. All quantities are given except for the velocity. The particle's velocity can be determined by employing the law of conservation of energy. The period of revolution can be determined from the particle's radius of curvature and velocity. $m_\alpha = 6.64\cdot 10^{-27}$ kg, $q_\alpha = +2e$, $\vec{B} = 0.340$ T, and $\Delta V = 2700$ V.
SKETCH:



RESEARCH: The radius of curvature is given by $r = mv/|q|B$. By conservation of kinetic energy, $|q|V = \frac{1}{2}mv^2$. The period of revolution is given by $T = \dfrac{2\pi r}{v}$.

SIMPLIFY: $|q|V = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\dfrac{2|q|V}{m}}$

CALCULATE: $r = \dfrac{(6.64\cdot 10^{-27}\text{ kg})(5.105\cdot 10^{5}\text{ m/s})}{2|1.602\cdot 10^{-19}\text{ C}|0.340\text{ T}} = 0.03111\text{ m}$

$$v = \sqrt{\frac{2|2(1.602\cdot 10^{-19}\text{ C})|2700}{6.64\cdot 10^{-27}\text{ kg}}} = 5.105\cdot 10^{5}\text{ m/s}$$

$$T = \frac{2\pi(0.03111\text{ m})}{5.105\cdot 10^{5}\text{ m/s}} = 3.829\cdot 10^{-7}\text{ s}$$

ROUND: Rounding to three significant figures, $r = 0.0311$ m and $T = 3.83\cdot 10^{-7}$ s.

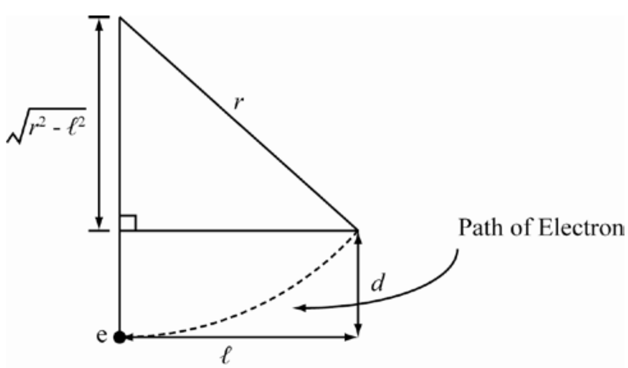
DOUBLE CHECK: All calculated values have the correct units. The numerical values are appropriate to the scale of the particle.

27.66. **THINK:** The electric field component and the vertical component must cancel each other.
SKETCH: Not necessary.
RESEARCH: It is required that $\vec{v}\times\vec{B}=-\vec{E} \Rightarrow \left(\vec{v}\times\vec{B}\right)\bullet\vec{B}=-\vec{E}\bullet\vec{B} \Rightarrow 0=\vec{E}\bullet\vec{B}$ (since for any vector, $\left(\vec{A}\times\vec{B}\right)\bullet\vec{B}=0$). But since $\vec{E}$ is not perpendicular to $\vec{B}$, $\vec{E}\bullet\vec{B}\neq 0$. Note that $\vec{E}=-150.\hat{z}$ N/C and $\vec{B}=\left(50.0\hat{y}-20.0\hat{z}\right)$ This scenario cannot occur.
SIMPLIFY: Nothing to simplify.
CALCULATE: No calculations are necessary.
ROUND: There are no values to round.
DOUBLE-CHECK: No Lorenz force can counteract an electric force in *z*-direction, if the particle is also traveling in *z*-direction, because the Lorenz force is always perpendicular to the velocity vector.

27.67. **THINK:** Determine the velocity in terms of the mass and see how this changes the answer.
SKETCH: Not necessary.
RESEARCH: $v=qBr/m$, $B=0.150$ T, $r=0.0500$ m and $m=6.64\cdot 10^{-27}$ kg.
SIMPLIFY: It is not necessary to simplify.
CALCULATE: $v=\dfrac{\left(1.602\cdot 10^{-19}\text{ C}\right)\left(0.150\text{ T}\right)\left(0.0500\text{ m}\right)}{6.64\cdot 10^{-27}\text{ kg}}=1.809\cdot 10^{5}\text{ m/s}$

Note that for $m'=\dfrac{3}{4}m$, $v'=\dfrac{qBr}{3m/4}=\dfrac{4}{3}v$; the velocity increases by a factor of $4/3$.
ROUND: $v=1.81\cdot 10^{5}$ m/s
DOUBLE-CHECK: Since there is an inverse relationship between *v* and *m*, it makes sense that decreasing *m* by a factor of 3/4 increases *v* by factor of $\left(3/4\right)^{-1}=4/3$.

27.68. **THINK:** First determine the radius of curvature and then determine the amount the electron deviates over a distance of $l=1.00$ m. Use $B=0.300$ G as the Earth's magnetic field. Convert the energy into Joules.
SKETCH:



The dashed line is the path of the electron.

RESEARCH: Geometry gives $d=r-\sqrt{r^2-l^2}$. $r=\dfrac{mv}{qB}=\dfrac{p}{qB}=\dfrac{\sqrt{2mE}}{qB}$, $E=7500\text{ eV}\left(1.609\cdot 10^{-19}\text{ J/eV}\right)$.
SIMPLIFY: It is not necessary to simplify.
CALCULATE: $r=\dfrac{\sqrt{2\left(9.11\cdot 10^{-31}\text{ kg}\right)\left(7.50\cdot 10^{3}\text{ eV}\right)\left(1.602\cdot 10^{-19}\text{ J/eV}\right)}}{\left(1.602\cdot 10^{-19}\text{ C}\right)\left(0.300\cdot 10^{-4}\text{ T}\right)}=9.7354\text{ m}$

$d = (9.7354\text{ m}) - \sqrt{(9.7354\text{ m})^2 - (1.00\text{ m})^2} = 0.051495\text{ m},$ upward from the ground.

ROUND: To three significant figures, the answer should be rounded to: d = 0.0515 m upward.

DOUBLE-CHECK: Note that d << 1.00 m, as is expected since the magnetic field of the Earth is fairly weak.

27.69. **THINK:** Since the electric field from the plates will cause the proton to move in the negative y-direction, the magnetic field must apply a force in the positive y-direction. It is determined from the right-hand rule that $\vec{B}$ must be in the negative z-direction.

SKETCH:

RESEARCH: $v = 1.35 \cdot 10^6$ m/s, $V = 200.$ V, $d = 35.0 \cdot 10^{-3}$ m , and $E = V / d$. It is required that $vB = E$.

SIMPLIFY: $B = \dfrac{E}{v} = \dfrac{V}{vd}$

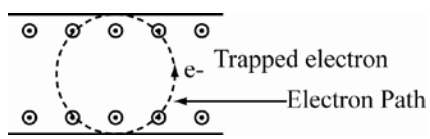
CALCULATE: $B = \dfrac{(200.\text{ V})}{(1.35 \cdot 10^6\text{ m/s})(35.0 \cdot 10^{-3}\text{ m})} = 4.23 \cdot 10^{-3}\text{ T}$

ROUND: To three significant figures, $B = -4.23 \cdot 10^{-3}\,\hat{z}$ T

DOUBLE-CHECK: The final expression for B makes sense. If the applied voltage is larger, one needs a larger magnetic field.

27.70. **THINK:** Determine the radius of curvature. This distance will allow the electron to be trapped in the field.

SKETCH:



RESEARCH: The magnitude is given by $F_B = qv_0B,$ in the positive y-direction (by the right-hand rule).

$d = r = \dfrac{mv_0}{|q|B}$

SIMPLIFY: $v_0 = \dfrac{eBd}{m}$

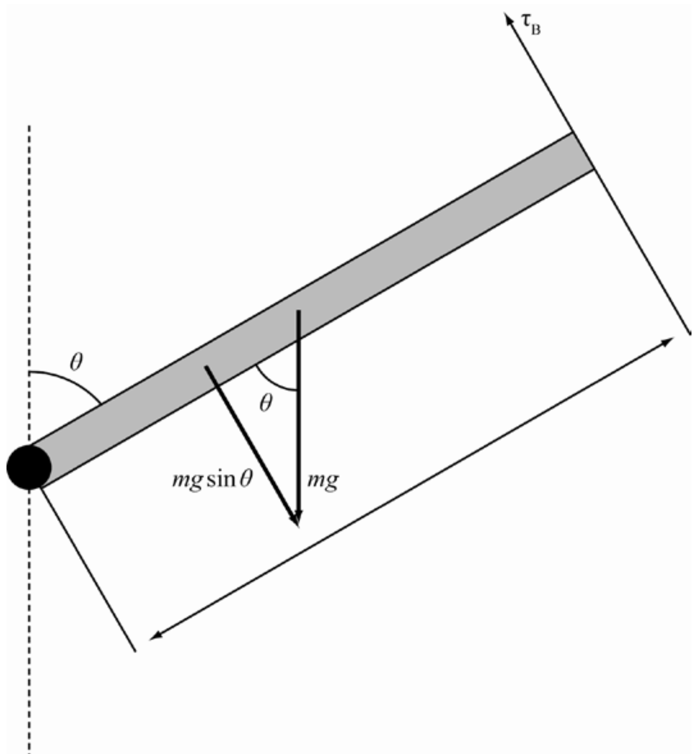
CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: It makes sense that for larger magnetic fields and for larger widths, the escape velocity is larger.

27.71. **THINK:** Equilibrium occurs when the net torque on the coil is zero. Use the values: $A = d^2$, $d = 0.200$ m, i= 5.00 A, m = 0.250 kg, B = 0.00500 T, and N = 30.

SKETCH:



RESEARCH: $\tau_g = mg\sin\theta(d/2)$, $\tau_B = NiAB\cos\theta$. It is required that $\tau_g = \tau_B$.

SIMPLIFY: $\tau_g = \tau_B \Rightarrow \frac{1}{2}mgd\sin\theta = Nid^2B\cos\theta$

$$\frac{\sin\theta}{\cos\theta} = \frac{2NdiB}{mg} \Rightarrow \tan\theta = \frac{2NdiB}{mg} \Rightarrow \theta = \tan^{-1}\left(\frac{2NdiB}{mg}\right)$$

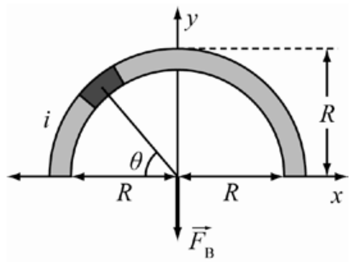
CALCULATE: $\theta = \tan^{-1}\left[\frac{2(30)(0.200\text{ m})(5.00\text{ A})(0.00500\text{ T})}{(0.250\text{ kg})(9.81\text{ m/s}^2)}\right] = 6.9740°$

ROUND: The number of turns is precise, so it does not limit the precision of the answer. The rest of the values are given to three significant figures of precision, so it is appropriate to round the final answer to: $\theta = 6.97°$.

DOUBLE-CHECK: It makes sense that θ is inversely proportional to m, since the less the coil weighs, the more vertical it must be.

27.72. **THINK:** It can be deduced from symmetry that the net force is in the negative y-direction. Therefore, $F_B = F_y$.

SKETCH:



RESEARCH: From the book, $F_B = iLB\sin\theta$. The objective is to sum up the forces due to each point of the semi-circle. This means we will integrate F_B over the length of the wire.

SIMPLIFY: $F_y = \int_{l_0}^{l_1} iLB\sin\theta\, dL = \int_0^{\pi} iLB\sin\theta\left(Rd\theta\right) = iLBR\int_0^{\pi}\sin\theta\, d\theta$ in the negative y-direction.

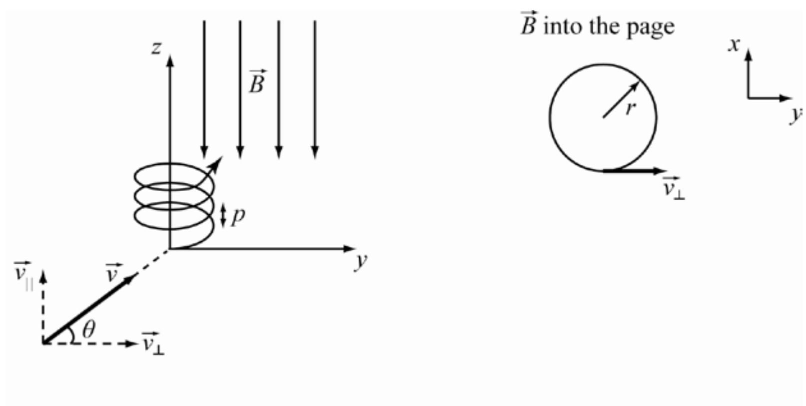
CALCULATE: $F_y = iLRB\cos\theta\big|_0^{\pi} = -iLRB\left(-1-1\right) = 2iLRB$ in the negative y-direction.

ROUND: Not applicable.

DOUBLE-CHECK: Note that $F_x = \int_0^{\pi} iR\cos\theta B d\theta = 0$. This confirms the analysis based on symmetry.

27.73. **THINK:** The magnetic force on the proton due to the presence of the magnetic field will affect only the component of the proton's velocity that is perpendicular to the magnetic field.

SKETCH:



Note: Take region of magnetic field as $y > 0$.

RESEARCH:

(a) When the proton enters the magnetic field, the component of its velocity that is parallel to the field will be unaffected, so the proton advances along the z-axis at a constant speed. The component of the particle's velocity that is perpendicular to the field will be forced into a circular path in the xy-plane. Thus, the trajectory of the proton will be a helix, as shown in the figure. The magnetic force on the proton is given by:

$$\vec{F}_B = q\vec{v}\times\vec{B} = q\left(\vec{v}_{||}+\vec{v}_{\perp}\right)\times\vec{B} = q\vec{v}_{||}\times\vec{B} + q\vec{v}_{\perp}\times\vec{B} = q\vec{v}_{\perp}\times\vec{B}.$$

(b) The magnitude of the magnetic force is given by $\left|\vec{F}_B\right| = \left|q\vec{v}_{\perp}\times\vec{B}\right| = qv_{\perp}B$. By Newton's Second Law, $\left|\vec{F}_B\right| = ma \Rightarrow qv_{\perp}B = mv_{\perp}^2 / r$.

(c) The period of the circular motion in the xy-plane projection is given by $T = \dfrac{2\pi r}{v_{\perp}}$. The frequency is given by $f = 1/T$.

(d) The pitch of the motion is $p = v_{\parallel}T$.

SIMPLIFY:

(b) The radius of the trajectory projected onto the *xy*-plane is given by

$$r = \frac{mv_{\perp}}{qB} = \frac{mv\sin(90^\circ + \theta)}{qB} = \frac{mv\cos\theta}{qB}.$$

(c) $T = \frac{2\pi}{v_{\perp}}\left(\frac{mv_{\perp}}{qB}\right) = \frac{2\pi m}{qB}.$

(d) $p = (v\sin\theta)T.$

CALCULATE:

(b) $r = \frac{(1.67\cdot 10^{-27}\text{ kg})(1.00\cdot 10^{6}\text{ m/s})\cos(60.0^\circ)}{(1.60\cdot 10^{-19}\text{ C})(0.500\text{ T})} = 10.44\text{ mm}.$

(c) $T = \frac{2\pi(1.67\cdot 10^{-27}\text{ kg})}{(1.60\cdot 10^{-19}\text{ C})(0.500\text{ T})} = 1.312\cdot 10^{-7}\text{ s},\;\; f = \frac{1}{(1.312\cdot 10^{-7}\text{ s})} = 7.624\cdot 10^{6}\text{ Hz}.$

(d) $p = (1.00\cdot 10^{6}\text{ m/s})\sin(60.0^\circ)(1.312\cdot 10^{-7}\text{ s}) = 113.6\text{ mm}.$

ROUND: Rounding to three significant figures,

(b) 10.4 mm

(c) $T = 1.31\cdot 10^{-7}$ s, $f = 7.62\cdot 10^{6}$ Hz

(d) 114 mm

DOUBLE CHECK: All calculated values have correct units. The magnitudes are appropriate for subatomic particles.

Multi-Version Exercises

Exercises 27.74–27.76 For the ball to travel in a circle with radius *r*, we have $r = \frac{mv}{|q|B}$.

27.74. $r = \frac{mv}{|q|B}$

$$B = \frac{mv}{|q|r} = \frac{(5.063\cdot 10^{-3}\text{ kg})(3079\text{ m/s})}{(11.03\text{ C})(2.137\text{ m})} = 0.6614\text{ T}$$

27.75. $r = \frac{mv}{|q|B}$

$$m = \frac{r|q|B}{v} = \frac{(2.015\text{ m})(11.17\text{ C})(0.8000\text{ T})}{3131\text{ m/s}} = 0.005751\text{ kg} = 5.751\text{ g}$$

27.76. $r = \frac{mv}{|q|B}$

$$|q| = \frac{mv}{rB} = \frac{(3.435\cdot 10^{-3}\text{ kg})(3183\text{ m/s})}{(1.893\text{ m})(0.5107\text{ T})} = 11.31\text{ C}$$

Exercises 27.77–27.79 The electric force is given by $F_E = qE.$ The magnetic force is given by $F_B = vBq.$ Setting these forces equal to each other gives us

$$qE = vBq$$
$$v = \frac{E}{B}.$$

27.77. $v = \frac{E}{B} = \frac{1.749 \cdot 10^4 \text{ V/m}}{46.23 \cdot 10^{-3} \text{ T}} = 3.783 \cdot 10^5 \text{ m/s}.$

27.78. $v = \frac{E}{B}$

$$B = \frac{E}{v} = \frac{2.207 \cdot 10^4 \text{ V/m}}{4.713 \cdot 10^5 \text{ m/s}} = 0.04683 \text{ T} = 46.83 \text{ mT}$$

27.79. $v = \frac{E}{B}$

$$E = vB = \left(5.616 \cdot 10^5 \text{ m/s}\right)\left(47.45 \cdot 10^{-3} \text{ T}\right) = 2.665 \cdot 10^4 \text{ V/m}$$

Chapter 28: Magnetic Fields of Moving Charges

Concept Checks

28.1. c **28.2.** a **28.3.** a **28.4.** b **28.5.** e **28.6.** d **28.7.** d **28.8.** d

Multiple-Choice Questions

28.1. b **28.2.** c **28.3.** c **28.4.** a **28.5.** d **28.6.** a **28.7.** c **28.8.** c **28.9.** a **28.10.** d **28.11.** a **28.12.** a **28.13.** d

28.14. b

Conceptual Questions

28.15. The wires are twisted in order to cancel out the magnetic fields generated by these wires.

28.16. Since the currents running through the wire generate magnetic fields, these fields may overpower the magnetic field of the Earth and make the compass give a false direction.

28.17. No, an ideal solenoid cannot exist, since we cannot have an infinitely long solenoid. To a certain extent, yes, it renders the derivation void. However, the derivation is an approximation and is an important theoretical example.

28.18. In Example 28.1, the right hand rule implies that the magnetic dipole of the loop points out of the page. Application of the right hand rule to the straight wire tells us that the magnetic field produced by wire points out of the page. Assume the angle between the dipole moment and the field remains fixed. Since the dipole strength is also constant, the only quantity left to vary is field strength. If the potential energy is to be reduced, the loop must move towards a region of smaller magnetic field strength. That is, the loop must move away from the straight current-carrying wire.

28.19. By Coulomb's Law, the electric force between the particles has magnitude $|F_\text{e}| = q^2/\left(4\pi\varepsilon_0 d^2\right)$. For the magnetic force, the version of the Biot-Savart Law given in the text can be adapted to describe the magnetic field produced by a moving particle via the replacement $Idl \Rightarrow \left(dq/dt\right)dl \Rightarrow dq\left(dl/dt\right) \Rightarrow qv_\text{s}$ with q charge and v_s, the velocity of the source particle. The magnetic field produced by one particle at the location of the other can be written as $B = \mu_0 qvd/\left(4\pi d^3\right)$ with v, common velocity and d, the separate of the particles. The magnitude of the magnetic force one particle is given by $|F_\text{e}| = |qvB| = \left(\mu_0/\left(4\pi\right)\right)\left|qv\cdot(qvd)/d^3\right| = \mu_0 q^2 v^2/\left(4\pi d^2\right)$. Since the vectors v, d and $v\cdot d$ are mutually perpendicular (the site of the angle between any two of them is unity) the ratio of forces is $|F_\text{m}/F_\text{e}| = \mu_0\varepsilon_0 v^2$ which also $|F_\text{m}/F_\text{e}| = v^2/c^2$, where c is the speed of light.

28.20. The field is given by Ampere's law $B\left(2\pi\right)\left(a+b\right)/2 = \mu_0 i_\text{enc}$. Current density is then given by:

$$J = i/\left(\pi\left(b^2 - a^2\right)\right)$$

The area of interest is:

$$\pi\left[\left(a+b\right)/2\right]^2 = A$$
$$B(2\pi)\left[\left(a+b\right)/2\right] = \mu_0 AJ$$

$$B = \frac{\mu_0}{\pi(a+b)}\pi\left[\left(\frac{a+b}{2}\right)^2 - a^2\right]\cdot\frac{i}{\pi\left(b^2-a^2\right)}$$
$$= \frac{\mu_0 i}{\pi\left(a+b\right)}\cdot\frac{\left(\frac{a+b}{2}\right)^2 - a^2}{\left(b^2-a^2\right)}.$$

28.21. The magnetic field at point *P* would be zero. The contribution from part A would be zero since *P* lies along the axis of A. The currents through B and C points in opposite directions and yield magnetic fields that cancel out at *P* .

28.22. Ampere's law states that, $\oint_C Bdl = \mu_0 i,$ but since B is constant the integral must be zero. If so, i is zero everywhere and consequently $J = 0$ everywhere.

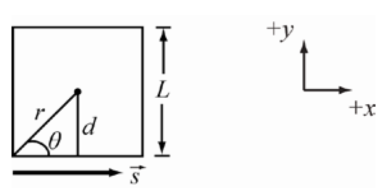
28.23. (a) Since molecular hydrogen is diamagnetic, the molecules must have no intrinsic dipole moment. Since the nuclear spins cannot cancel the electron spins, the electron spins must be opposite to cancel each other.
(b) With only a single electron, the hydrogen atoms must have an intrinsic magnetic moment. Atomic hydrogen gas, if it could be maintained, would have to exhibit paramagnetic or ferromagnetic behavior. But ferromagnetism would require inter atomic interactions strong enough to align the atoms in domains, which is not consistent with the gaseous state. Hence one would expect atomic hydrogen to be paramagnetic.

28.24. The saturation of magnetizations for paramagnetic and ferromagnetic materials is of comparable magnitude. In both types of materials the intrinsic magnetic moments of the atoms arise from a few unpaired electron spins. Magnetization effects in ferromagnetic materials are more pronounced at low applied fields because the atoms come pre-aligned in their domains, but once both types of atoms have been forced into essentially uniform alignment, the magnetization they produce is comparable. For either type of material maximum magnetizations of order 10^6 A m^2 / m^3 $= 10^6$ A/m magnetic dipole moment per unit volume are typical.

28.25. The wire carries a current which produces a magnetic field. This magnetic field will deflect the electron by the Lorentz force in the left direction.

28.26. Each side of the loop will create the same magnetic field at the center of the loop. The total field is 4 times the field of one side. The field at the center is given by the Biot-Savart Law:

$$\left|d\vec{B}\right| = \left|\frac{\mu_0 i}{4\pi}\frac{d\vec{s}\times\vec{r}}{r^3}\right| = \frac{\mu_0 i}{4\pi r^2}\sin\theta ds.$$



Since $\sin\theta = d/r$, the differential element of magnetic field is

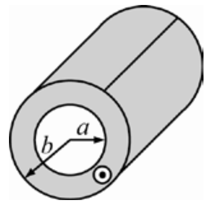
$$dB = \frac{\mu_0 i}{4\pi}\frac{d}{r^3}ds = \frac{\mu_0 id}{4\pi}\frac{ds}{(d^2+s^2)^{3/2}}.$$

Integration gives

$$B = \frac{\mu_0 id}{4\pi}\int_{-d}^{d}\frac{ds}{(d^2+s^2)^{3/2}} = \frac{2\mu_0 id}{4\pi}\int_0^d\frac{ds}{(d^2+s^2)^{3/2}} = \frac{\mu_0 id}{2\pi}\left[\frac{s}{d^2\sqrt{d^2+s^2}}\right]_0^d = \frac{\mu_0 i}{2\pi d}\frac{d}{\sqrt{2}d} = \frac{\mu_0 i}{2\sqrt{2}\pi d}.$$

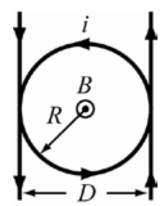
The total field is then $B_{\text{tot}} = 4B = \dfrac{\sqrt{2}\mu_0 i}{\pi d} = \dfrac{\sqrt{2}\mu_0 i}{\pi(L/2)} = \dfrac{2\sqrt{2}\mu_0 i}{\pi L}.$

28.27.



The current that flows through a ring of radius r which lies in the region $a < r < b$ is given by $i = \int J_0 dA = \int_a^r J_0 2\pi\rho d\rho = J_0 \pi\rho^2 \big|_a^r = J_0 \pi(r^2 - a^2)$. To find the magnetic field employ Ampere's Law $\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enclosed}}$. For a cylinder this becomes $B(2\pi r) = \mu_0 i_{\text{enc}}$ or $B = \mu_0 i_{\text{enc}} / (2\pi r)$. If $r < a$ then $B_{r<a} = 0$, thus if $a < r < b$ then $i_{\text{enc}} = J_0 \pi(r^2 - a^2)$ and $B_{a<r<b} = \dfrac{\mu_0 J_0 \pi(r^2 - a^2)}{2\pi r} = \dfrac{\mu_0 J_0 (r^2 - a^2)}{2r}$. If $r > b$ then $i_{\text{enc}} = J_0 \pi(b^2 - a^2)$ and $B_{r>b} = \dfrac{\mu_0 J_0 (b^2 - a^2)}{2r}$. Note that if $r = b$ then $B_{a<r<b} = \dfrac{\mu_0 J_0 (b^2 - a^2)}{2b} = B_{r>b}$.

28.28. The loop creates a magnetic field of $B_1 = \mu_0 i / (2R)$ at its center and is directed upwards. Out of the page. Both wires contribute a magnetic field of $B_{\text{w}} = \mu_0 i / (2\pi R)$ pointing out of the page. The total fields is then $B_{\text{tot}} = B_1 + 2B_{\text{w}} = \dfrac{\mu_0 i}{2R}(1 + \dfrac{2}{\pi})$, and points out of the page.



28.29. **THINK:** Ampere's Law can be used to determine the magnitude of the magnetic field in the two regions.
SKETCH: A sketch is included at the end of the SIMPLIFY step, once the two equations have been found.
RESEARCH: The current with the conductor is given by $i = \int J(r) dA$. The magnetic field is found using Ampere's Law $\oint \vec{B} \bullet d\vec{s} = \mu_0 i_{\text{enclosed}}$ or $B = \dfrac{\mu_0 i_{\text{enc}}}{2\pi r}$.

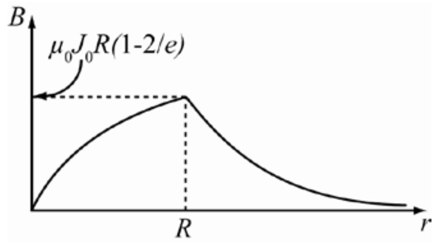
SIMPLIFY: $i = \int J(r) dA = 2\pi \int_0^r J(r') r' dr' = 2\pi J_0 \int_0^r r' e^{-r'/R} dr' = 2\pi J_0 \left[-R(R + r') e^{-r'/R} \right] \Big|_0^r$

$$= \left(\left(-R(R + r) e^{-r/R} \right) - \left(-R(R + 0) e^{-0/R} \right) \right) 2\pi J_0$$

$$= \left(R^2 (1 - e^{-r/R}) - Rre^{-r/R} \right) 2\pi J_0$$

If $r < R$ then $B_{r<R} = \dfrac{\mu_0}{2\pi r} \left[R^2 - R(R + r) e^{-r/R} \right] 2\pi J_0 = \dfrac{\mu_0 J_0}{r} [R^2 - R(R + r) e^{-r/R}]$.

If $r > R$ then $B_{r>R} = \dfrac{\mu_0}{2\pi r} \left[R^2 - R(R + R) e^{-R/R} \right] 2\pi J_0 = \dfrac{\mu_0 J_0}{r} \left[R^2 - 2R^2 e^{-1} \right] = \dfrac{\mu_0 J_0 R^2}{r} [1^2 - 2e^{-1}]$.



CALCULATE: There are no values to substitute.
ROUND: There are no values to round.
DOUBLE-CHECK: Note that the two computed formulas agree when $r = R$.

Exercises

28.30. The force of wire 1 on wire 2 is $F_{1\to 2} = i_2 LB = i_2 L\left[\mu_0 i_1 / (2\pi d)\right] = \mu_0 i_1 i_2 L / (2\pi d)$. Since $2i_1 = i_2$, $F_{1\to 2} = \mu_0 i_1^2 L / (\pi d)$. Solving for the current i_1 gives $i_1 = \sqrt{\dfrac{\pi d F_{1\to 2}}{\mu_0 L}} = \sqrt{\dfrac{\pi(0.0030 \text{ m})(7.0\cdot 10^{-6} \text{ N})}{(4\pi \cdot 10^{-7} \text{ T m/A})(1.0 \text{ m})}} = 0.23$ A.

The current on the other wire is $i_2 = 0.46$ A.

28.31. The magnetic field created by the wire is given by the Biot-Savart Law $B = \mu_0 i / (2\pi r)$. The force on the electron is given by the Lorentz force $F = qvB = qv\mu_0 i / (2\pi r)$. The acceleration of the electron is

$$a = \frac{F}{m} = \frac{qv\mu_0 i}{2\pi m r} = \frac{(1.602\cdot 10^{-19} \text{ C})(4.0\cdot 10^5 \text{ m/s})(4\pi\cdot 10^{-7} \text{ T m/A})(15 \text{ A})}{2\pi(9.109\cdot 10^{-31} \text{ kg})(0.050 \text{ m})} = 4.2\cdot 10^{12} \text{ m/s}^2$$

The direction of the acceleration is radially away from the wire.

28.32. The magnitude of the magnetic field created by a moving charge along it is line of motion is zero. By the Biot-Savart Law,

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{i d\vec{s} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{q d\vec{v} \times \hat{r}}{r^2} = 0,$$

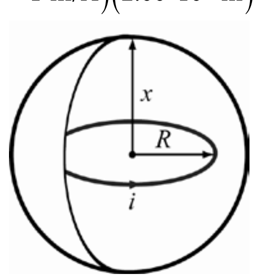
since the angle between the angle between the velocity and the position vector $\hat{r}$ is zero. The situation is the same for an electron and a proton.

28.33. The field along the axis of a current loop of radius R as measured at a distance x from the center of the loop is

$$B = \frac{\mu_0 i}{2}\frac{R^2}{\left(x^2 + R^2\right)^{3/2}}.$$

The current of the loop must be

$$i = \frac{2\left(x^2 + R^2\right)^{3/2} B}{\mu_0 R^2} = \frac{2\left[\left(2.00\cdot 10^6 \text{ m}\right)^2 + \left(6.38\cdot 10^6 \text{ m}\right)^2\right]^{3/2}}{\left(4\pi\cdot 10^{-7} \text{ T m/A}\right)\left(2.00\cdot 10^6 \text{ m}\right)^2}\left(6.00\cdot 10^{-5} \text{ T}\right) = 7.14\cdot 10^9 \text{ A}.$$



28.34. What does it mean to have an "average value of the magnetic field measured in the sides"? The answer is that the average value is: $\bar{B} = \oint \vec{B} \cdot d\vec{s} / \oint ds.$ And $\oint ds$ is just the total length of the closed path around the loop, in this case $\oint ds = 4l.$ For the integral above we can simple use Ampere's Law and find (see equation 28.10):

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}$$

We found above that $\bar{B} = \oint \vec{B} \cdot d\vec{s} / \oint ds = \oint \vec{B} \cdot d\vec{s} / 4l.$ Inserting Ampere's Law and solving for the enclosed current then yields:

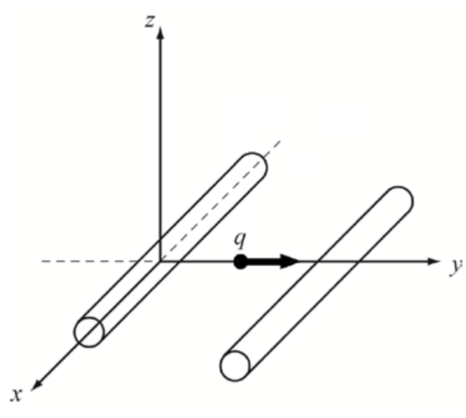
$$\bar{B} = \mu_0 i_{\text{enc}} / 4l \Rightarrow i_{\text{enc}} = 4l\bar{B} / \mu_0$$

Numerically we find $i_{\text{enc}} = 4(0.0300 \text{ m})(3.00 \cdot 10^{-4} \text{ T}) / (4\pi \cdot 10^{-7}) = 28.64789 \text{ A},$ which we round to $i_{\text{enc}} = 28.6 \text{ A}.$

We can also see if our solution makes sense. We have calculated the magnetic field from a long straight wire as a function of the distance to the wire in equation 28.4 and found $B = \mu_0 i / 2\pi r_\perp$. With our value of the current computed above, we can calculate the value of the magnetic field at the corners of the loop (furthest from the wire) and middle of the sides (closest to the wire) and see that these two values of the magnetic field are below and above the average value of *B* that was given in the problem. For the middle of the sides we find ($r_\perp = l/2$): $B = 3.82 \cdot 10^{-4} \text{ T}$, and for the corners we find ($r_\perp = l/\sqrt{2}$): $B = 2.70 \cdot 10^{-4} \text{ T}$. This gives us confidence that we have the right solution.

28.35. **THINK:** A force due to the magnetic field generated by a current carrying wire acts on a moving particle. In order for the net force on the particle to be zero, a second force of equal magnitude and opposite direction must act on the particle. Such a force can be generated by another current carrying wire placed near the first wire. Assume the second wire is to be parallel to the first and has the same magnitude of current. The wire along the *x*-axis has a current of 2 A oriented along the *x*-axis. The particle has a charge of $q = -3 \text{ μC}$ and travels parallel to the *y*-axis through point $(x, y, z) = (0, 2, 0).$

SKETCH:



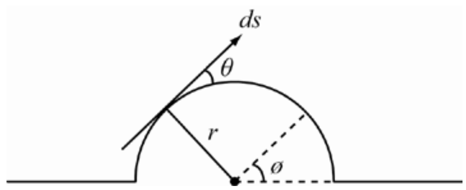
RESEARCH: The magnetic field produced by the current is given by the $B = \mu_0 i / (2\pi r).$ The force on the particle is given by the Lorentz force, $F = qv_0 B.$

SIMPLIFY: If the wires carry the same current then the new wire must be equidistant from the point that the particle passes through the *xy*-plane. Only then will the magnetic force on the particle due to each wire be equal. By the right hand rule, the currents will be in the same direction. This means that $r_1 = r_2.$

CALCULATE: The requirement $r_1 = r_2$ means that the second wire should be placed parallel to the first wire (parallel to the *x*-axis) so that it passes through the point $(x, y, z) = (0, 4, 0).$

ROUND: Not necessary.
DOUBLE-CHECK: It is reasonable that two wires carrying the same current need to be equidistant from a point in order for the magnitude of the force to be the same.

28.36. **THINK:** The current through the wire creates a magnetic field by the Biot-Savart Law. The straight part of the wire only creates a magnetic field at points perpendicular to it. Therefore this part of the wire can be ignored. The magnetic field at the center of the semicircle is created by the charge moving through the semicircle.
SKETCH:



RESEARCH: The Biot-Savart Law can be employed in the form $dB = \frac{\mu_0}{4\pi}\frac{i\sin\theta}{r^2}ds$. Going around the semicircle, the angle ϕ can be related to the current element by $ds = rd\phi$.
SIMPLIFY: Performing the integration gives

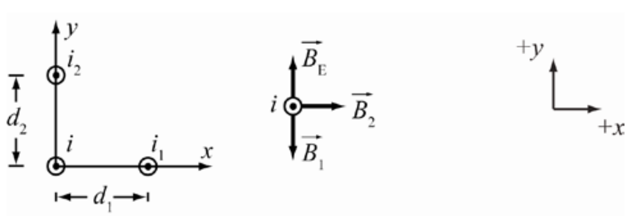
$$B = \frac{\mu_0 i}{4\pi}\int_0^{\pi}\frac{\sin\theta}{r^2}Rd\phi = \left[\frac{\mu_0 i\sin\theta}{4\pi r}\phi\right]_0^{\pi} = \frac{\mu_0 i\sin\theta\pi}{4\pi r} = \frac{\mu_0 i\sin\theta}{4r}.$$

The angle θ between the current and the radial vector $\hat{r}$ is $90°$ for the loop, thus $B = \mu_0 i/(4r)$.

CALCULATE: $B = \frac{\left(4\pi\cdot 10^{-7}\text{ T m/A}\right)\left(12.0\text{ A}\right)}{4\left(0.100\text{ m}\right)} = 3.76991\cdot 10^{-5}\text{ T}$

ROUND: The values are given to three significant figures, thus the magnetic field produced by the wire is $B = 3.77\cdot 10^{-5}\text{ T}$ and points into the page.
DOUBLE-CHECK: The magnetic field is very small, as would be expected from a real-world point of view.

28.37. **THINK:** Each of the wires creates a magnetic field at the origin. The sum of these fields and the Earth's magnetic field will produce a force on the compass, causing it to align with the total field. The wires carry a current of $i_1 = i_2 = 25.0$ A. The Earth's magnetic field is $\vec{B}_E = 2.6\cdot 10^{-5}\,\hat{y}$ T.
SKETCH:



RESEARCH: The magnetic field produced by a wire is $B = \mu_0 i/(2\pi d)$.

SIMPLIFY: The magnetic field of wire 1 is $\vec{B}_1 = \mu_0 i_1(-\hat{y})/(2\pi d_1)$. Wire 2 produces a magnetic field of $\vec{B}_2 = \mu_0 i_2\hat{x}/(2\pi d_2)$. The sum of the magnetic fields is $\vec{B}_{\text{net}} = \vec{B}_1 + \vec{B}_2 + \vec{B}_E = -\frac{\mu_0 i_1}{2\pi d_1}\hat{y} + \frac{\mu_0 i_2}{2\pi d_2}\hat{x} + \vec{B}_E$.

CALCULATE:
$$\vec{B}_{\text{net}} = -\frac{\left(4\pi\cdot 10^{-7}\text{ T m/A}\right)\left(25.0\text{ A}\right)}{2\pi\left(0.15\text{ m}\right)}\hat{y} + \frac{\left(4\pi\cdot 10^{-7}\text{ T m/A}\right)\left(25.0\text{ A}\right)}{2\pi\left(0.090\text{ m}\right)}\hat{x} + 2.6\cdot 10^{-5}\text{ T}\hat{y}$$
$$= 5.5555\cdot 10^{-5}\text{ T}\hat{x} - 7.3333\cdot 10^{-6}\text{ T}\hat{y}$$

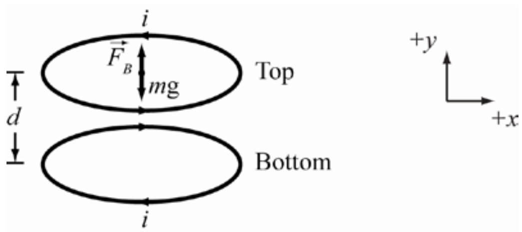
The direction of the field is $\theta = \tan^{-1}\left(\dfrac{-7.3333\cdot 10^{-6}\ \text{T}}{5.5555\cdot 10^{-5}\ \text{T}}\right) = -7.5196°$.

ROUND: The angle is accurate to two significant figures. The compass points $7.5°$ below the x-axis.

DOUBLE-CHECK: This is a reasonable answer. The compass points towards the east if $\hat{y}$ is north.

28.38. **THINK:** The coil will levitate if the force from the magnetic field cancels the force of gravity. The coils have radii of $R = 20.0$ cm. The current of the bottom coil is i and travels in the clockwise direction. By the right hand rule the top coil has a current of the same magnitude, moving in a counter clockwise direction. The mass of the coils is $m = 0.0500$ kg. The distance between the coils is $d = 2.00$ mm.

SKETCH:



RESEARCH: The force of gravity is $F_g = mg$. The magnetic force on the top coil due to the bottom coil is $F_B = \mu_0 i_1 i_2 L/(2\pi d) = \mu_0 i_1 i_2 2\pi R/(2\pi d)$.

SIMPLIFY: Equating the two forces give $mg = \dfrac{\mu_0 i_1 i_2 2\pi R}{2\pi d} = \dfrac{\mu_0 i^2 R}{d}$. The amount of current is $i^2 = \dfrac{mgd}{\mu_0 R}$ or $i = \sqrt{\dfrac{mgd}{\mu_0 R}}$.

CALCULATE: $i = \sqrt{\dfrac{(0.0500\text{ kg})(9.81\text{ m/s}^2)(0.00200\text{ m})}{(4\pi\cdot 10^{-7}\text{ T m/A})(0.200\text{ m})}} = 62.476\text{ A}$

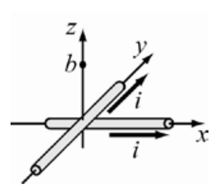
ROUND: Reporting to 3 significant figures, the current in the coils is 62.5 A and travel in opposite directions.

DOUBLE-CHECK: Dimensional analysis provides a check:

$$i = \sqrt{\frac{[\text{kg}][\text{m/s}^2][\text{m}]}{[T][\text{m/A}][\text{m}]}} = \sqrt{\frac{[\text{kg}][\text{m}][\text{m}][\text{A}]}{[\text{N/(A m)}][\text{m}][\text{m}][\text{s}^2]}} = \sqrt{\frac{[\text{kg}][\text{m}][\text{m}][\text{A}][\text{A}][\text{s}^2][\text{m}]}{[\text{kg}][\text{m}][\text{m}][\text{m}][\text{s}^2]}} = [\text{A}].$$

28.39. **THINK:** The current carrying wires along the x- and y-axes will each generate a magnetic field. The superposition of these fields generates a net field. The magnitude and direction of this net field at a point on the z-axis is to be determined.

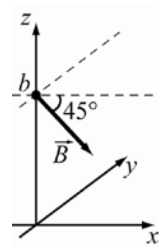
SKETCH:



RESEARCH: Both currents produce a magnetic field with magnitude $B = \mu_0 i/(2\pi r)$. The magnetic field produced by the wire along the x-axis gives $\overline{B}_1 = \mu_0 i(-\hat{y})/(2\pi b)$. The wire along the y-axis creates a magnetic field of $\overline{B}_2 = \mu_0 i\hat{x}/(2\pi b)$.

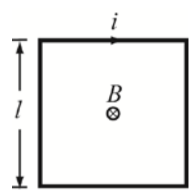
SIMPLIFY: The total magnetic field is then $\overline{B}_{\text{net}} = \overline{B}_1 + \overline{B}_2 = \frac{\mu_0 i}{2\pi b}\hat{x} - \frac{\mu_0 i}{2\pi b}\hat{y}$. The magnitude of the field is $B = \frac{\mu_0 i}{2\pi b}\sqrt{1^2 + (-1)^2} = \frac{\sqrt{2}\mu_0 i}{2\pi b} = \frac{\mu_0 i}{\sqrt{2}\pi b}$. The direction of the field is $\theta = \tan^{-1}\left(\frac{-\mu_0 i}{\sqrt{2}\pi b} \Big/ \frac{\mu_0 i}{\sqrt{2}\pi b}\right)$ in the *x*-*y* plane at a height of *b*.

CALCULATE: $\tan^{-1}(-1) = -45°$ in the *x*-*y* plane at point *b*.



ROUND: Not applicable.
DOUBLE CHECK: Both the right hand rule and the symmetry of the problem indicates that the net field should be in the fourth quadrant.

28.40. **THINK:** The loop creates a magnetic field at its center by the Biot-Savart Law. The loop has side length $l = 0.100$ m and carries a current of $i = 0.300$ A.
SKETCH:



RESEARCH: The Biot-Savart Law states $dB = \frac{\mu_0}{4\pi} \cdot \frac{i \sin\theta ds}{r^2}$. The angle θ is found by using the equations: $\sin\theta = d/r$, $r = \sqrt{s^2 + d^2}$, and $d = l/2$.

SIMPLIFY: The field due to one side of the loop is $dB = \frac{\mu_0 i}{4\pi} \cdot \frac{d}{r^2} ds = \frac{\mu_0 i d}{4\pi} \frac{ds}{\left(s^2 + d^2\right)^{3/2}}$. Since there are four sides, the total loop is four times this value. The total magnetic field is then

$$B = \int dB = 4\int_{-d}^{d} \frac{\mu_0 i d}{4\pi} \cdot \frac{ds}{\left(s^2 + d^2\right)^{3/2}} = 4\int_0^d \frac{\mu_0 i d}{4\pi} \frac{2ds}{\left(s^2 + d^2\right)^{3/2}}$$

$$= \frac{2\mu_0 i d}{\pi}\left[\frac{s}{d^2\sqrt{s^2 + d^2}}\right]_0^d = \frac{2\mu_0 i}{\pi d}\left(\frac{d}{\sqrt{2d^2}} - \frac{0}{\sqrt{d^2}}\right) = \frac{2\mu_0 i}{\sqrt{2}\pi d} = \frac{\sqrt{8}\mu_0 i}{\pi l}$$

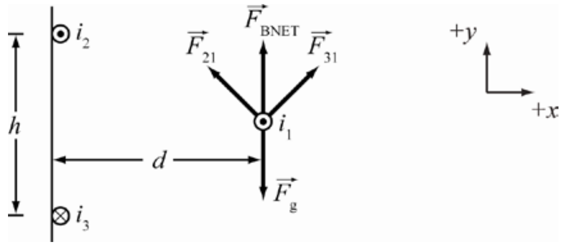
CALCULATE: $B = \frac{\sqrt{8}\left(4\pi \cdot 10^{-7} \text{ T m/A}\right)(0.300 \text{ A})}{\pi(0.100 \text{ m})} = 3.394 \cdot 10^{-6} \text{ T}$

ROUND: To three significant figures, the magnetic field at the center of the loop is $B = 3.39 \cdot 10^{-6}$ T.
DOUBLE-CHECK: The current is small, so the magnetic field it generates is expected to be small. This is a reasonable value.

28.41. **THINK:** In order for wire 1 to levitate, the forces on it must cancel. Both wire 2 and 3 will create magnetic fields that will interact with wire 1. Both wires create forces with horizontal and vertical components. The horizontal components will add destructively. The vertical components however will add constructively.

Therefore, only the vertical components need be calculated. Wires 2 and 3 each carry a current of $i = 600.$ A. All three wires have a linear mass density of $\lambda = 100.$ g/m. The wires are arranged as shown in the figure.

SKETCH:



RESEARCH: The force of gravity on the wire is $F_g = mg$. The force between two wires carrying current is $F_{21} = \mu_0 i_1 i_2 L / (2\pi d)$.

SIMPLIFY: The vertical component of the magnetic force for one wire is $F_{31} = \dfrac{\mu_0 i_3 i_1 L}{2\pi(h/2)} = \dfrac{\mu_0 i_3 i_1 L}{\pi h}$. The total force due to the wires is then $F_B = 2F_{31} = \dfrac{2\mu_0 i_3 i_1 L}{\pi h}$. Equating this to the force of gravity gives: $mg = \lambda L g = \dfrac{2\mu_0 i_3 i_1 L}{\pi h}$. Solving for the current i_1 gives: $i_1 = \dfrac{\pi h \lambda g}{2\mu_0 i_3}$.

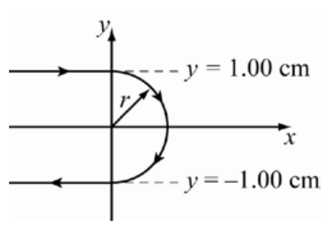
CALCULATE: $i_1 = \dfrac{\pi(0.100 \text{ m})(100.\cdot 10^{-3} \text{ kg/m})(9.81 \text{ m/s}^2)}{2\left(4\pi \cdot 10^{-7} \text{ T m/A}\right)(600. \text{ A})} = 204.375 \text{ A}$

ROUND: The current of wire 1 required to levitate is $i_1 = 204$ A.

DOUBLE-CHECK: The current in wire 1 is on the same order of magnitude as the other currents. This is a reasonable answer.

28.42. **THINK:** The net field is a superposition of the fields created by the top wire, the bottom wire and the loop. The wires are 2.00 cm apart and carry a current of $i = 3.00$ A. The radius of the loop is $r = 1.00$ cm.

SKETCH:



RESEARCH: The magnetic field produced by an infinite wire is $B = \mu_0 i / (2\pi r)$. A semi-infinite wire is half this value, $B = \mu_0 i / (4\pi r)$. A full loop produces a magnetic field of $B = \mu_0 i / (2r)$. The half loop produces half of this, $B = \mu_0 i / (4r)$.

SIMPLIFY: By the right hand rule, the magnetic field points into the page. The magnetic field is the sum of all the fields.

$$B_{\text{net}} = B_{\text{top}} + B_{\text{bottom}} + B_{\text{loop}} = \frac{\mu_0 i}{4\pi r} + \frac{\mu_0 i}{4\pi r} + \frac{\mu_0 i}{4r} = \frac{\mu_0 i}{4r}\left(\frac{2}{\pi} + 1\right)$$

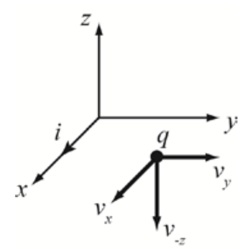
CALCULATE: $B_{\text{net}} = \dfrac{\left(4\pi \cdot 10^{-7} \text{ T m/A}\right)(3.00 \text{ A})}{4(0.0100 \text{ m})}\left(\dfrac{2}{\pi} + 1\right) = 1.54 \cdot 10^{-4}$ T. It is directed in the negative z direction.

ROUND: To 3 significant figures, the magnetic field at the origin is $-1.54 \cdot 10^{-4}$ T$\hat{z}$.

DOUBLE-CHECK: The field due to a single infinite wire similar to the wires in the problem would be $B=\left(4\pi\cdot 10^{-7}\text{ T m/A}\right)\left(3.00\text{ A}\right)/\left(2\pi\left(0.0100\text{ m}\right)\right)=6.00\cdot 10^{-5}\text{ T},$ which is similar to the result. Therefore, the result is reasonable.

28.43. **THINK:** The wire creates a magnetic field that produces a Lorentz force on the moving charged particle. The question asked for the force if the particle travels in various directions. The velocity is 3000 m/s in various directions.

SKETCH:



RESEARCH: The magnetic field produced by an infinite wire is $B=\mu_0 i/\left(2\pi d\right)$. By the right hand rule the field points in the positive z-direction. The force produced by the magnetic field is $\vec{F}=q\vec{v}\times\vec{B}$.

SIMPLIFY: The force is given by $\vec{F}=q\vec{v}\times\vec{B}=\frac{q\mu_0 i}{2\pi d}\vec{v}\times\hat{z}=\frac{q\mu_0 i}{2\pi d}\left(\left|\vec{v}\right|\cdot\hat{n}\times\hat{z}\right)$ where $\hat{n}$ is the direction of the particle.

CALCULATE: $\vec{F}=\frac{\left(9.00\text{ C}\right)\left(4\pi\cdot 10^{-7}\text{ T m/A}\right)\left(7.00\text{ A}\right)}{2\pi\left(2.00\text{ m}\right)}\left(3000.\text{ m/s}\cdot\hat{n}\times\hat{z}\right)=1.89\cdot 10^{-2}\text{ N}\left(\hat{n}\times\hat{z}\right)$

Note that $\hat{x}\times\hat{z}=-\hat{y},\ \hat{y}\times\hat{z}=\hat{x},$ and $-\hat{z}\times\hat{z}=0.$

ROUND: The force should be reported to 3 significant figures.

(a) The force is $\vec{F}=-1.89\cdot 10^{-2}\text{ N}\,\hat{y}$ if the particle travels in the positive x-direction.

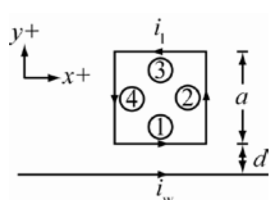
(b) The force is $\vec{F}=1.89\cdot 10^{-2}\text{ N}\,\hat{x}$ if the particle travels in the positive y-direction.

(c) The force is $F=0$ if the particle travels in the negative z-direction.

DOUBLE-CHECK: The right hand rule confirms the directions of the forces for each direction of motion of the particle.

28.44. **THINK:** The wire produces a magnetic field that creates a force on the loop. The wire has current of $i_{\text{w}}=10.0\text{ A}$ and is $d=0.500\text{ m}$ away from the bottom wire of the loop. The loop carries a current of $i_1=2.00\text{ A}$ and has sides of length $a=1.00$ m.

SKETCH:



RESEARCH: The force on two wires carrying a current is $F=\mu_0 i_1 i_2 L/\left(2\pi d\right)$. The torque is given by $\vec{\tau}=\vec{r}\times\vec{F}$.

SIMPLIFY: The forces on part ② and ④ cancel each other. The force on ① is $F_1=\mu_0 i_{\text{w}} i_1 a/\left(2\pi d\right)$ and points towards the long wire. The force on ③ is $F_3=\mu_0 i_{\text{w}} i_1 a/\left[2\pi(d+a)\right]$ and points away from the long wire. The total force is then $F_{\text{net}} = \vec{F}_1 + \vec{F}_3 = \frac{\mu_0 i_{\text{w}} i_1 a}{2\pi}\left(\frac{1}{d} - \frac{1}{d+a}\right)$ and points towards the long wire. Because the force and the length between the axis of rotation are parallel there is no torque on the loop.

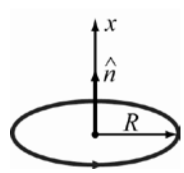
CALCULATE:

$$F_{\text{net}} = \frac{\left(4\pi\cdot 10^{-7}\ \text{Tm/A}\right)(10.0\ \text{A})(2.00\ \text{A})\left(1.00\ \text{m}\right)}{2\pi}\left(\frac{1}{0.500\ \text{m}} - \frac{1}{1.50\ \text{m}}\right) = -5.33333\cdot 10^{-6}\,\hat{y}\ \text{N}$$

ROUND: The force is reported to three significant figures. (a) The net force between the loop and the wire is $F = -5.33\cdot 10^{-6}\,\hat{y}$ N. (b) There is no net torque on the loop.

DOUBLE-CHECK: The force between the long wire and the lower arm of the loop is attractive, because the currents are in the same direction. The currents of the long wire and the upper arm of the loop are in opposite directions, therefore the force is repulsive. Since the lower arm is closer to the long wire, the attractive force dominates, and the net force is in the negative y direction, as calculated.

28.45. The magnetic field at the center of the box is the sum of the fields produced by the coils. A coil produces a magnetic field of $B = \frac{\mu_0 N i R^2}{2\left(x^2 + R^2\right)^{3/2}}\hat{n}.$

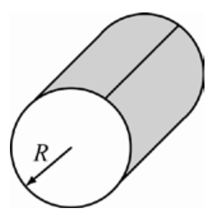


The magnetic field produced by the coil on the $x-z$ plane is

$$B_{\text{xz}} = \frac{\left(4\pi\cdot 10^{-7}\ \text{T m/A}\right)(30.0)(5.00\ \text{A})(0.500\ \text{m})^2}{2\left[(0.500\ \text{m})^2 + (0.500\ \text{m})^2\right]^{3/2}}(+\hat{y}) = 6.66\cdot 10^{-5}\ \text{T}\hat{y}$$

The magnetic field produced by the other coil has the same magnitude but points in the negative *x*-direction. Therefore $B_{\text{tot}} = 6.66\cdot 10^{-5}\ \text{T}[-\hat{x} + \hat{y}]$. The magnitude of the field is $\sqrt{2}\cdot 6.66\cdot 10^{-5}$ T, or $9.42\cdot 10^{-5}$ T. The direction of the field is at an angle of 45° from the negative *x*-direction towards the positive *y*-axis.

28.46.



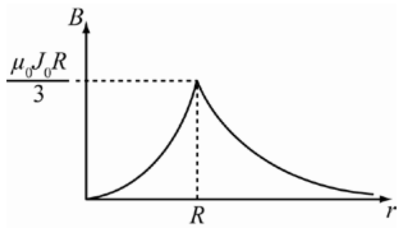
The current within a loop of radius $\rho \le R$ is given by

$$i = \int J(r)dA = 2\pi\int_0^r J(r')r'dr' = 2\pi J_0\int_0^r \frac{r'}{R}r'dr' = \frac{2\pi J_0}{R}\int_0^r r'^2 dr' = \frac{2\pi J_0}{R}\frac{r'^3}{3}\Bigg|_0^r = \frac{2\pi J_0 r^3}{3R}.$$

The magnetic field is given by Ampere's Law

$$\oint \vec{B}\cdot d\vec{s} = B\left(2\pi r\right) = \mu_0 i_{\text{enc}} \ \Rightarrow\ B = \frac{\mu_0 i_{\text{enc}}}{2\pi r}.$$

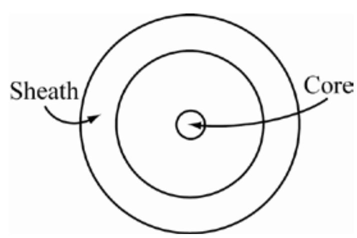
The magnetic field in the region $r<R$ is $B=\frac{\mu_0}{2\pi r}\left(\frac{2\pi J_0 r^3}{3R}\right)=\frac{\mu_0 J_0 r^2}{3R}$. The magnetic field in the region $r>R$ is $B=\frac{\mu_0}{2\pi r}\left(\frac{2\pi J_0 R^3}{3R}\right)=\frac{\mu_0 J_0 R^2}{3r}$.



28.47. Using Ampere's Law, the magnetic field at various points can be determined. $\oint \vec{B}\bullet d\vec{s}=\mu_0 i_{\text{enclosed}}$. For the cylinder, assuming the current is distributed evenly, $B2\pi r=\mu_0 i_{\text{enc}}$ or $B=\mu_0 i_{\text{enc}}/(2\pi r)$. The field at $r=r_{\text{a}}=0$ is zero since is does not enclose any current $B_{\text{a}}=0$. The field at $r=r_{\text{b}}<R$ is $B_{\text{b}}=\frac{\mu_0 i_{\text{enc}}}{2\pi r_{\text{b}}}=\frac{\mu_0}{2\pi r_{\text{b}}}\left(i_{\text{tot}}\frac{\pi r_{\text{b}}^2}{\pi R^2}\right)=\frac{\mu_0 i r_{\text{b}}}{2\pi R^2}=\frac{(4\pi\cdot 10^{-7}\text{ T m/A})(1.35\text{ A})(0.0400\text{ m})}{2\pi(0.100\text{ m})^2}=1.08\cdot 10^{-6}\text{ T}$. Note that i is equal to the fraction of total area of the conductor's cross section and the total current. The field at $r_{\text{c}}=R$ is $B_{\text{c}}=\frac{\mu_0 i_{\text{enc}}}{2\pi r_{\text{c}}}=\frac{\mu_0 i_{\text{tot}}}{2\pi R}=\frac{(4\pi\cdot 10^{-7}\text{ T m/A})(1.35\text{ A})}{2\pi(0.100\text{ m})}=2.70\cdot 10^{-6}\text{ T}$. The field at $r_{\text{d}}>R$ is $B_{\text{d}}=\frac{\mu_0 i_{\text{enc}}}{2\pi r_{\text{d}}}=\frac{(4\pi\cdot 10^{-7}\text{ T m/A})(1.35\text{ A})}{2\pi(0.160\text{ m})}=1.69\cdot 10^{-6}\text{ T}$. By inspection it can be seen that the magnetic field at r_b, r_c and r_d the magnetic field will point to the right.

28.48. **THINK:** The magnetic field is the sum of the field produced by the wire core B_{C} and the sheath B_{S}. The wire has a radius of $a=1.00$ mm. The sheath has an inner radius of $b=1.50$ mm and outer radius of $c=2.00$ mm. The current of the outer sheath opposes the current in the core.

SKETCH:



RESEARCH: The current density of the core is $J_{\text{C}}=i/(\pi a^2)$ and the current density of the sheath is $J_{\text{S}}=-i/[\pi(c^2-b^2)]$. The enclosed current is calculated by $i_{\text{enclosed}}=\int JdA$. The magnetic field is derived using Ampere's Law: $\oint B\cdot ds=\mu_0 i_{\text{enclosed}}$.

SIMPLIFY: When the radius is within the core , $r\le a$, the magnetic field is

$$\oint B\cdot ds=B_{r\le a}2\pi r=\mu_0 i_{\text{enc}}=\mu_0\int JdA=\mu_0\frac{i}{\pi a^2}\int_0^r\int_0^{2\pi} rd\theta dr$$
$$=\frac{\mu_0 i}{\pi a^2}\frac{2\pi r^2}{2}=\mu_0 i\frac{r^2}{a^2}$$

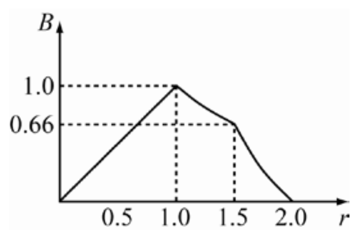
or $B_{r\le a}=\dfrac{\mu_0 i}{2\pi r}\dfrac{r^2}{a^2}=\dfrac{\mu_0 ir}{2\pi a^2}$. If the radius is between the core and the sheath, $a<r<b$, $\oint \vec{B}\bullet d\vec{s}=B_{a<r\le b}2\pi r=\mu_0 i$ or $B_{a<r\le b}=\mu_0 i/(2\pi r)$. Within the sheath, $b<r<c$, the magnetic field is $\oint \vec{B}\bullet d\vec{s}=B_{b<r<c}2\pi r=\mu_0 i_{\text{enc}}=\mu_C\left(i+\int_b^r\int_0^{2\pi}\dfrac{-i}{\pi(c^2-b^2)}rd\theta dr\right)=\mu_0\left(i-\dfrac{i}{(c^2-b^2)}\dfrac{2\pi r^2}{\pi 2}\Big|_b^r\right)=\mu_0 i\left[1-\dfrac{r^2-b^2}{c^2-b^2}\right]$

$$B_{b<r<c}=\frac{\mu_0 i}{2\pi r}\left[1-\frac{r^2-b^2}{c^2-b^2}\right]$$

If the radius is outside of the cable, $r\ge c$, then the magnetic field is $\oint B\cdot ds = B_{r\ge c}2\pi r=\mu_0 i_{\text{enc}}=\mu_0(i-i)=0$ or $B_{r\ge c}=0$. In summary the magnetic fields of various regions are

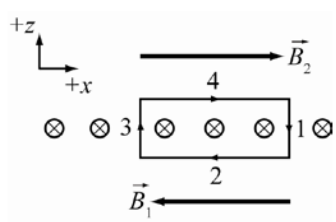
$$B_{r\le a}=\frac{\mu_0 ir}{2\pi a^2},\ B_{a<r\le b}=\frac{\mu_0 i}{2\pi r},\ B_{b<r<c}=\frac{\mu_0 i}{2\pi r}\left[1-\frac{r^2-b^2}{c^2-b^2}\right],\ B_{r\ge c}=0.$$

CALCULATE: In order to graph the behavior of the magnetic field as a function of the radius, set the magnetic field in units of $\dfrac{u_0 i}{2\pi a}$.



ROUND: There is no need to round.
DOUBLE-CHECK: Note that the magnetic field outside of the coaxial cable is zero. These cables are used when equipment that is sensitive to magnetic fields needs current.

28.49. **THINK:** To find the magnetic field above the center of the surface of a current carrying sheet, use Ampere's Law. The path taken should be far from the edges and should be rectangular as shown in the diagram. The current density of the sheet is $J=1.5$ A/cm.
SKETCH:



RESEARCH: The direction of the magnetic field is found using the right hand rule to be $+x$ above the surface of the conductor. Ampere's Law states $\oint \vec{B}\bullet d\vec{s}=B2\pi r=\mu_0 i_{\text{enclosed}}$.

SIMPLIFY: Note that sections 1 and 3 are perpendicular the field. $B\cdot ds=0$ for these two sections. If the path of 4 and 2 has a length of L, then by Ampere's Law, $\oint B\cdot ds=B_1L+B_2L=\mu_0 i_{\text{enclosed}}=\mu_0 JL$. By symmetry $B_1=B_2$. Thus, $2B_1=\mu_0 J$ or $B_1=\mu_0 J/2$.

CALCULATE: $B_1=\dfrac{\left(4\pi\cdot 10^{-7}\text{ T m/A}\right)(1.5\text{ A/cm})(100\text{ cm/m})}{2}=9.42478\cdot 10^{-5}\text{ T}$

ROUND: The magnetic field is accurate to two significant figures. The magnetic field near the surface of the conductor is $B_1=9.4\cdot 10^{-5}$ T.

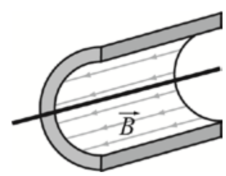
DOUBLE-CHECK: The form for the magnetic field is similar to that of a solenoid. It is divided by a factor of 2, which makes sense when considering the setup of a solenoid. The form of the equation is similar to that of question 28.12. This makes sense because the magnetic field inside a solenoid is generated by a current carrying wire on both sides of the Amperian loop, whereas the field generated by the flat conducting surface originates on one side of the Amperian loop only. In effect, the flat conductor can be seen as similar to half a solenoid, flattened out. See figure 28.21 in the text for a visual.

28.50. The magnetic field in a solenoid is given by the equation:

$$B = \mu_0 ni = \left(4\pi \cdot 10^{-7} \text{ T m/A}\right)\left(\frac{1000}{0.400 \text{ m}}\right)\left(2.00 \text{ A}\right) = 6.28 \cdot 10^{-3} \text{ T.}$$

28.51. The magnetic field in a solenoid is given by $B = \mu_0 in.$ Let the magnetic field of solenoid B be $B_B = \mu_0 in.$ The magnetic field of solenoid A is $B_A = \mu_0 i\left(4N\right)/\left(3L\right) = \left(4/3\right)\mu_0 in = \left(4/3\right)B_B$. The ratio of solenoid A magnetic field to that of solenoid B is 4:3.

28.52. The magnetic field at a point $r = 1.00$ cm from the axis of the solenoid will be the sum of the field due to the solenoid and the field produced by the wire. The solenoid has a magnetic field of $B_S = \mu_0 i_S n$ along the axis of the solenoid.



The wire produces a field which is perpendicular to the radial vector of $B_w = \mu_0 i_w / \left(2\pi r\right)$. The magnitude of the field is then

$$B_{\text{tot}} = \sqrt{B_S^{\ 2} + B_w^{\ 2}} = \mu_0 \sqrt{(i_S n)^2 + (i_w / 2\pi r)^2}$$

$$B_{\text{tot}} = \left(4\pi \cdot 10^{-7} \text{ T m/A}\right)\sqrt{\left(\left(0.250 \text{ A}\right)\left(1000 \text{ m}^{-1}\right)\right)^2 + \left(\frac{\left(10.0 \text{ A}\right)}{2\pi\left(0.0100 \text{ m}\right)}\right)^2} = 3.72 \cdot 10^{-4} \text{ T.}$$

28.53. (a) The magnetic field produced by the wire is

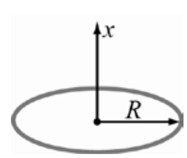
$$B = \mu_0 i / \left(2\pi r\right) = \left(4\pi \cdot 10^{-7} \text{ T m/A}\right)\left(2.5 \text{ A}\right) / \left(2\pi\left(0.039 \text{ m}\right)\right) = 1.3 \cdot 10^{-5} \text{ T.}$$

(b) The magnetic field of the solenoid is

$$B = \mu_0 in = \left(4\pi \cdot 10^{-7} \text{ T m/A}\right)\left(2.5 \text{ A}\right)\left(\frac{32}{0.01 \text{ m}}\right) = 0.010 \text{ T} = 1.0 \cdot 10^{-2} \text{ T.}$$

This field is much larger for the solenoid than the wire.

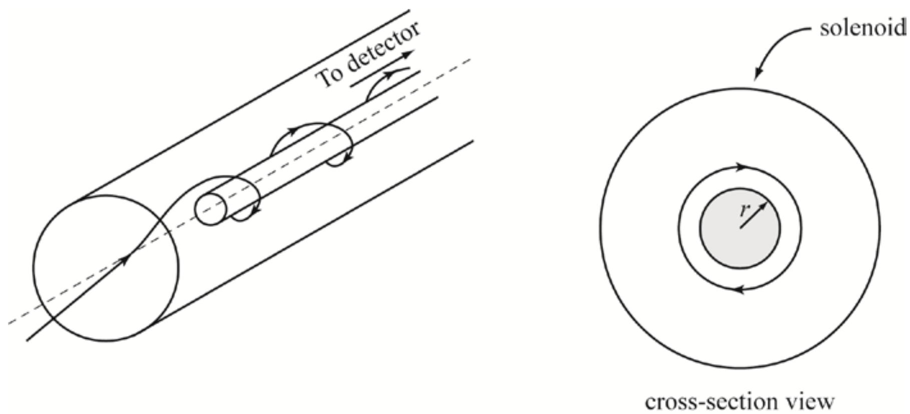
28.54. The magnetic field of a loop is $B = \frac{\mu_0 i}{2} \frac{R^2}{(x^2 + R^2)^{3/2}}$.



Therefore a coil of N loops produces a field of $B=\frac{\mu_0 iN}{2}\frac{R^2}{(x^2+R^2)^{3/2}}$. Let $x=R/2$ gives $B=\frac{\mu_0 iN}{2}\frac{R^2}{(x^2+R^2)^{3/2}}=\frac{\mu_0 iN}{2R^3}\frac{R^2}{(5/4)^{3/2}}=\left(\frac{4}{5}\right)^{3/2}\frac{\mu_0 iN}{2R}$. The field at the center of the coils is then $B_{\text{tot}}=2B=\left(\frac{4}{5}\right)^{3/2}\frac{\mu_0 iN}{R}=\left(\frac{4}{5}\right)^{3/2}\frac{\left(4\pi\cdot 10^{-7}\text{ T m/A}\right)(0.123\text{ A})(15)}{(0.750\text{ m})}=2.21\cdot 10^{-6}\text{ T}.$

28.55. **THINK:** If the perpendicular momentum of a particle is not large enough, its radius of motion will not be large enough to enter the detector. The minimum momentum perpendicular to the axis of the solenoid is determined by a condition such that the centripetal force is equal to the force due to the magnetic field.

SKETCH:



RESEARCH: Since the particle originates from the axis of the detector, the minimum radius of the circular motion of the particle must be equal to the radius of the detector as shown above. The magnetic force on the particle is $F=qvB$. Centripetal acceleration is $a_C=v^2/r$. The magnetic field due to the solenoid is $B=\mu_0 in$.

SIMPLIFY: Using Newton's Second Law, the momentum is $qvB=mv^2/r \Rightarrow mv=p=qrB$. Therefore, the minimum momentum is $p=\mu_0 qrin$.

CALCULATE: Substituting the numerical values yields.

$$p=\left(4\pi\cdot 10^{-7}\text{ T m/A}\right)\left(1.602\cdot 10^{-19}\text{C}\right)\left(0.80\text{ m}\right)\left(22\text{ A}\right)\left(550\cdot 10^{2}\text{ m}^{-1}\right)=1.949\cdot 10^{-19}\text{ kg m/s}$$

ROUND: Rounding the result to two significant figures gives $p=1.9\cdot 10^{-19}$ kg m/s.

DOUBLE-CHECK: This is a reasonable value.

28.56. The magnetic potential energy of a magnetic dipole in an external magnetic field is given by $U=-\vec{\mu}\bullet\vec{B}$. Therefore, the magnitude of the difference in energy for an electron "spin up" and "spin down" is $\Delta U=\left|U_{\text{up}}-U_{\text{down}}\right|=2\mu B$. This means the magnitude of the magnetic field is $B=\Delta U/2\mu$.

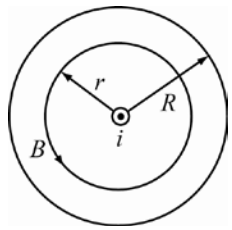
Putting in the numerical values gives $B=\frac{9.460\cdot 10^{-25}\text{ J}}{2\left(9.285\cdot 10^{-24}\text{ A m}^2\right)}=0.05094\text{ T}.$

28.57. The energy of a dipole in a magnetic field is $U=-\vec{\mu}\bullet\vec{B}$. The dipole has its lowest energy $U_{\text{min}}=-\vec{\mu}\bullet\vec{B}=-\mu B$, and its highest energy $U_{\text{max}}=\mu B$. The energy required to rotate the dipole from its lowest energy to its highest energy is $\Delta U=2\mu B$. This means that the thermal energy needed is ΔU which corresponds to a temperature $T=\Delta U/k_B=2\mu B/k_B$.

Substituting the numerical values of the dipole moment of hydrogen atom and $B = 0.15$ T yields

$$T = \frac{2\left(9.27 \cdot 10^{-24} \text{ J/T}\right)\left(0.15 \text{ T}\right)}{\left(1.38 \cdot 10^{-23} \text{ J/K}\right)} = 0.20 \text{ K}.$$

28.58.



The magnetic permeability of aluminum is $\mu = \left(1 + \chi_{\text{Al}}\right)\mu_0$. Applying Ampere's Law around an Amperian loop of radius r gives

$$\oint \vec{B} \bullet d\vec{s} = B\oint ds = B(2\pi r) = \mu i_{\text{enc}}.$$

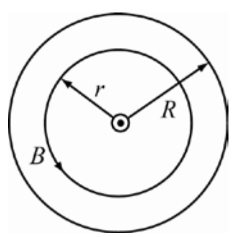
The current enclosed by the Amperian loop is $i_{\text{enc}} = i\frac{\pi r^2}{\pi R^2}$. Therefore, the magnetic field inside a wire is given by $B = \frac{\mu i r}{2\pi R^2}$. This means the maximum magnetic field is located at the surface of the wire where the magnitude is $B = \frac{\mu i}{2\pi R}$. Thus, the maximum current is

$$i_{\max} = \frac{2\pi R B_{\max}}{\left(1 + \chi_{\text{Al}}\right)\mu_0} = \frac{2\pi\left(1.0 \cdot 10^{-3} \text{ m}\right)\left(0.0105 \text{ T}\right)}{\left(1 + \left(2.2 \cdot 10^{-5}\right)\right)\left(4\pi \cdot 10^{-7} \text{ T m/A}\right)} = 52 \text{ A}.$$

28.59. The magnitude of the magnetic field inside a solenoid is given by $B = \mu i n = \kappa_{\text{m}}\mu_0 i(N/L)$. Thus the relative magnetic permeability κ_{m} is given by the equation:

$$\kappa_{\text{m}} = \frac{BL}{\mu_0 i N} = \frac{\left(2.96 \text{ T}\right) \cdot \left(3.50 \cdot 10^{-2} \text{ m}\right)}{\left(4\pi \cdot 10^{-7} \text{ T m/A}\right) \cdot \left(3.00 \text{ A}\right) \cdot \left(500.\right)} = 54.96 \approx 55.0.$$

28.60.



The magnetic permeability of tungsten is $\mu = \left(1 + \chi_{\text{W}}\right)\mu_0$. Applying Ampere's Law around an Amperian loop of radius r gives

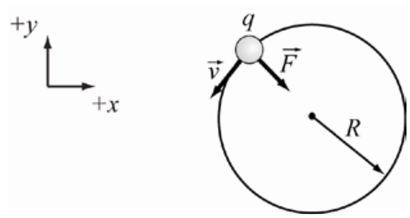
$$\oint \vec{B} \bullet d\vec{s} = B\oint ds = B(2\pi r) = \mu i_{\text{enc}}.$$

The current enclosed by the Amperian loop is $i_{\text{enc}} = i\frac{\pi r^2}{\pi R^2}$. Therefore, the magnetic field is

$$B = \left(\frac{\left(1 + \chi_{\text{W}}\right)\mu_0 i}{2\pi R^2}\right) r = \frac{\left(1 + 6.8 \cdot 10^{-5}\right)\left(4\pi \cdot 10^{-7} \text{ T m/A}\right)\left(3.5 \text{ A}\right)\left(0.60 \cdot 10^{-3} \text{ m}\right)}{2\pi\left(1.2 \cdot 10^{-3} \text{ m}\right)^2} = 2.9 \cdot 10^{-3} \text{ T}.$$

28.61. **THINK:** To determine the magnetic moment, the effective current of the system is needed. This implies the speed of the ball is required.

SKETCH:



RESEARCH: The ball travels in a circular orbit and it travels a distance of $2\pi R$ in time T, where T is the time for one revolution. The effective current is given by $i = q/T$. Since $T = 2\pi R/v$, this becomes $i = qv/(2\pi R)$. The effective magnetic moment is $\mu = iA = qv\pi R^2/(2\pi R) = qvR/2$. From the centripetal force, it is found that the speed is $mv^2/R = F \Rightarrow v = \sqrt{FR/m}$.

SIMPLIFY: Combining the above results yields $\mu = \frac{1}{2}q\sqrt{\frac{FR}{m}}R.$

CALCULATE: Putting in the numerical values gives

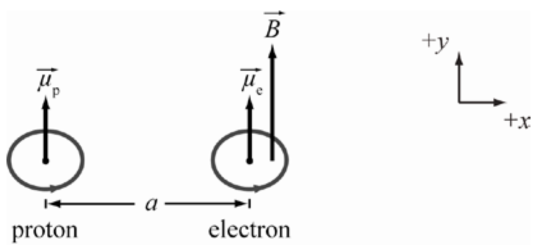
$$\mu = \frac{1}{2}\left(2.00\cdot 10^{-6}\ C\right)\sqrt{\frac{(25.0\text{ N})(1.00\text{ m})}{0.200\text{ kg}}}\left(1.00\text{ m}\right) = 1.118\cdot 10^{-5}\text{ A m}^2.$$

ROUND: Keeping 3 significant figures gives $\mu = 1.12\cdot 10^{-5}\text{ A m}^2$.

DOUBLE-CHECK: This magnetic moment is appropriately small for a small charge moving at a low velocity.

28.62. **THINK:** The magnetic field due to a proton is modeled as a dipole field. Using the value of the magnetic field, the potential energy of an electron spin in the magnetic field is $U = -\vec{\mu}\cdot\vec{B}$.

SKETCH:



RESEARCH: The electron field due to an electric dipole is given by $\vec{E} = \vec{P}/\left(2\pi\varepsilon_0 R^3\right)$. The corresponding magnetic field is obtained by replacing $1/\left(4\pi\varepsilon_0\right)$ with $\mu_0/\left(4\pi\right)$ and $\vec{P}$ with $\vec{\mu}$. Thus, $\vec{B} = \mu_0\vec{\mu}/\left(2\pi R^3\right)$.

SIMPLIFY: The energy difference between two electron-spin configurations is

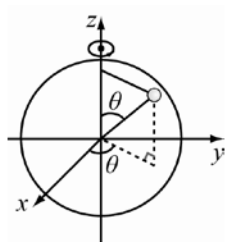
$$\begin{aligned}\Delta U &= U_{\text{anti}} - U_{\text{parallel}} \\ &= -(-\vec{\mu}_e)\bullet\vec{B} - (-\vec{\mu}_e\bullet\vec{B}) \\ &= 2\vec{\mu}_e\bullet\vec{B} = 2\vec{\mu}_e\bullet\frac{\mu_0\vec{\mu}_P}{2\pi a_0^3} \\ &= \frac{\mu_0\mu_e\mu_P}{\pi a_0^3}\end{aligned}$$

CALCULATE: Inserting all the numerical values yields

$$\Delta U = \frac{\left(4\pi \cdot 10^{-7}\ \text{T m/A}\right)\left(9.27 \cdot 10^{-24}\ \text{J/T}\right)\left(1.41 \cdot 10^{-26}\ \text{J/T}\right)}{\pi\left(5.292 \cdot 10^{-11}\ \text{m}\right)^3} = 3.528 \cdot 10^{-25}\ \text{J} = 2.204 \cdot 10^{-6}\ \text{eV}.$$

ROUND: Rounding the result to three significant digits produces $\Delta U = 2.20 \cdot 10^{-6}$ eV.
DOUBLE-CHECK: This is reasonable. A small difference in potential is expected for these small particles.

28.63. **THINK:** The classical angular momentum of rotating object is related to its moment of inertia. To get the magnetic dipole of a uniformly changed sphere, the spherical volume is divided into small elements. Each element produces a current and a magnetic dipole moment. The dipole moment of all elements is then added to get the net dipole moment.
SKETCH:



RESEARCH:
(a) The classical angular momentum of the sphere is given by $L = I\omega = (2/5)mR^2\omega.$
(b) The current produced by a small volume element dV is $i = \rho dV\omega/(2\pi)$. Thus the magnetic dipole moment of this element is $d\mu = \dfrac{\rho\omega dV}{2\pi}\pi(r\sin\theta)^2$. Integrating all the elements gives

$$\mu = \int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{\rho\omega r^2}{2}\left(\sin^2\theta\right)\left(r^2\sin\theta\right)dr\,d\theta\,d\phi.$$

(c) The gyromagnetic ratio is simply the ratio of the results from parts (a) and (b): $\gamma_e = \mu/L$.
SIMPLIFY:

(b) $\mu = \dfrac{\rho\omega}{2}\cdot 2\pi\int_0^{\pi}\int_0^{R} r^4\sin^3\theta drd\theta$

$= \rho\pi\omega\int_0^{\pi}\sin^3\theta d\theta \cdot \int_0^{R} r^4 dr$

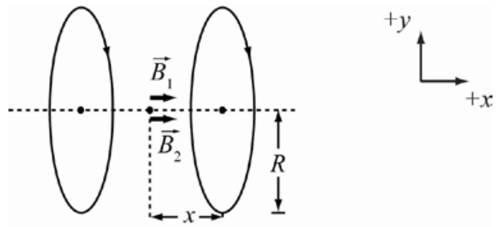
$= \rho\pi\omega\left[\int_{\cos 0}^{\cos\pi} -(1-\cos^2\theta)d\cos\theta\right]\dfrac{R^5}{5} = \rho\pi\omega\left[-x+\dfrac{x^3}{3}\right]_1^{-1}\dfrac{R^5}{5} = \rho\pi\omega\left(\dfrac{4}{3}\right)\dfrac{R^5}{5}$

Since $\rho\dfrac{4}{3}\pi R^3 = q$, the magnetic moment becomes $\mu = q\omega R^2/5$.
(c) Taking the ratio of the magnetic dipole moment and the angular momentum yields:

$\gamma_e = \dfrac{\mu}{L} = \dfrac{\dfrac{q\omega R^2}{5}}{\dfrac{2}{5}mR^2\omega} = \dfrac{q}{2m}$. Substituting $q = -e$ gives: $\gamma_e = -e/(2m)$.

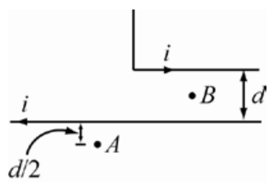
CALCULATE: Not required
ROUND: Not required
DOUBLE-CHECK: The magnetic dipole and the angular momentum should both be quadratic in *R*, so it is logical that the ratio of these two quantities is independent of *R*.

28.64.



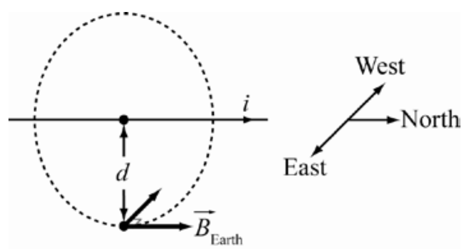
The magnitude of magnetic field due to one of the coils is $B_1 = \frac{\mu_0 i N_1}{2} \frac{R^2}{(x^2+R^2)^{3/2}}$. Since $B_1 = B_2$, the net magnetic field is $B = B_1 + B_2 = \frac{\mu_0 i N R^2}{(x^2+R^2)^{3/2}}$. Putting in $x = 0.500$ m, $R = 2.00$ m, $i = 7.00$ A and $N = 50$ yields $B = \frac{(4\pi \cdot 10^{-7} \text{ T m/A})(7.00 \text{ A})(50)(2.00 \text{ m})^2}{\left[(0.500 \text{ m})^2 + (2.00 \text{ m})^2\right]^{3/2}} = 2.01 \cdot 10^{-4}$ T.

28.65.



Since the horizontal distance between points *A* and *B* is large compared to *d*, the magnetic field at point *B* can be approximated by two parallel wires carrying opposite currents. By the right hand rule, the magnetic field at point *B* is directed into the page from both currents. Since point *B* is a distance of *d*/2 away from each wire, the magnitude of magnetic field at point *B* is twice that at point *A*. So, the strength of the magnetic field at point *B* is $B = 2(2.00 \text{ mT}) = 4.00$ mT.

28.66.



Applying the right hand rule gives the direction of the magnetic field due to the wire at the compass needle in the westward direction. The magnitude of B_{wire} is

$$B = \frac{\mu_0 I}{2\pi d} = \frac{(4\pi \cdot 10^{-7} \text{ T m/A}) \cdot 500.0 \text{ A}}{2\pi \cdot 12.0 \text{ m}} = 8.33 \text{ μT}.$$

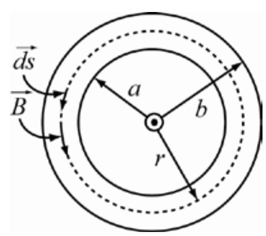
The deflection of the compass needle is $\delta\theta = \arctan\left(\frac{B_{\text{wire}}}{B_{\text{Earth}}}\right) = \arctan\left(\frac{8.33 \text{ μT}}{40.0 \text{ μT}}\right) = 11.8°$. The deflection is westward.

28.67. The magnetic dipole moment is defined as $\mu = iA = i\pi R^2$. This means the current that produces this magnetic dipole moment is $i = \mu / \left(\pi R^2\right)$. Substituting the numerical values gives the current of $i = \dfrac{8.0 \cdot 10^{22} \text{ A m}^2}{\pi (2.5 \cdot 10^6 \text{ m})^2} = 4.07 \cdot 10^9 \text{ A} \approx 4.1 \cdot 10^9 \text{ A}.$

28.68. The potential energy of a current loop in a magnetic field is given by $U = -\vec{\mu} \bullet \vec{B}$. The magnitude of the magnetic dipole moment is $\mu = iA = i\pi R^2$. The direction of the magnetic dipole moment can be determined using the right hand rule. In this case, the magnetic dipole is in the positive z-direction. Therefore, it follows that $\vec{\mu} = i\pi R^2 \hat{z} = 0.10 \text{ A} \cdot \pi \cdot \left(0.12 \text{ m}\right)^2 = 4.5 \cdot 10^{-3} \hat{z} \text{ A m}^2$. The energy is given by $U = -\vec{\mu} \bullet \vec{B} = \left(4.5 \cdot 10^{-3} \hat{z} \text{ A m}^2\right) \bullet \left(-1.5\hat{z} \text{ T}\right) = 6.8 \cdot 10^{-3} \text{ J}.$ If the loop can move freely, the loop will rotate such that its magnetic dipole moment aligns with the direction of the magnetic field. This means the magnetic dipole moment is $\vec{\mu} = 4.5 \cdot 10^{-3} (-\hat{z}) \text{ A m}^2$. Thus the minimum energy is $U = -4.5 \cdot 10^{-3} \text{ A m}^2 (-\hat{z}) \bullet (-1.5\hat{z} \text{ T}) = -6.8 \cdot 10^{-3} \text{ J}.$

28.69. The magnitude of magnetic field inside a solenoid is given by $B = \mu_0 in = \mu_0 i\left(N / L\right)$. Simplifying this, the number of turns of the wire is $N = BL / \left(\mu_0 i\right)$. Putting in the numerical values, $i = 0.20$ A, $L = 0.90$ m and $B = 5.0 \cdot 10^{-3}$ T yields $N = \dfrac{\left(5.0 \cdot 10^{-3} \text{ T}\right)\left(0.90 \text{ m}\right)}{\left(4\pi \cdot 10^{-7} \text{ T m/A}\right)\left(0.20 \text{ A}\right)} = 17904 \approx 18000$ turns.

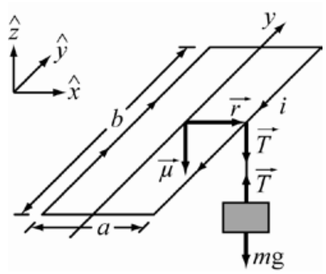
28.70.



Applying Ampere's Law around a loop as shown in the figure gives $\oint \vec{B} \bullet d\vec{s} = B \oint ds = \mu_0 i_{\text{enc}}$. Thus, the magnetic field is $B = \mu_0 i_{\text{enc}} / \left(2\pi r\right)$. The enclosed current is given by $i_{\text{enc}} = i - \dfrac{A_{\text{enc}} i}{A_{\text{total}}}$ when A_{enc} is cross sectional area of the shield that is enclosed by the loop and A_{total} is the cross sectional area shield. This means the areas are $A_{\text{enc}} = \pi(r^2 - a^2)$ and $A_{\text{total}} = \pi(b^2 - a^2)$. Thus the magnetic field inside the shield is $B = \dfrac{\mu_0 i}{2\pi r}\left(1 - \dfrac{r^2 - a^2}{b^2 - a^2}\right) = \dfrac{\mu_0 i}{2\pi r}\left(\dfrac{b^2 - r^2}{b^2 - a^2}\right).$

28.71. **THINK:** The torque due to the current in a loop of wire in a magnetic field must balance the torque due to weight.

SKETCH:



RESEARCH: The torque on a current loop in a uniform magnetic field is given by $\tau_B = \vec{\mu} \times \vec{B} = iN\vec{A} \times \vec{B} = iNA(-\vec{z}) \times \vec{B}$. Using Newton's Second Law, the torque due to the weight is found to be $\tau_W = \vec{r} \times \vec{T} = \left(\frac{1}{2}a\hat{x}\right) \times mg\left(-\hat{z}\right) = -\frac{1}{2}amg(\hat{x} \times \hat{z})$.

SIMPLIFY: Since the system is in equilibrium, the net torque must be zero: $\sum \tau = \tau_B + \tau_w = 0$.

Thus,

$$\tau_B = -\tau_w$$
$$-iNA\hat{z} \times \vec{B} = \frac{1}{2}amg(\hat{x} \times \hat{z}) = -\frac{1}{2}amg(\hat{z} \times \hat{x}).$$

This means that the magnetic field vector is in positive $\hat{x}$. Substituting $\vec{B} = B\hat{x}$ gives $iNAB = \frac{1}{2}amg$. After simplifying and using $A = ab$, $\vec{B} = \frac{1}{2}\frac{amg}{iNA}\hat{x} = \frac{1}{2}\frac{amg}{iN(ab)}\hat{x} = \frac{mg}{2iNb}\hat{x}$.

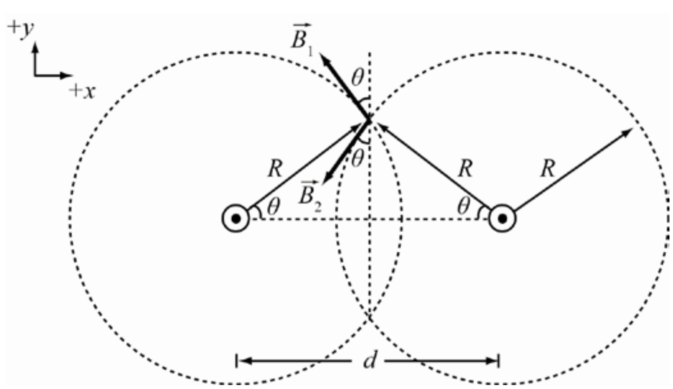
CALCULATE: Substituting the numerical values produces $\vec{B} = \frac{(0.0500 \text{ kg})(9.81 \text{ m/s}^2)}{2(1.00 \text{ A}) \cdot 50 \cdot (0.200 \text{ m})}\hat{x} = 0.02453 \text{ T}$.

ROUND: Three significant figures yields, $\vec{B} = 24.5 \text{ mT}$.

DOUBLE-CHECK: The magnetic force must be in the positive z-direction to balance gravity. By the right hand rule, it can be seen that the magnetic field must point in the positive x-direction for this to occur. This is consistent with the result calculated above. The result is reasonable.

28.72. **THINK:** In this problem, the net magnetic field due to two parallel wires is determined by adding the contributions from the wire.

SKETCH:



RESEARCH: The magnitude of the magnetic field of a long wire is given by $B = \mu_0 i/(2\pi R)$. The net magnetic field is $\vec{B}_{\text{net}} = \vec{B}_1 + \vec{B}_2$. Because of the symmetry of this problem, the y-component of the magnetic fields cancel out and only the x-component remains. Thus, the net magnetic field becomes

$$\vec{B} = -B\sin\theta\hat{x} - B\sin\theta\hat{x} = -2B\sin\theta\hat{x}$$

$$\vec{B} = \frac{-\mu_0 i}{\pi R}\sin\theta\hat{x}$$

SIMPLIFY: Since $\sin\theta = \frac{\sqrt{R^2 - (d/2)^2}}{R}$, the magnitude of the magnetic field simplifies to

$$B = \frac{\mu_0 i}{\pi R^2}\sqrt{R^2 - \frac{d^2}{4}}$$

CALCULATE: Inserting the numerical values of the parameters gives

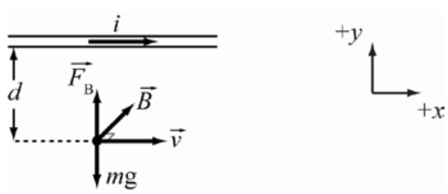
$$B = \frac{(4\pi\cdot 10^{-7}\text{ T m/A})(10.0\text{ A})}{\pi(12.0\cdot 10^{-2}\text{ m})^2}\sqrt{(12.0\cdot 10^{-2}\text{ m})^2 - \frac{(20.0\cdot 10^{-2}\text{ m})^2}{4}} = 1.843\cdot 10^{-5}\text{ T}.$$

ROUND: Keeping three significant figures, $B = 18.4\ \mu\text{T}$.

DOUBLE-CHECK: The magnetic field due to one wire at the same position is 16.7μT. It is therefore reasonable that the answer for two wires is slightly larger than this, considering that the y-components cancel out.

28.73. **THINK:** In this problem the force on a particle due to a magnetic field must balance the force due to gravity.

SKETCH:



RESEARCH: The force acting on the particle due to the magnetic field is $F_B = qvB\sin\theta$. Since the angle between $\vec{v}$ and $\vec{B}$ is 90.0°, the force due to the magnetic field becomes $F_B = qvB$. This force must balance the gravitational force which is given by $F_g = mg$. Therefore $F_B = F_g$ or $qvB = mg$.

SIMPLIFY: The magnetic field due to the current in the wire is $B = \mu_0 i/(2\pi d)$. The change of the particle is then found to be $q = mg/(vB) = mg2\pi d/(v\mu_0 i)$.

CALCULATE: Inserting the numerical values gives a charge of

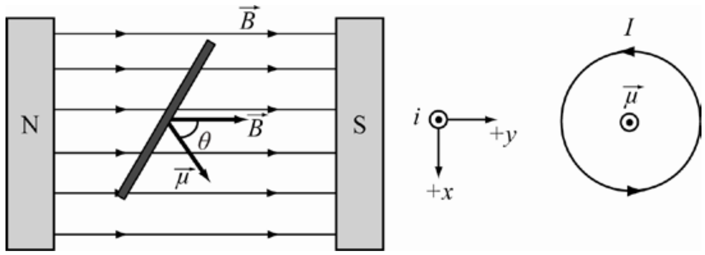
$$q = \frac{(1.00\cdot 10^{-6}\text{ kg})(9.81\text{ m/s}^2)2\pi(0.100\text{ m})}{(1000.\text{ m/s})(4\pi\cdot 10^{-7}\text{ T m/A})(10.0\text{ A})} = 4.905\cdot 10^{-4}\text{ C}.$$

ROUND: Rounding the result to 3 significant figures gives $q = 4.91\cdot 10^{-4}$ C.

DOUBLE-CHECK: Dimensional analysis confirms the calculation provided the answer in the correct units: $q = \frac{[\text{kg}][\text{m/s}^2][\text{m}]}{[\text{m/s}][\text{T}][\text{m/A}][\text{A}]} = \frac{[\text{kg}][\text{m/s}^2]}{[\text{m/s}][\text{N/(A m)}]} = \frac{[\text{A}][\text{m}]}{[\text{m/s}]} = [\text{A}][\text{s}] = [\text{C}]$.

28.74. **THINK:** The torque on a loop of wire in a magnetic field is given by $\vec{\tau} = \vec{\mu} \times \vec{B}$, where $\vec{\mu}$ is the magnetic dipole moment of the wire.

SKETCH:



RESEARCH:

(a) Using the right hand rule, the direction of current is counterclockwise as seen by an observer looking in the negative $\vec{\mu}$ direction as shown in the above figure.

(b) Using the magnetic dipole moment $\vec{\mu} = iNA\hat{n}$, the torque on the wire is $\vec{\tau} = iNA\hat{n} \times \vec{B}$, where $\hat{n}$ is a unit vector normal to the loop. Since $\left|\hat{n} \times \vec{B}\right| = B\sin\theta$ and $A = \pi R^2$, the magnitude of the torque is $\tau = iN\pi R^2 B\sin\theta$.

SIMPLIFY: From the equation, the number of turns needed to produce τ is $N = \dfrac{\tau}{\pi i R^2 B\sin\theta}$.

CALCULATE:

(b) Substituting the numerical values of the parameters yields

$$N_1 = \frac{(3.40 \text{ N m})}{\pi(5.00 \text{ A})(5.00 \cdot 10^{-2} \text{ m})^2 (2.00 \text{ T})\sin(60.0°)} = 49.98 = 50. \text{ turns.}$$

(c) Replacing the values of the above R with $R = 2.5 \cdot 10^{-2}$ m gives the number of turns

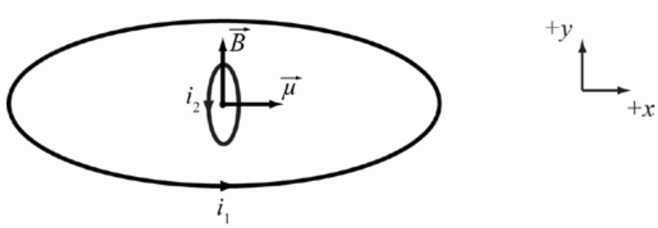
$$N_2 = \frac{(3.40 \text{ Nm})}{\pi(5.00 \text{ A})(2.50 \cdot 10^{-2} \text{ m})^2 (2.00 \text{ T})\sin(60.0°)} = 100. \text{ turns.}$$

ROUND: Not needed.

DOUBLE-CHECK: Since N is inversely proportional to R^2, the ratio of the results in (b) and (c) is $\dfrac{N_1}{N_2} = \dfrac{R_2^{\,2}}{R_1^{\,2}} = \dfrac{(R_1/2)^2}{R_1^{\,2}} = \dfrac{1}{4} = \dfrac{50}{200}$.

28.75. **THINK:** Assuming the inner loop is sufficiently small such that the magnetic field due to the larger loop is same across the surface of the smaller loop, the torque on the small loop can be determined by its magnetic moment.

SKETCH:



RESEARCH: The torque experienced by the small loop is given by $\vec{\tau} = \vec{\mu} \times \vec{B}$. The magnetic field in the center of the loop is given by $\vec{B} = \frac{\mu_0 i_1}{2R}\hat{y}$. The magnetic dipole moment of the small loop is $\vec{\mu} = i_2 \vec{A}_2 = i_2 \pi r^2 \hat{x}$.

SIMPLIFY: Combining all the above expressions yields the torque.

$$\tau = \left|\vec{\tau}\right| = \left|\left(i_2 \pi r^2 \hat{x}\right) \times \left(\frac{\mu_0 i_1}{2R}\hat{y}\right)\right| = \frac{\pi \mu_0 i_1 i_2 r^2}{2R}\left|\hat{x} \times \hat{y}\right| = \frac{\pi \mu_0 i_1 i_2 r^2}{2R}$$

CALCULATE: Putting in all the numerical values gives

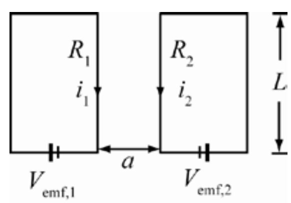
$$\tau = \frac{\pi\left(4\pi \cdot 10^{-7} \text{ T m/A}\right)(14.0 \text{ A})(14.0 \text{ A})(0.00900 \text{ m})^2}{2\left(0.250 \text{ m}\right)} = 1.254 \cdot 10^{-7} \text{ N m}.$$

ROUND: Rounding to 3 significant figures gives, $\tau = 1.25 \cdot 10^{-7}$ N m.

DOUBLE-CHECK: The units are correct: $\tau = \frac{\left[\text{T m/A}\right]\left[\text{A}\right]\left[\text{A}\right]\left[\text{m}^2\right]}{\left[\text{m}\right]} = \frac{\left[\text{N}\right]\left[\text{A}\right]\left[\text{m}^2\right]}{\left[\text{A}\right]\left[\text{m}\right]} = \left[\text{N m}\right].$

28.76. **THINK:** Two parallel wires carrying currents in the same direction have an attractive force. Two parallel wires carrying currents in opposite directions have a repulsive force.

SKETCH:



RESEARCH: By considering the direction of *emf* potentials, the currents in the wires have the same direction. Therefore the force between the wires is attractive. The force between the two wires is given by

$$F = \frac{\mu_0 i_1 i_2 L}{2\pi a}.$$

SIMPLIFY: The currents through the wires are given by $i_1 = \frac{V_{\text{emf},1}}{R_1}$ and $i_2 = \frac{V_{\text{emf},2}}{R_2}$. Thus, the force becomes $F = \frac{\mu_0 V_{\text{emf},1} V_{\text{emf},2} L}{2\pi a R_1 R_2}$. Solving for R_2 gives: $R_2 = \frac{\mu_0 V_{\text{emf},1} V_{\text{emf},2} L}{2\pi a R_1 F}$.

CALCULATE: Substituting the numerical values gives

$$R_2 = \frac{\left(4\pi \cdot 10^{-7} \text{ T m/A}\right)\left(9.00 \text{ V}\right)\left(9.00 \text{ V}\right)\left(0.250 \text{ m}\right)}{2\pi\left(0.00400 \text{ m}\right)\left(5.00\ \Omega\right)\left(4.00 \cdot 10^{-5} \text{ N}\right)} = 5.063\ \Omega.$$

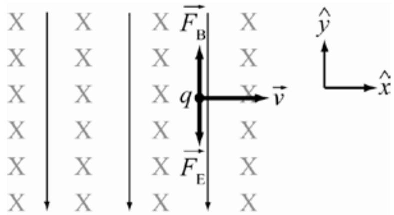
ROUND: Rounding the result to 3 significant figures gives $R_2 = 5.06\ \Omega$.

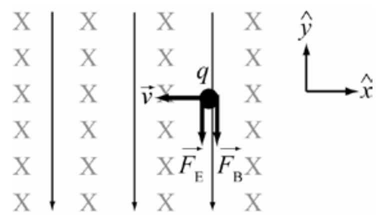
DOUBLE-CHECK: To 1 significant figure, the value of R_2 is the same as R_1. This is reasonable.

28.77. THINK: To solve this problem, the forces due to an electric field and a magnetic field are computed separately. The forces are added as vectors to get a net force.

SKETCH:

(a) (b)





RESEARCH: Using the right hand rule and since the charge of proton is positive, the directions of forces are shown above. The magnitude of the electric force on the proton is $F_E = qE$, and the magnitude of the magnetic force is $F_B = qvB$.

SIMPLIFY:

(a) The acceleration of the proton is $a = \frac{F_{\text{net}}}{m} = \frac{qvB - qE}{m} = \frac{q}{m}(vB - E)$.

(b) The acceleration of the proton if the velocity is reversed is

$$a = \frac{F_{\text{net}}}{m} = -F_B - F_E = \frac{-qvB - qE}{m} = -\frac{q}{m}(vB + E).$$

CALCULATE: Substituting the numerical values yields the acceleration

(a) $a = \frac{1.60 \cdot 10^{-19} \text{ e}}{1.67 \cdot 10^{-27} \text{ kg}}\left((200. \text{ m/s})(1.20 \text{ T}) - 1000. \text{ V/m}\right) = -7.28 \cdot 10^{10} \text{ m/s}^2$

(b) $a = -\frac{1.60 \cdot 10^{-19} \text{ e}}{1.67 \cdot 10^{-27} \text{ kg}}\left((200. \text{ m/s})(1.20 \text{ T}) + 1000. \text{ V/m}\right) = -1.19 \cdot 10^{11} \text{ m/s}^2$

ROUND:

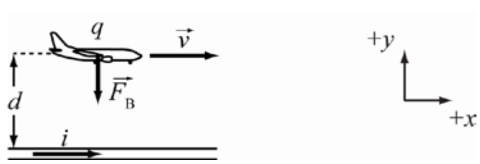
(a) $a = -7.28 \cdot 10^{10} \text{ m/s}^2$

(b) $a = -1.19 \cdot 10^{11} \text{ m/s}^2$

DOUBLE-CHECK: It is expected that the result in (b) is larger than in (a). This is consistent with the calculated values.

28.78. THINK: The net acceleration of a toy airplane is due to the gravitational acceleration and the magnetic field of a wire. However for this problem, the gravitational force is ignored.

SKETCH:



RESEARCH: Using a right hand rule, the magnetic force on the plane is directed toward the wire. The net acceleration of the plane due to the magnetic field is $a = F_B / m = qvB / m$.

SIMPLIFY: Substituting the magnetic field of the wire $B = \mu_0 i / (2\pi d)$ yields $a = \frac{qv\mu_0 i}{2\pi md}$.

CALCULATE: Putting in the numerical values gives the acceleration:

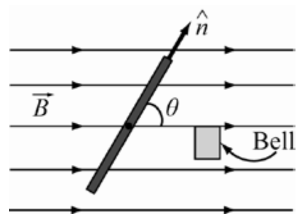
$$a = \frac{\left(36 \cdot 10^{-3} \text{ C}\right) \cdot (2.8 \text{ m/s})\left(4\pi \cdot 10^{-7} \text{ T m/A}\right) \cdot (25 \text{ A})}{2\pi\left(0.175 \text{ kg}\right)\left(0.172 \text{ m}\right)} = 1.674 \cdot 10^{-5} \text{ m/s}^2.$$

ROUND: Rounding the result to two significant figures gives $a = 1.7 \cdot 10^{-5} \text{ m/s}^2$.

DOUBLE-CHECK: It is expected that the result will be much less than the value of the gravitational acceleration.

28.79. **THINK:** To do this problem, the inertia of a long thin rod is required. The torque on a wire is also needed. The measure of the angle θ is 25.0°, and the current is $i = 2.00$ A. Let $A = 0.200 \cdot 10^{-4}\ \text{m}^2$ and $B = 9.00 \cdot 10^{-2}$ T.

SKETCH:



RESEARCH: The magnetic dipole moment of the wire is given by $\vec{\mu} = NiA\hat{n}$.

(a) The torque on the wire is $\vec{\tau} = \vec{\mu} \times \vec{B}$. The magnitude of this torque is $\tau = \mu B \sin\theta = NiAB\sin\theta$.

(b) The angular velocity of the rod when it strikes the bell is determined by using conservation of energy, that is, $E_\text{i} = E_\text{f}$ or $U_\text{i} + K_\text{i} = U_\text{f} + K_\text{f}$.

SIMPLIFY:

(a) $\tau = \mu B \sin\theta = NiAB\sin\theta$.

(b) Since $K_\text{i} = 0$, the final kinetic energy is

$$K_\text{f} = U_\text{i} - U_\text{f}$$

$$\frac{1}{2}I\omega^2 = -\mu B\cos\theta + \mu B\cos(0°) = -\mu B\cos\theta + \mu B = \mu B(1-\cos\theta)$$

Thus the angular velocity is $\omega = \sqrt{\dfrac{2\mu B(1-\cos\theta)}{I}} = \sqrt{\dfrac{2NiAB(1-\cos\theta)}{(1/12)mL^2}}$, using $I = \dfrac{1}{12}mL^2$, the inertia of a thin rod.

CALCULATE: Putting in the numerical values gives the following values.

(a) $\tau = (70)(2.00\ \text{A})(0.200 \cdot 10^{-4}\ \text{m}^2)(9.00 \cdot 10^{-2}\ \text{T})\sin(25.0°) = 1.06 \cdot 10^{-4}\ \text{N m}$

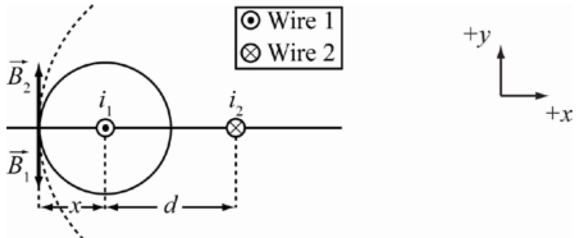
(b) $$\omega = \left[\frac{2(70)(2.00\ \text{A})(0.200 \cdot 10^{-4}\ \text{m}^2)(9.00 \cdot 10^{-2}\ \text{T})(1-\cos 25.0°)}{(1/12)(0.0300\ \text{kg})(0.0800\ \text{m})^2}\right]^{1/2} = 1.72\ \text{rad/s}$$

ROUND: Rounding to 3 significant figures yields $\tau = 1.06 \cdot 10^{-4}$ N m, $\omega = 1.72$ rad/s.

DOUBLE-CHECK: The torque should have units of Newton-meters, while the angular velocity should have units of radians per second.

28.80. **THINK:** Using a right hand rule, the sum of the magnetic fields of two parallel wires carrying opposite currents cannot be zero between the two wires.

SKETCH:



RESEARCH: The magnitude of the magnetic field of a long wire is $B=\mu_0 i/(2\pi R)$. Since $i_1 < i_2$ and i_1 is in an opposite direction to i_2, using the right hand rule, it is found that the location of the zero magnetic field must be to the left of the left-hand wire, as shown in the figure. Assuming the location is a distance x to the left of the left-hand wire, then the net magnetic field is $B_{\text{net}}=B_2-B_1=\dfrac{\mu_0 i_2}{2\pi(x+d)}-\dfrac{\mu_0 i_1}{2\pi x}=0.$

SIMPLIFY: Solving for x yields

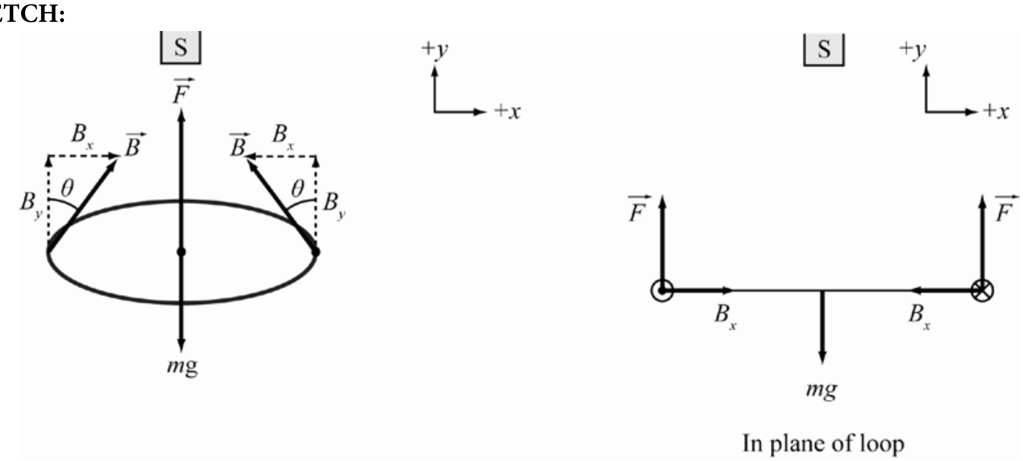
$$\frac{i_2}{x+d}=\frac{i_1}{x} \Rightarrow xi_2=i_1x+i_1d \Rightarrow x=\frac{i_1d}{i_2-i_1}.$$

Since $i_2=2i_1$,

$$x=\frac{i_1d}{2i_1-i_1}=d.$$

CALCULATE: Not required.
ROUND: Not required.
DOUBLE-CHECK: This result is expected since the ratio of $i_2/i_1=2.$ This means the ratio of distances is $\dfrac{d_2}{d_1}=\dfrac{2d}{d}=2$ also.

28.81. **THINK:** In order for a coil to float in mid-air, the downward force of gravity must be balanced an upward force due to the current loop in the magnetic field.
SKETCH:



RESEARCH: By using right-hand rule 1, the direction of the forces can be determined. For the y-component B_y of the magnetic field the force due to the current is in the radial direction of the coil. Therefore, this component cannot be responsible for levitating the coil. For the x-component B_x of the magnetic field, with a counterclockwise current as viewed from the bar magnet, the resulting force is in the y-direction, towards the bar magnet (see figure on right). This is the correct direction for balancing the weight of the coil. The magnitude of the y-component of the force on an element dl is $dF_y=Ni\left|d\vec{l}\times\vec{B}_x\right|\sin\theta=NiB\sin\theta dl.$ Thus the total magnetic force on the current loop is $F_y=\int_0^{2\pi R}NiB\sin\theta dl.$ Newton's Second Law requires that $F_y=mg.$

SIMPLIFY: The integral simplifies to: $F_y=2\pi RNiB\sin\theta.$ Therefore,

$$2\pi RNiB\sin\theta=mg \Rightarrow i=\frac{mg}{2\pi RNB\sin\theta}.$$

CALCULATE: Substituting in the numerical values yields

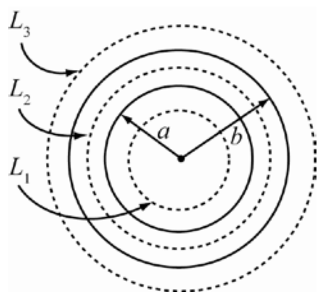
$$i = \frac{\left(10.0\cdot 10^{-3}\text{ kg}\right)(9.81\text{ m/s}^2)}{2\pi\left(5.00\cdot 10^{-2}\text{ m}\right)10.0(0.0100\text{ T})\sin(45.0°)} = 4.416\text{ A}.$$

ROUND: To 3 significant figures, the current is $i = 4.42$ A, counterclockwise as viewed from the bar magnet.

DOUBLE-CHECK: It takes large currents to generate strong magnetic forces. A current of 4 A is realistic to levitate a 10 g mass.

28.82. **THINK:** In this problem, Ampere's Law is applied on three different circular loops.

SKETCH:



RESEARCH: Loops L_1, L_2 and L_3 are Amperian loops.

(a) For distances $r < a$, applying Ampere's Law on the loop L_1, gives $\oint \vec{B}\cdot d\vec{s} = B(2\pi r) = \mu_0 i_{\text{enc}}$. Since $i_{\text{enc}} = 0$, the field is also zero, $B = 0$.

(b) For distances r between a and b, applying Ampere's law on the loop L_2 yields $\oint \vec{B}\cdot d\vec{s} = B(2\pi r) = \mu_0 i_{\text{enc}}$. The enclosed current is given by $i_{\text{enc}} = A_{\text{enc}} i / A$ or $i_{\text{enc}} = \dfrac{\pi\left(r^2 - a^2\right)}{\pi\left(b^2 - a^2\right)} i = \dfrac{r^2 - a^2}{b^2 - a^2} i.$

(c) For distances $r > b$, applying Ampere's Law on L_3 gives $B = \dfrac{\mu_0 i_{\text{enc}}}{2\pi r} = \dfrac{\mu_0 i}{2\pi r}$, since $i_{\text{enc}} = i.$

SIMPLIFY: Thus, the magnetic field is $B = \dfrac{\mu_0 i}{2\pi r}\dfrac{\left(r^2 - a^2\right)}{\left(b^2 - a^2\right)}.$

CALCULATE: Putting in the numerical values gives

(a) $B = 0$

(b) $$B = \frac{\left(4\pi\cdot 10^{-7}\text{ T m/A}\right)\cdot(0.100\text{ A})}{2\pi\left(6.50\cdot 10^{-2}\text{ m}\right)}\left[\frac{\left(6.50\text{ cm}\right)^2 - \left(5.00\text{ cm}\right)^2}{\left(7.00\text{ cm}\right)^2 - \left(5.00\text{ cm}\right)^2}\right] = 2.212\cdot 10^{-7}\text{ T}$$

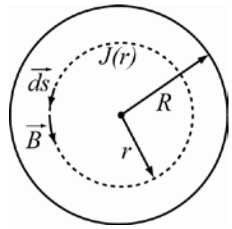
(c) $$B = \frac{\left(4\pi\cdot 10^{-7}\text{ T m/A}\right)\cdot(0.100\text{ A})}{2\pi\left(9.00\cdot 10^{-2}\text{ m}\right)} = 2.222\cdot 10^{-7}\text{ T}$$

ROUND: Keeping 3 significant figures yields the following results for (b) and (c). Note that the value found in (a) is precise. (a) $B = 0$ (b) $B = 2.21\cdot 10^{-7}$ T (c) $B = 2.22\cdot 10^{-7}$ T

DOUBLE-CHECK: The units of the calculated values are T, which is appropriate for magnetic fields.

28.83. **THINK:** To solve this problem, the current enclosed by an Amperian loop must be determined.

SKETCH:



RESEARCH: Applying Ampere's Law on a loop as shown above gives $\oint \vec{B}\cdot d\vec{s} = B(2\pi r) = \mu_0 i_{\text{enc}}$. i_{enc} is the current enclosed by the Amperian loop, that is $i_{\text{enc}} = \iint J(r)dA = \int_0^{2\pi}\int_0^r J(r')r'dr'd\theta.$

SIMPLIFY: Since $J(r)$ is a function of r only, the above integral becomes $i_{\text{enc}} = 2\pi\int_0^r J(r')r'dr'$. Substituting $J(r) = J_0(1-r/R)$ yields

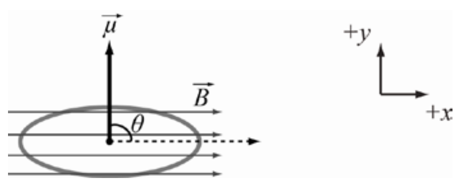
$$i_{\text{enc}} = 2\pi J_0\int_0^r\left[r' - \frac{r'^2}{R}\right]dr' = 2\pi J_0\left[\frac{r'^2}{2} - \frac{r'^3}{3R}\right]_0^r = 2\pi J_0\left[\frac{r^2}{2} - \frac{r^3}{3R}\right].$$

Thus, the magnetic field is $B = \dfrac{\mu_0 2\pi J_0}{2\pi r}\left[\dfrac{r^2}{2} - \dfrac{r^3}{3R}\right] = \mu_0 J_0\left[\dfrac{r}{2} - \dfrac{r^2}{3R}\right]$.

CALCULATE: Not required.
ROUND: Not required.
DOUBLE-CHECK: The form of the answer is reasonable.

28.84. **THINK:** The maximum torque on a circular wire in a magnetic field is when its magnetic moment is perpendicular to the magnetic field vector.

SKETCH:



RESEARCH: The torque on the circular wire is given by $\vec{\tau} = \vec{\mu}\times\vec{B}$. The magnitude of the torque is $\tau = \mu B\sin\theta$ where θ is the angle between $\vec{\mu}$ and $\vec{B}$.

SIMPLIFY:

(a) The maximum torque is when $\theta = 90°$, that is, $\tau = \mu B$. Using $\mu = iA = i\pi R^2$, the torque becomes $\tau = i\pi R^2 B$.

(b) The magnetic potential energy is given by $U = -\mu B\cos\theta$. The maximum and the minimum potential energies are when $\theta = 180°$ and $\theta = 0°$, that is, $U_{\text{max}} = +\mu B$ and $U_{\text{min}} = -\mu B$.

CALCULATE: (a) Inserting the numerical values gives the torque:

$$\tau = (3.0\text{ A})\pi\left(5.0\cdot 10^{-2}\text{ m}\right)^2\left(5.0\cdot 10^{-3}\text{ T}\right) = 1.18\cdot 10^{-4}\text{ N m}.$$

(b) Since the values of μB is the same as in (a), the range of the potential energy is

$\Delta U = U_{\text{max}} - U_{\text{min}} = 2\mu B = 2\cdot 1.2\cdot 10^{-4}\text{ J} = 2.4\cdot 10^{-4}\text{ J}.$

ROUND: Keeping only two significant figures yields $\tau = 1.2\cdot 10^{-4}$ N m and $\Delta U = 2.4\cdot 10^{-4}$ J.

DOUBLE-CHECK: The change in potential is a change in energy, so it is appropriate that the final answer have joules as units.

Multi-Version Exercises

Exercises 28.85–28.87 The magnetic field at the center of an arc of radius R subtended by an angle Φ is

$$B_\Phi = \int dB = \int_0^\Phi \frac{\mu_0}{4\pi}\frac{iRd\phi}{R^2} = \frac{\mu_0 i\Phi}{4\pi R}.$$

In this loop we have three sections:

$1:\ R = r,\ \Phi = \pi/2$

$2:\ R = 2r,\ \Phi = \pi/2$

$3:\ R = 3r,\ \Phi = \pi.$

The segments running directly toward/away from point *P* have no effect. So the magnetic field at *P* is

$$B = B_1 + B_2 + B_3 = \frac{\mu_0 i\left(\frac{\pi}{2}\right)}{4\pi(r)} + \frac{\mu_0 i\left(\frac{\pi}{2}\right)}{4\pi(2r)} + \frac{\mu_0 i(\pi)}{4\pi(3r)} = \frac{\mu_0 i}{8r} + \frac{\mu_0 i}{16r} + \frac{\mu_0 i}{12r} = \frac{6\mu_0 i}{48r} + \frac{3\mu_0 i}{48r} + \frac{4\mu_0 i}{48r} = \frac{13\mu_0 i}{48r}.$$

28.85. $B = \frac{13\mu_0 i}{48r} = \frac{13(4\pi\cdot 10^{-7}\ \text{T m/A})(3.857\ \text{A})}{48(1.411\ \text{m})} = 9.303\cdot 10^{-7}\ \text{T}$

28.86. $B = \frac{13\mu_0 i}{48r}$

$$r = \frac{13\mu_0 i}{48B} = \frac{13(4\pi\cdot 10^{-7}\ \text{T m/A})(3.961\ \text{A})}{48(7.213\cdot 10^{-7}\ \text{T})} = 1.869\ \text{m}$$

28.87. $B = \frac{13\mu_0 i}{48r}$

$$i = \frac{48rB}{13\mu_0} = \frac{48(2.329\ \text{m})(5.937\cdot 10^{-7}\ \text{T})}{13(4\pi\cdot 10^{-7}\ \text{T m/A})} = 4.063\ \text{A}$$

Exercises 28.88–28.90 The magnetic field inside a toroidal magnet is given by $B = \frac{\mu_0 Ni}{2\pi r}$.

28.88. $B = \frac{\mu_0 Ni}{2\pi r}$

$$N = \frac{2\pi rB}{\mu_0 i} = \frac{2\pi(1.985\ \text{m})(66.78\cdot 10^{-3}\ \text{T})}{(4\pi\cdot 10^{-7}\ \text{T m/A})(33.45\ \text{A})} = 19{,}814$$

To four significant figures, the toroid has 19,810 turns.

28.89. $B = \frac{\mu_0 Ni}{2\pi r}$

$$i = \frac{2\pi rB}{\mu_0 N} = \frac{2\pi(1.216\ \text{m})(78.30\cdot 10^{-3}\ \text{T})}{(4\pi\cdot 10^{-7}\ \text{T m/A})(22{,}381)} = 21.27\ \text{A}$$

28.90. $B = \frac{\mu_0 Ni}{2\pi r} = \frac{(4\pi\cdot 10^{-7}\ \text{T m/A})(24{,}945)(49.13\ \text{A})}{2\pi(1.446\ \text{m})} = 0.1695\ \text{T} = 169.5\ \text{mT}$

Chapter 29: Electromagnetic Induction

Concept Checks

29.1. c **29.2.** a **29.3.** c **29.4.** c **29.5.** a **29.6.** a **29.7.** a **29.8.** e

Multiple-Choice Questions

29.1. d **29.2.** c **29.3.** a **29.4.** a **29.5.** a **29.6.** c **29.7.** d **29.8.** b **29.9.** a **29.10.** d **29.11.** c **29.12.** d **29.13.** e **29.14.** a

Conceptual Questions

29.15. A refrigerator's electrical circuit contains a motor with a large number of winding coils, making it highly inductive. The electromagnetic induction due to the coil can create a large voltage, on the order of kV between the prongs. This voltage is great enough to ionize the air and the process of ionization produces light, creating a visible spark.

29.16. Large machinery and motors often convert electrical energy to mechanical energy or vice-versa to complete a task. The conversion from electrical energy to mechanical energy requires the creation of magnetic fluxes. Changes in the magnetic flux reaching a pacemaker, due to movements of the machine or the person, will create currents in the circuitry of the pacemaker, changing its behavior; this can be dangerous.

29.17. As the metal moves through the non-uniform magnetic field, it experiences a changing magnetic flux. The flux induces an emf in the metal, if it is a conductor, and produces eddy currents. Lenz's law states that the induced currents create a force to oppose the movement of the metal through the field. This action is analogous to the drag force or force of friction used to create the damping of a harmonic oscillator.

29.18. Lenz's law requires that as the magnet moves down the cylinder, a current is produced in the aluminum cylinder, which in turn creates a magnetic field that opposes the magnet's motion. The force of the currents on the magnet is proportional to the velocity of the magnet. Thus, the magnet will continue to accelerate until it reaches a terminal speed that creates a force equal and opposite to the force of gravity.

29.19. (a) The currents produced in the aluminum, by induction, create a force that opposes the motion of the magnet. The magnet falling in the glass tube does not create a current since glass is an insulator. Thus, the magnet in the glass tube falls faster since there is no magnetic field produced to oppose the force of gravity.
(b) Because the glass has nearly infinite resistance, no eddy currents are created as the magnet passes through it. The aluminum being a good conductor does produce eddy currents as the magnet falls through it. Thus, the aluminum tube has a larger eddy current.

29.20. (a) The *B* field inside the solenoid is uniform and equal to $B_{\text{i}} = \mu_0 ni.$ Outside the solenoid, the field is zero, $B_{\text{o}} = 0.$ The *B* field through the ring is only that of the field inside the solenoid of radius, *a*. The flux is then $\Phi = BA = \mu_0 ni\pi a^2 = \mu_0 n\pi a^2 Ct^2.$ Thus, the emf is $\left|\Delta V_{\text{ind}}\right| = \frac{d\Phi}{dt} = 2\mu_0 n\pi a^2 Ct.$

(b) The magnitude of the electric field is then $2\pi rE = \Delta V = 2\mu_0 n\pi a^2 Ct$ or $E = \frac{\mu_0 na^2 Ct}{r}.$

(c) The ring is not necessary for the induced electric field to exist. The solenoid will produce a magnetic field from the current being passed through the wire inducing an electric field on each concurrent loop of wire.

29.21. Lenz's law requires that the induced current opposes the change in the magnetic field. Therefore, the *B* field created by the induced current is downward. To produce a magnetic field in this direction, the current must flow clockwise as seen from above.

29.22. The area of the loop perpendicular to the field is given by $A = L^2\cos(\omega t)$. The potential difference is:

$$\Delta V_{\text{in,L}} = -\frac{d\Phi}{dt} = -\frac{d(AB)}{dt} = -B\frac{dA}{dt} = -B\frac{d}{dt}\left(L^2\cos(\omega t)\right) = -BL^2\left(-\omega\sin(\omega t)\right) = BL^2\omega\sin(\omega t).$$

29.23. The emf produced by a loop is given by $\Delta V_{\text{ind}} = vBL$, where L is the length of the moving conductor. By taking a differentially small element of the disk, we convert L into the differential, dr, and integrate from the center of the disk to the edge for the emf of the disk: $\Delta V_{\text{ind}} = \int_0^R vBdr$. The velocity of an element, dr, is given by $v = r\omega$. The emf is then:

$$\Delta V_{\text{ind}} = \int_0^R r\omega Bdr = \frac{1}{2}\omega R^2 B.$$

29.24. Separation of charge due to the magnetic force, $q\vec{v}\times\vec{B}$, engenders a compensating electric field of magnitude $E = \vec{v}\times\vec{B} = vB$. The corresponding potential difference across height, l, is:
$V = lE = lvB = (1.80\text{ m})(2.00\text{ m/s})(25.0\text{ T}) = 90.0\text{ V}$. In equilibrium this drives no current. However, such a large magnetic field offers further hazards due to any metal objects about the man's body and to stress on blood vessels, which are carrying conducting fluids in motion like iron.

29.25. The flux through the inside copper cylinder is constant during the process, so:

$$\Phi_{\text{i}} = \Phi_{\text{f}} \Rightarrow B_{\text{i}}A_{\text{i}} = B_{\text{f}}A_{\text{f}} \Rightarrow B_{\text{i}}\pi r_{\text{i}}^2 = B_{\text{f}}\pi r_{\text{f}}^2.$$

The final magnetic field is given by:

$$B_{\text{f}} = \left(\frac{r_{\text{i}}}{r_{\text{f}}}\right)^2 B_{\text{i}}.$$

If the initial B field is 1.0 T and the radius compresses by a factor of 14, then final field is given by:

$$B_{\text{f}} = \left(\frac{r_{\text{i}}}{r_{\text{i}}/14}\right)^2 B_{\text{i}} = (14)^2 B_{\text{i}} = (14)^2(1.0\text{ T}) = 2.0\cdot 10^2\text{ T}.$$

Experimental magnetic fields are typically lower than 10 T. This is a huge magnetic field.

29.26. Lenz's law requires that the induced current opposes the change in the magnetic field. Therefore, the B field created by the induced current is downward. To produce a magnetic field in this direction, the current must flow clockwise as seen from above.

29.27. The inductance of a solenoid is given by $L = \mu_0 n^2 lA$. Let d denote the length of the wire. The number of turns in each case is $N = d/2\pi r$. The inductance is then:

$$L = \mu_0 n^2 lA = \mu_0 n(nl)A = \mu_0 nNA = \mu_0 n\left(\frac{d}{2\pi r}\right)\pi r^2 = \frac{1}{2}\mu_0 ndr.$$

For both solenoids, the number of turns per unit length is equal, and the distance of the wire is the same. Therefore, the ratio of the inductances is:

$$\frac{L_1}{L_2} = \frac{\mu_0 ndr/2}{\mu nd2r/2} = \frac{1}{2}.$$

Thus, the inductance of the second solenoid is twice that of the first solenoid.

Exercises

29.28. The magnetic flux through the coil is given by:

$$\Phi = NBA\cos\theta = 20(5.00\text{ T})\pi(0.400\text{ m})^2\cos(90° - 25.8°) = 21.9\text{ T m}^2$$

29.29. The potential difference around the loop is:

$$V_{\text{emf}}=-\frac{d\Phi}{dt}\approx-\frac{\Delta\Phi}{\Delta t}=-\frac{\Delta(AB)}{\Delta t}=-A\frac{\Delta B}{\Delta t}=-\pi r^2\frac{\Delta B}{\Delta t}=-\pi(0.0100\text{ m})^2\left(\frac{0\text{ T}-1.20\text{ T}}{20.0\text{ s}}\right)=1.89\cdot10^{-5}\text{ V}.$$

Note that the area of the ring is perpendicular to the field. Thus, the normal of the area is parallel to the field and $\cos\theta=1.$

29.30. If the angle between the B-field and the plane of the loop is $40°,$ then the angle between the B-field and the normal to the loop is $90°-40°=50°,$ and so the voltage across the loop is given by:

$$V_{\text{ind}}=-\frac{d\Phi_{\text{B}}}{dt}=-\frac{d}{dt}(NAB\cos\theta)=-NA\cos\theta\frac{dB}{dt}=-NA\cos\theta\frac{d(1.50t^3)}{dt}=-NL^2(\cos 50.0°)4.50t^2.$$

The current induced if the loop has a resistance of $R=3.00\ \Omega$ is:

$$i=\frac{|V_{\text{ind}}|}{R}=\frac{NL^2(\cos 50.0°)4.50t^2}{R}=\frac{(8)(0.200\text{ m})^2(\cos 50.0°)4.50(2.00\text{ s})^2}{(3.00\ \Omega)}=1.23\text{ A}.$$

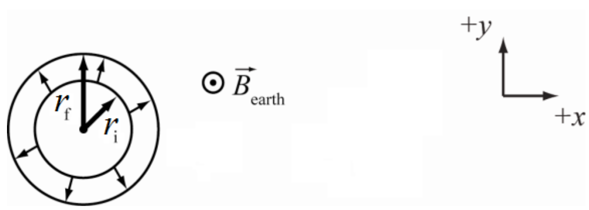
29.31. Because the magnetic field is perpendicular to the normal of the loop, there is no flux through the loop:

$$\Phi=AB\cos 90°=0.$$

Since there is no flux through the loop, there is no induced voltage: $V=-\frac{d\Phi}{dt}=-\frac{d(0)}{dt}=0.$

29.32. **THINK:** The change in the area of the loop creates a change in the magnetic flux through the loop. The change in flux produces a current. The loop has a resistance of $R=30.0\ \Omega$ and a radius which changes from $r_{\text{i}}=20.0\text{ cm}$ to $r_{\text{f}}=25.0\text{ cm}$ in 1.00 s. The magnetic field of the Earth is about $4.26\cdot10^{-5}$ T.

SKETCH:



RESEARCH: The flux through the loop is $\Phi_{\text{B}}=AB\cos\theta$ or $\Phi_{\text{B}}=AB,$ since the B field is perpendicular to the surface of the loop. The induced potential difference is given by $V_{\text{ind}}=-d\Phi_{\text{B}}/dt.$ This potential must also satisfy $V=iR.$

SIMPLIFY: The induced current in the loop is:

$$i=\frac{V_{\text{ind}}}{R}=\frac{1}{R}\left(-\frac{d\Phi_{\text{B}}}{dt}\right)=-\frac{1}{R}\left(\frac{dAB}{dt}\right)=-\frac{B}{R}\left(\frac{dA}{dt}\right)=-\frac{B}{R}\frac{d\pi r^2}{dt}=-\frac{B\pi}{R}\left(\frac{dr^2}{dt}\right)\approx-\frac{B\pi}{R}\left(\frac{r_{\text{f}}^2-r_{\text{i}}^2}{\Delta t}\right).$$

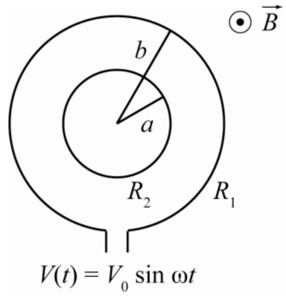
CALCULATE: $i=-\frac{(4.26\cdot10^{-5}\text{ T})\pi}{30.0\ \Omega}\left(\frac{(0.250\text{ m})^2-(0.200\text{ m})^2}{1.00\text{ s}}\right)=-1.00374\cdot10^{-7}\text{ A}$

ROUND: The induced current in the loop is $i=-1.00\cdot10^{-7}$ A.

DOUBLE-CHECK: This current is very small, as one would expect. The negative sign indicates that the direction of the induced current is such that the magnetic field due to the induced current opposes the change in magnetic flux that induces the current.

29.33. **THINK:** The current in the outer loop generates a magnetic field. Because the magnitude of the current in the outer loop changes with time, the magnetic field it generates also changes. The changing magnetic field, in turn, induces a potential difference and thus a current in the inner loop. Let I be the current in the outer loop and i be the induced current in the inner loop.

SKETCH:



RESEARCH: The current through the large loop is $I = \dfrac{V_0 \sin \omega t}{R_1}$. This creates a magnetic field at the center of the loop of:

$$B_1 = \frac{\mu_0 I}{2b}$$

which is derived from the Biot-Savart Law. Since the radius of the inner loop is much smaller than the radius of the outer loop, the magnetic field through the inner loop is $B_1 = \mu_0 V_0 \sin \omega t / 2bR_1$. This magnetic field creates a flux of:

$$\Phi_B = B_1 A = B_1 \pi a^2 = \frac{\mu_0 \pi a^2 V_0}{2bR_1} \sin \omega t.$$

The induced potential across the inner loop is then:

$$\Delta V_{\text{ind}} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left(\frac{\mu_0 \pi a^2 V_0}{2bR_1} \sin \omega t \right).$$

This voltage corresponds to a current in the inner loop of:

$$i = \frac{\Delta V_{\text{ind}}}{R_2} = -\frac{1}{R_2}\frac{d}{dt}\left(\frac{\mu_0 \pi a^2 V_0}{2bR_1} \sin \omega t \right).$$

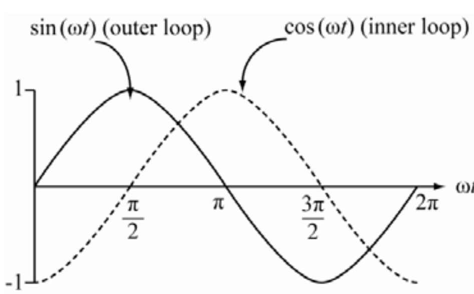
SIMPLIFY: The potential difference induced in the inner loop is:

$$\Delta V_{\text{ind}} = -\frac{d}{dt}\left(\frac{\mu_0 \pi a^2 V_0}{2bR_1} \sin \omega t \right) = -\frac{\mu_0 \pi a^2 V_0}{2bR_1}\frac{d}{dt}\left(\sin \omega t \right) = -\frac{\mu_0 \pi a^2 V_0 \omega}{2bR_1} \cos \omega t,$$

and the induced current in the inner loop is:

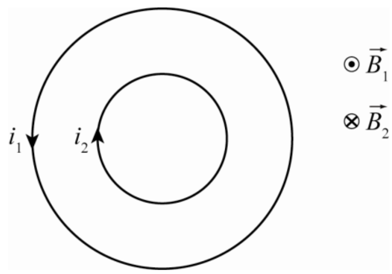
$$i = \frac{\Delta V_{\text{ind}}}{R_2} = -\frac{1}{R_2}\frac{d}{dt}\left(\frac{\mu_0 \pi a^2 V_0}{2bR_1} \sin \omega t \right) = -\frac{\mu_0 \pi a^2 V_0}{2bR_1 R_2}\frac{d}{dt}\left(\sin \omega t \right) = -\frac{\mu_0 \pi a^2 V_0 \omega}{2bR_1 R_2} \cos \omega t.$$

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE CHECK: The time dependence on the current for the outer loop and inner loop is shown in the plot below. For example, for $\omega t < \pi / 2$ (taking positive values to be the counterclockwise direction) if the current in the outer loop is moving counterclockwise and increasing then the current in the inner loop is increasing in the clockwise direction. This is consistent with Lenz's Law.



29.34. **THINK:** The varying current, i_1, through the outer solenoid creates a varying magnetic field, B_1, within the coil. This varying B field creates a flux in the inner solenoid, which in turn creates an induced *emf.*

SKETCH:



RESEARCH: The magnetic field generated by the outer solenoid is given by $B_1 = \mu_0 n i_1 = \mu_0 n i_0 \cos\omega t$. The flux generated in the inner solenoid is given by $\Phi = A_2 B_1$. The induced emf in the inner solenoid is given by $\Delta V_{\text{ind}} = -\frac{d\Phi}{dt} = -A_2 \frac{dB_1}{dt}$.

SIMPLIFY: $\Delta V_{\text{ind}} = -A_2 \frac{dB_1}{dt} = -A_2 \frac{d}{dt}\left(\mu_0 n i_0 \cos\omega t\right) = -A_2 \mu_0 n i_0 \frac{d}{dt}\left(\cos\omega t\right) = A_2 \mu_0 n i_0 \omega \sin\omega t$. This corresponds to a current of $i_2 = \frac{\Delta V}{R} = \frac{A_2 \mu_0 n i_0 \omega \sin\left(\omega t\right)}{R}$, in the inner solenoid. The current of the inner solenoid induces a B field of:

$$B_2 = \mu_0 n i_2 = \frac{\mu_0 n\left(A_2 \mu_0 n i_0 \omega \sin\omega t\right)}{R} = \frac{\mu_0{}^2 n^2 A_2 i_0 \omega \sin\omega t}{R}.$$

CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE CHECK: The induced magnetic field of the inner solenoid must oppose the change in flux of the outer solenoid. It can be seen from the expressions for B_2 and B_1 the two fields will always have opposite directions, satisfying this requirement.

29.35. (a) The decreasing B field creates a changing flux through the loop, confined to the area of the dotted circle of radius, $r = 3.00$ cm. The varying flux creates an emf of:

$$\Delta V_{\text{ind}} = -\frac{d\Phi_{\text{B}}}{dt} = -\frac{d\left(AB\right)}{dt} = -\frac{d\left(\pi r^2 B\right)}{dt} = -\pi r^2 \frac{dB}{dt} \approx -\pi r^2 \frac{\Delta B}{\Delta t} = -\pi r^2 \left(\frac{B_{\text{f}} - B_{\text{i}}}{\Delta t}\right).$$

This corresponds to a current of:

$$i = \frac{V}{R} = -\frac{\pi r^2}{R}\left(\frac{B_{\text{f}} - B_{\text{i}}}{\Delta t}\right) = -\frac{\pi\left(0.0300 \text{ m}\right)^2}{0.200\ \Omega}\left(\frac{1.00 \text{ T} - 2.00 \text{ T}}{2.00 \text{ s}}\right) = 0.00707 \text{ A} = 7.07 \text{ mA}$$

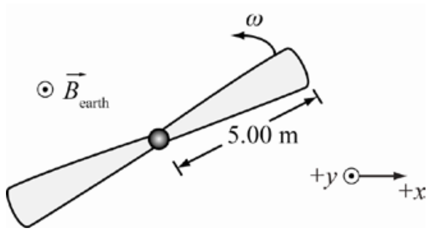
(b) The B field points into the page, thus a decrease in the B field will induce a current corresponding to a B field which points into the page. By the right-hand rule, the induced current flows clockwise.

29.36. The airplane's wings are approximated by a straight wire. The voltage across a wire moving in a B field is:

$$V = vLB = 3v_{\text{mach}} LB = 3\left(340. \text{ m/s}\right)\left(10.0 \text{ m}\right)\left(0.500 \cdot 10^{-4} \text{ T}\right) = 0.510 \text{ V}.$$

29.37. **THINK:** As a conductor travels through a magnetic field, perpendicular to the ground, of intensity $B = 0.426$ G, it creates a voltage difference between its ends. The length of metal of interest is L = 5.00 m and rotates at $1.00 \cdot 10^4$ rpm.

SKETCH:



RESEARCH: The potential difference across a wire moving in a magnetic field is $\Delta V_{\text{ind}} = vLB.$ Each element of the blade travels at a different speed, $v = r\omega.$ To calculate the potential difference, the length must be divided into pieces of length, *dl*, which travel at $v = l\omega.$ The value should be integrated over the total length, from 0 to *L*.

SIMPLIFY: $\int \Delta V = \int_0^L vBdl = \int_0^L l\omega Bdl = \frac{1}{2}\omega BL^2$

In terms of the blade's rpm, the potential difference is $V = \frac{1}{2}\left(\frac{2\pi(\text{rpm})}{60\text{ s}}\right)BL^2.$

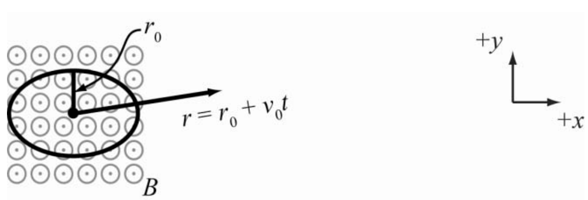
CALCULATE: $V = \left(\frac{\pi\left(1.00\cdot 10^4\text{ rpm}\right)}{60.0\text{ s/rpm}}\right)\left(0.426\cdot 10^{-4}\text{ T}\right)\left(5.00\text{ m}\right)^2 = 0.557633\text{ V} \approx 0.558\text{ V}$

ROUND: The potential difference from the hub of the helicopter's blade to its far end is $\Delta V_{\text{ind}} = 0.558$ V.

DOUBLE-CHECK: We can double-check this result by assuming that the blade moves with a constant speed equal to the speed of the middle of the blade, $v = \left(\frac{L}{2}\right)\omega = \left(\frac{L}{2}\right)2\pi f = 2\pi Lf.$ The induced potential difference would be $\Delta V = vB\frac{L}{2} = \left(2\pi Lf\right)B\frac{L}{2} = \pi fBL^2,$ which is the same answer we got by integrating over the length of the blade.

29.38. **THINK:** The expanding loop creates a changing flux through the loop. Lenz's law implies that the changing flux induces a current in the loop. This is similar to increasing the magnetic field within the loop. To counteract the increase in flux, the current must create a magnetic field opposite to the *B* field. By the right-hand rule, the current must flow clockwise. The radius of the loop expands by $r = r_0 + vt,$ where $r_0 = 0.100$ m and v = 0.0150 m/s. The resistance of the wire is $R = 12.0\ \Omega.$ The *B* field has a uniform value of $B_0 = 0.750$ T upward. The problem asks for the induced current at the time, *t* = 5.00 s.

SKETCH:



RESEARCH: The flux through the loop is $\Phi_B = AB = \pi r^2 B.$ The induced current of the loop is $i = V/R,$ where the voltage is given by $V = -d\Phi_B/dt.$

SIMPLIFY: The induced current in the wire is:

$$i = \frac{V}{R} = -\frac{1}{R}\frac{d\Phi_B}{dt} = -\frac{1}{R}\frac{d}{dt}\left(r^2\right) = -\left(\frac{\pi B}{R}\right)\frac{d}{dt}\left(r_0 + vt\right)^2 = -\frac{\pi B2\left(r_0+vt\right)v}{R} = -\frac{2\pi B}{R}v\left(r_0 + vt\right).$$

CALCULATE: The magnitude of the induced current at *t* = 5.00 s is:

$$i = -\frac{2\pi\left(0.750\text{ T}\right)}{12.0\ \Omega}\left(0.0150\text{ m/s}\right)\left[0.100\text{ m} + \left(0.0150\text{ m/s}\right)\left(5.00\text{ s}\right)\right] = 0.0010308\text{ A}.$$

ROUND: $i = 1.03$ mA at 5.00 s, travelling clockwise through the loop.

DOUBLE-CHECK: $[i] = \frac{[\text{T}]}{[\Omega]}[\text{m/s}]([\text{m}] + [\text{m/s}][\text{s}]) = \frac{[\text{T}][\text{m}^2]}{[\Omega][\text{s}]} = \frac{[\text{V}][\text{s}][\text{m}^2][\text{A}]}{[\text{m}^2][\text{V}][\text{s}]} = [\text{A}]$

29.39. **THINK:** Terminal velocity will be reached when the force due to the changing magnetic flux cancels the weight of the bar.

SKETCH: A sketch is not necessary.

RESEARCH: $\Delta V_{\text{ind}} = -\frac{d\Phi_{\text{B}}}{dt} = -B\frac{dA}{dt} == -Bw\frac{dy}{dt} = Bwv_{\text{term}}$

$i = \frac{\Delta V_{\text{ind}}}{R}$, $F_{\text{B}} = iLB = iwB$, $F_{\text{B}} = F_{\text{gravity}} = mg$.

SIMPLIFY: $iBw = mg \Rightarrow \frac{\Delta V_{\text{ind}}}{R}Bw = mg \Rightarrow \frac{Bwv_{\text{term}}}{R}Bw = mg \Rightarrow v_{\text{term}} = \frac{mgR}{w^2B^2}$.

CALCULATE: No calculations are necessary.

ROUND: Rounding is not necessary.

DOUBLE-CHECK: It makes sense the larger m is, the higher v_{term} has to be to compensate for the greater gravitational force.

29.40. **THINK:**

(a) The change in area causes an induced voltage.

(b) After finding the induced voltage, the induced current can be determined.

(c) The induced current will cause a force opposite to the direction of motion (from Lenz's law) which requires F_{ext} compensating for it.

(d) Determine W_{ext} and P_{ext} from F_{ext}.

SKETCH: Provided with the question.

RESEARCH:

(a) $|\Delta V_{\text{ind}}| = \left|-\frac{d\Phi_{\text{B}}}{dt}\right| = B\frac{dA}{dt} = BvL$

(b) $i_{\text{ind}} = \frac{\Delta V}{R}$, in the clockwise direction.

(c) $F_{\text{B}} = i_{\text{ind}}LB = F_{\text{ext}}$

(d) $W_{\text{ext}} = F_{\text{ext}}\Delta y$, $P_{\text{ext}} = Fv$

(e) $P_{\text{ext}} = P_{\text{R}} = i_{\text{ind}}^2 R$

SIMPLIFY:

(a) $|\Delta V_{\text{ind}}| = BvL$

(b) $i_{\text{ind}} = \frac{BvL}{R}$

(c) $|F_{\text{B}}| = |F_{\text{ext}}| = \frac{L^2B^2v}{R}$

(d) $W_{\text{ext}} = \frac{L^2B^2v}{R}\Delta y$, $P_{\text{ext}} = \frac{L^2B^2v^2}{R}$

(e) $P_{\text{R}} = \frac{L^2B^2v^2}{R}$

CALCULATE: Not necessary.

ROUND: Not necessary.

DOUBLE-CHECK:
(e) This is due to the law of conservation of energy. The work done has to go somewhere, and in this case is dissipated by the resistor as heat.

29.41. **THINK:** The current in the wire will cause a magnetic field. The changing current will cause a changing flux through the loop, inducing a potential.
SKETCH: Provided with question.
RESEARCH: For a wire: $B=\frac{\mu_0}{4\pi}\left(\frac{2i}{r}\right)$. $\Delta V_{\text{ind}}=\frac{d\Phi_{\text{B}}}{dt}$, $i=2.00\text{ A}+(0.300\text{ A/s})t$, A = 7.00 m by 5.00 m, $\Phi_{\text{B}}=\oiint \vec{B}\bullet d\vec{A}$.

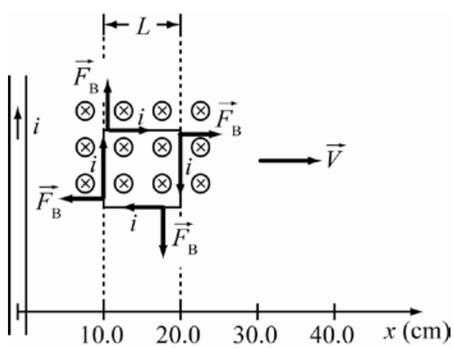
SIMPLIFY: $\Phi_{\text{B}}=(5.00\text{ m})\int_{1\text{ m}}^{8\text{ m}}\frac{\mu_0}{4\pi}\left(\frac{2i}{r}\right)dr=(5.00\text{ m})\left(\frac{\mu_0 i}{2\pi}\right)\ln\left(\frac{8.00\text{ m}}{1.00\text{ m}}\right)=(5.00\text{ m})\left(\frac{\mu_0 i}{2\pi}\right)\ln 8.00$

$$\Delta V_{\text{ind}}=\frac{d\Phi_{\text{B}}}{dt}=(5.00\text{ m})\left(\frac{\mu_0}{2\pi}\right)(\ln 8.00)\left(\frac{di}{dt}\right)=(5.00\text{ m})\left(\frac{\mu_0}{2\pi}\right)(\ln 8.00)(0.300\text{ A/s})$$

CALCULATE: $\Delta V_{\text{ind}}=(5.00\text{ m})\left(\frac{4\pi\cdot 10^{-7}\text{ H/m}}{2\pi}\right)(\ln 8.00)(0.300\text{ A/s})=6.238\cdot 10^{-7}\text{ V}$

ROUND: $\Delta V_{\text{ind}}=6.24\cdot 10^{-7}\text{ V}$
DOUBLE-CHECK: It makes sense that the larger the rate of change of the current, the larger the induced voltage.

29.42. **THINK:**
(a) By the right-hand rule, the flux is into the page. Since the square is moving away from the wire, the flux is decreasing. Lenz's law states that the current is moving clockwise.
(c) The top and bottom parts have the same contributions and cancel each other.
SKETCH:
(b)



RESEARCH: Use $x_2=20.0\text{ cm}$ and $x_1=10.0\text{ cm}$ as the end points. $B=\frac{\mu_0}{4\pi}\left(\frac{2i}{r}\right)$, $\Phi_{\text{B}}=\oiint \vec{B}\bullet d\vec{A}$,

$\Delta V_{\text{ind}}=-\frac{d\Phi}{dt}$, $i_{\text{ind}}=\frac{\Delta V_{\text{ind}}}{R}$, r = 10.0 cm, i = 1.00 A, v = 10.0 cm/s, $R=0.0200\ \Omega$, L = 10.0 cm, $F_{\text{left}}=i_{\text{ind}}LBx_1$, $F_{\text{right}}=i_{\text{ind}}LBx_2$, and $F_{\text{net}}=F_{\text{right}}-F_{\text{left}}$.

SIMPLIFY: $\Phi_B=L\int_{x_1+vt}^{x_2+vt}\frac{\mu_0}{4\pi}\frac{2i}{r}dr=L\frac{\mu_0}{2\pi}i\left[\ln(x_2+vt)-\ln(x_1+vt)\right]$

$$\Delta V_{\text{ind}}=-\frac{d\Phi_{\text{B}}}{dt}=-\frac{Li\mu_0}{2\pi}\left(\frac{v}{x_2+vt}-\frac{v}{x_1+vt}\right)$$

$$F_{\text{net}} = i_{\text{ind}}LBx_2 - i_{\text{ind}}LBx_1 = i_{\text{ind}}LB(x_2 - x_1) = \frac{\Delta V_{\text{ind}}}{R}LB(x_2 - x_1)$$
$$= \left(-\frac{Li\mu_0}{2\pi R}v\left(\frac{1}{x_2+vt} - \frac{1}{x_1+vt}\right)\right)L\left(\frac{\mu_0}{4\pi}\left(\frac{2i}{r}\right)\right)(x_2 - x_1)$$
$$= -\frac{L^2i^2\mu_0 v}{2\pi R}\left(\frac{1}{x_2+vt} - \frac{1}{x_1+vt}\right)\left(\frac{\mu_0}{2\pi}\right)\left(\frac{x_2 - x_1}{r}\right)$$
$$= -\frac{L^2i^2\mu_0^2 v}{(2\pi)^2 R}\left(\frac{1}{x_2+vt} - \frac{1}{x_1+vt}\right)\left(\frac{x_2 - x_1}{r}\right)$$

CALCULATE: At time $t = 0$:

$$F_{\text{net}} = -\frac{(0.100 \text{ m})^2(1.00 \text{ A})^2(4\pi\cdot 10^{-7} \text{ H/m})^2(0.100 \text{ m/s})}{(2\pi)^2\, 0.0200\ \Omega}\left(\frac{1}{20.0 \text{ cm}} - \frac{1}{10.0 \text{ cm}}\right)\left(\frac{20.0 \text{ cm} - 10.0 \text{ cm}}{10.0 \text{ cm}}\right)$$
$$= 1.00\cdot 10^{-16} \text{ N}$$

ROUND: $F_{\text{net}} = 1.00\cdot 10^{-16}$ N

DOUBLE-CHECK: It makes sense that for larger velocities and currents through the wire, the induced force is larger. This is in some ways analogous to how a car traveling faster than another has a larger drag force.

29.43. $\Phi(t) = BA\cos(2\pi ft)$, $\Delta V_{\text{ind}} = \frac{d\Phi}{dt} = -2\pi fBA\sin(2\pi ft)$. The maximum occurs when $|\sin(2\pi ft)| = 1.$

$\Delta V_{\text{ind,max}} = 2\pi fBA = 110.$ V. Substitute the values to obtain: $f = \frac{110. \text{ V}}{2\pi BA} = \frac{110. \text{ V}}{2\pi(1.00 \text{ T})(1.00 \text{ m}^2)} = 17.5$ Hz.

29.44. **THINK:** First relate the magnetic flux to the angular speed and then determine the maximum angular speed. Use the values $B = 0.87$ T, $A = 0.0300 \text{ m}^2$.

SKETCH: A sketch is not necessary.

RESEARCH: For a single loop: $\Phi(t) = BA\cos(\omega t)$. $\Delta V_{\text{ind}} = -\frac{d\Phi}{dt} = \omega BA\sin(\omega t)$

$\Delta V_{\text{ind,max}} = 170 \text{ V} = \omega BA,$ since the maximum occurs when $|\sin(\omega t)| = 1.$

SIMPLIFY: $\omega = \frac{\Delta V_{\text{ind,max}}}{BA}$

CALCULATE: $\omega = \frac{170 \text{ V}}{0.87 \text{ T}(0.0300 \text{ m}^2)} = 6513$ Hz

ROUND: $\omega = 6500$ Hz

DOUBLE-CHECK: It is reasonable that the higher the applied voltage, the higher the angular speed.

29.45. **THINK:** First determine an expression for the magnetic flux, and then use Faraday's law to determine the induced voltage.

SKETCH: A sketch is not necessary.

RESEARCH: $B_{\text{Earth}} = 0.300 \text{ G} = 0.300\cdot 10^{-4}$ T, $\Phi_{\text{B}} = NBA\cos(\omega t)$, $A = \pi r^2$, $r = 0.250$ m, $N = 1.00\cdot 10^5$,

$\omega = 2\pi(150. \text{ Hz})$, $i_{\text{ind}} = \frac{\Delta V_{\text{ind}}}{R} = -\left(\frac{1}{R}\right)\frac{d\Phi_{\text{B}}}{dt}$, $i_{\text{ind,peak}} = -\left(\frac{1}{R}\right)\frac{d\Phi_{\text{B}}}{dt}_{\text{peak}}$, $R = 1500.\ \Omega$

SIMPLIFY:

(a) $i_{\text{ind}} = -\left(\frac{1}{R}\right)(-NBA\omega\sin(\omega t)) = \frac{NBA\omega}{R}\sin(\omega t)$; The peak occurs at $|\sin(\omega t)| = 1$: $i_{\text{ind,peak}} = \frac{NBA\omega}{R}.$

(b) $i_{avg} = 0.7071\left(i_{ind,peak}\right), \quad P_{avg} = i_{avg}^2 R$

CALCULATE:

(a) $i_{ind,peak} = \dfrac{\left(1.00\cdot 10^5\right)\left(0.300\cdot 10^{-4}\text{ T}\right)\left(0.250\text{ m}\right)^2 2\pi^2\left(150.\text{ Hz}\right)}{\left(1500.\ \Omega\right)} = 0.3701\text{ A}$

(b) $i_{avg} = 0.7071\left(0.3701\text{ A}\right) = 0.2617\text{ A}, \quad P_{avg} = \left(0.2617\text{ A}\right)^2\left(1500.\ \Omega\right) = 102.7\text{ W}$

ROUND:

(a) $i_{ind,peak} = 0.370\text{ A}$

(b) $i_{avg} = 0.262\text{ A}, \ P_{avg} = 103\text{ W}$

DOUBLE-CHECK: The answer seems reasonable since there are a very large number of turns for the generator turning at a very fast rate.

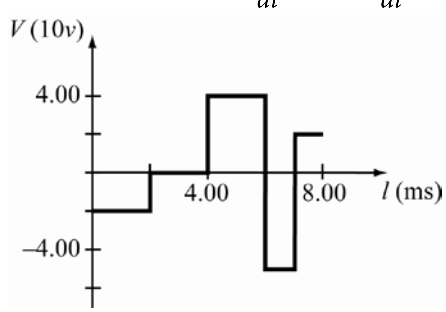
29.46. First solve for n: $B = \mu_0 n i \ \Rightarrow \ n = \dfrac{B}{\mu_0 i} = \dfrac{0.025\text{ T}}{\left(1.2566\cdot 10^{-6}\text{ m kg s}^{-2}\text{ A}^{-2}\right)\left(0.60\text{ A}\right)} = 33158.$

$$M = N_1 \pi \mu_0 n_2 r_1^2 = 200\pi\left(1.2566\cdot 10^{-6}\text{ m kg s}^{-2}\text{ A}^{-2}\right)\left(33158\right)\left(0.034\text{ m}\right)^2 = 0.0302\text{ H}, \ i(t) = i_0\left(1+\left(2.4\text{ s}^{-2}\right)t^2\right)$$

$$V = -M\frac{di}{dt} = -\left(0.0302\text{ H}\right)\left(2\right)\left(0.60\text{ A}\right)\left(2.4\text{ s}^{-2}\right)\left(2.0\text{ s}\right) = -0.17\text{ V}$$

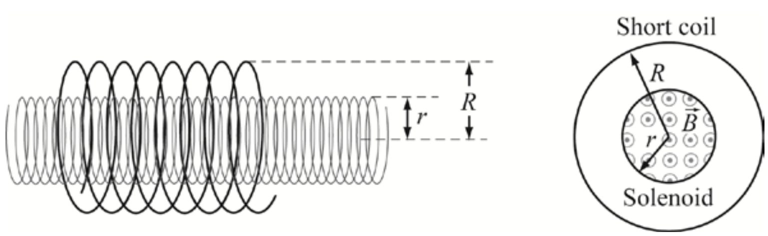
The results match those of the example.

29.47. The potential across an inductor is given by: $\Delta V_{ind} = -L\dfrac{di}{dt}$, where $\dfrac{di}{dt}$ is the slope.



29.48. **THINK:** The potential difference induced in the solenoid is due to the changing current in the coil. Using the mutual inductance of the solenoid due to the coil, the potential difference induced in the solenoid can be calculated. Assume the magnetic field of the short coil is uniform. This is not strictly accurate, but necessary to answer the question and will give a reasonable approximation.

SKETCH:



RESEARCH: The mutual inductance between the coil and the solenoid is

$$M = \frac{N_s \Phi_{c\to s}}{i_c},$$

where N_s is the number of turns in the solenoid, $\Phi_{c\to s}$ is the flux in the solenoid resulting from the magnetic field through the coil, and i_c is the current in the coil. The flux is given by

$$\Phi_{c\to s} = in\pi\mu_0 r^2.$$

$\Delta V_{\text{ind}} = M\dfrac{di}{dt}$, $N_s = 30$, $n = 60/\text{cm} = 6000/\text{m}$, $r = 0.0800$ m, $\dfrac{di}{dt} = \dfrac{2.00\text{ A}}{12.0\text{ s}}$.

SIMPLIFY: $\Delta V_{\text{ind}} = N_s n\pi\mu_0 r^2 \dfrac{di}{dt}$

CALCULATE: $\Delta V_{\text{ind}} = (30)(6000/m)\pi\left(4\pi\cdot10^{-7}\text{ H/m}\right)(0.0800\text{ m})^2\left(\dfrac{2.00\text{ A}}{12.0\text{ s}}\right) = 7.57986\cdot10^{-4}\text{ V}$

ROUND: $\Delta V_{\text{ind}} = 7.58\cdot10^{-4}\text{ V}$

DOUBLE-CHECK: It makes sense that for larger changes in current, larger potential differences are induced.

29.49. (a) $\tau_L = \dfrac{L}{R} = \dfrac{1.00\text{ H}}{1.00\text{ M}\Omega} = 1.00\ \mu\text{s}$

(b) $i(t) = \dfrac{V_{\text{emf}}}{R}\left(1-e^{-t/\tau_L}\right)$. At $t = 0$, $i(t) = 0$. At $t = 2.00\ \mu$s, $i(t) = \dfrac{10.0\text{ V}}{1.00\text{ M}\Omega}\left(1-e^{-(2.00\ \mu\text{s})/(1.00\ \mu\text{s})}\right) = 8.65\ \mu\text{A}.$

At steady state. $t\to\infty$: $i(\infty) = \dfrac{V_{\text{emf}}}{R} = 10.0\ \mu A.$

29.50. For an *RL* circuit: $i(t) = \dfrac{V_{\text{emf}}}{R}\left(1-e^{-t/\tau}\right)$, where $\tau = \dfrac{L}{R} = 0.0250$ s.

$$\frac{i(t)R}{V_{\text{emf}}} = 1-e^{-t/\tau} \Rightarrow -\tau\ln\left(1-\frac{i(t)R}{V_{\text{emf}}}\right) = t \Rightarrow t = -(0.0250\text{ s})\ln\left(1-\frac{(0.300\text{ A})(120.\ \Omega)}{40.0\text{ V}}\right) = 0.0576\text{ s}$$

29.51. The potential drop is the sum of the potential drop across the resistor and the inductor:

$$\Delta V = iR + L\frac{di}{dt} = (3.0\text{ A})(3.25\ \Omega) + (0.440)(3.6\text{ A/s}) = 11 \approx 11.3\text{ V}.$$

29.52. **THINK:** In a circuit containing only a resistor, the current would be established almost instantaneously. However, with the RL circuit, the current must increase exponentially from zero to the steady state. $V_{\text{emf}} = 18$ V, $R_1 = R_2 = 6.0\ \Omega$, $L = 5.0$ H.

SKETCH: Provided with question.

RESEARCH:

(a) The inductor functions as an open-circuit, so $i = V_{\text{emf}}/R_2 = 18\text{ V}/6.0\ \Omega = 3.0$ A.

(b) The inductor acts as an open-circuit, so there is no current across it and hence no current across R_1.

(c) The current across R_2 is given by Ohm's Law, $i = V_{\text{emf}}/R$.

(d) The potential difference across a resistor is also given by Ohm's Law, $\Delta V = iR$.

(e) Same as (d).

(f) The sum of the voltages around any loop is zero.

(g) The rate of current change across R_1 is the same as that of L.

SIMPLIFY:

(a) $i = V/R_2$

(b) Not applicable.

(c) $i_{R_2} = V_{\text{emf}}/R_2$

(d) $\Delta V_{R_1} = i_{R_1}R_1$

(e) $\Delta V_{R_2} = i_{R_2}R_2$

(f) $V_{\text{emf}} - V_L - V_{R_1} = 0 \Rightarrow V_L = V_{\text{emf}} - V_{R_1}$

(g) $V_L = L\dfrac{di}{dt} \;\Rightarrow\; \dfrac{di}{dt} = \dfrac{V_L}{L}$

CALCULATE:

(a) $i = 18\text{ V} / 6.0\ \Omega = 3.0\text{ A}$

(b) $i_{R_1} = 0$

(c) $i_{R_2} = 18\text{ V} / 6.0\ \Omega = 3.0\text{ A}$

(d) $\Delta V_{R_1} = (0\text{ A})(6.0\ \Omega) = 0$

(e) $\Delta V_{R_2} = (3.0\text{ A})(6.0\ \Omega) = 18\text{ V}$

(f) $V_L = 18\text{ V} - 0 = 18\text{ V}$

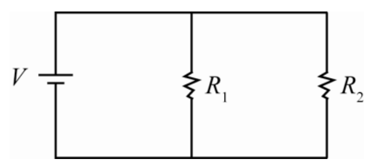
(g) $\dfrac{di}{dt} = \dfrac{18\text{ V}}{5.0\text{ H}} = 3.6\text{ A/s}$

ROUND: Not necessary. The values are already to the correct number of significant figures.

DOUBLE CHECK: The branch of the circuit which contains only a resistor and a source of emf behaves as a simple resistor circuit, with the current being established almost instantaneously. For the branch of the circuit which contains a resistor and an inductor, equation 29.29 states $i(t) = \dfrac{V_{emf}}{R}\left[1 - e^{-t/(L/R)}\right]$. When $t = 0$, $i(t) = 0$, as found above.

29.53. **THINK:** After a long time, the inductor acts like a short-circuit. The circuit is in steady state, so the current is no longer changing. $V_{emf} = 18$ V, $R_1 = R_2 = 6.0\ \Omega$, $L = 5.0$ H.

SKETCH: An equivalent sketch when the circuit is in steady-state is as follows.



RESEARCH: The current from the battery is given by $i_{tot} = \dfrac{V_{emf}}{R_{net}}$, where $R_{net} = \left(\dfrac{1}{R_2} + \dfrac{1}{R_1}\right)^{-1}$. The current through each resistor is given by Ohm's Law, $i = V / R$. The sum of the potentials around any loop must be zero: $V_{emf} + V_{R_1} = 0$, $V_{emf} + V_{R_2} + V_L = 0$.

SIMPLIFY:

(a) $i_{tot} = \dfrac{V_{emf}}{R_1 R_2}(R_1 + R_2)$

(b) $i_{R_1} = \dfrac{V_{R_1}}{R_1}$

(c) $i_{R_2} = \dfrac{V_{R_2}}{R_2}$

(d) $V_{emf} + V_{R_1} = 0 \;\Rightarrow\; V_{R_1} = -V_{emf}$

(e) $V_{emf} + V_{R_2} + V_L = 0 \;\Rightarrow\; V_{R_2} = -V_{emf} - V_L = -V_{emf} - L\dfrac{di}{dt}$

(f) $V_L = L\dfrac{di}{dt}$

(g) $\dfrac{di_{R_1}}{dt} = \dfrac{di_L}{dt} = \dfrac{V_L}{L}$

CALCULATE:

(a) $i_{\text{tot}} = \dfrac{18\text{ V}}{(6.0\ \Omega)(6.0\ \Omega)}(6.0\ \Omega + 6.0\ \Omega) = 6.0\text{ A}$

(b) $i_{\text{R}_1} = \dfrac{18\text{ V}}{6.0\ \Omega} = 3.0\text{ A}$

(c) $i_{\text{R}_2} = \dfrac{18\text{ V}}{6.0\ \Omega} = 3.0\text{ A}$

(d) $V_{\text{R}_1} = -18\text{ V}$

(e) $V_{\text{R}_2} = -18\text{ V} - (5.0\text{ H})(0) = -18\text{ V}$

(f) $V_{\text{L}} = 5.0\text{ H}(0) = 0$

(g) $\dfrac{di_{\text{R}_1}}{dt} = \dfrac{0}{5.0\text{ H}} = 0$

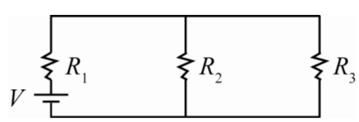
ROUND: Not necessary.

DOUBLE CHECK: Evaluating the loop containing the inductor using equation 29.29 shows that after a long time, $i_2(t) = \dfrac{V_{\text{emf}}}{R_2}\left(1 - e^{-t/(L/R)}\right) = \dfrac{V_{\text{emf}}}{R_2}$, as found above. Kirchoff's rules can be used to show that $i_{\text{R}_1} = i_{\text{R}_2}$, also as found above.

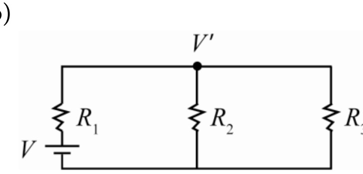
29.54. **THINK:** As the current begins to flow through the circuit, the self induced potential difference in the inductor opposes the change in current. As the change in current decreases, the self induced potential difference also decreases until the current reaches the steady state given by Ohm's Law, $i = V_{\text{emf}} / R$. When the switch is opened, the current will continue to flow, at a decreasing rate, through the loop composed of R_3, L, and R_2 until the energy which has been stored in the inductor is dissipated.

SKETCH:

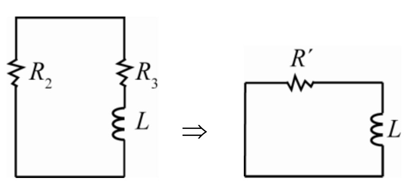
(a)



(b)



(c)



RESEARCH:

(a) Immediately after the switch is closed, the inductor is like an open-circuit. Clearly, $i_{\text{R}_3} = 0$, and $V + i_{\text{R}_1}R_1 + i_{\text{R}_2}R_2 + i_{\text{R}_3}R_3 = 0$. so $i_{\text{R}_2} = i_{\text{R}_1} = \dfrac{V}{R_1 + R_2}$.

(b) After a long time, the inductor acts like a short-circuit. $R_{\text{tot}} = R_1 + \dfrac{R_2R_3}{R_2 + R_3}$, $\quad V' = V - R_1\left(\dfrac{V}{R_{\text{tot}}}\right)$

(c) When the switch is opened, $i_{\text{L}} = i_{\text{R}_2} = i_{\text{R}_3}$. In fact, the equivalent resistance of this circuit is $R' = R_2 + R_3$ and the circuit can be redrawn accordingly. Since the current in an inductor cannot change instantaneously, from part (b): $i_{\text{initial}}=\dfrac{V'}{R_3}=\dfrac{V-R_1\left(V/R_{\text{tot}}\right)}{R_3}$, $R_{\text{tot}}=R_1+\dfrac{R_2R_3}{R_2+R_3}$ and $\tau=\dfrac{L}{R'}$. The current for an *RL* circuit is $i(t)=i_{\text{initial}}\left(e^{-t/\tau}\right)$. Immediately after opening the switch, $t\approx 0$ and $i(t_0)=i_{\text{initial}}\left(e^0\right)=i_{\text{initial}}$.

SIMPLIFY:

(a) $V+i_{R_1}R_1+i_{R_2}R_2+0=0 \Rightarrow i_{R_2}=i_{R_1}=\dfrac{V}{R_1+R_2},\ i_{R_3}=0$

(b) $i_{R_1}=\dfrac{V}{R_{\text{tot}}}=\dfrac{V}{R_1+\dfrac{R_2R_3}{R_2+R_3}}=\dfrac{V(R_2+R_3)}{R_1R_2+R_1R_3+R_2R_3}$

$$i_{R_2}=\frac{V'}{R_2}=\frac{V}{R_2}-\frac{R_1}{R_2}\left(\frac{V(R_2+R_3)}{R_1R_2+R_1R_3+R_2R_3}\right)=\frac{V(R_1R_2+R_1R_3+R_2R_3)-V(R_1R_2+R_1R_3)}{R_2(R_1R_2+R_1R_2+R_2R_3)}=\frac{V(R_3)}{(R_1R_2+R_1R_2+R_2R_3)}$$

$$i_{R_3}=\frac{V'}{R_3}=\frac{V}{R_3}-\frac{R_1}{R_3}\left(\frac{V(R_2+R_3)}{R_1R_2+R_1R_3+R_2R_3}\right)=\frac{V(R_1R_2+R_1R_3+R_2R_3)-V(R_1R_2+R_1R_3)}{R_3(R_1R_2+R_1R_2+R_2R_3)}=\frac{V(R_2)}{R_1R_2+R_1R_2+R_2R_3}$$

(c) $i(t)=i_{\text{initial}}\left(e^{-t/\tau}\right)$, where $i_{R_1}=0,\ i_{R_3}=-i_{R_2}=i(t), i_{\text{initial}}=\dfrac{V-R_1\left(V/R_{\text{tot}}\right)}{R_3}=\dfrac{V(R_2)}{R_1R_2+R_1R_3+R_2R_3}$.

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE CHECK: When the switch is closed and the current is increasing according to $i(t)=\dfrac{V_{\text{emf}}}{R}\left(1-e^{-t/\tau}\right)$, as $t\to\infty$, $i(t)=\dfrac{V_{\text{emf}}}{R}$, which agrees with the result.

29.55. The energy density is given by $u_B=\dfrac{B^2}{2\mu_0}$. Determine the volume that gives $Vu_B=1$ J:

$$V=\frac{2\mu_0(1\text{ J})}{B^2}=\frac{2\left(4\pi\cdot 10^{-7}\text{ H}\cdot\text{m}^{-1}\right)(1\text{ J})}{\left(5.0\cdot 10^{-5}\text{ T}\right)^2}=1.01\cdot 10^3\text{ m}^3.$$

This volume is equivalent to a 10 m by 10 m by 10 m cube. This is a fraction of the size of a house.

29.56. (a) The magnetic energy density is given by: $u_B=\dfrac{1}{2\mu_0}B^2=\dfrac{1}{2\left(4\pi\cdot 10^{-7}\text{ H}\cdot\text{m}^{-1}\right)}(3.00\text{ T})^2=3.58\cdot 10^6\text{ J/m}^3$.

(b) The total energy is given by $U_B=Vu_B$. $V=\pi R^2L$, $R=0.500$ m, $L=1.50$ m.

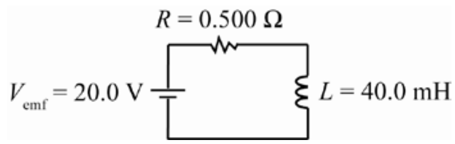
$$\Rightarrow U_B=\pi(0.500\text{ m})^2(1.50\text{ m})\left(3.58\cdot 10^6\text{ J/m}^3\right)=4.22\cdot 10^6\text{ J}$$

29.57. (a) $u_B=\dfrac{1}{2\mu_0}B^2=\dfrac{1}{2\left(4\pi\cdot 10^{-7}\text{ H}\cdot\text{m}^{-1}\right)}\left(4.00\cdot 10^{10}\text{ T}\right)^2=6.366\cdot 10^{26}\text{ J/m}^3$

(b) The associated mass density is then: $\dfrac{u_B}{c^2}=\rho_{\text{rest}}=\dfrac{6.366\cdot 10^{26}\text{ J/m}^3}{(3.00\cdot 10^8\text{ m/s}^2)^2}=7.07\cdot 10^9\text{ kg/m}^3$

29.58. **THINK:** The emf potential and the resistance can be used to find the maximum current. Then the energy stored in the magnetic field of the inductor at one fourth of this current can be found. The equation for the rise in current as a function of time can be used to find the time for the circuit to reach a current of one fourth of its maximum value. The inductance of the inductor is $L=40.0$ mH, the resistance of the resistor is $R=0.500\ \Omega$, and the emf potential is $V_{\text{emf}}=20.0$ V.

SKETCH:



RESEARCH:
(a) At steady-state,

$$i_{\text{max}} = \frac{V_{\text{emf}}}{R}.$$

The time of interest is when $i = i_{\text{max}}/4$. Use the equation $U_{\text{B}} = \frac{1}{2}Li^2$.

(b) $i(t) = i_{\text{max}}\left(1 - e^{-t/\tau_{\text{RL}}}\right) = \frac{1}{4}i_{\text{max}}, \quad \tau_{\text{RL}} = \frac{L}{R}$

SIMPLIFY:

(a) $U_{\text{B}} = \frac{1}{2}L\left(\frac{1}{4}\frac{V_{\text{emf}}}{R}\right)^2 = \left(\frac{1}{32}\right)\frac{LV_{\text{emf}}^{\ 2}}{R^2}$

(b) $\frac{1}{4}i_{\text{max}} = i_{\text{max}}\left(1 - e^{-t/\tau_{\text{RL}}}\right) \Rightarrow \ln\left(\frac{3}{4}\right) = -\frac{t}{\tau_{\text{RL}}} \Rightarrow t = -\tau_{\text{RL}}\ln\left(\frac{3}{4}\right) = -\frac{L}{R}\ln\left(\frac{3}{4}\right)$

CALCULATE:

(a) $U = \left(\frac{1}{32}\right)\frac{(0.0400 \text{ H})(20.0 \text{ V})^2}{(0.500\ \Omega)^2} = 2.00 \text{ J}$

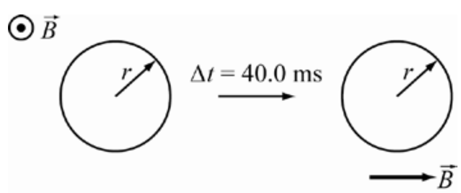
(b) $t = -\left(\frac{0.0400 \text{ H}}{0.500\ \Omega}\right)\ln\left(\frac{3}{4}\right) = 0.0230 \text{ s}$

ROUND:
(a) $U = 2.00$ J
(b) $t = 0.0230$ s

DOUBLE-CHECK: It makes sense that the time it takes to reach one fourth of the maximum value is comparable to the time constant, τ_{RL}.

29.59. **THINK:** The equation for the rate of energy production due to a potential across a resistance can be used to determine the heat generated. The induced potential can be found by using Faraday's Law. Then the rise in temperature due to this heat can be found for the ring of mass $m = 0.0150$ kg and specific heat capacity $c = 129$ J/kg °C. The strength of the magnetic field is $B = 0.0800$ T, the radius of the ring is $r = 0.00750$ m, the time change between maximum and zero magnetic flux is $\Delta t = 0.0400$ s, and the resistance of the ring is $R = 61.9 \cdot 10^{-6}\ \Omega$.

SKETCH:



RESEARCH: The induced potential in the ring is given by: $\Delta V_{\text{ind}} = -\frac{d\Phi}{dt} = -\frac{BA}{\Delta t}$. The rate of energy production as heat is given by

$$P = \frac{\left(\Delta V_{\text{ind}}\right)^2}{R}.$$

The power produced multiplied by the time difference is equal to the heat generated:

$$P\Delta t = Q = mc\Delta T.$$

SIMPLIFY: The temperature rise is

$$\Delta T = \frac{\left(\Delta V_{\text{ind}}\right)^2 \Delta t}{mcR} = \frac{\Delta t}{mcR}\left(-\frac{BA}{\Delta t}\right)^2 = \frac{\left(\pi B r^2\right)^2}{mcR\Delta t}$$

CALCULATE: $\Delta T = \frac{\left(\pi(0.0800\text{ T})(0.00750\text{ m})^2\right)^2}{(0.0150\text{ kg})(129\text{ J/kg }^\circ\text{C})(61.9\cdot 10^{-6}\ \Omega)(0.0400\text{ s})} = 4.1715\cdot 10^{-5}\ ^\circ\text{C}$

ROUND: To three significant figures, the temperature rise is $\Delta T = 4.17\cdot 10^{-5}\ ^\circ\text{C}$.
DOUBLE-CHECK: It makes sense that for larger fields, ΔT is larger, and for larger masses, ΔT is smaller since it would take more work to heat up the ring. As expected, the temperature increase is quite small.

29.60. **THINK:** Consider the energy of the dipole before and after the flip and relate this to the work done.
SKETCH: A sketch is not necessary.
RESEARCH: When the dipole is in alignment: $U = -NiAB$. When the dipole is anti-parallel to the field: $U = NiAB$.
SIMPLIFY: The work done must therefore be $W = \Delta U = 2NiAB$.
CALCULATE: No calculations are necessary.
ROUND: Rounding is not necessary.
DOUBLE-CHECK: Larger fluxes (larger *NAB*) yield more work for the power supply.

29.61. **THINK:** Determine the energy density of the electric field and the magnetic field separately.
SKETCH: A sketch is not necessary.

RESEARCH: $u_{\text{B}} = \frac{1}{2\mu_0}\left|B^2\right|$, $u_{\text{E}} = \frac{1}{2}\varepsilon_0\left|E^2\right|$, $\left|\vec{B}_0\right| = \left|\frac{k\times\vec{E}_0}{\omega}\right|$, $\omega = \frac{\left|\vec{k}\right|}{\sqrt{\mu_0\varepsilon_0}}$, $\vec{E}(\vec{x},t) = \vec{E}_0\cos\left(\vec{k}\bullet\vec{x}-\omega t\right)$, $\vec{B}(\vec{x},t) = \vec{B}_0\cos\left(\vec{k}\bullet\vec{x}-\omega t\right)$.

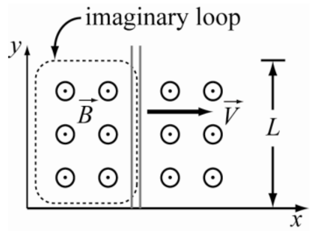
SIMPLIFY:

$$\frac{u_{\text{B}}}{u_{\text{E}}} = \frac{\left|B^2\right|}{2\mu_0}\left(\frac{1}{2}\varepsilon_0\left|E^2\right|\right)^{-1} = \frac{1}{\mu_0\varepsilon_0}\frac{\left|B^2\right|}{\left|E^2\right|} = \frac{1}{\mu_0\varepsilon_0}\left[\frac{\left|\vec{B}_0\right|^2\cos^2\left(\vec{k}\bullet\vec{x}-\omega t\right)}{\left|\vec{E}_0\right|^2\cos^2\left(\vec{k}\bullet\vec{x}-\omega t\right)}\right] = \frac{1}{\mu_0\varepsilon_0}\left(\frac{1}{\omega^2}\right)\frac{\left|k\times\vec{E}_0\right|^2}{\left|\vec{E}_0\right|^2} = \frac{\left|k\times\vec{E}_0\right|^2}{\left|\vec{k}\right|^2\left|\vec{E}_0\right|^2}$$

Note that $\vec{k}$ is perpendicular to $\vec{E}_0$ so $\left|\vec{k}\times\vec{E}_0\right|^2 = \left|\vec{k}\right|^2\left|\vec{E}_0\right|^2$, so the above expression becomes $\frac{u_{\text{B}}}{u_{\text{E}}} = 1$.

CALCULATE: No calculations are necessary.
ROUND: Rounding is not necessary.
DOUBLE-CHECK: This result shows that the energy in this type of wave is partitioned equally between the electric and magnetic fields.

29.62.



The induced voltage is given by:

$$\left|\Delta V_{\text{ind}}\right| = \left|-\frac{d\Phi}{dt}\right| = vLB = 2.00\text{ V} \Rightarrow v = \frac{\Delta V}{BL} = \frac{2.00\text{ V}}{(0.100\text{ m})(1.00\text{ T})} = 20.0\text{ m/s}.$$

29.63. The potential difference is given by Faraday's law:

$$\left|\Delta V_{\text{ind}}\right| = \left|-\frac{d\Phi}{dt}\right| = A\frac{dB}{dt} = \pi R^2\frac{dB}{dt} = \pi(0.0400\text{ m})^2(1.50\text{ T/s}) = 7.54\cdot 10^{-3}\text{ V}$$

Note that the radius of the coil is irrelevant.

29.64. The inductor cannot have the current jump instantaneously. From Kirchoff's loop law:

$$V_{\text{emf}} - L\frac{di}{dt} = 0 \Rightarrow \frac{V_{\text{emf}}}{L} = \frac{di}{dt}.$$

$$\frac{V_{\text{emf}}}{L} = \frac{di}{dt} \Rightarrow di = \frac{V_{\text{emf}}}{L}dt \Rightarrow \int_0^i di = \int_0^t \frac{V_{\text{emf}}}{L}dt \Rightarrow i(t) = \frac{V_{\text{emf}}}{L}t + C$$

Since $i(0) = 0$, $C = 0$. The expression is then $i(t) = \frac{V_{\text{emf}}}{L}t$.

29.65. The energy stored in a solenoid is given by $U_{\text{B}} = Li^2/2$. The energy is dependent only on the magnitude, not the direction of the current. Therefore the energy stored in the magnetic field does not change.

29.66. Use the formulas: $u_{\text{B}} = \frac{1}{2\mu_0}B^2$, $u_{\text{E}} = \frac{1}{2}\varepsilon_0 E^2$ and $\frac{u_{\text{B}}}{u_{\text{E}}} = \frac{1}{\mu_0\varepsilon_0}\frac{B^2}{E^2}$. In particular, the values of the energy densities are:

$$u_{\text{B}} = \frac{1}{2\left(1.257\cdot 10^{-6}\ \frac{\text{m kg}}{\text{s}^2\ \text{A}^2}\right)}(50.0\ \mu\text{T})^2 = 9.94\cdot 10^{-4}\text{ J/m}^3,$$

and

$$u_{\text{E}} = \frac{1}{2}\left(8.842\cdot 10^{-12}\ \frac{\text{s}^4\ \text{A}^2}{\text{m}^3\ \text{kg}}\right)(150.\text{ N/C})^2 = 9.95\cdot 10^{-8}\text{ J/m}^3.$$

To compute the ratios, it is useful to remember that $1/\mu_0\varepsilon_0 = c^2$. This is a result from light being an electromagnetic wave where *c* is the speed of light in a vacuum.

$$\frac{u_{\text{B}}}{u_{\text{E}}} = c^2\frac{B^2}{E^2} = (3.00\cdot 10^8\text{ m/s})^2\left(\frac{50.0\cdot 10^{-6}\text{ T}}{150.\text{ N/C}}\right)^2 = 1.00\cdot 10^5$$

The energy density of the magnetic field is much larger than that of the electric field.

29.67. The current of an RL circuit is given by: $i(t) = \frac{V_{\text{emf}}}{R}\left(1 - e^{-t/\tau}\right)$, where $\tau = L/R$. For $t = 20.0\ \mu\text{s}$:

$$\frac{1}{2}\left(\frac{V_{\text{emf}}}{R}\right) = \frac{V_{\text{emf}}}{R}\left(1 - e^{-t/\tau}\right) \Rightarrow 1 - \frac{1}{2} = e^{-t/\tau} \Rightarrow \tau\ln\left(\frac{1}{2}\right) = -t \Rightarrow \frac{L}{R}\ln\left(\frac{1}{2}\right) = -t \Rightarrow L = -\frac{Rt}{\ln(1/2)}$$

$$\Rightarrow L = -\frac{(3.00\cdot 10^3\ \Omega)(20.0\cdot 10^{-6}\ \text{s})}{\ln(1/2)} = 0.0866\ \text{H}.$$

29.68. The current of an RL circuit is given by $i(t) = i_{\text{max}}\left(1-e^{-t/\tau}\right)$, where $\tau = L/R$.

$$0.995 i_{\text{max}} = i_{\text{max}}\left(1-e^{-t/\tau}\right) \Rightarrow e^{-t/\tau} = 0.00500 \Rightarrow$$

$$t = -\frac{L}{R}\ln(0.00500) = -\frac{0.200\cdot 10^{-6}\ \text{H}}{500.\ \Omega}\ln(0.00500) = 2.12\ \text{ns}$$

It is interesting to note that the voltage of the battery is irrelevant to the result of the problem.

29.69. For a single loop of wire (N = 1), the induced potential difference is:

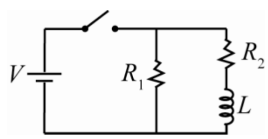
$$\Delta V_{\text{ind}} = -\frac{d\Phi_{\text{B}}}{dt} = -\frac{d}{dt}(BA\cos\theta).$$

Since the normal vector of the loop and the magnetic field is parallel, $\cos\theta = 1$. The negative sign can be dropped and ΔV_{ind} becomes:

$$\Delta V_{\text{ind}} = A\frac{dB}{dt} = A\frac{d}{dt}(3.00\ \text{T} + 2.00t\ \text{T/s}) = A(2.00\ \text{T/s}) = (5.00\ \text{m}^2)(2.00\ \text{T/s}) = 10.0\ \text{V}.$$

Note the magnetic field, $\vec{B}$, is increasing, and it is directed into the page. By Lenz's law, the induced magnetic field, $\vec{B}_{\text{i}}$, opposes the change in magnetic flux, Φ_{B}. In this case, $\vec{B}_{\text{i}}$ is directed out of the page to oppose the increasing field, $\vec{B}$, directed into the page. The induced current is therefore counterclockwise.

29.70. The following circuit has values: V = 9.00 V, $R_1 = R_2 = 100.\ \Omega$, and L = 3.00 H.



(a) When the switch is closed at t = 0 s, the current through R_1 is: $i_1 = \frac{V}{R_1} = \frac{9.00\ \text{V}}{100.\ \Omega} = 0.0900\ \text{A}$. The current through R_2 is $i_2(t) = \frac{V}{R_2}\left[1-e^{-t/(l/R)}\right] \Rightarrow i_2(0) = 0.$

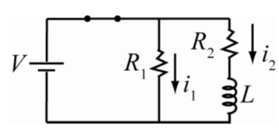
(b) At t = 50.0 ms = 0.0500 s, i_1 is still 0.0900 A, while i_2 is:

$$i_2(0.0500\ \text{s}) = \frac{9.00\ \text{V}}{100.\ \Omega}\left[1-\exp\left\{(-0.0500\ \text{s})\left(\frac{100.\ \Omega}{3.00\ \text{H}}\right)\right\}\right] = 0.0900(1-0.189)\ \text{A} = 0.0730\ \text{A}$$

(c) At t = 500. ms = 0.500 s, i_1 is still 0.0900 A, and i_2 is:

$$i_2(0.500\ \text{s}) = \frac{9.00\ \text{V}}{100.\ \Omega}\left[1-\exp\left\{(-0.500\ \text{s})\left(\frac{100.\ \Omega}{3.00\ \text{H}}\right)\right\}\right] = 0.0900\left(1-5.78\cdot 10^{-8}\right)\ \text{A} = 0.0900\ \text{A}.$$

(d) After 10.0 s, the equilibrium current of 0.0900 A has long since been reached. Before the switch is opened, the currents i_1 and i_2 oppose each other in the right-most loop, as shown below.



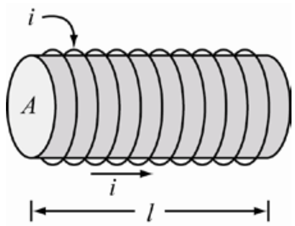
When the switch is opened (after achieving an equilibrium current in the circuit), $i_1 = -i_2$. After opening the switch, Kirchhoff's loop rule becomes $Ldi/dt + iR_1 + iR_2 = 0$. With $R_1 = R_2 = R$, this expression becomes $Ldi/dt + i(2R) = 0.$ Solving for i yields: $i(t) = i_0 e^{-t/\tau_L}$, $\tau_L = L/2R$, and i_0 is the achieved equilibrium current, i = 0.0900 A. At t = 0 s, $-i_1(0) = i_2(0) = i_0 e^{-0/\tau_L} = i_0 = 0.0900$ A.

(e) At t = 50.0 ms = 0.0500 s, $i_1(0.0500\text{ s}) = -i_2(0.0500\text{ s}) = (0.0900\text{ A})e^{-2(100.\ \Omega)(0.0500\text{ s})/3.00\text{ H}} = 0.00321$ A.

(f) At t = 500. ms = 0.500 s, $i_1(0.500\text{ s}) = -i_2(0.500\text{ s}) = (0.0900\text{ A})e^{-2(100.\ \Omega)(0.500\text{ s})/3.00\text{ H}} \approx 0$ A.

29.71. **THINK:** A solenoid of length, l = 3.0 m, and n = 290 turns/m has a current of i = 3.0 A, and stores an energy of $U_B = 2.8$ J. Find the cross-sectional area, A, of the solenoid.

SKETCH:



RESEARCH: The energy stored in the magnetic field of an ideal solenoid is $U_B = \mu_0 n^2 l A i^2 / 2.$

SIMPLIFY: Solving for A yields: $A = \dfrac{2U_B}{\mu_0 n^2 l i^2}.$

CALCULATE:

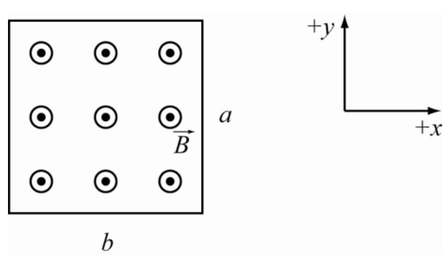
$$A = \frac{2(2.80\text{ J})}{(4\pi\cdot 10^{-7}\text{ T m/A})(290\text{ m}^{-1})^2(3.00\text{ m})(3.00\text{ A})^2} = 1.9625\text{ J/T A} = 1.9625\ \frac{\text{N m}}{(\text{V s/m}^2)(\text{J/V s})} = 1.9625\text{ m}^2$$

ROUND: Rounding to three significant figures, $A = 1.96\text{ m}^2$.

DOUBLE-CHECK: Considering the length, l, of the solenoid, this is a reasonable cross-sectional area. The units of the result are also correct.

29.72. **THINK:** The rectangular loop has dimensions a by b and resistance R. It is placed on the xy-plane. The magnetic field direction points out of the page and varies in time according to $B = B_0(l + c_1 t^3)$. Determine the direction of the current induced in the loop, i_{ind}, and its value at t = 1 s (in terms of a, b, R, B_0 and c_1).

SKETCH:



RESEARCH: Since the magnetic field is increasing as it comes out of the page, the induced magnetic field, B_i, points into the page according to Lenz's law. The induced current flows clockwise. The current is found from $V_{\text{ind}} = i_{\text{ind}} R$, where $V_{\text{ind}} = -d\Phi_B/dt = -dBA\cos\theta/dt$.

SIMPLIFY: With $\cos\theta = \cos(0°) = 1$, and A constant:

$$V_{\text{ind}} = -A\frac{dB}{dt} = -A\frac{d}{dt}\left[B_0(l + c_1 t^3)\right] = -3AB_0 c_1 t^2.\ \text{Then, } i_{\text{ind}} = \frac{|V_{\text{ind}}|}{R} = \frac{3AB_0 c_1 t^2}{R} = \frac{3abB_0 c_1 t^2}{R}.$$

CALCULATE: At $t = 1$ s, $i_{\text{ind}} = \dfrac{3abB_0c_1}{R}$, clockwise

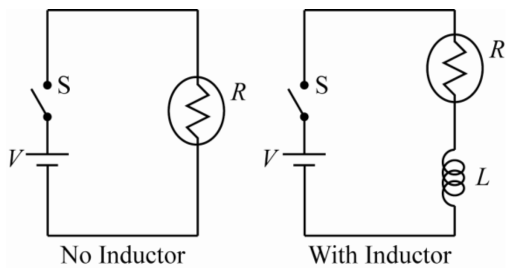
ROUND: Not applicable.

DOUBLE-CHECK: By dimensional analysis, the result has units of current:

$$\left[\frac{abB_0c_1t^2}{R}\right]=\left[\frac{\text{m}^2\ \text{T s}^2}{\text{s}^3\ \Omega}\right]=\left[\frac{\text{m}^2\ \text{V s/m}^2\ \text{s}^2}{\text{s}^3\ \text{V/A}}\right]=\left[\text{A}\right].$$

29.73. **THINK:** The battery with $V_{\text{emf}} = 12.0$ V, is connected in series with a switch and a light-bulb. When the light-bulb draws a current of $i = 0.100$ A, its glow becomes visible. This bulb draws $P = 2.00$ W when it has been connected and when the switch has been closed for a long time. When an inductor is put in series with the bulb and the rest of the circuit, the light-bulb begins to glow $t = 3.50$ ms after the switch is closed. Find the size of the inductor, L.

SKETCH:



RESEARCH: The resistance of the light-bulb can be determined from $P = V^2/R$. When the inductor is attached, the current is given by $i(t) = i_0\left(1-\text{e}^{-tR/L}\right) = \dfrac{V_{\text{emf}}}{R}\left(1-\text{e}^{-tR/L}\right)$.

SIMPLIFY: $R = \dfrac{V^2}{P} = \dfrac{V_{\text{emf}}^2}{P}$. Substitute this expression into the equation for the current to get:

$$i(t)=\frac{V_{\text{emf}}}{R}\left(1-\text{e}^{-tR/L}\right)=\frac{V_{\text{emf}}}{V_{\text{emf}}^2/P}\left(1-\text{e}^{-tR/L}\right)=\frac{P}{V_{\text{emf}}}\left(1-\text{e}^{-tR/L}\right)\ \Rightarrow\ \text{e}^{-tR/L}=1-\frac{V_{\text{emf}}i(t)}{P}$$

$$\Rightarrow\ -tR/L=\ln\left[1-\frac{V_{\text{emf}}i(t)}{P}\right]\ \Rightarrow\ L=-\frac{tR}{\ln\left[1-V_{\text{emf}}i(t)/P\right]}.$$

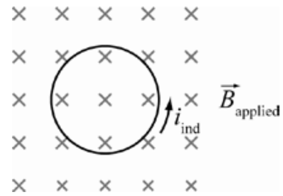
CALCULATE: $R = \dfrac{(12.0\text{ V})^2}{2.00\text{ W}} = 72.0\ \Omega,\quad L = -\dfrac{(0.00350\text{ s})(72.0\ \Omega)}{\ln\left\{1-\left[(12.0\text{ V})(0.100\text{ A})/2.00\text{ W}\right]\right\}} = 0.27502\text{ H}$

ROUND: $L = 0.275$ H.

DOUBLE-CHECK: An inductor of this capacity in this circuit is capable of storing energy $U_{\text{B}} = \frac{1}{2}Li^2 = \frac{1}{2}(0.300\text{ H})(0.100\text{ A})^2 = 1.50$ mJ. This is sufficient energy to power a 2.00 W light bulb for 0.750 ms. This is a reasonable value for L in this light-bulb circuit.

29.74. **THINK:** A circular loop of cross-section A is placed perpendicular to a time-varying magnetic field of $B(t) = B_0 + at + bt^2$, where B_0, a, and b are constants. For purposes of making a sketch, view the loop so that the field points into the plane of the page. Determine (a) the magnetic flux, Φ_{B}, through the loop at $t = 0$, (b) an equation for the induced potential difference, V_{ind}, in the loop as a function of time, and (c) the magnitude and direction of the induced current if the resistance of the loop is R.

SKETCH:



RESEARCH:

(a) Since the loop is perpendicular to the field, the magnetic flux is given by $\Phi_B = BA$.

(b) From Faraday's law, $V_{ind} = -d\Phi_B / dt$. Since A is constant while B varies with time, this expression becomes $V_{ind} = -A(dB/dt)$.

(c) The magnitude of the induced current is found from $V = iR$. With the applied magnetic field directed into the page and increasing in time, the induced magnetic field will point out of the page to oppose the change in magnetic flux. The induced current flows counterclockwise.

SIMPLIFY:

(a) $\Phi_B(t) = BA = (B_0 + at + bt^2)A$. At $t = 0$, $\Phi_B(0) = B_0 A$.

(b) $V_{ind}(t) = -A\dfrac{d}{dt}(B_0 + at + bt^2) = -A(a + 2bt)$

(c) The magnitude of i_{ind} is given by: $i_{ind} = \left|\dfrac{V_{ind}}{R}\right| = \dfrac{A(a+2bt)}{R}$, counterclockwise.

CALCULATE: Not applicable.

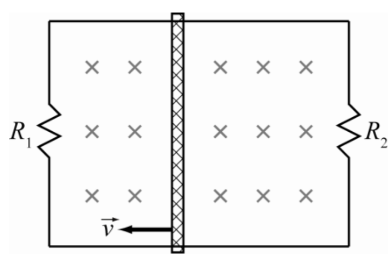
ROUND: Not applicable.

DOUBLE-CHECK: By dimensional analysis, the units are correct:

$[\Phi] = [BA] \Rightarrow \text{Wb} = \text{T m}^2$; $[V] = [AB/t] \Rightarrow \text{V} = \text{m}^2 \text{ T/s}$; $[i] = [V/R] \Rightarrow \Omega$ $[.\text{A}] = [\text{V}/\ ]$

29.75. **THINK:** A conducting rod of length, L = 0.500 m, slides over a frame of two metal bars placed in a magnetic field of strength, B = 1000. gauss = 0.1000 T. The ends of the rods are connected by two resistors, $R_1 = 100.\ \Omega$ and $R_2 = 200.\ \Omega$. The conducting rod moves with a constant velocity of v = 8.00 m/s. Determine (a) the current flowing through the two resistors, i_1 and i_2, (b) the power, P, delivered to the resistors, and (c) the force, F, needed for the motion of the rod with constant velocity.

SKETCH:



RESEARCH:

(a) The induced potential difference across the resistors is $V_{ind} = -d\Phi_B / dt$. Since B is constant while A varies in time at a velocity of v, this expression becomes $V_{ind} = -B(dA/dt) = -BLv$. The current in each resistor can be determined from $V_{ind} = i_{ind}R$.

(b) The power delivered to the resistors is $P = i_1^2 R_1 + i_2^2 R_2$.

(c) The force needed to move the rod with a constant velocity is obtained by calculating the total force acting on the rod. The magnetic force on the rod, F_{mag}, is given by $F_{mag} = BiL = B(V/R_{eq})L$, where R_{eq} is the equivalent resistance.

Note for n resistors in parallel, the equivalent resistance is:

$$\frac{1}{R_{\text{eq}}}=\frac{1}{R_1}+\frac{1}{R_2}+\ldots+\frac{1}{R_n}.$$

SIMPLIFY:

(a) $V_{\text{ind}}=-BLv,\quad i_1=\frac{|V_{\text{ind}}|}{R_1},\quad i_2=\frac{|V_{\text{ind}}|}{R_2}$

(b) $P=i_1^2R_1+i_2^2R_2$

(c) $F_{\text{mag}}=BV_{\text{ind}}L\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=B^2L^2v\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$

CALCULATE:

(a) $V_{\text{ind}}=-(0.100\text{ T})(0.500\text{ m})(8.00\text{ m/s})=-0.400\text{ V},\quad i_1=\frac{|-0.400\text{ V}|}{100.\ \Omega}=0.00400\text{ A},$

$i_2=\frac{|-0.400\text{ V}|}{200.\ \Omega}=0.00200\text{ A}$

(b) $P=(0.00400\text{ A})^2(100.\ \Omega)+(0.00200\text{ A})^2(200.\ \Omega)=0.00240\text{ W}$

(c) $F_{\text{mag}}=(0.100\text{ T})^2(0.500\text{ m})^2(8.00\text{ m/s})\left(\frac{1}{100.\ \Omega}+\frac{1}{200.\ \Omega}\right)=0.000300\text{ N}$

ROUND:

(a) $i_1=4.00\text{ mA}$, $i_2=2.00\text{ mA}$

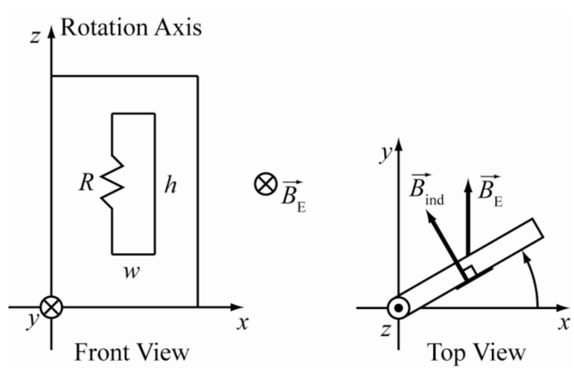
(b) $P=2.40\text{ mW}$

(c) $F_{\text{mag}}=0.300\text{ mN}$

DOUBLE-CHECK: The calculated values are consistent with the given values. Dimensional analysis confirms all the units are correct.

29.76. **THINK:** The loop on the door has dimensions h = 0.150 m, w = 0.0800 m and resistance, $R=5.00\ \Omega$. When the door is closed, it is perpendicular $(\theta=0°)$ to the Earth's uniform magnetic field, $B_{\text{E}}=2.6\cdot10^{-5}$ T. At time, t = 0 s, the door is opened (right edge moving into the page in the figure below) at a constant rate, with an opening angle of $\theta(t)=\omega t$, where $\omega=3.5$ rad/s. Determine the direction and magnitude of the induced current, $i(t)$, at t = 0.200 s.

SKETCH:



RESEARCH: The induced current, i, is found from $i=V_{\text{ind}}/R$, where V_{ind} is given by $V_{\text{ind}}=-d\Phi_{\text{B}}/dt$, and $\Phi_{\text{B}}=BA\cos\theta$. As the door opens, the B field through the loop decreases; by Lenz's law the induced B field points into the page, at an angle of $\theta(t)$ from the plane of the page. The induced current flows clockwise.

SIMPLIFY: $\Phi_B = BA\cos(\theta(t)) = whB_E\cos\omega t, \quad V_{ind} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(whB_E\cos\omega t) = whB_E\omega\sin\omega t$

The magnitude of i is $i(t) = \frac{|V_{ind}|}{R} = \frac{whB_E\omega\sin\omega t}{R}.$

CALCULATE: At t = 0.200 s:

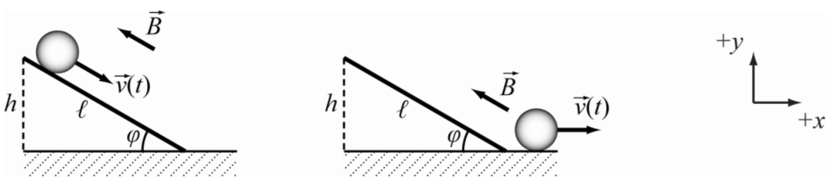
$$i(0.200\text{ s}) = \frac{(0.0800\text{ m})(0.150\text{ m})(2.6\cdot 10^{-5}\text{ T})(3.5\text{ rad/s})\sin[(3.5\text{ rad/s})(0.200\text{ s})]}{5.00\ \Omega} = 1.407\cdot 10^{-7}\text{ A.}$$

ROUND: Rounding to two significant figures, $i(0.200\text{ s}) = 140$ nA clockwise.

DOUBLE-CHECK: This induced current is reasonable for a loop with such a small cross-sectional area in the Earth's magnetic field.

29.77. **THINK:** The steel cylinder has radius r = 2.5 cm = 0.025 m and length L = 10.0 cm = 0.100 m. The ramp is inclined at $\phi = 15.0°$ and has a length l = 3.00 m. Determine the induced potential difference, V_{ind} between the ends at the bottom of the ramp if the ramp points in the direction of magnetic North. Use $0.426\cdot 10^{-4}$ T as the magnetic field of the Earth.

SKETCH:



RESEARCH: The magnetic field of the Earth lies at an angle to the surface of the Earth (i.e. it is usually not parallel or perpendicular to the Earth's surface). Generally, as a charge q moves with velocity through a magnetic field, the magnetic force acting on the electrons in the conductor is $F_{mag} = q|\vec{v}\times\vec{B}| = qvB\sin\theta,$ where θ is the angle between the velocity $\vec{v}$ of the charge and the magnetic field vector $\vec{B}$. When the electric force, $F_E = qE$, and the magnetic force on the electrons are in equilibrium, then $E = vB\sin\theta$. This means that the induced potential difference, V_{ind}, between the ends of the conductor is given by $V_{ind} = EL = vBL\sin\theta$. As the cylinder rolls down the ramp (in the direction of the Earth's magnetic field, B), the angle between the cylinder's velocity vector and the Earth's magnetic field vector is zero, so the induced voltage between the ends of the cylinder is zero. At the bottom of the ramp, the cylinder changes direction, and the induced potential difference between the ends is $V_{ind} = vBL\sin\theta$. To determine the speed, v, of the cylinder, recall that the cylinder rolls without slipping so the change in potential energy for the cylinder is equal in magnitude to the change in its kinetic energy: $\Delta K = -\Delta U \Rightarrow mv^2/2 + I\omega^2/2 = mgh$. Here I is the moment of inertia for the cylinder: $I = mr^2/2$, and $\omega = v/r$.

SIMPLIFY: Determine v:

$$\Delta K = -\Delta U \Rightarrow \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh \Rightarrow \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\frac{v^2}{r^2}$$
$$\Rightarrow \frac{1}{2}v^2 + \frac{1}{4}v^2 = gh \Rightarrow v = \sqrt{\frac{4}{3}gh} = \sqrt{\frac{4}{3}gl\sin\varphi}$$

At the bottom of the ramp, the angle between $\vec{B}$ and $\vec{v}$ is $\theta = \varphi$.

$$V_{ind} = vBL\sin\theta = \sqrt{\frac{4}{3}gl\sin\varphi}\,[BL\sin(\varphi)]$$

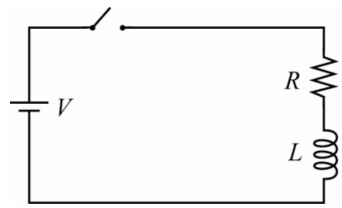
CALCULATE: $V_{ind} = \sqrt{\frac{4}{3}(9.81\text{ m/s}^2)(3.00\text{ m})\sin 15.0°}\,\left[(0.426\cdot 10^{-4}\text{ T})(0.100\text{ m})\sin(15°)\right] = 3.514\cdot 10^{-6}$ V

ROUND: $V_{\text{ind}} = 3.51\ \mu\text{V}.$

DOUBLE-CHECK: Considering the given values for this problem, this result is reasonable and also has the correct units: $V_{\text{ind}} = \sqrt{[\text{m/s}^2][\text{m}]}[\text{T}][\text{m}] \Rightarrow \sqrt{\frac{[\text{m}][\text{m}]}{[\text{s}^2]}} \cdot \frac{[\text{V}][\text{s}][\text{m}]}{[\text{m}^2]} \Rightarrow [\text{V}].$

29.78. **THINK:** The battery is connected to a resistor and an inductor in series. Determine (a) the current, $i(t)$, across the circuit after the switch is closed, (b) the total energy, U, provided by the battery from time, $t = 0$ to $t = L/R$, (c) the total energy, U_{R}, dissipated from the resistor, R, for the same time period, and (d) discuss the conservation of energy in this circuit.

SKETCH:



RESEARCH:

(a) In this RL circuit, current at any given time, t, is given by equation 29.29 in the text, namely $i = i_0\left(1-e^{-t/\tau}\right)$, where $i_0 = V/R$, and the time constant is $\tau = L/R$.

(b) The power provided by the battery is $P = Vi$. In the given time period, the total energy provided by the battery is $U = \int Vi dt$.

(c) The power dissipated in the resistor is $P = i^2 R$. In the given time period, the total energy dissipated in the resistor is $U_{\text{R}} = \int i^2 R dt$.

(d) Any discrepancy between the energy provided by the battery and the energy dissipated in the resistor is due to the fact that there is energy stored in the inductor, $U_{\text{L}} = Li^2/2$.

SIMPLIFY:

(a) $i(t) = \frac{V}{R}\left(1-e^{-tR/L}\right)$

(b) $U = \int_{t=0}^{t=\tau} Vi(t)dt = \int_0^\tau \frac{V^2}{R}\left(1-e^{-tR/L}\right)dt = \frac{V^2}{R}[t]_0^\tau + \frac{V^2}{R}\left(\frac{L}{R}\right)\left[e^{-tR/L}\right]_0^\tau = \frac{V^2}{R}\tau + \frac{V^2 L}{R^2}e^{-\tau R/L} - \frac{V^2 L}{R^2} = \frac{V^2 L}{R^2}e^{-1}$

$= \frac{V^2 L}{R^2}(0.368)$

(c) $U_{\text{R}} = \int_{t=0}^{t=\tau} i^2 R dt = \int_0^\tau \frac{V^2}{R}\left(1-e^{-tR/L}\right)^2 dt = -\frac{V^2 L}{2R^2}\left[\left(e^{-tR/L}\right)^2\right]_0^\tau + \left[\frac{2V^2 L e^{-tR/L}}{R^2}\right]_0^\tau - \frac{V^2 L}{R^2}\left[\ln\left(e^{-tR/L}\right)\right]_0^\tau$

$= \frac{V^2 L}{R^2}\left(-\frac{e^{-2}}{2} + \frac{1}{2} + 2e^{-1} - 2 + 1 + 0\right) = 0.168\frac{V^2 L}{R^2}$

(d) At time, $t = \tau = L/R$: $U_{\text{L}} = \frac{1}{2}Li^2\tau = \frac{LV^2}{2R^2}\left(1-e^{-1}\right)^2 = (0.200)\frac{V^2 L}{R^2}.$

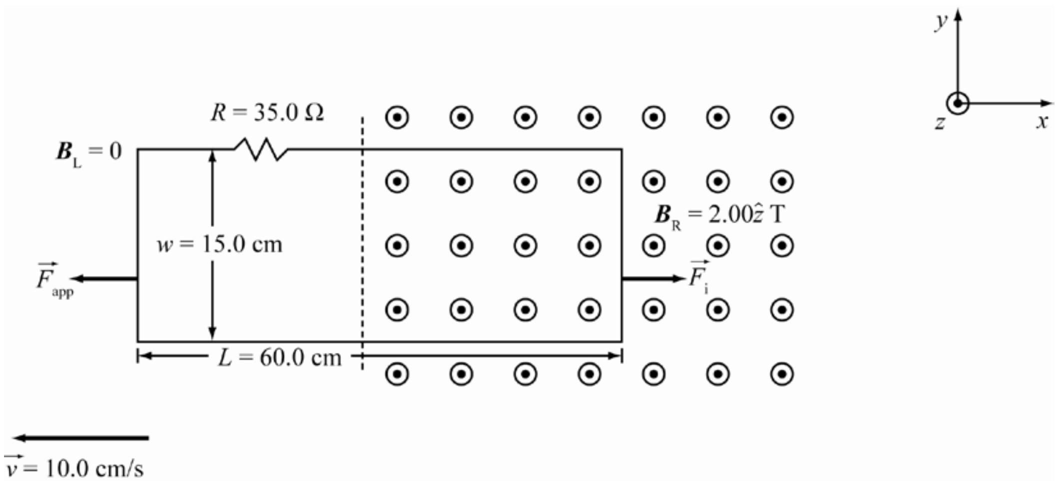
CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: The total energy of the battery is the sum of the energy dissipated in the resistor and the energy stored in the inductor; energy is conserved.

29.79. **THINK:** The rectangular circuit loop has length, $L = 0.600$ m, and width, $w = 0.150$ m, with resistance, $R = 35.0\ \Omega$. It is held parallel to the xy-plane with one end inside a uniform magnetic field as shown in the figure. The magnetic field is $\vec{B}_\text{R} = 2.00\hat{z}$ T along the positive z-axis to the right of the dotted line; $\vec{B}_\text{L} = 0$ T to the left of the dotted line. Determine the magnitude of the force, F_app, required to move the loop to the left at a constant speed of $v = 0.100$ m/s, while the right end of the loop is still in the magnetic field. Determine the power, P, used by an agent to pull the loop out of the magnetic field at this speed, and the power, P_R, dissipated by the resistor.

SKETCH:



RESEARCH: The magnitude of the force required to move the loop will be equal to the magnitude of the force, F_i, on the current induced in the segment of the loop that lies along the y-axis in the magnetic field. That is, $F_\text{app} = F_2 = iwB\sin\theta$. Since the angle, θ, between the loop segment of length, w, and the magnetic field is $90°$: $\sin\theta = 1$. The induced current, i, is $i = V_\text{ind}/R$, where $V_\text{ind} = vwB$ (see equation 29.15). The power, P, used by an agent to pull the loop out of the magnetic field is given by $P = F_\text{app}v$. The power dissipated by the resistor is given by $P_\text{R} = i^2R$.

SIMPLIFY: The current is $i = \dfrac{V_\text{ind}}{R} = \dfrac{vwB}{R}$.

(a) $F_\text{app} = iwB$

(b) $P = F_\text{app}v$

(c) $P_\text{R} = i^2R$

CALCULATE:

(a) $i = \dfrac{(0.100\text{ m/s})(0.150\text{ m})(2.00\text{ T})}{35.0\ \Omega} = 0.85714\text{ mA}$

$F_\text{app} = (0.85714\text{ mA})(0.150\text{ m})(2.00\text{ T}) = 0.25714\text{ mN}$

(b) $P = (0.25714\text{ m N})(0.100\text{ m/s}) = 25.714\ \mu\text{W}$

(c) $P_\text{R} = (0.85714\text{ mA})^2(35.0\ \Omega) = 25.714\ \mu\text{W}$

ROUND:

(a) $F_\text{app} = 0.257$ mN

(b) $P = 25.8\ \mu$W

(c) $P_\text{R} = 25.7\ \mu$W

DOUBLE-CHECK: All the power used to move the loop while in the magnetic field is dissipated in the resistor: $P = P_\text{R}$.

Multi-Version Exercises

29.80. $i(t) = i_\text{max}\left(1-e^{-t/\tau}\right), \;\; \tau = L/R$

$$\frac{3}{4}i_\text{max} = i_\text{max}\left(1-e^{-t/\tau}\right) \;\Rightarrow\; -\frac{t}{\tau} = -\frac{tR}{L} = \ln\left(\frac{1}{4}\right)$$

$$\Rightarrow\; R = -\frac{L}{t}\ln\left(\frac{1}{4}\right) = \frac{L}{t}\ln(4) = \frac{33.03\cdot 10^{-3}\text{ H}}{3.35\cdot 10^{-3}\text{ s}}\ln(4) = 13.7\ \Omega$$

29.81. $i(t) = i_\text{max}\left(1-e^{-t/\tau}\right), \;\; \tau = L/R$

$$\frac{3}{4}i_\text{max} = i_\text{max}\left(1-e^{-t/\tau}\right) \;\Rightarrow\; -\frac{t}{\tau} = -\frac{tR}{L} = \ln\left(\frac{1}{4}\right)$$

$$\Rightarrow\; L = -tR/\ln\left(\frac{1}{4}\right) = tR/\ln(4) = (3.45\cdot 10^{-3}\text{ s})(17.88\ \Omega)/\ln(4) = 44.5\text{ mH}$$

29.82. $i(t) = i_\text{max}\left(1-e^{-t/\tau}\right), \;\; \tau = L/R$

$$\frac{3}{4}i_\text{max} = i_\text{max}\left(1-e^{-t/\tau}\right) \;\Rightarrow\; -\frac{t}{\tau} = -\frac{tR}{L} = \ln\left(\frac{1}{4}\right)$$

$$\Rightarrow\; t = -\frac{L}{R}\ln\left(\frac{1}{4}\right) = \frac{L}{R}\ln(4) = \frac{55.93\cdot 10^{-3}\text{ H}}{21.84\ \Omega}\ln(4) = 3.55\text{ ms}$$

29.83. $\Phi_\text{B} = NAB\cos(2\pi ft)$, which means that $V_\text{ind} = -d\Phi_\text{B}/dt = NAB(2\pi f)\sin(2\pi ft)$.

The maximum occurs when $\sin(2\pi ft) = 1$. $N = 1$, so

$$V_\text{max} = AB(2\pi f) = (0.25\pi d^2)B(2\pi f) = 0.5\pi^2\left(0.0195\text{ m}\right)^2\left(4.77\cdot 10^{-5}\text{ T}\right)\left(13.3\text{ s}^{-1}\right) = 1.19\cdot 10^{-6}\text{ V.}$$

29.84. $\Phi_\text{B} = NAB\cos(2\pi ft)$, which means that $V_\text{ind} = -d\Phi_\text{B}/dt = NAB(2\pi f)\sin(2\pi ft)$.

The maximum occurs when $\sin(2\pi ft) = 1$. $N = 1$, so

$$V_\text{max} = AB(2\pi f) = (0.25\pi d^2)B(2\pi f)$$

$$\Rightarrow d = \sqrt{2V_\text{max}/Bf\pi^2} = \sqrt{2\left(1.446\cdot 10^{-6}\text{ V}\right)/\left(4.97\cdot 10^{-5}\text{ T}\right)\left(13.5\text{ s}^{-1}\right)\pi^2} = 2.09\text{ cm.}$$

29.85. $\Phi_\text{B} = NAB\cos(2\pi ft)$, which means that $V_\text{ind} = -d\Phi_\text{B}/dt = NAB(2\pi f)\sin(2\pi ft)$.

The maximum occurs when $\sin(2\pi ft) = 1$. $N = 1$, so

$$V_\text{max} = AB(2\pi f) = (0.25\pi d^2)B(2\pi f)$$

$$\Rightarrow B = 2V_\text{max}/(f\pi^2 d^2) = 2\left(6.556\cdot 10^{-7}\text{ V}\right)/\left((13.7\text{ s}^{-1})\pi^2(1.63\text{ cm})^2\right) = 3.65\cdot 10^{-5}\text{ T.}$$

Chapter 30: Alternating Current Circuits

Concept Checks

30.1. b **30.2.** b **30.3.** a **30.4.** a **30.5.** d **30.6.** b **30.7.** d **30.8.** c

Multiple-Choice Questions

30.1. d **30.2.** c **30.3.** b **30.4.** a **30.5.** d **30.6.** b **30.7.** d **30.8.** c **30.9.** a **30.10.** b

Conceptual Questions

30.11. The impedance of an RLC circuit in series is

$$Z=\sqrt{R^2+\left(X_L-X_C\right)^2}.$$

At resonance, $X_L=X_C$. Therefore, the impedance is at is minimum value of $Z=R$.

30.12. The total energy stored in a magnetic field is the magnetic energy density times the volume of the field, $U_B=u_BV$, where $u_B=B^2/\left(2\mu_0\right)$ (Equation 29.35). For the $d=5.00$ km thick shell above the Earth's surface, the volume is, $V=\left(4/3\right)\pi\left(R_{\text{E}}+d\right)^3-\left(4/3\right)\pi R_{\text{E}}^3$, where $R_{\text{E}}=6378$ km is the Earth's radius. The total energy stored in this field above the Earth's surface is

$$U_B=\frac{4\pi}{6}\frac{B^2}{\mu_0}\left(\left(R_{\text{E}}+d\right)^3-R_{\text{E}}^3\right)=\frac{2\pi}{3}\left(\frac{\left(0.500\cdot 10^{-4}\text{ T}\right)^2}{\left(4\pi\cdot 10^{-7}\text{ T m/A}\right)}\right)\left(\left(6.383\cdot 10^6\text{ m}\right)^3-\left(6.378\cdot 10^6\text{ m}\right)^3\right)=2.54\cdot 10^{12}\text{ kJ}.$$

30.13. No, charges are not crossing the gap (dielectric) of the capacitor. It simply means that, because of the periodic change in the polarity of the emf source, the capacitor is being periodically charged and discharged. No charge crosses the gap of the capacitor, whether in a DC or AC circuit. (This presumes that the potential on the capacitor does not get so big that the electric field exceeds the dielectric strength.)

30.14. In an RL circuit with alternating current, the expression "the current lags behind the voltage" means that the current achieves its maximum value at a delayed time compared to the time when the applied voltage achieves its maximum value. This is due to the phenomenon of self-induction: the changing current through the coil of the inductor creates a changing magnetic flux through the coil. Faraday's law will result in an induced emf in the coil, which will oppose the externally applied emf, in compliance with Lenz's law. The net effect is that the current will always try to "catch up" with the applied voltage, but will "lag behind" because of the self induced emf of the inductor.

30.15. The voltages given in the problem are root-mean-square values. Of course, Kirchhoff's rules will be obeyed at any instant of time, but not when using root-mean-square values since the voltages are out of phase with each other. This circuit does not violate Kirchhoff`s rules.

30.16. The power depends upon the voltage. For an AC circuit, the voltage oscillate about zero, so the average voltage is zero. Thus, the average power for any AC circuit would be zero regardless of amplitude, which would not be very informative as an average value. Hence, RMS power is used instead.

30.17. Each device has its own specific operating current and voltage. Each needs its own transformer with a specific ratio of primary to secondary coils to convert the normal household current and voltage into the required current and voltage in order to prevent damage to the device.

30.18. When electrons arrive at one plate of the capacitor, they repel an equal number of electrons off the opposite plate. For this reason, the amount of current flowing into one plate of the capacitor is equal to the amount of current flowing out of the other plate, despite the fact that charge does not actually flow across the gap.

30.19. Any surrounding high-frequency electromagnetic waves can induce unwanted signals. In parallel pairs, noise sources may affect one wire more than the other and this can be disruptive. Twisting the pairs minimizes this effect, since for each half twist the wire nearest to the noise source is exchanged.

30.20. (a) In this circuit capacitive reactance can be neglected since there is no capacitor. The root-mean-square input voltage V_{rms} and the root-mean-square current I_{rms} are related by $I_{\text{rms}} = V_{\text{rms}} / Z = V_{\text{rms}} / \left(R^2 + \omega^2 L^2\right)^{1/2}$, where R is the resistance in the circuit and L is the inductance of the solenoid. The effect of the ferromagnetic core is to increase the inductance of the solenoid by the factor κ_{m}. If the inductive reactance ωL is substantially greater than the resistance R, inserting the core will greatly increase the total impedance. Thus, the root-mean-square current will decrease.

(b) With a DC power source $(\omega = 0)$, the current would have fluctuated as the magnet was being inserted due to induction. Once the insertion was complete the current would return to its original value since the resistance of the circuit is unaffected by the presence of the core.

30.21. The response of the tuner at any frequency is related to the amplitude of the input signal and how close to resonance the input signal is. By design the response should be dominated by the signal that is at resonance. However, if the signal at some other non-resonant frequency is large and somewhat close to the resonant frequency, noticeable crosstalk can occur.

30.22. A sine or cosine signal is termed "pure" or "monochromatic"; it consists of a single frequency. Any other periodic signal is a superposition of harmonics. For simplicity, consider a square wave $S(t)$ which takes the value $+1$ ("on") for $0 \le t < (T/2)$ and -1 ("off") for $(T/2) \le t < T$, repeating periodically for all time, t, in both directions. As this is an odd function, it can be written as a sum of sine functions of period T: $S(t) = \sum_{n=1}^{\infty} b_n \sin\left((2\pi n / T)t\right)$. From Fourier analysis (see below), the coefficients b_n can be determined by multiplying both sides by $\sin\left((2\pi k / T)t\right)$ and integrating from $t = 0$ to $t = T$.

$$
\begin{aligned}
b_k &= \frac{1}{T/2} \int_0^T S(t) \sin\left(\frac{2\pi k}{T} t\right) dt \\
&= \frac{2}{T} \left(\int_0^{T/2} (+1) \sin\left(\frac{2\pi k}{T} t\right) dt - \int_{T/2}^{T} (-1) \sin\left(\frac{2\pi k}{T} t\right) dt \right) \\
&= \frac{2}{T} \left[\frac{-T}{2\pi k} \cos\left(\frac{2\pi k}{T} t\right) \right]_{t=0}^{t=T/2} - \frac{2}{T} \left[\frac{-T}{2\pi k} \cos\left(\frac{2\pi k}{T} t\right) \right]_{t=T/2}^{t=T} \\
&= \frac{1}{\pi k} \Big(\big((-\cos(\pi k)) - (-\cos(0)) \big) + \big(\cos(2\pi k) - \cos(\pi k) \big) \Big) \\
&= \frac{1}{\pi k} \left(2 - 2\cos(\pi k) \right) \\
&= \begin{cases} \dfrac{4}{\pi k} & \text{if } k \text{ is odd;} \\ 0 & \text{if } k \text{ is even.} \end{cases}
\end{aligned}
$$

Hence, this square wave can be written as $S(t) = \dfrac{4}{\pi} \sum_{j=0}^{\infty} \dfrac{1}{2j+1} \sin\left(\dfrac{2\pi (2j+1)}{T} t \right)$, where the harmonic frequencies are all odd multiples of the fundamental frequency $2\pi / T$. *Fourier analysis* is a very important mathematical tool, named for Jean Baptist Joseph Fourier (1768-1830), with a long and curious history. If such a square wave is applied as the driving voltage of an RLC circuit, the response of the circuit will be the sum of the responses to the fundamental $(j=0)$ term. But if the frequency of the square wave is any odd submultiple (one-third, one-fifth, etc.) of the resonant frequency, the square wave will contain a harmonic at the resonant frequency and the circuit will resonate in response to that term. The frequency response curve of the circuit with square-wave input will contain a sequence of resonance peaks, at its resonant frequency and all odd submultiples of it.

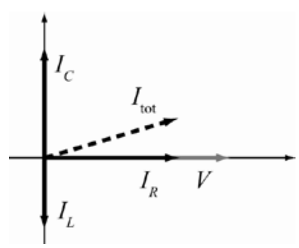
30.23. Yes, it is possible to have the voltage amplitude across the inductor exceed the voltage amplitude of the voltage supply. Since the voltage across each component is related by $V_{\text{m}}^2 = \sqrt{(V_R)^2 + (V_L - V_C)^2}$, the voltage across the battery is equal to the voltage across the resistor at resonance. Therefore, at resonance, the voltage across the inductor could be anything since it is countered by the same voltage across the capacitor.

30.24. The transformer works on the principle of mutual inductance, and depends on the (back) emf that is generated in the set of two coils in the transformer due to changing magnetic flux within the coils. If the current is DC, then there is no change in flux, and therefore, no possibility of operating the two coils as a transformer.

30.25. There can be no non-zero steady state current though the circuit. The current will vary until the capacitor is fully charged. Since the battery remains in the circuit, the current is not allowed to oscillate as it would in an LC circuit. The current will rise to a maximum and fall to zero eventually, where a steady state is achieved.

30.26. **THINK:** In this RLC circuit, L, C and R are connected in parallel, as shown in the provided figure. The circuit is connected to an AC source providing V_{rms} at frequency f. A phasor diagram can be used to find an expression for I_{rms} in terms of V_{rms}, f, L, R and C.

SKETCH: Phasor diagram for current across each component in the circuit:



I_R is in phase with V, and I_C leads I_L.

RESEARCH: Since this is a parallel circuit, the voltage V_{rms} is the same across all components. The current, however, has different phases and amplitudes within each component, as shown in the above phasor diagram. By a similar analysis to determining the voltage in a series RLC circuit, the phasor diagram above shows that the current is:

$$I_{\text{tot}} = I_{\text{rms}} = \sqrt{I_R^2 + (I_L - I_C)^2}.$$

Since V_{rms} is the same in each component, $V_{\text{rms}} = I_L X_L = I_C X_C = I_R R$, where $X_L = \omega L$, $X_C = 1/\omega C$ and $\omega = 2\pi f$.

SIMPLIFY: $I_{\text{rms}} = \sqrt{I_R^2 + (I_L - I_C)^2} = \sqrt{\frac{V_{\text{rms}}^2}{R^2} + \left(\frac{V_{\text{rms}}}{X_L} - \frac{V_{\text{rms}}}{X_C}\right)^2} = V_{\text{rms}}\sqrt{\frac{1}{R^2} + \left(\frac{1}{2\pi fL} - 2\pi fC\right)^2}$

CALCULATE: Not required.

ROUND: Not required.

DOUBLE-CHECK: By dimensional analysis of the above expression, the units are correct:

$$[I_{\text{rms}}]=\text{V}\left(\frac{1}{\Omega^2}+\left(\frac{\text{s}}{\text{H}}-\frac{\text{F}}{\Omega\text{s}}\right)^2_{\Omega}\right)^{1/2}=\text{V}\left(\frac{1}{\Omega^2}+\left(\frac{1}{\Omega}-\frac{1}{\Omega}\right)^2\right)^{1/2}=\frac{\text{V}}{\Omega}=\text{A}.$$

Exercises

30.27. From the inductance and capacitance, $L=32.0\text{ mH}$ and $C=45.0\ \mu\text{F}$, the frequency of oscillation is $\omega_0=(LC)^{-1/2}$. The total energy is constant at $U=q_0^2/2C$ where $q_0=10.0\ \mu\text{C}$, and the charge varies as $q=q_0\cos(\omega_0 t)$. Since energy remains constant, when the energy in both is the same, it is $(1/2)U$.

$$U_{\text{E}}=\frac{1}{2}U\ \Rightarrow\ \frac{q_0^2\cos^2(\omega_0 t)}{2C}=\frac{1}{2}\left(\frac{q_0^2}{2C}\right)\ \Rightarrow\ \cos^2(\omega_0 t)=\frac{1}{2}\ \Rightarrow\ t=\frac{1}{\omega_0}\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow\ t=\sqrt{LC}\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\sqrt{(32.0\text{ mH})(45.0\ \mu\text{F})}\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=9.42\cdot 10^{-4}\text{ s}$$

Note that the result does not depend on the original amount of charge, q_0. Dimensional analysis shows that the result has the correct units: $\sqrt{\left[\frac{\text{s}^2}{\text{F}}\right]\cdot[\text{F}]}=[\text{s}]$.

30.28. (a) From conservation of energy, $U_E=U_B\ \Rightarrow\ q_{\text{max}}^2/(2C)=Li_{\text{max}}^2/2$, where $q_{\text{max}}=CV$, with $C=2.00\ \mu\text{F}$, $L=0.250\text{ H}$ and $V=12.0\text{ V}$. Therefore,

$$\frac{q_{\text{max}}^2}{2C}=\frac{(CV)^2}{2C}=\frac{1}{2}CV^2=\frac{Li_{\text{max}}^2}{2}\ \Rightarrow\ i_{\text{max}}=\sqrt{\frac{C}{L}}V=\sqrt{\frac{(2.00\ \mu\text{F})}{(0.250\text{ H})}}(12.0\text{ V})=33.9\text{ mA}.$$

(b) The angular frequency of the current is $\omega_0=(LC)^{-1/2}$, so that the frequency of the oscillation is $f=\frac{\omega_0}{2\pi}=\frac{1}{2\pi\sqrt{LC}}=\frac{1}{2\pi\sqrt{(0.250\text{ H})(2.00\ \mu\text{F})}}=225\text{ Hz}.$

30.29. In general, the frequency of oscillation is $\omega_0=(LC)^{-1/2}$, where $L=1.00\text{ mH}$. The maximum energy in the capacitor is $U_{\text{E}}=q_{\text{max}}^2/(2C)$. Since the charge varies as $q=q_{\text{max}}\cos(\omega_0 t)$, and at time $t=2.10\text{ ms}$ the energy on capacitor is half the maximum value, $U_{\text{E}}=(1/2)U$. This means

$$U_{\text{E}}=\frac{1}{2}U\ \Rightarrow\ \frac{q_{\text{max}}^2\cos^2(\omega_0 t)}{2C}=\frac{1}{2}\left(\frac{q_{\text{max}}^2}{2C}\right)\ \Rightarrow\ \cos^2(\omega_0 t)=\frac{1}{2}\ \Rightarrow\ \omega_0=\left(\frac{1}{t}\right)\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow\ \frac{1}{\sqrt{LC}}=\frac{\cos^{-1}(1/\sqrt{2})}{t}\ \Rightarrow\ C=\frac{1}{L}\left[\frac{t}{\cos^{-1}(1/\sqrt{2})}\right]^2=\frac{1}{(1.00\cdot 10^{-3}\text{ H})}\left(\frac{(2.10\cdot 10^{-3}\text{ s})}{\cos^{-1}(1/\sqrt{2})}\right)^2=7.15\text{ mF}.$$

30.30. (a) Since the current is proportional to $\sin(\omega_0 t)$ where $\omega_0=1200.\text{ s}^{-1}$, this is at a maximum when $\omega_0 t=\pi/2$; therefore, $t=\pi/(2\omega_0)=\pi/\left(2(1200.\text{ s}^{-1})\right)=1.31\text{ ms}.$

(b) The total energy in the circuit is $U=Li_{\text{max}}^2/2$, where $i_{\text{max}}=1.00\text{ A}$. Since the angular frequency is $\omega_0=(LC)^{-1/2}$, then $L=1/(\omega_0^2 C)$, where $C=10.0\ \mu\text{F}$. The total energy of the circuit is then

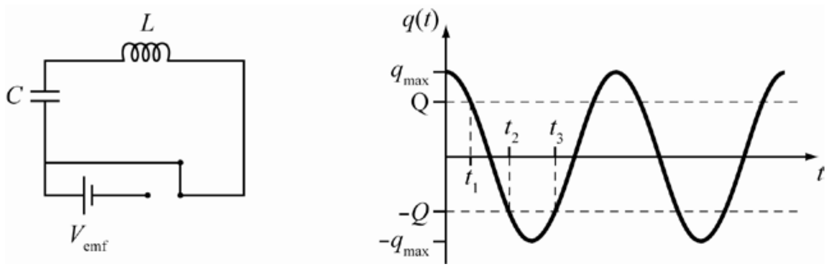
$$U=\frac{Li_{\text{max}}}{2}=\frac{i_{\text{max}}}{2\omega_0^2 C}=\frac{(1.00\text{ A})}{2(1200.\text{ s}^{-1})^2(10.0\ \mu\text{F})}\approx 34.7\text{ mJ}.$$

(c) The inductance is

$$L = \frac{1}{\omega_0^2 C} = \frac{1}{\left(1200.\ \text{s}^{-1}\right)^2 \left(10.0\ \mu\text{F}\right)} = 69.4\ \text{mH}.$$

30.31. **THINK:** The charge on the capacitor will oscillate with time as a cosine function with a period determined by the inductance, $L = 0.200$ H, and capacitance, $C = 10.0\ \mu\text{F}$. The potential, $V_{\text{emf}} = 12.0$ V, will give the initial charge on the capacitor. Ignoring the sign of the charge, the charge on the capacitor will equal $Q = 80.0\ \mu\text{C}$ periodically.

SKETCH:



RESEARCH: The initial (and maximum) charge on the capacitor is $q_{\text{max}} = CV$. The charge will oscillate as $q = q_{\text{max}} \cos(\omega_0 t)$, where $\omega_0 = 1/\sqrt{LC}$.

SIMPLIFY: The first time t_1 when the charge on the capacitor is equal to Q is

$$q = \text{Q} = q_{\text{max}} \cos(\omega_0 t_1) \Rightarrow t_1 = \left(\frac{1}{\omega_0}\right)\cos^{-1}\left(\frac{\text{Q}}{q_{\text{max}}}\right);$$

By symmetry, the second time t_2 and the third time t_3 are given by

$$\omega_0 t_2 = \pi - \omega_0 t_1 = \omega_0\left(\frac{\pi}{\omega_0} - t_1\right) \Rightarrow t_2 = \frac{\pi}{\omega_0} - t_1 \text{ and}$$

$$\omega_0 t_3 = \pi + \omega_0 t_1 = \omega_0\left(\frac{\pi}{\omega_0} + t_1\right) \Rightarrow t_3 = \frac{\pi}{\omega_0} + t_1.$$

CALCULATE: $\omega_0 = \dfrac{1}{\sqrt{(0.200\ \text{H})(10.0\ \mu\text{F})}} = 707.107\ \text{rad/s}$

$q_{\text{max}} = (10.0\ \mu\text{F})(12.0\ \text{V}) = 120.\ \mu\text{C}$

$$t_1 = \left(\frac{1}{(707.107\ \text{rad/s})}\right)\cos^{-1}\left(\frac{80.0\ \mu\text{C}}{120.\ \mu\text{C}}\right) = 0.0011895\ \text{s}$$

$$t_2 = \frac{\pi}{(707.107\ \text{rad/s})} - 0.0011895\ \text{s} = 0.0032534\ \text{s}$$

$$t_3 = \frac{\pi}{(707.107\ \text{rad/s})} + 0.0011895\ \text{s} = 0.0056323\ \text{s}$$

ROUND: Rounding the times to three significant figures gives: $t_1 = 1.19$ ms, $t_2 = 3.25$ ms, and $t_3 = 5.63$ ms.

DOUBLE-CHECK: Given the high frequency, small times are expected, so the answers are reasonable.

30.32. **THINK:** The current, $i_{\text{max}} = 3.00$ A, will oscillate with time as a sine function with a period determined by the inductance, $L = 7.00$ mH, and capacitance, $C = 4.00$ mF. The charge will also vary with the same period as the current, but as a cosine function. The maximum charge on the capacitor is related to the energy of the system.

SKETCH: Not required.
RESEARCH: The total energy in circuit is found either when the current is at a maximum, $E=(1/2)Li_{max}^2$, or when charge on capacitor is a maximum, $E=q_{max}^2/(2C)$. The charge varies as $q=q_{max}\cos(\omega_0 t)$, where $\omega_0=1/\sqrt{LC}$.
SIMPLIFY:
(a) The energy in the circuit is $U=\frac{1}{2}Li_{max}^2$.
(b) The maximum charge on the capacitors is found by $U=\frac{q_{max}^2}{2C} \Rightarrow q_{max}=\sqrt{2CU}=i_{max}\sqrt{LC}$. Therefore, the expression for charge is $q=i_{max}\sqrt{LC}\cos\left(\frac{t}{\sqrt{LC}}\right)$.
CALCULATE:
(a) $U=\frac{1}{2}(7.00\text{ mH})(3.00\text{ A})^2=0.0315\text{ J}$
(b) $q=(3.00\text{ A})\sqrt{(7.00\text{ mH})(4.00\text{ mF})}\cos\left(\frac{t}{\sqrt{(7.00\text{ mH})(4.00\text{ mF})}}\right)=0.01587\cos(189.0t)\text{ C}$
ROUND:
(a) $U=31.5\text{ mJ}$
(b) It is best not to round the coefficients of the expression. Keep the answer as $q=0.01587\cos(189.0t)$, with the intention of rounding after particular values of t are substituted.
DOUBLE-CHECK: The equation for the charge has the proper units since $\sqrt{LC}$ has units of s and i_{max} has units of C/s.

30.33. The angular frequency without the resistor is $\omega_0=1/\sqrt{LC}$, and with the resistor it is $\omega=\sqrt{\omega_0^2-(R/2L)^2}$. The fractional change is

$$\frac{\Delta\omega}{\omega_0}=\frac{\omega-\omega_0}{\omega_0}=\frac{\omega}{\omega_0}-1=\sqrt{1-\left(\frac{R}{2L\omega_0}\right)^2}-1=\sqrt{1-\frac{R^2C}{4L}}-1\,.$$

Inserting the numbers, we find

$$\frac{\Delta\omega}{\omega_0}=\sqrt{1-\frac{(1.00\cdot 10^3\ \Omega)^2(4.5\cdot 10^{-9}\ \text{F})}{4(4.00\cdot 10^{-3}\ \text{H})}}-1=-0.152\,.$$

This means that the fractional change is a drop of 15.2%.

30.34. **THINK:** For an RLC circuit, the charge on the capacitor is described by equation 30.6. The capacitance is found by equating the term in the exponential to the period determined from the term in the cosine. The resistance and inductance are $R=50.0\ \Omega$ and $L=1.00$ mH.
SKETCH: Not required.
RESEARCH: The general expression for an RLC circuit is

$$q=q_{max}e^{-Rt/2L}\cos(\omega t),$$

where $\omega=\sqrt{\omega_0^2-(R/2L)^2}$ and where $\omega_0=1/\sqrt{LC}$. The period of the oscillation is $T=2\pi/\omega$. The decay rate of the exponential is $\tau=2L/R$.

SIMPLIFY: Capacitance is found by letting $\tau = T$, which gives the result:

$$\frac{2L}{R} = \frac{2\pi}{\omega} \Rightarrow \left(\frac{R}{2L}\right)^2 = \left(\frac{\sqrt{\omega_0^2 - (R/2L)^2}}{2\pi}\right)^2 \Rightarrow \frac{R^2}{4L^2} = \frac{\omega_0^2 - \frac{R^2}{4L^2}}{4\pi^2} \Rightarrow \left(4\pi^2 + 1\right)\frac{R^2}{4L^2} = \omega_0^2 = \frac{1}{LC}.$$

Therefore, $C = \dfrac{4L}{R^2\left(4\pi^2 + 1\right)}$.

CALCULATE: $C = \dfrac{4\left(1.00 \cdot 10^{-3} \text{ H}\right)}{(50.0\ \Omega)^2\left(4\pi^2 + 1\right)} = 3.9527 \cdot 10^{-8} \text{ F}$

ROUND: Rounding to three significant figures, $C = 39.5$ nF.

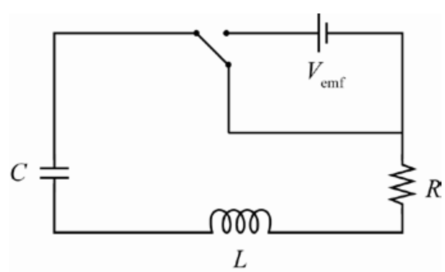
DOUBLE-CHECK: The time constant is $\tau = 2L/R = 4 \cdot 10^{-5}$ s. The angular frequency, ω, is

$$\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = 1.5708 \cdot 10^5 \text{ s}^{-1},$$

and the period is $T = 2\pi/\omega = 4.00 \cdot 10^{-5}$ s. So $\tau = T$, as required.

30.35. **THINK:** The frequency of the damped oscillation is independent of the initial charge, and hence potential. It then only depends on the inductance, $L = 0.200$ H, the resistance, $R = 50.0\ \Omega$, capacitance $C = 2.00\ \mu\text{F}$.

SKETCH:



RESEARCH: In general the charge on the capacitor is

$$q = q_{\max} e^{-Rt/2L} \cos(\omega t),$$

where $\omega = \sqrt{\omega_0^2 - (R/2L)^2}$ and where $\omega_0 = 1/\sqrt{LC}$. The frequency of oscillation is $f = \omega/(2\pi)$.

SIMPLIFY: $f = \dfrac{\omega}{2\pi} = \dfrac{\sqrt{\left(\omega_0^2 - R/2L\right)^2}}{2\pi} = \dfrac{\sqrt{1/(LC) - R^2/\left(4L^2\right)}}{2\pi}$

CALCULATE: $f = \dfrac{\sqrt{1/\left((0.200 \text{ H})\left(2.00 \cdot 10^{-6} \text{ F}\right)\right) - (50.0\ \Omega)^2/\left(4(0.200 \text{ H})^2\right)}}{2\pi} = 250.858 \text{ Hz}$

ROUND: Rounding to three significant figures, $f = 251$ Hz.

DOUBLE-CHECK: This frequency is typical of RLC circuits. Additionally, dimensional analysis yields:

$$\left[\left[\frac{\text{F}}{\text{s}^2}\right]\frac{1}{[\text{F}]} - \frac{[\Omega]^2}{[\Omega \cdot \text{s}]^2}\right]^{\frac{1}{2}} = \left[\frac{1}{\left[\text{s}^2\right]}\right]^{\frac{1}{2}} = \frac{1}{[\text{s}]}.$$

30.36. **THINK:** The angular frequency of the RLC is 20% less than the angular frequency of the LC circuit where the inductance is $L = 4.0$ mH and the capacitance is $C = 2.50\ \mu\text{F}$. Even though the current is actually a damped oscillation, the magnitude of the oscillation is describe simply by the exponential decay, similar to that of the charge. Therefore, the current is at half of its maximum when the exponential is at a half. The number of periods in a given time is the number of oscillations.

SKETCH: Not required.
RESEARCH: The RLC frequency is related to LC frequency by $\omega_{\text{RLC}} = 0.8\omega_{\text{LC}}$. The RLC frequency is $\omega_{\text{RLC}} = \sqrt{\omega_0^2 - \left(R/(2L)\right)^2}$, where $\omega_0 = \omega_{\text{LC}} = 1/\sqrt{LC}$. The current varies as $i = i_{\max}e^{-Rt/2L}$. A wave goes through n oscillations in time $t = nT$, where T is the period.
SIMPLIFY:
(a) For the oscillation frequency,

$$\omega_{\text{RLC}} = 0.8\omega_{\text{LC}} \Rightarrow \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2} = 0.8\omega_0 \Rightarrow \omega_0^2 - \left(\frac{R}{2L}\right)^2 = 0.64\omega_0^2 \Rightarrow 0.36\omega_0^2 = \frac{R^2}{4L^2}$$

$$\Rightarrow R = 1.2L\omega_0 = 1.2\left(\frac{L}{\sqrt{LC}}\right) = 1.2\sqrt{\frac{L}{C}}.$$

(b) $i = \left(\frac{1}{2}\right)i_{\max} = i_{\max}e^{-Rt/2L} \Rightarrow \frac{1}{2} = e^{-Rt/2L} \Rightarrow t = \frac{2L}{R}\ln 2$

(c) The period is $T = \frac{2\pi}{\omega_{\text{RLC}}}$. Therefore, the number of oscillations is $n = \frac{t}{T} = \frac{t\omega_{\text{RLC}}}{2\pi}$.

CALCULATE:
(a) $R = 1.2\sqrt{\frac{\left(4.0 \cdot 10^{-3}\text{ H}\right)}{\left(2.50 \cdot 10^{-6}\text{ F}\right)}} = 48\ \Omega$

(b) $t = \frac{2\left(4.0 \cdot 10^{-3}\text{ H}\right)}{\left(48\ \Omega\right)}\ln(2) = 0.000115525\text{ s}$

(c) $\omega_{\text{RLC}} = \sqrt{\frac{1}{\left(4.0 \cdot 10^{-3}\text{ H}\right)\left(2.50 \cdot 10^{-6}\text{ F}\right)} - \left(\frac{\left(48\ \Omega\right)}{2\left(4.0 \cdot 10^{-3}\text{ H}\right)}\right)^2} = 8000\text{ rad/s},$ and

$n = \frac{\left(0.000115525\text{ s}\right)\left(8000\text{ rad/s}\right)}{2\pi} = 0.14709\text{ cycles}.$

ROUND: To three significant figures,
(a) $R = 48.0\ \Omega$
(b) $t = 116\ \mu\text{s}$
(c) $n = 0.147$ cycles
DOUBLE-CHECK: Since the resistance is large compared to the inductance, the current will die off quickly, so a complete oscillations will not occur before it is at half its maximum value.

30.37. The capacitive reactance, $X_C = 200.\ \Omega$, is given by $X_C = 1/(\omega C)$, where $C = 10.0\ \mu\text{F}$; therefore, the frequency is

$$\omega = \frac{1}{X_C C} = \frac{1}{(200.\ \Omega)(10.0\ \mu\text{F})} = 500.\text{ rad/s}.$$

30.38. The capacitive reactance is given by $X_C = 1/(\omega C)$, where $C = 5.00 \cdot 10^{-6}$ F and $\omega = 2\pi f$, where $f = 100.$ Hz. Therefore, the capacitive reactance is

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi\left(100.\text{ Hz}\right)\left(5.00 \cdot 10^{-6}\text{ F}\right)} = 318\ \Omega.$$

The maximum current through the capacitor, I_C, is given by $I_C = V_C / X_C$, where $V_C = 10.0$ V. Therefore,

$$I_C = \frac{V_C}{X_C} = \frac{\left(10.0\text{ V}\right)}{\left(318\ \Omega\right)} = 31.4\text{ mA}.$$

30.39. (a) The resonant angular frequency of an RLC circuit is $\omega_0 = 1/\sqrt{LC}$ where $L = 0.500\text{ H}$ and $C = 0.400\ \mu\text{F}$. Therefore,

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\left(0.400 \cdot 10^{-6}\text{ F}\right)\left(0.500\text{ H}\right)}} = 2240\text{ rad/s}$$

(b) At resonance, only the resistor contributes to the overall impedance, so $I = V_{\text{emf}} / R$ where $V_{\text{emf}} = 40.0\text{ V}$ and $R = 100.0\ \Omega$. Therefore, $I = \left(40.0\text{ V}\right)/\left(100.0\ \Omega\right) = 0.400\text{ A}$.

30.40. In general, the resonant frequency is determined by $f_0 = \dfrac{\omega_0}{2\pi} = \dfrac{1}{2\pi\sqrt{LC_0}}$, when $f_0 = 5.0\text{ MHz}$ and $C_0 = 15\text{ pF}$, the circuit has an inductance of L that satisfies the given relation. When $C_1 = 380\text{ pF}$, the same L will give another resonant frequency. Therefore,

$$f_0 = \frac{1}{2\pi\sqrt{LC_0}} \Rightarrow L = \frac{1}{4\pi^2 f_0^2 C_0},$$

$$f_1 = \frac{1}{2\pi\sqrt{LC_1}} = \frac{1}{2\pi\sqrt{\dfrac{C_1}{4\pi^2 f_0^2 C_0}}} = f_0\sqrt{\frac{C_0}{C_1}} = f_1 = \left(5.0\text{ MHz}\right)\sqrt{\frac{\left(15\text{ pF}\right)}{\left(380\text{ pF}\right)}} = 1.0\text{ MHz}.$$

30.41. Given the frequency, $f = 1.00\text{ kHz}$, the angular frequency is $\omega = 2\pi f$. The phase constant for an RLC circuit is given by

$$\tan\phi = \frac{X_L - X_C}{R} = \frac{\omega L - 1/\left(\omega C\right)}{R} \Rightarrow \phi = \tan^{-1}\left(\frac{2\pi f \cdot L - 1/\left(2\pi f \cdot C\right)}{R}\right)$$

For the values $R = 100.\ \Omega$, $L = 10.0\text{ mH}$ and $C = 100.\ \mu\text{F}$, the phase constant is

$$\phi = \tan^{-1}\frac{\left(\left(2\pi\right)\left(1.00 \cdot 10^3\text{ Hz}\right)\left(10.0 \cdot 10^{-3}\text{ H}\right) - \left(\left(2\pi\right)\left(1.00 \cdot 10^3\text{ Hz}\right)\left(100. \cdot 10^{-6}\text{ F}\right)\right)^{-1}\right)}{\left(100.\ \Omega\right)} = 0.549\text{ rad}.$$

The impedance for this circuit is

$$Z = \sqrt{R^2 + \left(\omega L - 1/\left(\omega C\right)\right)^2} = \sqrt{R^2 + \left(2\pi f \cdot L - 1/\left(2\pi f \cdot C\right)\right)^2}$$
$$= \sqrt{\left(100.\ \Omega\right)^2 + \left(\left(2\pi\right)\left(1.00 \cdot 10^3\text{ Hz}\right)\left(10.0 \cdot 10^{-3}\text{ H}\right) - \left(\left(2\pi\right)\left(1.00 \cdot 10^3\text{ Hz}\right)\left(100. \cdot 10^{-6}\text{ F}\right)\right)^{-1}\right)^2} = 117\ \Omega.$$

30.42. For an RLC circuit with $L = 5.00\text{ mH}$ and $C = 4.00\ \mu\text{F}$, the resonant frequency, ω_0, is $\omega_0 = 1/\sqrt{LC}$. Therefore,

$$\omega_0 = \frac{1}{\sqrt{\left(5.00 \cdot 10^{-3}\text{ H}\right)\left(4.00 \cdot 10^{-6}\text{ F}\right)}} = 7070\text{ rad/s}.$$

At resonance, the resistor with $R = 1.00\text{ k}\Omega$, is the only contribution to the impedance. For a peak voltage of $V_{\text{m}} = 10.0\text{ V}$, the maximum current is

$$I_{\text{m}} = \frac{V_{\text{m}}}{R} = \frac{\left(10.0\text{ V}\right)}{\left(1000.\ \Omega\right)} = 10.0\text{ mA}.$$

30.43. **THINK:** Given the equation, $V = \left(12.0\text{ V}\right)\sin\left(\omega t\right)$ the peak voltage is clearly $V_{\text{m}} = 12.0\text{ V}$. When the circuit is in resonance, the current is dictated solely by the resistance, $R = 10.0\ \Omega$. This means the inductor, $L = 2.00\text{ H}$, and the capacitor, $C = 10.0\ \mu\text{F}$ will not influence the current. However, the current will dictate the voltage drop across each.

SKETCH: Not required.
RESEARCH: At resonance, impedance is $Z = R$. The maximum current at resonance is $I_m = V_m / R$. The resonant frequency is $\omega_0 = 1/\sqrt{LC}$. The voltage drop across the inductor is $V_L = IX_L$, where $X_L = 1/\omega_0 C$.
SIMPLIFY: The voltage drop across the inductor is,

$$V_L = IX_L = \frac{V_m}{R}\omega_0 L = \frac{V_m}{R}\frac{L}{\sqrt{LC}} = \frac{V_m}{R}\sqrt{\frac{L}{C}}.$$

CALCULATE: $V_L = \frac{(12.0 \text{ V})}{(10.0 \ \Omega)}\sqrt{\frac{(2.00 \text{ H})}{(10.0 \cdot 10^{-6} \text{ F})}} = 536.66 \text{ V}$

ROUND: To three significant figures, $V_L = 537$ V.
DOUBLE-CHECK: Even though the answer seems large compared to V_m, the voltage drop across the capacitor is the exact same. Since the voltage across each component is related by $V_m^{\ 2} = \sqrt{(V_R)^2 + (V_L - V_C)^2}$, the voltage across the battery is equal to the voltage across the resistor at resonance. Therefore, at resonance, the voltage across the inductor could be anything since it is countered by the same voltage across the capacitor.

30.44. **THINK:** The inductive reactance and the capacitive reactance are needed to find the impedance of the RLC circuit. The AC power source oscillates with frequency $f = 60.0$ Hz and has an amplitude of $V_m = 220$ V. The resistance is $R = 50.0 \ \Omega$, the inductance is $L = 0.200$ H and the capacitance is $C = 0.040$ mF.
SKETCH: Not required.
RESEARCH: The angular frequency of oscillation is $\omega = 2\pi f$. The inductive reactance is $X_L = \omega L$. The capacitive reactance is $X_C = 1/\omega C$. The impedance of circuit is $Z = \sqrt{R^2 + (X_L - X_C)^2}$. The maximum current through the circuit is $I_m = V_m / Z$. The maximum potential drop across each component is $V_i = I_m X_i$, where i denotes either R, C or L.
SIMPLIFY:
(a) $X_L = \omega L = 2\pi f L$
(b) $X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$
(c) $Z = \sqrt{R^2 + (X_L - X_C)^2}$
(d) $I_m = \frac{V_m}{Z}$
(e) $V_R = I_m R$, $V_C = I_m X_C$ and $V_L = I_m X_L$
CALCULATE:
(a) $X_L = 2\pi(60.0 \text{ Hz})(0.200 \text{ H}) = 75.398 \ \Omega$
(b)
(c) $Z = \sqrt{(50.0 \ \Omega)^2 + (75.398 \ \Omega - 66.315 \ \Omega)^2} = 50.818 \ \Omega$
(d)
(e) $V_R = (4.329 \text{ A})(50.0 \ \Omega) = 216.457 \text{ V}$, $V_C = (4.329 \text{ A})(66.315 \ \Omega) = 287.085 \text{ V}$ and $V_L = (4.329 \text{ A})(75.398 \ \Omega) = 326.409 \text{ V}$.
ROUND:
(a) $X_L = 75.4 \ \Omega$
(b) $X_C = 66.3 \ \Omega$

(c) $Z = 50.8\ \Omega$

(d) $I_{\text{m}} = 4.33$ A

(e) $V_R = 220$ V, $V_C = 290$ V and $V_L = 330$ V.

DOUBLE-CHECK: In order to satisfy Kirchhoff's loop rule, the vector phasors must sum as vectors to match $\vec{V}_{\text{m}}$: $V_{\text{m}}^2 = V_R^2 + (V_L - V_C)^2$. Therefore,

$$V_{\text{m}} = \sqrt{(216.457\text{ V})^2 + (326.409\text{ V} - 287.085\text{ V})^2} = 220\text{ V},$$

as required.

30.45. **THINK:** The maximum current occurs for when the AC voltage is at its peak, $V_{\text{m}} = 110$ V. The angular frequency of the oscillation is $\omega = 377$ rad/s. The voltage acts across the total impedance of the circuit where $R = 2.20\ \Omega$, $L = 9.30$ mH and $C = 2.27$ mF. The maximum current I'_{m} occurs for a capacitance C' that puts the RLC circuit in resonance with the supplied voltage. At resonance, the phase constant is zero and only the resistor influences the current.

SKETCH: Not required.

RESEARCH: The impedance of inductor and capacitor are $X_L = \omega L$ and $X_C = 1/\omega C$. The maximum current is $I_{\text{m}} = V_{\text{m}}/Z$, where $Z = \sqrt{R^2 + (X_L - X_C)^2}$. The phase angle for the circuit is $\phi = \tan^{-1}((X_L - X_C)/R)$. When at resonance, $\omega = \omega_0 = 1/\sqrt{LC}$. The maximum current at resonance is $I_{\text{m}} = V_{\text{m}}/R$.

SIMPLIFY:

(a) The maximum current is $I_{\text{m}} = \dfrac{V_{\text{m}}}{Z} = \dfrac{V_{\text{m}}}{\sqrt{R^2 + (X_L - X_C)^2}} = \dfrac{V_{\text{m}}}{\sqrt{R^2 + \left(\omega L - (\omega C)^{-1}\right)^2}}$.

(b) The phase constant is $\phi = \tan^{-1}\left(\dfrac{X_L - X_C}{R}\right) = \tan^{-1}\left(\dfrac{\omega L - (\omega C)^{-1}}{R}\right)$.

(c) The required capacitance for resonance is

$$\omega = \frac{1}{\sqrt{LC'}} \Rightarrow C' = \frac{1}{\omega^2 L},$$

and maximum current at resonance is $I_{\text{m}} = \dfrac{V_{\text{m}}}{R}$.

CALCULATE:

(a) $$I_{\text{m}} = \frac{(110\text{ V})}{\sqrt{(2.20\ \Omega)^2 + \left((377\text{ rad/s})(9.30\cdot 10^{-3}\text{ H}) - \left((377\text{ rad/s})(2.27\cdot 10^{-3}\text{ F})\right)^{-1}\right)^2}} = 34.2675\text{ A}$$

(b) $$\phi = \tan^{-1}\frac{(377\text{ rad/s})(9.30\cdot 10^{-3}\text{ H}) - \left((377\text{ rad/s})(2.27\cdot 10^{-3}\text{ F})\right)^{-1}}{(2.20\ \Omega)} = 0.8157\text{ rad}$$

(c) $$C' = \frac{1}{(377\text{ rad/s})^2(9.30\cdot 10^{-3}\text{ H})} = 7.565\cdot 10^{-4}\text{ F} \text{ and } I'_{\text{m}} = \frac{(110\text{ V})}{(2.20\ \Omega)} = 50.0\text{ A}$$

ROUND: To three significant figures:

(a) $I_{\text{m}} = 34.3$ A

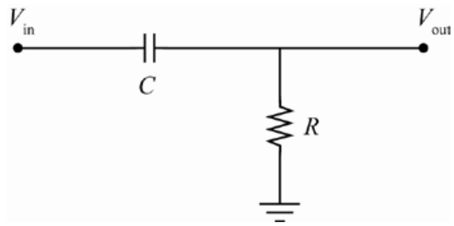
(b) $\phi = 0.816$ rad

(c) $C' = 757\ \mu\text{F}$, $I'_{\text{m}} = 50.0$ A, $\phi' = 0$ rad

DOUBLE-CHECK: The current is at a maximum when at resonance, so having $I'_{\text{m}} > I_{\text{m}}$ is reasonable.

30.46. **THINK:** Although not explicitly stated, since the circuit should pass a frequency of $f = 5.00$ kHz, it is natural to assume that it is a high-pass filter. The ratio of the voltages is $V_{\text{out}} / V_{\text{in}} = 1/2$.

SKETCH:



RESEARCH: The ratio of the voltages for the high-pass RC filter is given by $V_{\text{out}} / V_{\text{in}} = \left[1 + \frac{1}{\omega^2 R^2 C^2}\right]^{-\frac{1}{2}}$, where $\omega = 2\pi f$. The phase constant for the circuit is $\phi = \tan^{-1}\left(\left(X_L - X_C\right)/R\right)$, with $X_L = 0$ and $X_C = 1/\left(\omega C\right)$.

SIMPLIFY:

(a) $\frac{V_{\text{out}}}{V_{\text{in}}} = 0.500 = \frac{1}{\sqrt{1 + 1/\left(\omega^2 R^2 C^2\right)}} \Rightarrow 1 + \frac{1}{\omega^2 R^2 C^2} = 4.00 \Rightarrow \omega^2 R^2 C^2 = \frac{1}{3.00}$

$\Rightarrow C = \frac{1}{\sqrt{3.00}} \frac{1}{\omega R} = \frac{1}{\sqrt{3.00}} \frac{1}{2\pi f R}$.

(b) The phase constant is $\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{0 - \left(1/\left(\omega C\right)\right)}{R}\right) = \tan^{-1}\left(-\frac{1}{2\pi f R C}\right)$.

CALCULATE:

(a) $C = \frac{1}{\sqrt{3.00}} \frac{1}{2\pi\left(5.00 \cdot 10^3 \text{ Hz}\right)\left(1.00 \cdot 10^3 \ \Omega\right)} = 1.838 \cdot 10^{-8}$ F

(b) $\phi = \tan^{-1}\left(-\frac{1}{2\pi\left(5.00 \cdot 10^3 \text{ Hz}\right)\left(1.00 \cdot 10^3 \ \Omega\right)\left(1.838 \cdot 10^{-8} \text{ F}\right)}\right) = -1.0472$ rad

ROUND:

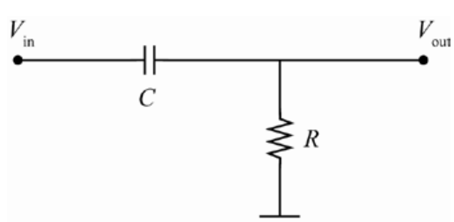
(a) $C = 18.4$ nF

(b) $\phi = -1.05$ rad

DOUBLE-CHECK: The negative phase constant simply means the current lends the voltage, which makes sense since the current must first reach the capacitor before it can charge it. The values for the capacitor and phase constant seem reasonable.

30.47. **THINK:** For signals of frequency $f = 60.0$ Hz the required noise factor is $V_{\text{out}} / V_{\text{in}} = 0.00100$. The given total impedance $R = 20.0$ kΩ. Since the ratio of voltages will increase as the frequency does, the lowest frequency that has 90.0% signal strength is when $V_{\text{out}} / V_{\text{in}} = 0.900$.

SKETCH:



RESEARCH: The signal ratio is $V_{\text{out}}/V_{\text{in}} = 1/\sqrt{1+1/\left(\omega^2R^2C^2\right)}$, where the angular frequency is $\omega = 2\pi f$.

SIMPLIFY:

(a) When $V_{\text{out}}/V_{\text{in}} = 0.001000 = 1.00\cdot 10^{-3}$, then

$$\frac{V_{\text{out}}}{V_{\text{in}}} = 1.00\cdot 10^{-3} = \frac{1}{\sqrt{1+1/\left(\omega^2R^2C^2\right)}} \Rightarrow 1+\frac{1}{\omega^2R^2C^2} = 10^6 \Rightarrow \omega^2R^2C^2 = \left(10^6-1\right)^{-1}$$

$$\Rightarrow C = \frac{1}{\sqrt{\left(10^6-1\right)}\left(2\pi fR\right)}.$$

(b) When $\dfrac{V_{\text{out}}}{V_{\text{in}}} = 0.900$, then

$$\frac{V_{\text{out}}}{V_{\text{in}}} = 0.900 = \frac{1}{\sqrt{1+1/\left(\omega^2R^2C^2\right)}} \Rightarrow 1+\frac{1}{\omega^2R^2C^2} = \frac{1}{0.81} \Rightarrow \omega^2R^2C^2 = \left(\frac{1}{0.81}-1\right)^{-1} = \frac{81}{19}$$

$$\Rightarrow \omega = \sqrt{\frac{81}{19}}\left(\frac{1}{RC}\right) \Rightarrow f = \sqrt{\frac{81}{19}}\left(\frac{1}{2\pi RC}\right).$$

CALCULATE:

(a) $$C = \frac{1}{\sqrt{\left(1.00\cdot 10^6-1\right)}\left(2\pi\left(60.0\text{ Hz}\right)\left(2.00\cdot 10^3\ \Omega\right)\right)} = 1.3263\cdot 10^{-9}\text{ F}$$

(b) $$f = \sqrt{\frac{81}{19}}\left(\frac{1}{2\pi\left(2.00\cdot 10^3\ \Omega\right)\left(1.3263\cdot 10^{-9}\text{ F}\right)}\right) = 123884\text{ Hz}$$

ROUND: To three significant figures,

(a) $C = 1.33$ nF

A capacitor of capacitance 1.00 nF needs to be used in this high-pass filter.

(b) $f = 124$ kHz

Frequencies of 120 kHz and higher will be passed with at least 90% of their amplitude.

DOUBLE-CHECK: Since frequencies higher than 120 kHz will pass with 90% of their strength, this seems like a reasonable high-pass filter.

30.48. In general, $V_{\text{rms}} = V_{\text{m}}/\sqrt{2}$; therefore, $V_{\text{m}} = \sqrt{2}V_{\text{rms}}$.

(a) For $V_{\text{rms}} = 110$ V, $V_{\text{m}} = \sqrt{2}\left(110\text{ V}\right) = 160$ V.

(b) For $V_{\text{rms}} = 220$ V, $V_{\text{m}} = \sqrt{2}\left(220\text{ V}\right) = 310$ V.

30.49. The quality factor for an RLC circuit is defined as $Q = \omega_0\left(\text{Energy stored/Power lost}\right)$. For the RLC circuit, the resonant frequency is $\omega_0 = 1/\sqrt{LC}$. In general, the energy stored in the circuit is $U = U_0e^{-Rt/L}$. The power lost is defined as $P = -\left|dU/dt\right|$. Therefore,

$$Q = \frac{\omega_0\left(U_0e^{-Rt/L}\right)}{-\dfrac{d}{dt}\left(U_0e^{-Rt/L}\right)} = \frac{\left(U_0e^{-Rt/L}\right)}{\sqrt{LC}\left(R/L\right)\left(U_0e^{-Rt/L}\right)} = \frac{1}{R}\sqrt{\frac{L}{C}}.$$

30.50. On any product label the voltage and power displayed are the rms voltage and average power. Therefore, take $V_{\text{rms}} = 110$ V and $\langle P\rangle = 1250$ W. The peak value of current is related to rms current by $I_{\text{rms}} = I_{\text{m}}/\sqrt{2}$. In general, $\langle P\rangle = I_{\text{rms}}V_{\text{rms}}\cos\phi$.

Assuming that the hair dryer acts like a resistor, $\cos\phi = R/Z = 1$, so

$$I_{\text{rms}} = \frac{\langle P \rangle}{V_{\text{rms}}} \Rightarrow I_{\text{m}} = \sqrt{2}\frac{\langle P \rangle}{V_{\text{rms}}} = \sqrt{2}\frac{(1250\text{ W})}{(110\text{ V})} = 16\text{ A}.$$

30.51. (a) The resonant frequency of the radio tuner is related to the inductance, $L = 3.00$ mH and capacitance $C = 25.0$ nF, by $\omega_0 = 1/\sqrt{LC}$. Keeping in mind that $\omega_0 = 2\pi f_0$,

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(3.00\cdot 10^{-3}\text{ H})(25.0\cdot 10^{-9}\text{ F})}} = 18.4\text{ kHz}.$$

(b) At resonance, the total impedance is solely that from the resistance, $R = 1.00\ \mu\Omega$. Given that the voltage drop across the resistor is $V_{\text{rms}} = 1.50$ mV, the power in the circuit is

$$P = \frac{V_{\text{rms}}^2}{R} = \frac{(1.50\cdot 10^{-3}\text{ V})^2}{(1.00\cdot 10^{-6}\ \Omega)} = 2.25\text{ W}.$$

30.52. **THINK:** The current through the circuit is driven by the AC potential, $V_{\text{rms}} = 50.0$ V and $f = 2000.$ Hz, which has a total impedance from the resistance, $R = 100.\ \Omega$, capacitance, $C = 0.400\ \mu$F and inductance, $L = 0.0500$ H. The average power lost over the circuit is determined by the rms voltage and rms current.
SKETCH: Not required.
RESEARCH: The rms current is given by

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}.$$

The rms voltage drop across each component is $V_i = I_{\text{rms}} X_i$, where i denotes R, L or C. The average power drawn from the circuit is

$$\langle P \rangle = I_{\text{rms}}^2 R.$$

SIMPLIFY:
(a) Since the angular frequency is $\omega = 2\pi f$,

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{\sqrt{R^2 + \left(2\pi fL - \dfrac{1}{2\pi fC}\right)^2}}.$$

(b) The rms voltage drop across the resistor is $V_R = I_{\text{rms}} R$. The rms voltage drop across the capacitor is $V_C = I_{\text{rms}} X_C = \dfrac{I_{\text{rms}}}{2\pi fC}$. The rms voltage drop across the inductor is $V_L = I_{\text{rms}} X_L = 2\pi I_{\text{rms}} fL$.

(c) $\langle P \rangle = I_{\text{rms}}^2 R$

CALCULATE:

(a) $$I_{\text{rms}} = \frac{(50.0\text{ V})}{\sqrt{(100.\ \Omega)^2 + \left(2\pi(2000.\text{ Hz})(0.0500\text{ H}) - \dfrac{1}{2\pi(2000.\text{ Hz})(0.400\cdot 10^{-6}\text{ F})}\right)^2}} = 0.1134\text{ A}$$

(b) $V_R = (0.1134\text{ A})(100.\ \Omega) = 11.34\text{ V}$

$$V_C = \frac{(0.1134\text{ A})}{2\pi(2000.\text{ Hz})(0.400\cdot 10^{-6}\text{ F})} = 22.56\text{ V}$$

$V_L = 2\pi(0.1134\text{ A})(2000.\text{ Hz})(0.0500\text{ H}) = 71.26\text{ V}$

(c) $\langle P \rangle = (0.1134 \text{ A})^2 (100. \ \Omega) = 1.286 \text{ W}$

ROUND: To three significant figures,

(a) $I_{\text{rms}} = 0.113 \text{ A}$

(b) $V_R = 11.3 \text{ V}, \ V_C = 22.6 \text{ V}, \ V_L = 71.3 \text{ V}$

(c) $\langle P \rangle = 1.29 \text{ W}$

DOUBLE-CHECK: For an RLC circuit, the maximum voltage is given by $V_{\text{m}}{}^2 = V_R^2 + (V_L - V_C)^2$; therefore, as a check $V_{\text{m}} = \sqrt{(11.34 \text{ V})^2 + (71.26 \text{ V} - 22.56 \text{ V})^2}$, which equals 50.003 V, which is V_{m} within rounding errors, so the answer is reasonable.

30.53. **THINK:** In order to receive the best signal, the radio should be tuned at resonance with the incoming frequency, $f_0 = 88.7 \text{ MHz}$. The inductance of the radio receiver is $L = 8.22 \ \mu\text{H}$. Signal strength is $V_{\text{m}} = 12.9 \ \mu\text{V}$. The similar signal with frequency $f = 88.5 \text{ MHz}$, is not at resonance, so its total impedance is influenced by the resistor, capacitor and inductor such that its current is half that of the current for the frequency at resonance.

SKETCH: Not required.

RESEARCH: At resonance, $\omega = \omega_0 = 1/\sqrt{LC}$, where the angular frequency is $\omega_0 = 2\pi f_0$. The impedances of the inductor and capacitor are $X_L = \omega L$ and $X_C = 1/(\omega c)$, respectively. The impedance of the RLC circuit is $Z = \sqrt{R^2 + (X_L - X_C)^2}$. At resonance, the current amplitude is $I_{\text{m}} = V_{\text{m}} / R$, and when not at resonance, the current amplitude is $I'_{\text{m}} = V_{\text{m}} / Z$.

SIMPLIFY:

(a) At resonance, $\omega_0 = \dfrac{1}{\sqrt{LC_0}} = 2\pi f_0 \ \Rightarrow \ C_0 = \dfrac{1}{4\pi^2 f_0^2 L}$.

(b) When $I'_{\text{m}} = \dfrac{1}{2} I_{\text{m}}$, then

$$\frac{V_{\text{m}}}{Z} = \frac{1}{2}\frac{V_{\text{m}}}{R_0} \ \Rightarrow \ 2R_0 = Z = \sqrt{R_0{}^2 + (X_L - X_C)^2}$$

$$4R_0^2 = R_0^2 + (X_L - X_C)^2 \ \Rightarrow \ R_0 = \left| \frac{(X_L - X_C)}{\sqrt{3}} \right| = \left| \frac{2\pi f L - (2\pi f c)^{-1}}{\sqrt{3}} \right|.$$

CALCULATE:

(a) $C = \dfrac{1}{4\pi^2 (88.7 \cdot 10^6 \text{ Hz})^2 (8.22 \cdot 10^{-6} \text{ H})} = 3.9167 \cdot 10^{-13} \text{ F}$

(b) $R_0 = \left| \dfrac{2\pi(88.5 \cdot 10^6 \text{ Hz})(8.22 \cdot 10^{-6} \text{ H}) - \left(2\pi(88.5 \cdot 10^6 \text{ Hz})(3.9167 \cdot 10^{-13} \text{ F})\right)^{-1}}{\sqrt{3}} \right| = 11.941 \ \Omega$

ROUND: To three significant figures,

(a) $C = 0.392 \text{ pF}$

(b) $R_0 = 11.9 \ \Omega$

DOUBLE-CHECK: This is an RLC circuit, so the current across the circuit decays exponentially with a time constant $\tau = 2L/R = 1.38 \ \mu\text{s}.$ Assuming that the time constant represents the delay from when the radio picks up the signal to when it transmits it as sound, it is reasonable that the value is small.

30.54. If a power station provides power P, and delivers it over potential difference V, then the current out of the station is $I = P/V$. The power loss over the lines is $P' = I^2R = (P/V)^2 R$. Therefore, if $V \to 10V$, the power loss over the lines is now $P'' = (P/10V)^2 R = (P/V)^2 R/100 = P'/100$. Therefore, if voltage is raised by a factor of 10, the power loss is 100 times smaller.

30.55. (a) From solved problem 29.1, the coil has $N_C = 31$ turns, and the solenoid has $n = 290$ turns/cm and length of $l = 12.0$ cm. Therefore, the number of turns in the solenoid is $N_S = nl$. If the voltage across the solenoid is $V_S = 120$ V, and the coil and solenoid act as a transformer, then

$$\frac{V_C}{N_C} = \frac{V_S}{N_S} \Rightarrow V_C = \frac{V_S N_C}{N_S} = \frac{V_S N_C}{nl} = \frac{(120\text{ V})(31)}{(290\text{ cm}^{-1})(12.0\text{ cm})} = 1.1\text{ V}.$$

(b) With DC current, no magnetic flux is created within the solenoid so there is no voltage in the coil.

30.56. The primary coil has $N_P = 800$ turns, and the secondary coil has $N_S = 40$ turns.

(a) The voltages across each coil is given by $V_P / N_P = V_S / N_S$, so when $V_P = 100.$ V, the voltage on the secondary coil is

$$V_S = N_S V_P / N_P = (40)(100.\text{ V})/(800) = 5.00\text{ V}.$$

(b) The current on the secondary coil when $I_P = 5.00$ A is

$$I_S = I_P V_P / V_S = (5.00\text{ A})(100.\text{ V})/(5.00\text{ V}) = 100.\text{ A}.$$

(c) When the voltage is DC, no magnetic flux is created within the secondary coil, so there is no voltage on the secondary coil.

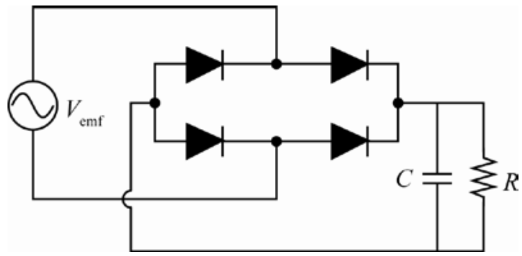
(d) When the voltage is DC, the voltage on the secondary coil is zero, so the secondary coil does not carry a current.

30.57. The primary coil has $N_P = 200$ turns and the secondary coil has $N_S = 120$ turns. The secondary coil drives a current *I* through a resistance of $R = 1.00$ kΩ. The input voltage applied across the primary coil is $V_{rms} = 75.0$ V. The voltage across the secondary coil is $V_S = V_{rms} N_S / N_P$. The power dissipated in the resistor is $P = \frac{(V_{rms} N_S / N_P)^2}{R} = \frac{((75.0\text{ V})(120)/(200))^2}{(1.00 \cdot 10^3\ \Omega)} = 2.03\text{ W}.$

30.58. The frequency of the source is $f_S = 60.$ Hz. The full wave rectifier is a configuration of diodes that allows all of the current in the circuit to flow in one direction. This is illustrated in Figure 30.35 of the textbook. Comparing the plot of emf as a function of the time (Figure 30.35 (a)) to the rectified current (Figure 30.35 (d)), it can be seen that the frequency is doubled, $f = 2f_S = 120$ Hz. The capacitor minimizes the size of the ripples, but the frequency stays the same.

30.59. **THINK:** The voltage applied to the primary side of the transformer is $V_{rms} = V_P = 110$ V and its frequency is $f = 60.$ Hz. The ratio of primary coil turns to secondary coil turns in the transformer is $N_P / N_S = 11$. The secondary coil voltage, V_S is used as the source voltage for the fullwave rectifier shown in Problem 30.56. Using the equations for transformers, the rms voltage in the secondary coil can be found. To find the DC voltage V_{DC} provided to the resistor, it is necessary to integrate over the AC voltage to obtain the time-averaged value.

SKETCH:



RESEARCH:
(a) The secondary rms voltage is given by $V_S = V_P N_S / N_P$. For any rms voltage, the maximum voltage is given $V_m = \sqrt{2} V_{rms}$.
(b) The DC voltage V_{DC} is the time-averaged value of the rectified $V_{emf.}$. The time average is found from the equation

$$V_{DC} = \frac{2}{T}\int_0^{T/2} V_{emf}(t)\,dt,$$

where $V_{emf}(t) = V_m \sin(\omega t)$.
SIMPLIFY:
(a) $V_m = \sqrt{2}\frac{V_P N_S}{N_P}$

(b) Substitute $T = \frac{2\pi}{\omega}$ into the equation:

$$V_{DC} = \frac{\omega}{\pi} V_m \int_0^{\pi/\omega} \sin(\omega t)\,dt = -\frac{V_m}{\pi}\cos(\omega t)\Big|_0^{\pi/\omega} = -\frac{V_m}{\pi}(-1-1) = \frac{2}{\pi}V_m.$$

CALCULATE:
(a) $V_m = \sqrt{2}(110 \text{ V})\left(\frac{1}{11}\right) = 14.14 \text{ V}$

(b) $V_{DC} = \frac{2}{\pi}(14.14 \text{ V}) = 9.003 \text{ V}$

ROUND: To two significant figures,
(a) $V_m = 14 \text{ V}$
(b) $V_{DC} = 9.0 \text{ V}$
DOUBLE-CHECK: The number of turns in the primary is greater than the number of turns in the secondary so it is expected that the voltage in the secondary is lower.

30.60. The given quantities are: the inductance, $L = 100.$ mH; the frequency, $f = 60.0$ Hz; and the rms voltage, $V_{rms} = 115$ V. The average power is given by $\langle P \rangle = I_{rms} V_{rms} R / Z$, so to maximize the power output the impedance must be minimized. The impedance is $Z = \sqrt{R^2 + \left(\omega L - 1/(\omega C)\right)^2}$. To minimize Z, the expression in the brackets must equal zero:

$$\omega L - \frac{1}{\omega C} = 0 \Rightarrow C = \frac{1}{\omega^2 L} = \frac{1}{(2\pi f)^2 L} = \frac{1}{\left(2\pi(60.0 \text{ Hz})\right)^2 (0.100 \text{ H})} = 7.04 \cdot 10^{-5} \text{ F}.$$

30.61. This question deals with an LC circuit. The given quantities are the frequency, $f = 1000.$ kHz and the inductance, $L = 10.0$ mH. What is the capacitance, C, of the capacitor when the station is properly tuned? Equating the expressions $\omega = 2\pi f$ and $\omega = 1/\sqrt{LC}$ and solving for C: $2\pi f = 1/\sqrt{LC}$,

$$2\pi f = \frac{1}{\sqrt{LC}} \Rightarrow C = \frac{1}{(2\pi f)^2 L}$$

$$C = \frac{1}{(2\pi f)^2 L} = \frac{1}{\left((2\pi)(1.000 \cdot 10^{6}\ \text{Hz})\right)^2 (1.00 \cdot 10^{-2}\ \text{H})} = 2.53 \cdot 10^{-12}\ \text{F}.$$

30.62. This question deals with an RLC circuit. The circuit is driven by a generator with $V_{\text{rms}} = 12.0\ \text{V}$ and frequency, f_0. The inductance is $L = 7.00\ \text{mH}$, the resistance is $R = 100.\ \Omega$ and the capacitance is $C = 0.0500\ \text{mF}$.

(a) The angular resonant frequency of the RLC circuit is given by $\omega_0 = 1/\sqrt{LC}$, where $\omega_0 = 2\pi f_0$. Therefore, the resonant frequency is

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(7.00 \cdot 10^{-3}\ \text{H})(0.0500 \cdot 10^{-3}\ \text{F})}} = 269\ \text{Hz}.$$

(b) The average power dissipated in the resistor at the resonant frequency is

$$\langle P \rangle = \frac{V_{\text{rms}}^2}{R} = \frac{(12.0\ \text{V})^2}{(100.\ \Omega)} = 1.44\ \text{W}.$$

30.63. A "60-W light bulb" dissipates power at $\langle P \rangle = 60.\ \text{W}$. The question gives $V_{\text{rms}} = 110\ \text{V}$.

(a) The maximum current is $I_{\text{m}} = \sqrt{2} I_{\text{rms}}$. The average power is $\langle P \rangle = I_{\text{rms}} V_{\text{rms}}$, so the maximum current is

$$I_{\text{m}} = \frac{\sqrt{2}\langle P \rangle}{V_{\text{rms}}} = \frac{\sqrt{2}(60.\ \text{W})}{(110\ \text{V})} = 0.77\ \text{A}.$$

(b) The maximum voltage is $V_{\text{m}} = \sqrt{2} V_{\text{rms}} = \sqrt{2}(110\ \text{V}) = 160\ \text{V}$.

30.64. The given quantities are the frequency, $f = 360\ \text{Hz}$, the inductance $L = 25\ \text{mH}$ and the resistance $R = 0.80\ \Omega$. In order for the current and voltage to be in phase, the circuit must be in resonance. This occurs when the inductive reactance and the capacitive reactance are equal (Note that this is result is independent of the resistor value):

$$X_L = X_C \ \Rightarrow\ \omega L = \frac{1}{\omega C} \ \Rightarrow\ \mu\text{F.} = \frac{1}{\omega^2 L} = \frac{1}{(2\pi f)^2 L} = \frac{1}{\left(2\pi(360\ \text{Hz})\right)^2 (25 \cdot 10^{-3}\ \text{H})} = 7.8$$

30.65. In an RLC circuit, the inductance is $L = 65.0\ \text{mH}$ and the capacitance is $C = 1.00\ \mu\text{F}$. The circuit loses electromagnetic energy at a rate of $\Delta U = -3.50\%$ per cycle. The energy stored in the electric field of the capacitor is expressed by $\Delta U_{\text{E}} = q_{\text{max}}^2 e^{-Rt/L} \cos(\omega_0 t)$. The rate of energy loss is $\frac{\Delta U_{\text{E}}}{U_{\text{E}}} = \frac{U_{\text{E, final}} - U_{\text{E, initial}}}{U_{\text{E, initial}}}$, where time t_{initial} is zero and t_{final} is the time to complete one cycle, $t_{\text{final}} = 2\pi/\omega_0$. The rate of energy loss per cycle can now be written as

$$\frac{\Delta U_{\text{E}}}{U_{\text{E}}} = -0.035 = \frac{\frac{q_{\text{max}}^2}{2C} e^{-2\pi R/\omega_0 L} \cos^2(2\pi) - \frac{q_{\text{max}}^2}{2C} e^{-0} \cos^2(0)}{\frac{q_{\text{max}}^2}{2C} e^{-0} \cos^2(0)} = e^{-2\pi R/\omega L} - 1.$$

Since $\omega_0 = 1/\sqrt{LC}$,

$$1-0.0350 = e^{-\left(2\pi R\sqrt{LC}\right)/L} \Rightarrow \ln(0.9650) = -\frac{2\pi R\sqrt{LC}}{L}$$

$$R = -\frac{\ln(0.9650)}{2\pi}\sqrt{\frac{L}{C}} = -\frac{\ln(0.9650)}{2\pi}\sqrt{\frac{(65.0\cdot 10^{-3}\text{ H})}{(1.00\cdot 10^{-6}\text{ F})}} = 1.45\ \Omega.$$

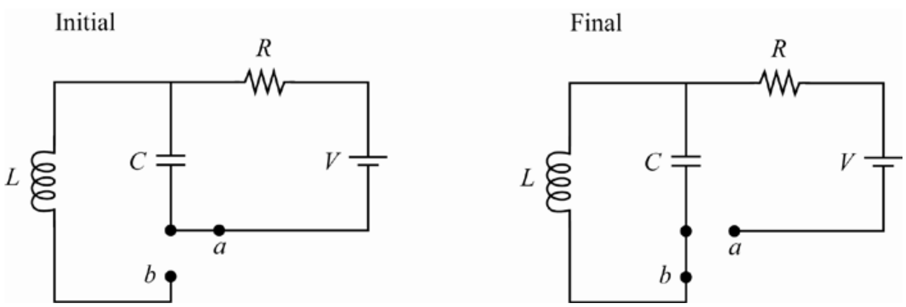
30.66. The transformer has $N_\text{P} = 400$ turns on its primary coil and $N_\text{S} = 20$ turns on its secondary coil. The average power output from the secondary coil is $\langle P\rangle = 1200.$ W and the maximum voltage output from the secondary coil is $V_\text{S, m} = 60.0$ V. The rms current in the primary coil is given by,

$$I_\text{P, rms} = \frac{I_\text{S, rms}N_\text{S}}{N_\text{P}} = \frac{\langle P\rangle N_\text{S}}{V_\text{S, rms}N_\text{P}}.$$

Substituting the maximum values for the rms values gives

$$\frac{I_\text{P, m}}{\sqrt{2}} = \frac{\langle P\rangle N_\text{S}}{\left(\frac{V_\text{S, m}}{\sqrt{2}}\right)N_\text{P}} \Rightarrow I_\text{P, m} = 2\frac{\langle P\rangle N_\text{S}}{V_\text{S, m}N_\text{P}} = 2\frac{(1200.\text{ W})(20)}{(60.0\text{ V})(400)} = 2.00\text{ A}.$$

30.67. The given quantities are the capacitance, $C = 5.00\ \mu\text{F}$, the resistance, $R = 4.00\ \Omega$ and the battery voltage, $V = 9.00$ V. The capacitor is charged for a long time by closing the switch to position *a*. At time $t = 0$ the switch is closed at position *b* and the capacitor is discharged through an inductor with $L = 40.0$ mH.



(a) The maximum current through the inductor is given by $i_\text{max} = \omega_0 q_\text{max}$, where q_max is the maximum charge on the capacitor, and $\omega_0 = 1/\sqrt{LC}$. The fully charged capacitor has charge $q_\text{max} = CV$. Substituting for ω_0 and q_max gives

$$i_\text{max} = \frac{CV}{\sqrt{LC}} = \sqrt{\frac{C}{L}}V = \sqrt{\frac{(5.00\cdot 10^{-6}\text{ F})}{(40.0\cdot 10^{-3}\text{ H})}}(9.00\text{ V}) = 0.101\text{ A}.$$

(b) The current is given by $i = -i_\text{max}\sin(\omega_0 t)$. For the current to be a maximum $\sin(\omega_0 t) = 1$, or when $\omega_0 t = \pi/2$. This occurs at time

$$t = \frac{\pi}{2}\sqrt{LC} = \frac{\pi}{2}\sqrt{(40.0\cdot 10^{-3}\text{ H})(5.00\cdot 10^{-6}\text{ F})} = 7.02\cdot 10^{-4}\text{ s}.$$

30.68. **THINK:** The resistance is $R = 10.0\text{ k}\Omega$ and the capacitance is $C = 0.0470\ \mu\text{F}$ in the RC high-pass filter. To find the frequency f where the ratio of the output voltage to the input voltage gives $20\log(V_\text{out}/V_\text{in}) = -3.00$, use the voltage ratio for an RC high-pass filter.

SKETCH: A sketch of the circuit is provided in the problem.

RESEARCH: For an RC high-pass filter, the ratio V_{out}/V_{in} is given by:

$$\frac{V_{out}}{V_{in}}=\frac{1}{\sqrt{1+\frac{1}{\omega^2R^2C^2}}}.$$

The frequency is $f=\omega/2\pi$. From the properties of logarithms, if $y=\log_b(x)$ then $x=b^y$.

SIMPLIFY: $20\log\left(\frac{V_{out}}{V_{in}}\right)=-3.00\Rightarrow\log\left(\frac{V_{out}}{V_{in}}\right)=-0.150.$ This can be rewritten as $\frac{V_{out}}{V_{in}}=10^{-0.150}.$

Substituting for $\frac{V_{out}}{V_{in}}$ gives:

$$\frac{1}{\sqrt{1+\frac{1}{\omega^2R^2C^2}}}=10^{-0.150}\Rightarrow\frac{1}{1+\frac{1}{\omega^2R^2C^2}}=10^{-0.300}\Rightarrow\frac{1}{\omega^2R^2C^2}+1=10^{0.300}\Rightarrow\omega^2R^2C^2=\frac{1}{10^{0.300}-1}$$

$$\omega=\sqrt{\frac{1}{\left(10^{0.300}-1\right)}}\frac{1}{RC}\Rightarrow f=\sqrt{\frac{1}{\left(10^{0.300}-1\right)}}\left(\frac{1}{2\pi RC}\right).$$

CALCULATE: $f=\sqrt{\frac{1}{\left(10^{0.300}-1\right)}}\left(\frac{1}{2\pi\left(10.0\cdot10^3\ \Omega\right)\left(0.0470\cdot10^{-6}\right)}\right)=339.43\text{ Hz}$

ROUND: To three significant figures, $f=339$ Hz.

DOUBLE-CHECK: The breakpoint frequency for this RC high-pass filter is $f_B=1/\left(2\pi RC\right)=338$ Hz. Since the calculated frequency is larger than this value, the answer is reasonable.

30.69. **THINK:** The unknown wire-wound resistor R is initially connected to a DC power supply. When there is a voltage of $V_{emf}=10.0$ V across the resistor, the current is $I=1.00$ A. Next the resistor is connected to an AC power source with $V_{rms}=10.0$ V. When the AC power source is operated at frequency $f=20.0$ kHz, a current of $I_{rms}=0.800$ A is measured. Find:

(a) the resistance, R;

(b) the inductive reactance, X_L, of the resistor;

(c) the inductance, L, of the resistor; and

(d) the frequency, f', of the AC power source at which $X_L=R$.

SKETCH: Not required.

RESEARCH:

(a) The resistance of the resistor when used with the DC source can be found using Ohm's law, $R=V/I$.

(b) When connected to the AC power source, the resistor can be treated as an RL series circuit. The impedance of the RL circuit is $Z=\sqrt{R^2+X_L^2}=V_{rms}/I_{rms}$.

(c) The inductance can be found with $X_L=\omega L$ and $\omega=2\pi f$.

(d) $\omega L=R$, $\omega=2\pi f'$

SIMPLIFY:

(a) $R=\frac{V}{I}$

(b) $Z=\sqrt{R^2+X_L^2}\Rightarrow X_L^2=Z^2-R^2\Rightarrow$ Substituting $Z=\frac{V_{rms}}{I_{rms}}$ gives, $X_L=\sqrt{\left(\frac{V_{rms}}{I_{rms}}\right)^2-R^2}$.

(c) $L = \frac{X_L}{\omega} = \frac{X_L}{2\pi f}$

(d) $f' = \frac{R}{2\pi L}$

CALCULATE:

(a) $R = \frac{(10.0 \text{ V})}{(1.00 \text{ A})} = 10.0 \ \Omega$

(b) $X_L = \sqrt{\left(\frac{10.0 \text{ V}}{0.800 \text{ A}}\right)^2 - (10.0 \ \Omega)^2} = 7.50 \ \Omega$

(c) $L = \frac{(7.50 \ \Omega)}{2\pi(20.0 \cdot 10^3 \text{ Hz})} = 5.968 \cdot 10^{-5} \text{ H}$

(d) $f' = \frac{10.0 \ \Omega}{2\pi(5.968 \cdot 10^{-5} \text{ H})} = 26667 \text{ Hz}$

ROUND: The answers should be reported to three significant figures.

(a) $R = 10.0 \ \Omega$

(b) $X_L = 7.50 \ \Omega$

(c) $L = 5.97 \cdot 10^{-5} \text{ H}$

(d) $f' = 26.7 \text{ kHz}$

DOUBLE-CHECK: Since the current decreased when the power supply was changed from DC to AC, the resistance must have increased. This additional resistance is explained by the inductive reactance of the resistor. The units for all calculated values are correct.

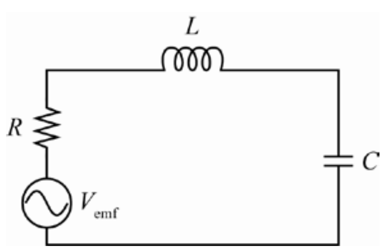
30.70. **THINK:** The components of the RLC circuit are connected in series, and their values are $R = 20.0 \ \Omega$, $L = 10.0 \text{ mH}$ and $C = 5.00 \cdot 10^{-6} \text{ F}$. They are connected to an AC source of peak voltage $V = 10.0 \text{ V}$ and frequency $f = 100. \text{ Hz}$. Find:

(a) The amplitude of the current I.

(b) The phase difference ϕ between the current and the voltage.

(c) The maximum voltage across R, L, and C.

SKETCH:



RESEARCH:

(a) The amplitude of the current in an RLC series circuit is

$$I_m = \frac{V_m}{\sqrt{R^2 + (X_L - X_C)^2}},$$

where $X_L = \omega L$ and $X_C = 1/\omega C$. Recall that $\omega = 2\pi f$ and $V_m = \sqrt{2} V_{rms}$.

(b) The phase difference between the current and the voltage is $\phi = \tan^{-1}\left(\frac{(X_L - X_C)}{R}\right)$.

(c) For the maximum voltage across each circuit component, $V_L = I_m X_L$, $V_C = I_m X_C$ and $V_R = I_m R$.

SIMPLIFY:

(a) $I_{\text{m}} = \dfrac{\sqrt{2}V_{\text{rms}}}{\sqrt{R^2 + \left(2\pi fL - 1/(2\pi fC)\right)^2}}$

(b) $\phi = \tan^{-1}\left(\dfrac{2\pi fL - 1/(2\pi fC)}{R}\right)$

(c) $V_L = 2\pi fI_{\text{m}}L$, $V_C = \dfrac{I_{\text{m}}}{2\pi fC}$ and $V_R = I_{\text{m}}R$.

CALCULATE:

(a) $I_{\text{m}} = \dfrac{\sqrt{2}(10.0 \text{ V})}{\sqrt{(20.0 \ \Omega)^2 + \left((200.\pi \text{ rad/s})(0.0100 \text{ H}) - \dfrac{1}{(200\pi \text{ rad/s})(5.00\cdot 10^{-6} \text{ F})}\right)^2}} = 0.04523 \text{ A}$

(b) $\phi = \tan^{-1}\left(\dfrac{(200.\pi \text{ rad/s})(0.0100 \text{ H}) - 1/\left((200\pi \text{ rad/s})(5.00\cdot 10^{-6} \text{ F})\right)}{(20.0 \ \Omega)}\right) = -1.507 \text{ rad}$

(c) $V_L = (0.04523 \text{ A})(200.\pi \text{ rad/s})(0.0100 \text{ H}) = 0.2842 \text{ V}$

$V_C = \dfrac{(0.04523 \text{ A})}{(200\pi \text{ rad/s})(5.00\cdot 10^{-6} \text{ F})} = 14.40 \text{ V}$

$V_R = (0.04523 \text{ A})(20.0 \ \Omega) = 0.9046 \text{ V}$

ROUND: To three significant figures,

(a) $I_{\text{m}} = 45.2 \text{ mA}$

(b) $\phi = -1.51 \text{ rad}$

(c) $V_L = 0.284 \text{ V}$, $V_C = 14.4 \text{ V}$, $V_R = 0.905 \text{ V}$

DOUBLE-CHECK: It must be true that $V_{\text{m}}^2 = V_R^2 + (V_L - V_C)^2$. To check this, plug in the calculated values for each side of the equation:

$$V_{\text{m}}^2 = \left(\sqrt{2}(V_{\text{rms}})\right)^2 = 2(10.0 \text{ V})^2 = 200. \text{ V}$$

$$V_R^2 + (V_L - V_C)^2 = (0.9046 \text{ V})^2 + \left((0.2842 \text{ V}) - (14.40 \text{ V})\right)^2 = 200. \text{ V},$$

as required.

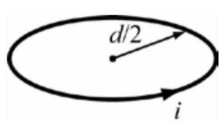
30.71. **THINK:**

(a) The loop of wire has a diameter of $d = 5.00 \text{ cm} = 0.0500 \text{ m}$ and it carries current $i = 2.00 \text{ A}$. Find the magnetic energy density, u_{B}, at the loop's center.

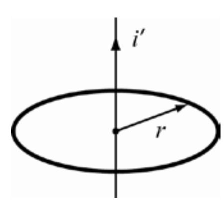
(b) Find the current, i' in a straight wire that gives the same value of u_{B} at a point, $r = 4.00 \text{ cm} = 0.0400 \text{ m}$ from the wire.

SKETCH:

(a)



(b)

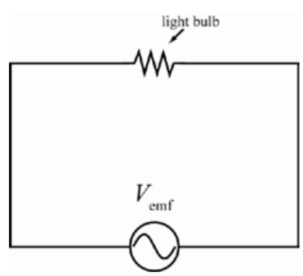


RESEARCH:
(a) The magnetic energy density is $u_B = (1/(2\mu_0))B^2$. The magnetic field, B for the wire loop is given by Equation 28.8 in the textbook, $B_{\text{loop}} = \dfrac{\mu_0 i}{2(d/2)}$.
(b) The magnetic field due to current traveling in a straight wire is given by Equation 28.4, $B_{\text{wire}} = \mu_0 i'/(2\pi r)$.
SIMPLIFY:
(a) $u_B = \left(\dfrac{1}{2\mu_0}\right)B_{\text{loop}}^2$, substituting for B_{loop} gives: $u_B = \left(\dfrac{1}{2\mu_0}\right)\left(\dfrac{\mu_0 i}{2(d/2)}\right)^2 = \dfrac{1}{2}\dfrac{\mu_0 i^2}{d^2}$.
(b) $u_B = \left(\dfrac{1}{2\mu_0}\right)B_{\text{wire}}^2$, substituting for B_{wire} gives:

$$u_B = \left(\frac{1}{2\mu_0}\right)\frac{\mu_0^2 i'^2}{4\pi^2 r^2} \Rightarrow i'^2 = \frac{8\pi^2 r^2 u_B}{\mu_0} \Rightarrow i' = \sqrt{\frac{8\pi^2 r^2 u_B}{\mu_0}}.$$

CALCULATE:
(a) $u_B = \dfrac{1}{2}\dfrac{(4\pi \cdot 10^{-7}\text{ T m/A})(2.00\text{ A})^2}{(0.0500\text{ m})^2} = 1.005 \cdot 10^{-3}\text{ J/m}^3$
(b) $i' = \sqrt{\dfrac{8\pi^2(0.0400\text{ m})^2(1.005 \cdot 10^{-3}\text{ J/m}^3)}{4\pi \cdot 10^{-7}\text{ T m/A}}} = 10.053\text{ A}$
ROUND: The answers should be reported to three significant figures.
(a) $u_B = 1.01 \cdot 10^{-3}\text{ J/m}^3$
(b) $i' = 10.1\text{ A}$
DOUBLE-CHECK: The calculated values have the proper units. It is expected that the current required to generate a given magnetic field would be much larger for a straight wire than for a loop.

30.72. **THINK:** The bulb of average power $\langle P \rangle = 75000$ W operates at a current of $I_{\text{rms}} = 200.$ A, a voltage of $V_{\text{rms}} = 440.$ V, and a frequency of $f = 60.0$ Hz. The inductive reactance of the bulb is not negligible so its impedance needs to be considered. The inductive reactance can be neglected, so $X_C = 0$.
SKETCH:



RESEARCH: The average power of the bulb is $\langle P \rangle = I_{\text{rms}}^2 R$. The rms voltage is $V_{\text{rms}} = I_{\text{rms}} Z$, where $Z = \sqrt{R^2 + X_L^2}$ and $X_L = \omega L$.
SIMPLIFY: The resistance of the bulb is

$$R = \frac{\langle P \rangle}{I_{\text{rms}}^2}.$$

The inductance of the bulb is

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{\sqrt{R^2 + (\omega L)^2}} \;\Rightarrow\; \left(R^2 + (\omega L)^2\right)\left(I_{\text{rms}}^2\right) = V_{\text{rms}}^2 \;\Rightarrow\; (\omega L)^2 = \left(\frac{V_{\text{rms}}}{I_{\text{rms}}}\right)^2 - R^2$$

$$\Rightarrow\; L = \pm\frac{1}{\omega}\sqrt{\left(\frac{V_{\text{rms}}}{I_{\text{rms}}}\right)^2 - R^2} = \frac{1}{2\pi f}\sqrt{\left(\frac{V_{\text{rms}}}{I_{\text{rms}}}\right)^2 - R^2}.$$

CALCULATE: $R = \dfrac{(75000 \text{ W})}{(200. \text{ A})^2} = 1.875\ \Omega$

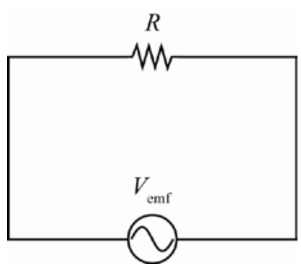
$$L = \frac{1}{2\pi(60.0 \text{ Hz})}\sqrt{\left(\frac{(440. \text{ V})}{(200. \text{ A})}\right)^2 - (1.875\ \Omega)^2} = 0.003053 \text{ H}$$

ROUND: To three significant figures, $R = 1.88\ \Omega$ and $L = 3.05$ mH.

DOUBLE-CHECK: These are reasonable values. If this was a DC source Ohm's Law would give, $R = V / I = 440. \text{ V} / 200. \text{ A} = 2.20\ \Omega$ which is comparable to the total calculated impedance. All values have the correct units.

30.73. **THINK:** A resistor R is connected across an AC source which oscillates at angular frequency ω. Show that the power dissipated in R oscillates with frequency 2ω.

SKETCH:



RESEARCH: The power is $P = i_R^2 R$. For an AC power supply, $i_R = I_R \sin(\omega t)$. Also useful is the trigonometric identity $\cos(2\theta) = 1 - 2\sin^2\theta$.

SIMPLIFY: $P = i_R^2 R = I_R^2 \sin^2(\omega t) R = I_R^2 \dfrac{1 - \cos(2(\omega t))}{2} R = \dfrac{1}{2} I_R^2 R(1 - \cos(2\omega t))$. It can be seen from the above equation that the power oscillates with a frequency twice that of the voltage.

CALCULATE: Not required.

ROUND: Not required.

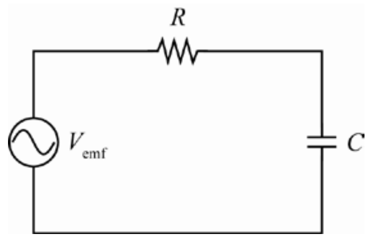
DOUBLE-CHECK: Power, P, is proportional to i^2. Since i varies proportionately with V, it must be the case that i^2 varies proportionately with V^2. Since V varies proportionately with ω, it must be the case that V^2 varies proportionately with ω^2. Therefore by transitivity, P is proportional to ω^2. Therefore, there exists a constant, c, such that $P = c\omega^2$. So the change in P with respect to time, dP/dt, will be proportional to $d\omega^2/dt$, or 2ω.

30.74. **THINK:** The resistor has a resistance of $R = 300.\ \Omega$ and is connected in series with a capacitor with $C = 4.00\ \mu\text{F}$. The AC power supply has $V_{\text{rms}} = 40.0$ V. Find:

(a) the frequency f at which $V_C = V_R$;

(b) the current I_{rms} at which this occurs.

SKETCH:



RESEARCH:
(a) $V_C = iX_C$, and $X_C = 1/\omega C$, while $V_R = iR$. Recall $\omega = 2\pi f$.

(b) In this circuit, $V_{\text{rms}} = I_{\text{rms}}Z$, where $Z = \sqrt{R^2 + X_C^2}$ (as there is no inductor).

SIMPLIFY:

(a) $V_C = V_R \Rightarrow iX_C = iR \Rightarrow \dfrac{1}{\omega C} = R \Rightarrow \dfrac{1}{2\pi fC} = R \Rightarrow f = \dfrac{1}{2\pi RC}$

(b) $I_{\text{rms}} = \dfrac{V_{\text{rms}}}{Z} = \dfrac{V_{\text{rms}}}{\sqrt{R^2 + X_C^2}} = \dfrac{V_{\text{rms}}}{\sqrt{R^2 + R^2}} = \dfrac{1}{\sqrt{2}}\dfrac{V_{\text{rms}}}{R}$

CALCULATE:

(a) $f = \dfrac{1}{2\pi(300.\ \Omega)(4.00\ \mu\text{F})} = 132.6\text{ Hz}$

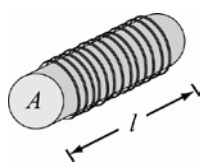
(b) $I_{\text{rms}} = \dfrac{1}{\sqrt{2}}\dfrac{(40.0\text{ V})}{(300.\ \Omega)} = 0.09428\text{ A}$

ROUND: To three significant figures, $f = 133\text{ Hz}$ and $I_{\text{rms}} = 0.0943\text{ A}$.

DOUBLE-CHECK: These values are reasonable given the initial conditions. Dimensional analysis confirms the units are correct.

30.75. **THINK:** The electromagnet has $N = 200$ loops, a length $l = 0.100\text{ m}$, and a cross-sectional area $A = 5.00\text{ cm}^2$. Find its resonant frequency f_0 when it is attached to the Earth.

SKETCH:



RESEARCH: The resonant frequency is $\omega_0 = 1/\sqrt{LC}$. Recall $\omega = 2\pi f$. The radius of the Earth is $r = 6.38 \cdot 10^6\text{ m}$ and for a spherical capacitor, $C = 4\pi\varepsilon_0 r$. From Chapter 29, $L = \mu_0 n^2 lA$ for a solenoid, where $n \equiv N/l$.

SIMPLIFY: $f_0 = \dfrac{1}{2\pi\sqrt{\mu_0 (N/l)^2 lA(4\pi\varepsilon_0 r)}} = \dfrac{1}{4N}\sqrt{\dfrac{l}{\pi^3 \mu_0 \varepsilon_0 Ar}}$

CALCULATE:

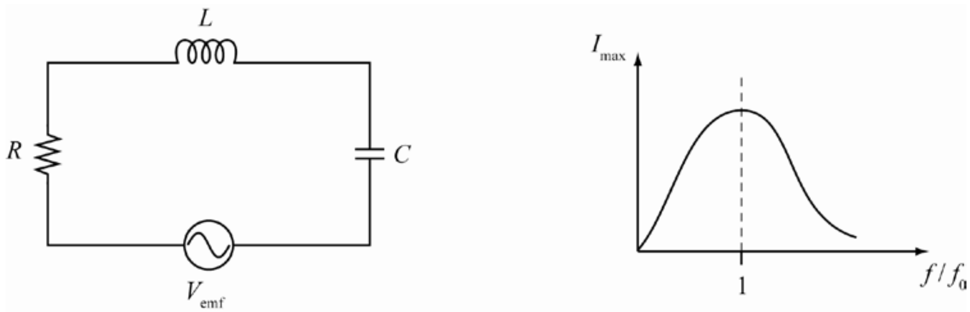
$$f_0 = \frac{1}{4(200)}\sqrt{\frac{(0.100\text{ m})}{\pi^3(4\pi\cdot 10^{-7}\text{ T m/A})(8.854\cdot 10^{-12}\text{ F/m})(5.00\cdot 10^{-4}\text{ m}^3)(6.38\cdot 10^6\text{ m})}} = 376.8\text{ Hz}$$

ROUND: To three significant figures, $f_0 = 377\text{ Hz}$.

DOUBLE-CHECK: This is a reasonable frequency for an electromagnet. Dimensional analysis confirms the units are correct.

30.76. **THINK:** The inductance of the inductor is $L = 1.00$ H. The resonance of the series RLC circuit is to occur at frequency $f_0 = 60.0$ Hz. The voltage across the capacitor (or inductor), V_C (or V_L) is to be 20.0 times that across the resistor, V_R. Find the capacitance C and the resistance R.

SKETCH:



RESEARCH: At resonance, the angular frequency is $\omega_0 = 1/\sqrt{LC}$, where $\omega_0 = 2\pi f_0$. At resonance, $V_R = V_{\text{emf}}$ and $V_L = -V_C$, where $V_L = IX_L$. Recall $V = IR$ and $X_L = \omega_0 L$.

SIMPLIFY: $C = \dfrac{1}{\omega_0^2 L} = \dfrac{1}{4\pi^2 f_0^2 L}$, $V_L = IX_L = \dfrac{V_{\text{emf}}}{R}\omega_0 L = \dfrac{2\pi V_R f_0 L}{R} \Rightarrow R = \dfrac{2\pi V_R f_0 L}{V_L}$

Since $V_L = 20.0V_R$, $R = \dfrac{2\pi V_R f_0 L}{(20.0V_R)} = \dfrac{\pi f_0 L}{10.0}$.

CALCULATE: The capacitance of the capacitor must be $C = \dfrac{1}{4\pi^2(60.0\text{ Hz})^2(1.00\text{ H})} = 7.0362\cdot 10^{-6}$ F.

The resistance of the resistor must be $R = \dfrac{\pi(60.0\text{ Hz})(1.00\text{ H})}{10.0} = 18.8496\ \Omega$.

ROUND: Rounding to three significant figures, $C = 7.04\ \mu\text{F}$ and $R = 18.8\ \Omega$.

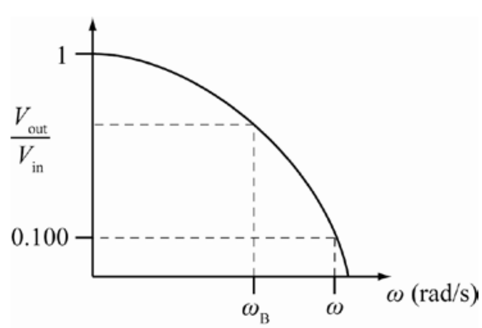
DOUBLE-CHECK: At resonance, $X_L - X_C = 0$. To check this, plug in the value found for the capacitance:

$$X_L - X_C = \omega_0 L - \frac{1}{\omega_0 C} = 2\pi f_0 L - \frac{1}{2\pi f_0 C} = 2\pi(60.0\text{ Hz})(1.00\text{ H}) - \frac{1}{2\pi(60.0\text{ Hz})(7.0362\cdot 10^{-6}\text{ F})} = 0,$$

as required.

30.77. **THINK:** The RC low-pass filter has a breakpoint frequency of $f_B = 200.$ Hz. Find the frequency at which the output voltage divided by the input voltage is $V_{\text{out}}/V_{\text{in}} = 0.100$.

SKETCH:



RESEARCH: For a RC low-pass filter, the breakpoint frequency is: $\omega_B = 1/(RC)$, where $\omega_B = 2\pi f_B$. The ratio of the input voltage to output voltage is

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}.$$

SIMPLIFY: $\omega_{\mathrm{B}} = 2\pi f_{\mathrm{B}} = \dfrac{1}{RC} \Rightarrow RC = \dfrac{1}{2\pi f_{\mathrm{B}}}$

$$\frac{V_{\text{in}}}{V_{\text{out}}} = \sqrt{1+\omega^2 (RC)^2} = \sqrt{1+\omega^2\left(\frac{1}{2\pi f_{\mathrm{B}}}\right)^2} \Rightarrow \left(\frac{V_{\text{in}}}{V_{\text{out}}}\right)^2 = 1+\left(\frac{\omega}{2\pi f_{\mathrm{B}}}\right)^2 \Rightarrow \left(\frac{V_{\text{in}}}{V_{\text{out}}}\right)^2 - 1 = \frac{\omega^2}{(2\pi f_{\mathrm{B}})^2}$$

$$\omega = 2\pi f_{\mathrm{B}}\sqrt{\left(\frac{V_{\text{in}}}{V_{\text{out}}}\right)^2 - 1} \Rightarrow f = f_{\mathrm{B}}\sqrt{\left(\frac{V_{\text{in}}}{V_{\text{out}}}\right)^2 - 1}$$

CALCULATE: $f = (200.\text{ Hz})\sqrt{\left(\dfrac{1}{0.100}\right)^2 - 1} = 1989.97\text{ Hz}$

ROUND: To three significant figures, $f = 1990$ Hz.

DOUBLE-CHECK: Since $V_{\text{out}} / V_{\text{in}}$ is less than $1/\sqrt{2}$ (the value associated with the breakpoint frequency), by the above sketch, the frequency f must be greater than the breakpoint frequency f_{B}.

Multi-Version Exercises

30.78. $X_L = 2\pi fL = 2\pi(605\text{ Hz})(42.1\text{ mH}) = 160.\ \Omega$

30.79. $I_L = V_L / X_L = V_L / (2\pi fL) = (19.9\text{ V}) / [2\pi(669\text{ Hz})(52.5\text{ mH})] = 90.2\text{ mA}$

30.80. $X_L = 2\pi fL \Rightarrow L = X_L / (2\pi f) = (81.52\ \Omega) / [2\pi(733\text{ Hz})] = 17.7\text{ mH}$

30.81. $I_L = V_L / (2\pi fL) \Rightarrow L = V_L / (2\pi f I_L) = (21.5\text{ V}) / [2\pi(797\text{ Hz})(0.1528\text{ A})] = 28.1\text{ mH}$

Chapter 31: Electromagnetic Waves

Concept Checks

31.1. e **31.2.** c **31.3.** e **31.4.** c **31.5.** a **31.6.** d **31.7.** b

Multiple-Choice Questions

31.1. c **31.2.** b, c, e **31.3.** b **31.4.** a **31.5.** a **31.6** a **31.7.** c **31.8.** b **31.9.** c **31.10.** b

Conceptual Questions

31.11. (a) The intensity of the light passing through the first polarizer is $I_1 = I_0/2$. The angle between the transmission axis of the first and second polarizer as a function of time is $\theta_{1,2}(t) = 45° + \omega t$, where *t* is in seconds and ω is in rad/s. The intensity of the light passing the second polarizer is then $I_2 = I_1\cos^2\theta_{1,2} = \frac{1}{2}I_0\cos^2(45° + \omega t) = I_2(t)$. The angle between the transmission axis of the second and third polarizer as a function of time is $\theta_{2,3}(t) = 45° - \omega t$. The intensity of the light passing the third polarizer is:

$$I_3 = I_2\cos^2\theta_{2,3} = \frac{1}{2}I_0\cos^2(45° + \omega t)\cos^2(45° - \omega t) = \frac{1}{2}I_0\left[\cos(45° + \omega t)\cos(45° - \omega t)\right]^2 = I_3(t).$$

Now, use the trigonometric identity, $\cos u\cos v = \frac{1}{2}\left[\cos(u+v) + \cos(u-v)\right]$:

$$\begin{aligned}\cos(45° + \omega t)\cos(45° - \omega t) &= \frac{1}{2}\left[\cos(45° + \omega t + 45° - \omega t) + \cos(45° + \omega t - 45° + \omega t)\right] \\ &= \frac{1}{2}(\cos 90° + \cos 2\omega t) = \frac{1}{2}\cos 2\omega t.\end{aligned}$$

Therefore, $I_3 = \frac{1}{2}I_0\left[\frac{1}{2}\cos 2\omega t\right]^2 = \left(\frac{1}{2}I_0\right)\frac{1}{4}\cos^2(2\omega t) = \frac{1}{8}I_0\cos^2(2\omega t)$. Next, make use of the identity $\cos^2 u = \frac{1+\cos 2u}{2}$, which in this case means: $\cos^2(2\omega t) = \frac{1+\cos(4\omega t)}{2}$. So:

$$I_3(t) = \frac{1}{8}I_0\left(\frac{1+\cos(4\omega t)}{2}\right) = \frac{1}{16}I_0 + \frac{1}{16}I_0\cos(4\omega t).$$

So, the intensity of the light making it past the third polarizer will oscillate about the value of $I_0/16$ as a cosine function with amplitude of $I_0/16$ and an angular frequency that is four times the angular frequency at which the polarizer is rotating. Thus, the intensity oscillates between a minimum value of zero (when polarizer 2 is parallel to either polarizer 1 or polarizer 3) and a maximum value of $I_0/8$ when polarizer 2 is at $45°$ with polarizer 1 and 3. The result is thus consistent with the $I_0/8$ result of Example 31.4, where polarizer 2 was at a fixed angle of $45°$.

(b) The transmission axis of a polarizer is a direction in the plane of the polarizer, not a single specific line in the plane of the polarizer. Therefore, moving the second polarizer parallel to itself in any direction will not change anything for the light passing through the second polarizer. Light that is incident on the first polarizer but that does not pass through the second polarizer will not pass through the third polarizer at all. In other words, if the light is initially incident on the total surface area of the first polarizer, the total amount of light (i.e. number of photons) that passes through the third polarizer after the second polarizer is displaced by a distance $d < R$, will be proportional to the fraction of surface area of overlap between all three polarizers.

31.12. Charge moving up and down along the antenna creates an electric dipole on the antenna. This produces an electric field along $\pm\vec{z}$ at point *A* (parallel to the antenna). From Ampere's law, the current produces a magnetic field along $\pm\vec{x}$ at point *A*. Since radiation is moving along the $\pm\vec{y}$ direction away from the antenna, the possible directions for $\vec{E}$ and $\vec{B}$ are:

(a) $\vec{E}$ in positive $\vec{z}-$direction and $\vec{B}$ in the positive $\vec{x}-$direction.

(b) $\vec{E}$ in the negative $\vec{z}-$direction and $\vec{B}$ in the negative $\vec{x}-$direction.

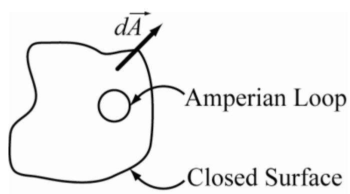
See Figure 31.16 as a visual aid.

31.13. Assuming that the randomly polarized light source (the sun) was replaced by a polarized source, the results are still correct since for randomly polarized light, the average of E^2 is the same as the average of E^2 for a polarized light.

31.14. A magnetic monopole is a magnet with only one pole. The magnetic field produced by the monopole is similar to the electric field produced by an electric charge. The magnetic field vector is directed radially outward, that is, $\vec{B} \propto \vec{r}$. If a charged particle is moving parallel to $\vec{r}$, its motion is not affected by the magnetic field. However, if the particle is moving perpendicular to $\vec{r}$, then its motion is affected and there is a perpendicular force to its velocity producing a helical motion.

31.15. Both signals would arrive at the same time, since the speed of both signals is the speed of light. This is correct provided there is no medium between the Earth and the Moon. It is known that the speed of light in a medium depends on its refractive index. The refractive index of the medium depends also on the frequency of light.

31.16.



Ampere's law is defined as:

$$\oint \vec{B} \bullet d\vec{S} = \mu_0 \oint_S \vec{J} \bullet d\vec{A},$$

where $\vec{J}$ is a charge density within the surface. Since the surface is closed, the Amperian loop can be made very small, such that $\oint \vec{B} \bullet d\vec{S} = 0.$ The integral $\oint \vec{J} \bullet d\vec{A}$ represents the net rate of charge transport out of the region bounded by the surface, *S*. In a static situation, the charge in the region is constant and the integral is $\oint \vec{J} \bullet d\vec{A} = 0.$ For a dynamical situation, the integral must be equal to the negative rate of change of charge in the region, that is,

$$\oint \vec{J} \bullet d\vec{A} = -\frac{dQ_{\text{in}}}{dt} \neq 0.$$

Therefore, there is inconsistency in Ampere's law. The Maxwell-Ampere law, however, take the form:

$$\oint \vec{B} \bullet d\vec{S} = \mu_0 \oint_S \vec{J} \bullet d\vec{A} + \mu_0 \varepsilon_0 \frac{d}{dt} \oint_S \vec{E} \bullet d\vec{A}.$$

Applying Gauss's law, it is found that: $-\mu_0 \frac{dQ_{\text{in}}}{dt} + \mu_0 \frac{dQ_{\text{in}}}{dt} = 0.$ Therefore, the Ampere-Maxwell law is satisfied. For Faraday's law: $\oint \vec{E} \bullet d\vec{S} = -\frac{d}{dt} \oint_S \vec{B} \bullet d\vec{A}.$ Both sides give zero since Gauss' law states:

$\oint_S \vec{B} \bullet d\vec{A} = 0$. That is, there is no magnetic charge in the region enclosed by *S*. Therefore, there is no inconsistency.

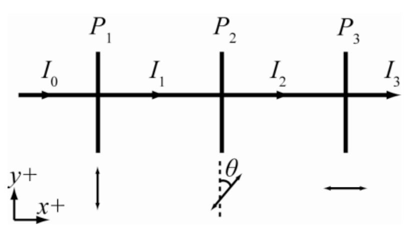
31.17. According to Maxwell's equations, the velocity of light always has a fixed value regardless of the observer's speed. This is in direct contradiction to Newton's laws of motion based on the Galilean addition law for velocities. According to the Galilean addition law, the velocity of light should not be the same in all inertial frames. Therefore, Maxwell's equations and Newton's laws of motion are mutually inconsistent.

31.18. Our vision begins with chemical reactions in the rod and cone cells of the retinas of our eyes, which release neurotransmitters in response to electromagnetic radiation in the visible range. These are resonant processes –the transfer of energy from the electromagnetic field to the nervous system is enhanced by a matching of frequencies. As in all resonances, high amplification can only be achieved in a narrow resonance peak. In order to achieve the sensitivity necessary for seeing, a narrow bandwidth is needed. Therefore, the narrow frequency band is necessary for our eyes to have sufficient sensitivity. This sensitivity allows seeing in high and low intensity situations. This is the reason that it is impossible to have a wide range of frequencies that can be seen.

31.19. (a) From energy conservation, the power per unit area or intensity of radiation from a point source must be inversely proportional to r^2.
(b)The radiation field falls off with distance at the same rate as the electrostatic field of a point charge which falls off according to $E = kQ / r^2$.

31.20. As discussed in Section 31.10, LCD displays use polarizing filters in one of their display components. Therefore, the light emitted by the LCD is a polarized light. Since some sunglasses also have polarizing filters for their lenses, the intensity of LCD light passing through the glass varies as the sunglasses are rotated. It can be concluded that the sunglasses must be polarizing the light.

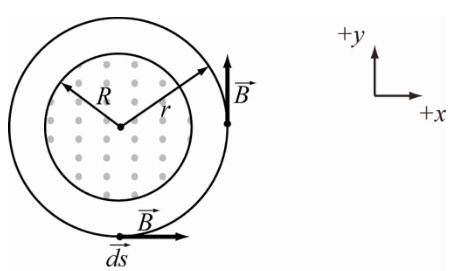
31.21.



Since the light reaching P_2 is polarized, the transmitted intensity is given by $I_2 = I_1 \cos^2\theta$. Similarly, the intensity passing through P_3 is given by $I_3 = I_2 \cos^2(90° - \theta) = I_1 \cos^2\theta \sin^2\theta = \sin^2(2\theta)/4$. Therefore, as the intermediate filter is rotated, the intensity of light passing through the polarizers will increase to a maximum at $\theta = 45°$ and then it will decrease to zero at $\theta = 90°$ and it will continue in this pattern every $45°$ increase in the rotation of the second polarizer.

Exercises

31.22.



Applying Maxwell's law of induction along a closed loop of radius, r, gives:

$$\oint \vec{B} \bullet d\vec{S} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}, \text{ where } \Phi_E \text{ is given by } \Phi_E = EA = E\pi R^2.$$

Substituting the expression for Φ_E into the equation for Maxwell's law gives:

$$B\oint dS = \mu_0 \varepsilon_0 \pi R^2 \frac{dE}{dt} \Rightarrow B(2\pi r) = \mu_0 \varepsilon_0 \pi R^2 \frac{dE}{dt}.$$

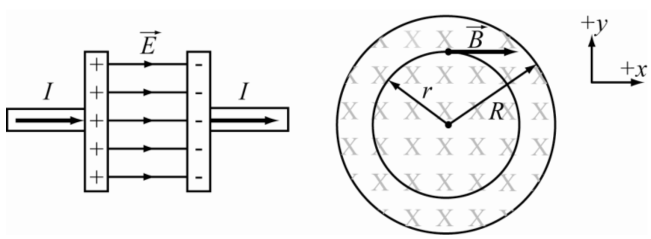
Thus, the magnetic field is: $B = \frac{1}{2}\mu_0 \varepsilon_0 \frac{R^2}{r}\left(\frac{dE}{dt}\right)$. Substituting R = 0.0600 m, r = 0.100 m and $dE/dt = 10.0$ V/m s yields:

$$B = \frac{1}{2}\left(4\pi \cdot 10^{-7} \text{ H/m}\right)\left(8.85 \cdot 10^{-12} \text{ F/m}\right)\frac{(0.0600 \text{ m})^2}{0.100 \text{ m}}(10.0 \text{ V/m s}) = 2.00 \cdot 10^{-18} \text{ T}.$$

Because dE/dt is positive, the direction of $\vec{B}$ is counterclockwise, as shown in the figure above.

31.23. **THINK:** A magnetic field can be produced by a current and by induction due to a change in an electric flux. To solve this problem, use the Maxwell-Ampere law. There is no current between the plates, but there is a change in the electric flux. The wire carries a current $i = 20.0$ A. The parallel plate capacitor has radius $R = 4.00$ cm, and separation $s = 2.00$ mm. The radius of interest is $r = 1.00$ cm from the center of the parallel plates.

SKETCH:



RESEARCH: Since there is no current between the capacitor plates, the Maxwell-Ampere law becomes:

$$\oint \vec{B} \bullet d\vec{S} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}.$$

SIMPLIFY: Applying this law along a circular Amperian loop with a radius, $r \leq R$, as shown above. Since $\vec{B}$ is parallel to $d\vec{S}$, the left-hand side of the above equation is $\oint \vec{B} \bullet d\vec{S} = B\oint dS = B2\pi r$. Assuming the electric field, $\vec{E}$, is uniform between the capacitor plates and directed perpendicular to the plates, the electric flux through the loop is $\Phi_E = EA_r = E\pi r^2$. Thus, the Ampere-Maxwell law becomes:

$$B(2\pi r) = \mu_0 \varepsilon_0 \pi r^2 \frac{dE}{dt}.$$

Therefore, the magnetic field is: $B = \left(\frac{\mu_0 \varepsilon_0 r}{2}\right)\frac{dE}{dt}$. Since the electric field of the capacitor is $E = \sigma/\varepsilon_0$, the rate of change of the electric field is given by:

$$\frac{dE}{dt} = \frac{d}{dt}(\sigma/\varepsilon_0) = \left(\frac{1}{\varepsilon_0}\right)\frac{d\sigma}{dt} = \left(\frac{1}{\varepsilon_0}\right)\frac{d}{dt}(q/A_R) = \left(\frac{1}{\varepsilon_0 A_R}\right)\frac{dq}{dt}.$$

Since $i = dq/dt$, $\frac{dE}{dt} = \frac{i}{\varepsilon_0 \pi R^2}$. Using this result, the magnetic field is: $B = \left(\frac{\mu_0 \varepsilon_0 r}{2}\right)\frac{i}{\varepsilon_0 \pi R^2} = \left(\frac{\mu_0 i}{2\pi R^2}\right)r.$

CALCULATE: $B = \frac{\left(4\pi \cdot 10^{-7} \text{ H/m}\right)(20.0 \text{ A})}{2\pi(0.0400 \text{ m})^2}(0.0100 \text{ m}) = 2.50 \cdot 10^{-5} \text{ T}$

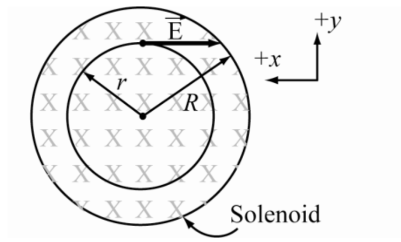
ROUND: Three significant figures are required: $B = 2.50 \cdot 10^{-5}$ T.

DOUBLE-CHECK: This is the same as calculating a magnetic field inside a wire with radius, R. Applying Ampere's law gives:

$$B = \frac{\mu_0}{2\pi r}\left(\frac{\pi r^2}{\pi R^2}\right)i = \left(\frac{\mu_0 i}{2\pi R^2}\right)r. \text{ This is the same result as above.}$$

31.24. **THINK:** To determine the electric field, apply Faraday's law of induction. The solenoid is 20.0 cm long, 2.00 cm in radius, and has 500. turns. The current varies from 3.00 A to 1.00 A in 0.100 s.

SKETCH:



RESEARCH: The magnetic field inside a solenoid is given by $B = \mu_0 ni = \mu_0 Ni / L$. Applying Faraday's law along a loop with radius, r, gives:

$$E = -\left(\frac{1}{2\pi r}\right)\frac{d\Phi_{\text{B}}}{dt}.$$

SIMPLIFY: Substituting $\Phi_{\text{B}} = BA = \mu_0 Ni\pi r^2 / L$ into the above equation yields:

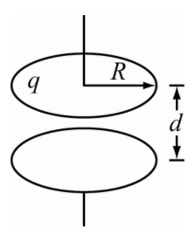
$$E = -\left(\frac{1}{2\pi r}\right)\left(\frac{\mu_0 N\pi r^2}{L}\right)\frac{di}{dt} = -\left(\frac{\mu_0 Nr}{2L}\right)\frac{di}{dt} = -\left(\frac{\mu_0 Nr}{2L}\right)\left(\frac{i_2 - i_1}{\Delta t}\right).$$

CALCULATE: $E = -\left(\frac{\left(4\pi \cdot 10^{-7} \text{ H/m}\right)(500.)(0.0100 \text{ m})}{2(0.200 \text{ m})}\right)\left(\frac{1.00 \text{ A} - 3.00 \text{ A}}{0.100 \text{ s}}\right) = 3.142 \cdot 10^{-4} \text{ V/m}$

ROUND: Keeping three significant figures: $E = 3.14 \cdot 10^{-4}$ V/m.

DOUBLE-CHECK: The direction of the induced electric field must be such that the magnetic field induced by the current opposes the change in magnetic flux. Because the magnetic flux is decreasing, the induced magnetic field will be in the same direction as the original magnetic field. The fact that the calculated electric field is positive confirms that this requirement is satisfied.

31.25.

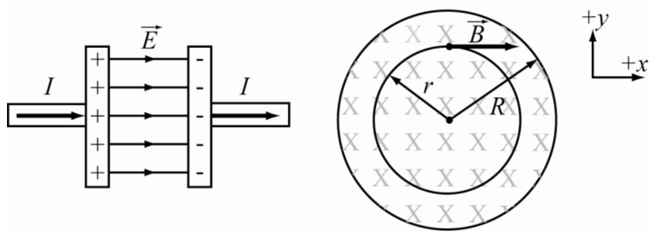


The displacement current i_{d} produced by a rate of change in the electric field of a parallel plate capacitor is $i_{\text{d}} = \varepsilon_0 d\Phi_{\text{E}} / dt$. The flux Φ_{E} is given by $\Phi_{\text{E}} = \sigma A / \varepsilon_0 = q / \varepsilon_0$. Therefore, the displacement current is:

$$i_{\text{d}} = \varepsilon_0 \frac{1}{\varepsilon_0}\frac{dq}{dt} = 10.0 \ \mu\text{A}.$$

31.26. **THINK** This problem is similar to problem 31.21 except that here, the rate of change of the potential difference across the capacitor is given. In order to get the induced magnetic field, apply Maxwell's law of induction. The parallel plate capacitor has radius $R = 10.0$ cm and separation $d = 5.00$ mm. The potential in increasing at a rate of 1.20 kV/s. The radius of interest is $r = 4.00$ cm from the center of the capacitor.

SKETCH:



RESEARCH: Applying Maxwell's law of induction along a circular loop with a radius, $r \le R$, and assuming a uniform electric field yields $\oint \vec{B} \bullet d\vec{S} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$.

SIMPLIFY: $\oint \vec{B} \bullet d\vec{S} = B2\pi r = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \varepsilon_0 \frac{dEA}{dt} = \mu_0 \varepsilon_0 A \frac{d(V/d)}{dt} = \frac{\mu_0 \varepsilon_0 A}{d}\frac{dV}{dt}$

The magnetic field is: $B = \left(\frac{\mu_0 \varepsilon_0 \pi r^2}{2\pi r d}\right) \cdot \frac{d(\Delta V)}{dt} = \left(\frac{\mu_0 \varepsilon_0 r}{2d}\right)\frac{d(\Delta V)}{dt}$.

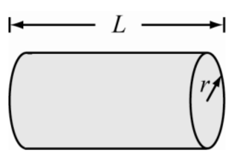
CALCULATE: $B = \left(\frac{(4\pi \cdot 10^{-7} \text{ H/m})(8.85 \cdot 10^{-12} \text{ F/m})(0.0400 \text{ m})}{2(5.00 \cdot 10^{-3} \text{ m})}\right) 1.20 \cdot 10^3 \text{ V/s} = 5.338 \cdot 10^{-14} \text{ T}$

ROUND: Rounding to three significant figure gives $B = 5.34 \cdot 10^{-14}$ T.

DOUBLE-CHECK: Treating the area between the parallel plates as a solid conductor carrying a current of magnitude equal to the displacement current, $i_d = \varepsilon_0 AdE/dt$, the problem becomes one of finding the magnetic field inside a current carrying wire. Applying Ampere's Law, $B = \left(\frac{\mu_0 i_d}{2\pi R^2}\right) r = \left(\frac{\mu_0 \varepsilon_0 AdE/dt}{2\pi R^2}\right) r = \left(\frac{\mu_0 \varepsilon_0 (\pi R^2) dV/dt}{2\pi R^2 d}\right) r = \frac{\mu_0 \varepsilon_0}{2d}\frac{dV}{dt} r$, which is the same result as that obtained by applying Maxwell's law of induction.

31.27. **THINK:** To determine the displacement current, the electric field inside the conductor is needed.

SKETCH:



RESEARCH: The displacement current is defined as: $i_d = \varepsilon_0 d\Phi_E / dt$. The electric flux inside the conductor is: $\Phi_E = EA = (V/L)A$.

SIMPLIFY: Since $V = iR$, the electric flux becomes $\Phi = iRA/L$. Therefore, the displacement current is:

$$i_d = \varepsilon_0 R\left(\frac{A}{L}\right)\frac{di}{dt}.$$

Using $R = \rho L/A$ or $\rho = RA/L$, the displacement current simplifies to: $i_d = \varepsilon_0 \rho \frac{di}{dt}$.

CALCULATE: Not required.

ROUND: Not required.

DOUBLE-CHECK: Since the current depends in part on the resistance of the current carrying conductor, and the resistance depends on the geometry and resistivity of the material, it makes sense that the current is some function of the resistivity.

31.28. The amplitude of the B field of an electromagnetic field is related to the electric field by $B = E/c$. Therefore,

$$B = \frac{250.\text{ V/m}}{3.00\cdot 10^{8}\text{ m/s}} = 8.33\cdot 10^{-7}\text{ T.}$$

31.29. The distance traveled by light is given by:

$$x = c\Delta t = 3.00\cdot 10^{8}\text{ m/s}\left(1.00\cdot 10^{-9}\text{ s}\right) = 0.300\text{ m} = 0.300\text{ m}\left(3.28\ \frac{\text{ft}}{\text{m}}\right) = 0.984\text{ ft.}$$

31.30. The interval of time taken by light to travel:

(a) from the Moon to the Earth is: $\Delta t = \dfrac{d}{c} = \dfrac{3.84\cdot 10^{8}\text{ m}}{3.00\cdot 10^{8}\text{ m/s}} = 1.28\text{ s},$

(b) from the Sun to the Earth is: $\Delta t = \dfrac{1.50\cdot 10^{11}\text{ m}}{3.00\cdot 10^{8}\text{ m/s}} = 500\text{ s} = 8.33\text{ min},$

(c) Here we first calculate the time from Jupiter to the Sun:

Perihelion: $\Delta t = \dfrac{7.41\cdot 10^{11}\text{ m}}{3.00\cdot 10^{8}\text{ m/s}} = 2470\text{ s}$

Apehelion: $\Delta t = \dfrac{8.17\cdot 10^{11}\text{ m}}{3.00\cdot 10^{8}\text{ m/s}} = 2723\text{ s}$

The shortest time is then the time from Jupiter at perihelion to the Earth when it is on the same side, which is $2470 - 500 = 1970\text{ s} = 32.8\text{ min}$. The longest time is the time from Jupiter at aphelion to the Earth on the other side, which is $2723 + 500 = 3223\text{ s} = 53.7\text{ min}$.

31.31. (a) The time delay from New York to Baghdad by cable is $\Delta t = \dfrac{d}{c} = \dfrac{1\cdot 10^{7}\text{ m}}{3.00\cdot 10^{8}\text{ m/s}} = 0.03\text{ s}.$

(b) The time delay via satellite is given by $\Delta t = d/c$. The distance, d, is given by twice the distance from New York to the satellite, that is, $d = 2\sqrt{\left(36000\text{ km}\right)^{2} + \left(5000\text{ km/2}\right)^{2}} = 2\cdot 36345\text{ km} = 72691\text{ km}$. The time delay is:

$$\Delta t = \frac{7.269\cdot 10^{7}\text{ m}}{3.00\cdot 10^{8}\text{ m/s}} = 0.24\text{ s.}$$

When the signal travels by the cable, the time delay is very short, so it is not noticeable. However, the time delay for the signal traveling via satellite is about a quarter of a second. This means in a conversation, Alice will find that she receives a response from her fiancé after 0.5 s, which is quite noticeable.

31.32. **THINK:** The speed of electromagnetic waves in a vacuum is different from the speed of such waves in different media. The difference depends on the dielectric constant, κ, and the relative permeability, κ_m, of the material.

SKETCH: Not required.

RESEARCH: The speed of electromagnetic waves in a material is $v = 1/\sqrt{\mu\varepsilon}$ and the speed in a vacuum is $c = 1/\sqrt{\mu_0\varepsilon_0}$. The permittivity is $\varepsilon = \kappa\varepsilon_0$, and the permeability is $\mu = \kappa_{\text{m}}\mu_0$.

SIMPLIFY: The ratio of the speed of electromagnetic waves in a vacuum to the speed in a material is: $\dfrac{c}{v} = \dfrac{1/\sqrt{\mu_0\varepsilon_0}}{1/\sqrt{\mu\varepsilon}} = \sqrt{\dfrac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \sqrt{\kappa\kappa_{\text{m}}}$. This ratio is the index of refraction.

CALCULATE: Not applicable.

ROUND: Not applicable.
DOUBLE CHECK: The calculated ratio is the index of refraction, which is a measure of how much the speed of light, or other electromagnetic waves, is reduced in a medium compared to the speed in a vacuum.

31.33. The relation between the wavelength of light and the frequency is $\lambda f = c.$ Therefore, the frequencies for the wavelengths of 400 nm and 700 nm are:

$$f_1 = \frac{3.00 \cdot 10^8 \text{ m/s}}{400 \cdot 10^{-9} \text{ m}} = 7.5 \cdot 10^{14} \text{ Hz and } f_2 = \frac{3.00 \cdot 10^8 \text{ m/s}}{700 \cdot 10^{-9} \text{ m}} = 4.3 \cdot 10^{14} \text{ Hz.}$$

The range of frequencies is $4 \cdot 10^{14}$ Hz to $8 \cdot 10^{14}$ Hz.

31.34. Using the relation between frequency and wavelength, the operating frequency of the signal of a cell phone is $f = c / \lambda.$ Since $L = \lambda / 4,$ the frequency is:

$$f = \frac{c}{4L} = \frac{3.00 \cdot 10^8 \text{ m/s}}{4(0.080 \text{ m})} = 9.4 \cdot 10^8 \text{ Hz} = 940 \text{ MHz.}$$

31.35. **THINK:** To solve this problem, the frequency of oscillation of an RLC circuit must be determined. The circuit has a capacitor $C = 2.0 \cdot 10^{-12}$ F, and must have a resonance frequency such that it will generate a radio wave with wavelength $\lambda = 150$ m.
SKETCH: A sketch is not required.
RESEARCH: The angular frequency of the *RLC* circuit in resonance is $\omega_0 = 1/\sqrt{LC}.$
SIMPLIFY: Using $\omega_0 = 2\pi f$ and $f = c / \lambda,$ the above equation becomes: $\frac{2\pi c}{\lambda} = \frac{1}{\sqrt{LC}}.$ The inductance required in the *RLC* circuit is: $L = \frac{\lambda^2}{(2\pi c)^2 C}.$

CALCULATE: $L = \frac{(150 \text{ m})^2}{\left[2\pi(3.00 \cdot 10^8 \text{ m/s})\right]^2 (2.0 \cdot 10^{-12} \text{ F})} = 0.00317 \text{ H}$

ROUND: Rounding to two significant figures yields $L = 3.2$ mH.
DOUBLE-CHECK: A wavelength of 150 m corresponds to a frequency of $2 \cdot 10^6$ Hz. Such a large frequency necessarily requires a fairly small inductance.

31.36. **THINK:** The radio frequencies given are: $f_1 = 91.1$ MHz, $f_2 = 91.3$ MHz, and $f_3 = 91.5$ MHz. To determine the wavelength width of the band-pass filter used in a radio receiver, the wavelengths of the three radio frequencies are required.
SKETCH: A sketch is not required.
RESEARCH: Wavelength is related to frequency by $\lambda = c / f.$ The maximum bandwidth required to distinguish between two adjacent frequencies is given by $\lambda_{12} = 2(\lambda_1 - \lambda_2)$ for f_1 and f_2, and $\lambda_{23} = 2(\lambda_2 - \lambda_3)$ for f_2 and f_3. Thus, the maximum allowable bandwidth to distinguish all three frequencies is $\Delta\lambda = \min(\Delta\lambda_{12}, \Delta\lambda_{23}).$
SIMPLIFY: Simplification is not necessary.
CALCULATE: The corresponding wavelengths of the three radio frequencies are given by: $\lambda_1 = \frac{c}{f_1} = \frac{3.00 \cdot 10^8 \text{ m/s}}{91.1 \cdot 10^6 \text{ Hz}} = 3.293 \text{ m},$ $\lambda_2 = \frac{c}{f_2} = \frac{3.00 \cdot 10^8 \text{ m/s}}{91.3 \cdot 10^6 \text{ Hz}} = 3.286 \text{ m},$ and finally, $\lambda_3 = \frac{c}{f_3} = \frac{3.00 \cdot 10^8 \text{ m/s}}{91.5 \cdot 10^6 \text{ Hz}} = 3.279 \text{ m}.$

The differences of two adjacent wavelengths are:

$$\Delta\lambda_{12} = 2(3293 \text{ mm} - 3286 \text{ mm}) = 14 \text{ mm} \text{ and } \Delta\lambda_{23} = 2(3286 \text{ mm} - 3279 \text{ mm}) = 14 \text{ mm}.$$

Therefore, the maximum allowable wavelength bandwidth is $\Delta\lambda = 14$ mm.

ROUND: Rounding is not necessary.

DOUBLE-CHECK: A larger wavelength width in the band pass filter would allow overlap between two signals, resulting in interference. This result is reasonable.

31.37. The magnitude of a Poynting vector is given by: $S = \dfrac{\text{Power}}{\text{Spherical Area}} = \dfrac{\text{Power}}{4\pi R^2}$. Therefore, the magnitudes of the Poynting vectors are:

(a) $S = \dfrac{1.5 \text{ W}}{4\pi(0.30 \text{ m})^2} = 1.3 \text{ W/m}^2$,

(b) $S = \dfrac{1.5 \text{ W}}{4\pi(0.32 \text{ m})^2} = 1.2 \text{ W/m}^2$,

(c) $S = \dfrac{1.5 \text{ W}}{4\pi(1.00 \text{ m})^2} = 0.12 \text{ W/m}^2$.

31.38. (a) The electric field experienced by an electron is:

$$E = \frac{kq}{r^2} = \frac{(8.99 \cdot 10^9 \text{ N m}^2/\text{C}^2)(1.602 \cdot 10^{-19} \text{ C})}{(0.050 \cdot 10^{-9} \text{ m})^2} = 5.761 \cdot 10^{11} \text{ V/m} \approx 5.8 \cdot 10^{11} \text{ V/m}.$$

(b) The intensity of a laser beam is related to the rms electric field by:

$$I = \frac{1}{\mu_0 c} E_{\text{rms}}^2 = \frac{(5.761 \cdot 10^{11} \text{ V/m})^2}{(4\pi \cdot 10^{-7} \text{ H/m})(3.00 \cdot 10^8 \text{ m/s})} = 8.8 \cdot 10^{20} \text{ W/m}^2.$$

31.39. The intensity of the laser beam is $I = P/A$. This intensity is related to the amplitude of the electric field by $I = E^2/(2\mu_0 c)$. Therefore, the amplitude of the electric field in the beam is:

$$E = \sqrt{\frac{2\mu_0 cP}{A}} = \sqrt{\frac{2(4\pi \cdot 10^{-7} \text{ H/m})(3.00 \cdot 10^8 \text{ m/s})(3.00 \cdot 10^3 \text{ W})}{\pi(0.500 \cdot 10^{-3} \text{ m})^2}} = 1.697 \cdot 10^6 \text{ V/m} \approx 1.70 \cdot 10^6 \text{ V/m}.$$

31.40. The electric field of an electromagnetic radiation is related to its magnetic field by $E = cB$. Therefore, the maximum E in the region is $E_{\text{m}} = cB_{\text{m}} = (3.00 \cdot 10^8 \text{ m/s})(0.00100 \text{ T}) = 3.00 \cdot 10^5 \text{ V/m}$. The period of oscillation is: $T = \dfrac{1}{f} = \dfrac{1}{1 \text{ Hz}} = 1$ s. The magnitude of the Poynting vector is:

$$S_{\text{m}} = \frac{1}{\mu_0} E_{\text{m}} B_{\text{m}} = \frac{(3.00 \cdot 10^5 \text{ V/m})(0.001 \text{ T})}{4\pi \cdot 10^{-7} \text{ H/m}} = 2.3873 \cdot 10^8 \text{ W/m}^2 = 2.39 \cdot 10^8 \text{ W/m}^2.$$

31.41. The average value of the Poynting vector, S_{ave}, is:

$$S_{\text{ave}} = \frac{1}{2\mu_0 c} E_{\text{m}}^2 = \frac{(100. \text{ V/m})^2}{2(4\pi \cdot 10^{-7} \text{ H/m})(3.00 \cdot 10^8 \text{ m/s})} = 13.3 \text{ W/m}^2.$$

(a) The average energy density is: $u = \dfrac{1}{2}\varepsilon_0 E^2 = \dfrac{1}{2}(8.85 \cdot 10^{-12} \text{ C}^2/\text{N m})(100. \text{ V/m})^2 = 4.43 \cdot 10^{-8} \text{ J/m}^3$.

(b) The amplitude of the magnetic field is: $B = \dfrac{E}{c} = \dfrac{100. \text{ V/m}}{3.00 \cdot 10^8 \text{ m/s}} = 3.33 \cdot 10^{-7}$ T.

31.42. **THINK:** The maximum electric field of a beam of light is given as $E_{\text{m}} = 3.0 \cdot 10^6$ V/m.

SKETCH: A sketch is not necessary.

RESEARCH:

(a) The magnitude of a magnetic field is related to the magnitude of an electric field by $B = E / c$.

(b) The intensity of the wave is given by $I = E_{\text{m}}^2 / \left(2\mu_0 c\right)$.

(c) If the electric field is above this maximum value, the air will be ionized by the presence of the electric field. The energy of the wave will be dissipated in the ionized air.

SIMPLIFY: Simplification is not necessary.

CALCULATE:

(a) $B = \dfrac{3.0 \cdot 10^6 \text{ V/m}}{3.00 \cdot 10^8 \text{ m/s}} = 1.0 \cdot 10^{-2} \text{ T}$

(b) $I = \dfrac{\left(3.0 \cdot 10^6 \text{ V/m}\right)^2}{2\left(4\pi \cdot 10^{-7} \text{ H/m}\right)\left(3.00 \cdot 10^8 \text{ m/s}\right)} = 1.19 \cdot 10^{10} \text{ W/m}^2$

ROUND: Round the results to two significant figures.

(a) $B = 1.0 \cdot 10^{-2} \text{ T}$

(b) $I = 1.2 \cdot 10^{10} \text{ W/m}^2$

DOUBLE-CHECK: This is a very large magnetic field, and a very high intensity, as expected for a field at the breakdown threshold. The results are reasonable.

31.43. **THINK:** A laser beam has a power of 10.0 W and a beam diameter of 1.00 mm. Assume the intensity of the beam is the same throughout the cross section of the beam.

SKETCH: A sketch is not required.

RESEARCH:

(a) The intensity of the laser beam is given by $I = P/A$. Area $A = \pi r^2$.

(b) The intensity is related to the rms electric field by $I = E_{\text{rms}}^2 / \left(\mu_0 c\right) \Rightarrow E_{\text{rms}} = \sqrt{\mu_0 c I}$.

(c) The time-averaged Poynting vector is equal the intensity of the beam, $S_{\text{ave}} = I$.

(d) $S\left(x,t\right) = \dfrac{\left[E\left(x,t\right)\right]^2}{\mu_0 c}$ and $E\left(x,t\right) = E_{\text{m}} \sin\left(kx - \omega t + \phi\right)$.

(e) The rms magnetic field is $B_{\text{rms}} = E_{\text{rms}} / c$.

SIMPLIFY:

(d) Substituting the expression for $E\left(x,t\right)$ gives: $S\left(x,t\right) = \dfrac{1}{\mu_0 c} E_{\text{m}}^2 \sin^2\left(kx - \omega t + \phi\right)$. Because $S\left(0,0\right) = 0$, take $\phi = 0$. Therefore, $S\left(x,t\right) = 2I \sin^2\left(kx - \omega t\right)$. Note that $\omega = 2\pi f = 2\pi c / \lambda$ and $k = 2\pi / \lambda$.

CALCULATE:

(a) $I = \dfrac{10.0 \text{ W}}{\pi\left(0.500 \cdot 10^{-3} \text{ m}\right)^2} = 1.2732 \cdot 10^7 \text{ W/m}^2$

(b) $E_{\text{rms}} = \sqrt{\left(4\pi \cdot 10^{-7} \text{ H/m}\right)\left(3.00 \cdot 10^8 \text{ m/s}\right)\left(1.2732 \cdot 10^7 \text{ W/m}^2\right)} = 6.932809 \cdot 10^4 \text{ V/m}$

(c) $S_{\text{ave}} = 1.2732 \cdot 10^7 \text{ W/m}^2$

(d) $$S\left(x,t\right) = 2\left(1.2732 \cdot 10^7 \text{ W/m}^2\right)\sin^2\left[\left(\frac{2\pi}{514.5 \cdot 10^{-9} \text{ m}}\right)x - \left(\frac{2\pi\left(3.00 \cdot 10^8 \text{ m/s}\right)}{514.5 \cdot 10^{-9} \text{ m}}\right)t\right]$$

$$= 2.5464 \cdot 10^7 \text{ W/m}^2 \sin^2\left[\left(1.22122 \cdot 10^7 \text{ m}^{-1}\right)x - \left(3.66366 \cdot 10^{15} \text{ Hz}\right)t\right]$$

(e) $B_{\text{rms}} = \dfrac{6.932809 \cdot 10^4 \text{ V/m}}{3.00 \cdot 10^8 \text{ m/s}} = 2.30936 \cdot 10^{-4} \text{ T}$

ROUND:

(a) $I = 1.27 \cdot 10^7 \text{ W/m}^2$. This intensity is much larger than the intensity of sunlight on Earth $\left(1400 \text{ W/m}^2\right)$.

(b) $E_{\text{rms}} = 6.93 \cdot 10^4 \text{ V/m}$

(c) $S_{\text{ave}} = 1.27 \cdot 10^7 \text{ W/m}^2$

(d) $S(x,t) = 2.5464 \cdot 10^7 \text{ W/m}^2 \sin^2\left(\left(1.22122 \cdot 10^7 \text{m}^{-1}\right)x - \left(3.66366 \cdot 10^{15} \text{Hz}\right)t\right)$.

Rounding the coefficients to three significant figures,

$S(x,t) = 2.55 \cdot 10^7 \text{ W/m}^2 \sin^2\left(\left(1.22 \cdot 10^7 \text{m}^{-1}\right)x - \left(3.66 \cdot 10^{15} \text{Hz}\right)t\right)$.

Note that for given values for x and t, it would be better to keep the unrounded coefficients and then round the calculted value of S.

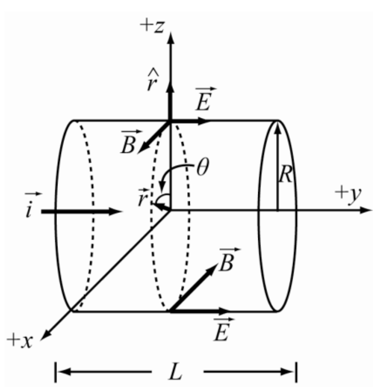
(e) $B_{\text{rms}} = 2.31 \cdot 10^{-4} \text{ T}$

DOUBLE-CHECK: The laser has a very high power output in a very narrow beam. This is a desirable property in a laser. The results make sense.

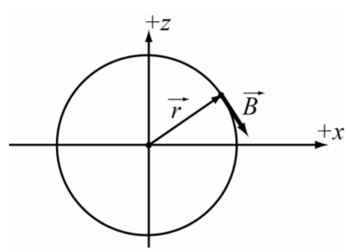
31.44. **THINK:** The Poynting vector is proportional to $\vec{E} \times \vec{B}$. Assume that the electric field in a conductor is uniform. The conductor is placed along the y-axis and the current is flowing along the positive y-direction. This means the electric field is in the positive y-direction.

SKETCH:

(a)



(b)



Cross Section of Cylinder

RESEARCH: The Poynting vector is defined as $\vec{S} = \vec{E} \times \vec{B} / \mu_0$.

(a) The electric field on the surface of the conductor is $\vec{E} = (V/L)\hat{y}$. Assume a long cylindrical conductor and the magnetic field on the surface is:

$$\vec{B} = \frac{\mu_0 I}{2\pi R}\hat{\theta} = \frac{\mu_0 I}{2\pi R}\left(\cos\theta\hat{x} - \sin\theta\hat{z}\right).$$

(b) The integral of $\vec{S} \bullet d\vec{A}$ is: $\oint \vec{S} \bullet d\vec{A} = -S\oint dA$, since $d\vec{A} = dA\hat{r}$.

SIMPLIFY:

(a) The Poynting vector is the cross product:

$$\vec{S} = \frac{1}{\mu_0}\left(\frac{V}{L}\right)\left(\frac{\mu_0 I}{2\pi R}\right)\hat{y} \times \left(\cos\theta\hat{x} - \sin\theta\hat{z}\right) = \left(\frac{VI}{2\pi RL}\right)\left(-\cos\theta\hat{z} - \sin\theta\hat{x}\right) = \frac{VI}{2\pi RL}\left(-\hat{r}\right).$$

This means that the Poynting vector is directed toward the cylindrical conductor with a magnitude of $S = VI/(2\pi RL)$.

(b) Taking the integral over the surface of the cylinder:

$$\oint dA = 2\pi RL \;\Rightarrow\; \oint \vec{S}\bullet d\vec{A} = -S(2\pi RL) = -\frac{VI}{2\pi RL}(2\pi RL) = -VI = -IR_{\text{R}}I = -I^2R_{\text{R}}.$$

Note that the subscript, R, is to distinguish that R_{R} means resistance.

CALCULATE: Not necessary.

ROUND: Not necessary.

DOUBLE-CHECK: Dimensional analysis shows that the units of the calculated result are W/m^2, as required for the Poynting vector.

31.45. (a) The intensity above the Earth's atmosphere is $I = 1.40 \text{ kW/m}^2$.

$$I = \frac{1}{c\mu_0}E_{\text{rms}}^2 \;\Rightarrow\; E_{\text{rms}} = \sqrt{Sc\mu_0} = \sqrt{\left(1.40\cdot 10^3 \text{ W/m}^2\right)\left(3.00\cdot 10^8 \text{ m/s}\right)\left(4\pi\cdot 10^{-7} \text{ T m/A}\right)}$$

$$E_{\text{rms}} = 726.49 \text{ V/m} \rightarrow E_{\max} = \sqrt{2}\,(726.49 \text{ V/m}) = 1027.4 \text{ V/m}.$$

$$B = \frac{E}{c} = \frac{1027.4 \text{ V/m}}{3.00\cdot 10^8 \text{ m/s}} = 3.4247\cdot 10^{-6} \text{ T}.$$

$$E_{\max} = 1030 \text{ V/m} = 1.03 \text{ kV/m},\; B_{\max} = 3.42\cdot 10^{-6} \text{ T}.$$

(b) $P_{\text{r}} = \dfrac{I}{c} = \dfrac{1.00\cdot 10^3 \text{ W/m}^2}{3.00\cdot 10^8 \text{ m/s}} = 3.33333\ \mu\text{Pa} \approx 3.33\ \mu\text{Pa},$

$$P_{\text{r}} = \frac{F}{A} \;\Rightarrow\; F = P_{\text{r}}A = \left(3.33333\cdot 10^{-6} \text{ Pa}\right)\left(0.750 \text{ m}^2\right) = 2.50\ \mu\text{N}.$$

31.46. (a) First, determine the force needed to accelerate a 10.0 ton spaceship by 1 m/s^2. Newton's second law, $F = ma$, gives: $F = \left(10.0\cdot 10^3 \text{ kg}\right)\left(1.00 \text{ m/s}^2\right) = 1.00\cdot 10^4 \text{ N}$. Now, the radiation pressure is given by $P_{\text{r}} = F/A$. The radiation pressure is related to the intensity of the radiation by $P_{\text{r}} = I/c$ (total absorption). Comparing the two equations for the radiation pressure gives:

$$\frac{F}{A} = \frac{I}{c} \;\Rightarrow\; A = \frac{Fc}{I} = \frac{\left(1.00\cdot 10^4 \text{ N}\right)\left(3.00\cdot 10^8 \text{ m/s}\right)}{1.40\cdot 10^3 \text{ W/m}^2} = 2.14\cdot 10^9 \text{ m}^2.$$

This area is large. Moreover, even if the scientists are able to get the astronauts to another planet, how do they get them back to Earth?

(b) For perfect reflection: $P_{\text{r}} = \dfrac{2I}{c} = \dfrac{F}{A} \;\Rightarrow\; A = \dfrac{Fc}{2I} = \dfrac{1}{2}\left(2.14\cdot 10^9 \text{ m}^2\right) = 1.07\cdot 10^9 \text{ m}^2.$

31.47. The net force on the sail is $F = A\Delta P$. The area, A, is $A = \pi R^2 = \pi\left(10.0\cdot 10^3 \text{ m}\right)^2 = 3.142\cdot 10^8 \text{ m}^2$. The differential pressure, ΔP, is:

$$\Delta P = \frac{2I}{c} - \frac{I}{c} = \frac{I}{c}.$$

The intensity, I, is given by the Stefan-Boltzman law as:

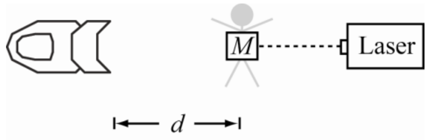
$$I = \sigma T^4 = \left(5.67\cdot 10^{-8} \text{ W/m}^2 \text{ K}^4\right)\left(2.725 \text{ K}\right)^4 = 3.126\cdot 10^{-6} \text{ W/m}^2.$$

$$\Rightarrow\; \Delta P = \frac{3.126\cdot 10^{-6} \text{ W/m}^2}{3.00\cdot 10^8 \text{ m/s}} = 1.042\cdot 10^{-14} \text{ Pa}$$

$$\Rightarrow\; F = \left(3.142\cdot 10^8 \text{ m}^2\right)\left(1.042\cdot 10^{-14} \text{ Pa}\right) = 3.27\cdot 10^{-6} \text{ N}$$

31.48. **THINK:** There will be a constant force on the astronaut due to the radiation pressure from the laser. This force can be determined from the laser power, and then the time required to reach the shuttle can be determined. d = 20.0 m, m = 100.0 kg and P = 100.0 W.

SKETCH:



RESEARCH: $P_r = \dfrac{2I}{c}$ (totally reflecting), $I = \dfrac{P}{A}$, $P_r = \dfrac{F}{A} = \dfrac{ma}{A}$, $x = \dfrac{1}{2}at^2$ (constant acceleration)

SIMPLIFY: $P_r = \dfrac{ma}{A} \Rightarrow a = \dfrac{P_r A}{m}$, $P_r = \dfrac{2I}{c} = \dfrac{2(P/A)}{c} \Rightarrow a = \dfrac{A}{m}\left(\dfrac{2P}{Ac}\right) = \dfrac{2P}{mc}$

$x = d = \dfrac{1}{2}at^2 \Rightarrow t = \sqrt{\dfrac{2d}{a}} = \sqrt{2d\left(\dfrac{mc}{2P}\right)} = \sqrt{\left(\dfrac{dmc}{P}\right)}$

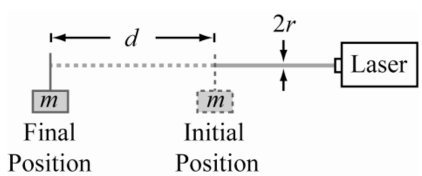
CALCULATE: $t = \sqrt{\dfrac{(20.0 \text{ m})(100.0 \text{ kg})(3.00 \cdot 10^8 \text{ m/s})}{100.0 \text{ W}}} = 7.746 \cdot 10^4 \text{ s}$

ROUND: $t = 7.75 \cdot 10^4 \text{ s} = 21.5 \text{ h}$

DOUBLE-CHECK: The time decreases as the laser power increases or the mass decreases. This is what would be expected.

31.49. **THINK:** The applied force can be determined from the given data. From the force, the radiation pressure, and then the laser power used in the demonstration can be determined. d = 2.00 mm, t = 63.0 s, m = 0.100 g and $2r$ = 1.00 mm.

SKETCH:



RESEARCH: $I = P/A$, $A = \pi r^2$, $P_r = \dfrac{2I}{c} = \dfrac{ma}{A}$, $d = \dfrac{1}{2}at^2$

SIMPLIFY: $P = IA = I\pi r^2$, $I = \dfrac{mca}{2A} = \dfrac{mca}{2\pi r^2}$, $a = \dfrac{2d}{t^2} \Rightarrow P = \dfrac{mca}{2\pi r^2}\pi r^2 = \dfrac{mca}{2} = \dfrac{mc}{2}\left(\dfrac{2d}{t^2}\right) = \dfrac{mcd}{t^2}$

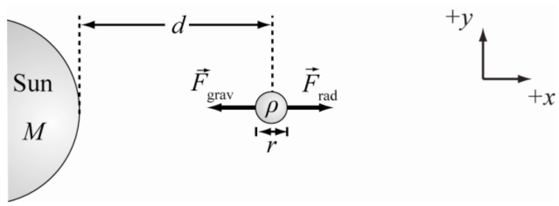
CALCULATE: $P = \dfrac{(0.100 \cdot 10^{-3} \text{ kg})(3.00 \cdot 10^8 \text{ m/s})(2.00 \cdot 10^{-3} \text{ m})}{(63.0 \text{ s})^2} = 1.512 \cdot 10^{-2} \text{ W}$

ROUND: $P = 1.51 \cdot 10^{-2} \text{ W} = 15.1 \text{ mW}$

DOUBLE-CHECK: Note that the result does not depend on the spot size of the laser. The power is all that matters. This result is reasonable.

31.50. **THINK:** The radius of the particle can be determined if the required mass of the particle is known because the density is given. The mass of the particle can be determined by comparing the radiation force and the gravitational force. $\rho = 2000. \text{ kg/m}^3$, $d = 1.50 \cdot 10^{11} \text{ m}$, $M = 2.00 \cdot 10^{30} \text{ kg}$, $F_{\text{rad}} / F_{\text{grav}} = 1.00\% = 0.0100$, $I = 1400. \text{ W/m}^2$.

SKETCH:



RESEARCH: $F_{grav} = \frac{GMm}{d^2}$, $m = \frac{4}{3}\pi r^3 \rho$, $F_{rad} = P_r A$, $A = \pi r^2$, $P_r = 2I/c$.

SIMPLIFY: $F_{grav} = \frac{GMm}{d^2} = \frac{GM}{d^2}\left(\frac{4}{3}\pi r^3 \rho\right)$, $F_{rad} = P_r A = \frac{2I}{c}\left(\pi r^2\right)$

$$F_{rad} = 0.01 F_{grav} \Rightarrow \frac{2\pi r^2 I}{c} = 0.01\frac{\left(\frac{4}{3}\pi r^3 \rho GM\right)}{d^2} \Rightarrow r = \left(\frac{2\pi I}{c}\right)\frac{d^2}{0.01\left(\frac{4}{3}\pi\rho GM\right)} = \frac{3Id^2}{(0.01)2c\rho GM}$$

CALCULATE:

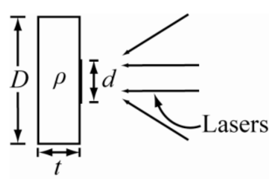
$$r = \frac{3\left(1400.\text{ W/m}^2\right)\left(1.500\cdot 10^{11}\text{ m}\right)^2}{(0.0100)2\left(3.00\cdot 10^8\text{ m/s}\right)\left(2000.\text{ kg/m}^3\right)\left(6.673\cdot 10^{-11}\text{ m}^3\text{ kg}^{-1}\text{ s}^{-2}\right)\left(2.00\cdot 10^{30}\text{ kg}\right)} = 5.901\cdot 10^{-5}\text{ m}$$

ROUND: $r = 59.0\ \mu\text{m}$

DOUBLE-CHECK: If the radiation pressure on the particle is to be equal to the gravitational force on it, its radius would have to be smaller by a factor of 100. This indicates that very small particles would be pushed away from the sun, while more massive objects would be pulled toward the sun. This is consistent with observation. The result makes sense.

31.51. **THINK:** Given the density and volume, the mass can be determined. Given the power and spot size of the laser, the intensity and the radiation pressure of the laser can be determined. To determine how many lasers are needed, calculate the total force required and divide this by the force per laser applied. $\rho = 1.00\text{ mg/cm}^3$, D = 2.00 mm, t = 0.100 mm, P = 5.00 mW, d = 2.00 mm.

SKETCH:



RESEARCH:

(a) The weight is given by $w = mg$, where $m = \pi\left(D/2\right)^2 t\rho$. Note that $1\text{ mg/cm}^3 = 1\text{ kg/m}^3$.

(b) $P_r = \frac{I}{c}$ (absorbing material), $I = \frac{P}{A}$, $A = \pi\left(\frac{d}{2}\right)^2$.

(c) $F_{las} = P_r A$. The number of lasers needed is given by $N = w/F_{las}$.

SIMPLIFY:

(a) $w = \frac{\pi D^2 t\rho g}{4}$

(b) $I = P\frac{4}{\pi d^2} = \frac{4P}{\pi d^2}$, $P_r = \frac{I}{c} = \frac{4P}{\pi d^2 c}$

(c) $F_{las} = P_r A = \frac{IA}{c} = \frac{P}{cA}A = \frac{P}{c}$, $N = \frac{w}{F_{las}} = \frac{wc}{P}$

CALCULATE:

(a) $w = \dfrac{\pi\left(2.00\cdot 10^{-3}\text{ m}\right)^2\left(0.100\cdot 10^{-3}\text{ m}\right)\left(1.00\text{ kg/m}^3\right)\left(9.81\text{ m/s}^2\right)}{4} = 3.082\cdot 10^{-9}\text{ N}$

(b) $I = \dfrac{4\left(5.00\cdot 10^{-3}\text{ W}\right)}{\pi\left(2.00\cdot 10^{-3}\text{ m}\right)^2} = 1.592\cdot 10^3\text{ W/m}^2,\ P_\text{r} = \dfrac{I}{c} = \dfrac{1.592\cdot 10^3\text{ W/m}^2}{3.00\cdot 10^8\text{ m/s}} = 5.30510^{-6}\text{ N/m}^2$

(c) $N = \dfrac{\left(3.082\cdot 10^{-9}\text{ N}\right)\left(3.00\cdot 10^8\text{ m/s}\right)}{\left(5.00\cdot 10^{-3}\text{ W}\right)} = 184.9$

ROUND:

(a) $w = 3.08\cdot 10^{-9}\text{ N} = 3.08\text{ nN}$

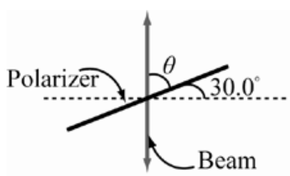
(b) $I = 1.59\text{ kW/m}^2,\ P_\text{r} = 5.31\ \mu\text{N/m}^2$

(c) $N = 185$ lasers

DOUBLE-CHECK: Even though the object is very light, it would still require a large power output to produce enough radiation pressure to overcome the force of gravity.

31.52. The first filter is out of alignment by $15.0°$ with the incident light. The second filter is out of alignment by $30.0°$ with the incident light. The intensity of the transmitted light is $I = I_0\cos^2\theta_1\cos^2\theta_2 = (1.00)\cos^2(15.0°)\cos^2(30.0°) = 0.69976\text{ W/m}^2 \approx 0.700\text{ W/m}^2$.

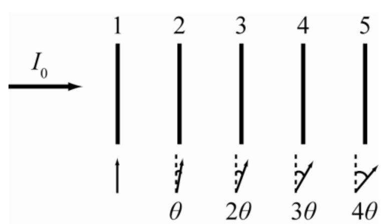
31.53.



$I = I_0\cos^2\theta = \left(10.0\cdot 10^{-3}\text{ W}\right)\cos^2(90.0° - 30.0°) = \left(10^{-2}\text{ W}\right)\cos^2 60.0° = 2.50\text{ mW}$

31.54. **THINK:** Only half the intensity gets through the first polarizer since the incident light is un-polarized. After this, multiply the transmission for each polarizer to obtain the final intensity. $\theta = 10°$.

SKETCH:



RESEARCH: $I_1 = I_0/2,\ I_n = I_{n-1}\cos^2\theta\ (n = 2, 3, 4, 5)$

SIMPLIFY: $I_2 = I_1\cos^2\theta = \left(I_0\cos^2\theta\right)/2,\quad I_3 = I_2\cos^2\theta = \left(I_0\cos^4\theta\right)/2,\quad I_4 = I_3\cos^2\theta = \left(I_0\cos^6\theta\right)/2,$

$I_5 = I_4\cos^2\theta = \left(I_0\cos^8\theta\right)/2$

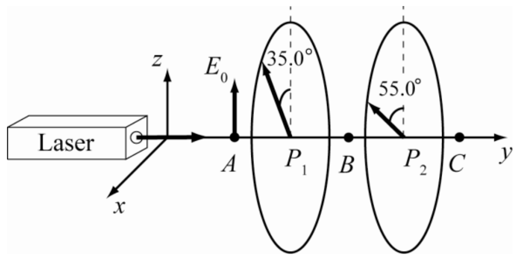
CALCULATE: $I_5 = \left(I_0\cos^8 10°\right)/2 = 0.4424 I_0$

ROUND: $I_5 = 0.442 I_0$

DOUBLE-CHECK: Only 44.2% of the original intensity passes through the polarizers. The first polarizer decreases the intensity by 50%, but the subsequent polarizers allow the majority of the light to pass through due to the smaller angles. This is a reasonable result.

31.55. **THINK:** First calculate the intensity of the light after it first passes through the two polarizers. Once the intensity is calculated, the magnitude of the electric and magnetic fields can be determined. The angles of the first and second polarizers are $\theta_1 = 35°$ and $\theta_2 = 55°$, respectively. The laser spot size diameter is $d =$ 1.00 mm and the laser power is P = 15.0 mW.

SKETCH:



RESEARCH: $I_1 = I_0 \cos^2\theta_1,\ I_2 = I_1\cos^2(\theta_2-\theta_1),\ I_0 = \dfrac{P}{A},\ A = \pi\left(\dfrac{d}{2}\right)^2,\ I = \dfrac{1}{2}\left(\dfrac{E^2}{c\mu_0}\right),\ \dfrac{E}{B} = c$

SIMPLIFY: $I_2 = I_1\cos^2(\theta_2-\theta_1) = I_0\cos^2\theta_1\cos^2(\theta_2-\theta_1),\ I_0 = \dfrac{P}{A} = \dfrac{4P}{\pi d^2} \Rightarrow I_2 = \dfrac{4P}{\pi d^2}\cos^2\theta_1\cos^2(\theta_2-\theta_1)$

$E = \sqrt{2Ic\mu_0}$, $B = E/c$

CALCULATE: $I_2 = \dfrac{4(15.0\cdot 10^{-3}\text{ W})}{\pi(1.00\cdot 10^{-3}\text{ m})^2}\cos^2(35°)\cos^2(55°-35°) = 1.132\cdot 10^4\text{ W/m}^2$

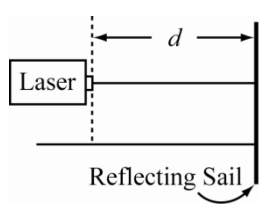
$E = \sqrt{2(1.132\cdot 10^4\text{ W/m}^2)(3.00\cdot 10^8\text{ m/s})(4\pi\cdot 10^{-7}\text{ T m A}^{-1})} = 2.921\cdot 10^3\text{ V/m}$

$B = \dfrac{2.921\cdot 10^3\text{ V/m}}{3.00\cdot 10^8\text{ m/s}} = 9.737\cdot 10^{-6}\text{ T}$

ROUND: $I_2 = 1.13\cdot 10^4\text{ W/m}^2,\ E = 2.92\cdot 10^3\text{ V/m},\ B = 9.74\cdot 10^{-6}\text{ T}$

DOUBLE-CHECK: The initial intensity of the laser light is about $1.9\cdot 10^4\text{ W/m}^2$. The initial electric and magnetic fields are also significantly larger. It is expected that some of the intensity of the laser beam would be blocked by the polarizers.

31.56.



The total distance traveled by the beam is 2*d*.

$$v = c = \frac{2d}{t} \Rightarrow d = \frac{ct}{2} = \frac{(3.00\cdot 10^8\text{ m/s})(50.0\cdot 10^{-3}\text{ s})}{2} = 7.50\cdot 10^6\text{ m}$$

31.57. The total average power on all of the photovoltaic panels is equal to the average power times the area:

$$(300.\text{ W/m}^2)(3.00\text{ m})(8.00\text{ m}) = 7.20\text{ kW}.$$

The total electrical energy for 30 days is the product of the total power times 30 days times the efficiency for converting the solar power into electricity: $E_{\text{total}} = (7.20\text{ kW})(720\text{ h})(0.100) = 518\text{ kW h}$. This is enough for a small energy-efficient home.

31.58. Use the fact that radiation pressure scales with intensity, and intensity goes linearly with the intrinsic power of the star, and inversely as the square of the distance away. Thus, if it is known that at the Earth's orbit, the intensity is 1.35 kW/m^2, this intensity can be related to the radiation pressure at Earth's orbit, and then to the hypothetical distance of Uranus' orbit away from Betelgeuse. Note that the radius of the earth's orbit is $r_E \approx 1.50\cdot 10^{11}\text{ m} = 1\text{ AU}$ and Uranus's orbit is $r_U \approx 2.88\cdot 10^{12}\text{ m} = 19.2\text{ AU}$.

$$P_{\text{rE}} \propto \frac{\text{P}_\text{S}}{r_\text{E}^2},\quad P_{\text{rB}} \propto \frac{\text{P}_B}{r_U^2}$$

$$R = \frac{P_{\text{rU}}}{P_{\text{rE}}} = \frac{\text{P}_B r_\text{E}^2}{r_\text{U}^2 \text{P}_S} = \frac{\text{P}_B}{\text{P}_S}\left(\frac{r_\text{E}}{r_U}\right)^2 = 10^4\left(\frac{1}{19.2}\right)^2 = 27.1 \Rightarrow \text{I}_{\text{rU}} = 27.1\cdot \text{I}_{\text{rE}} = (27.1)\left(1350\text{ W/m}^2\right)$$
$$= 3.66\cdot 10^4\text{ W/m}^2$$

For a perfect absorber we have: $P_{rU} = \dfrac{I_{rU}}{c} = \dfrac{36600\text{ W/m}^2}{3.00\cdot 10^8\text{ m/s}} = 1.22\cdot 10^{-4}\text{ N/m}^2$, and for a perfect reflector we have: $P_{rU} = \dfrac{2I_{rU}}{c} = \dfrac{2\cdot 36600\text{ W/m}^2}{3.00\cdot 10^8\text{ m/s}} = 2.44\cdot 10^{-4}\text{ N/m}^2$. Since no information is provided on the reflectivity of the surface being acted on by the radiation, the final solution is: $1\cdot 10^{-4}\text{ N/m}^2 < P_{rU} < 2\cdot 10^{-4}\text{ N/m}^2$,

31.59. $$S = \left(\frac{\text{power}}{\text{area}}\right) = \left(\frac{200.\text{ W}}{1.00\cdot 10^6\text{ m}^2}\right) = 2.00\cdot 10^8\text{ W/m}^2 = \frac{E^2}{2c\mu_0} \quad\Rightarrow\quad E = \sqrt{2c\mu_0 S} = 3.88\cdot 10^5\text{ V/m}$$

Note the 2 is in the denominator from: $E_{rms} = \dfrac{E}{\sqrt{2}}$. Therefore, $E_{rms}^2 = \dfrac{E^2}{2}$ and $\text{S} = \dfrac{E_{rms}^2}{c\mu_0} = \dfrac{E^2}{2c\mu_0}$. The wavelength has nothing to do with the solution.

31.60. $$f\lambda = c \Rightarrow \lambda = \frac{c}{f} = \frac{2.9979\cdot 10^8\text{ m/s}}{848.97\cdot 10^6\text{ Hz}} = 0.34901\text{ m} = 34.901\text{ cm.}$$

31.61. $P = IA\varepsilon,\ I = \dfrac{1}{2}\left(\dfrac{E^2}{c\mu_0}\right),\ A = LW,$ and $\varepsilon = 0.18.$

$$\Rightarrow P = \frac{E^2 LW}{2c\mu_0}\varepsilon = \frac{(673\text{ V/m})^2(1.40\text{ m})(0.900\text{ m})(0.180)}{2\left(3.00\cdot 10^8\text{ m/s}\right)\left(4\pi\cdot 10^{-7}\text{ T m/A}\right)} = 136\text{ W}$$

31.62. The displacement current between the capacitor plates is the same as the conventional current in the rest of the circuit.

$$i = \frac{V_{\text{emf}}}{R}e^{-t/RC} = \left(\frac{25.0\text{ V}}{24{,}300\ \Omega}\right)\exp\left[\frac{-0.3621\text{ s}}{24{,}300\ \Omega\left(14.9\cdot 10^{-6}\text{ F}\right)}\right] = \left(1.0288\cdot 10^{-3}\right)(0.36785) = 0.37845\text{ mA}$$

$$i_\text{d} = \varepsilon_0 A\frac{dE}{dt} \Rightarrow \frac{dE}{dt} = \frac{i_\text{d}}{\varepsilon_0 A} = \frac{0.37845\text{ mA}}{\left(8.85\cdot 10^{-12}\text{ C}^2/\left(\text{N m}^2\right)\right)\left(1.00\cdot 10^{-4}\text{ m}^2\right)} = 4.28\cdot 10^{11}\text{ V/(m s)}$$

31.63. $$E_{\text{rms}} = \sqrt{c\mu_0 I} = \sqrt{c\mu_0\left(\frac{P}{A}\right)} = \sqrt{\frac{c\mu_0 P}{\pi(d/2)^2}} = \sqrt{\frac{4c\mu_0 P}{\pi d^2}}$$
$$= \sqrt{\frac{4\left(3.00\cdot 10^8\text{ m/s}\right)\left(4\pi\cdot 10^{-7}\text{ T m/A}\right)(0.40\cdot 300.\text{ W})}{\pi(2\text{ m})^2}} = 100\text{ V/m.}$$

31.64. $$\frac{E}{B} = c \Rightarrow E = cB = \left(3.00\cdot 10^8\text{ m/s}\right)\left(5.00\cdot 10^{-3}\text{ T}\right) = 1.50\text{ MV/m}$$

31.65. The antinodes are spaced half a wavelength apart, $d=\lambda/2,$ where $\lambda=c/f.$

$$\Rightarrow d=\frac{c}{2f}=\frac{3.00\cdot10^{8}\text{ m/s}}{2\left(2.4\cdot10^{9}\text{ Hz}\right)}=6.25\cdot10^{-2}\text{ m}=6.3\text{ cm}$$

31.66. $I=1400\text{ W/m}^2$

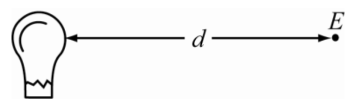
(a) $I=\dfrac{E_{\text{rms}}^2}{c\mu_0}\Rightarrow E_{\text{rms}}=\sqrt{Ic\mu_0}=\sqrt{\left(1400\text{ W/m}^2\right)\left(3.00\cdot10^8\text{ m/s}\right)\left(4\pi\cdot10^{-7}\text{ T m/A}\right)}=726.5\text{ V/m}$

$E_{\max}=\sqrt{2}E_{\text{rms}}=\sqrt{2}\left(726.5\text{ V/m}\right)=1027\text{ V/m}=1.0\cdot10^3\text{ V/m}$

(b) $B_{\max}=\dfrac{E_{\max}}{c}=\dfrac{1027\text{ V/m}}{3.00\cdot10^8\text{ m/s}}=3.425\cdot10^{-6}\text{ T}=3.4\cdot10^{-6}\text{ T}$

31.67. **THINK:** The peak magnetic field can be determined from the speed of light and the peak electric field. The power of the bulb can be determined from its intensity, which can be determined from the electric and magnetic fields. Use the values *r* = 2.25 m and *E* = 21.2 V/m.

SKETCH:



RESEARCH: $\dfrac{E}{B}=c,\ I=\dfrac{P}{A},\ I=\dfrac{E^2}{2c\mu_0}$

SIMPLIFY:

(a) $B=E/c$

(b) $P=IA=\dfrac{E^2}{2c\mu_0}A.$ Light from a light-bulb is emitted isotropically, that is equally in all directions. To determine the power a distance, *d*, away from the light-bulb, the intensity at all points a distance, *d*, from the light-bulb must be summed. Hence, *A* should be the surface area of a sphere of radius, *r*:

$$A=4\pi r^2\Rightarrow P=\frac{4\pi r^2E^2}{2c\mu_0}.$$

CALCULATE:

(a) $B=\dfrac{21.2\text{ V/m}}{3\cdot10^8\text{ m/s}}=7.067\cdot10^{-8}\text{ T}$

(b) $P=\dfrac{4\pi\left(2.25\text{ m}\right)^2\left(21.2\text{ V/m}\right)^2}{2\left(3.00\cdot10^8\text{ m/s}\right)\left(4\pi\cdot10^{-7}\text{ T m/A}\right)}=37.92\text{ W}$

ROUND:

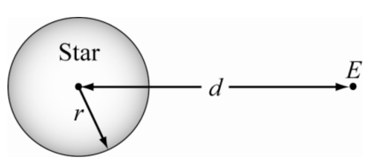
(a) $B=7.07\cdot10^{-8}\text{ T}$

(b) *P* = 37.9 W

DOUBLE-CHECK: These values are consistent with the power output for a regular household light bulb.

31.68. **THINK:** To determine the temperature of the star, the power radiated by the star must be known. To determine the power, the intensity is needed. The intensity can be determined from the electric field and the distance. *D* = 15 AU, *E* = 0.015 V/m, $r=2r_{\text{S}},\ r_{\text{S}}=6.955\cdot10^5\text{ km}.$

SKETCH:



RESEARCH: $P = A\sigma T^4 = 4\pi r^2 \sigma T^4$, $I = \dfrac{P}{4\pi d^2} = \dfrac{E^2}{2\mu_0 c}$

SIMPLIFY: $P = \dfrac{4\pi d^2 E^2}{2\mu_0 c} = \dfrac{2\pi d^2 E^2}{\mu_0 c} = 4\pi\left(2r_S\right)^2 \sigma T^4$, $T = \left[\dfrac{2\pi d^2 E^2}{8\pi r_S^2 \sigma \mu_0 c}\right]^{1/4} = \left[\dfrac{d^2 E^2}{4 r_S^2 \sigma \mu_0 c}\right]^{1/4}$

CALCULATE:

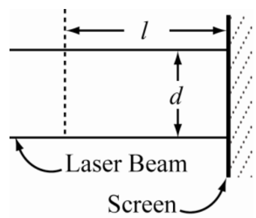
$$T = \left[\frac{\left((15)1.49598\cdot 10^{11}\text{ m}\right)^2 \left(44.0\text{ V/m}\right)^2}{4\left(6.955\cdot 10^5\cdot 10^3\text{ m}\right)^2\left(5.670\cdot 10^{-8}\text{ W m}^{-2}\text{ K}^{-4}\right)\left(4\pi\cdot 10^{-7}\text{ N/A}^{-2}\right)\left(3.00\cdot 10^8\text{ m/s}\right)}\right]^{1/4} = 3918\text{ K}$$

ROUND: T = 3920 K

DOUBLE-CHECK: This temperature is realistic for a K-class star.

31.69. **THINK:** From the power and the spot size, the intensity of the beam can be determined. From the intensity, the electric field can be determined. For the total energy, multiply the energy density by the volume of the beam. P = 5.00 mW, d = 2.00 mm, l = 1.00 m.

SKETCH:



RESEARCH: $I = \dfrac{E_{\text{rms}}^2}{\mu_0 c} = \dfrac{P}{A}$, $A = \pi d^2$, $u_E = \dfrac{1}{2}\varepsilon_0 E_{\text{max}}^2$

SIMPLIFY:

(a) $E_{\text{rms}} = \sqrt{\dfrac{\mu_0 c P}{\pi (d/2)^2}}$

(b) $U = u_E V = u_E A l = \dfrac{1}{2}\varepsilon_0 E_{\text{max}}^2 A l = \varepsilon_0 E_{\text{rms}}^2 A = \varepsilon_0\left(P\mu_0 c\right) l = \left(\varepsilon_0 \mu_0 c\right) P l.$

$c^2 = \dfrac{1}{\varepsilon_0 \mu_0} \rightarrow \varepsilon_0 \mu_0 = \dfrac{1}{c^2}.$

$U = Pl/c.$

CALCULATE:

(a) $E_{\text{rms}} = \sqrt{\dfrac{\left(4\pi\cdot 10^{-7}\text{ N/A}^2\right)\left(3.00\cdot 10^8\text{ m/s}\right)\left(5.00\cdot 10^{-3}\text{ W}\right)}{\pi\left(1.00\cdot 10^{-3}\text{ m}\right)^2}} = 774.6\text{ V/m}$

(b) $U = \dfrac{\left(5.00\cdot 10^{-3}\text{ W}\right)\left(1.00\text{ m}\right)}{3.00\cdot 10^8\text{ m/s}} = 1.6667\cdot 10^{-11}\text{ J}.$

ROUND:

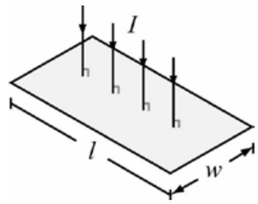
(a) $E_{\text{rms}} = 775\text{ V/m}.$

(b) $U = 1.67\cdot 10^{-11}\text{ J}.$

DOUBLE-CHECK: It is expected that a laser pointer would have small electric field and generate a small amount of energy, considering it's intended use. A laser pointer with more power would be dangerous.

31.70. **THINK:** The total power incident on the roof is the intensity of the light times the area of the roof. From the intensity, the radiation pressure can be determined, and from this the force can be determined. $I = 1.00\text{ kW/m}^2$, l = 30.0 m, w = 10.0 m.

SKETCH:



RESEARCH: $P = IA,\; A = lw,\; p_{\text{r}} = I/c.$

SIMPLIFY:

(a) $P = Ilw$

(b) $p_{\text{r}} = I/c$

CALCULATE:

(a) $P = \left(1.00\cdot 10^3 \text{ W/m}^2\right)\left(30.0 \text{ m}\right)\left(10.0 \text{ m}\right) = 3.00\cdot 10^5 \text{ W}$

(b) $p_{\text{r}} = \left(\dfrac{1.00\cdot 10^3 \text{ W/m}^2}{3.00\cdot 10^8 \text{ m/s}}\right) = 3.33\cdot 10^{-6} \text{ N/m}^2$

ROUND:

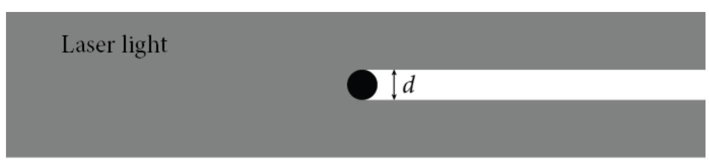
(a) $P = 3.00\cdot 10^5 \text{ W}$

(b) $p_{\text{r}} = 3.33\cdot 10^{-6} \text{ N/m}^2$

DOUBLE-CHECK: The radiation force is small, as expected, while the amount of power incident on the roof is fairly large. This large source of energy can be harnessed by the use of solar panels.

31.71. **THINK:** The laser will apply a force to the particle. Assume the particle starts from rest and 2.00% of the laser light is absorbed. The laser applies a force to a known mass for a time interval Δt from which we can calculate the impulse applied by the laser. Use the values: $P = 500./192 \text{ TW}$, $d = 2.00 \text{ mm}$, $\rho = 2.00 \text{ g/cm}^3 = 2.00\cdot 10^3 \text{ kg/m}^3$, $\Delta t = 1.00\cdot 10^{-9} \text{ s}$, and $\varepsilon = 0.0200.$

SKETCH:



RESEARCH: $F\Delta t = m\Delta v = p_r A\Delta t \Rightarrow \dfrac{\Delta v}{\Delta t} = \dfrac{p_r A}{m} = a,\; p_r = \dfrac{\varepsilon I}{c},\; I = \dfrac{P}{A},\; m = \dfrac{4}{3}\pi\left(\dfrac{d}{2}\right)^3 \rho = \dfrac{\pi d^3 \rho}{6}$

SIMPLIFY: $a = \dfrac{\Delta v}{\Delta t} = \dfrac{p_r A}{m} = \dfrac{A}{m}\left(\dfrac{\varepsilon I}{c}\right) = \dfrac{A\varepsilon}{mc}\left(\dfrac{P}{A}\right) = \dfrac{\varepsilon P}{mc} = \dfrac{\varepsilon P}{c}\left(\dfrac{6}{\pi d^3 \rho}\right) = \dfrac{6\varepsilon P}{\pi d^3 \rho c}.$

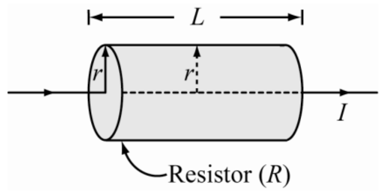
CALCULATE: $a = \dfrac{6\left(0.0200\right)\left(500.\cdot 10^{12} \text{ W/192}\right)}{\pi\left(2.00\cdot 10^{-3} \text{ m}\right)^3\left(2.00\cdot 10^3 \text{ kg/m}^3\right)\left(3.00\cdot 10^8 \text{ m/s}\right)} = 2.0723\cdot 10^7 \text{ m/s}^2.$

ROUND: To three significant figures, $a = 2.07\cdot 10^7 \text{ m/s}^2.$

DOUBLE-CHECK: This is a reasonable result for 2% of the power of one very powerful laser.

31.72. **THINK:** Power will be dissipated out of the curved surface of the resistor. The result should be $P = i^2 R.$ This can be derived by determining the expressions for the electric and magnetic fields at the surface of the resistor and using the Poynting vector definition of power/area.

SKETCH:



RESEARCH: $S = \dfrac{EB}{\mu_0}$, $P = SA$, $A = 2\pi rL$, $V = iR$, $E = \dfrac{V}{L}$, $B = \dfrac{\mu_0 i}{2\pi r}$

SIMPLIFY: $P = SA = \dfrac{EB}{\mu_0}(2\pi rL) = \dfrac{2\pi rL}{\mu_0}\left(\dfrac{V}{L}\right)\left(\dfrac{\mu_0 i}{2\pi r}\right) = Vi = (iR)i = i^2 R$

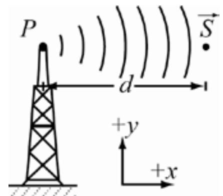
CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: This is consistent with previous results for the power dissipated by a resistor.

31.73. **THINK:** The direction of the Poynting vector is the direction in which energy is transported, in this case radially away from the antenna. From the definition of the Poynting vector as power/area, the magnitude can be calculated given the power and the radial distance. Once the Poynting vector is known, the electric field can be determined. The power emitted toward the ground is reflected so we assume that the power is emitted into a hemisphere rather than a sphere. $P = 3.00 \cdot 10^4$ W, d = 12.0 km.

SKETCH:



RESEARCH: $\vec{S} = \dfrac{1}{\mu_0}\vec{E} \times \vec{B}$, $\left|\vec{S}\right| = S = \dfrac{P}{A} = \dfrac{E^2}{\mu_0 c}$, $E_{\text{rms}} = \dfrac{E}{\sqrt{2}}$, $A = 2\pi d^2$, $F_{rms} = qE_{rms}$

SIMPLIFY: (a) $S = \dfrac{P}{2\pi d^2}$

(b) $E = \sqrt{\mu_0 cS} \;\Rightarrow\; E_{\text{rms}} = \sqrt{\mu_0 cS/2}$, $F_{rms} = q_e\sqrt{\mu_0 cS/2}$

CALCULATE

(a) $S = \dfrac{3.00 \cdot 10^4 \text{ W}}{2\pi\left(12.0 \cdot 10^3 \text{ m}\right)^2} = 3.3157 \cdot 10^{-5} \text{ W/m}^2.$

(b) $F_{\text{rms}} = \left(1.602 \cdot 10^{-19} \text{ C}\right)\sqrt{\left(4\pi \cdot 10^{-7} \text{ T m/A}\right)\left(3.00 \cdot 10^8 \text{ m/s}\right)\left(3.3157 \cdot 10^{-5} \text{ W/m}^2\right)/2} = 1.26649 \cdot 10^{-20} \text{ N}.$

ROUND:

(a) $S = 3.32 \cdot 10^{-5} \text{ W/m}^2.$

(b) $F_{\text{rms}} = 1.27 \cdot 10^{-20}$ N.

DOUBLE-CHECK: Dimensional analysis shows the results all have the correct units.

31.74. **THINK:** To answer these questions, use the classical equation for the momentum and angular momentum, and use the quantum equation for the energy.
SKETCH:

RESEARCH: $p = \frac{U}{c} = \frac{E}{c}$, $L = \frac{U}{\omega} = \frac{E}{\omega}$

SIMPLIFY:

(a) $p = \frac{E}{c} = \frac{\hbar\omega}{c}$

From the dispersion relation of light (see Exercise 29.61): $\left|\vec{k}\right| = \omega\sqrt{\mu_0\varepsilon_0} = \omega / c$. Then the vector momentum can be written:

$$\left|\vec{p}\right| = \hbar\left|\vec{k}\right| \Rightarrow \vec{p} = \hbar\vec{k}.$$

(b) $L = \frac{E}{\omega} = \frac{\hbar\omega}{\omega} = \hbar = \left|\vec{L}\right|$

The angular momentum of a photon is constant!

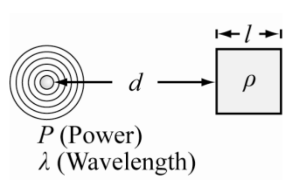
(c) The spin quantum number is given by: $s = \pm L / \hbar \Rightarrow s = \pm 1$.

CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: These results are classically unreasonable. However, they are correct nonetheless. They all stem from the purely quantum fact that light is quantized in units of $\hbar$. For light of frequency, ω, $\hbar\omega$ is the smallest amount of energy the electromagnetic wave can be measured to have.

31.75. **THINK:** To determine how long it takes the ice to melt, first determine how much total energy is required to melt the ice cube, and then determine the intensity of the microwaves at the location of the ice cube. To determine the number of photons hitting the ice per second, the energy of one photon must be calculated and compared to the total radiation power incident on the ice. $P_0 = 250.$ W, l = 2.00 cm, d = 10.0 cm, $\rho = 0.960$ g/cm^3, $\lambda = 10.0$ cm. The fraction of incident light absorbed by ice is $\varepsilon = 0.100$.
SKETCH:



RESEARCH: The energy required to melt the ice is $c_{\text{f}} = 334$ J/g. The intensity of light at the cube is $I = P_0 / 4\pi d^2$. The radiation power incident on the cube is Il^2. The power absorbed by the cube is $P = \varepsilon Il^2 = E / t$. The mass of the ice is given by $m = \rho l^3$. The energy of one photon is given by $E_{\text{ph}} = hf = hc / \lambda$.

SIMPLIFY: The energy required to melt the ice is given by $E_{\text{m}} = mc_{\text{f}} = \rho l^3 c_{\text{f}}$. The power absorbed by the cube is given by:

$$P = \varepsilon Il^2 = \varepsilon l^2 \left(\frac{P_0}{4\pi d^2}\right) = \frac{E}{t}.$$

The time required to melt the cube can be determined as follows:

$$\frac{E}{t}=\frac{\varepsilon l^2 P_0}{4\pi d^2} \Rightarrow t=\frac{4\pi E d^2}{\varepsilon l^2 P_0}, \quad E=E_{\text{m}}=\rho l^3 c_{\text{f}} \quad \Rightarrow t=\frac{4\pi\rho l^3 c_{\text{f}} d^2}{\varepsilon l^2 P_0}=\frac{4\pi\rho l d^2 c_{\text{f}}}{\varepsilon P_0}.$$

The total power incident on the cube is given by: $Il^2 = P_0 l^2 / 4\pi d^2 = x$ J/s. x J/s is supplied by N photons of energy E_{ph} every second:

$$NE_{\text{ph}}=\frac{Nhc}{\lambda} \Rightarrow \frac{Nhc/\lambda}{\text{s}}=x \text{ J/s} \Rightarrow \frac{Nhc}{\lambda}=x \text{ J} \Rightarrow N=(x \text{ J})\frac{\lambda}{hc},$$

$$x \text{ J}=\frac{P_0 l^2}{4\pi d^2} \text{ s} \Rightarrow N=\frac{P_0 l^2 \lambda}{4\pi h d^2 c} \text{ s}.$$

CALCULATE: $t=\dfrac{4\pi\left(0.960 \text{ g/cm}^3\right)(2.00 \text{ cm})(10.0 \text{ cm})^2(334 \text{ J/g})}{(0.100)250. \text{ J/s}}=3.223\cdot 10^4 \text{ s}=8.954 \text{ h}$

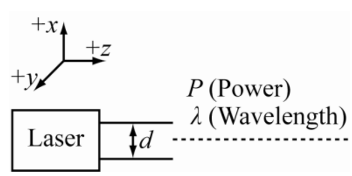
$$N=\frac{(250. \text{ J/s})(2.00 \text{ cm})^2(10.0 \text{ cm}) \text{ s}}{4\pi\left(6.626\cdot 10^{-34} \text{ J s}\right)(10.0 \text{ cm})^2\left(3.00\cdot 10^{10} \text{ cm/s}\right)}=4.003\cdot 10^{23}$$

ROUND: $t = 8.95$ h (or 8 hours 57 minutes), $N = 4.00\cdot 10^{23}$

DOUBLE-CHECK: The number of photons per second, *N*, is reasonable. The time is correct, although a real microwave will work much faster. This is because a real microwave is not a single point source. Also, a microwave has shielding which serves to reflect all waves hitting the walls, which keeps the intensity of the radiation high.

31.76. **THINK:** The Poynting vector is directed along the propagation axis, $\hat{z}$. Its magnitude is given by the laser power and the beam's cross-sectional area. The electric field is along the polarization direction, and the magnetic field is directed in the *y*-direction, perpendicular to both $\vec{E}$ and $\vec{S}$. P = 6.00 kW, $\lambda = 10.6$ μm and d = 100 μm. Note that due to the properties of electromagnetic waves the relative phase between the magnetic field and electric field is 0 degrees.

SKETCH:



RESEARCH: $\vec{E}=E_0\sin(kz-\omega t)\hat{x},\ k=2\pi/\lambda,\ \vec{B}=B_0\sin(kz-\omega t)\hat{y},\ \omega/k=c,\ S=\dfrac{E_0 B_0}{\mu_0}=\dfrac{2P}{A},$ where P is the average power, $A=\pi\left(\dfrac{d}{2}\right)^2$, and $\dfrac{E_0}{B_0}=c$.

SIMPLIFY: $S=\dfrac{2P}{A}=\dfrac{8P}{\pi d^2}$ and $S=\dfrac{E_0(E_0/c)}{\mu_0}=\dfrac{E_0^2}{\mu_0 c}$. Set these expressions equal to each other to get:

$$\frac{8P}{\pi d^2}=\frac{E_0^2}{\mu_0 c} \Rightarrow E_0=\sqrt{\frac{8P\mu_0 c}{\pi d^2}},\quad B_0=\frac{E_0}{c}=\sqrt{\frac{8P\mu_0}{\pi d^2 c}}.$$

Then, with $k=\dfrac{2\pi}{\lambda},\quad \omega=ck=\dfrac{2\pi c}{\lambda}$: $\vec{E}=\sqrt{\dfrac{8P\mu_0 c}{\pi d^2}}\sin\left(\dfrac{2\pi z}{\lambda}-\dfrac{2\pi ct}{\lambda}\right)\hat{x}$ and $\vec{B}=\sqrt{\dfrac{8P\mu_0}{\pi d^2 c}}\sin\left(\dfrac{2\pi z}{\lambda}-\dfrac{2\pi ct}{\lambda}\right)\hat{y}.$

CALCULATE: $E_0=\sqrt{\dfrac{8\left(6.00\cdot 10^3 \text{ W}\right)\left(4\pi\cdot 10^{-7} \text{ T m/A}\right)\left(3.00\cdot 10^8 \text{ m/s}\right)}{\pi\left(1.000\cdot 10^{-4} \text{ m}\right)^2}}=2.400\cdot 10^7 \text{ V/m}$

$B_0=\dfrac{2.400\cdot 10^7\text{ V/m}}{3\cdot 10^8\text{ m/s}}=8\cdot 10^{-2}\text{ T},\ \dfrac{2\pi}{\lambda}=\dfrac{2\pi}{10.6\cdot 10^{-6}\text{ m}}=5.928\cdot 10^5\text{ m}^{-1}$

$\dfrac{2\pi c}{\lambda}=5.928\cdot 10^5\text{ m}^{-1}\left(3\cdot 10^8\text{ m/s}\right)=1.778\cdot 10^{14}\text{ s}^{-1}$

ROUND: $\vec{E}=\left(2.40\cdot 10^7\text{ V/m}\right)\sin\left(5.93\cdot 10^5 z\text{ m}^{-1}-1.78\cdot 10^{14}t\text{ s}^{-1}\right)\hat{x}$

$\vec{B}=\left(8.00\cdot 10^{-2}\text{ T}\right)\sin\left(5.93\cdot 10^5 z\text{ m}^{-1}-1.78\cdot 10^{14}t\text{ s}^{-1}\right)\hat{y}$

DOUBLE-CHECK: Check that $\dfrac{E_0}{B_0}=c$: $\dfrac{2.40\cdot 10^7\text{ V/m}}{8.00\cdot 10^{-2}\text{ T}}=3.00\cdot 10^8\text{ m/s}$. Also, it is necessary that $\dfrac{\omega}{k}=c$: $\dfrac{1.78\cdot 10^{14}\text{ s}^{-1}}{5.93\cdot 10^5\text{ m}^{-1}}=3.00\cdot 10^8\text{ m/s}$. These results are reasonable.

Multi-Version Exercises

31.77. $I=\dfrac{E_{\text{rms}}^2}{c\mu_0}\Rightarrow E_{\text{rms}}=\sqrt{Ic\mu_0}=\sqrt{(182.9\text{ W/m}^2)(2.998\cdot 10^8\text{ m/s})(4\pi\cdot 10^{-7}\text{ Tm/A})}=262.5\text{ V/m}$

31.78. $I=B_{\text{rms}}^2c/\mu_0\Rightarrow B_{\text{rms}}=\sqrt{I\mu_0/c}=\sqrt{(191.4\text{ W/m}^2)(4\pi\cdot 10^{-7}\text{ Tm/A})/(2.998\cdot 10^8\text{ m/s})}=8.957\cdot 10^{-7}\text{ T}$

31.79. $I=B_{\text{rms}}^2c/\mu_0=(9.142\cdot 10^{-7}\text{ T})^2(2.998\cdot 10^8\text{ m/s})/(4\pi\cdot 10^{-7}\text{ Tm/A})=199.3\text{ W/m}^2$

31.80. $I=E_{\text{rms}}^2/\left(c\mu_0\right)=(279.9\text{ V/m})^2/\left[(2.998\cdot 10^8\text{ m/s})(4\pi\cdot 10^{-7}\text{ Tm/A})\right]=208.0\text{ W/m}^2$

31.81. If I_0 is the intensity of the incoming sunlight, then the light passing through the first polarizer has intensity $I_1=\frac{1}{2}I_0$. The intensity of the light passing through the second polarizer is given by $I_2=I_1\cos^2(\theta_2-\theta_1)$, so that $I_2=\frac{1}{2}I_0\cos^2(\theta_2-\theta_1)$. The reduction in intensity, then, is

$$R=\frac{I_0-I_2}{I_0}=1-\tfrac{1}{2}\cos^2(\theta_2-\theta_1)=1-\tfrac{1}{2}\cos^2(88.6°-28.1°)=87.9\%.$$

31.82. As in the preceding problem, reduction of initial intensity $R=1-\frac{1}{2}\cos^2(\theta_2-\theta_1)$.

$$\theta_2=\theta_1+\cos^{-1}\left(\sqrt{2-2R}\right)=38.3°+\cos^{-1}\left(\sqrt{2-2\cdot 0.7584}\right)=84.3°$$

31.83. As above, reduction of initial intensity $R=1-\frac{1}{2}\cos^2(\theta_2-\theta_1)$.

$$\theta_1=\theta_2-\cos^{-1}\left(\sqrt{2-2R}\right)=110.6°-\cos^{-1}\left(\sqrt{2-2\cdot 0.7645}\right)=63.9°$$

Chapter 32: Geometric Optics

Concept Checks

32.1 a **32.2** e **32.3** b **32.4** c **32.5.** b **32.6** b **32.7.** c **32.8.** c **32.9** b

Multiple-Choice Questions

32.1. c **32.2.** c **32.3.** a **32.4.** c **32.5.** b **32.6.** a **32.7.** e **32.8.** d **32.9.** b **32.10.** c **32.11.** a **32.12.** a

Conceptual Questions

32.13. In a step-index fiber, there is a discontinuity of the index of refraction at the core-cladding interface. Consequently, light will undergo total internal reflection at the core-cladding interface, and will propagate through the fiber in a zigzag path. By contrast, in a graded-index fiber, the refractive index changes gradually in moving from core to cladding. A light ray entering the core of a graded-index fiber will continuously change its direction as the refractive index continuously changes. Consider light incident at the core at an angle of θ_{i}. The incident ray moves from the core, of refractive index n_{core}, to a refractive index of $n_{\text{core}} - \Delta n$, where Δn is a small change in the index of refraction. For subsequent refractions, assume that the index of refraction changes by the same amount and the new incident angle is the previous final angle. The first three refractions are given by:

$$n_{\text{core}} \sin\theta_{\text{i}} = \left(n_{\text{core}} - \Delta n\right)\sin\theta_{\text{f}}^{(1)} \Rightarrow \sin\theta_{\text{f}}^{(1)} = \left(\frac{n_{\text{core}}}{n_{\text{core}} - \Delta n}\right)\sin\theta_{\text{i}}$$

$$\left(n_{\text{core}} - \Delta n\right)\sin\theta_{\text{f}}^{(1)} = \left(n_{\text{core}} - 2\Delta n\right)\sin\theta_{\text{f}}^{(2)} \Rightarrow \sin\theta_{\text{f}}^{(2)} = \left(\frac{n_{\text{core}} - \Delta n}{n_{\text{core}} - 2\Delta n}\right)\sin\theta_{\text{f}}^{(1)}$$

$$\left(n_{\text{core}} - 2\Delta n\right)\sin\theta_{\text{f}}^{(2)} = \left(n_{\text{core}} - 3\Delta n\right)\sin\theta_{\text{f}}^{(3)} \Rightarrow \sin\theta_{\text{f}}^{(3)} = \left(\frac{n_{\text{core}} - 2\Delta n}{n_{\text{core}} - 3\Delta n}\right)\sin\theta_{\text{f}}^{(2)}.$$

Combining these equations gives:

$$\begin{aligned}\sin\theta_{\text{f}}^{(3)} &= \left(\frac{n_{\text{core}} - 2\Delta n}{n_{\text{core}} - 3\Delta n}\right)\sin\theta_{\text{f}}^{(2)} = \left(\frac{n_{\text{core}} - 2\Delta n}{n_{\text{core}} - 3\Delta n}\right)\left(\frac{n_{\text{core}} - \Delta n}{n_{\text{core}} - 2\Delta n}\right)\sin\theta_{\text{f}}^{(1)} \\ &= \left(\frac{n_{\text{core}} - 2\Delta n}{n_{\text{core}} - 3\Delta n}\right)\left(\frac{n_{\text{core}} - \Delta n}{n_{\text{core}} - 2\Delta n}\right)\left(\frac{n_{\text{core}}}{n_{\text{core}} - \Delta n}\right)\sin\theta_{\text{i}} = \left(\frac{n_{\text{core}}}{n_{\text{core}} - 3\Delta n}\right)\sin\theta_{\text{i}}.\end{aligned}$$

For *N* refractions,

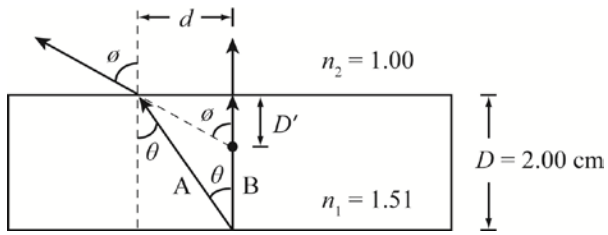
$$\sin\theta_{\text{f}}^{(N)} = \left(\frac{n_{\text{core}}}{n_{\text{core}} - N\Delta n}\right)\sin\theta_{\text{i}}.$$

As the light approaches the cladding, $n_{\text{core}} - N\Delta n \to n_{\text{cladding}}$, so

$$\sin\theta_{\text{f}} = \left(\frac{n_{\text{core}}}{n_{\text{cladding}}}\right)\sin\theta_{\text{i}}.$$

In the limit, Snell's Law is recovered. If n_{core} and n_{cladding} are chosen correctly, then total internal reflection will occur. The difference between the two fibers is that for a step-index fiber, the reflection occurs instantly (zigzag path), but for a graded-index fiber, the reflection occurs gradually (sinusoidal path) since the light gets refracted along the way.

32.14. Ray A will leave the plexiglass at some angle, ϕ. This ray extrapolated back will reconnect with ray B, which does not get refracted since it is normal to the interface. The location where the rays reconnect is where the image will appear to form.



From the diagram, two equations are apparent:

$$d = D\tan\theta \text{ and } d = D'\tan\phi.$$

Equating these gives,

$$D\tan\theta = D'\tan\phi \Rightarrow D' = \frac{\tan\theta}{\tan\phi}D.$$

Therefore, the apparent height of the text is:

$$y = D - D' = D\left(1 - \frac{\tan\theta}{\tan\phi}\right).$$

The angle, ϕ, is given by Snell's Law:

$$n_1 \sin\theta = n_2 \sin\phi \Rightarrow \phi = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta\right) = \sin^{-1}\left(\frac{(1.51)}{(1.00)}\sin(25.0^\circ)\right) = 39.65^\circ.$$

Therefore, the height is:

$$y = (2.00 \text{ cm})\left(1 - \frac{\tan(25.0^\circ)}{\tan(39.65^\circ)}\right) = 0.875 \text{ cm}.$$

32.15. Due to spherical aberration, light rays a distance d from the center axis of a mirror of curvature R will cross the optical axis a distance y from the center of the mirror, where y is approximated as:

$$y \approx \frac{R}{2}\left(1 - \frac{d^2}{2R^2}\right).$$

When there is no spherical aberration, the image is formed at the focal point; that is, $y = f = R/2$. Therefore, to reduce the spherical aberration produced by the mirror, $d^2/(2R^2) \to 0$. Since the mirror cannot be changed (R cannot change), the only way to reduce the spherical aberration is to make the height of the object, d, small.

32.16. (a) If one were to look straight down, the cone of rays coming from a point at the bottom of the pool would refract away from the normal when they cross the water/air interface. If the rays were then extrapolated back, they would intersect at a point above the bottom of the pool. Therefore, the pool would appear to be less deep than it actually is. This finding can be confirmed by considering Example 32.3 for very small angles θ_1 and θ_2: the formula for apparent depth becomes

$$d_{\text{apparent}} = d_{\text{actual}} \frac{n_{\text{air}}}{n_{\text{water}}}.$$

The result can also be confirmed by examining two drinking glasses side by side, one empty and one containing water.

(b) If one were to look at an angle, the cone of rays coming from a point at the bottom of the pool would refract away from the normal when they cross the water/air interface. If the rays were then extrapolated back, they would intersect at a point above the bottom of the pool. Therefore, the pool would appear to be less deep than it actually is. Figure 32.40 illustrates this for a similar situation.

32.17. When the light enters a medium, it will interact with the atoms that make up the material. Electrons will vibrate and then re-emit the light, which will then go along and interact with another electron. This effect will continue all the way through the material. The interactions with the electrons take a finite amount of time, causing delays along the way through the material. This gives the appearance that the light is traveling at a different speed.

32.18. If the fiber is bent too much, then the incident angle of light on the edge of the fiber exceeds the critical angle, and light is lost, reducing signal transmission.

32.19. The drum acts like a concave or converging mirror. From Table 32.1, it can be seen that when the object moves from just inside the focal point of a converging mirror to just outside the focal point, the image changes from an upright image to an inverted image. Hence, if she starts with her finger close enough to the drum that she sees an upright image and slowly moves it away, the point at which the image flips is the focal point, and the distance from her finger to the mirror at that point is the focal length. Twice the focal length is the radius of curvature.

32.20. True. The speed of the light wave in vacuum is given by $v = c = f\lambda.$ In a medium, the speed is given by $v = c/n = f\lambda'.$ Since the frequency *f* is constant, these equations can be solved in terms of wavelength: $\lambda' = \lambda / n.$ Therefore, the wavelength of any type of light in water $n = 1.33$ is less than its wavelength in air $n = 1.00$

32.21. A ray of light incident at any angle on a corner cube of mirrors will always be reflected back at the same angle as the incident ray due to the law of reflection. The array of cubes is then necessary to provide a larger surface to strike. The smaller the corner cube, the closer to the original path the reflected path will be, minimizing any possible change in path length between the two rays. Therefore, when light is incident on the array, the reflected beam will be sent back virtually along the same path as the incident beam. The change in path length of the two beams will be insignificant compared to the large distance between the Earth and the Moon, thus still providing excellent precision in the measurement.

32.22. (a) Inside the prism, the reflected beam would hit each leg of the prism at $45°$. For the prism-water interface, the critical angle is:

$$\sin\theta_c = \frac{n_{H_2O}}{n_{glass}} \Rightarrow \theta_c = \sin^{-1}\left(\frac{n_{H_2O}}{n_{glass}}\right) = \sin^{-1}\left(\frac{(1.333)}{(1.520)}\right) = 61.28°.$$

Therefore, when the light strikes the legs at $45°$ when in water, total internal reflection will not occur and most of the light will exit the prism. Some of the light will get reflected, but its intensity will be low, rendering the prism ineffective.

(b) Prisms are used for several reasons: (1) Cost; quality mirrors require expensive reflecting coatings while prisms do not. (2) Quality; total internal reflection reflects all of the light while all mirrors reflect with some loss in intensity. (3) Durability; there is no reflective coating on the prisms to corrode or degrade over time. It is only necessary to keep it clean and dry.

32.23. It the upper half of the mirror is covered then only the bottom portion of the mirror will focus the rays. The image will be at the same location, but will appear dimmer since fewer light rays will be brought to a focus.

32.24. The light entering the water will bend towards the vertical, so you will observe the light at a steeper angle than you would if you were not under water. Therefore, the Sun will appear to be higher in the sky than it actually is.

32.25. The spoon can be treated as a spherical convex mirror. Decent estimates for the radius of curvature of the spoon and object distance are $R = 5.0$ cm and $d_o = 15.0$ cm, respectively. Since the spoon is a convex mirror, the radius of curvature is negative. The basic mirror equation then gives the image distance:

$$\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f}=\frac{2}{R} \Rightarrow d_i=\left(\frac{2}{R}-\frac{1}{d_o}\right)^{-1}=\left(\frac{2}{(-5.0\text{ cm})}-\frac{1}{(15.0\text{ cm})}\right)^{-1}=-2.14\text{ cm}\approx -2.1\text{ cm}.$$

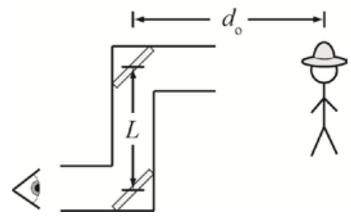
The magnification is: $m=-\frac{d_i}{d_o}=-\frac{(-2.14\text{ cm})}{(15.0\text{ cm})}=0.14.$

32.26. No, it is not possible to attain temperatures exceeding the temperature of the photosphere of the Sun. The second law of thermodynamics prevents this from happening. For a perfectly efficient process, the maximum attainable temperature is 6000 K.

Exercises

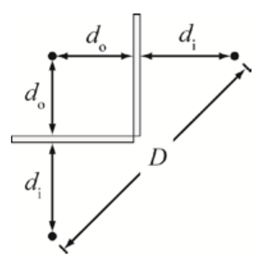
32.27. For plane mirrors, the object distance, $d_o=1.00$ m, is always equal to the image distance, but the image is located behind the mirror. Therefore, the location of the image is $d_i=-d_o=-1.00$ m.

32.28.



The distance D of the final image of the yellow hat from the lower mirror is $D=d_o+L=3.00\text{ m}+0.400\text{ m}=3.40\text{ m}.$

32.29.

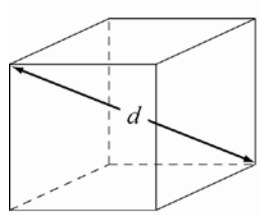


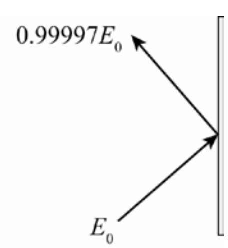
The object has an image in each mirror equidistant from the mirror and the object, so $|d_o|=|d_i|$. This means the distance between the two images is the hypotenuse of a triangle, where each leg is a length, $|d_o|+|d_i|=2d_o$. So, the distance between the images is

$$D=\sqrt{(2d_o)^2+(2d_o)^2}=\sqrt{8d_o^2}=\sqrt{8(2.00\text{ m})^2}=5.66\text{ m}.$$

32.30. **THINK:** If a photon bounces once, it has 99.997% of its energy, so after n bounces it has $(0.99997)^n$ of its original energy left. The longest length in the cube room, sides l = 3.00 m, is the diagonal along the cube. The average distance a photon goes in the room is half the longest distance.

SKETCH:





RESEARCH: The maximum distance of the room is $d=\sqrt{3}l.$ The average distance is then $\langle d\rangle = d/2.$ The energy after n bounces is $E'=(0.99997)^n E_0.$ The time for one bounce is $\Delta t=\langle d\rangle/c.$ The total time is $t=n\Delta t.$

SIMPLIFY: The number of bounces to reduce to $E'=0.0100$ is:

$$E'=(0.99997)^n E_0 = 0.0100E_0 \Rightarrow n=\frac{\ln(0.0100)}{\ln(0.99997)}.$$

The total time of light dissipation is:

$$t=n\Delta t=n\frac{\langle d\rangle}{c}=\frac{nd}{2c}=\frac{\sqrt{3}l}{2c}n.$$

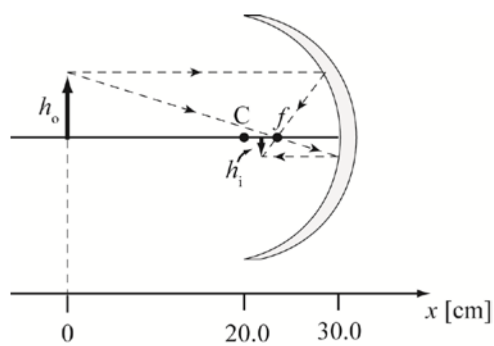
CALCULATE: $t=\dfrac{\sqrt{3}(3.00\text{ m})}{2(3.00\cdot 10^8\text{ m/s})}\left(\dfrac{\ln(0.0100)}{\ln(0.99997)}\right)=0.001329\text{ s}$

ROUND: To three significant figures, $t=1.33$ ms.

DOUBLE-CHECK: Such a small time would be undetectable for an average human. This is expected, considering the speed of light.

32.31. The focal length is given by $f=\dfrac{R}{2}.$ For $R=-25.0$ cm, the focal length is: $f=\dfrac{R}{2}=\dfrac{(-25.0\text{ cm})}{2}\approx -12.5\text{ cm}.$

32.32.



For a radius of curvature of $R=10.0$ cm and an object distance of $d_\text{o}=30.0$ cm, the image distance is:

$$\frac{1}{d_\text{o}}+\frac{1}{d_\text{i}}=\frac{1}{f}=\frac{2}{R} \Rightarrow d_\text{i}=\left(\frac{2}{R}-\frac{1}{d_\text{o}}\right)^{-1}=\left(\frac{2}{(10.0\text{ cm})}-\frac{1}{(30.0\text{ cm})}\right)^{-1}=6.00\text{ cm}.$$

Therefore, the position of the image is

$$x_\text{i}=d_\text{o}-d_\text{i}=(30.0\text{ cm})-(6.00\text{ cm})=24.0\text{ cm}.$$

The magnification is $m=h_\text{i}/h_\text{o}=-d_\text{i}/d_\text{o}.$ For an object height of $h_\text{o}=5.00$ cm, the image height is:

$$h_\text{i}=mh_\text{o}=-\frac{d_\text{i}h_\text{o}}{d_\text{o}}=-\frac{(6.00\text{ cm})(5.00\text{ cm})}{(30.0\text{ cm})}=-1.00\text{ cm}.$$

Since $h_\text{i}<0,$ the image is inverted. Since $d_\text{i}>0,$ the image is real.

32.33. For a radius of curvature of $R=-14.0$ m and an object distance of $d_\text{o}=11.0$ m, the image distance is:

$$\frac{1}{d_\text{o}}+\frac{1}{d_\text{i}}=\frac{1}{f}=\frac{2}{R} \Rightarrow d_\text{i}=\left(\frac{2}{R}-\frac{1}{d_\text{o}}\right)^{-1}=\left(\frac{2}{(-14.0\text{ m})}-\frac{1}{(11.0\text{ m})}\right)^{-1}=-4.28\text{ m}.$$

The magnification of the mirror is: $m=-\dfrac{d_\text{i}}{d_\text{o}}=-\dfrac{(-4.28\text{ m})}{(11.0\text{ m})}=0.389.$

32.34. For a focal length of $f=-10.0\text{ cm}$ and an object distance of $d_\text{o}=30.0\text{ cm},$ the image distance is:

$$\frac{1}{d_\text{o}}+\frac{1}{d_\text{i}}=\frac{1}{f}\ \Rightarrow\ d_\text{i}=\left(\frac{1}{f}-\frac{1}{d_\text{o}}\right)^{-1}=\left(\frac{1}{(-10.0\text{ cm})}-\frac{1}{(30.0\text{ cm})}\right)^{-1}=-7.50\text{ cm}.$$

For an object height of $h_\text{o}=5.00\text{ cm},$ the image height is:

$$h_\text{i}=-\frac{d_\text{i}h_\text{o}}{d_\text{o}}=-\frac{(-7.50\text{ cm})(5.00\text{ cm})}{(30.0\text{ cm})}=1.25\text{ cm}.$$

Since $d_\text{i}<0,$ the image is virtual. Since $h_\text{i}>0,$ the image is upright.

32.35. For an object a distance of $d_\text{o}=2.0\text{ m}$ in front of a convex mirror with magnification $m=0.60,$ the image distance is

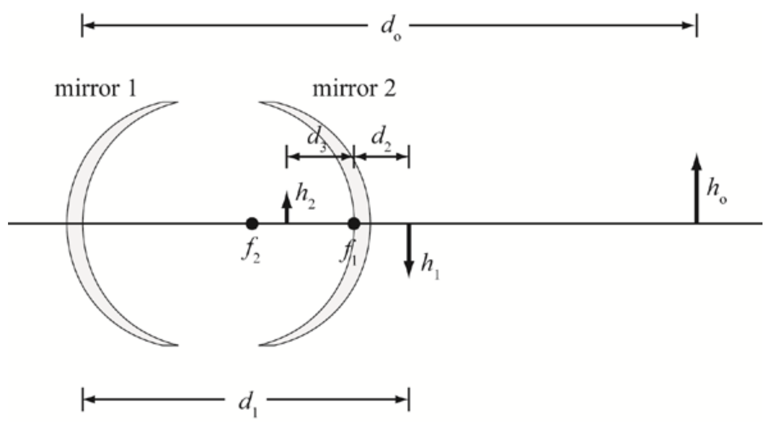
$$m=-\frac{d_\text{i}}{d_\text{o}}\ \Rightarrow\ d_\text{i}=-md_\text{o}.$$

The focal length is:

$$\frac{1}{f}=\frac{1}{d_\text{o}}+\frac{1}{d_\text{i}}\ \Rightarrow\ \frac{1}{f}=\frac{1}{d_\text{o}}-\frac{1}{md_\text{o}}=\frac{m-1}{md_\text{o}}\ \Rightarrow\ f=\frac{md_\text{o}}{m-1}=\frac{(0.60)(2.0\text{ m})}{(0.60)-1}=-3.0\text{ m}.$$

32.36. **THINK:** The object, at $d_\text{o}=100.\text{ cm},$ is behind the second mirror (which is located at the focal point of the first mirror of focal length $f_1=20.0\text{ cm}$) of focal length $f_2=5.00\text{ cm}.$ For simplicity, ignore the fact that the second mirror sits between the first mirror and the object. That is, assume the second mirror is a two-way mirror so that the light rays from the object go through it, reflect from the first mirror, and then reflect from the second mirror. The reflections will continue until the final image is formed outside of both mirrors.

SKETCH:



RESEARCH: The relevant equations are the mirror equation and the magnification equation:

$$\frac{1}{d_\text{o}}+\frac{1}{d_\text{i}}=\frac{1}{f},\quad m=-\frac{d_\text{i}}{d_\text{o}}.$$

SIMPLIFY: When the object is first reflected by mirror 1, the image distance and magnification are:

$$\frac{1}{d_\text{o}}+\frac{1}{d_1}=\frac{1}{f_1}\ \Rightarrow\ d_1=\left(\frac{1}{f_1}-\frac{1}{d_\text{o}}\right)^{-1}\text{ and }\ m_1=-\frac{d_1}{d_\text{o}}.$$

This image must form behind mirror 2 since mirror 2 is at the focal point of mirror 1. Therefore, the image is a virtual object for mirror 2 at a distance of $d_2 = -(d_1 - f_1)$. The image distance and magnification are:

$$\frac{1}{d_2} + \frac{1}{d_3} = \frac{1}{f_2} \Rightarrow d_3 = \left(\frac{1}{f_2} - \frac{1}{d_2}\right)^{-1} \text{ and } m_2 = -\frac{d_3}{d_2}.$$

The total magnification of the system is then $m = m_1 m_2 = \dfrac{d_1 d_3}{d_o d_2}$.

CALCULATE: $d_1 = \left(\dfrac{1}{(20.0\text{ cm})} - \dfrac{1}{(100.\text{ cm})}\right)^{-1} = 25.0\text{ cm},\ \ d_2 = -\big((25.0\text{ cm}) - (20.0\text{ cm})\big) = -5.00\text{ cm}$

The final image distance and total magnification is:

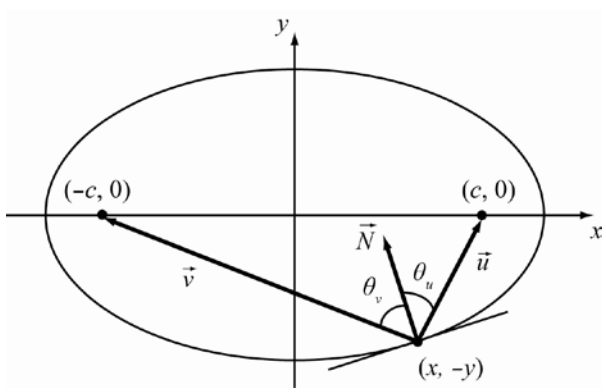
$$d_3 = \left(\frac{1}{(5.00\text{ cm})} - \frac{1}{(-5.00\text{ cm})}\right)^{-1} = 2.50\text{ cm and } m = \frac{(25.0\text{ cm})(2.50\text{ cm})}{(100.\text{ cm})(-5.00\text{ cm})} = -0.125$$

ROUND: Remaining at 3 significant figures, the final image location is between the two mirrors, a distance of $d_3 = 2.50\text{ cm}$ from mirror 2. The total magnification is $m = -0.125$.

DOUBLE-CHECK: For two plane mirrors facing each other, an infinite number of virtual images are formed. For two concave mirrors, it is expected that a real image must form after some number of reflections.

32.37. **THINK:** An arbitrary point on the elliptical mirror can be chosen: $p(+x,-y)$. Two ray vectors exist that point from p to $(\pm c, 0)$, where $c = \sqrt{a^2 - b^2}$. The normal line, which is perpendicular to the surface of the elliptical mirror, can be determined. The dot product can be used to determine the angle between the two ray vectors and the normal vector. If perfect reflection occurs, the angles between the normal vector and the two ray vectors should be the same.

SKETCH:



RESEARCH: The two ray vectors are defined as $\vec{u} = -(x-c)\hat{x} + y\hat{y}$ and $\vec{v} = -(x+c)\hat{x} + y\hat{y}$, and they make an angles of θ_u and θ_v with $\vec{N}$. The normal vector to a surface is defined as:

$$\vec{N} = \frac{\partial f(x,y)}{\partial x}\hat{x} + \frac{\partial f(x,y)}{\partial y}\hat{y}.$$

SIMPLIFY: First, determine the normal vector:

$$\vec{N} = \frac{\partial f(x,y)}{\partial x}\hat{x} + \frac{\partial f(x,y)}{\partial y}\hat{y} = \frac{\partial}{\partial x}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\hat{x} + \frac{\partial}{\partial y}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\hat{y} = \frac{2x}{a^2}\hat{x} + \frac{2y}{b^2}\hat{y}.$$

For point $(x,-y)$:

$$\vec{N}=\frac{2x}{a^2}\hat{x}-\frac{2y}{b^2}\hat{y}.$$

The unit vectors of $\vec{u}$ and $\vec{v}$ are given by:

$$\hat{u}=\frac{\vec{u}}{|\vec{u}|}=\frac{-(x-c)\hat{x}+y\hat{y}}{\sqrt{(x-c)^2+y^2}} \text{ and } \hat{v}=\frac{\vec{v}}{|\vec{v}|}=\frac{-(x+c)\hat{x}+y\hat{y}}{\sqrt{(x+c)^2+y^2}}.$$

The dot product of $\vec{N}$ with the two unit vectors gives

$$\vec{N}\bullet\hat{u}=|\vec{N}||\hat{u}|\cos\theta_u=|\vec{N}|\cos\theta_u \text{ and } \vec{N}\bullet\hat{v}=|\vec{N}||\hat{v}|\cos\theta_v=|\vec{N}|\cos\theta_v.$$

It is known from the law of reflection that the two angles must be equal. Therefore, if $\vec{N}\bullet\hat{u}=\vec{N}\bullet\hat{v}$ is shown, then the proof will be complete.

$$\left(\vec{N}\bullet\hat{u}\right)^2=\frac{\left[\left(\frac{2x}{a^2}\hat{x}-\frac{2y}{b^2}\hat{y}\right)\bullet\left(-(x-c)\hat{x}+y\hat{y}\right)\right]^2}{(x-c)^2+y^2}=\frac{4\left[-\frac{x^2}{a^2}+\frac{xc}{a^2}-\frac{y^2}{b^2}\right]^2}{x^2-2xc+c^2+y^2}$$

Recall that $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2}$. Also, $y^2=b^2\left(1-\frac{x^2}{a^2}\right)=\left(a^2-c^2\right)\left(1-\frac{x^2}{a^2}\right)$. Substitution gives:

$$\left(\vec{N}\bullet\hat{u}\right)^2=\frac{4\left(-1+\frac{xc}{a^2}\right)^2}{x^2-2xc+c^2+\left(a^2-c^2\right)\left(1-\frac{x^2}{a^2}\right)}=\frac{4\left(1-\frac{2xc}{a^2}+\frac{x^2c^2}{a^4}\right)}{x^2-2xc+c^2+a^2-c^2-x^2+\frac{x^2c^2}{a^2}}$$

$$=\frac{\frac{4}{a^2}\left(a^2-2xc+\frac{x^2c^2}{a^2}\right)}{a^2-2xc+\frac{x^2c^2}{a^2}}=\frac{4}{a^2}.$$

$$\left(\vec{N}\bullet\hat{v}\right)^2=\frac{\left[\left(\frac{2x}{a^2}\hat{x}-\frac{2y}{b^2}\hat{y}\right)\bullet\left(-(x+c)\hat{x}+y\hat{y}\right)\right]^2}{(x+c)^2+y^2}=\frac{4\left[-\frac{x^2}{a^2}-\frac{xc}{a^2}-\frac{y^2}{b^2}\right]^2}{x^2+2xc+c^2+y^2}$$

Making the same substitutions as above gives:

$$\left(\vec{N}\bullet\hat{v}\right)^2=\frac{4\left(-1-\frac{xc}{a^2}\right)^2}{x^2+2xc+c^2+\left(a^2-c^2\right)\left(1-\frac{x^2}{a^2}\right)}=\frac{4\left(1+\frac{2xc}{a^2}+\frac{x^2c^2}{a^4}\right)}{x^2+2xc+c^2+a^2-c^2-x^2+\frac{x^2c^2}{a^2}}$$

$$=\frac{\frac{4}{a^2}\left(a^2+2xc+\frac{x^2c^2}{a^2}\right)}{a^2+2xc+\frac{x^2c^2}{a^2}}=\frac{4}{a^2}.$$

$\vec{N}\bullet\hat{u}=\vec{N}\bullet\hat{v}=\frac{4}{a^2}, \Rightarrow \cos\theta_{\text{u}}=\cos\theta_{\text{v}} \Rightarrow \theta_{\text{u}}=\theta_{\text{v}}$, since the angles are both in the same quadrant.

CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: The above derivation proves that, given the law of reflection, the vector that goes through a focal point will be reflected through the other focal point of an elliptical mirror.

32.38. The velocity of light in a medium with refractive index n is $v = c/n$. For crown glass with index of refraction of $n = 1.52$, the velocity is:

$$v = \frac{c}{n} = \frac{\left(3.00 \cdot 10^8 \text{ m/s}\right)}{(1.52)} = 1.97 \cdot 10^8 \text{ m/s}.$$

32.39. The critical angle is given by $\sin\theta_c = n_2 / n_1$. The critical angles of the optical fiber in air, water and oil are:

$$\theta_{\text{c,air}} = \sin^{-1}\left(\frac{1.000}{1.50}\right) = 41.8°, \quad \theta_{\text{c,water}} = \sin^{-1}\left(\frac{1.333}{1.50}\right) = 62.7° \text{ and } \theta_{\text{c,oil}} = \sin^{-1}\left(\frac{1.50}{1.50}\right) = 90.0°.$$

32.40. The helium-neon laser light of wavelength $\lambda_{\text{vac}} = 632.8 \text{ nm}$ is in water with an index of refraction of $n = 1.333$.

(a) The velocity is:

$$v = \frac{c}{n} = \frac{\left(2.998 \cdot 10^8 \text{ m/s}\right)}{(1.333)} = 2.249 \cdot 10^8 \text{ m/s}.$$

(b) The frequency remains unchanged (it is independent of the medium), so using values in a vacuum gives:

$$c = f\lambda_{\text{vac}} \Rightarrow f = \frac{c}{\lambda_{\text{vac}}} = \frac{\left(2.998 \cdot 10^8 \text{ m/s}\right)}{\left(632.8 \cdot 10^{-9} \text{ m}\right)} = 4.738 \cdot 10^{14} \text{ Hz}.$$

(c) The wavelength is:

$$\lambda = \frac{\lambda_{\text{vac}}}{n} = \frac{(632.8 \text{ nm})}{(1.333)} = 474.7 \text{ nm}.$$

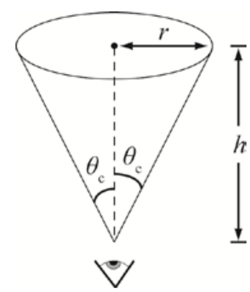
(d) Technically the color does not change because it is the frequency of light that our eyes receive and interpret. Therefore, the color is still that of $4.738 \cdot 10^{14}$ Hz on the spectrum (red-like).

32.41. To get fully polarized light, the incident light must strike the water-plate glass interface at the Brewster angle:

$$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right) = \tan^{-1}\left(\frac{1.73}{1.33}\right) = 52.4°.$$

32.42. **THINK:** Regardless of the angle of incidence, light rays from the air will enter the water. However, some light rays coming from underwater will hit the surface at or above the critical angle and will undergo total internal reflection, creating a virtual mirror. For light incident from water to air, the indices of refraction are $n_1 = 1.333$ and $n_2 = 1.000$. This means that the only clear window is the circle created by the cone, with the angle from the vertical equal to the critical angle of the water/air interface. The tip of the cone is $h = 2.00$ m below the surface of the water.

SKETCH:



RESEARCH: The critical angle is defined as $\sin\theta_c = n_2 / n_1$. The radius of the circle is $r = h\tan\theta_c$. The diameter of the window is $d = 2r$.

SIMPLIFY: The diameter of the window is:

$$d = 2r = 2h\tan\theta_c = 2h\tan\left[\sin^{-1}\left(\frac{n_2}{n_1}\right)\right].$$

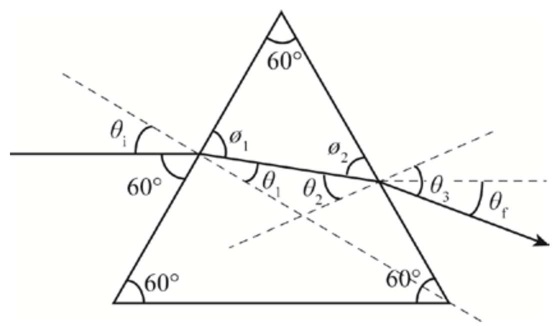
CALCULATE: $d = 2(2.00 \text{ m})\tan\left[\sin^{-1}\left(\frac{1.000}{1.333}\right)\right] = 4.538 \text{ m}$

ROUND: To 3 significant figures, $d = 4.54$ m.

DOUBLE-CHECK: This value is reasonable considering the depth of the observer.

32.43. **THINK:** Since the normal line of the first surface bisects the opposite angle, the refracted ray must hit the other angled surface. Simple geometry must be utilized to determine all the angles involved. The index of refraction of air and the prism are $n_a = 1.00$ and $n_p = 1.23$, respectively.

SKETCH:



RESEARCH: Since the incident beam is parallel to the base, the incident angle is $\theta_i = 30°$. Snell's Law is used to determine refracted angles: $n_i \sin\theta_i = n_j \sin\theta_j$.

SIMPLIFY: At the first interface:

$$n_a \sin\theta_i = n_p \sin\theta_1 \Rightarrow \theta_1 = \sin^{-1}\left(\frac{n_a}{n_p}\sin\theta_i\right).$$

Based on the geometry shown in the figure above, $\phi_1 = 90° - \theta_1$ and $\phi_2 = 180° - (60° + \phi_1)$. Therefore,

$$\phi_2 = 120° - \phi_1 = 120° - (90° - \theta_1) = 30° + \theta_1.$$

Also, $\theta_2 = 90° - \phi_2$. Therefore,

$$\theta_2 = 90° - (30° + \theta_1) = 60° - \sin^{-1}\left(\frac{n_a}{n_p}\sin\theta_i\right).$$

At the second interface, Snell's Law is reapplied as the light exits the prism:

$$n_p \sin\theta_2 = n_a \sin\theta_3 \Rightarrow \theta_3 = \sin^{-1}\left(\frac{n_p}{n_a}\sin\theta_2\right) = \sin^{-1}\left[\frac{n_p}{n_a}\sin\left(60° - \sin^{-1}\left(\frac{n_a}{n_p}\sin\theta_i\right)\right)\right].$$

The change in direction is equal to the sum of the changes in angle at each interface:

$$\theta_f = (\theta_i - \theta_1) + (\theta_3 - \theta_2) = \theta_i - \theta_1 + \theta_3 - (60° - \theta_1) = \theta_i - 60° + \theta_3,$$

$$\theta_f = \theta_i - 60° + \sin^{-1}\left[\frac{n_p}{n_a}\sin\left(60° - \sin^{-1}\left(\frac{n_a}{n_p}\sin\theta_i\right)\right)\right].$$

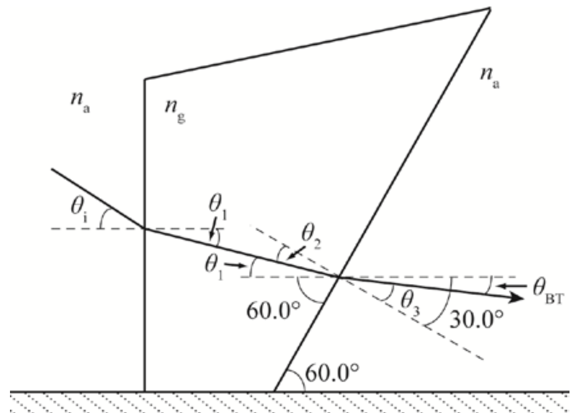
CALCULATE: $\theta_f = (30°) - 60° + \sin^{-1}\left[\frac{(1.23)}{(1.00)}\sin\left(60° - \sin^{-1}\left(\frac{(1.00)}{(1.23)}\sin(30°)\right)\right)\right] = 16.322°$

ROUND: Rounding to three significant figures, $\theta_f = 16.3°$.

DOUBLE-CHECK: The change in direction depends on the initial incident angle, the refractive index of air and the refractive index of the prism, as expected. This is a reasonable angle for the ray of light to be deflected after going through a prism.

32.44. **THINK:** The light is refracted as it crosses the air-glass interface and the glass-air interface. The air and the glass have a refractive index of $n_a = 1.00$ and $n_g = 1.55$, respectively.

SKETCH:



RESEARCH: The angle of refraction at each interface can be determined using Snell's Law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

SIMPLIFY: At the first interface,

$$n_a \sin\theta_i = n_g \sin\theta_1 \;\Rightarrow\; \theta_1 = \sin^{-1}\left(\frac{n_a}{n_g}\sin\theta_i\right).$$

Based on the geometry of the glass block,

$$\theta_2 = 30.0° - \theta_1 = 30.0° - \sin^{-1}\left(\frac{n_a}{n_g}\sin\theta_i\right).$$

At the second interface,

$$n_g \sin\theta_2 = n_a \sin\theta_3 \;\Rightarrow\; \theta_3 = \sin^{-1}\left(\frac{n_g}{n_a}\sin\theta_2\right) = \sin^{-1}\left[\frac{n_g}{n_a}\sin\left(30.0° - \sin^{-1}\left(\frac{n_a}{n_g}\sin\theta_i\right)\right)\right].$$

The angle from the horizontal is

$$\theta_{BT} = 30.0° - \theta_3 = 30.0° - \sin^{-1}\left[\frac{n_g}{n_a}\sin\left(30.0° - \sin^{-1}\left(\frac{n_a}{n_g}\sin\theta_i\right)\right)\right].$$

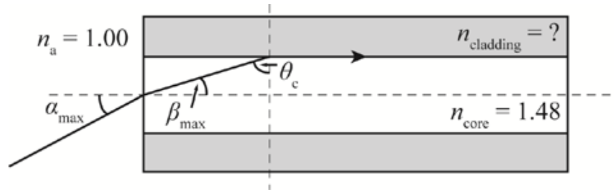
CALCULATE: $\theta_{BT} = 30.0° - \sin^{-1}\left[\frac{(1.55)}{(1.00)}\sin\left(30.0° - \sin^{-1}\left(\frac{(1.00)}{(1.55)}\sin(20.0°)\right)\right)\right] = 2.632°$

ROUND: Rounding to three significant figures, $\theta_{BT} = 2.63°$.

DOUBLE-CHECK: This result is reasonable.

32.45. **THINK:** The maximum incident angle $\alpha_{max} = 14.033°$ corresponds to the light ray that reaches the core-cladding interface at an angle equal to the critical angle. Knowing the critical angle and the refractive index of $n_{core} = 1.48$, the index of refraction of the cladding can be determined using Snell's Law.

SKETCH:



RESEARCH: Snell's Law is given by $n_1 \sin\theta_1 = n_2 \sin\theta_2$, and the critical angle is given by $\sin\theta_c = n_2 / n_1$.

SIMPLIFY: By Snell's Law, $n_a \sin\alpha_{max} = n_{core} \sin\beta_{max} \Rightarrow \beta_{max} = \sin^{-1}\left(\frac{n_a}{n_{core}}\sin\alpha_{max}\right)$. The critical angle is

$\theta_c = 90° - \beta_{max} = 90° - \sin^{-1}\left(\frac{n_a}{n_{core}}\sin\alpha_{max}\right)$. At the core-cladding interface:

$$n_{cladding} = n_{core}\sin\theta_c = n_{core}\sin\left(90° - \sin^{-1}\left(\frac{n_a}{n_{core}}\sin\alpha_{max}\right)\right).$$

The percent difference between the index of refraction of the core and the index of refraction of the cladding is:

$$\% \text{ difference} = \left(1 - \frac{n_{cladding}}{n_{core}}\right)(100\%) = \left(1 - \sin\left(90° - \sin^{-1}\left(\frac{n_a}{n_{core}}\sin\alpha_{max}\right)\right)\right)(100\%).$$

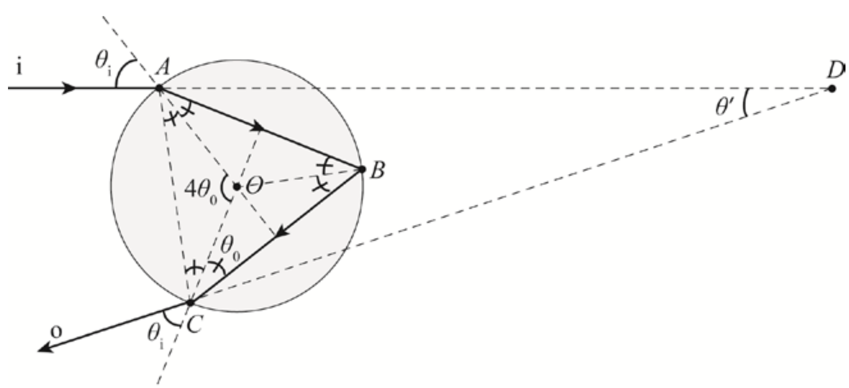
CALCULATE: $\% \text{ difference} = \left(1 - \sin\left(90° - \sin^{-1}\left(\frac{(1.00)}{(1.48)}\sin(14.033°)\right)\right)\right)(100\%) = 1.3513\%$

ROUND: To three significant figures, the percent difference between the index of refraction of the core and the index of refraction of the cladding is 1.35%.

DOUBLE-CHECK: This result is reasonable. One would expect the difference to be small.

32.46. **THINK:** The colors of a rainbow occur because white light from the sun is refracted into its component colors by water droplets in the atmosphere. A rainbow is observed at an angle of $42°$ from the direction of the sunlight, because at this angle, the intensity of the various colors is greatest. This occurs because, for angles less than $42°$, the separation of the colors is less pronounced and rays merge to form white light. The angle of $42°$ represents the maximum angle at which light rays exit a spherical water droplet.

SKETCH:



RESEARCH: The path of the ray inside the water droplet can be determined using Snell's Law, the law of reflection, $\theta_i = \theta_r$, and the geometry of circles and triangles.

SIMPLIFY: The angle, θ_0, is given by Snell's Law: $n_\text{a}\sin\theta_\text{i} = n_\text{w}\sin\theta_0 \Rightarrow \theta_0 = \sin^{-1}\left(\frac{n_\text{a}}{n_\text{w}}\sin\theta_\text{i}\right)$. Due to the geometry of a circle, the incident angle at point *B* is equal to θ_0. This is true of the incident angle at *C* as well. Therefore, the refracted ray leaving at *C* is θ_i by Snell's Law. The angle θ' is the change in direction of the light ray. For the two refractions and the one reflection, the total change in direction is:

$$\theta' = (\theta_0 - \theta_\text{i}) + 2\theta_0 + (\theta_0 - \theta_\text{i}) = 4\theta_0 - 2\theta_\text{i} = 4\sin^{-1}\left(\frac{n_\text{a}}{n_\text{w}}\sin\theta_\text{i}\right) - 2\theta_\text{i}.$$

The maximum value of θ' occurs when $d\theta'/d\theta_\text{i} = 0$. The following derivatives can be found in a table of derivatives:

$$\frac{d}{dx}\left(\sin^{-1}u\right) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \text{ and } \frac{d}{dx}\sin x = \cos x.$$

The value of θ_i for the maximum value of θ' is given by:

$$\frac{d\theta'}{d\theta_\text{i}} = \frac{4}{\sqrt{1-\left(\frac{n_\text{a}}{n_\text{w}}\right)^2\sin^2\theta_\text{i}}}\left(\frac{n_\text{a}}{n_\text{w}}\right)\cos\theta_\text{i} - 2 = 0 \Rightarrow 2\left(\frac{n_\text{a}}{n_\text{w}}\right)\cos\theta_\text{i} = \sqrt{1-\left(\frac{n_\text{a}}{n_\text{w}}\right)^2\sin^2\theta_\text{i}}$$

$$4\left(\frac{n_\text{a}}{n_\text{w}}\right)^2\cos^2\theta_\text{i} = 1 - \left(\frac{n_\text{a}}{n_\text{w}}\right)^2\sin^2\theta_\text{i} \Rightarrow 4\cos^2\theta_\text{i} + \sin^2\theta_\text{i} = \left(\frac{n_\text{w}}{n_\text{a}}\right)^2.$$

Using the trigonometric identity $\sin^2 x = 1 - \cos^2 x$ gives:

$$4\cos^2\theta_\text{i} + \left(1-\cos^2\theta_\text{i}\right) = \left(\frac{n_\text{w}}{n_\text{a}}\right)^2 \Rightarrow 3\cos^2\theta_\text{i} = \left(\frac{n_\text{w}}{n_\text{a}}\right)^2 - 1 = \frac{n_\text{w}^2 - n_\text{a}^2}{n_\text{a}^2}$$

$$\theta_\text{i} = \cos^{-1}\left(\sqrt{\frac{n_\text{w}^2 - n_\text{a}^2}{3n_\text{a}^2}}\right)$$

CALCULATE: $\theta_\text{i} = \cos^{-1}\left(\sqrt{\frac{(1.333)^2 - (1.000)^2}{3(1.000)^2}}\right) = 59.4105°$

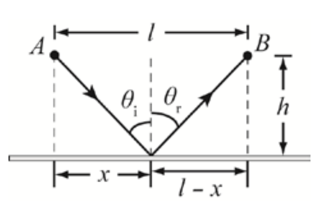
$$\theta'_\text{max} = 4\sin^{-1}\left(\frac{(1.000)}{(1.333)}\sin(59.4105°)\right) - 2(59.4105°) = 42.078°$$

ROUND: Rounding to four significant figures, the maximum angle is $42.08°$. Therefore, the observed angle for a rainbow is $42.08°$ from the direction of the sunlight.

DOUBLE-CHECK: This is the angle that the question asked to derive.

32.47. **THINK:** Fermat's Principle states that the path taken by a ray between two points in space is the path that takes the least amount of time. The law of reflection can be found by using this principle. To accomplish this, determine the time it takes for a ray to travel from one point to another by hitting the mirror. Using calculus, this time can be minimized and the law of reflection is recovered.

SKETCH:



RESEARCH: The time it takes the ray to reach the mirror is $t = d/v$. To minimize the time, set $dt/dx = 0$.

SIMPLIFY: $t=\frac{d_1}{v}+\frac{d_2}{v}=\frac{1}{v}\left(d_1+d_2\right)=\frac{n}{c}\left(\sqrt{h^2+x^2}+\sqrt{h^2+\left(l-x\right)^2}\right)$

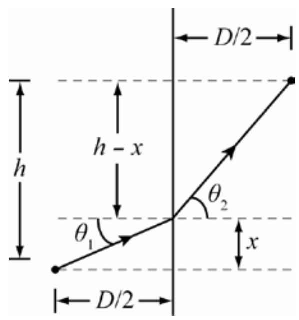
The path of least time is determined from:

$$\frac{dt}{dx}=0=\frac{n}{c}\left[\frac{(1/2)2x}{\sqrt{h^2+x^2}}-\frac{(1/2)2(l-x)}{\sqrt{h^2+(l-x)^2}}\right]=\frac{n}{c}\left[\frac{x}{\sqrt{h^2+x^2}}-\frac{(l-x)}{\sqrt{h^2+(l-x)^2}}\right]=\frac{n}{c}\left(\sin\theta_{\text{i}}-\sin\theta_{\text{r}}\right)$$

$$\sin\theta_{\text{i}}-\sin\theta_{\text{r}}=0\Rightarrow\theta_{\text{i}}=\theta_{\text{r}}$$

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: The law of reflection was recovered using Fermat's Principle.

32.48. **THINK:** Fermat's Principle states that the path taken by a ray between two points in space is the path that takes the least amount of time. Snell's Law can be determined by using this principle. To accomplish this, determine the time it takes for a ray to travel from one point to another by traveling through both materials. Using calculus, this time can be minimized and the Snell's Law is recovered.
SKETCH:



RESEARCH: The time it takes the ray to reach the point is $t=d/v$. To minimize the time, set $dt/dx=0$.

SIMPLIFY: $t=\frac{d_1}{v_1}+\frac{d_2}{v_2}=\frac{n_1}{c}d_1+\frac{n_2}{c}d_2=\frac{n_1}{c}\sqrt{x^2+D^2/4}+\frac{n_2}{c}\sqrt{\left(h-x\right)^2+D^2/4}$

The path of least time is determined from:

$$\frac{dt}{dx}=0=\frac{n_1}{c}\left[\frac{(1/2)2x}{\sqrt{x^2+D^2/4}}\right]-\frac{n_2}{c}\left[\frac{(1/2)2(h-x)}{\sqrt{(h-x)^2+D^2/4}}\right]=\frac{n_1}{c}\left(\frac{x}{\sqrt{x^2+D^2/4}}\right)-\frac{n_2}{c}\left(\frac{(h-x)}{\sqrt{(h-x)^2+D^2/4}}\right)$$

$$\frac{n_1}{c}\sin\theta_1-\frac{n_2}{c}\sin\theta_2=0\Rightarrow n_1\sin\theta_1=n_2\sin\theta_2$$

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: Fermat's Principle was used to derive an equation involving the indices of refraction and angles from the horizontal, as desired. The resulting equation is Snell's Law.

32.49. In the case of a plane mirror, we have that $h_0=h_i$ and that $|d_0|=|d_i|$. Therefore:
(a) The image distance is 50.0 cm behind the mirror.
(b) The image has the same height, h = 2.00 m.
(c) The image is upright.
(d) The image is virtual.

32.50. (a) The frequency of the ray does not change in the medium, so:

$$f=\frac{c}{\lambda_{\text{air}}}=\frac{\left(3.00\cdot 10^{8}\text{ m/s}\right)}{\left(7.00\cdot 10^{-7}\text{ m}\right)}=4.29\cdot 10^{14}\text{ Hz}.$$

(b) The speed inside the liquid is: $v=\frac{c}{n_2}=\frac{\left(3.00\cdot 10^{8}\text{ m/s}\right)}{(1.63)}=1.84\cdot 10^{8}\text{ m/s}.$

(c) The wavelength of the refracted ray is: $\lambda=\frac{v}{f}=\frac{c}{n_2}\left(\frac{1}{c/\lambda_{\text{air}}}\right)=\frac{\lambda_{\text{air}}}{n_2}=\frac{(700.\text{ nm})}{(1.63)}=429\text{ nm}.$

32.51. For the image to be twice the size of the object, the magnification is:

$$m=2=\left|\frac{d_{\text{i}}}{d_{\text{o}}}\right|\Rightarrow d_{\text{i}}=\pm 2d_{\text{o}}.$$

The spherical mirror equation is:

$$\frac{1}{d_{\text{o}}}+\frac{1}{d_{\text{i}}}=\frac{1}{f}=\frac{2}{R}\Rightarrow\frac{1}{d_{\text{o}}}\pm\frac{1}{2d_{\text{o}}}=\frac{2}{R}$$

$$\frac{(2\pm 1)}{2d_{\text{o}}}=\frac{2}{R}\Rightarrow d_{\text{o}}=\frac{(2\pm 1)R}{4}$$

The object can be placed at:

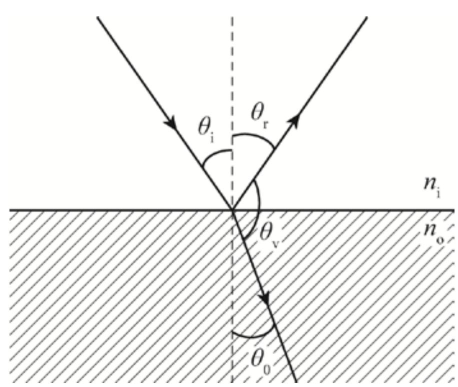
$$d_{\text{o}}=\frac{3}{4}R=\frac{3}{4}(20.0\text{ cm})=15.0\text{ cm or } d_{\text{o}}=\frac{R}{4}=\frac{(20.0\text{ cm})}{4}=5.00\text{ cm},$$

to produce an image that is twice the size of the object. If the object is placed at 15.0 cm, the image distance will be $d_{\text{i}}=2(15.0\text{ cm})=30.0\text{ cm}$. Since $d_{\text{i}}>0$, this image will be real. If the object is placed at 5.00 cm, the image distance will be $d_{\text{i}}=-2(5.00\text{ cm})=-10.0\text{ cm}$. Since $d_{\text{i}}<0$, this image will be virtual.

32.52. The critical angle is given by $\sin\theta_{\text{c}}=n_2/n_1$. Thus, the critical angle for a water-air interface is:

$$\theta_{\text{c}}=\sin^{-1}\left(\frac{n_2}{n_1}\right)=\sin^{-1}\left(\frac{1.000}{1.333}\right)=48.61°.$$

32.53.



The reflected ray has the same angle to the normal as the incident ray. The refracted ray has an angle given by Snell's Law:

$$n_{\text{i}}\sin\theta_{\text{i}}=n_0\sin\theta_0\Rightarrow\theta_0=\sin^{-1}\left(\frac{n_{\text{i}}}{n_0}\sin\theta_{\text{i}}\right)=\sin^{-1}\left(\frac{(1.000)}{(1.333)}\sin 30.0°\right)=22.0°.$$

The angle between the reflected and refracted rays is $\theta_{\text{v}}=180.0°-\theta_{\text{r}}-\theta_0=180.0°-30.0°-22.0°=128.0°.$

32.54. The focal point of the ornament is $f = R/2 = d/4$. By convention, a convex mirror has a negative value for *R*, so *d* is negative. Using the mirror equation, the image distance is:

$$\frac{1}{d_\text{i}} = \frac{1}{f} - \frac{1}{d_\text{o}} = \frac{4}{d} - \frac{1}{d_\text{o}} \Rightarrow d_\text{i} = \left(\frac{4}{d} - \frac{1}{d_\text{o}}\right)^{-1}.$$

Saint Nicholas will see his reflection at:

$$d_\text{i} = \left(\frac{4}{(-8.00 \text{ cm})} - \frac{1}{(156 \text{ cm})}\right)^{-1} = -1.97 \text{ cm}.$$

The image is virtual since $d_\text{i} < 0$.

32.55. The critical angle is given by:

$$\sin\theta_\text{c} = \frac{n_2}{n_1} \Rightarrow \theta_\text{c} = \sin^{-1}\left(\frac{n_2}{n_1}\right).$$

The critical angle for the diamond-air interface is:

$$\theta_\text{c, a} = \sin^{-1}\left(\frac{1.000}{2.417}\right) = 24.44°.$$

The critical angle for the diamond-water interface is:

$$\theta_\text{c, w} = \sin^{-1}\left(\frac{1.333}{2.417}\right) = 33.47°.$$

Therefore, the critical angle in water is $9.03°$ greater than the critical angle in air.

32.56. Since $f = R/2$, Table 32.1 shows that

(a) for $d_\text{o} > R$, the image is real,

(b) for $R/2 < d_\text{o} < R$, the image is real,

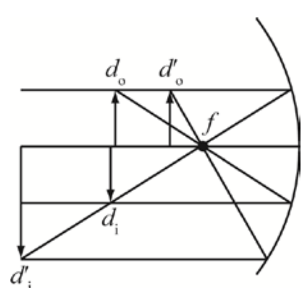
(c) and for $d_\text{o} < R/2$, the image is virtual.

32.57. Since the incident angle is equal to the reflected angle, $\theta_1 = 40.0°$. The refracted angle θ is given by Snell's Law:

$$n_1 \sin\theta_1 = n_2 \sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right) = \sin^{-1}\left(\frac{(1.000)}{(1.333)}\sin(40.0°)\right) = 28.8°.$$

32.58. **THINK:** The object is moved around from one point to another. Using the magnification of the two points and the change in the image distance, the focal point of the mirror and the change in the distance of object can be determined. The magnification of the image at the first position is $m = 2$ and the magnification of the image at the second position is $m' = 3$. The difference between the image distances is $\Delta d_\text{i} = d_\text{i}' - d_\text{i} = 75$ cm.

SKETCH:



RESEARCH: The magnification is given by $m = |d_\text{i} / d_\text{o}|$. The mirror equation is: $\frac{1}{d_\text{o}} + \frac{1}{d_\text{i}} = \frac{1}{f}$.

SIMPLIFY: The object distances are $d_\text{o} = d_\text{i} / m$ and $d_\text{o}' = d_\text{i}' / m'$. The mirror equation gives:

$$\frac{1}{f} = \frac{1}{d_\text{o}} + \frac{1}{d_\text{i}} = \frac{m}{d_\text{i}} + \frac{1}{d_\text{i}} = \frac{(m+1)}{d_\text{i}} \Rightarrow d_\text{i} = (m+1)f,$$

$$\frac{1}{f} = \frac{1}{d_\text{o}'} + \frac{1}{d_\text{i}'} = \frac{m'}{d_\text{i}'} + \frac{1}{d_\text{i}'} = \frac{(m'+1)}{d_\text{i}'} \Rightarrow d_\text{i}' = (m'+1)f.$$

The focal length is given by

$$\Delta d_\text{i} = d_\text{i}' - d_\text{i} = (m'+1)f - (m+1)f = (m'-m)f \Rightarrow f = \frac{\Delta d_\text{i}}{(m'-m)}.$$

The change in the object distance is:

$$\Delta d_\text{o} = d_\text{o}' - d_\text{o} = \frac{d_\text{i}'}{m'} - \frac{d_\text{i}}{m} = \frac{m(m'+1)f - m'(m+1)f}{mm'} = \frac{(m-m')}{mm'}f = \frac{(m-m')}{(m'-m)mm'}\Delta d_\text{i}.$$

CALCULATE: $f = \frac{(75 \text{ cm})}{(3-2)} = 75 \text{ cm}$

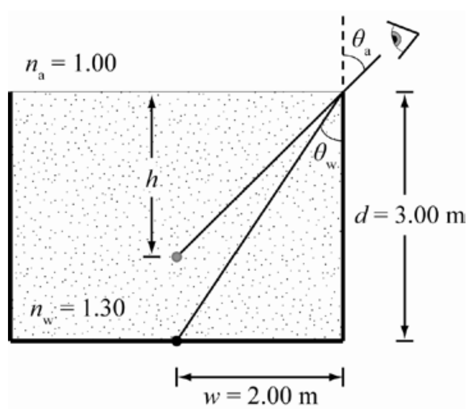
$$\Delta d_\text{o} = \frac{(2-3)}{(3-2)(2)(3)}(75 \text{ cm}) = -12.5 \text{ cm}$$

ROUND: To two significant figures, the focal length of the mirror is $f = 75 \text{ cm}$ and the object was moved $|\Delta d_\text{o}| = 13 \text{ cm}$.

DOUBLE-CHECK: Since the image is larger after it is moved, the object should be moved towards the mirror. This is indicated by the negative value for the change in the object distance.

32.59. **THINK:** The rays of light from the point are refracted before they reach the person, according to Snell's Law. Because the index of refraction of air is less than that of water, the image appears shallower. The point is $d = 3.00 \text{ m}$ from the surface and $w = 2.00 \text{ m}$ from the edge of the pool.

SKETCH:



RESEARCH: The angle of the ray is given by Snell's Law: $n_\text{w} \sin\theta_\text{w} = n_\text{a} \sin\theta_\text{a}$. The triangles also relate the angles to the lengths:

$$\sin\theta_\text{w} = \frac{w}{\sqrt{w^2 + d^2}} \text{ and } \sin\theta_\text{a} = \frac{w}{\sqrt{w^2 + h^2}}.$$

SIMPLIFY: Combining the above equations gives:

$$n_\text{w} \sin\theta_\text{w} = \frac{n_\text{w} w}{\sqrt{w^2 + d^2}} = n_\text{a} \sin\theta_\text{a} = \frac{n_\text{a} w}{\sqrt{w^2 + h^2}}.$$

Solving for the apparent depth gives:

$$n_w\sqrt{w^2+h^2}=n_a\sqrt{w^2+d^2} \;\Rightarrow\; n_w^2\left(w^2+h^2\right)=n_a^2\left(w^2+d^2\right)$$

$$\Rightarrow\; n_w^2h^2=\left(n_a^2-n_w^2\right)w^2+n_a^2d^2 \;\Rightarrow\; h=\frac{1}{n_w}\sqrt{\left(n_a^2-n_w^2\right)w^2+n_a^2d^2}.$$

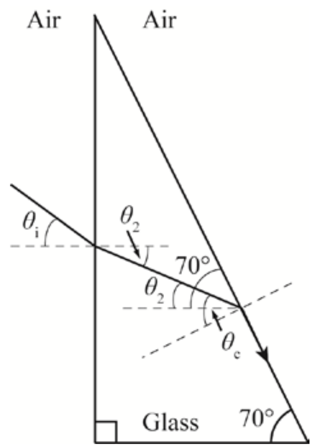
CALCULATE: $h=\frac{1}{(1.30)}\sqrt{\left((1.00)^2-(1.30)^2\right)(2.00\text{ m})^2+(1.00)^2(3.00\text{ m})^2}=1.92\text{ m}$

ROUND: Remaining at 3 significant figures, the apparent depth of the pool is $h=1.92$ m.

DOUBLE-CHECK: The apparent depth is less than the true depth of the pool, as expected.

32.60. **THINK:** For the smallest incident angle, total internal reflection at the surface occurs at the critical angle. Snell's Law and the geometry of the prism can be used to find this incident angle.

SKETCH:



RESEARCH: Snell's Law at the air-glass interface is $n_a\sin\theta_i=n_g\sin\theta_2$ and the critical angle at the glass-air interface is given by $\sin\theta_c=\frac{n_a}{n_g}$. The angles are related by $90°=70°+\theta_c-\theta_2 \;\Rightarrow\; \theta_2=\theta_c-20°$.

SIMPLIFY: The critical angle is given by $\theta_c=\sin^{-1}\left(\frac{n_a}{n_g}\right)$. Therefore, $\theta_2=\sin^{-1}\left(\frac{n_a}{n_g}\right)-20°$. The incident angle is given by:

$$\sin\theta_i=\frac{n_g}{n_a}\sin\theta_2=\frac{n_g}{n_a}\sin\left(\sin^{-1}\left(\frac{n_a}{n_g}\right)-20°\right) \;\Rightarrow\; \theta_i=\sin^{-1}\left[\frac{n_g}{n_a}\sin\left(\sin^{-1}\left(\frac{n_a}{n_g}\right)-20°\right)\right].$$

CALCULATE: $\theta_i=\sin^{-1}\left\{\frac{(1.500)}{(1.000)}\sin\left[\sin^{-1}\left(\frac{(1.000)}{(1.500)}\right)-20°\right]\right\}=33.87°$

ROUND: The angle $70°$ was given in a geometric figure, so treat it as having two significant figures. To two significant figures, the smallest incident angle is $\theta_i=34°$.

DOUBLE-CHECK: This result is reasonable.

32.61. (a) Time reversal leaves the charge and electric field the same, but reverses the current and magnetic field. The time reversal solution is obtained with:

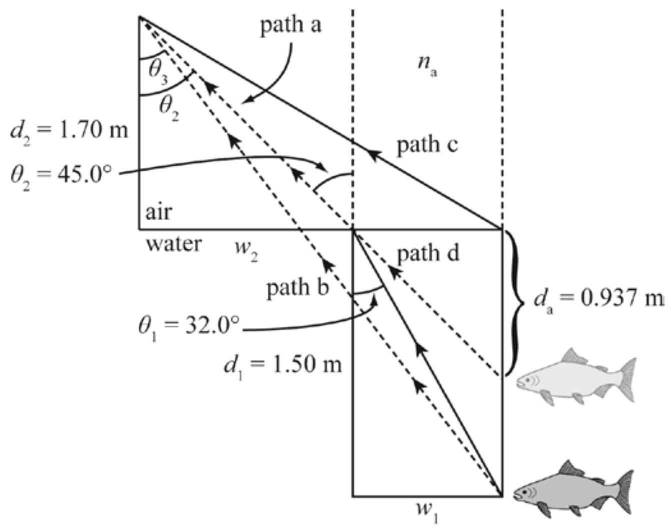
$$\rho(t)\to\rho(-t),\; \vec{j}(t)\to-\vec{j}(-t),\; \vec{E}(t)\to\vec{E}(-t),\text{ and }\vec{B}(\vec{x},t)\to-\vec{B}(\vec{x},-t).$$

By plugging these transformations in Maxwell's equations in Table 31.1, it is seen that the negative signs cancel out in each of the equations, and the original equations are recovered.

(b) One-way mirrors do not violate Maxwell's equations since light can go both ways through a one-way mirror. A one-way mirror is a partially silvered mirror mounted between a brightly lit room and a darkened room. The mirror is partially reflective and partially transparent. The key to its operation is the difference in lighting between the two rooms. In the brightly lit room, reflected light overwhelms light transmitted from the dark room and the one-way mirror looks like a mirror. Seen from the dark room, the light transmitted from the bright room overwhelms reflected light and the one-way mirror looks like a window into the bright room. Ordinary windows demonstrate the same effect. A window in a brightly lit room looks like a mirror from the room side at night, but a window to the outdoors in daylight.

32.62. **THINK:** The path a ray of light takes between two points minimizes the time required for the trip. This problem asks for the time it takes light to travel between two points via various paths. The situation is depicted below.

SKETCH:



RESEARCH: The time it takes the light ray to travel its path is $t = d/v$. The speed of the ray in a medium is $v = c/n$.

SIMPLIFY:

(a) The time of travel for light along path *a* is:

$$t_a = \frac{l_1}{v_1} + \frac{l_2}{v_2} = \frac{1}{c}\left(\frac{n_w d_1}{\cos\theta_1} + \frac{n_a d_2}{\cos\theta_2}\right).$$

(b) The time of travel for light along path *b* is:

$$t_b = \frac{l_1}{v_1} + \frac{l_2}{v_2} = \frac{1}{c}\left(\frac{n_w d_1}{\cos\theta_3} + \frac{n_a d_2}{\cos\theta_3}\right) = \frac{n_w d_1 + n_a d_2}{c\cos\theta_3},$$

where the angle θ_3 is given by:

$$\tan\theta_3 = \frac{w_1 + w_2}{d_1 + d_2} \Rightarrow \tan\theta_3 = \frac{d_1\tan\theta_1 + d_2\tan\theta_2}{d_1 + d_2}.$$

Thus

$$t_b = \frac{n_w d_1 + n_a d_2}{\cos\left[\tan^{-1}\left(\dfrac{d_1\tan\theta_1 + d_2\tan\theta_2}{d_1 + d_2}\right)\right]c}.$$

(c) The time of travel for light along path *c* is:

$$t_c = \frac{n_w d_1}{c} + \frac{n_a}{c}\sqrt{d_2^2 + \left(w_1 + w_2\right)^2} = \frac{1}{c}\left(n_w d_1 + n_a\sqrt{d_2^2 + \left(d_1\tan\theta_1 + d_2\tan\theta_2\right)^2}\right).$$

(d) The time of travel for light along path d is:

$$t_{\text{d}} = \frac{l_1}{v_1} + \frac{l_2}{v_2} = \frac{1}{c}\left(\frac{n_{\text{w}} d_{\text{a}}}{\cos\theta_2} + \frac{n_{\text{a}} d_2}{\cos\theta_2}\right) = \frac{n_{\text{w}} d_{\text{a}} + n_{\text{a}} d_2}{c\cos\theta_2}.$$

CALCULATE:

(a) $t_{\text{a}} = \dfrac{1}{\left(3.00\cdot 10^8 \text{ m/s}\right)}\left(\dfrac{(1.333)(1.50 \text{ m})}{\cos(32.0°)} + \dfrac{(1.000)(1.70 \text{ m})}{\cos(45.0°)}\right) = 1.5873\cdot 10^{-8} \text{ s}$

(b) $t_{\text{b}} = \dfrac{(1.333)(1.50 \text{ m}) + (1.000)(1.70 \text{ m})}{\cos\left(\tan^{-1}\left[\dfrac{(1.50 \text{ m})\tan(32.0°) + (1.70 \text{ m})\tan(45.0°)}{(1.50 \text{ m}) + (1.70 \text{ m})}\right]\right)\left(3.00\cdot 10^8 \text{ m/s}\right)} = 1.5980\cdot 10^{-8} \text{ s}$

(c) $t_{\text{c}} = \dfrac{(1.333)(1.50 \text{ m}) + (1.000)\sqrt{(1.70 \text{ m})^2 + \left((1.50 \text{ m})\tan(32.0°) + (1.70 \text{ m})\tan(45.0°)\right)^2}}{\left(3.00\cdot 10^8 \text{ m/s}\right)}$

$= 1.7109\cdot 10^{-8} \text{ s}$

(d) $t_{\text{d}} = \dfrac{(1.333)(0.937 \text{ m}) + (1.000)(1.70 \text{ m})}{\left(3.00\cdot 10^8 \text{ m/s}\right)\cos(45.0°)} = 1.3902\cdot 10^{-8} \text{ s}$

ROUND: Round the results to three significant figures.

(a) $t_{\text{a}} = 1.59\cdot 10^{-8} \text{ s}$

(b) $t_{\text{b}} = 1.60\cdot 10^{-8} \text{ s}$

(c) $t_{\text{c}} = 1.71\cdot 10^{-8} \text{ s}$

(d) $t_{\text{d}} = 1.39\cdot 10^{-8} \text{ s}$

(e) Path *d* has the shortest travel time, but the rays are not actually starting at the location where the fish appears to be. Path *a*, given by Fermat's Principle (using Snell's Law), has the smallest travel time for light from the fish to the observer. Therefore, Fermat's Principle holds since path *a* is the actual path taken by the light.

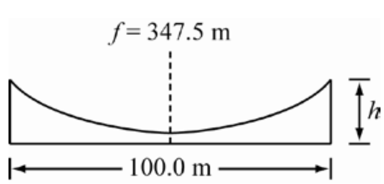
DOUBLE-CHECK: The path given by Fermat's principle (i.e. Snell's Law) does take the least amount of time, as expected.

32.63. The focal length of a liquid mirror is $f = \dfrac{g}{2\omega^2}$, where ω is the angular velocity of the rotating mirror. The angular velocity is:

$$\omega = \sqrt{\frac{g}{2f}} = \sqrt{\frac{\left(9.81 \text{ m/s}^2\right)}{2(2.50 \text{ m})}} = 1.40 \text{ rad/s}.$$

32.64. **THINK:** A proposal for a space telescope is to place a rotating liquid mirror, of focal length $f = 347.5$ m and diameter $d = 100.0$ m, on the Moon, where the gravitational acceleration is $g_{\text{M}} = 1.62 \text{ m/s}^2$.

SKETCH:



RESEARCH: The focal length of a rotating mirror is given by $f = g/2\omega^2$. The linear speed of a rotating point a distance *r* from the axis of rotation is $v = r\omega$. The height of the liquid can be determined by considering the conservation of energy. The kinetic energy of the liquid is $K=(1/2)mv^2$ and the potential energy is $U=mg_{\mathrm{M}}h$.

SIMPLIFY:

(a) The angular velocity of the liquid is given by

$$\omega=\sqrt{\frac{g_{\mathrm{M}}}{2f}}.$$

(b) The linear speed of a point on the perimeter is

$$v=r\omega=\frac{d}{2}\sqrt{\frac{g_{\mathrm{M}}}{2f}}.$$

(c) The height of the liquid at any point on the mirror is given when the potential energy and kinetic energy are equal:

$$K=U\;\Rightarrow\;\frac{1}{2}mv^2=\frac{1}{2}m\omega^2r^2=mg_{\mathrm{M}}h\;\Rightarrow\;h=\frac{\omega^2r^2}{2g_{\mathrm{M}}}=\left(\frac{g_{\mathrm{M}}}{2f}\right)\frac{(d/2)^2}{2g_{\mathrm{M}}}=\frac{d^2}{16f}.$$

CALCULATE:

(a) $\omega=\sqrt{\frac{\left(1.62\text{ m/s}^2\right)}{2\left(347.5\text{ m}\right)}}=4.82798\cdot10^{-2}\text{ rad/s}$

(b) $v=\frac{\left(100.0\text{ m}\right)}{2}\sqrt{\frac{\left(1.62\text{ m/s}^2\right)}{2\left(347.5\text{ m}\right)}}=2.41399\text{ m/s}$

(c) $h=\frac{\left(100.0\text{ m}\right)^2}{16\left(347.5\text{ m}\right)}=1.79856\text{ m}$

ROUND: Round the answers for parts (a) and (b) to three significant figures, and the answer for part (c) to four significant figures.

(a) The angular velocity of the mirror is $\omega=4.83\cdot10^{-2}$ rad/s.

(b) The linear speed of a point on the perimeter of the mirror is $v=2.41\text{ m/s}$.

(c) The perimeter is at a height of $h=1.799\text{ m}$ above the center of the mirror.

DOUBLE-CHECK: Each result has the appropriate units.

Multi-Version Exercises

32.65. $\frac{1}{f}=\frac{1}{d_{\mathrm{i}}}+\frac{1}{d_{\mathrm{o}}}\Rightarrow d_{\mathrm{i}}=\frac{d_{\mathrm{o}}f}{d_{\mathrm{o}}-f}$

$$f=R/2\quad\text{and}\quad d_{\mathrm{o}}=R+x_{\mathrm{o}}\Rightarrow d_{\mathrm{i}}=\frac{(R+x_{\mathrm{o}})R/2}{R+x_{\mathrm{o}}-R/2}=\frac{R^2+Rx_{\mathrm{o}}}{R+2x_{\mathrm{o}}}$$

$$d_{\mathrm{i}}=R+x_{\mathrm{i}}\Rightarrow x_{\mathrm{i}}=d_{\mathrm{i}}-R=\frac{R^2+Rx_{\mathrm{o}}}{R+2x_{\mathrm{o}}}-R=-\frac{Rx_{\mathrm{o}}}{R+2x_{\mathrm{o}}}=-11.7\text{ cm}$$

32.66. $\frac{1}{f}=\frac{1}{d_{\mathrm{i}}}+\frac{1}{d_{\mathrm{o}}}\Rightarrow d_{\mathrm{i}}=\frac{d_{\mathrm{o}}f}{d_{\mathrm{o}}-f}$

$$f=R/2\quad\text{and}\quad d_{\mathrm{o}}=R+x_{\mathrm{o}}\Rightarrow d_{\mathrm{i}}=\frac{(R+x_{\mathrm{o}})R/2}{R+x_{\mathrm{o}}-R/2}=\frac{R^2+Rx_{\mathrm{o}}}{R+2x_{\mathrm{o}}}$$

$$m=-\frac{d_{\mathrm{i}}}{d_{\mathrm{o}}}=-\frac{R^2+Rx_{\mathrm{o}}}{(R+2x_{\mathrm{o}})(R+x_{\mathrm{o}})}=-\frac{R}{R+2x_{\mathrm{o}}}=-0.429$$

32.67. $\frac{1}{f}=\frac{1}{d_{\text{i}}}+\frac{1}{d_{\text{o}}}\Rightarrow f=\frac{d_{\text{o}}d_{\text{i}}}{d_{\text{o}}+d_{\text{i}}}$

$f=R/2$ and $d_{\text{o}}=R+x_{\text{o}}$ and $d_{\text{i}}=R+x_{\text{i}}$

$\Rightarrow\frac{R}{2}=\frac{(R+x_{\text{o}})(R+x_{\text{i}})}{R+x_{\text{o}}+R+x_{\text{i}}}=\frac{R^2+R(x_{\text{o}}+x_{\text{i}})+x_{\text{o}}x_{\text{i}}}{2R+x_{\text{o}}+x_{\text{i}}}$

$\Rightarrow 2R^2+R(x_{\text{o}}+x_{\text{i}})=2R^2+2R(x_{\text{o}}+x_{\text{i}})+2x_{\text{o}}x_{\text{i}}$

$\Rightarrow R=-\frac{2x_{\text{o}}x_{\text{i}}}{x_{\text{o}}+x_{\text{i}}}=55.5\text{ cm}$

32.68. $\frac{1}{f}=\frac{1}{d_{\text{i}}}+\frac{1}{d_{\text{o}}}\Rightarrow f=\frac{d_{\text{o}}d_{\text{i}}}{d_{\text{o}}+d_{\text{i}}}$

$f=R/2$ and $d_{\text{o}}=R+x_{\text{o}}$ and $d_{\text{i}}=R+x_{\text{i}}$

$\Rightarrow\frac{R}{2}=\frac{(R+x_{\text{o}})(R+x_{\text{i}})}{R+x_{\text{o}}+R+x_{\text{i}}}=\frac{R^2+R(x_{\text{o}}+x_{\text{i}})+x_{\text{o}}x_{\text{i}}}{2R+x_{\text{o}}+x_{\text{i}}}$

$\Rightarrow 2R^2+R(x_{\text{o}}+x_{\text{i}})=2R^2+2R(x_{\text{o}}+x_{\text{i}})+2x_{\text{o}}x_{\text{i}}$

$\Rightarrow R=-\frac{2x_{\text{o}}x_{\text{i}}}{x_{\text{o}}+x_{\text{i}}}\Rightarrow f=-\frac{x_{\text{o}}x_{\text{i}}}{x_{\text{o}}+x_{\text{i}}}=28.3\text{ cm}$

32.69. This is simply Snell's Law, but with a slight twist that the angles are measured relative to the interface between the two media, not relative to the normal.

$\theta_1=90°-\varphi_1$

$\theta_2=\sin^{-1}(n_1\sin(\theta_1)/n_2)$

$\varphi_2=90°-\theta_2=90°-\sin^{-1}(n_1\sin(90°-\varphi_1)/n_2)$

$=90°-\sin^{-1}(1.329\sin(90°-61.07°)/1.310)=60.61°$

32.70. This is simply Snell's Law, but with a slight twist that the angles are measured relative to the interface between the two media, not relative to the normal.

$\theta_1=90°-\varphi_1;\quad \theta_2=90°-\varphi_2$

$n_1\sin(\theta_1)=n_2\sin(\theta_2)$

$\Rightarrow n_1=n_2\frac{\sin(90°-\varphi_2)}{\sin(90°-\varphi_1)}=1.111$

32.71. This is simply Snell's Law, but with a slight twist that the angles are measured relative to the interface between the two media, not relative to the normal.

$\theta_2=90°-\varphi_1$

$\theta_1=\sin^{-1}(n_2\sin(\theta_2)/n_1)$

$\varphi_1=90°-\theta_1=90°-\sin^{-1}(n_2\sin(90°-\varphi_2)/n_1)$

$=90°-\sin^{-1}(1.310\sin(90°-72.06°)/1.00045)=66.21°$

32.72. This is simply Snell's Law, but with a slight twist that the angles are measured relative to the interface between the two media, not relative to the normal.

$\theta_1=90°-\varphi_1;\quad \theta_2=90°-\varphi_2$

$n_1\sin(\theta_1)=n_2\sin(\theta_2)\Rightarrow n_2=n_1\frac{\sin(90°-\varphi_1)}{\sin(90°-\varphi_2)}$

$v=\frac{c}{n_2}=\frac{c\sin(90°-\varphi_2)}{n_1\sin(90°-\varphi_1)}=\frac{(2.9979\cdot 10^8\text{ m/s})\sin(90°-72.98)}{1.333\sin(90°-68.77)}=1.818\cdot 10^8\text{ m/s}$

Chapter 33: Lenses and Optical Instruments

Concept Checks

33.1. c **33.2.** b **33.3.** d **33.4.** b **33.5.** c **33.6.** e **33.7.** d **33.8.** a **33.9.** e **33.10.** a **33.11.** a **33.12.** d

Multiple-Choice Questions

33.1. b **33.2.** b **33.3.** c **33.4.** d **33.5.** a **33.6.** d **33.7.** b **33.8.** d **33.9.** b **33.10.** b **33.11.** b **33.12.** a **33.13.** d **33.14.** e **33.15.** e **33.16.** b **33.17.** d

Conceptual Questions

33.18. The dots are on the lens of the glasses, so they are too close to be brought into focus by the eyes of the painter. Since they are so small, they will not appear in what the painter sees. However, the dots will block the light coming from other objects, reducing the brightness of other objects.

33.19. When the diver is wearing the mask, light rays enter the eye from the air (index of refraction is 1) so the diver's vision is normal. When the mask is removed light rays enter the eye from the water (having an index of refraction of 1.33). As a result, the strength of the lens of the eye decreases and objects that are near will not be able to be brought into focus and the diver becomes farsighted. As the index of refraction of the medium approaches that of the lens (in this case, $n = 1.40$), the focal length of the lens approaches infinity and even distant objects will appear blurred.

33.20. In order to focus light properly, the index of refraction of the lenses of his eyes must be greater than the index of refraction of the surrounding medium. Since his lens has the same index of refraction as that of air, the focal length of the lens will be at infinity. This means that everything will be totally unfocused and he will only be able to detect changes in brightness and color.

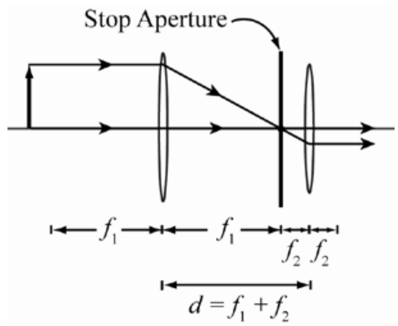
33.21. A lens cannot focus all colors to the same point, due to chromatic aberration. The index of refraction of a lens varies with the wavelength of light. By allowing only one wavelength to pass through their telescopes, astronomers eliminate chromatic aberration. The disadvantage is that the intensity of the light is reduced and images appear fainter.

33.22. In order to start a fire the image of the Sun must be focused to a small area. Focusing the light concentrates the energy of the Sun's rays, creating a large amount of heat at that point and making it possible to start a fire. If the glasses are for myopia (nearsightedness) then they are diverging lenses. Since diverging lenses only produce virtual images, light cannot be focused on a point. If the glasses are for hyperopia (farsightedness) then they are converging lenses. Since converging lenses can create real images, light can be focused to a point. Therefore, it is possible to start a fire with eye glasses, but only if they are for correcting hyperopic vision.

33.23. The magnification produced by the lens is due to its ability to refract light. Since the difference between the index of refraction of water and glass is less than the difference between that of air and glass, light will refract less at a water/glass boundary. Hence, when the lens is submerged in water, the magnification will decrease.

33.24. Light is reflected in all directions from each point of an object. In order to create an image of an object, the light arriving at one point of an image must be originating from one point on the object. Imagine what is involved in seeing, the light from each object in the field of view enters the eye and is projected onto a particular spot on the retina by using a lens to focus the light. Without the lens in our eye, all of the rays diverging from a particular point on any object would not be focused and would be projected over the entire retina. Without using optical elements, an image can be made by allowing light to pass through a very small hole. Such a device is called a "pinhole camera" where light passes in a straight line from a point on an object through the hole and then onto one point on the image. Essentially, a pinhole camera eliminates the angular spread of light reaching the image from a point on an object. The drawback is that only the light along a straight path enters so the image will be faint since only a small amount of light can enter through the hole.

33.25. (a) A ray diagram through the system is presented below:



From the ray diagram it can be seen that in a telecentric system, due to the stop aperture being at the common focal point, only the rays that are parallel (or near parallel) to the axis of the system will contribute to the image formed. The image magnification does not depend on the distance from the system.

(b) Based on the geometry of the system, the magnification is given by $|m| = \dfrac{f_2}{f_1}$.

(c) To achieve the maximum resolution, the image of the circular object must cover the entire short dimension (5.00 mm) of the CCD detector. Therefore,

$$|m| = \frac{h_i}{h_o} = \frac{5.00 \text{ mm}}{50.0 \text{ mm}} = 0.100 = \frac{f_2}{f_1} \Rightarrow f_2 = 0.100 f_1.$$

No specific values for f_1 and f_2 can be determined, but the first lens will have to have a focal length ten times longer than the second lens. In addition, to accept only parallel rays, the first lens must have a diameter larger than the diameter of the disk to be imaged, and the second lens a diameter larger than the diagonal dimension of the CCD detector.

33.26. (a) The "speed" of a lens is directly connected to the speed with which a photographic exposure can be made in any given lighting situation. The amount of light through the lens per unit time is proportional to the area of the lens aperture, i.e. the square of the aperture diameter or the inverse square of the *f*- number. Hence the exposure time or shutter speed required is inversely proportional to the aperture area, or proportional to the square of the *f*-number. A "fast" lens (low *f*-number) requires a faster shutter speed or shorter exposure than a "slower" lens of larger *f*-number. The traditional values for *f*-numbers correspond to factors of two in aperture area or inverse factors of two in shutter speed.

(b) The Keck Observatory document *Interfacing Visitor Instruments to the Keck Telescopes* gives the maximum diameter of the Keck II primary mirror as 10.95 m; it has an area equal to a circular aperture 9.96 m. The focal length of the primary is 17.5 m. So the *f*-number of the primary mirror is:

$$f\text{-number} = \frac{f}{D} = \frac{17.5 \text{ m}}{9.96 \text{ m}} = 1.76,$$

which is fairly fast in comparison with ordinary camera lenses. The text gives $D = 2.40$ m and $f = 57.6$ m for the primary mirror of the Hubble Space Telescope. These imply

$$f\text{-number} = \frac{f}{D} = \frac{57.6 \text{ m}}{2.40 \text{ m}} = 24.0,$$

which is slow compared to an ordinary camera lens. The National Astronomy and Ionosphere Center/Arecibo Observatory document *The 305 meter Radio Telescope* gives the diameter of the Arecibo radio telescope as 305 m, and its focal length, the height of the receiving platform above the dish as 450 feet or 137 m. The *f*-number of this primary mirror is

$$f\text{-number} = \frac{f}{D} = \frac{137\text{ m}}{305\text{ m}} = 0.449,$$

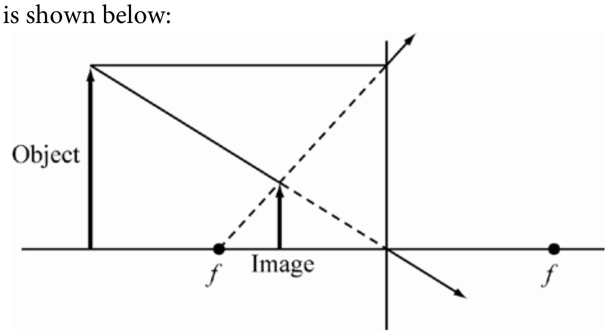
which is very fast compared to an ordinary camera lens.

33.27. In an image, the portion of a scene within the depth of field of a lens appears in focus. Of course, a lens can only focus at one distance, but the decrease in sharpness away from this point on the image may be gradual enough so that it appears in focus to the human eye. If a large aperture is placed in front of a lens, rays reaching the lens far from the optical axis (small *f*-number) will be bent through large angles. Therefore, rays exiting the lens will intercept the optical axis at large angles and the range of distances over which an image will be in focus will be small. That is, the depth of field is small for large apertures. Conversely, a lens with a small aperture (high *f*-number) excludes highly diverging rays so that the rays exiting the lens approach the optical axis at shallow angles. Thus, the range of distances over which an image will be in focus will be larger. In this case, the depth of field is large. The limiting case of a very small aperture or high *f*-number approximates the pinhole camera, which forms images by excluding all rays except those passing through the pinhole. It has no focal length, and can form images of objects at any distance in any plane beyond the pinhole.

33.28. For astronomical mirrors the accuracy and precision of the reflection properties of the mirror are paramount. First-surface mirrors are used for astronomical instruments to avoid refraction through the glass before and after reflection off of the coating, and the accompanying distortion and dispersion ("chromatic aberration," as the refraction would be different for different wavelengths of light). For household mirrors such precision is not required. Second-surface mirrors are used because of their greater durability since the reflective coating is protected by the glass covering.

33.29. When your friend adjusts the binoculars to his vision, the light rays exiting them are focused by his eyes onto his retina. However, since your friend wears glasses and you do not, the lenses in your eyes are different from his. Therefore, when you use the binoculars on his setting, the light rays are not properly focused onto your retina, so a re-adjustment is required to suit your eyes.

33.30. The ray tracing diagram is shown below:



Since the light rays diverge from the lens and it is the extrapolated rays that actually produce the image, the image is virtual. It is seen from the diagram that the image height is less than the object height.

33.31. If the objective lens intercepts the converging rays from the objective lens before they fully converge, then they will converge sooner, still beyond the eyepiece lens and on the same side of the optical axis as the image from the objective lens. So the final image is real and inverted.

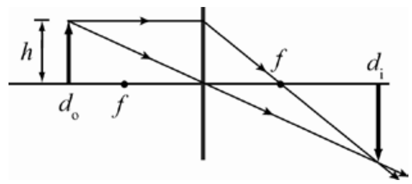
33.32. (a) A nearsighted person can only focus on objects that are near. Without corrective lenses, light rays from farther away come to a focus at a point before the retina. Diverging lenses are required, to separate the rays before they enter the eye so that the focal point advances to the retina.

(b) A farsighted person can only focus on objects that are far away. Without corrective lenses, light rays from a near object come to a focus at a point after the retina. Converging lenses are required, to converge the rays more sharly before entering the eye so that the focal point recedes onto the retina.

33.33. The make-shift microscope has converging lenses, one with focal length $f_1 = 6.0\text{ cm}$ and the other with $f_2 = 3.0\text{ cm}$. The lenses are separated by a distance $L = 20.\text{ cm}$. The magnification of a microscope is given by: $m = -(0.25\text{ m})L/(f_0 f_e)$. It therefore does not matter which lens is used as the eyepiece and which is used as the objective. However, it is more practical to use the lens with the smaller focal length as the objective in order to bring the object closer to the microscope.

Exercises

33.34. The setup is as shown:



(a) Assume the lens is a thin lens. For the image to form at a distance $d_i = 3f$ on the right side of the lens, d_o must be: $\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f} \Rightarrow \frac{1}{d_o}=\frac{1}{f}-\frac{1}{d_i} \Rightarrow d_o=\frac{fd_i}{d_i-f}=\frac{(f)(3f)}{3f-f}=1.5f.$

(b) The magnification m must be $m=-\frac{d_i}{d_o}=-\frac{3f}{1.5f}=-2,$ where the negative sign denotes that the image is inverted.

33.35. The distance to the image d_i is:

$$\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f} \Rightarrow \frac{1}{d_i}=\frac{1}{f}-\frac{1}{d_o} \Rightarrow d_i=\frac{fd_o}{d_o-f}$$

Therefore, the magnification is

$$m=-\frac{\left(\frac{fd_o}{d_o-f}\right)}{d_o}=-\frac{f}{d_o-f}=-\frac{9.0\text{ cm}}{6.0\text{ cm}-9.0\text{ cm}}=3.0.$$

33.36. The radius of curvature of the front surface of the ice lens is $R_1 = 15.0\text{ cm}$ and that for the back is $R_2 = -20.0\text{ cm}$. To start a fire with the ice lens the twigs would need to be placed at the location where the light rays are focused. Presume the source of the light rays (the Sun) is at infinity; $d_o = \infty$. Use the Lens-Maker's formula in the following form: $\frac{1}{d_o}+\frac{1}{d_i}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$. Since ice has an index of refraction of $n = 1.31$, this becomes:

$$\frac{1}{\infty}+\frac{1}{d_i}=(1.31-1)\left[\left(\frac{1}{15.0\text{ cm}}\right)-\left(\frac{1}{-20.0\text{ cm}}\right)\right] \Rightarrow d_i=\left\{(0.31)\left[\left(\frac{1}{15.0\text{ cm}}\right)+\left(\frac{1}{20.0\text{ cm}}\right)\right]\right\}^{-1}=27.6\text{ cm}$$

It would be best to put the twigs about 27.6 cm from the ice lens in order to create a fire.

33.37. For the purposes of this question, the laser can be treated as an object at a distance d_o with height $h_o = 1.06\cdot 10^{-3}\text{ m}$. The image height is $h_i = 10.0\cdot 10^{-6}\text{ m}$ and the distance to the image is $d_i = 20.0\text{ cm} = 0.200\text{ m}$. Since the image is to be formed behind the lens and reduced in size, the lens must be a converging lens, and the object should be greater than $2f$ away from the lens, where f is the focal length. This means that both the object distance d_o and the image distance d_i are positive. From the magnification equation $|m| = |d_\text{i} / d_\text{o}| = h_\text{i} / h_\text{o}$, the object distance must be

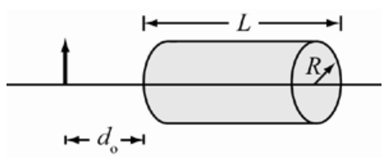
$$d_\text{o} = \frac{d_\text{i} h_\text{o}}{h_\text{i}} = \frac{(0.200 \text{ m})(1.06 \cdot 10^{-3} \text{ m})}{(10.0 \cdot 10^{-6} \text{ m})} = 21.2 \text{ m}.$$

This large value for d_o is consistent with how a laser beam is highly collimated, that is the rays are almost parallel. From the thin equation $1/d_\text{o} + 1/d_\text{i} = 1/f$, the focal length is

$$\frac{1}{f} = \frac{1}{21.2 \text{ m}} + \frac{1}{0.200 \text{ m}} = 5.0472 \text{ m}^{-1} \Rightarrow f = 0.198 \text{ m}.$$

For incoming rays that are parallel with the optical axis, the focal point is the focus. Since these rays are highly collimated it is reasonable that image location is near the focal point.

33.38. The plastic cylinder, shown below, has length $L = 30.$ cm, and the radius of curvature of each end is $R = 10.$ cm. The index of refraction of the plastic is $n = 1.5$. The object distance is $d_\text{o} = 10.$ cm from the left end.



Assume the object is in a medium with an index of refraction of $n = 1$ (like air, or a vacuum). The image distance from the left end of the plastic cylinder is:

$$\frac{1}{d_\text{o,1}} + \frac{n}{d_\text{i,1}} = \frac{(n-1)}{R} \Rightarrow d_\text{i,1} = \left(\frac{nRd_\text{o,1}}{d_\text{o,1}(n-1) - R} \right) = \left(\frac{(1.5)(10. \text{ cm})(10. \text{ cm})}{(10. \text{ cm})(1.5-1) - (10. \text{ cm})} \right) = -30. \text{ cm}.$$

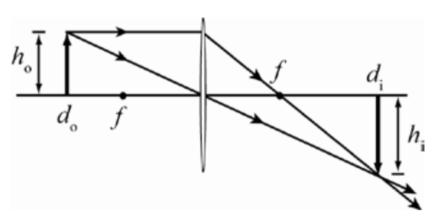
The negative sign indicates that the image is to the left of the cylinder. Therefore, the object for the right end is at a distance $d_\text{o,2} = |d_\text{i,1}| + L = 60.$ cm. The image distance from the right end of the cylinder is:

$$\frac{n}{d_\text{o,2}} + \frac{1}{d_\text{i,2}} = \frac{(1-n)}{R} \Rightarrow d_\text{i,2} = \left(\frac{Rd_\text{o,2}}{d_\text{o,2}(1-n) - nR} \right) = \left(\frac{(-10. \text{ cm})(60. \text{ cm})}{(60. \text{ cm})(1-1.5) - (1.5)(-10. \text{ cm})} \right) = 40. \text{ cm}.$$

Therefore, a real image is formed 40. cm to the right of the right end of the cylinder.

33.39. **THINK:** The object height is $h_\text{o} = 2.5$ cm, and is $d_\text{o} = 5.0$ cm from a converging lens of focal length $f = 3.0$ cm. The thin lens equation can be used to find the image distance and the magnification can be found from this.

SKETCH:



RESEARCH: The magnification m is: $m = -d_\text{i} / d_\text{o}$. The thin lens equation is: $1/d_\text{o} + 1/d_\text{i} = 1/f$.

SIMPLIFY: The image distance is: $\dfrac{1}{d_\text{o}} + \dfrac{1}{d_\text{i}} = \dfrac{1}{f} \Rightarrow \dfrac{1}{d_\text{i}} = \dfrac{1}{f} - \dfrac{1}{d_\text{o}} \Rightarrow d_\text{i} = \dfrac{fd_\text{o}}{d_\text{o} - f}$. Therefore, the magnification is: $m = -\dfrac{f}{d_\text{o} - f}$.

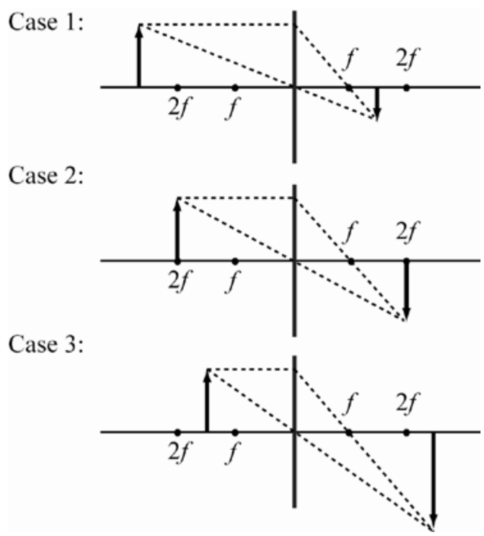
CALCULATE: The magnification is:] $m=-\frac{3.0\text{ cm}}{5.0\text{ cm}-3.0\text{ cm}}=-1.5.$ Since the magnification is negative, the image is inverted and since $|m|>1,$ the image is enlarged.

ROUND: To two significant figures, the magnification of the image is $m=-1.5.$

DOUBLE-CHECK: The ray tracing shown above confirms that the image is inverted and enlarged. As seen in Table 33.1, this is what it is expected for $f<d_o<2f.$

33.40. **THINK:** There are three different locations for placing a real object in front of a thin convex lens which results in a real image. Consider each case separately: *Case 1*: The object distance is $d_o>2f$. *Case 2:* The object distance is $d_o=2f$. *Case 3:* The object distance is $2f>d_o>f$. (Note that when $d_o=f$ no image is formed and when $d_o<f,$ the image is virtual.) The thin lens equation can be used to find the minimum distance between a real object and a real image.

SKETCH:



RESEARCH: For a thin lens, $1/f=1/d_o+1/d_i$. When the image of a real object is on the opposite side of the lens, the image is real and both d_o and d_i are positive by convention.

SIMPLIFY: For separation distance L between the object and the image, write $d_o+d_i=L.$ Then $d_i=L-d_o$. The thin lens equation becomes:

$$\frac{1}{f}=\frac{1}{d_o}+\frac{1}{L-d_o}\ \Rightarrow f=\frac{d_o\left(L-d_o\right)}{L-d_o+d_o}=\frac{d_o\left(L-d_o\right)}{L}.$$

Solving for L gives: $Lf=d_oL-d_o^2\ \Rightarrow\ L=\frac{d_o^2}{d_o-f}.$

Case 1: If $\infty>d_o>2f,$ the distance L lies between $\lim_{(d_o\to\infty)}\frac{d_o^2}{d_o-f}>L>\frac{(2f)^2}{(2f)-f}\ \Rightarrow \infty>L>4f.$

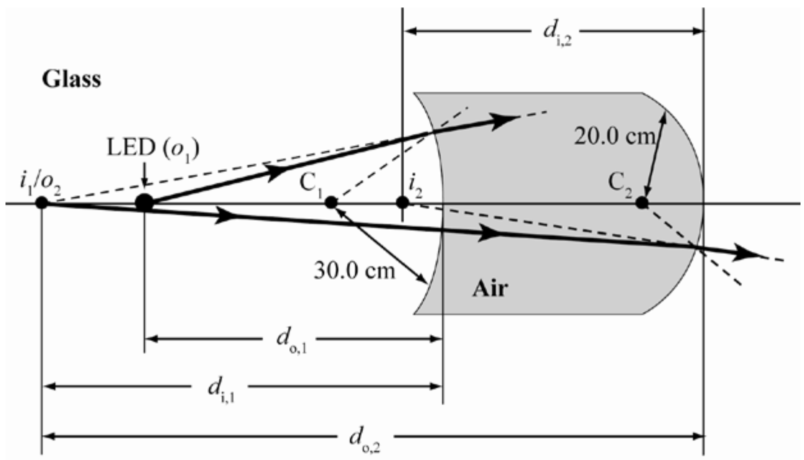
Case 2: If $d_o=2f,$ the distance L is $L=\frac{(2f)^2}{(2f)-f}=4f.$

Case 3: If $2f>d_o>f,$ the distance L lies between $\frac{(2f)^2}{(2f)-f}>L>\frac{(f)^2}{(f)-f}\ \Rightarrow\ 4f>L>\infty.$ So the minimum separation distance between a real object and a real image for a thin convex lens is $L=4f.$

CALCULATE: Not required.

ROUND: Not required.
DOUBLE-CHECK: This is consistent with the ray diagrams.

33.41. THINK:
(a) An air-filled cavity bound by two spherical surfaces is created inside a glass block. The two spherical surfaces have radii of curvatures of $R_1 = -30.0$ cm and $R_2 = -20.0$ cm. Both values are negative because each surface is concave. The LED is a distance $d_{o,1} = 60.0$ cm from the cavity. The thickness of the cavity is $d = 40.0$ cm. The index of refraction for glass and air is $n_g = 1.50$ and $n_a = 1.00$, respectively. The equations for thick lenses can be used to find the final position of the image of the LED through the cavity. The image formed by the first (left) surface will act as the object for the second (right) surface.
SKETCH:



RESEARCH: In the paraxial approximation, $\dfrac{n_1}{d_o} + \dfrac{n_2}{d_i} = \dfrac{n_2 - n_1}{R}$.

SIMPLIFY: For the first surface the interface is glass to air, so the image distance is:

$$\frac{n_g}{d_{o,1}} + \frac{n_a}{d_{i,1}} = \frac{n_a - n_g}{R_1} \Rightarrow d_{i,1} = \frac{n_a d_{o,1} R_1}{d_{o,1}\left(n_a - n_g\right) - R_1 n_g}.$$

For the second surface the interface is air to glass, so the image distance is:

$$\frac{n_a}{d_{o,2}} + \frac{n_g}{d_{i,2}} = \frac{n_g - n_a}{R_2} \Rightarrow d_{i,2} = \frac{n_g d_{o,2} R_2}{d_{o,2}\left(n_g - n_a\right) - R_2 n_a}.$$

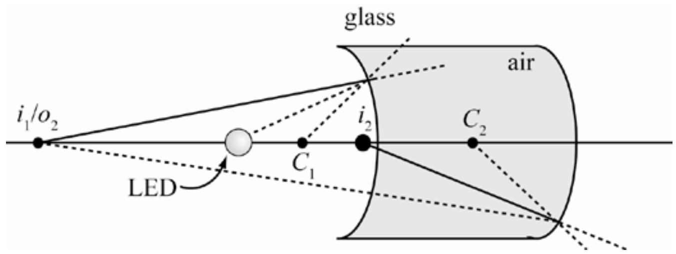
CALCULATE: The image for the first lens is: $d_{i,1} = \dfrac{(1.00)(60.0 \text{ cm})(-30.0 \text{ cm})}{(60.0 \text{ cm})(1.00 - 1.50) - (-30.0 \text{ cm})(1.50)} = -120. \text{ cm}.$

The negative sign indicates that the image is to the left of the first surface, so object distance for the second surface is given by $d_{o,2} = d + \left|d_{i,1}\right| = 40.0 \text{ cm} + \left|-120. \text{ cm}\right| = 160. \text{ cm}$. Therefore, the image formed is at a distance of

$$d_{i,2} = \frac{(1.50)(160. \text{ cm})(-20.0 \text{ cm})}{(160. \text{ cm})(1.50 - 1.00) - (-20.0 \text{ cm})(1.00)} = -48.0 \text{ cm}.$$

The negative sign indicates that the image is to the left of the second surface and that it is virtual.
ROUND: All values are given to two significant figures. The final position of the virtual image of the LED through the cavity is 48.0 cm to the left of the second surface (or 8.00 cm to the left of the first surface).

DOUBLE-CHECK: The ray diagram below is consistent with the calculated position of the final image:



The dotted lines drawn from the center of each curve to the respective curve's surface makes a normal line to that surface. The dashed lines represent rays being extrapolated back to find the location of the virtual image. Note the light rays refract at each interface, bending away from the normal when going from glass to air, and bending toward the normal when going from air to glass.

33.42. The magnification of a lens is given by $m = -d_{\text{i}} / d_{\text{o}}$. The object distance is given as $d_{\text{o}} = 3.00$ cm. From the thin lens equation:

$$\frac{1}{f} = \frac{1}{d_{\text{o}}} + \frac{1}{d_{\text{i}}} \Rightarrow d_{\text{i}} = \frac{fd_{\text{o}}}{d_{\text{o}} - f}.$$

Therefore, the magnification is:

$$m = -\frac{f}{d_{\text{o}} - f} = -\frac{5.00 \text{ cm}}{3.00 \text{ cm} - 5.00 \text{ cm}} = 2.50.$$

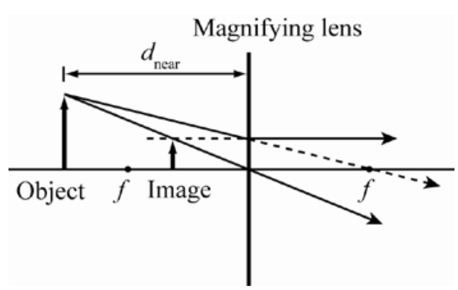
33.43. The angular magnification of a magnifying glass is approximately $m_\theta \approx d_{\text{near}} / f$. With a given focal length of $f = 5.0$ cm, and assuming a near point of $d_{\text{near}} = 25$ cm, the magnifying power of this lens with the object placed at the near point is $m_\theta = \dfrac{25 \text{ cm}}{5.0 \text{ cm}} = 5.0.$

33.44. Generally, magnification is defined as the ratio of image height to object height, $m = h_{\text{i}} / h_{\text{o}}$. With an object height of $h_{\text{o}} = 1.0$ mm and an image height of $h_{\text{i}} = 10.$ mm, the magnification is $m = 10.\text{ mm} / 1.0 \text{ mm} = 10.$ The angular magnification of a magnifying glass is approximately $m_\theta \approx (25 \text{ cm})/f$, where a near point of 25 cm is assumed. The focal length of the magnifying glass is:

$$f \approx \frac{25 \text{ cm}}{m} = \frac{25 \text{ cm}}{10.} = 2.5 \text{ cm}.$$

33.45. **THINK:** The person's near-point distance is $d_{\text{near}} = 24.0$ cm. The magnifying glass gives a magnification m_{near} that is 1.25 times larger when the image of the magnifier is at the near point that when the image is at infinity, that is $m_{\text{near}} = 1.25 m_\infty$. Find the focal length of the magnifying glass, f.

SKETCH:



RESEARCH: When the image is place at infinity, the angular magnification of a magnifying glass is: $m_\infty = d_{\text{near}} / f$. When the image is placed at the near-point, the text shows that the equation becomes $m_{\text{near}} = \left(d_{\text{near}} / f\right) + 1$.

SIMPLIFY: $m_{\text{near}} = 1.25 m_\infty \Rightarrow \dfrac{d_{\text{near}}}{f} + 1 = \dfrac{1.25 d_{\text{near}}}{f} \Rightarrow 1 = \dfrac{0.25 d_{\text{near}}}{f} \Rightarrow f = 0.25 d_{\text{near}}$

CALCULATE: The focal length is $f = 0.25\left(24.0 \text{ cm}\right) = 6.00$ cm.

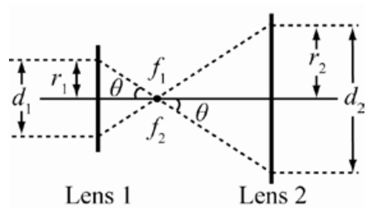
ROUND: To three significant figures, the focal length of the magnifying glass is $f = 6.00$ cm.

DOUBLE-CHECK: The focal length should be less than d_{near} if the image is to form at d_{near}.

33.46. A beam of light parallel to the optic axis has a diameter $d_1 = 1.00$ mm. It passes through the first lens of focal length $f_1 = 10.0$ cm, then a second lens of focal length $f_2 = 20.0$ cm. The emerging light is again parallel.

(a) Light from the first lens is focused at its focal point f_1. Since the light exiting the second lens is parallel to the optic axis, the object location for the second lens must be located a distance in front of that lens equal to its focal length, f_2. The total separation between the two lenses is therefore the sum of their focal lengths: $L = 10.0 \text{ cm} + 20.0 \text{ cm} = 30.0$ cm.

(b) A triangle can be formed for the original lens with a height of 0.50 mm (the radius of the beam) and length of 10.0 cm $\left(f_1\right)$. For the second lens a triangle can be drawn whose height is the outgoing beam's radius and whose length is 20.0 cm $\left(f_2\right)$. See the diagram below:



Since these triangles are similar triangles (same angles), the ratio of length to height must be the same for both: $\dfrac{r_1}{f_1} = \dfrac{r_2}{f_2} \Rightarrow r_2 = r_1 \dfrac{f_2}{f_1} = 0.50 \text{ mm}\left(\dfrac{20.0 \text{ cm}}{10.0 \text{ cm}}\right) = 1.00$ mm. And the width of the outgoing beam is $d_2 = 2\left(1.00 \text{ mm}\right) = 2.00$ mm.

33.47. The total magnification is the product of the magnification after passing through the first lens, m_1, and the magnification of the second lens, m_2. Magnification is $m = -d_{\text{i}} / d_{\text{o}} = h_{\text{i}} / h_{\text{o}}$. The focal length of each lens is $f = 5.0$ cm, and the distance that the insect is from the first lens is $d_{\text{o},1} = 10.0$ cm. Using the thin lens equation the image distance from the first lens is:

$$d_{\text{i},1} = \frac{f d_{\text{o},1}}{d_{\text{o},1} - f} = \frac{\left(5.0 \text{ cm}\right)\left(10.0 \text{ cm}\right)}{(10.0 \text{ cm} - 5.0 \text{ cm})} = 10. \text{ cm.}$$

Then $m_1 = -d_{\text{i},1} / d_{\text{o},1} = -\left(10.0 \text{ cm}\right) / \left(10.0 \text{ cm}\right) = -1.00$. This image is inverted, but the size does not change. This image acts as an object for the second lens, and is a distance $d_{\text{o},2} = L - d_{\text{i},1}$ from the second lens, where L is the separation distance of the two lenses, $L = 12$ cm. Using the thin lens equation, the image distance from the second lens is:

$$d_{\text{i},2} = \frac{f d_{\text{o},2}}{d_{\text{o},2} - f} = \frac{\left(5.0 \text{ cm}\right)\left(12.0 \text{ cm} - 10.0 \text{ cm}\right)}{\left(\left(12.0 \text{ cm} - 10.0 \text{ cm}\right) - 5.0 \text{ cm}\right)} = -3.333 \text{ cm.}$$

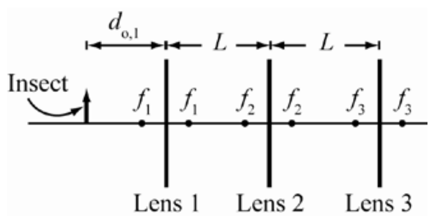
Then $m_2 = -d_{i,2}/d_{o,2} = -(-3.333 \text{ cm})/(12.0 \text{ cm} - 10.0 \text{ cm}) = 1.667$. This image is oriented the same way as the object (inverted). The final magnification of the insect is $m = m_1 m_2 = (-1.0)(1.667) = -1.667$. Therefore, the final image of the insect has a size of

$$h_i = mh_o = -1.667(5.0 \text{ mm}) = -8.333 \text{ mm} \approx -8.3 \text{ mm}.$$

With respect to the original insect, the final image is enlarged, inverted (since magnification is negative) and virtual (since $d_{i,2}$ is negative).

33.48. **THINK:** Three converging lenses of focal length $f = 5.0$ cm are arranged with a spacing of $L = 20.$ cm between them. They are used to image an insect $d_{o,1} = 20.$ cm away. In each case, the image formed by the preceding lens will act as the object for the next lens. (a) To find the location and orientation of the image, the thin lens equation can be applied consecutively for the different lenses. (b) If the final image is to the right of the third lens then the image is real, and if it is to the left then it is virtual. (c) Every time a real image is formed by a convex lens, the image is inverted.

SKETCH:



RESEARCH: In each case, the image formed by the preceding lens will act as the object for the next lens. The thin lens equation is $1/f = 1/d_o + 1/d_i$.

SIMPLIFY:

(a) Find the first image location, $d_{i,1}$:

$$\frac{1}{f} = \frac{1}{d_{o,1}} + \frac{1}{d_{i,1}} \Rightarrow \frac{1}{d_{i,1}} = \frac{1}{f} - \frac{1}{d_{o,1}} \Rightarrow d_{i,1} = \frac{d_{o,1}f}{d_{o,1} - f}.$$

This image acts as the object for the second lens. The second image location, $d_{i,2}$, is:

$$\frac{1}{f} = \frac{1}{d_{o,2}} + \frac{1}{d_{i,2}} \Rightarrow \frac{1}{d_{i,2}} = \frac{1}{f} - \frac{1}{d_{o,2}} \Rightarrow d_{i,2} = \frac{d_{o,2}f}{d_{o,2} - f}.$$

The final image location, $d_{i,3}$, is:

$$\frac{1}{f} = \frac{1}{d_{o,3}} + \frac{1}{d_{i,3}} \Rightarrow \frac{1}{d_{i,3}} = \frac{1}{f} - \frac{1}{d_{o,3}} \Rightarrow d_{i,3} = \frac{d_{o,3}f}{d_{o,3} - f}.$$

CALCULATE:

(a) The first image is at location: $d_{i,1} = \frac{(20. \text{ cm})(5.0 \text{ cm})}{(20. \text{ cm}) - (5.0 \text{ cm})} = 6.667$ cm. This image acts as the object for the second lens. Since $d_{i,1}$ is positive, $d_{o,2} = 20. \text{ cm} - 6.667 \text{ cm} = 13.33$ cm. The second image is at location: $d_{i,2} = \frac{(13.33 \text{ cm})(5.0 \text{ cm})}{(13.33 \text{ cm}) - (5.0 \text{ cm})} = 8.00$ cm. This image acts as the object for the third lens. Since $d_{i,2}$ is positive $d_{o,3} = 20. \text{ cm} - 8.00 \text{ cm} = 12$ cm. The final image is at location:

$$d_{i,3} = \frac{(12 \text{ cm})(5.0 \text{ cm})}{(12 \text{ cm}) - (5.0 \text{ cm})} = 8.57 \text{ cm}.$$

(b) Since $d_{i,3}$ is positive, the final image is on the right side of the third lens, so the image is real.

(c) Since the image of each object is inverted, and there are an odd number of lenses, the final image is inverted.

ROUND:

(a) To two significant figures, the image is located $d_{i,3} = 8.6\text{ cm}$ to the right of the third lens.

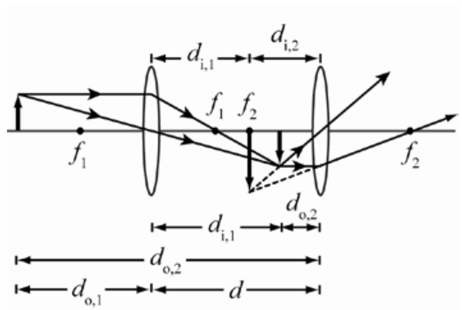
DOUBLE-CHECK: It is reasonable that the image due to each lens is real since each object is outside of the focal length of each lens. Using the equation for magnification the orientation of the final image is verified (recall d_o and d_i are positive in each case):

$$m = m_1 m_2 m_3 = \left(-\frac{d_{i,1}}{d_{o,1}}\right)\left(-\frac{d_{i,2}}{d_{o,2}}\right)\left(-\frac{d_{i,3}}{d_{o,3}}\right) < 0.$$

Therefore, the final image is inverted.

33.49. **THINK:** Two identical thin convex lenses, each of focal length, f, are separated by a distance $d = 2.5f$. An object is placed in front of the first lens at a distance $d_{o,1} = 2f$. The thin lens equation can be used to find the location, orientation, and size of the final image. The image formed by the first lens will act as the object of the second lens.

SKETCH:



RESEARCH: The thin lens equation, $1/f = 1/d_o + 1/d_i$, can be used in succession to determine the final image location. The magnification is given by: $m = -d_i/d_o = h_i/h_o$. The total magnification is the product of the magnification of the first lens, m_1, and the magnification of the second lens, m_2; that is, $m = m_1 m_2$.

SIMPLIFY: For the first image: $\frac{1}{d_{i,1}} = \frac{1}{f} - \frac{1}{d_{o,1}} \Rightarrow d_{i,1} = \frac{d_{o,1} f}{d_{o,1} - f}$. Since $d_{o,1} > f$, $d_{i,1}$ is positive so the object distance for the second lens is $d_{o,2} = d - d_{i,1}$. For the final image:

$$\frac{1}{d_{i,2}} = \frac{1}{f} - \frac{1}{d_{o,2}} \Rightarrow d_{i,2} = \frac{d_{o,2} f}{d_{o,2} - f}.$$

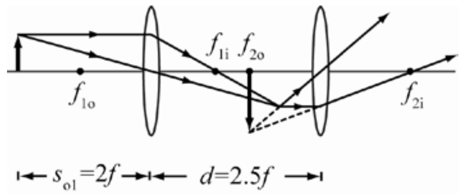
The magnification is $m = m_1 m_2 = \left(-\frac{d_{i,1}}{d_{o,1}}\right)\left(-\frac{d_{i,2}}{d_{o,2}}\right).$

CALCULATE:

(a) $d_{i,1} = \frac{(2f)f}{(2f) - f} = 2f$, $d_{o,2} = 2.5f - 2f = 0.5f$. Therefore, $d_{i,2} = \frac{(0.5f)f}{(0.5f) - f} = -f$. The final image is at the focal point on the left side of the second lens. It must be a virtual image.

(b) The total transverse magnification of the system is: $m = \left(-\frac{2f}{2f}\right)\left(-\frac{-f}{0.5f}\right) = -2.$

(c)



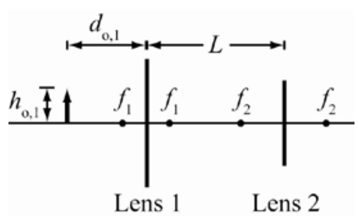
(d) Since $d_{i,2}$ is negative, the final image is virtual. Since $m<0,$ the final image is inverted. Since $|m|>1,$ the final image is enlarged.

ROUND: Not required.

DOUBLE-CHECK: The ray tracing diagram of part (c) is consistent with the calculations of part (a) and part (b).

33.50. **THINK:** Two converging lenses with focal lengths $f_1 = 5.00$ cm and $f_2 = 10.0$ cm are placed $L = 30.0$ cm apart. An object of height $h_{o,1} = 5.00$ cm is placed $d_{o,1} = 10.0$ cm to the left of the first lens. The thin lens equation can be used to. The image formed by the first lens will act as the object for the second lens. The thin lens equation can be used consecutively to find the position $d_{i,2}$ of the final image produced by this lens system. The magnification equation can be used to find the final image height $h_{i,2}$.

SKETCH:



RESEARCH: The thin lens equation is given by: $\dfrac{1}{f}=\dfrac{1}{d_o}+\dfrac{1}{d_i}.$ The total magnification is the product of the magnification after passing through the first lens m_1, and the magnification of the second lens, m_2; that is $m = m_1m_2$. The magnification is given by: $m = -d_i / d_o = h_i / h_o$.

SIMPLIFY: The first image is at location: $\dfrac{1}{d_{i,1}}=\dfrac{1}{f_1}-\dfrac{1}{d_{o,1}} \Rightarrow d_{i,1}=\dfrac{d_{o,1}f_1}{d_{o,1}-f_1}.$ Since $d_{o,1} > f_1$, $d_{i,1}$ is positive so the object distance for the second lens is $d_{o,2} = L - d_{i,1}$. For the final image: $\dfrac{1}{d_{i,2}}=\dfrac{1}{f_2}-\dfrac{1}{d_{o,2}} \Rightarrow d_{i,2}=\dfrac{d_{o,2}f_2}{d_{o,2}-f_2}.$ The magnification of the final image is $m = m_1m_2 = \left(-\dfrac{d_{i,1}}{d_{o,1}}\right)\left(-\dfrac{d_{i,2}}{d_{o,2}}\right).$ Therefore, the final image height is $h_{i,2} = h_{o,1}\left(-\dfrac{d_{i,1}}{d_{o,1}}\right)\left(-\dfrac{d_{i,2}}{d_{o,2}}\right).$

CALCULATE: The image distance of the first lens is $d_{i,1}=\dfrac{(10.0\text{ cm})(5.00\text{ cm})}{(10.0\text{ cm})-(5.00\text{ cm})}=10.0\text{ cm}.$ The object distance for the second lens is then: $d_{o,2} = 30.0\text{ cm} - 10.0\text{ cm} = 20.0\text{ cm}.$ The final image distance is $d_{i,2}=\dfrac{(20.0\text{ cm})(10.0\text{ cm})}{(20.0\text{ cm})-(10.0\text{ cm})}=20.0\text{ cm}.$ The final image is 20.0 cm to the right of the second lens. This image is real since $d_{i,2}>0.$ The final height is $h_{i,2}=5.00\text{ cm}\left(-\dfrac{10.0\text{ cm}}{10.0\text{ cm}}\right)\left(-\dfrac{20.0\text{ cm}}{20.0\text{ cm}}\right)=5.00\text{ cm}.$ The image is the same size as the object and since the height is positive, the image is upright.

ROUND: To three significant figures, the location of the final image is $d_{i,2} = 20.0$ cm to the right of the second lens and the final image height is $h_{i,2} = 5.00$ cm.

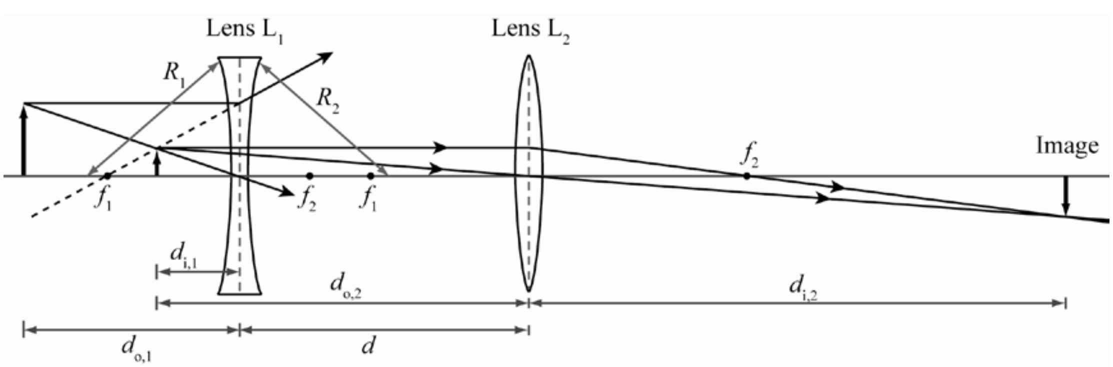
DOUBLE-CHECK: Since for each converging lens $d_o > f$, the image produced must be real. Upon each pass through a lens, the image is inverted. Thus after two lenses, the final image is upright.

33.51. The object is $h_{o,1} = 10.0$ cm tall and is located $d_{o,1} = 30.0$ cm to the left of the first lens. Lens L_1 is a biconcave lens with index of refraction $n = 1.55$ and has a radius of curvature of 20.0 cm for both surfaces. The first surface has negative radius of curvature as its surfaces is concave with respect to the object: $R_1 = -20.0$ cm. The second surface is convex with respect to the object, so its radius of curvature is positive: $R_2 = 20.0$ cm. Lens L_2 is $d = 40.0$ cm to the right of the first lens L_1. Lens L_2 is a converging lens with a focal length of $f_2 = 30.0$ cm. The image formed from the first lens acts as the object for the second lens. The position of the image formed by lens L_1 is found from the Lens Maker's Formula with the thin lens approximation: $\frac{1}{d_o} + \frac{1}{d_i} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$. Then the image distance is:

$$d_{i,1} = \left[(1.55-1)\left(\frac{1}{-20.0\text{ cm}} - \frac{1}{20.0\text{ cm}}\right) - \frac{1}{30.0\text{ cm}}\right]^{-1} = -11.32\text{ cm.}$$

This image is on the left side of lens L_1 and it acts as the object for lens L_2. The object distance for lens L_2 is $d_{o,2} = d + |d_{i,1}| = 40.0\text{ cm} + 11.32\text{ cm} = 51.32\text{ cm}$ from lens L_2. From the thin lens equation, the image distance of lens L_2 is: $\frac{1}{d_{i,2}} = \frac{1}{f_2} - \frac{1}{d_{o,2}} \Rightarrow d_{i,2} = \left(\frac{1}{30.0\text{ cm}} - \frac{1}{51.32\text{ cm}}\right)^{-1} = 72.2\text{ cm}$. Since this distance is positive, the final image is real and is 72.2 cm to the right of lens L_2, or $30.0 + 40.0 + 72.2 = 142$ cm to the right of the original object. The focal length of lens L_1 is required for a ray diagram. The Lens Maker's Formula gives:

$$f_1 = \left[(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right]^{-1} = \left[(1.55-1)\left(\frac{1}{-20.0\text{ cm}} - \frac{1}{20.0\text{ cm}}\right)\right]^{-1} \Rightarrow f_1 = -18.2\text{ cm.}$$



33.52. **THINK:** Light rays are described at any point along the axis of the system by a two-component column vector containing y, the distance of the ray from the optic axis, and y', the slope of the ray. Components of the system are described by 2×2 matrices which incorporate their effects on the ray; combinations of components are described by products of these matrices.

(a) Construct the matrix for a thin lens of focal length f.

(b) Write the matrix for a space of length x.

(c) Write the matrix for the two-lens "zoom lens" system described in the text.

SKETCH: Not required.

RESEARCH:
(a) As stated, a thin lens does not alter the position of a ray, but increases (diverging) or decreases (converging) its slope an amount proportional to the distance of the ray from the axis. The constant of proportionality between the distance of the ray from the optic axis and the change in its slope must be $-1/f$, so that a ray initially parallel to the axis (zero slope) will descend to the axis after traveling a distance f from the lens. So the matrix corresponding to a thin lens of focal length f is

$$L(f)=\begin{pmatrix}1 & 0\\ -1/f & 1\end{pmatrix},$$

where it is assumed that the displacement of a ray from the axis is the first component of the column vector describing the ray, and its slope is the second.
(b) A space between components does not alter the slope of a ray; the distance of the ray from the axis changes by the slope of the ray times the length of the space. As described, the matrix for a space of length x along the optic axis

$$S(x)=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}.$$

(c) The zoom lens described consists of a lens of focal length f_1 followed by a space of length x, then a second lens of focal length f_2. The corresponding matrix is $Z=L(f_2)S(x)L(f_1)$.
SIMPLIFY: For part (c),

$$Z=L(f_2)S(x)L(f_1)=\begin{pmatrix}1 & 0\\ -1/f_2 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1/f_1 & 1\end{pmatrix}=\begin{pmatrix}1-x/f_1 & x\\ x/(f_1f_2)-(1/f_1+1/f_2) & 1-x/f_2\end{pmatrix}.$$

CALCULATE: Not required.
ROUND: Not required.
DOUBLE-CHECK: A ray parallel to the optical axis will descend to the optical axis a distance $d_{i,2}$ from the second lens. Using the matrix above, the distance $d_{i,2}$ is given by the negative of the original distance from the axis divided by the effective slope of the two-lens system:

$$d_{i,2}=\frac{\dfrac{x}{f_1}-1}{\dfrac{x}{f_1f_2}-\left(\dfrac{1}{f_1}+\dfrac{1}{f_2}\right)}=\frac{f_2(x-f_1)}{x-(f_2+f_1)}.$$

In the text, the effective focal length f_{eff} of the combination is measured from the first lens. This result implies $f_{eff}=x+d_{i,2}=x+\dfrac{f_2(x-f_1)}{x-(f_2+f_1)}$, in exact agreement with the analysis in the text.

33.53. The typical length of a human eyeball is 2.50 cm. Use the thin lens equation: $\dfrac{1}{d_o}+\dfrac{1}{d_i}=\dfrac{1}{f}$.

(a) When d_o is large, $f\approx d_i$. The effective focal length for viewing objects at large distances is $f=2.50$ cm.
(b) When the object is at a typical near point, $d_o=d_{near}\approx 25$ cm and the image forms at the back of the eye at $d_i=2.50$ cm, the effective focal length is:

$$f=\left[\frac{1}{25\text{ cm}}+\frac{1}{2.50\text{ cm}}\right]^{-1}\approx 2.27\text{ cm}.$$

33.54. The effective focal length of two thin lenses placed close together is: $\frac{1}{f_{\text{eff}}}=\frac{1}{f_1}+\frac{1}{f_2}$. The cornea in a typical human eye has a fixed focal length of $f_1 = 2.33$ cm. For very distant objects the effective focal length of the lens-cornea system was found to be $f_{\text{eff}} = 2.50$ cm in the previous problem. In this case the focal length f_2 of the lens of the eye would have to be:

$$f_2 = \left[\frac{1}{f_{\text{eff}}}-\frac{1}{f_1}\right]^{-1} = \left[\frac{1}{2.50\text{ cm}}-\frac{1}{2.33\text{ cm}}\right]^{-1} = -34.3\text{ cm}.$$

For objects at the near point the effective focal length of the lens-cornea system was found to be $f_{\text{eff}} = 2.273$ cm in the previous problem. In this case the focal length f_2 of the lens of the eye would have to be:

$$f_2 = \left[\frac{1}{f_{\text{eff}}}-\frac{1}{f_1}\right]^{-1} = \left[\frac{1}{2.273\text{ cm}}-\frac{1}{2.33\text{ cm}}\right]^{-1} = 92.9\text{ cm}.$$

Therefore, the lens in the human eye must have a range of focal lengths between -34.3 cm and 92.5 cm.

33.55. Jane's near point is $d_{\text{near}} = 125$ cm and the computer screen is $d_{\text{o}} = 40.$ cm from her eye. Use the thin lens equation: $\frac{1}{d_{\text{o}}}+\frac{1}{d_{\text{i}}}=\frac{1}{f}$. Also, the power of a lens (in diopters) is $P=\frac{1}{f}$ where f is in meters.

(a) The object distance is just the distance to the computer screen: $d_{\text{o}} = 40.$ cm.

(b) The image distance is Jane's near point: $d_{\text{i}} = -d_{\text{near}} = -125$ cm. It is negative because the image appears on the same side of the eye as the object (the image is virtual).

(c) The focal length is $f = \left[\frac{1}{40.\text{ cm}}+\frac{1}{-125\text{ cm}}\right]^{-1} = 59$ cm.

(d) Jane's near point is 1.25 m; to read the computer screen at $d_{\text{o}} = 0.40$ m, the image must be located at the near point, $d_{\text{i}} = -d_{\text{near}}$. The power of this corrective lens would be:

$$P=\frac{1}{f}=\frac{1}{d_{\text{o}}}+\frac{1}{d_{\text{i}}}=\frac{1}{0.40\text{ m}}+\frac{1}{-1.25\text{ m}} = +1.7\text{ Diopter}.$$

(e) Since the focal length is positive, the corrective lens is converging.

33.56. Bill's far point is $d_{\text{far}} = 125$ cm. Use the thin lens equation: $\frac{1}{d_{\text{o}}}+\frac{1}{d_{\text{i}}}=\frac{1}{f}$. Also, the power of a lens (in diopters) is $P = 1/f$ where f is measured in meters.

(a) The objects he wishes to see are far away, so the object distance is $d_o = \infty$.

(b) The image distance is $d_{\text{i}} = -d_{\text{far}} = -125$ cm. It is negative because the image appears on the same side of the eye as the object (the image is virtual).

(c) The focal length is $f = \left[\frac{1}{\infty}+\frac{1}{-125\text{ cm}}\right]^{-1} \Rightarrow f = -125$ cm.

(d) Bill's far point is 1.25 m, so images of distant objects $(d_{\text{o}} = \infty)$ must be located at the far point, $d_{\text{i}} = -d_{\text{far}}$. The power of this corrective lens would be:

$$P=\frac{1}{f}=\frac{1}{d_{\text{o}}}+\frac{1}{d_{\text{i}}}=\frac{1}{\infty}+\frac{1}{-1.25\text{ m}} = -0.800\text{ diopter}.$$

(e) Since the focal length is negative, the corrective lens is diverging.

33.57. The newspaper is located at $d_o = 25$ cm. The converging part of the lens has a focal length of $f_c = 70.$ cm. The diverging part of the lens has a focal length of $f_d = -50.$ cm (it is negative because the lens is a diverging lens). The converging lens places the image at the near point. Since the image is on the same side of the lens as the object, $d_i = -d_{near}$. From the thin lens equation

$$\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f} \Rightarrow \frac{1}{d_o}-\frac{1}{d_{near}}=\frac{1}{f_c} \Rightarrow d_{near}=\left[\frac{1}{d_o}-\frac{1}{f_c}\right]^{-1}=\left[\frac{1}{25\text{ cm}}-\frac{1}{70.\text{ cm}}\right]=39\text{ cm};$$

The diverging lens places the image at the far point. Since the image is on the same side of the lens as the object, $d_{far} = -d_i$. The objects are at $d_o = \infty$. From the thin lens equation,

$$\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f} \Rightarrow \frac{1}{d_o}-\frac{1}{d_{far}}=\frac{1}{f_d} \Rightarrow d_{far}=\left[\frac{1}{d_o}-\frac{1}{f_d}\right]^{-1}=\left[\frac{1}{\infty}-\frac{1}{-50.\text{ cm}}\right]=50.\text{ cm};$$

33.58. The radius of curvature for the outer part of the cornea is $R_1 = 8.0\cdot 10^{-3}$ m, while the inner portion is relatively flat, so $R_2 = \infty$. The radius of curvature R_1 is positive because the surface facing the object is convex. The index of refraction of the cornea and the aqueous humor is $n = 1.34$.

(a) The power of the cornea is $P_1 = \frac{1}{f}$ where the focal length is measured in meters. From the Lens Maker's Formula, the power of the cornea is:

$$P_1=\frac{1}{f}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=(1.34-1)\left(\frac{1}{8.0\cdot 10^{-3}\text{ m}}-\frac{1}{\infty}\right)=42.5\text{ diopter}\approx 43\text{ diopter}.$$

(b) The combination of the lens and the cornea has a power of $P_{eff} = 50.$ diopter. For two adjacent lenses placed closely together, the effective focal length is $\frac{1}{f_{eff}}=\frac{1}{f_1}+\frac{1}{f_2}$. Rewriting in terms of power yields $P_{eff} = P_1 + P_2$. The power P_2 of the lens is $P_2 = P_{eff} - P_1 = 50.\text{ diopter} - 42.5\text{ diopter} = 7.5\text{ diopter}$.

33.59. **THINK:** As objects are moved closer to the human eye the focal length of the lens decreases. The shortest focal length is $f_{min} = 2.3$ cm. The thin lens equation can be used to determine the closest one can bring an object to a normal human eye, $d_{o,norm}$, and still have the image of the object projected sharply onto the retina, which is $d_{i,norm} = 2.5$ cm. behind the lens. A near sighted human eye has the same f_{min} but has a retina that is 3.0 cm behind the lens. The thin lens equation can be used to determine the closest one can bring an object to this nearsighted human eye, $d_{o,near}$, and still have the image of the object projected sharply on the retina at $d_{i,near} = 3.0$ cm.

SKETCH: Provided with the problem.

RESEARCH: In each case the object is in front of the lens, and the image is formed behind the lens, so both d_o and d_i are positive. The thin lens equation is: $1/d_o + 1/d_i = 1/f$. The angular magnification is given by $m_\theta \approx d_{near}/f$.

SIMPLIFY: $\frac{1}{d_o}=\frac{1}{f}-\frac{1}{d_i} \Rightarrow d_o=\left(\frac{1}{f}-\frac{1}{d_i}\right)^{-1}$. The ratio of angular magnifications is:

$$\frac{m_{norm}}{m_{near}}=\left(\frac{d_{near,\,norm}}{f_{norm}}\right)\left(\frac{f_{near}}{d_{near,\,near}}\right).$$

Since the object is placed at the near point for the image to form on the retina and $f_{near} = f_{norm} = f_{min}$, this becomes

$$\frac{m_{norm}}{m_{near}}=\left(\frac{d_{o,\,norm}}{d_{o,\,near}}\right)\left(\frac{f_{near}}{f_{norm}}\right)=\left(\frac{d_{o,\,norm}}{d_{o,\,near}}\right).$$

CALCULATE: For the normal eye the minimum distance is: $d_{\text{o, norm}} = \left(\dfrac{1}{2.3\text{ cm}} - \dfrac{1}{2.5\text{ cm}}\right)^{-1} = 28.75\text{ cm.}$

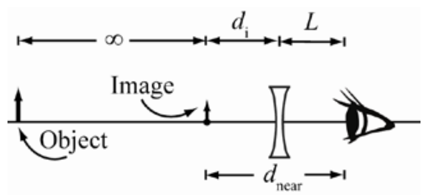
For the elongated eye the minimum distance is: $d_{\text{o, near}} = \left(\dfrac{1}{2.3\text{ cm}} - \dfrac{1}{3.0\text{ cm}}\right)^{-1} = 9.86\text{ cm.}$ The ratio of angular magnifications is $\dfrac{m_{\text{norm}}}{m_{\text{near}}} = \dfrac{28.75\text{ cm}}{9.86\text{ cm}} = 2.916.$

ROUND: To two significant figures, $d_{\text{o, norm}} = 29\text{ cm},$ $d_{\text{o, near}} = 10.\text{ cm},$ and $m_{\text{norm}} = 2.9\, m_{\text{near}}.$

DOUBLE-CHECK: The nearsighted eye should have a closer near point than the normal eye.

33.60. **THINK:** The power of the eyeglasses lens is $P = -5.75$ diopters. The negative power implies that the lenses are diverging lenses, and that the person is indeed nearsighted. Objects at a far distance must have an image formed at the person's near point to be resolved. The lenses are $L = 1.00\text{ cm}$ in front of his corneas. The thin lens equation can be used to find the prescribed power of his contact lenses.

SKETCH:



RESEARCH: The power of a lens is $P = 1/f$ where f is measured in meters. The thin lens equation is $\dfrac{1}{d_{\text{o}}} + \dfrac{1}{d_{\text{i}}} = \dfrac{1}{f}.$ Since the person is nearsighted it can be assumed that the objects are distant: $d_{\text{o}} = \infty.$ The near point d_{near} is $d_{\text{near}} = |d_{\text{i}}| + L$ (see sketch above). The image distance d_{i} is negative because the image forms on the same side of the lens as the object. The image formed from the contact lenses must be at the near point as well. In this case $d_{\text{i}} = -d_{\text{near}}$ since there is no space between the contacts and the cornea.

SIMPLIFY: With the glasses, $d_{\text{near}} = |d_{\text{i}}| + L = \left|\left(\dfrac{1}{f} - \dfrac{1}{d_{\text{o}}}\right)^{-1}\right| + L = \left|\left(\dfrac{1}{f} - \dfrac{1}{\infty}\right)^{-1}\right| + L = |f| + L = \left|\dfrac{1}{D_{\text{glasses}}}\right| + L.$

With $d_o = \infty,$ the power of the contacts is $P_{\text{contacts}} = -\dfrac{1}{d_{\text{near}}} = \dfrac{1}{\left|\dfrac{1}{P_{\text{glasses}}}\right| + L}.$

CALCULATE: $D_{\text{contacts}} = -\dfrac{1}{\left|1/\left(-5.75\text{ m}^{-1}\right)\right| + 1.00 \cdot 10^{-2}\text{ m}} = -5.437\text{ diopter.}$

ROUND: To three significant figures, the prescribed power of the contact lenses is $P_{\text{contacts}} = -5.44$ diopter.

DOUBLE-CHECK: The power of the contacts should be slightly less than the power of the glasses, since the contacts are on the eye.

33.61. Equation 33.7 gives the effective focal length of a two lens system as:

$$f_{\text{eff}} = x + \frac{f_2\left(x - f_1\right)}{x - \left(f_2 + f_1\right)}.$$

For a separation of $x = 50.\text{ mm}$ between the lenses, with focal lengths of $f_1 = 2.0 \cdot 10^2\text{ mm}$ and $f_2 = -3.0 \cdot 10^2\text{ mm}$ for the first and second lens, respectively, the effective focal length is:

$$f_{\text{eff}} = 50.\text{ mm} + \frac{\left(-3.0\cdot 10^2\text{ mm}\right)\left(50.\text{ mm} - 2.0\cdot 10^2\text{ mm}\right)}{50.\text{ mm} - \left(-3.0\cdot 10^2\text{ mm} + 2.0\cdot 10^2\text{ mm}\right)} = 350\text{ mm}.$$

For a separation of $x = 1.0\cdot 10^2$ mm between the lenses, the effective focal length is:

$$f_{\text{eff}} = 1.0\cdot 10^2\text{ mm} + \frac{\left(-3.0\cdot 10^2\text{ mm}\right)\left(1.0\cdot 10^2\text{ mm} - 2.0\cdot 10^2\text{ mm}\right)}{1.0\cdot 10^2\text{ mm} - \left(-3.0\cdot 10^2\text{ mm} + 2.0\cdot 10^2\text{ mm}\right)} = 250\text{ mm}.$$

33.62. The distance between the lens and the film is 10.0 cm. Initially, objects that are very far away appear properly focused on the film, so the distance from the lens to an object can be taken as $d_\text{o} = \infty$. Since the images form on the film, the image distance is $d_i = 10.0$ cm. Approximating the lens as a thin lens, the focal length of the lens is $1/d_\text{o} + 1/d_\text{i} = 1/f \;\Rightarrow f = \left(1/d_\text{o} + 1/d_\text{i}\right)^{-1} = \left(1/\infty + 1/10.0\text{ cm}\right)^{-1} = 10.0$ cm. To properly focus an object $d_\text{o} = 100.$ cm away, the film must lie at the location where the image forms, at d_i: $1/d_\text{i} = 1/f - 1/d_\text{o} \;\Rightarrow\; d_\text{i} = \left(1/10.0\text{ cm} - 1/100.\text{ cm}\right)^{-1} = 11.1$ cm. Therefore, you would have to move the lens about 1.1 cm in order for it to focus an object 1.00 m away.

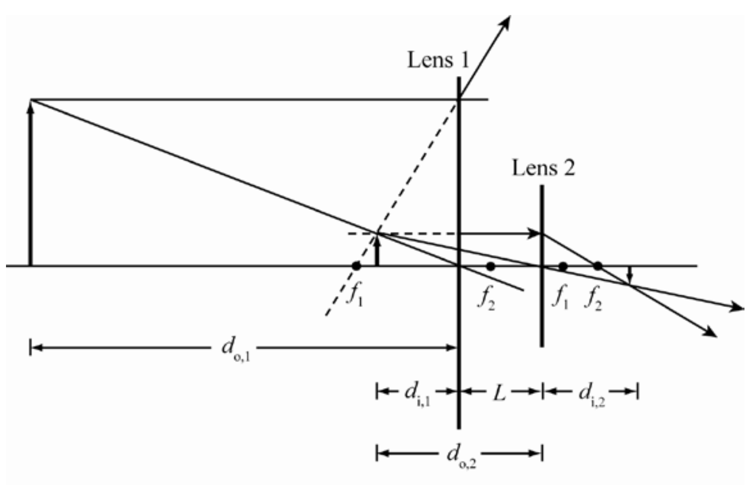
33.63. The focal length of the original lens is fixed at $f = 60.$ mm and the zoom lens has a variable focal length. The object is a distance $d_\text{o} = \infty$ from the lens. Using the thin lens equation for the original lens shows $\frac{1}{d_0} + \frac{1}{d_i} = \frac{1}{f} \Rightarrow \frac{1}{\infty} + \frac{1}{d_i} = \frac{1}{f} \Rightarrow f = d_i$, the image appears at $d_\text{i} = f = 60.$ mm. With the zoom lens set to a focal length of $f' = 240.$ mm, the image appears at $d_\text{i}' = f' = 240.$ mm. The ratio of magnifications of each lens is:

$$\frac{m_{\text{original}}}{m_{\text{zoom}}} = \frac{-d_\text{i}/d_\text{o}}{-d_\text{i}'/d_\text{o}} = \frac{d_\text{i}}{d_\text{i}'} = \frac{60.\text{ mm}}{240.\text{ mm}} = \frac{1}{4.0}.$$

The zoom lens (at $f' = 240.$ mm) produces an image that is 4.0 times the size of the image produced by the original $f = 60.$ mm lens.

33.64. **THINK:** The first lens is the diverging lens of focal length $f_1 = -10.0$ cm; the second lens is the converging lens of focal length $f_2 = 5.00$ cm. The two lenses are held $L = 7.00$ cm apart. A flower of length $h_{\text{o},1} = 10.0$ cm is held upright at a distance $d_{\text{o},1} = 50.0$ cm in front of the diverging lens. The thin lens equation can be used to find the location $d_{\text{i},2}$ of the final image, and the magnification equation can be used to find the orientation, size $h_{\text{i},2}$, and the magnification m of the final image.

SKETCH:



RESEARCH: The thin lens equation is: $\frac{1}{d_\text{o}}+\frac{1}{d_\text{i}}=\frac{1}{f}$. The magnification equation for each lens is: $m=h_\text{i}/h_\text{o}=-d_\text{i}/d_\text{o}$. For multiple lenses, the total magnification is the product of the magnification of each lens: $m=m_1m_2$.

SIMPLIFY: The image produced by the diverging lens is formed at position: $d_{\text{i},1}=\frac{d_{\text{o},1}f_1}{d_{\text{o},1}-f_1}$. For a diverging lens, $f<0$. Since $d_{\text{o},1}>0$, from the above equation $d_{\text{i},1}<0$; that is, the image is on the left side of the lens. This image acts as the object for the converging lens at a distance of $d_{\text{o},2}=L+\left|d_{\text{i},1}\right|$. The position of the image produced by the converging lens is: $d_{\text{i},2}=\frac{d_{\text{o},2}f_2}{d_{\text{o},2}-f_2}$. The final magnification is $m=\left(-\frac{d_{\text{i},1}}{d_{\text{o},1}}\right)\left(-\frac{d_{\text{i},2}}{d_{\text{o},2}}\right)$. The size of the final image is $h_{\text{i},2}=mh_{\text{o},1}$.

CALCULATE: The image formed by the first lens is at location:

$$d_{\text{i},1}=\frac{(50.0\text{ cm})(-10.0\text{ cm})}{(50.0\text{ cm})-(-10.0\text{ cm})}=-8.3333\text{ cm}.$$

The object distance for the second lens is: $d_{\text{o},2}=7.00\text{ cm}+\left|-8.3333\text{ cm}\right|=15.33\text{ cm}$. The final image formed by the second lens is at location:

$$d_{\text{i},2}=\frac{(15.33\text{ cm})(5.00\text{ cm})}{15.33\text{ cm}-5.00\text{ cm}}=7.4201\text{ cm}.$$

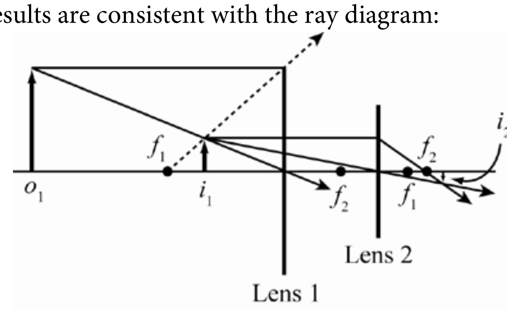
The total magnification is

$$m=\left(-\frac{-8.3333\text{ cm}}{50.0\text{ cm}}\right)\left(-\frac{7.4201\text{ cm}}{15.33\text{ cm}}\right)=-0.08067.$$

The size of the final image is $h_{\text{i},2}=(-0.08067)(10.0\text{ cm})=-0.8067\text{ cm}$. Since $m<0$, the final image is inverted.

ROUND: To three significant figures: the final image is $d_{\text{i},2}=7.42\text{ cm}$ to the right of the convex lens, the magnification of the final image is $m=-0.0807$ and the size of the final image is $h_{\text{i},2}=-0.807\text{ cm}$.

DOUBLE-CHECK: These results are consistent with the ray diagram:



33.65. The magnification of a microscope is given by the equation: $|m|=(25\text{ cm})L/f_\text{o}f_\text{e}$. The magnitude of the desired magnification is $|m|=3.0\cdot 10^2$. Treating the lens attached to the tube as the objective lens with focal length, $f_\text{o}=0.70\text{ cm}$, the focal length, f_e, of the eyepiece required, should be $|m|=\frac{(25\text{ cm})L}{f_\text{o}f_\text{e}}\Rightarrow f_\text{e}=\frac{(25\text{ cm})L}{f_\text{o}|m|}=\frac{(25\text{ cm})(20.\text{ cm})}{(0.70\text{ cm})(3.0\cdot 10^2)}=2.4\text{ cm}$. (Note that the designation of eyepiece and objective to the two lenses is independent of the magnification.)

33.66. The objective lens in a laboratory microscope has a focal length of $f_o = 3.00$ cm and provides an overall magnification of $|m| = 1.0 \cdot 10^2$. The distance between the two lenses is $L = 30.0$ cm. The focal length of the eyepiece, f_e, is given by:

$$|m| = \frac{(25 \text{ cm})L}{f_o f_e} \Rightarrow f_e = \frac{(25 \text{ cm})L}{f_o |m|} = \frac{(25 \text{ cm})(30.0 \text{ cm})}{(3.00 \text{ cm})\left|1.0 \cdot 10^2\right|} = 2.5 \text{ cm}.$$

33.67. The focal length of the objective lens is $f_o = 7.00$ mm. The distance between the objective lens and the eyepiece lens is $L = 20.0$ cm. The magnitude of the magnification is $|m| = 200$. The viewing distance to the image is $d_{i,2} = 25.0$ cm. The focal length of the eyepiece, f_e, can be found from the equation for the magnification of a microscope: $|m| = \frac{d_{i,1}d_{i,2}}{d_{o,1}d_{o,2}} = \frac{(25.0 \text{ cm})L}{f_o f_e}$. The focal length of the eyepiece is:

$$f_e = \frac{(0.250 \text{ m})L}{f_o |m|} = \frac{(25.0 \text{ cm})(20.0 \text{ cm})}{(0.700 \text{ cm})(200.)} = 3.57 \text{ cm}.$$

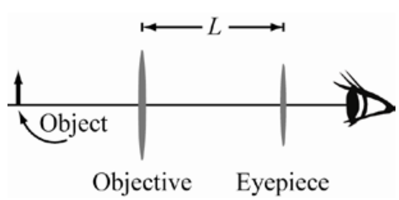
The best choice is the lens marked with a 4.00 cm focal length.

33.68. The focal length of the eyepiece is $f_e = 2.0$ cm. The focal length of the objective lens is $f_o = 0.800$ cm. The relaxed viewing distance is typically $d_{i,2} = 25$ cm. The distance between the lenses is $L = 16.2$ cm. In a microscope, the image of the objective lens forms approximately at the focal length of the eyepiece (see Figure 33.36 in the text) so that $L = d_{i,1} + d_{o,2} \approx d_{i,1} + f_e$. Then $d_{i,1} \approx L - f_e = 16.2 \text{ cm} - 2.0 \text{ cm} = 14.2 \text{ cm}$. The object distance from the objective lens, $d_{o,1}$ is given by the thin lens equation, $\frac{1}{f_o} = \frac{1}{d_{o,1}} + \frac{1}{d_{i,1}}$. Then

$$d_{o,1} = \left(\frac{1}{f_o} - \frac{1}{d_{i,1}}\right)^{-1} = \left(\frac{1}{0.80 \text{ cm}} - \frac{1}{14.2 \text{ cm}}\right)^{-1} = 0.85 \text{ cm}.$$

33.69. **THINK:** The distance between the two lenses of the microscope, L, is fixed. The magnitude of the magnification is to vary from $m_1 = 150$ to $m_2 = 450$ for substituted eyepieces of various focal lengths. The equation for the magnification of a microscope can be used to determine the focal length. The longest focal length eyepiece corresponds to the smallest magnification.

SKETCH:



RESEARCH: The equation for the magnification of a microscope is: $|m| = \frac{d_{i,1}d_{i,2}}{d_{o,1}d_{o,2}} = \frac{(25 \text{ cm})L}{f_o f_e}$.

SIMPLIFY:

(a) $|m| = \frac{(25 \text{ cm})L}{f_o f_e} \Rightarrow |m_1| = \frac{(25 \text{ cm})L}{f_o f_{e,1}}$ and $|m_2| = \frac{(25 \text{ cm})L}{f_o f_{e,2}}$

$$\frac{|m_2|}{|m_1|} = \left(\frac{(25 \text{ cm})L}{f_o f_{e,2}}\right) \div \left(\frac{(25 \text{ cm})L}{f_o f_{e,1}}\right) = \frac{f_{e,1}}{f_{e,2}} \Rightarrow f_{e,2} = f_{e,1}\frac{|m_1|}{|m_2|}$$

(b) $|m| = \frac{(25 \text{ cm})L}{f_o f_e} \Rightarrow f_o = \frac{(25 \text{ cm})L}{|m_1| f_{e,1}}$.

CALCULATE:

(a) $f_{e,2} = (60.\text{ mm})\dfrac{150}{450} = 20.0\text{ mm}$

(b) $f_o = \dfrac{(25\text{ cm})(35\text{ cm})}{(150)(6.0\text{ cm})} = 9.72\text{ mm}$

ROUND: To two significant figures, (a) the shortest focal length of the eyepiece is $f_{e,2} = 20.\text{ mm},$ and (b) the focal length of the objective lens should be $f_o = 9.7\text{ mm}.$

DOUBLE-CHECK: Using the calculated value of $f_{e,2} = 20\text{ mm}$ and its corresponding magnification $m_2 = 450$ yields a focal length for the objective lens of:

$$f_o = \frac{(25\text{ cm})L}{|m_2| f_{e,2}} = \frac{(25\text{ cm})(35\text{ cm})}{450(2.0\text{ cm})} = 9.7\text{ mm}.$$

33.70. The angular magnification of a refracting telescope is $m_\theta = -f_o / f_e.$ With an objective lens of focal length $f_o = 10.0\text{ m},$ and an eyepiece of focal length $f_e = 2.00\cdot 10^{-2}\text{ m},$ the magnification of this telescope is: $m_\theta = -10.0\text{ m} / 2.00\cdot 10^{-2}\text{ m} = -500.,$ where the negative sign indicates that the image is inverted.

33.71. The angular magnification of a refracting telescope is $m_\theta = -f_o / f_e.$ With an objective lens of focal length $f_o = 100.\text{ cm},$ and an eyepiece of focal length $f_e = 5.00\text{ cm},$ the magnification of this telescope is: $m_\theta = -100.\text{ cm} / 5.00\text{ cm} = -20.0,$ where the negative sign indicates that the image is inverted.

33.72. The angular magnification of a refracting telescope is $m_\theta = -\theta_e / \theta_o = -f_o / f_e.$ The telescope has an eyepiece focal length of $f_e = 25.0\text{ mm}$ and an objective focal length of $f_o = 80.0\text{ mm}.$ The magnification of this telescope is, therefore: $m = f_o / f_e = -80.0\text{ mm} / 25.0\text{ mm} = -3.20.$ Using the small angle approximation of $\tan\theta \approx \theta,$ the angle subtended by the moon (the object) when viewed by the unaided eye is (in radians) $\theta_o = \dfrac{2R_{\text{moon}}}{d_{\text{moon}}} = \dfrac{2(1.737\cdot 10^6\text{ m})}{3.844\cdot 10^8\text{ m}} = 9.037\cdot 10^{-3}\text{ rad}.$ Thus, the angle subtended by the image of the moon through the eyepiece is: $\theta_e = |m|\theta_o = 3.20(9.037\cdot 10^{-3}\text{ rad}) = 2.89\cdot 10^{-2}\text{ rad}.$

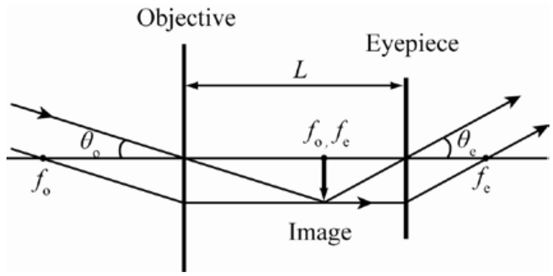
33.73. Galileo's telescope had an objective lens with a focal length of $f_o = 40.0\text{ inches}$ and an eyepiece lens with a focal length of $f_e = 2.00\text{ inches}.$ The angular magnification of the refracting telescope is $m_\theta = -f_o / f_e.$ Therefore, $m_\theta = -\dfrac{40.0\text{ inches}}{2.00\text{ inches}} = -20.0,$ where the negative sign indicates that the image is inverted.

33.74. The two distant stars are separated by an angle of $\theta_o = 35\text{ arcseconds}.$ The stars are observed to be separated by $\theta_e = 35\text{ arcminutes} = 2100\text{ arcseconds}$ through a refracting telescope. (There are 60 arcseconds in one arcminute.) This telescope has an objective lens of focal length $f_o = 3.5\text{ m}.$ The focal length of the eyepiece, $f_e,$ is found from the equation for the angular magnification of a refracting telescope, $m_\theta = -\theta_e / \theta_o = -f_o / f_e.$ Then f_e is

$$f_e = \frac{f_o\theta_o}{\theta_e} = \frac{(3.5\text{ m})(35\text{ arcseconds})}{2100\text{ arcseconds}} = 5.8\text{ cm}.$$

33.75. **THINK:** The telescope is a refracting telescope with a magnification of $|m| = 180.$ It is adjusted for a relaxed eye when the two lenses are $L = 1.30\text{ m}$ apart. The telescope is designed such that the image formed by the objective lens (which appears at its focal length f_o) lies at the focal length of the eyepiece. Then the distance L between the two lenses is the sum of the two focal lengths: $L = f_o + f_e.$ The magnification equation for a telescope can be used to find the focal length of each the objective lens, f_o, and the eyepiece lens, f_e.

SKETCH:



RESEARCH: The angular magnification of a refracting telescope is $m_\theta = -f_o / f_e$. With two equations and two unknowns, the two focal lengths f_o and f_e can be determined.

SIMPLIFY: $|m_\theta| = f_o / f_e \Rightarrow f_o = f_e|m_\theta|, \; L = f_o + f_e = f_e|m_\theta| + f_e \Rightarrow f_e = \dfrac{L}{1+|m_\theta|}.$

CALCULATE: $f_e = \dfrac{1.30 \text{ m}}{1+(180)} = 7.182 \text{ mm}, \; f_o = (7.182 \text{ mm})(180) = 1.293 \text{ m}.$

ROUND: Rounding to two significant figures, the focal length of the eyepiece is 7.2 mm. The focal length of the objective lens is 1.3 m.

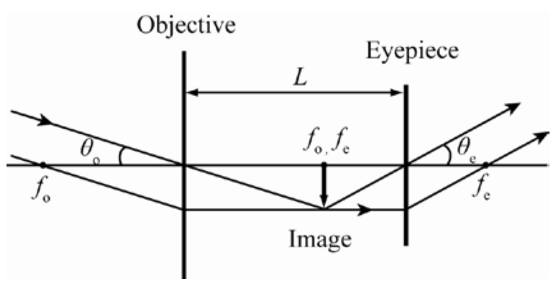
DOUBLE-CHECK: The focal point of the objective should be much greater the focal length of the eyepiece for a refracting telescope.

33.76. **THINK:** The objective focal length of both telescopes is $f_o = 95.0$ cm and the eyepiece focal length of both telescopes is $f_e = 3.80$ cm. Telescope A has an objective diameter of $D_{oA} = 10.0$ cm while telescope B has an objective diameter of $D_{oB} = 20.0$ cm, and for the eyepiece diameter, $D_{eB} = 2D_{eA}$.

(a) The angular magnifications of telescopes A and B can be found by using the magnification equation for telescopes. Both telescopes have the same angular magnification since both of their lenses have the same focal lengths.

(b) The brightness of an image is proportional to the area of the lenses.

SKETCH:



RESEARCH:

(a) The angular magnification of a refracting telescope is $m_\theta = -f_o / f_e$.

(b) The area of a lens is $A = \pi D^2 / 4$, where D is the diameter of the lens.

SIMPLIFY: Not required.

CALCULATE:

(a) $m_\theta = -\dfrac{95.0 \text{ cm}}{3.80 \text{ cm}} = -25.0,$ where the negative sign indicates that the image is inverted.

(b) $\frac{A_A}{A_B} = \left(\frac{D_A}{D_B}\right)^2 = \left(\frac{D_A}{2D_A}\right)^2 = \frac{1}{4}.$ Therefore, the images of telescope B are four times brighter than the images of telescope A.

ROUND:

(a) To three significant figures, the magnification of telescopes A and B is $m_\theta = -25.0.$

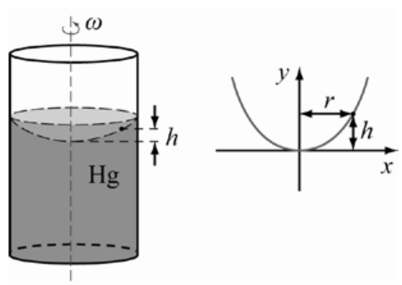
DOUBLE-CHECK:

(a) The magnification of the telescopes should have a magnitude greater than 1.

(b) It is reasonable that the images of telescope B are brighter since more light enters through its larger lens.

33.77. **THINK:** Some reflecting telescope mirrors utilize a rotating tub of mercury to produce a large parabolic surface. The tub is rotating on its axis with an angular frequency ω. Conservation of energy can be used to show that the focal length of the resulting mirror is: $f = g/2\omega^2$.

SKETCH:



RESEARCH: Consider a single drop of mercury in the rotating tub. The kinetic energy of this drop of mercury is given by:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega r)^2$$

The gravitational potential energy relative to the bottom of the lowest point of the surface is given by:

$$U_g = mgh$$

SIMPLIFY: By conservation of energy, $K = U_g$:

$$\frac{1}{2}m(\omega r)^2 = mgh \Rightarrow h = \frac{\omega^2 r^2}{2g}$$

Now, the equation of a parabola with its vertex at the origin is given by $x^2 = 4fy \Rightarrow r^2 = 4fh \Rightarrow h = \frac{r^2}{4f}$, where f is the focal length. Substitution gives: $\frac{r^2}{4f} = \frac{\omega^2 r^2}{2g} \Rightarrow f = \frac{g}{2\omega^2}$.

CALCULATE: Not required.

ROUND: Not required.

DOUBLE-CHECK: It is reasonable that the focal length is inversely proportional to the angular frequency of the tub since a faster rotation results in a steeper parabola.

33.78. The object height is $h_o = 4.0$ cm. It is projected onto a screen using a converging lens with a focal length of $f = 35$ cm. The image on the screen is $h_i = -56$ cm in size. (It is negative to represent the fact that the image has been inverted; the image must be real to be projected onto a screen, and for a converging lens a real image is always inverted). The distance from the lens to the screen is d_i and the distance from the object to the screen is $d_i + d_o$. The magnification is: $m = h_i / h_o = -d_i / d_o$. Then $d_o = -h_o d_i / h_i = -(4.0 \text{ cm})d_i / (-56 \text{ cm}) = d_i / 14$. From the thin lens equation, $d_i = d_o f / (d_o - f)$. Substitution for d_o gives the distance from the lens to the screen:

$$d_i = \frac{(d_i/14)f}{(d_i/14)-f} \Rightarrow \frac{d_i}{14} - f = \frac{f}{14} \Rightarrow d_i = 14f\left(\frac{1}{14}+1\right) = 14f\left(\frac{15}{14}\right) = 15f = 15(35 \text{ cm}) = 5.25 \text{ m} \approx 5.3 \text{ m}.$$

Therefore, the distance from the object to the screen is $d_i + d_o = 5.25 \text{ m} + (5.25 \text{ m}/14) = 5.6 \text{ m}$.

33.79. The eyeglasses of a near sighted person use diverging lenses and create virtual images of objects for the near sighted wearer. When a normal person wears these eyeglasses, the person with normal vision will only be able to focus on these virtual images if they fall within the focusable distances of a normal eye, which is from 25 cm out to infinity. Since only the most distant objects can be focused on, the objects at infinity must be making virtual images at the normal near point of 25 cm. This will happen when: $\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{\infty} + \frac{1}{-0.25 \text{ m}} = -4.0 \text{ m}^{-1}$. Note that d_i is negative because the image is virtual. The prescription strength of the eyeglasses is about -4.0 diopter.

33.80. The focal length of the spectacles is the reciprocal of the power, so the focal length is $f = \frac{1}{-0.20 \text{ diopter}} = -5.0 \text{ m}$. Therefore, light from a distant object will form a virtual image 5 m in front of the spectacles. Since this is a distance at which your eye can bring objects to a focus, you will still be able to focus on distant objects. The problem comes from near objects. This is a diverging lens (negative focal length), so light from nearby objects will be even more divergent, and therefore, more difficult for your eye to focus. Since the near point of your eye is 20. cm, virtual images formed by the spectacles cannot be closer than 20. cm. Otherwise, your eye will not be able to focus. From the thin lens equation

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \Rightarrow d_o = \left(\frac{1}{f} - \frac{1}{d_i}\right)^{-1} = \left(\frac{1}{-5.0 \cdot 10^2 \text{ cm}} - \frac{1}{(-20. \text{ cm})}\right)^{-1} = 21 \text{ cm}.$$

Thus, the range over which you will be able to see is from 21 cm to infinity. The spectacles have hardly changed your range because they are low in power.

33.81. In the derivation of the Lens Maker's Formula, the following relation can be inferred (in the text it was assumed that $n_1 = 1$): $\frac{n_1}{d_o} + \frac{n_2}{d_i} = \frac{n_2 - n_1}{R}$. In this case, the refracting surface is flat so R is infinite $(R = \infty)$. The equation can be rearranged as $d_o = -(n_1/n_2)d_i$. With the fish (the object) in water, $n_1 = 1.33$ and with you in air, $n_2 = 1$. The apparent depth of the fish is the virtual image distance, $d_i = -10.$ cm. (it is negative because it is on the same side of the surface as the object and therefore, a virtual image.) Then $d_o = -(1.33/1)(-10. \text{ cm}) = 13 \text{ cm}$. The fish is actually 13 cm under the surface of the water, and must be grabbed at this position.

33.82. The mirror has a focal length of $f = 40.0$ cm. To project the image onto a screen, the image must be real, and therefore, the mirror must be a concave mirror with $f > 0$. The bird has a height $h_o = 10.0$ cm and is $d_o = 100.$ m away from the mirror. From the mirror equation, $d_i = \frac{d_o f}{d_o - f}$. From the equation for magnification, $m = \frac{h_i}{h_o} = -\frac{d_i}{d_o}$, the image height is:

$$h_i = -\frac{d_i}{d_o}h_o = -\frac{d_o f}{d_o - f}\cdot\frac{1}{d_o}h_o = -\frac{fh_o}{d_o - f} = -\frac{(40.0 \text{ cm})(10.0 \text{ cm})}{(1.00 \cdot 10^4 \text{ cm}) - (40.0 \text{ cm})} = -0.402 \text{ mm}.$$

The image of the bird is inverted, but it is much smaller than one centimeter in size. Therefore, he will not make good on his claim.

33.83. The object is $d_o = 6.0 \text{ cm}$ away from a thin lens of focal length $f = 9.0 \text{ cm}$. The image distance d_i is determined from the thin lens equation: $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$. Therefore,

$$d_i = \left(\frac{1}{f} - \frac{1}{d_o}\right)^{-1} = \left(\frac{1}{9.0 \text{ cm}} - \frac{1}{6.0 \text{ cm}}\right)^{-1} = -18 \text{ cm}.$$

The image is 18 cm from the lens, and on the same side of the lens as the object (the negative sign indicates that it is a virtual image).

33.84. The spherical lens bulges outwards in the middle on both sides so it is a convex lens. The surfaces are ground to radii of 0.25 m and 0.30 m. The radii will have opposite signs, and since there will be an absolute value it does not matter which is taken to be negative. Take $R_1 = 0.25 \text{ m}$ and $R_2 = -0.30 \text{ m}$. Using the Lens Maker's Formula, the power of the lens is:

$$|P| = \frac{1}{f} = \left(n_{\text{glass}} - 1\right)\left|\frac{1}{R_1} - \frac{1}{R_2}\right| = (1.5-1)\left|\frac{1}{0.25 \text{ m}} - \frac{1}{-0.30 \text{ m}}\right| = 3.7 \text{ diopter}.$$

33.85. The convex surface is part of a sphere with radius $r = 0.45 \text{ m}$. The concave surface is part of a sphere with radius $R = 0.20 \text{ m}$, and r and R have the same sign. Using the Lens Makers Formula, the power of the lens is:

$$|P| = \frac{1}{f} = \left(n_{\text{glass}} - 1\right)\left|\frac{1}{r} - \frac{1}{R}\right| = (1.5-1)\left|\frac{1}{0.45 \text{ m}} - \frac{1}{0.20 \text{ m}}\right| = 1.4 \text{ diopter}.$$

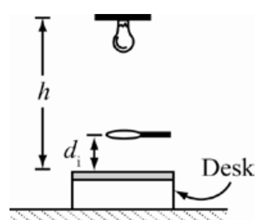
Since the lens is a diverging lens, the answer should be taken to be negative. The answer is $P = -1.4$ diopter.

33.86. The farsighted person can clearly see an object if it is at least 2.5 m away; therefore, for this person the image distance is $d_i = -2.5 \text{ m}$. Using the thin lens equation, the power of the lenses required to read a book a distance $d_o = 0.20 \text{ m}$ away is:

$$P = \frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{0.20 \text{ m}} + \frac{1}{-2.5 \text{ m}} = 4.6 \text{ diopter}.$$

Since the power is positive, the will require glasses with converging lenses.

33.87. The magnifying glass is a converging lens. If you hold the magnifying glass at $d_i = 9.20 \text{ cm}$ above your desk you can form a real image on the desk of a light directly overhead. The distance from the light to the table is $h = 235 \text{ cm}$.



Using the thin lens equation, where $d_o = h - d_i$, the focal length of the magnifying glass is:

$$f = \left(\frac{1}{d_i} + \frac{1}{d_o}\right)^{-1} = \left(\frac{1}{d_i} + \frac{1}{h - d_i}\right)^{-1} = \left(\frac{1}{9.20 \text{ cm}} + \frac{1}{235 \text{ cm} - 9.20 \text{ cm}}\right)^{-1} = 8.84 \text{ cm}.$$

33.88. The girl needs to hold the book at a distance 15 cm from her eyes to clearly see the print. This is her near point.

(a) The girl is nearsighted since she can see objects close to her eye. Therefore, she requires diverging lenses in order to see the book 25 cm away.

(b) The thin lens equation can be used to find the focal length of the lens: $\frac{1}{f}=\frac{1}{d_o}+\frac{1}{d_i}$. Substituting $d_o = 25$ cm and $d_i = -15$ cm gives a focal length of $f=\left(\frac{1}{25\text{ cm}}-\frac{1}{15\text{ cm}}\right)^{-1}=-38$ cm.

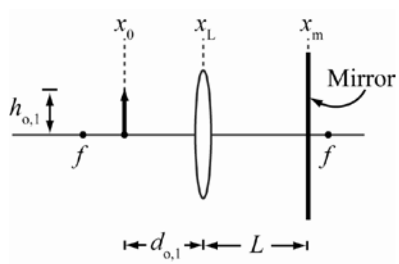
33.89. The focal length of the camera lens is $f = 38.0$ mm. The lens must be moved a distance Δd to change focus from a person at $d_o = 3.00\cdot 10^4$ mm to a person that is at $d_o' = 5.00\cdot 10^3$ mm, where $\Delta d = |d_i - d_i'|$. Using the thin lens equation, $d_i = \frac{fd_o}{d_o - f}$ and $d_i' = \frac{fd_o'}{d_o' - f}$. Therefore, the lens must be moved a distance

$$\Delta d = |d_i - d_i'| = \left|\frac{fd_o}{d_o - f}-\frac{fd_o'}{d_o' - f}\right| = \left|\frac{(38.0\text{ mm})(3.00\cdot 10^4\text{ mm})}{(3.00\cdot 10^4\text{ mm}-38.0\text{ mm})}-\frac{(38.0\text{ mm})(5.00\cdot 10^3\text{ mm})}{(5.00\cdot 10^3\text{ mm}-38.0\text{ mm})}\right| = 0.243\text{ mm.}$$

33.90. The magnitude of a telescope's magnification is $|m| = 41$. The focal length of the eyepiece is $f_e = 0.040$ m. The magnitude of the magnification is given by: $|m| = f_o / f_e$. Solving for f_o gives: $f_o = |m| f_e = (41)(0.040\text{ m}) = 1.6$ m.

33.91. **THINK:** The object is $h_{o,1} = 2.0$ cm high and is located at $x_0 = 0$ m. A converging lens with focal length $f = 50.$ cm is located at $x_L = d_{o,1} = 30.$ cm. A plane mirror is located at $x_m = 70.$ cm, so the distance between the lens and the mirror is $L = x_m - x_L = 40.$ cm. The image formed by the lens will act as the object for the plane mirror. The thin lens equation can be used to determine the position $x_{i,2}$ and the size $h_{i,2}$ of the final image.

SKETCH:



RESEARCH: The thin lens equation is $\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f}$. The magnification of a lens is $m = h_i / h_o = -d_i / d_o$. For plane mirrors, $|d_i| = |d_o|$ and $h_i = h_o$.

SIMPLIFY: When the thin lens equation is rearranged to solve for the image distance, it becomes $d_i = \frac{fd_o}{d_o - f}$. The image produced by the lens is located a distance of $d_{i,1} = \frac{fd_{o,1}}{d_{o,1} - f}$ from the lens. Since $f > d_{o,1}$, $d_{i,1}$ will be negative, and therefore, on the same side of the lens as the object. This image acts as the object for the mirror, and is a distance $d_{o,2} = L + |d_{i,1}|$ from the plane mirror. The final image is the image created by the plane mirror, and will appear $d_{i,2} = |d_{o,2}|$ to the right of the mirror. The final image position is given by $x_{i,2} = x_m + d_{i,2}$. Since the mirror does not change the height of the image, the magnification is due to the lens, and the final height of the image is $h_{i,2} = -\frac{d_{i,1}}{d_{o,1}}h_{o,1}$.

CALCULATE: The image distance for the lens is $d_{i,1}=\dfrac{(30.\text{ cm})(50.\text{ cm})}{(30.\text{ cm})-(50.\text{ cm})}=-75\text{ cm}.$ The object distance for the plane mirror is $d_{o,2}=40.\text{ cm}+\left|-75\text{ cm}\right|=115\text{ cm}.$ Therefore, the position of the final image is $x_{i,2}=70.\text{ cm}+115\text{ cm}=185\text{ cm}.$ The size of the final image is $h_{i,2}=-\dfrac{(-75\text{ cm})(2.0\text{ cm})}{(30.\text{ cm})}=5.0\text{ cm}.$

ROUND: To two significant figures, the final image is $x_{i,2}=190\text{ cm}$ to the right of the object and the size of the final image is $h_{i,2}=5.0\text{ cm}.$

DOUBLE-CHECK: Since $d_o<f$ for the converging lens, the image of the lens must be virtual, enlarged and upright. The plane mirror cannot change these attributes, so the calculated results agree with these expectations $\left(h_{i,2}>h_{o,1}>0\right)$.

33.92. The distance from the lens to the retina at the back of the eye is 2.0 cm. The focal length can be found with the thin lens equation: $f=\left(\dfrac{1}{d_o}+\dfrac{1}{d_i}\right)^{-1}$. (a) The focal length of the lens when viewing a distant object $\left(d_o=\infty\right)$ is $f=\left(\dfrac{1}{\infty}+\dfrac{1}{2.0\text{ cm}}\right)^{-1}=2.0\text{ cm}.$ (b) The focal length of the lens when viewing an object $d_o=25\text{ cm}$ from the front of the eye is $f=\left(\dfrac{1}{25\text{ cm}}+\dfrac{1}{2.0\text{ cm}}\right)^{-1}=1.9\text{ cm}.$

33.93. You require lenses of power $P=-8.4$ diopter. A negative power infers that the focal length is negative, so diverging lenses are being used. In a nearsighted eye, light comes to a focus before it reaches the retina and diverging lenses are required to correct this. Therefore, you are nearsighted. For nearsighted eyes, corrective lenses focus distant objects $\left(d_o=\infty\right)$ at the near point, so $d_i=-d_{\text{near}}$. Solving the thin lens equation for d_{near} gives:

$$D=1/f=1/d_o+1/d_i=1/\infty-1/d_{\text{near}}\;\Rightarrow\;-d_{\text{near}}=1/D=-\left(1/-8.4\text{ m}^{-1}\right)=0.12\text{ m}.$$

$$P=\frac{1}{f}=\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{\infty}-\frac{1}{d_{\text{near}}}\;\Rightarrow\;-d_{\text{near}}=\frac{1}{P}=-\left(\frac{1}{-8.4\text{ m}^{-1}}\right)=0.12\text{ m}.$$

Without glasses the book must be held 12 cm from your eye in order to read clearly.

33.94. Jack has a near point of $d_{\text{near}}=32\text{ cm}=0.32\text{ m}$ and the power of the magnifier is $P=25$ diopter.
(a) The focal length is given by $f=1/P$ and the angular magnification of a magnifier for an image formed at infinity is $m=\dfrac{d_{\text{near}}}{f}$. Therefore, $m=d_{\text{near}}P=(0.32\text{ m})\left(25\text{ m}^{-1}\right)=8.0.$

(b) If the final image is at the near point then $m=\dfrac{-d_i}{d_o}=-\left(\dfrac{-d_{\text{near}}}{d_o}\right)=\dfrac{d_{\text{near}}}{d_o}$. Using the thin lens equation:

$\dfrac{1}{f}=\dfrac{1}{d_i}+\dfrac{1}{d_o}\;\Rightarrow\;d_o=\left(\dfrac{1}{f}-\dfrac{1}{d_i}\right)^{-1}=\left(\dfrac{1}{f}+\dfrac{1}{d_{\text{near}}}\right)^{-1}=\dfrac{fd_{\text{near}}}{f+d_{\text{near}}}$. Therefore the magnification is:

$$m=\frac{d_{\text{near}}}{\dfrac{fd_{\text{near}}}{f+d_{\text{near}}}}=\frac{1}{f}\left(f+d_{\text{near}}\right)=1+\frac{d_{\text{near}}}{f}=1+Pd_{\text{near}}=1+(25\text{ m})(0.32\text{ m})=9.0.$$

33.95. The diameter of the glass marble $(n_g = 1.5)$ is $d = 2.0\text{ in} = 5.1\text{ cm}$. The radius of curvature of the marble is then $R = d/2$. Holding the marble a distance of $d_{o,1} = 1.0\text{ ft} = 30.\text{ cm}$ from your face, the distance of the image formed by the first side of the marble is:

$$\frac{1}{d_{o,1}} + \frac{n_g}{d_{i,1}} = \frac{2(n_g - 1)}{d} \Rightarrow d_{i,1} = \frac{n_g d d_{o,1}}{2d_{o,1}(n_g - 1) - d} = \frac{(1.5)(5.1\text{ cm})(30.\text{ cm})}{2(30.\text{ cm})(1.5 - 1) - (5.1\text{ cm})} = 9.217\text{ cm}.$$

This image acts as the object for the second surface, for which the radius of curvature is negative (concave), $d = -5.1\text{ cm}$. Since $d_{i,1} > d$, the image for the second surface appears past it, so $d_{o,2} = d_{i,1} - d$. Therefore, the final image distance can be computed as follows.

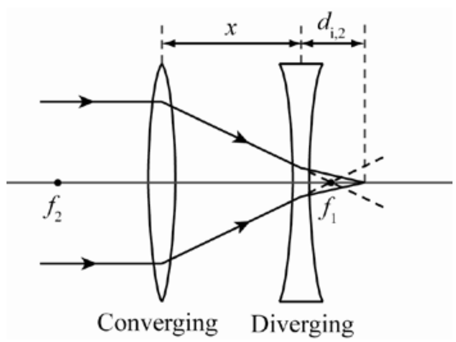
$$\frac{n_g}{d_{o,2}} + \frac{1}{d_{i,2}} = \frac{2(1 - n_g)}{d} \Rightarrow d_{i,2} = \frac{d d_{o,2}}{2d_{o,2}(1 - n_g) + dn_g} = \frac{d(d_{i,1} - d)}{2(d_{i,1} - d)(1 - n_g) + dn_g},$$

$$d_{i,2} = \frac{(-5.1\text{ cm})(9.217\text{ cm} + 5.1\text{ cm})}{2(9.217\text{ cm} + 5.1\text{ cm})(1 - 1.5) + (-5.1\text{ cm})(1.5)} = 3.324\text{ cm} = 1.3\text{ in}.$$

The magnification is $m = -\frac{d_{i,1} d_{i,2}}{d_{o,1} d_{o,2}} = -\frac{d_{i,1} d_{i,2}}{d_{o,1}(d_{i,1} - d)} = -\frac{(9.217\text{ cm})(3.324\text{ cm})}{(30.48\text{ cm})(9.217\text{ cm} + 5.1\text{ cm})} = -0.070,$ where the negative sign indicates that the image is inverted.

33.96. **THINK:** The diverging lens has a focal length of $f_2 = -30.0$ cm. It is placed a distance $x = 15.0$ cm behind a converging lens with focal length, $f_1 = 20.0$ cm. The thin lens equation can be used to find the image location for an object that is located at infinity in front of the converging lens. The image formed by the converging lens will act as the object for the diverging lens.

SKETCH:



RESEARCH: The thin lens equation is: $\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o}$.

SIMPLIFY: For the converging lens: $\frac{1}{f_1} = \frac{1}{d_{o,1}} + \frac{1}{d_{i,1}} = \frac{1}{\infty} + \frac{1}{d_{i,1}} \Rightarrow d_{i,1} = f_1$. The object distance of the diverging lens can now be written as: $d_{o,2} = x - d_{i,1} = x - f_1$. Substituting this into the thin lens equation for the diverging lens gives:

$$\frac{1}{f_2} = \frac{1}{d_{o,2}} + \frac{1}{d_{i,2}} = \frac{1}{(x - f_1)} + \frac{1}{d_{i,2}} \Rightarrow d_{i,2} = \left(\frac{1}{f_2} - \frac{1}{(x - f_1)}\right)^{-1}.$$

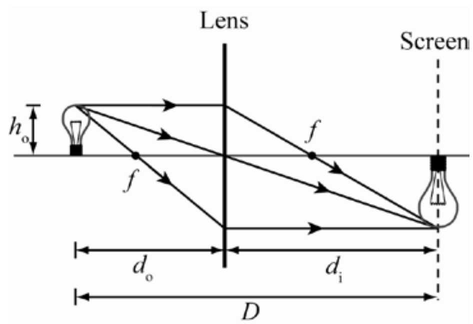
CALCULATE: $d_{i,2} = \left(\frac{1}{-30.0\text{ cm}} - \frac{1}{(15.0\text{ cm} - 20.0\text{ cm})}\right)^{-1} = 6.00\text{ cm}$

ROUND: To three significant figures, the object at infinity will be focused $d_{i,2} = 6.00$ cm to the right of the diverging lens.

DOUBLE-CHECK: This result agrees with the diagram shown above. It is expected that the diverging lens causes the focal point to be beyond the focus of the converging lens.

33.97. **THINK:** The instructor wants the lens to project a real image of a light bulb onto a screen a distance $D = 1.71$ m from the bulb. The thin lens equation can be used to find the focal length that is required to achieve a magnification of $|m| = 2.$

SKETCH:



RESEARCH: The image is real and enlarged; therefore, the focal length f must be smaller than the object distance d_o. The distance from the bulb to the screen is $D = d_o + d_i$, where d_i is the image distance. The magnitude of the magnification is $|m| = d_i / d_o$. The thin lens equation is: $\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o}$.

SIMPLIFY: Also, $|m| = 2 = d_i / d_o \Rightarrow d_i = 2d_o$. Therefore, $D = d_i + d_o = 2d_o + d_o = 3d_o$ or $d_o = D/3$.

From the thin lens equation, $f = \left(\frac{1}{d_i} + \frac{1}{d_o}\right)^{-1} = \left(\frac{1}{2(D/3)} + \frac{1}{(D/3)}\right)^{-1} = \frac{2}{9}D.$

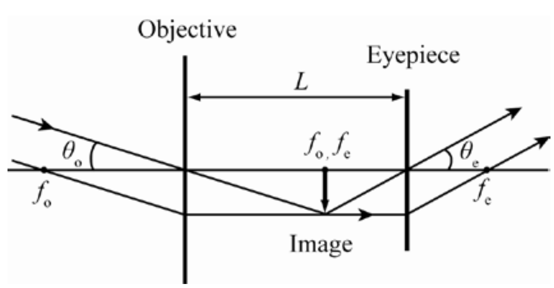
CALCULATE: $f = \frac{2}{9}(1.71 \text{ m}) = 0.380 \text{ m}$

ROUND: To three significant figures, the focal length required is $f = 38.0$ cm.

DOUBLE-CHECK: The calculated focal length has the correct units. The answer seems reasonable considering the values provided in the question.

33.98. **THINK:** The length of the refracting telescope is $L = 55$ cm and it has a magnification of $|m| = 45.$ The equation for the magnification of a telescope can be used to find the focal length of its objective, f_o and the focal length of its eye lens, f_e. The length of a refracting telescope is just the sum of the focal lengths, $L = f_o + f_e$.

SKETCH:



RESEARCH: The magnification of a refracting telescope is: $|m| = f_o / f_e$.

SIMPLIFY: The focal length of the eye lens is: $f_o = f_e|m| \Rightarrow L = f_e|m| + f_e = f_e(1+|m|) \Rightarrow f_e = \dfrac{L}{(1+|m|)}$.

The focal length of the objective lens is: $f_o = \dfrac{L|m|}{(1+|m|)}$.

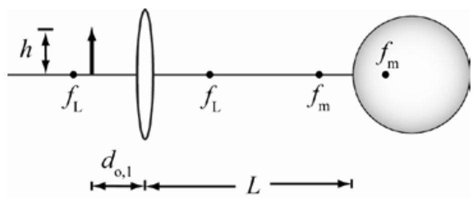
CALCULATE: $f_e = \dfrac{55\text{ cm}}{(1+45)} = 1.196\text{ cm}, \quad f_o = \dfrac{(55\text{ cm})(45)}{(1+45)} = 53.80\text{ cm}$

ROUND: To two significant figures, the focal length of the objective lens is $f_o = 54\text{ cm}$ and the focal length of the eye lens is $f_e = 1.2\text{ cm}$.

DOUBLE-CHECK: As shown in the diagram, it is expected that $f_o > f_e$.

33.99. **THINK:** The converging lens has a focal length $f_L = 50.0\text{ cm}$. It is $L = 175\text{ cm}$ to the left of a metallic sphere. This metallic sphere acts as a convex mirror of radius $R = -100.\text{ cm}$ (the radius of curvature of a diverging mirror is negative) and focal length $f_m = R/2 = -50.0\text{ cm}$. The object of height, $h = 20.0\text{ cm}$, is a distance $d_{o,1} = 30.0\text{ cm}$ to the left of the lens. The thin lens equation, the mirror equation, and the magnification for a system of optical elements can be used to find the height of the image formed by the metallic sphere, $h_{i,2}$. The image formed by the lens acts as the object for the mirror.

SKETCH:



RESEARCH: The thin lens equation is $\dfrac{1}{f} = \dfrac{1}{d_i} + \dfrac{1}{d_o}$, The magnification (for lenses and mirrors) is $m = h_i / h_o = -d_i / d_o$. The total magnification m is the product of the magnification of the lens and the mirror: $m = m_L m_m$.

SIMPLIFY: For the lens, the thin lens equation can be rearranged as: $d_{i,1} = \dfrac{d_{o,1} f_L}{d_{o,1} - f_L}$. Since $f_L > d_{o,1}$, $d_{i,1}$ is negative, so the image is on the same side as the object (the image is virtual). This image acts as the object for the mirror at a distance of $d_{o,2} = L + |d_{i,1}|$ from the metallic sphere. The location of the image produced from the sphere is $d_{i,2} = \dfrac{d_{o,2} f_m}{d_{o,2} - f_m}$. The final image height is

$$h_{i,2} = mh = (m_L)(m_m)h = \left(\frac{d_{i,1}}{d_{o,1}}\right)\left(\frac{d_{i,2}}{d_{o,2}}\right)h = \left(\frac{\dfrac{d_{o,1} f_L}{d_{o,1} - f_L}}{d_{o,1}}\right)\left(\frac{\dfrac{d_{o,2} f_m}{d_{o,2} - f_m}}{d_{o,2}}\right)h = \frac{f_L f_m h}{(d_{o,1} - f_L)\left(L + \left|\dfrac{d_{o,1} f_L}{d_{o,1} - f_L}\right| - f_m\right)}.$$

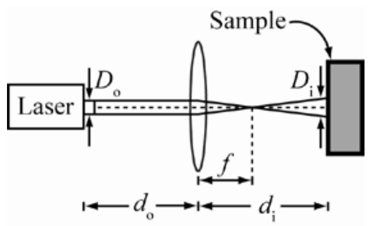
CALCULATE:
$$h_{i,2} = \frac{(50.0\text{ cm})(-50.0\text{ cm})(20.0\text{ cm})}{(30.0\text{ cm} - 50.0\text{ cm})\left(175\text{ cm} + \left|\dfrac{(30.0\text{ cm})(50.0\text{ cm})}{30.0\text{ cm} - 50.0\text{ cm}}\right| - (-50.0\text{ cm})\right)} = 8.3333\text{ cm}$$

ROUND: To three significant figures, the height of the image formed by the metallic sphere is $h_{i,2} = 8.33\text{ cm}$.

DOUBLE-CHECK: It is expected that $h_{i,2} < h$. For a converging lens, an image produced by an object placed within the focal length of the lens is enlarged, virtual and upright. For a diverging mirror, the image is always virtual, upright and reduced. Therefore, the height of the final image should be less than the height of the object since both the lens and the mirror act to reduce it.

33.100. **THINK:** The lens has a focal distance, $f = 10.0$ cm. The laser beam exits a pupil of diameter, $D_o = 0.200$ cm that is located a distance $d_o = 150.$ cm from the focusing lens. Consider the case when the image of the exit pupil forms on the sample. (a) The thin lens equation can be used to find the distance, d_i, from the sample to the lens and (b) the magnification equation can be used to find the diameter, D_i, of the laser spot on the sample (this is the image of the exit pupil).

SKETCH:



RESEARCH:

(a) The thin lens equation is: $\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o}$.

(b) The magnification is $m = \frac{D_i}{D_o} = -\frac{d_i}{d_o}$.

SIMPLIFY:

(a) $d_i = \left(\frac{1}{f} - \frac{1}{d_o} \right)^{-1}$

(b) $\frac{D_i}{D_o} = -\frac{d_i}{d_o} \Rightarrow D_i = -\frac{d_i D_o}{d_o}$

CALCULATE:

(a) $d_i = \left(\frac{1}{10.0 \text{ cm}} - \frac{1}{150. \text{ cm}} \right)^{-1} = 10.714 \text{ cm}$

(b) $D_i = -\frac{(10.714 \text{ cm})(0.200 \text{ cm})}{150. \text{ cm}} = -0.1429 \text{ mm},$ where the negative sign indicates that the image is inverted.

ROUND: Round to three significant figures.

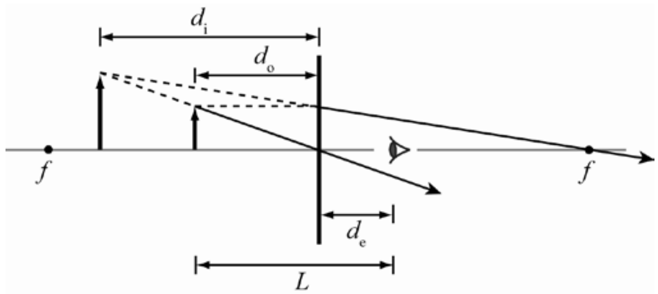
(a) The sample is located $d_i = 10.7$ cm past the lens.

(b) The image of the exit pupil has a diameter of $D_i = 0.143$ mm.

DOUBLE-CHECK: The laser beam is being focused on the sample so it is reasonable that the diameter of the laser sport on the sample is smaller than the exit pupil.

33.101. **THINK:** The computer monitor is at a distance of $L = 0.55$ m from his eyes. The image of the monitor must be located at his near point, $d_{near} = 1.15$ m. Since the image is located in front of the lens (the image is virtual), the image distance is $d_i = -(d_{near} - d_e)$. Since the lens-eye distance for his glasses is known to be $d_e = 0.020$ m, the object distance from the lens to the computer monitor is $d_o = L - d_e$. The thin lens equation can be used to find the lens power required.

SKETCH:



RESEARCH: The thin lens equation is given by $\frac{1}{f}=\frac{1}{d_o}+\frac{1}{d_i}$.

SIMPLIFY: The power of the lens is defined as $D=1/f$ where f is in meters. Therefore, $\frac{1}{f}=P=\frac{1}{L-d_e}-\frac{1}{d_{near}-d_e}$.

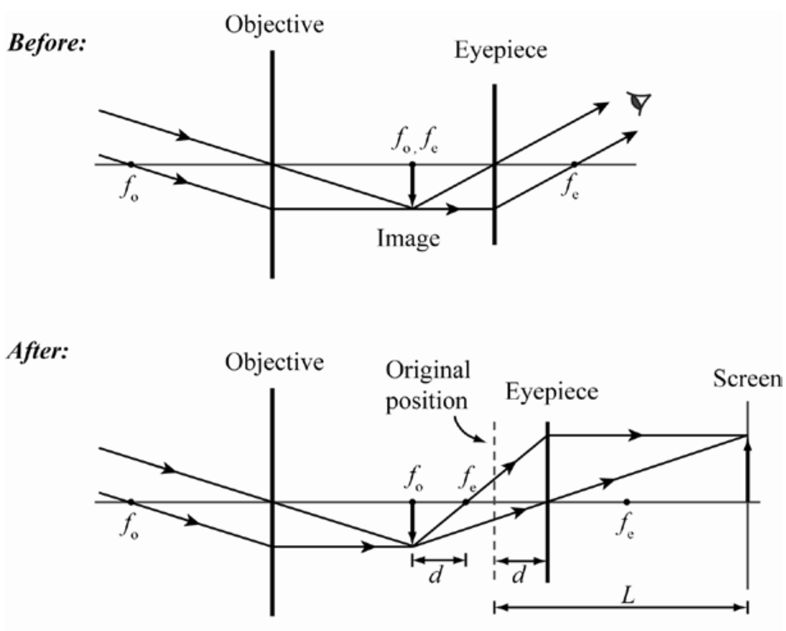
CALCULATE: $P=\frac{1}{0.55\text{ m}-0.020\text{ m}}-\frac{1}{1.15\text{ m}-0.020\text{ m}}=1.0018\text{ diopter}$

ROUND: To two significant figures, his optician should prescribe a power of $P=1.0$ diopter.

DOUBLE-CHECK: Since the power is positive, a converging lens must be used. Since the object is inside his near point a converging lens is expected in order to correct his vision.

33.102. **THINK:** An image of a far away object produced by an objective lens of a telescope is located at the focal point of the objective lens. This image becomes the object for the eyepiece. The focal length of the eyepiece is $f_e=8.0\text{ cm}$ and the image is to be projected on a screen that is a distance of $L=150\text{ cm}$ past the original location of the eyepiece. The thin lens equation can be used to determine how far the eyepiece must be moved.

SKETCH:



RESEARCH: The thin lens equation is given by: $\frac{1}{f}=\frac{1}{d_o}+\frac{1}{d_i}$.

SIMPLIFY: The distance from the object to the eyepiece is $d_o = f_e + d$ and the distance from the image to the eyepiece is $d_i = L - d$. Therefore, the thin lens equation becomes:

$$\frac{1}{f_e} = \frac{1}{f_e + d} + \frac{1}{L - d} = \frac{L - d + f_e + d}{(f_e + d)(L - d)} = \frac{L + f_e}{(f_e + d)(L - d)} \Rightarrow (f_e + d)(L - d) = f_e(L + f_e)$$

$$\Rightarrow -d^2 + (L - f_e)d + f_e L = f_e L + f_e^2 \Rightarrow d^2 + (f_e - L)d + f_e^2 = 0.$$

Solving this quadratic equation for d yields: $d = \dfrac{-(f_e - L) \pm \sqrt{(f_e - L)^2 - 4f_e^2}}{2}.$

CALCULATE: Substituting the numerical values gives:

$$d = \frac{-(8.0 \text{ cm} - 150 \text{ cm}) \pm \sqrt{(8.0 \text{ cm} - 150 \text{ cm})^2 - 4(8.0 \text{ cm})^2}}{2} = 0.452 \text{ cm or } 142 \text{ cm}.$$

The most realistic distance is $d = 0.452$ cm.

ROUND: To two significant figures, the eyepiece should be moved a distance of $d = 4.5$ mm towards the screen.

DOUBLE-CHECK: Since $L >> d$, the thin lens equation can be approximated by $\dfrac{1}{f_e} \approx \dfrac{1}{f_e + d} + \dfrac{1}{L}$. Solving this equation for d gives

$$f_e + d = \left(\frac{1}{f_e} - \frac{1}{L}\right)^{-1} \Rightarrow d = \left(\frac{1}{f_e} - \frac{1}{L}\right)^{-1} - f_e = \left(\frac{1}{8.0 \text{ cm}} - \frac{1}{150 \text{ cm}}\right)^{-1} - 8.0 \text{ cm} = 0.45 \text{ cm}.$$

This approximation is the same as what was obtained above.

Multi-Version Exercises

33.103. $P_{\text{water}} = \dfrac{1}{f} = \dfrac{n_{\text{lens}} - n_{\text{water}}}{n_{\text{water}}}\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$

$P_{\text{air}} = (n_{\text{lens}} - 1)\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$

Take the ratio:

$$\frac{P_{\text{water}}}{P_{\text{air}}} = \frac{n_{\text{lens}} - n_{\text{water}}}{n_{\text{water}}(n_{\text{lens}} - 1)}$$

Solve for power in water:

$$P_{\text{water}} = P_{\text{air}} \frac{n_{\text{lens}} - n_{\text{water}}}{n_{\text{water}}(n_{\text{lens}} - 1)} = (4.29 \text{ D}) \frac{1.723 - 1.333}{1.333(1.723 - 1)} = 1.74 \text{ D}$$

33.104. $P_{\text{water}} = \dfrac{1}{f} = \dfrac{n_{\text{lens}} - n_{\text{water}}}{n_{\text{water}}}\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$

$P_{\text{air}} = (n_{\text{lens}} - 1)\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$

Take the ratio:

$\dfrac{P_{\text{water}}}{P_{\text{air}}} = \dfrac{n_{\text{lens}} - n_{\text{water}}}{n_{\text{water}}(n_{\text{lens}} - 1)}$

Solve for the index of the lens:

$P_{\text{water}} n_{\text{water}} n_{\text{lens}} - P_{\text{water}} n_{\text{water}} = P_{\text{air}}(n_{\text{lens}} - n_{\text{water}}) \Rightarrow$

$n_{\text{lens}}\left(P_{\text{water}} n_{\text{water}} - P_{\text{air}}\right) = P_{\text{water}} n_{\text{water}} - P_{\text{air}} n_{\text{water}} \Rightarrow$

$n_{\text{lens}} = \dfrac{P_{\text{water}} n_{\text{water}} - P_{\text{air}} n_{\text{water}}}{P_{\text{water}} n_{\text{water}} - P_{\text{air}}}$

$= \dfrac{2.262 \cdot 1.333 - 5.55 \cdot 1.333}{2.262 \cdot 1.333 - 5.55} = 1.73$

33.105. $P_{\text{water}} = \dfrac{1}{f} = \dfrac{n_{\text{lens}} - n_{\text{water}}}{n_{\text{water}}}\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$

$P_{\text{air}} = (n_{\text{lens}} - 1)\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$

Take the ratio:

$\dfrac{P_{\text{air}}}{P_{\text{water}}} = \dfrac{n_{\text{water}}(n_{\text{lens}} - 1)}{n_{\text{lens}} - n_{\text{water}}}$

Solve for power in air:

$P_{\text{air}} = P_{\text{water}} \dfrac{n_{\text{water}}(n_{\text{lens}} - 1)}{n_{\text{lens}} - n_{\text{water}}} = (2.794\text{ D})\dfrac{1.333(1.735 - 1)}{1.735 - 1.333} = 6.810\text{ D}$

33.106. $|m_\theta| = \dfrac{f_o}{f_e} = \dfrac{P_e}{P_o} \Rightarrow P_e = |m_\theta| P_o = 81.4(0.234\text{ D}) = 19.0\text{ D}$

33.107. $|m_\theta| = \dfrac{f_o}{f_e} = \dfrac{1}{f_e P_o} = \dfrac{1}{(0.0533\text{ m})(0.186\text{ m}^{-1})} = 101\text{ times}$

33.108. $|m_\theta| = \dfrac{f_o}{f_e} = \dfrac{646.7\text{ cm}}{5.41\text{ cm}} = 120\text{ times}$

Chapter 34: Wave Optics

Concept Checks

34.1. c **34.2.** a **34.3.** b **34.4.** b **34.5.** b **34.6.** b **34.7.** a **34.8.** a **34.9.** c

Multiple-Choice Questions

34.1. c **34.2.** c **34.3.** d **34.4.** a **34.5.** a **34.6.** a **34.7.** b **34.8.** c **34.9.** d **34.10.** a **34.11.** b **34.12.** c

Conceptual Questions

34.13. The fringe width, defined as the distance between two bright or dark fringes, is given by $\Delta y = \lambda L / d.$

(a) If the wavelength is increased, the fringe width will increase, and thus the pattern will expand.

(b) If the separation distance between the slits is increased, the fringe width will decrease, and thus the pattern will shrink.

(c) If the apparatus is placed in water, the wavelength is decreased and the fringe width will decrease, causing the pattern to shrink.

34.14. Diffraction effects depend on the ratio between the size of an obstacle and the wavelength of light. If the diffraction effect for a sound wave is similar to that of light, the wavelength of a sound wave should be similar to light. Let us assume the wavelength of sound is about $\lambda = 500$ nm. Since the speed of sound is about 340 m/s, the frequency corresponding to this wavelength is

$$f = \frac{v}{\lambda} = \frac{340 \text{ m/s}}{500 \cdot 10^{-9} \text{ m}} = 680 \text{ MHz}.$$

34.15. A radio telescope is so much larger than an optical telescope because the wavelength of a radio wave is much larger than the wavelength of visible light. Since the resolution of a telescope is proportional to the ratio λ / D (D is the diameter of the telescope), in order to get similar resolution as the visible light, the diameter of the telescope must be larger. With similar reasoning, since the wavelength of x-ray's are much less than visible light, the diameter of an x-ray telescope can be smaller than a visible light telescope.

34.16. Yes, light can pass through such a slit. Using Huygens's principle, where each point on the wave front of light acts as a source of a spherical wave, the diffraction pattern of a very narrow slit is produced by a single spherical wave. The intensity as a function of angle from the direct beam is $I(\theta) = I_0 \frac{\sin^2\left((\pi a \sin\theta)/\lambda\right)}{\left((\pi a \sin\theta)/\lambda\right)^2}$, where I_0 is the intensity at $\theta = 0$. Since d is less than λ, the ratio a/λ is less than 1. As a consequence, the intensity falls off with the angle θ; but it never reaches the minimum.

34.17. (a) A hologram is an interference pattern produced by the interference of two light sources (object and reference sources). The recorded pattern acts as a diffraction grating for the light shining on it. The scale of a diffraction pattern is set by the wavelength of the light. The size of the image produced by the hologram is proportional to the wavelength of the light that produced the hologram. Therefore, if white light is used, it will produce a set of nested images of different colors, the size of each image is proportional to its wavelength.

(b) The size of each image is proportional to the wavelength. The longest wavelengths of the visible light, is those of red light, produce the largest images. Conversely, the violet light, the smallest wavelength, produces the smallest image.

34.18. No, it will not. No interference pattern will be produced since the light source is not a coherent light source.

34.19. There are two advantages:
(a) The intensity of the collected radio wave is increased.
(b) The effective diameter of the telescopes is increased producing a better resolution.

34.20. The maxima of a diffraction pattern are located at angles determined by the equation $\sin\theta = m\lambda / d$. For a maximum to be visible on screen, the angle must be less than $90°$ or $m\lambda / d < 1$. This means there is an upper limit on the value of m that satisfies the above equation. Therefore, the number of maxima is finite.

34.21. For a circular aperture telescope, the minimum angle resolvable or the limiting angle is given by Rayleigh's criterion, $\theta_R = \sin^{-1}(1.22\lambda / d)$, where λ is the wavelength and d is the diameter of the aperture. Since the blue light has smaller wavelength than the red light, the minimum angle for the blue light is also smaller than for the red light. Therefore, two blue stars are more resolvable than two red stars.

34.22. Bright spots on the screen behind a diffraction grating are produced when there is a constructive interference. The condition for the constructive interference is $d\sin\theta = m\lambda$ or $\theta = \sin^{-1}(m\lambda / d)$. Since green light has smaller wavelength than red light, it will produce bright spots at smaller angles. Therefore, the green bright spots will be closer together.

Exercises

34.23. The wavelength of EM radiation in a medium with a refractive index n is $\lambda = \lambda_o / n$ where λ_o is the wavelength of light in a vacuum. Similarly the speed of light in the medium is $v = c / n$.

(a) The wavelength of a helium-neon laser in Lucite is $\lambda = \dfrac{\lambda_o}{n} = \dfrac{632.8 \text{ nm}}{1.500} = 421.9 \text{ nm}$

(b) The speed of light in the Lucite is $v = \dfrac{c}{n} = \dfrac{2.998 \cdot 10^8 \text{ m/s}}{1.500} = 1.999 \cdot 10^8 \text{ m/s}.$

34.24. The wavelength of light in a medium is $\lambda = \lambda_o / n$. Thus, the wavelength of the light from a HeNe laser in water is $\lambda = \dfrac{632.8 \text{ nm}}{1.333} = 474.7 \text{ nm}$. The color of the light in water is the same as the color in the air, since the color of a light is determined from its frequency, not its wavelength. The frequency of light does not change as it passes into a different medium.

34.25. One wavelength corresponds to a phase difference of 2π. Therefore, the minimum path difference which causes a phase shift by $\pi / 4$ is $\Delta x = \lambda \dfrac{\Delta\theta}{2\pi} = \lambda \dfrac{\pi / 4}{2\pi} = \dfrac{1}{8}\lambda = \dfrac{1}{8}(700. \text{ nm}) = 87.5 \text{ nm}.$

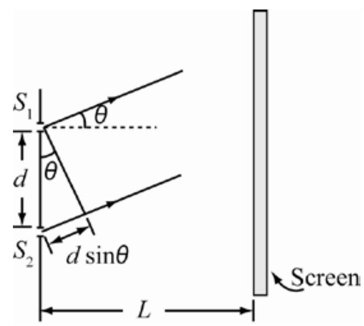
34.26. A constructive interference occurs when the path difference between two coherent light sources is a multiple of wavelength. A destructive interference occurs when the path difference is $\Delta x = (m + 1/2)\lambda$. By dividing the path difference by the wavelength, the properties of the interference can be determined. The ratio of the path difference and wavelength is $\text{ratio} = \dfrac{\Delta x}{\lambda} = \dfrac{20.25 \cdot 10^{-2} \text{ m}}{450.0 \cdot 10^{-9} \text{ m}} = 4.500 \cdot 10^5$. The ratio is a multiple of the wavelength. Therefore, the interference is constructive.

34.27. For a Young's interference experiment, the maxima of the interference pattern is located at $y = m\lambda L / d$. Substituting $m = 1$ for the first maximum intensity yields $y = \lambda L / d$. Therefore, the distance between the slits and the screen is $L = \dfrac{yd}{\lambda} = \dfrac{(5.40 \cdot 10^{-3} \text{ m})(0.100 \cdot 10^{-3} \text{ m})}{540 \cdot 10^{-9} \text{ m}} = 1.0 \text{ m}.$

34.28. The maxima of the fringe pattern is located at $y = m\lambda L/d$. The separation between the central maximum intensity $(m=0)$ to the next maximum intensity $(m=1)$ is $\Delta y = \lambda L/d$. Note that d is the distance between the centers of the two slits, that is, $d = 1.00\text{ mm} + 1.50\text{ mm} = 2.50\text{ mm}$. Thus, the separation between the maxima is $\Delta y = \dfrac{(633\cdot 10^{-9}\text{ m})(5.00\text{ m})}{2.50\cdot 10^{-3}\text{ m}} = 0.001266\text{ m} \approx 1.27\text{ mm}.$

34.29. **THINK:** The intensity of light is proportional to the square of the electric field. The light has wavelength $\lambda = 514$ nm and the slits are separated by a distance of $d = 0.500$ mm. The intensity of the radiation at the screen 2.50 m away from each slit is 180.0 W/cm^2 (not the maximum intensity, I_{max}). However, this intensity is not needed to find the position where $I = I_{\text{max}}/3$.

SKETCH:



RESEARCH: The intensity of the light produced by the interference from two narrow slits on a distant screen is given by:

$$I = 4I_{\text{max}}\cos^2\left(\frac{\pi d y}{\lambda L}\right).$$

SIMPLIFY: For $I = I_{\text{max}}/3,\ y \to y_{1/3}$:

$$\frac{I_{\text{max}}}{3} = 4I_{\text{max}}\cos^2\left(\frac{\pi d y_{1/3}}{\lambda L}\right) \Rightarrow \cos\left(\frac{\pi d y_{1/3}}{\lambda L}\right) = \frac{1}{\sqrt{12}} \Rightarrow y_{1/3} = \frac{\lambda L}{\pi d}\cos^{-1}\left(\frac{1}{\sqrt{12}}\right).$$

CALCULATE: Substituting the numerical values gives

$$y_{1/3} = \frac{(514\cdot 10^{-9}\text{ m})(2.50\text{ m})}{\pi(5.00\cdot 10^{-4}\text{ m})}\cos^{-1}\left(\frac{1}{\sqrt{12}}\right) = 0.001045\text{ m}.$$

ROUND: To three significant figures, $y_{1/3} = 1.05$ mm.

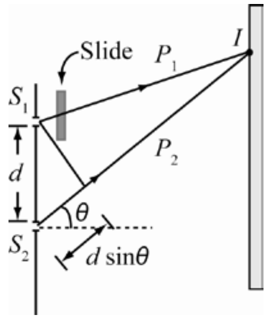
DOUBLE-CHECK: As a comparison the first minimum intensity is located at

$$y = \frac{(1/2)\lambda L}{d} = \frac{(1/2)(514\cdot 10^{-9}\text{ m})(2.50\text{ m})}{(5.00\cdot 10^{-4}\text{ m})} = 1.29\text{ mm}.$$

The result for $y_{1/3}$ is less than 1.29 mm, as expected.

34.30. **THINK:** The new wavelength as light passes through a medium of refractive index of n is given by $\lambda = \lambda_o / n$. The 10th dark fringe corresponds to a path difference of $\Delta x = (m + 1/2)\lambda$, with $m = 9$.

SKETCH:



RESEARCH: The path difference between two paths (P_1 and P_2) is given by $\Delta x = d\sin\theta - (n-1)t$, where t is the thickness of a glass slide. The central fringe is when $\Delta x = 0$, that is $d\sin\theta = (n-1)t$. This central fringe corresponds to the 10th dark fringe for the interference without the glass slide. The condition for the 10th dark fringe is $d\sin\theta = (9 + 1/2)\lambda$.

SIMPLIFY: From the equations in the Research step, it can be concluded that $(n-1)t = (9+1/2)\lambda$. Therefore, the refractive index is $n = \left(9 + \frac{1}{2}\right)\frac{\lambda}{t} + 1$.

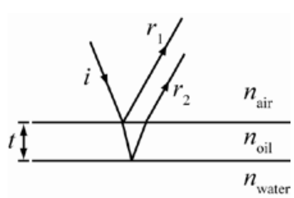
CALCULATE: Putting in the numerical values gives $n = \left(9 + \frac{1}{2}\right)\frac{633 \text{ nm}}{12000 \text{ nm}} + 1 = 1.5011$.

ROUND: Keeping three significant figures yields $n = 1.50$.

DOUBLE-CHECK: This value is within the expected range for glass.

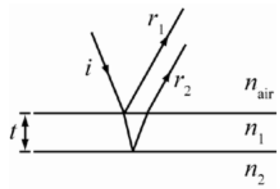
34.31. The minima of the interference pattern produced by a thin film is related to its thickness by $2t = m\lambda / n$. The first dark band which corresponds to the thinnest and is when $m = D$ or when the thickness is much less than λ. The next dark bands are for $m = 1$ and $m = 2$. Therefore, the thicknesses that produces the dark bands are $t_1 = \frac{1}{2}\frac{\lambda}{n} = \frac{550 \text{ nm}}{2(1.32)} = 208 \text{ nm} \approx 210 \text{ nm}$ and $t_2 = \frac{2}{2}\frac{\lambda}{n} = \frac{\lambda}{n} = \frac{550 \text{ nm}}{1.32} = 417 \text{ nm} \approx 420 \text{ nm}$.

34.32.



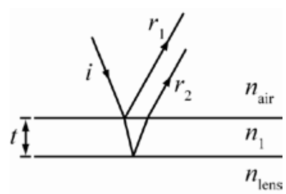
Since n_{air} is less than n_{oil}, there will be a phase change of $(1/2)\lambda$ and 180° in the light reflected by the air-oil interface. However, for the oil-water interface, there will be no phase change since $n_{oil} > n_{water}$. Therefore, in order to get a constructive interference, the path difference between two reflected light waves must be $\Delta x = \left(m + \frac{1}{2}\right)\frac{\lambda}{n_{oil}}$. Using $\Delta x = 2t$, it becomes $2t = \left(m + \frac{1}{2}\right)\frac{\lambda}{n_{oil}}$. The wavelength that satisfies this requirement is $\lambda = \frac{2tn_{oil}}{m + 1/2} = \frac{2(100.0 \text{ nm})(1.47)}{m + 1/2} = \frac{294 \text{ nm}}{m + 1/2}$. Since $m = 0, 1, 2...$, the only possible light within the given wavelength range that is reflected is for $m = 0$. Thus $\lambda = 2(294 \text{ nm}) = 588 \text{ nm}$.

34.33. Sketch the reflection, with incidence slightly away from normal, for ease of visualization.



The first interface (air-hafina) causes a phase change of $180°$ in the first reflected light wave (r_1). The second interface does not cause a phase change since $n_1 > n_2$. Therefore, to get a constructive interference in the reflected light, the path difference must be $\Delta x = 2t = \left(m + \frac{1}{2}\right)\frac{\lambda}{n_1}$. The minimum thickness of the thin film is when $m = 0$, that is, $t = \frac{1}{4}\frac{\lambda}{n_1} = \frac{1.06\ \mu\text{m}}{4(1.90)} = 0.139\ \mu\text{m} = 139\ \text{nm}$.

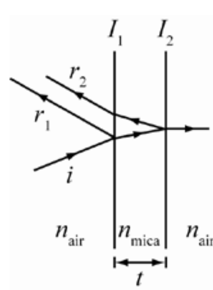
34.34.



A constructive interference is needed in the reflected light. There are two possible answers to this problem depending on the value of n_{lens}. If $n_1 > n_{lens}$, using similar reasoning as in problem 34.29, the minimum thickness is $t = \frac{1}{4}\frac{\lambda}{n_1} = \frac{800.0\ \text{nm}}{4(1.38)} = 145\ \text{nm}$. If $n_1 < n_{lens}$, there will be a phase change of $180°$ in the light reflected by the MgF_2-lens interface. Therefore, the condition for constructive interference is the path difference $\Delta x = 2t = m\lambda / n_1$. The minimum thickness $(m = 1)$ is $t = \frac{1}{2}\frac{\lambda}{n_1} = \frac{800.0\ \text{nm}}{2(1.38)} = 290.\ \text{nm}$. Since camera lenses (whether made of glass or another material) normally have a refractive index of 1.5 or greater, choose 290 nm as the final answer.

34.35. **THINK:** It is assumed that the refractive index of mica is independent of wavelength. In order to solve the problem, the condition for destructive interference of the reflected light is required. The film has thickness $t = 1.30\ \mu\text{m}$. The wavelengths of interest are 433.3 nm, 487.5 nm, 557.1 nm, 650.0 nm, and 780.0 nm.

SKETCH:



RESEARCH: Since $n_{air} < n_{mica}$, the light reflected by the first interface I_1 has a phase change of 180°. The light reflected by the second interface (I_2) has no phase change. The condition for destructive interference in the reflected light is

$$m\frac{\lambda_{\text{air}}}{n} = 2t \quad (m = 0, 1, 2, ...).$$

For two adjacent wavelengths with $\lambda_2 > \lambda_1$, $m_2 = m_1 - 1$. Therefore,

$$m_1 = \frac{2nt}{\lambda_1} \text{ and } m_1 - 1 = \frac{2nt}{\lambda_2}.$$

SIMPLIFY: Solving these two equations for the refractive index n gives:

$$\frac{2nt}{\lambda_1} = \frac{2nt}{\lambda_2} + 1 \Rightarrow 2nt\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) = 1 \Rightarrow n = \frac{\lambda_1 \lambda_2}{2t(\lambda_2 - \lambda_1)}.$$

CALCULATE: Choosing two adjacent wavelengths, $\lambda_1 = 433.3$ nm and $\lambda_2 = 487.5$ nm and substituting into the above equation yields $n = \frac{(433.3 \cdot 10^{-9}\text{ m})(487.5 \cdot 10^{-9}\text{ m})}{2(1.30 \cdot 10^{-6}\text{ m})((487.5 - 433.3) \cdot 10^{-9}\text{ m})} = 1.499.$

ROUND: To three significant figures, the refractive index of the mica is $n = 1.50$.

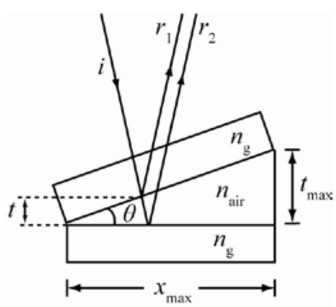
DOUBLE-CHECK: Choosing another two adjacent wavelengths, $\lambda_1 = 650.0$ nm and $\lambda_2 = 780.0$ nm, the refractive index is found to be

$$n = \frac{(650.0 \cdot 10^{-9}\text{ m})(780.0 \cdot 10^{-9}\text{ m})}{2(1.30 \cdot 10^{-6}\text{ m})((780.0 - 650.0) \cdot 10^{-9}\text{ m})} = 1.50.$$

This is in agreement with the previous result.

34.36. **THINK:** To determine the condition for a bright band (constructive interference), the phase shift at the interfaces and the path difference between the two exiting beams of light need to be determined. Since both Beam 1 and Beam 2 pass through the same thickness of glass, the refractive index of glass is not needed to solve the problem. This means that the location of the bright bands will be the same for any material.

SKETCH:



RESEARCH: Since n_g is larger than n_{air}, there is no phase change in the reflected light r_1. But for the reflected light r_2, there is a phase change of 180°. Therefore, the condition for constructive interference is,

$$2t = \left(m + \frac{1}{2}\right)\lambda_{\text{air}} \Rightarrow t = \frac{(2m+1)\lambda_{\text{air}}}{4} \quad (m = 0, 1, 2, ...).$$

This can be related to the location x of the bright fringes from the geometry of the set up. The air wedge has length $x_{max} = 8.00 \cdot 10^{-2}$ m and at this location it has thickness $t_{max} = 2.00 \cdot 10^{-5}$ m.

SIMPLIFY: If θ is the angle of the wedge:

$$\tan\theta = \frac{t_{max}}{x_{max}}.$$

In general, the location of bright fringes is:

$$x_{\text{bright}} = \frac{tx_{\max}}{t_{\max}} = \frac{(2m+1)\lambda_{\text{air}}x_{\max}}{4t_{\max}}.$$

The number of bright bands is found by setting $x_{\text{bright}} = x_{\max}$ and solving for m:

$$1 = \frac{(2m+1)\lambda_{\text{air}}}{4t_{\max}} \Rightarrow 2m+1 = \frac{4t_{\max}}{\lambda_{\text{air}}},$$

$$m = \frac{2t_{\max}}{\lambda_{\text{air}}} - \frac{1}{2}.$$

CALCULATE: The location of the bright bands as a function of m is:

$$x_{\text{bright}} = \frac{(633\cdot 10^{-9}\text{ m})(8.00\cdot 10^{-2}\text{ m})}{4(2.00\cdot 10^{-5}\text{ m})}(2m+1) = (2m+1)6.33\cdot 10^{-4}\text{ m}.$$

The number of bright bands is:

$$m = \frac{2(2.00\cdot 10^{-5}\text{ m})}{(633\cdot 10^{-9}\text{ m})} - \frac{1}{2} = 62.69.$$

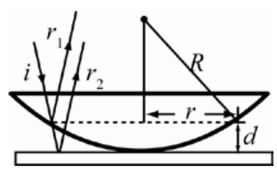
ROUND: To three significant figures, $x_{\text{bright}} = (2m+1)6.33\cdot 10^{-4}\text{ m}, (m = 0, 1, 2,...)$. The number of full bright bands is $m = 62$.

DOUBLE-CHECK: Setting $m = 62.69$ should give $x = 8.00$ cm:

$$x_{\text{bright}} = (2(62.69)+1)6.33\cdot 10^{-4}\text{ m} = 0.0800\text{ m} = 8.00\text{ cm, as required.}$$

34.37. **THINK:** The path length difference between the two beams and phase shifts at the interfaces need to be considered. For a plano-convex lens with focal length $f = 0.8000$ m and index of refraction $n_1 = 1.500$, the Lens-Maker's Formula can be used to determine the radius of curvature of the lens. The third bright circle is observed to have a radius of $r = 0.8487\cdot 10^{-3}$ m.

SKETCH:



RESEARCH: Since $n_1 > n_{\text{air,}}$ there is no phase change in the reflected beam r_1. However, there is a phase change of 180° for the beam reflected by the mirror. Because the path length difference between the two beams is $\Delta x = 2d$ and there is a phase change of 180° in one of the beams, the condition for constructive interference is $\Delta x = 2d = (m+1/2)\lambda$ with $m = 0,1,2...$. The Lens-Maker's Formula is given by:

$$\frac{1}{f} = (n_1 - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

SIMPLIFY: Using $d = R - \sqrt{R^2 - r^2}$, the wavelength is given by $\lambda = \dfrac{2\left(R - \sqrt{R^2 - r^2}\right)}{m+1/2}$. If $R >> r$, $\sqrt{R^2 - r^2}$ can be approximated by,

$$\sqrt{R^2 - r^2} = R\left(1 - \frac{r^2}{R^2}\right)^{1/2} \approx R - \frac{1}{2}\frac{r^2}{R}.$$

Therefore, the wavelength simplifies to

$$\lambda = \frac{r^2}{(m+1/2)R}.$$

An expression for R can be found by using the Lens-Maker's Formula using $R_1 = R$ for the radius of curvature of the bottom surface of the lens and $R_2 \to \infty$ for the plane surface:

$$\frac{1}{f} = (n_1 - 1)\left(\frac{1}{R} - \frac{1}{\infty}\right) = \frac{n_1 - 1}{R} \Rightarrow R = f(n_1 - 1).$$

The wavelength of light is therefore:

$$\lambda = \frac{r^2}{f(m+1/2)(n_1 - 1)}.$$

CALCULATE: Substituting $m = 2$ for the third bright circle ($m = 0$ corresponds to the first) yields:

$$\lambda = \frac{(0.8487 \cdot 10^{-3}\text{ m})^2}{(0.8000\text{ m})(2+1/2)(1.500-1)} = 720.29\text{ nm}.$$

ROUND: Rounding the answer to four significant figures gives $\lambda = 720.3$ nm.
DOUBLE-CHECK: This is within the range of wavelengths of visible light.

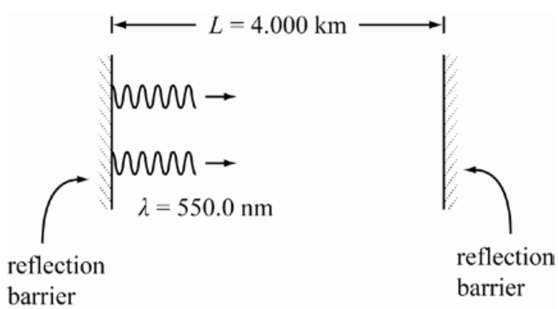
34.38. In a wavelength meter, the number of counted fringes corresponds to the number of wavelengths in the path difference. Since the path difference is $\Delta x = 2\Delta d$, the number of fringes is $\Delta N = \Delta x / \lambda = 2\Delta d / \lambda$. Therefore, the number of fringes for two wavelengths are $\Delta N_1 = 2\Delta d / \lambda_1$ and $\Delta N_2 = 2\Delta d / \lambda_2$.

(a) Taking a ratio of ΔN_1 and ΔN_2 gives $\Delta N_1 / \Delta N_2 = \lambda_2 / \lambda_1$. If λ_1 is a known wavelength, then the unknown wavelength is $\lambda_2 = \frac{\Delta N_1}{\Delta N_2}\lambda_1 = \frac{6.000 \cdot 10^4}{7.780 \cdot 10^4}(632.8\text{ nm}) = 488.0\text{ nm}.$

(b) The displacement, Δd, is $\Delta d = \frac{\Delta N_1 \lambda_1}{2} = \frac{6.000 \cdot 10^4 (632.8\text{ nm})}{2} = 0.01898\text{ m} \approx 18.98\text{ mm}.$

34.39. The number of fringes is given by the ratio of the path difference and the wavelength, that is, $N = \Delta x / \lambda = 2d / \lambda = 2(0.381 \cdot 10^{-3}\text{ m}) / 449 \cdot 10^{-9}\text{ m} = 1697 \approx 17.0 \cdot 10^2.$

34.40. **THINK:** The phase difference of two light beams is given by $\theta = 2\pi\Delta x / \lambda$ where Δx is the path difference between the two beams and $\lambda = 550.0 \cdot 10^{-9}$ m is the wavelength of each beam.
SKETCH:



RESEARCH: If the number of round trips is $N = 100$ and the length of the interferometer arm is denoted by $L = 4000.$ m then the total distance traveled by each beam is $L_{\text{total}} = 2NL.$

SIMPLIFY: If there is a decrease in the length of one path and an increase in the length of the other path due to gravitational waves, each by a fractional change of $\delta = 1.000 \cdot 10^{-21}$, then the net fractional change is 2δ. Therefore, the difference in path length between the two beams is $\Delta x = 4\delta NL.$

The phase difference is

$$\theta = \frac{2\pi(4\delta NL)}{\lambda} = \frac{8\pi\delta NL}{\lambda}.$$

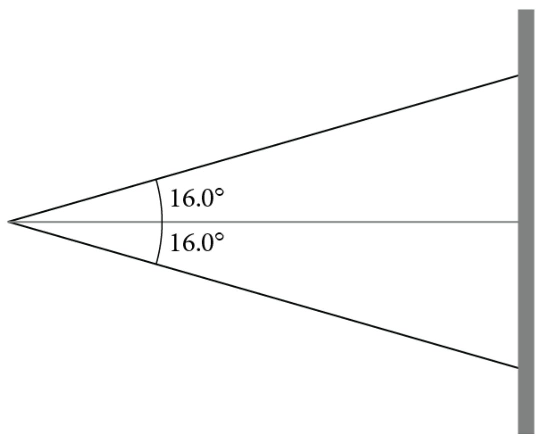
CALCULATE: Substituting in the numerical values yields

$$\theta = \frac{8\pi(1.000\cdot 10^{-21})(100)(4000.\text{ m})}{550.0\cdot 10^{-9}\text{ m}} = 1.8278\cdot 10^{-8}\text{ rad.}$$

ROUND: The value of $N = 100$ can be taken as an exact number. Rounding the answer to four significant figures gives $\theta = 1.828\cdot 10^{-8}$ rad.

DOUBLE-CHECK: A very small phase change is expected since the effect that gravitational waves have on the path length of light is always neglected.

34.41.



The first minima on either side of the central maximum are described by

$a\sin\theta = (1)\lambda,$ where $\theta = 32.0°/2 = 16.0°.$

The width of the slit is given by $a = \dfrac{(1)(653\text{ nm})}{\sin 16.0°} = 2370\text{ nm}.$

34.42. The width of the central maximum is given by: $w = 2\lambda L/a$ from problem 34.1.

$$L = \frac{wa}{2\lambda} = \frac{(0.0500\text{ m})(0.135\cdot 10^{-3}\text{ m})}{(2)(633\cdot 10^{-9}\text{ m})} = 5.33\text{ m}$$

34.43. The minima of a single slit width are given by: $a\sin\theta = m\lambda.$ The first minimum corresponds to $m = 1,$ $a\sin\theta = \lambda.$ Minima do not appear for $\theta = 90°$ or larger angles. Solving for a gives: $a = \lambda/\sin\theta \Rightarrow a = \lambda = 600.\text{ nm}.$ If a is any larger θ would be less than $90°,$ since $\sin\theta = \lambda/a.$

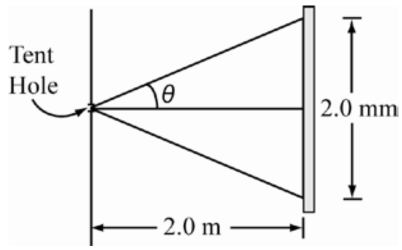
34.44. The dark fringes of a single slit are given by: $a\sin\theta = m\lambda.$ The second dark fringe corresponds to $m = 2,$

$$a\sin\theta = 2\lambda \Rightarrow \lambda = \frac{a\sin\theta}{2} = \frac{(0.0200\text{ m})\sin 43.0°}{2} = 0.682\cdot 10^{-2}\text{ m} = 0.682\text{ cm}.$$

34.45. Using Rayleigh's Criterion, the minimum angular resolution for green light is:

$$\theta_R = \sin^{-1}\left(\frac{1.22\lambda}{d}\right) = \sin^{-1}\left(\frac{1.22(550\cdot 10^{-9}\text{ m})}{14.4\text{ m}}\right) = 2.7\cdot 10^{-6}\text{ degrees.}$$

34.46. The first diffraction minimum is given by: $\sin\theta = 1.22\lambda / d$.



The angle θ is then given by $\tan^{-1}\left(\frac{2.0 \text{ mm}}{2}\frac{1}{2.0 \text{ m}}\right) = 0.0286°$ and $d = \frac{1.22\lambda}{\sin\theta} = \frac{1.22(570 \cdot 10^{-9} \text{ m})}{\sin(0.0286°)}$ where λ is taken to be 570 nm, the average wavelength of sunlight. $d = 1.39 \text{ mm} \approx 1.4 \text{ mm}$.

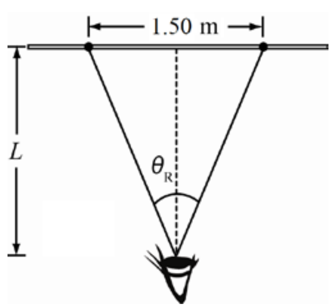
34.47. The angular resolution is given by Rayleigh's Criterion $\theta_R = \sin^{-1}(1.22\lambda / d)$. For the Hubble Space Telescope the value is $\theta_R = \sin^{-1}\left(1.22(450. \cdot 10^{-9} \text{ m}) / 2.40 \text{ m}\right) = 1.31 \cdot 10^{-5}$ degrees. For the Keck Telescope the value is $\theta_R = \sin^{-1}\left(1.22(450. \cdot 10^{-9} \text{ m}) / 10.0 \text{ m}\right) = 3.15 \cdot 10^{-6}$ degrees. For the Arecibo radio telescope, the value is $\theta_R = \sin^{-1}\left(1.22(0.210 \text{ m}) / 305 \text{ m}\right) = 0.0481$ degrees. The radio telescope is clearly worse than the other telescope in terms of angular resolution. The Keck Telescope is better than the Hubble Space Telescope due to its larger diameter.

34.48. Angular resolution is given by the Rayleigh Criterion $\sin\theta_R = 1.22\lambda / d$, which is $\theta_R = 1.22\lambda / d$ using the small angle approximation. Therefore $d = \frac{(1.22)(0.100 \text{ m})}{2.80 \cdot 10^{-7} \text{ radians}} = 4.357 \cdot 10^5 \text{ m} \approx 436 \text{ km}.$

34.49. (a) Rayleigh's Criterion is given by:

$$\theta_R = \sin^{-1}\left(\frac{1.22\lambda}{d}\right) = \sin^{-1}\left(\frac{1.22(550. \cdot 10^{-9} \text{ m})}{0.00500 \text{ m}}\right) = 0.007689° \approx 7.69 \cdot 10^{-3} \text{ degrees.}$$

(b)



From the diagram the distance is given by $L = \frac{1.50 \text{ m}}{2}\frac{1}{\tan\left(7.70 \cdot 10^{-3} \,°/2\right)} = 11{,}177 \text{ m} \approx 11.2 \text{ km}.$

34.50. For the first dark fringe due to double slit interference:

$$d\sin\theta = (m + 1/2)\lambda \;\Rightarrow\; d(y/L) = (m + 1/2)\lambda.$$

The width of the central maximum is twice y, so $w = 2y$. Using $m = 0$,

$$d = \frac{L\lambda}{2y} = \frac{L\lambda}{w} = \frac{(1.60 \text{ m})(635 \cdot 10^{-9} \text{ m})}{(0.0420 \text{ m})} = 2.42 \cdot 10^{-5} \text{ m.}$$

The missing bright fringe is due to single slit destructive interference, with $m=1$. The size of the individual slits is

$$a=\frac{L\lambda}{y'}, \text{ where } y'=\frac{m\lambda L}{d}$$

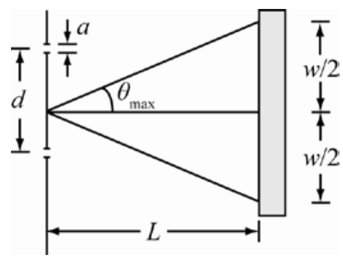
with $m=4$ for the forth bright spot due to double slit interference. Therefore,

$$a=\frac{d}{4}=\frac{\left(2.42\cdot 10^{-5}\text{ m}\right)}{4}=6.05\cdot 10^{-6}\text{ m}$$

The slit separation is 4 times the slit width causing the fourth double slit maximum to be missing due to single slit interference.

34.51. **THINK:** Light of wavelength $\lambda=600.\text{ nm}$ illuminates two slits. The slits are separated by a distance $d=24\mu\text{m}$ and the width of each slit is $a=7.2\mu\text{m}$. A screen $w=1.8\text{ m}$ wide is $L=2.0\text{ m}$ from the slits. The problem can be approached by determining the number of fringes that appear due to the double slit and eliminate those removed by the minima due to single-slit diffraction.

SKETCH:



RESEARCH: The maximum angle θ_{max} is given by $\tan\theta_{\text{max}}=w/2L.$ The bright fringes occur when $\sin\theta=m\lambda/d.$ The disallowed fringes occur when $\sin\theta=n\lambda/a.$

SIMPLIFY: The maximum number of bright fringes that can appear on the screen is

$$m_{\text{max}}=\frac{d\sin\theta_{\text{max}}}{\lambda}=\frac{d\sin\left(\tan^{-1}\left(\frac{w}{2L}\right)\right)}{\lambda}.$$

The disallowed fringes occur when

$$\frac{m\lambda}{d}=\frac{n\lambda}{a}\Rightarrow\frac{m}{n}=\frac{d}{a}.$$

CALCULATE: The number of bright fringes is:

$$m_{\text{max}}=\frac{d\sin\theta_{\text{max}}}{\lambda}=\frac{\left(24\text{ μm}\right)\sin\left(\tan^{-1}\left(\frac{1.8\text{ m}}{2\left(2.0\text{ m}\right)}\right)\right)}{\left(600.\text{ nm}\right)}=16.4.$$

The disallowed fringes occur when:

$$\frac{m}{n}=\frac{d}{a}=\frac{24\mu\text{m}}{7.2\mu\text{m}}=\frac{10}{3}.$$

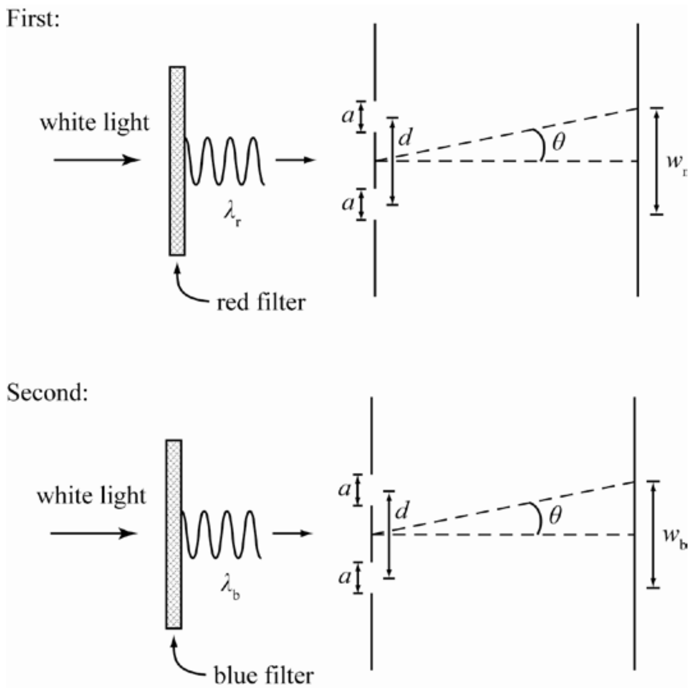
The only scenario this can occur for (since $m_{\text{max}}=16$) is $m=10$ and $n=3$. Therefore, the only disallowed value of m is 10, so there are 15 bright fringes on either side of the central maximum.

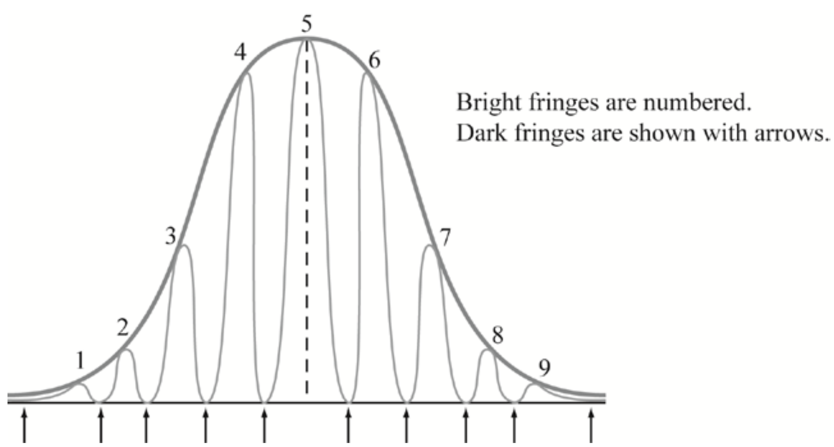
ROUND: To the nearest integer, there are 31 fringes on the screen.

DOUBLE-CHECK: Without the effects from single-slit diffraction there would be 33. It is expected that there would be fewer fringes due to the effects of single-slit diffraction.

34.52. **THINK:** Equations for the angular positions of the dark fringes due to single-slit and double-slit diffraction can be used to determine a relation between the slit width a and the slit separation d. Then this can be used to find the number of fringes present with the blue filter. The equation for the width of the central diffraction maximum is required to find the new width using blue light.

SKETCH:





RESEARCH: For the red light, the nine interference maxima correspond to four bright fringes (and five dark fringes) on either side of the central diffraction maximum. The angular positions of the dark fringes due to single-slit diffraction are given by:

$$\sin\theta = \frac{m\lambda}{a} \quad (m = 1, 2, 3, \ldots).$$

The angular positions of the dark fringes due to double-slit diffraction are given by:

$$\sin\theta = \left(m + \frac{1}{2}\right)\frac{\lambda}{d} \quad (m = 0, 1, 2, \ldots).$$

The width of the central diffraction maximum is given by:

$$w = \frac{2m\lambda L}{a} \quad (m = 1, 2, 3, \ldots).$$

SIMPLIFY: The angular position of the first $(m=1)$ dark fringe due to single-slit diffraction is equal to the angular position of the fifth $(m=4)$ dark fringe due to double-slit diffraction, so

$$\frac{\lambda_r}{a} = \left(4 + \frac{1}{2}\right)\frac{\lambda_r}{d} \Rightarrow d = \frac{9a}{2}.$$

Since the slit width stays constant:

$$a = \frac{2m\lambda_r L}{w_r} = \frac{2m\lambda_b L}{w_b} \Rightarrow w_b = \frac{\lambda_b}{\lambda_r} w_r.$$

CALCULATE: For blue light, the angular position of the first $(m=1)$ dark fringe due to single-slit diffraction is equal to the angular position of the fifth $(m=4)$ dark fringe due to double-slit diffraction, so

$$\frac{\lambda_b}{a} = \left(m + \frac{1}{2}\right)\frac{\lambda_b}{d} \Rightarrow \frac{\lambda_b}{a} = \left(m + \frac{1}{2}\right)\frac{\lambda_b}{9a/2} \Rightarrow m + \frac{1}{2} = \frac{9}{2} \Rightarrow m = 4$$

Therefore, the number of fringes is independent of wavelength. There will still be nine bright fringes. The width of the central diffraction maximum for blue light is:

$$w_b = \frac{(450 \text{ nm})}{(670 \text{ nm})}(4.50 \text{ cm}) = 3.02 \text{ cm}$$

ROUND: To two significant figures, $w_b = 3.0$ cm.

DOUBLE-CHECK: It is reasonable that the width of the central diffraction maximum will decrease slightly for the blue light.

34.53. (a) The first minimum on either side of the central maximum is given by: $a\sin\theta = m\lambda$, $m=1$, $\sin\theta \approx \theta$ for small angles $a\theta = \lambda$. From the graph $\theta \approx 0.1$, $a = \lambda / 0.1 = 10\lambda$.

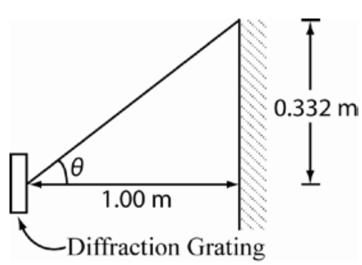
(b) Note that the m^{th} interference maxima for a double slit setup is given by: $d\sin\theta = m\lambda$, $\theta \approx \sin\theta$ for small angles $d\theta = m\lambda$. From 0 to 0.1 radians there are 10 interference maxima $d = m\lambda / \theta = 10\lambda / 0.1 = 100\lambda$.

(c) $a / d = 10\lambda / 100\lambda = 1/10$ so the ratio is 1:10.

(d) Without λ, there is insufficient information to find a or d.

34.54. Constructive interference of a grating is given by $m\lambda = d\sin\theta$ we have $3\lambda_{\text{unknown}} = d\sin\theta = 2(600.\text{ nm}) \Rightarrow \lambda_{\text{unknown}} = (2/3)600.\text{ nm} = 400.\text{ nm}$.

34.55.

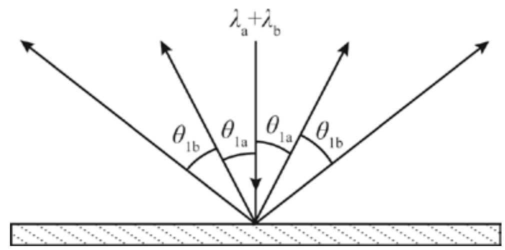


From the above diagram, $\tan\theta = \dfrac{0.332 \text{ m}}{1.00 \text{ m}}$. For a diffraction grating with $m=1$, the wavelength of light is

$$\lambda = d\sin\theta \Rightarrow \lambda = \frac{1}{(7.02 \cdot 10^5 / \text{m})}\sin\left(\tan^{-1}\left(\frac{0.332 \text{ m}}{1.00 \text{ m}}\right)\right) = 4.49 \cdot 10^{-7} \text{ m} = 449 \text{ nm}.$$

34.56. **THINK:** A diffraction grating with width $a = 5.000 \cdot 10^{-2}$ m and $N = 200$ grooves is used to resolve two beams of wavelength $\lambda_a = 629.8$ nm and $\lambda_b = 630.2$ nm. The condition for constructive interference of the grating is required to determine the angular position of the beams.

SKETCH:



RESEARCH: The expression for the angle of constructive interference from a diffraction grating is $d\sin\theta = m\lambda$. For first-order diffracted beams use $m = 1$. Resolving power is given by: $R = \dfrac{\lambda}{\Delta\lambda} = Nm$, where λ is the average wavelength.

SIMPLIFY: The spacing of the gratings is $d = a / N$. The angle of the first-order diffraction peak is

$$\theta = \sin^{-1}\left(\frac{\lambda}{d}\right) = \sin^{-1}\left(\frac{N\lambda}{a}\right).$$

The order of diffraction required to resolve the two lines is $m = \dfrac{\lambda}{\Delta\lambda}\dfrac{1}{N} = \dfrac{\lambda}{\lambda_b - \lambda_a}\dfrac{1}{N}$.

CALCULATE: $\theta_{1a} = \sin^{-1}\left(\dfrac{(200)(629.8 \cdot 10^{-9}\text{ m})}{(5.000 \cdot 10^{-2}\text{ m})}\right) = 0.144340°$

$$\theta_{1b} = \sin^{-1}\left(\frac{(200)(630.2 \cdot 10^{-9}\text{ m})}{(5.000 \cdot 10^{-2}\text{ m})}\right) = 0.144431°$$

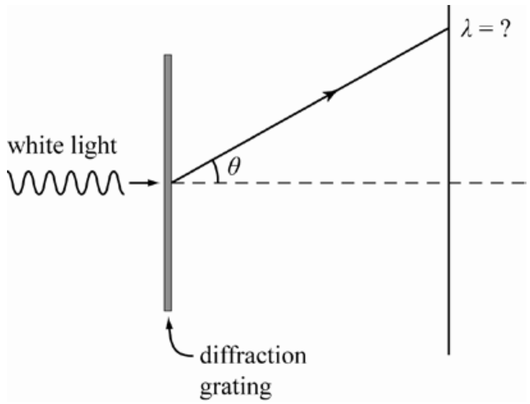
$$m = \frac{(630.0\text{ nm})}{(630.2\text{ nm} - 629.8\text{ nm})}\left(\frac{1}{200}\right) = 7.875$$

ROUND: Taking $N = 200$ to be an exact number, the angles should be rounded to four significant figures: $\theta_{1a} = 0.1443°$ and $\theta_{1b} = 0.1444°$. Since the order of diffraction must be an integer, rounding up is appropriate: $m = 8$, or the eighth-order diffracted beams.

DOUBLE-CHECK: It is reasonable that the angles of the first-order diffracted beams are very close since their wavelengths are very close. The high order of m is necessary due to the closeness of two spectral lines and is to be expected.

34.57. **THINK:** The condition for constructive interference for the grating is required. For each order of diffraction m, compute the wavelengths λ that fall into the range of visible light. The question gives the range for white light as the interval (400. nm - 700. nm). Wavelength is inversely proportional to *m,* and hence, an interval of allowable values for wavelength must correspond to an interval of allowable values for *m*. It is sufficient to find the least value of *m* for which the wavelength is in the interval, and then to increment *m* until the wavelength falls outside the given interval. Use the known values of $\theta = 45.0°$ and $d = \left(4.00 \cdot 10^{5}\text{ m}^{-1}\right)^{-1}$.

SKETCH:



RESEARCH: For constructive interference, $d\sin\theta = m\lambda.$
SIMPLIFY: The wavelength is given by

$$\lambda = \frac{d\sin\theta}{m}.$$

CALCULATE: Set m = 1, 2, 3, …,:

$$\lambda = \frac{\left(4.00\cdot 10^5 \text{ m}^{-1}\right)^{-1}\sin\left(45.0°\right)}{m}$$

$$m=1 \Rightarrow \lambda = 1767.8 \text{ nm}$$
$$m=2 \Rightarrow \lambda = 883.9 \text{ nm}$$
$$\left.\begin{array}{l} m=3 \Rightarrow \lambda = 589.3 \text{ nm} \\ m=4 \Rightarrow \lambda = 441.9 \text{ nm} \end{array}\right] \text{visible}$$
$$m=5 \Rightarrow \lambda = 353.6 \text{ nm}$$

ROUND: To three significant figures, the wavelengths that will be visible are 589 nm and 442 nm.
DOUBLE-CHECK: Constructive interference occurs at integral multiples of the wavelength. As m increases, λ decreases, so there will be no values of λ more than 400 when m is greater than 5.

34.58. Bragg's Law is given by $2a\sin\theta = m\lambda.$ First order implies:

$$m=1,\ \lambda = 2a\sin\theta / m = 2a\sin\theta = \left(0.256 \text{ nm}\right)2\sin\left(23.0°\right) = 0.200 \text{ nm}.$$

34.59. The number of lines per centimeter is related to the slit separation d: $d\sin\theta = mx.$ No second order spectrum occurs if for the smallest wavelength $\theta = 90°,$

$$d\sin 90° = 2\left(400. \text{ nm}\right) \Rightarrow d = 800. \text{ nm} \Rightarrow 1/d = 1.25\cdot 10^6 \text{ lines/m} = 1.25\cdot 10^4 \text{ lines/cm}.$$

34.60. This is similar to two slit interference where destructive interference is desired along the $45°$ line. $d\sin\theta = \left(m+1/2\right)\lambda$ for destructive interference. It is important to note that θ here is the angle to the bisector of the line joining the antennas. θ in this case is $\theta = 90.0° - 45.0° = 45.0°.$ Also $\lambda = c/f,$

$$d = \left(m+\frac{1}{2}\right)\left(\frac{c}{f}\right)\frac{1}{\sin\theta} = \frac{1}{2}\left(\frac{c}{f}\right)\frac{1}{\sin\theta} = \frac{1}{2}\left(\frac{3.00\cdot 10^8 \text{ m/s}}{88.1\cdot 10^6 \text{ / sec}}\right)\frac{1}{\sin 45.0°} = 2.41 \text{ m}.$$

34.61. The width of the central maximum is given by twice the distance of the first minima. $y/L = 1.22\lambda/d$ for the first diffraction minimum, where d is the diameter of the aperture, so $y = L\left(1.22\right)\lambda/d,$

$$2y = \frac{2L\lambda}{d}\left(1.22\right) = \text{width of central maximum } w. \text{ Therefore,}$$

$$d = \frac{2\left(1.22\right)L\lambda}{w} = \frac{2\left(1.22\right)\left(384\cdot 10^6 \text{ m}\right)\left(633\cdot 10^{-9} \text{ m}\right)}{1.00\cdot 10^3 \text{ m}} = 0.593 \text{ m}.$$

34.62. (a) The maximum occurs for $\theta = 90°$, $d\sin\theta = m\lambda \Rightarrow d\sin 90° = m\lambda$,

$$\frac{d}{\lambda} = m = \frac{1}{1000\text{ / cm}}\frac{1}{633\text{ nm}} = 15.79.\quad \text{The maximum is } m = 15.$$

(b) For 10000 / cm, $m = \dfrac{1}{10000\text{ / cm}}\dfrac{1}{633\text{ nm}} = 1.579.$ The maximum is $m = 1.$

34.63. The distance moved in an interferometer is given by $2d = N\lambda_{\text{water}}$,

$$n = 1.33 = \frac{c}{v_{\text{water}}} = \frac{c}{f_{\text{water}}\lambda_{\text{water}}} = \frac{f_{\text{air}}\lambda_{\text{air}}}{f_{\text{water}}\lambda_{\text{water}}},$$

since $f_{\text{air}} = f_{\text{water}}$, $\lambda_{\text{water}} = \lambda_{\text{air}} / n.$

$$2d = \frac{N\lambda_{\text{air}}}{n} \Rightarrow \lambda_{\text{air}} = \frac{2nd}{N} = \frac{2(1.33)(0.200\cdot 10^{-3}\text{ m})}{800} = 6.65\cdot 10^{-7}\text{ m} = 665\text{ nm}.$$

34.64. Destructive interference is given by $\dfrac{(m+1/2)\lambda_{\text{air}}}{n_{\text{coating}}} = 2t$ for $m = 0$ corresponding to minimum thickness

$$t = \frac{1}{4}\frac{\lambda_{\text{air}}}{n_{\text{coating}}} = \frac{1}{4}\frac{405\text{ nm}}{1.58} = 64.1\text{ nm}.$$

For CD illuminated with infrared light of wavelength 750 nm, $t = \dfrac{1}{4}\dfrac{\lambda_{\text{air}}}{n_{\text{coating}}} = \dfrac{1}{4}\dfrac{750\text{ nm}}{1.58} = 119\text{ nm}$, almost double the thickness.

34.65. It is assumed that the refractive index of the material that the body of the airplane is made from is greater than that of the polymer coating. For this case, there will be a phase change at both interfaces of the coating, so the condition for destructive interference is given by

$$\left(m+\frac{1}{2}\right)\frac{\lambda_{\text{air}}}{n} = 2t \Rightarrow \lambda_{\text{air}} = 2\left(m+\frac{1}{2}\right)^{-1} tn.$$

The maximum wavelength for which the plane is invisible occurs for $m = 0$, $\lambda_{\text{air, max}} = 4tn = 4(5.00\text{ mm})(1.50) = 30.0\text{ mm}.$ It makes sense to consider the maximum wavelength.

34.66. The bright spot from a double slit source is given by: $y = m\lambda L / d.$ So the distance between two consecutive bright spots is given by:

$$y_{\text{mt}} - y_{\text{m}} = \frac{\lambda L}{d} = 6.00\text{ cm} \Rightarrow \lambda = \frac{d}{L}6.00\text{ cm} = \frac{2.00\cdot 10^{-5}\text{ m}}{2.40\text{ m}}6.00\cdot 10^{-2}\text{ m} = 5.00\cdot 10^{-7}\text{ m} = 500.\text{ nm}.$$

34.67. Constructive interference for a thin film is given by $\dfrac{(m+1/2)\lambda_{\text{air}}}{n} = 2t.$ For the minimum thickness,

$m = 0:\ t = \dfrac{1}{4}\dfrac{\lambda_{\text{air}}}{n_{\text{coating}}} = \dfrac{1}{4}\dfrac{550.\text{ nm}}{1.32} = 104\text{ nm}.$

34.68. The angle of deflection is given by: $m\lambda = d\sin\theta$, $\sin\theta = y / L$ with the small angle approximation for $m = 1$, $\lambda = dy / L.$ The wavelengths to be resolved are 588.995 nm and $589.5924\text{ nm}.$

$$\Delta\lambda = \frac{d\Delta y}{L} \Rightarrow \frac{\Delta\lambda}{\Delta y}L = d = \frac{589.5924\text{ nm} - 588.9950\text{ nm}}{2.00\text{ mm}}(80.0\text{ cm}) = 238.96\text{ nm}$$

So the number of lines is given by $N = \dfrac{1.50\text{ cm}}{238.96\text{ nm}} = 62{,}800.$

34.69. The distance moved is related to the wavelength by:

$$N\lambda = 2d \;\Rightarrow\; N = \frac{2d}{\lambda} = \frac{2\left(200.\cdot 10^{-6}\text{ m}\right)}{600.\cdot 10^{-9}\text{ m}} = 666.7 \approx 667 \text{ fringes.}$$

34.70. The Rayleigh criterion is given by:

$$\theta_{\text{R}} = \sin^{-1}\left(\frac{1.22\lambda}{d}\right) = \sin^{-1}\left(\frac{(1.22)(400\text{ nm})}{3.5\text{ mm}}\right) = 7.99\cdot 10^{-3}\text{ degrees} = 1.39\cdot 10^{-4}\text{ rad}$$

For small angles $\theta_{\text{R}} \approx \tan\theta = \Delta y / L$ where Δy is the smallest object separation able to be resolved. Since Δy is to be as small as possible, L is chosen to be the near point: $\Delta y = \left(1.39\cdot 10^{-4}\text{ rad}\right)\left(25\cdot 10^{-2}\text{ m}\right) = 3.5\cdot 10^{-5}\text{ m.}$

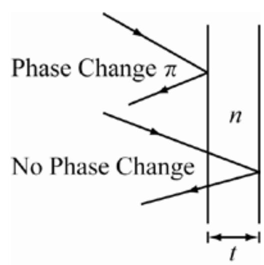
34.71. The Rayleigh criterion is given by:

$$\sin\theta_{\text{R}} = \frac{1.22\lambda}{d} \;\Rightarrow\; \frac{\Delta y}{L} = \frac{1.22\lambda}{d},$$

where $L = 384{,}000\text{ km}$ is the distance to the Moon.

$$\Delta y = \frac{1.22\lambda}{d}L = \frac{1.22\left(550.\cdot 10^{-9}\text{ m}\right)}{12.0\cdot 10^{-2}\text{ m}}\left(384\cdot 10^{6}\text{ m}\right) = 2147.2\text{ m} \approx 2.15\text{ km}$$

34.72.



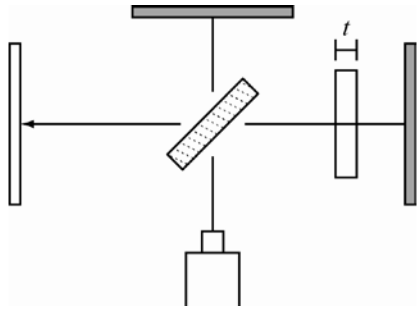
The angles are exaggerated. The first wave has a phase change of π. The second has a path difference of $2t$ and a phase change of $\frac{2nt}{\lambda}\pi$. The factor of n accounts for the difference of wavelength in air and in the soap bubble. $2tn - \frac{1}{2}\lambda = \left(m + \frac{1}{2}\right)\lambda,\; t = m\lambda / 2n$ for $m = 1,\; t = \frac{\lambda}{2n} = \frac{500.\text{ nm}}{2(1.420)} = 176\text{ nm.}$

34.73. The Rayleigh Criterion is given by: $\sin\theta_{\text{R}} = \frac{1.22\lambda}{d} \;\Rightarrow\; \frac{\Delta y}{L} = \frac{1.22\lambda}{d}$ with the small angle approximation where Δy is the minimum separation distance.

$$\Delta y = \frac{L}{d}1.22\lambda = \frac{100.\text{ mm}}{1.00\text{ mm}}1.22\left(1.00\text{ nm}\right) = 122\text{ nm}$$

34.74. **THINK:** The Michelson interferometer uses a light source with a wavelength of $\lambda_{\text{air}} = 600.\text{ nm}$ to measure the thickness t of a piece of glass with refractive index $n = 1.50$. Upon insertion of the glass, the fringe pattern shifts by $\Delta N = 1000$ fringes. The presence of the glass causes a change in number of wavelengths travelled by the light, which is equal to the number of fringes that the pattern is shifted by.

SKETCH:



RESEARCH: The number of wavelengths travelled by the light in a distance L is given by $N = L/\lambda$. The index of refraction of the glass can be expressed in terms of the speed of the light in air and glass:

$$n = \frac{c}{v} = \frac{f\lambda_{\text{air}}}{f\lambda_{\text{glass}}},$$

The wavelength of the light in the glass is $\lambda_{\text{glass}} = \lambda_{\text{air}} / n$.

SIMPLIFY: A factor two is needed to account for the light going through the section of air and glass twice:

$$\Delta N = 2\left(N_{\text{glass}} - N_{\text{air}}\right) = 2\left(\frac{L}{\lambda_{\text{glass}}} - \frac{L}{\lambda_{\text{air}}}\right) = 2\left(\frac{L}{\left(\lambda_{\text{air}}/n\right)} - \frac{L}{\lambda_{\text{air}}}\right) = \frac{2L}{\lambda_{\text{air}}}(n-1).$$

$$L = \frac{\lambda_{\text{air}}\Delta N}{2(n-1)}$$

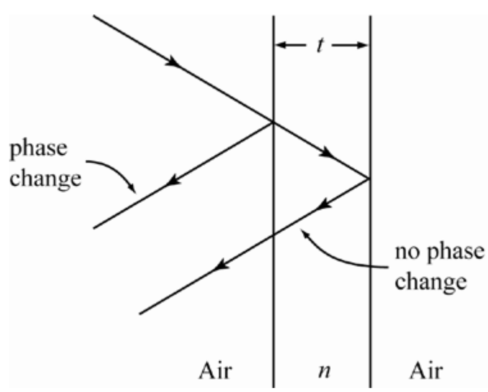
CALCULATE: $L = \dfrac{(600.\text{ nm})(1000.)}{2(1.50-1)} = 6.00 \cdot 10^{-4}\text{ m}$

ROUND: To three significant figures, $L = 600.\ \mu\text{m}$.

DOUBLE-CHECK: The final expression indicates that the width of the glass is proportional to the increase in the number of fringes which is reasonable, since as the glass gets thicker we expect the phase change to be larger.

34.75. **THINK:** Upon reflection, light undergoes a phase change of half a wavelength at the first interface, but not at the second interface. Since maxima are seen for two adjacent wavelengths, the layer thickness can be found by using the conditions for constructive interference.

SKETCH:



RESEARCH: Since $n_{\text{air}} < n_{\text{mica}}$, the light reflected by the first interface has a phase change of 180°. The light reflected by the second interface has no phase change. The condition for constructive interference in the reflected light is

$$\left(m + \frac{1}{2}\right)\frac{\lambda_{\text{air}}}{n} = 2t \quad (m = 0, 1, 2, \ldots).$$

For two adjacent wavelengths with $\lambda_2 > \lambda_1$, $m_2 = m_1 - 1$. Therefore,

$$m_1 = \frac{2nt}{\lambda_1} - \frac{1}{2} \text{ and } m_2 = m_1 - 1 = \frac{2nt}{\lambda_2} - \frac{1}{2}.$$

SIMPLIFY: Solving these two equations for the thickness t gives:

$$\frac{2nt}{\lambda_1} - \frac{1}{2} = \frac{2nt}{\lambda_2} + \frac{1}{2} \Rightarrow 2nt\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) = 1 \Rightarrow t = \frac{\lambda_1 \lambda_2}{2n(\lambda_2 - \lambda_1)}.$$

CALCULATE: $t = \frac{(516.9 \text{ nm})(610.9 \text{ nm})}{2(1.57)(610.9 \text{ nm} - 516.9 \text{ nm})} = 1070.8 \text{ nm} = 1.07 \ \mu\text{m}.$

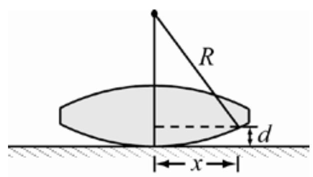
ROUND: To three significant figures, the thickness of the mica layer is $t = 1.07 \ \mu\text{m}$.

DOUBLE-CHECK: As expected, the layer thickness is much larger than the observed wavelengths.

34.76. **THINK:** Show (a) $x^2 = 2Rd$ and (b) $x_n = \left[\left(n + \frac{1}{2}\right)\lambda R\right]^{1/2}$ for Newton's ring apparatus. (c) For $R = 10.0$ m, and a plane glass disk of diameter $D = 5.00$ cm, with light of wavelength $\lambda = 700.$ nm, find the number of bright fringes observed. Note that maximum radial distance $x_{\text{max}} = D/2 = 0.0250$ m.

SKETCH:

(a)



RESEARCH:

(a) x, R and d are related by the Pythagorean Theorem, $x^2 + (R-d)^2 = R^2$.

(b) Since there is a phase shift from a reflected light from the plane glass disk, it needs an additional phase shift by an angle of π (half wavelength) due to path difference. The condition for constructive interference (bright fringes) is $2d = (n + 1/2)\lambda$, $(n = 0,1,2...)$.

(c) Use the result from (b) to find the number of fringes.

SIMPLIFY:

(a) $x^2 + (R-d)^2 = R^2 \Rightarrow x^2 + R^2 - 2Rd + d^2 = R^2$, neglecting the d^2 term, which is very small, gives the equation, $x^2 = 2Rd$.

(b) $2d = \left(n + \frac{1}{2}\right)\lambda$, but since $x^2 = 2Rd \Rightarrow d = \frac{x^2}{2R}$:

$$2\left(\frac{x^2}{2R}\right) = \left(n + \frac{1}{2}\right)\lambda \Rightarrow x^2 = \left(n + \frac{1}{2}\right)\lambda R \Rightarrow x = \left[\left(n + \frac{1}{2}\right)\lambda R\right]^{1/2}.$$

(c) Solving for n from the result of (b): $(x_{\text{max}})^2 = \left(n + \frac{1}{2}\right)\lambda R \Rightarrow n = \frac{(x_{\text{max}})^2}{\lambda R} - \frac{1}{2}.$

CALCULATE:

(c) $n = \frac{(0.0250 \text{ m})^2}{(700.\cdot 10^{-9} \text{ m})(10.0 \text{ m})} - \frac{1}{2} = 88.8.$ The outermost visible fringe corresponds to n = 88. Since the innermost bright fringe corresponds to n = 0, there are 89 bright fringes.

ROUND:

(c) 89 bright fringes.

DOUBLE-CHECK: In parts (a) and (b), the appropriate equations have been derived. In part (c), the quantity found is unitless, as would be expected.

Multi-Version Exercises

34.77. $\Delta y = \dfrac{\lambda L \Delta m}{d} = \dfrac{(477 \cdot 10^{-9}\text{ m})(1.23\text{ m}) \cdot 1}{(2.49 \cdot 10^{-5}\text{ m})} = 2.36\text{ cm}$

34.78. $\Delta y = \dfrac{\lambda L \Delta m}{d} \Rightarrow \lambda = \dfrac{d \Delta y}{L \Delta m} \dfrac{(3.41 \cdot 10^{-5}\text{ m})(2.30\text{ cm})}{(1.63\text{ m}) \cdot 1} = 4.81 \cdot 10^{-7}\text{ m} = 481.\text{ nm}$

34.79. $\Delta y = \dfrac{\lambda L \Delta m}{d} \Rightarrow d = \dfrac{\lambda L \Delta m}{\Delta y} = \dfrac{(485 \cdot 10^{-9}\text{ m})(2.01\text{ m}) \cdot 2}{(4.50\text{ cm})} = 4.33 \cdot 10^{-5}\text{ m} = 43.3\ \mu\text{m}$

34.80. $\Delta y = \dfrac{\lambda L \Delta m}{d} \Rightarrow L = \dfrac{d \Delta y}{\lambda \Delta m} = \dfrac{(1.25 \cdot 10^{-5}\text{ m})(0.2805\text{ m})}{(489 \cdot 10^{-9}\text{ m}) \cdot 3} = 2.39\text{ m}$

34.81. $w = \dfrac{2\lambda L}{a} = \dfrac{2(495 \cdot 10^{-9}\text{ m})(2.77\text{ m})}{(0.487 \cdot 10^{-3}\text{ m})} = 5.63\text{ mm}$

34.82. $w = \dfrac{2\lambda L}{a} \Rightarrow \lambda = \dfrac{wa}{2L} = \dfrac{(5.81 \cdot 10^{-3}\text{ m})(0.555 \cdot 10^{-3}\text{ m})}{2(3.17\text{ m})} = 509\text{ nm}$

34.83. $w = \dfrac{2\lambda L}{a} \Rightarrow a = \dfrac{2\lambda L}{w} = \dfrac{2(503 \cdot 10^{-9}\text{ m})(3.55\text{ m})}{(5.71 \cdot 10^{-3}\text{ m})} = 0.625\text{ mm}$

34.84. $w = \dfrac{2\lambda L}{a} \Rightarrow L = \dfrac{wa}{2\lambda} = \dfrac{(5.75 \cdot 10^{-3}\text{ m})(0.693 \cdot 10^{-3}\text{ m})}{2(507 \cdot 10^{-9}\text{ m})} = 3.93\text{ m}$

Chapter 35: Relativity

Concept Checks

35.1. c, d, e **35.2.** a, c, d, e **35.3.** e **35.4.** a) True b) False c) True **35.5.** a **35.6.** b and c **35.7.** b, c, d **35.8.** a **35.9.** d

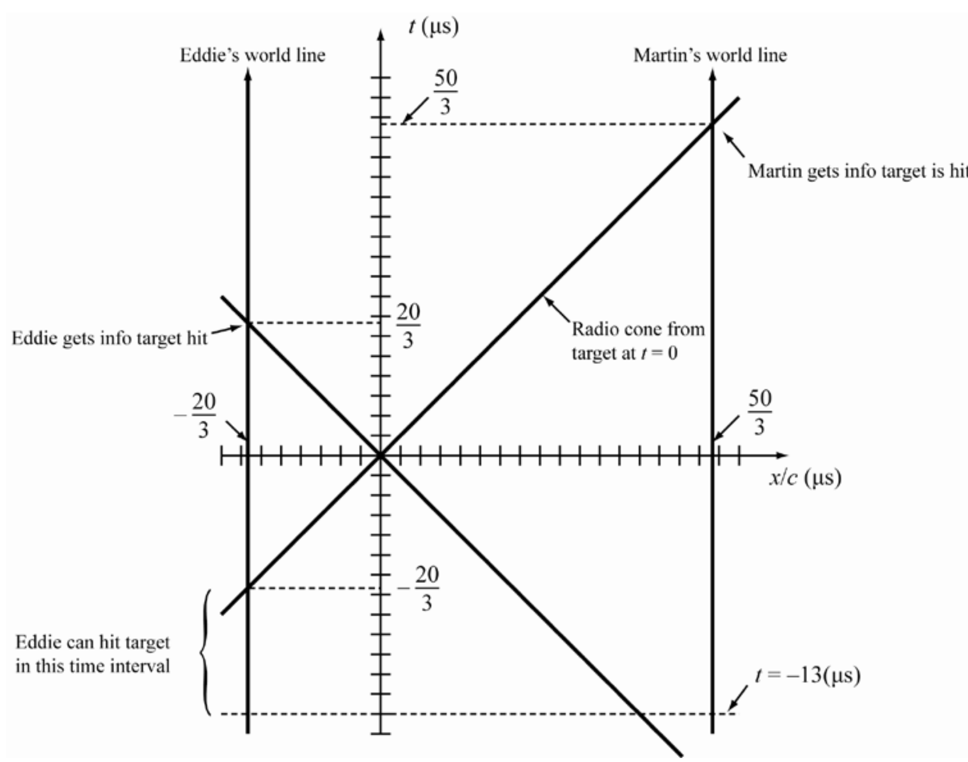
Multiple-Choice Questions

35.1. a **35.2.** d **35.3.** c **35.4.** d **35.5.** a **35.6.** d **35.7.** c **35.8.** c **35.9.** a **35.10.** c **35.11.** c **35.12.** b **35.13.** d **35.14.** d

Conceptual Questions

35.15. A direct corollary of Einstein's special theory of relativity postulates that no entity or interaction in the universe can propagate with a speed greater than the speed of light in vacuum. Therefore, instantaneous effects of events at one point in space on another point in space are impossible. The translational motion of a perfectly rigid object would imply that, by moving one end of the object, the other end of the object would also move instantaneously, without any time delay. This contradicts Einstein's theory.

35.16. The *y*-axis is the time given in μs and the *x*-axis is the 'distance' (x/c) given in units of μs also, since the speed of light can be written $c = 3.00\cdot 10^8\text{ m/s} = 3.00\cdot 10^2\text{ m/μs} = 0.300\text{ km/μs}$. To hit the target, the world line from $t = -13\text{ μs}$ of the person (Eddie and/or Martin) must lie inside the past light cone of the target at $x = 0$ and $t = 0$. As seen in the diagram, Eddie's world line is inside the past light cone of the target from $t = -13\text{ μs}$ to $t = -2\text{ km}/0.3\text{ km/μs} = -20/3\text{ μs}$ and so Eddie could hit the target. However, Martin's world line lies outside of the light cone for all time after $t = -13\text{ μs}$ and so he could not have hit the target. Eddie and Martin find out the target has been hit at the point where their individual world lines intersect the light cone from the target at the origin at some time after the target is hit at $t = 0$. As shown in the diagram, Eddie finds out the target has been hit at $t = 20/3\text{ μs}$ and Martin finds out it has been hit at $t = -5\text{ km}/0.3\text{ km/μs} = -50/3\text{ μs}$.



35.17. If the lens was situated symmetrically about the mass, there would be indeed be a halo, but since the alignment is typically not exact, we see arcs instead. Likewise, the curvature is a result of the object's mass, so if the object does not have a uniform mass distribution, different rays would be affected non-uniformly.

35.18. In the relativistic limit, velocities must be added relativistically (using the Lorentz transformation) rather than classically (using the Galilean transformation), as your friend is suggesting. Let F' be the frame of the rocket and F be the frame of the Earth. The torpedo has a speed of $u' = 2c/3$ with respect to the rocket (frame F') and the rocket travels at a speed of $v = 2c/3$ with respect to Earth (frame F). According to the Lorentz transformation the velocity, u, of the torpedo in the Earth's frame is

$$u = \frac{u' + v}{1 + vu'/c^2} = \frac{(2c/3) + (2c/3)}{1 + (4c^2/9)/c^2} = \frac{12}{13}c.$$

This is less than the speed of light, so no violation of the theory of relativity occurs.

35.19. Yes, the observer still sees the positive charge attracted to the wire. If the positive charge is moving, with velocity $\vec{v}$ in the lab frame, parallel to the current, then it is actually moving anti-parallel to electrons, which have velocity $-\vec{u}$ in the lab frame. Since the positive charge sees only a magnetic field, this must mean that the wire is electrically neutral, i.e. there are equal positive charges (ion cores) per unit length as there are negative charges per unit length. When the wire is seen in the reference frame of the positive charge, the positive charge is stationary while the ion cores are moving away from the positive charge with velocity $-\vec{v}$. The electrons are also moving away from the positive charge with a velocity

$$u' = \frac{-u - v}{1 + vu/c^2} < -v.$$

Both the electrons and ion cores have their separation contracted due to their velocities. Since the electrons are; however, moving faster than the ion cores, their separation is smaller than the separation of the ion cores, meaning the positive charge now sees a net electric charge in any given length of wire and is therefore, attracted to the wire via an electric force instead of the magnetic force in the lab frame.

35.20. The pilot of the rocket sees the garage length contracted. At the speed of the rocket the value of γ is:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \left(1 - \frac{(0.866c)^2}{c^2}\right)^{-1/2} = 2.$$

The rocket pilot therefore thinks that the garage has a length that is reduced by the γ factor of 2; that is, $(L/2)/\gamma = L/4$.

35.21. Since the rod makes an angle with the x-axis, it has a projected length on both the x and y axes. Since the velocity is in the x-direction, only the projection of the length on the x-axis will be contracted, meaning the y-projection length remains unchanged. Since the angle is given by $\theta = \tan^{-1}(y/x)$, as x decreases, the angle increases as viewed by an observer on the ground.

35.22. The primary reason that this presents no contradiction is that the two observations are made in reference frames that are not equivalent. As such, the measurements cannot be directly compared simply by making comparison of observed dimensions. The Earth's shape is distorted from the usual spherical shape due to the fact that length contraction that occurs in the direction of the observers motion only – perpendicular to the axis of rotation for the first astronaut and along the axis of rotation for the second astronaut. If the two observers really want to compare what they've seen, they must exchange information that includes their own relative speed and direction with respect to the Earth.

35.23. The Lorentz transformation for the positions relating the coordinates in the moving frame (primed coordinates) to our reference frame (unprimed coordinates) takes the form

$$x' = \gamma(x - vt),$$

with γ and z-coordinates unchanged, and γ given by

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

Hence, the moving clock at $x' = 0$ has coordinate $x = vt$ and the clock at $x' = l$ has coordinate $x = vt + l/\gamma$. The time readings are then related by the Lorentz transformation,

$$t' = \gamma\left(t - \frac{vx}{c^2}\right).$$

For the clock at $x' = 0$ the reading is

$$t'_0 = \gamma\left(t - \frac{v(vt)}{c^2}\right) = \gamma\left(t - \frac{v^2}{c^2}t\right) = \gamma\left(t - \left(1 - \frac{1}{\gamma^2}\right)t\right) = \gamma\left(t - t + \frac{t}{\gamma^2}\right) = \frac{t}{\gamma}.$$

For the clock at $x' = l$ the reading is

$$t'_1 = \gamma\left(t - \frac{v(vt + l/\gamma)}{c^2}\right) = \gamma\left(t - \frac{v^2}{c^2}t - \frac{vl}{\gamma c^2}\right) = \gamma\left(t - \left(1 - \frac{1}{\gamma^2}\right)t - \frac{vl}{\gamma c^2}\right) = \frac{t}{\gamma} - \frac{lv}{c^2}.$$

These results display two important effects. First, time dilation is apparent, as the advance of the t' values is slowed compared to the advance of t by factor $1/\gamma$. Second, relativity of simultaneity is also manifest, as the readings on the moving clocks – which are synchronized in their own reference frame – differ by lv/c^2 at fixed time t in our reference frame. The clock behind in position is "ahead" in time reading. That is, "the same time" at different positions is a reference-frame-dependent notion. This effect is often overlooked, but most purported relativistic kinematics are resolved unambiguously once it is take into account.

35.24. Velocities are added using the relativistic velocity transformation. Assume that the velocities are along the x-axis. Then the transformation equation is

$$u' = \frac{u - v}{1 - uv/c^2} = \frac{x - y}{1 - xy}c,$$

where x and y represent the fractions of the speed of light of the two sub-light velocities being added. Now, since $|x| < 1$, it follows that $x^2 < 1$. Multiply both sides of this inequality by $1 - y^2$ (which is positive since $|y| < 1$), to obtain $x^2(1 - y^2) < 1 - y^2$. Expand, and add the negative terms to the opposite sides to get $x^2 + y^2 < 1 + x^2y^2$. Subtract $2xy$ from both sides, to yield: $x^2 - 2xy + y^2 < 1 - 2xy + x^2y^2$. Factoring both sides as squares gives the inequality: $(x - y)^2 < (1 - xy)^2$. Divide both sides by the right-hand side (which is positive since $|xy| < 1$) which results in the inequality $\frac{(x-y)^2}{(1-xy)^2} < 1$. Taking square roots of both sides preserves the inequality (with absolute values), so $\left|\frac{x-y}{1-xy}\right| < 1$. It follows that the velocity added relativistically is still less than c, since $|u'| = \left|\frac{x-y}{1-xy}c\right| = \left|\frac{x-y}{1-xy}\right|c < c$.

35.25. Classically, conservation of kinetic energy in an elastic collision for identical particles of mass m means that

$$\frac{1}{2}mv_{1i}^{\ 2}+0=\frac{1}{2}mv_{1f}^{\ 2}+\frac{1}{2}mv_{2f}^{\ 2},$$

Where v_{1i} is the velocity before the collision and v_{2i} and v_{2f} are the velocities after the collision. If the particles have the same mass this reduces to $v_{1i}^{\ 2}=v_{1f}^{\ 2}+v_{2f}^{\ 2}$, which can only be true if the velocities are perpendicular (since conservation of momentum requires also that $\vec{v}_{1i}=\vec{v}_{1f}+\vec{v}_{2f}$). Let the energy and momentum of the originally moving particle be E and p. Let the two particles have total energies after the collision of E_1 and E_2, and momenta after the collision of p_1 and p_2, respectively. Energy-momentum conservation implies the relationships:

$$E+mc^2=E_1+E_2$$
$$p=p_1+p_2.$$

The term $E^2-p^2c^2$ is a scalar invariant so it is the same before and after the collision, implying:

$$\left(E+mc^2\right)^2-p^2c^2=\left(E_1+E_2\right)^2c^2-\left(p_1+p_2\right)^2c^2$$
$$E^2+2Emc^2+m^2c^4-p^2c^2=E_1^2+2E_1E_2+E_2^2-p_1^2c^2-p_2^2c^2-2p_1p_2c^2$$
$$\left(E^2-p^2c^2\right)+2Emc^2+m^2c^4=\left(E_1^2-p_1^2c^2\right)+\left(E_2^2-p_2^2c^2\right)+2E_1E_2-2p_1p_2c^2$$

Using the term $E^2-p^2c^2=m^2c^4$, this reduces to

$$2m^2c^4+2Emc^2=2m^2c^4+2E_1E_2-2p_1p_2c^2$$
$$Emc^2=E_1E_2-p_1p_2c^2$$

Hence, the dot product of the momenta p_1 and p_2 is given by

$$p_1p_2c^2=E_1E_2-Emc^2$$
$$=E_1\left(E+mc^2-E_1\right)-Emc^2.$$

Energy E_1 can take values from mc^2 to E (as can E_2). Therefore, the function on the right-hand side of this equation increases monotonically from zero to the value

$$\frac{1}{4}\left(E-mc^2\right)^2 \text{ for } mc^2\le E_1\le\frac{1}{2}\left(E+mc^2\right),$$

and decreases monotonically back to zero for $\frac{1}{2}\left(E+mc^2\right)\le E_1\le E.$ It is never negative over the allowed range of E_1. This implies $p_1p_2\ge 0,$ with equality only for $E_1=mc^2$ or $E_1=E,$ i.e., only if one of the particles remains at rest after the collision. Otherwise the dot product is positive, meaning the two particles emerge from the collision on trajectories forming an acute angle. Therefore, it is not necessary for the velocities of the two particles to be perpendicular.

35.26. The spaceship is accelerating, and since special relativity deals only with objects moving with constant velocity, one might think that general relativity is required to solve this problem. However, the fact that the spaceship is accelerating is irrelevant since at any point in the trajectory, its velocity is constant. Since the direction of the speed is constantly changing, the length will also appear to be warped along the curvature of the orbit. The observed length of the spaceship is

$$L=\frac{L_0}{\gamma}=L_0\sqrt{1-\left(v/c\right)^2}=L_0\sqrt{1-\left(0.800\right)^2}=0.600L_0.$$

So, the length would look to be 60.0% of the original length.

Exercises

35.27. The speed of light converted from SI to ft/ns is:

$$c = 2.9979\cdot 10^{8}\text{ m/s} = 2.9979\cdot 10^{8}\text{ m/s}\left(\frac{1\text{ s}}{10^{9}\text{ ns}}\right)\left(\frac{3.2808\text{ ft}}{1\text{ m}}\right) = 0.984\text{ ft/ns}.$$

You can see that our result is quite close to 1 foot per nanosecond, which makes this a great way to visualize the speed of light: light moves about a foot in a time interval of a billionth of a second!

35.28. Convert the acceleration due to gravity from SI units into units of ly/year^2.

$$1\text{ year} = 1\text{ year}\left(\frac{365.25\text{ days}}{1\text{ year}}\right)\left(\frac{24\text{ hours}}{1\text{ day}}\right)\left(\frac{3600\text{ s}}{1\text{ hour}}\right) = 3.1556\cdot 10^{7}\text{ s}$$

$$g = 9.81\text{ m/s}^2\left(\frac{1\text{ ly}}{9.461\cdot 10^{15}\text{ m}}\right)\left(\frac{3.1556\cdot 10^{7}\text{ s}}{1\text{ year}}\right)^2 = 1.03\text{ ly/year}^2$$

Just as in Exercise 35.28, the numerical coefficient comes out to be very close to 1. However, unlike the answer in 35.28, the answer to the present problem is more of a curiosity than a useful number for any practical purposes.

35.29. The boat has a velocity of v with respect to the water. The velocity of the water is u downstream. So in order for the boat to directly cross the river, the boat must be headed upstream at an angle such that the velocity of the boat with respect to the ground is $\sqrt{v^2 - u^2}$. The cross-stream time across the river of width D with this velocity is

$$t_{\text{cs}} = \frac{2D}{\sqrt{v^2 - u^2}}.$$

Going upstream, the boat has velocity $v - u$, and going downstream it is $v + u$. Over a distance D, the upstream-downstream time is:

$$t_{\text{ud}} = \frac{D}{v-u} + \frac{D}{v+u} = \frac{D(v+u) + D(v-u)}{(v-u)(v+u)} = \frac{2Dv}{v^2 - u^2}.$$

The ratio of times is then: $\dfrac{t_{\text{cs}}}{t_{\text{ud}}} = \dfrac{2D/\sqrt{v^2-u^2}}{2Dv/\left(v^2-u^2\right)} = \dfrac{\sqrt{v^2-u^2}}{v}.$

35.30. For $v = 0.800c$,

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-(0.800)^2}} = 1.6667.$$

For three significant figures, we have $\gamma = 1.67$.

35.31. (a) Another astronaut on the ship sees the meter stick in the same (rest) frame as the astronaut holding the stick and so its length remains unchanged at one meter.
(b) For a ship moving at $v = 0.50c$, the length of the meter stick as measured by an observer on Earth is

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1-(v/c)^2} = (1.00\text{ m})\sqrt{1-(0.50c/c)^2} = 0.87\text{ m}.$$

35.32. (a) According to a clock on Earth the trip takes

$$\Delta t = \frac{L_0}{v} = \frac{\left(3.84\cdot 10^{8}\text{ m}\right)}{0.50\left(3.00\cdot 10^{8}\text{ m/s}\right)} = 2.6\text{ s}.$$

(b) According to a clock on the spaceship the trip takes,

$$\Delta t_0 = \frac{\Delta t}{\gamma} = \Delta t\sqrt{1-(v/c)^2} = (2.56\text{ s})\sqrt{1-(0.50c/c)^2} = 2.2\text{ s}.$$

(c) On the ship, the distance to the Moon is contracted to L:

$$L = \frac{L_0}{\gamma} = D\sqrt{1-(v/c)^2} = (3.84\cdot 10^8\text{ m})\sqrt{1-(0.50c/c)^2} = 3.3\cdot 10^8\text{ m}.$$

35.33. The time that passes in the rest frame of the Earth is $\Delta t = 30.$ yr. The time that passes in the mother's frame is $\Delta t_0 = 10.$ yr. Therefore,

$$\Delta t = \gamma\Delta t_0 \Rightarrow \frac{\Delta t_0}{\Delta t} = \frac{1}{\gamma} = \sqrt{1-(v/c)^2} \Rightarrow \left(\frac{\Delta t_0}{\Delta t}\right)^2 = 1-(v/c)^2$$

$$\Rightarrow v = \left(1-\left(\frac{\Delta t_0}{\Delta t}\right)^2\right)^{1/2} c = \left(1-\left(\frac{10.}{30.}\right)^2\right)^{1/2} c = 0.94c.$$

35.34. The muon's lifetime Δt when it is moving at $v = 0.90c$ will be longer than $\Delta t = 2.2\mu\text{s}$ when it is at rest in the laboratory frame due to time dilation:

$$\Delta t = \gamma\Delta t_0 = \frac{\Delta t_0}{\sqrt{1-(v/c)^2}} = \frac{(2.2\cdot 10^{-6}\text{ s})}{\sqrt{1-(0.90\,c/c)^2}} = 5.0\cdot 10^{-6}\text{ s}.$$

35.35. The fire truck of length $L_0 = 10.0$ m is traveling fast enough so a stationary observer sees its length contracted to $L = 8.00$ m. Therefore,

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1-(v/c)^2} \Rightarrow \left(\frac{L}{L_0}\right)^2 = 1-(v/c)^2 \Rightarrow v = \left(1-\left(\frac{L}{L_0}\right)^2\right)^{1/2} c = \left(1-\left(\frac{8.00\text{ m}}{10.0\text{ m}}\right)^2\right)^{1/2} c = 0.600c.$$

(a) The time taken from the garage's point of view is

$$t_g = \frac{L}{v} = \frac{(8.00\text{ m})}{0.600(3.00\cdot 10^8\text{ m/s})} = 4.44\cdot 10^{-8}\text{ s}.$$

(b) From the fire truck's perspective the length of the garage will be contracted to

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1-(v/c)^2} = (8.00\text{ m})\sqrt{1-(0.600c/c)^2} = 6.40\text{ m}.$$

Therefore, the truck will not fit inside the garage from the fire truck's point of view since the length of the truck from its rest frame is 10.0 m.

35.36. The rest frame time taken by Phileas Fogg is $\Delta t_0 = 80$ days, while time dilation makes the time seem like $\Delta t = 81$ days. Therefore

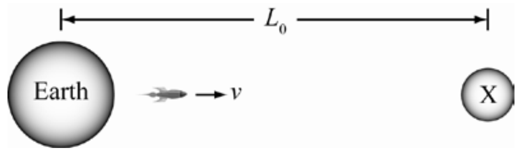
$$\Delta t = \gamma\Delta t_0 \Rightarrow \frac{\Delta t_0}{\Delta t} = \frac{1}{\gamma} = \left(1-(v/c)^2\right)^{1/2} \Rightarrow \left(\frac{\Delta t_0}{\Delta t}\right)^2 = 1-(v/c)^2.$$

Therefore,

$$v = \left(1-\left(\frac{\Delta t_0}{\Delta t}\right)^2\right)^{1/2} c = \left(1-\left(\frac{80\text{ days}}{81\text{ days}}\right)^2\right)^{1/2} c = 0.16c.$$

35.37. **THINK:** The planet is $L_0 = 35$ ly away, but the astronauts cannot travel as fast as c and hence will take longer than 35 years in the NASA (Earth) reference frame while it will take only $\Delta t_0 = 25$ years in the astronauts' reference frame. The astronauts will see the distance as being contracted.

SKETCH:



RESEARCH: The time it takes to reach the planet as observed from Earth is $\Delta t = L_0 / v$. The relationship between Δt and Δt_0 is $\Delta t = \gamma \Delta t_0$, where $\gamma = \left(1-(v/c)^2\right)^{-1/2}$.

SIMPLIFY:

(a) $\Delta t = \dfrac{L_0}{v} = \gamma \Delta t_0 \Rightarrow \left(\dfrac{L_0}{v}\right)^2 = \dfrac{\Delta t_0^2}{1-(v/c)^2}$

$$v^2 = \left(\frac{L_0}{\Delta t_0}\right)^2\left(1-\left(\frac{v}{c}\right)^2\right) \Rightarrow v^2 + \left(\frac{L_0}{\Delta t_0 c}\right)^2 v^2 = \left(\frac{L_0}{\Delta t_0}\right)^2 \Rightarrow v = \frac{1}{\sqrt{1+\left(\dfrac{\Delta t_0 c}{L_0}\right)^2}}c$$

(b) $L = \dfrac{L_0}{\gamma}$.

CALCULATE:

(a) Since $L_0 / c = 35$ years, $v = \left(1+\left(\dfrac{25 \text{ years}}{35 \text{ years}}\right)^2\right)^{-1/2} c = 0.81373c$.

(b) $L = (35 \text{ ly})\sqrt{1-(0.81373)^2} = 20.343 \text{ ly}$

ROUND: The answers should be given to two significant figures.

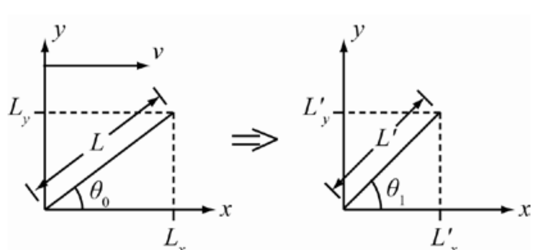
(a) $v = 0.81c$

(b) $L = 20.$ ly

DOUBLE-CHECK: The velocity found for the astronauts is less than the speed of light and the distance of the planet from the perspective of the astronauts does contract; so these values are reasonable. Also, the astronauts believe that 25 years pass during their trip. Their length contracted distance to the planet is 20.343 ly. This means their speed in terms of *c* during the trip is $(20.343 \text{ ly})/(25 \text{ yr}) = 0.81c$ which agrees with the value found.

35.38. **THINK:** Since the velocity of frame *F* is in the *x*-direction, the projection of the length of the rod on the *x*-axis will experience a contraction, while the projection on the *y*-axis will remain unchanged. The angle that the meter stick makes with the *x*-axis changes from $\theta_0 = 37°$ to $\theta_1 = 45°$ in frame F'. Trigonometry can give equations relating the angles to the speed and length.

SKETCH:



RESEARCH: In both frames, $L_y = L'_y \Rightarrow L\sin\theta_0 = L'\sin\theta_1$. In frame F, $L_x = L\cos\theta_0$, and in frame F', $L'_x = L'\cos\theta_1$. The *x*-axis contraction is given by $L'_x = L_x / \gamma$.

SIMPLIFY:

(a) $L\sin\theta_0 = L'\sin\theta_1 \Rightarrow L' = L\dfrac{\sin\theta_0}{\sin\theta_1}$. In frame F', the x-axis projection is

$$L'_x = L'\cos\theta_1 \Rightarrow \frac{L\sin\theta_0}{\tan\theta_1} = \frac{L_x}{\gamma} \Rightarrow \frac{L\sin\theta_0}{\tan\theta_1} = \frac{L\cos\theta_0}{\gamma} \Rightarrow \left(\frac{1}{\gamma}\right)^2 = \left(\frac{\tan\theta_0}{\tan\theta_1}\right)^2$$

$$1-(v/c)^2 = \left(\frac{\tan\theta_0}{\tan\theta_1}\right)^2 \Rightarrow v = \left(1-\left(\frac{\tan\theta_0}{\tan\theta_1}\right)^2\right)^{1/2} c.$$

(b) The length of the rod in frame F' is $L' = L\dfrac{\sin\theta_0}{\sin\theta_1}$.

CALCULATE:

(a) $v = \left(1-\left(\dfrac{\tan(37°)}{\tan(45°)}\right)^2\right)^{1/2} c = 0.6574c$

(b) $L' = (1.00 \text{ m})\dfrac{\sin(37°)}{\sin(45°)} = 0.8511 \text{ m}$

ROUND: The answers should be rounded to two significant figures.

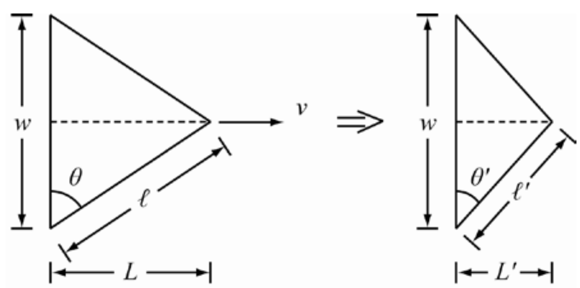
(a) $v = 0.66c$

(b) $L' = 0.85 \text{ m}$

DOUBLE-CHECK: The velocity does not exceed the speed of light and the length does contract; therefore, the answers are reasonable.

35.39. **THINK:** The tip of the triangle is the direction of the speed, $v = 0.400c$, so that only the length, $L = 50.0$ m, will be contracted and the width, $w = 20.0$ m, is not affected. The length of the ship L is not the same as the length of a side of the ship l. Relate the observed angle θ' to the speed of the ship.

SKETCH:



RESEARCH: The lengths are related to the angles, in both frames, by $l\cos\theta = w/2$, $l'\cos\theta' = w/2$, $L = l\sin\theta$, $L' = l'\sin\theta'$, and $\tan\theta = 2L/w$. The length of the ship contracts by $L' = L/\gamma$.

SIMPLIFY: Determine l' in terms of l:

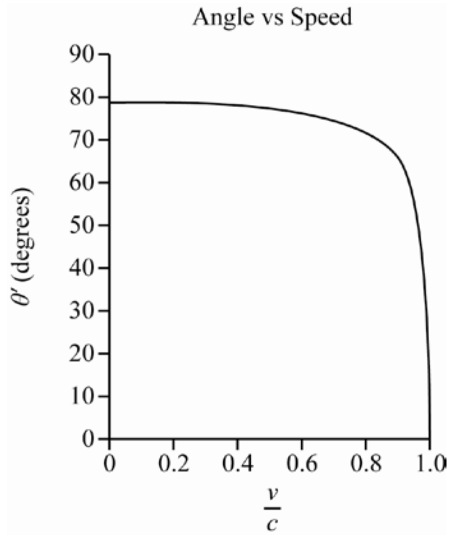
$$\frac{w}{2} = l'\cos\theta' = l\cos\theta \Rightarrow l' = \frac{\cos\theta}{\cos\theta'}l.$$

The contracted length is then

$$L' = l'\sin\theta' = l\cos\theta\tan\theta' = \frac{L}{\gamma} = \frac{l\sin\theta}{\gamma} \Rightarrow \tan\theta' = \frac{\tan\theta}{\gamma} = \frac{2L}{w}\sqrt{1-(v/c)^2}.$$

Therefore, $\theta'(v) = \tan^{-1}\left(\dfrac{2L}{w}\sqrt{1-(v/c)^2}\right)$.

The plot of the angle between the base and side of the ship as a function of the speed of the ship as measured by a stationary observer is shown below.



CALCULATE: $\theta'(v=0.40c)=\tan^{-1}\left(\frac{2(50.\text{ m})}{(20.\text{ m})}\sqrt{1-(0.40c/c)^2}\right)=77.69°$

ROUND: To three significant figures, $\theta'(v=0.400c)=77.7°$.

DOUBLE-CHECK: As v approaches c, the expression under the square root approaches zero and hence the angle will also approach zero. This agrees with the graph where the angle is smaller at higher velocities. When $v=c$, the side of the ship would effectively contract to zero, thus making an angle of zero with the width.

35.40. Since the light whose rest wavelength, $\lambda_0=480$ nm, appears as $\lambda=660$ nm, it is red-shifted, so you must be travelling away from the light.

$$\lambda=\lambda_0\sqrt{\frac{c+v}{c-v}}\ \Rightarrow\ \lambda^2=\lambda_0^2\frac{c+v}{c-v}\ \Rightarrow\ \lambda^2(c-v)=\lambda_0^2(c+v)$$

$$v=\left(\frac{\lambda^2-\lambda_0^2}{\lambda^2+\lambda_0^2}\right)c=\left(\frac{(660\text{ nm})^2-(480\text{ nm})^2}{(660\text{ nm})^2+(480\text{ nm})^2}\right)c=0.31c$$

35.41. The light with wavelength $\lambda_0=650$ nm is blue-shifted and appears as $\lambda=520$ nm, as expected since the driver is travelling towards the light. Therefore,

$$\lambda=\lambda_0\sqrt{\frac{c-v}{c+v}}\ \Rightarrow\ \lambda^2=\lambda_0^2\frac{c-v}{c+v}\ \Rightarrow\ \lambda^2(c+v)=\lambda_0^2(c-v)$$

$$v=\left(\frac{\lambda_0^2-\lambda^2}{\lambda_0^2+\lambda^2}\right)c\ \Rightarrow\ v=\left(\frac{(650\text{ nm})^2-(520\text{ nm})^2}{(650\text{ nm})^2+(520\text{ nm})^2}\right)c=0.22c.$$

You would have been traveling 0.22*c*, or 22% of the speed of light. This explanation would likely result in a speeding ticket!

35.42. Since the light has a rest wavelength of $\lambda_0=532$ nm and must appear to have $\lambda=560$ nm, it must be red-shifted and therefore must travel away from the meteor.

$$\lambda=\lambda_0\sqrt{\frac{c+v}{c-v}}\ \Rightarrow\ \lambda^2=\lambda_0^2\frac{c+v}{c-v}\ \Rightarrow\ \lambda^2(c-v)=\lambda_0^2(c+v)$$

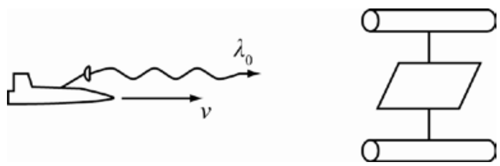
$$v=\left(\frac{\lambda^2-\lambda_0^2}{\lambda^2+\lambda_0^2}\right)c=\left(\frac{(560\text{ nm})^2-(532\text{ nm})^2}{(560\text{ nm})^2+(532\text{ nm})^2}\right)c=0.051c$$

35.43. Since the car, moving with a speed $v=32.0\text{ km/h}\left(\frac{1000\text{ m}}{1\text{ km}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right)=8.889\text{ m/s},$ is moving away from the radar of frequency $f_0=10.6\text{ GHz},$ the shift in frequency is,

$$\Delta f=f-f_0=f_0\left(\sqrt{\frac{c-v}{c+v}}-1\right)=\left(10.6\cdot 10^9\text{ Hz}\right)\left(\sqrt{\frac{\left(3.00\cdot 10^8\text{ m/s}\right)-\left(8.889\text{ m/s}\right)}{\left(3.00\cdot 10^8\text{ m/s}\right)+\left(8.889\text{ m/s}\right)}}-1\right)=-314.078\text{ Hz}.$$

Therefore, the frequency is red-shifted by 314 Hz.

35.44. **THINK:** Since the spaceship is moving towards the station, the wavelength will be blue-shifted, resulting in the original wavelength of $\lambda_0=632.8\text{ nm}$ being reduced to $\lambda=514.5\text{ nm}$. Using the relativistic formula for wavelength shift the speed of the ship can be deduced.
SKETCH:



RESEARCH: Since the ship is moving towards the station, the relevant formula for wavelength shift is $\lambda=\lambda_0\sqrt{\frac{c-v}{c+v}}$. The shift parameter is by definition: $z=\frac{\lambda-\lambda_0}{\lambda_0}$.

SIMPLIFY: $\lambda=\lambda_0\sqrt{\frac{c-v}{c+v}}\Rightarrow\lambda^2=\lambda_0^2\frac{c-v}{c+v}\Rightarrow\lambda^2(c+v)=\lambda_0^2(c-v)\Rightarrow v=\left(\frac{\lambda_0^2-\lambda^2}{\lambda_0^2+\lambda^2}\right)c$

CALCULATE: $v=\left(\frac{(632.8\text{ nm})^2-(514.5\text{ nm})^2}{(632.8\text{ nm})^2+(514.5\text{ nm})^2}\right)c=0.20405c,\ z=\frac{(514.5\text{ nm})-(632.8\text{ nm})}{(632.8\text{ nm})}=-0.186946$

ROUND: To four significant figures, $v=0.2041c$ and $z=-0.1869.$
DOUBLE-CHECK: The velocity is less than the speed of light and the shift parameter is negative, which is what it should be for blue shift, so it makes sense.

35.45. In Sam's reference frame, each event occurs at the following points: $x_A=0\text{ m},\ t_A=0\text{ s},\ x_B=500.\text{ m}$ and $t_B=0\text{ s}.$ To find the timing of the events in Tim's reference frame, use the Lorentz transformation $t'=\gamma\left(t-vx/c^2\right)$. Therefore, since $x=0$, $t'_A=0\text{ s}$ and

$$t'_B=\frac{-\gamma vx}{c^2}=\frac{-(0.999)(500.\text{ m})}{\left(2.9979\cdot 10^8\text{ m/s}\right)\sqrt{1-(0.999)^2}}=-3.73\cdot 10^{-5}\text{ s}.$$

(a) Therefore, Tim experiences event *B* before event *A*.

(a) For Tim, event *A* occurs $3.73\cdot 10^{-5}$ s after event *B*.

35.46. Let an inertial reference frame *F* be at rest and let another inertial reference frame F' move at a constant speed *v* along a common *x*-axis with respect to reference frame *F*. According to the relativistic velocity addition formula,

$$u' = \frac{u-v}{1-vu/c^2}$$

$$u = c \;\Rightarrow\; u' = \frac{c-v}{1-\dfrac{vc}{c^2}} = \frac{c-v}{1-\dfrac{v}{c}} = \frac{c(c-v)}{c-v} = c,$$

as required. Thus, the result is independent of the specific value of v.

35.47. Let all speeds be in a common x-direction. Let frame F be the ground and frame F' be the frame of your car. The speed of your car with respect to the ground is $v = 50.0$ m/s and the speed of the oncoming car is $u = -50.0$ m/s in frame F. Using the relativistic velocity transformation, the relative speed of the oncoming car is

$$u' = \frac{u-v}{1+uv/c^2} = \frac{(-50.0\text{ m/s})-(50.0\text{ m/s})}{\sqrt{1+(-50.0\text{ m/s})(50.0\text{ m/s})/(2.9979\cdot 10^8\text{ m/s})^2}} = -99.99999999999862\text{ m/s} \approx -100.\text{ m/s}.$$

The relative velocity is about the same as a Galilean velocity transformation $u' = u - v = 2u = -100$ m/s, since the speed of the cars is so small compared to the speed of light. In order to detect a difference, fourteen significant figures would need to be kept. This shows how close the values are.

35.48. Assuming all speeds are measured along the same direction, let $v = 0.90c$ be the speed of the ship (frame F') relative to Earth (frame F) and let $u' = 0.50c$ be the speed of the missile relative to the ship. The speed of the missile as seen from the Earth is given by

$$u = \frac{u'+v}{1+vu'/c^2} = \frac{(0.50c)+(0.90c)}{1+(0.90c)(0.50c)/c^2} = 0.97c.$$

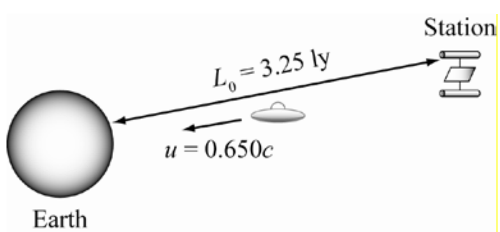
35.49. (a) The total distance travelled, as measured by Alice is

$$L = \frac{L_0}{\gamma} = 2(3.25\text{ ly})\sqrt{1-(0.65c/c)^2} = 4.940\text{ ly} \approx 4.9\text{ ly}.$$

(b) The total time duration for the trip as measured by Alice is $t = \dfrac{L}{v} = \dfrac{(4.940\text{ ly})}{(0.65c)} = 7.6$ years.

35.50. **THINK:** The spaceship that Alice boards travels at a speed of $u = 0.650c$ to a station $L_0 = 3.25$ ly away. The question asks for the speed v Alice must travel so that she measures a relative speed of $u = 0.650c$ on the return journey. In Alice's frame, the distance of the return flight will be length contracted. The relativistic velocity transformation and length contraction formulae can be used to solve the problem.

SKETCH:



RESEARCH:

(a) The relativistic velocity transforms as

$$u' = \frac{u-v}{1-vu/c^2}.$$

(b) The time of the return flight as measured by Alice is $t = L/v$, where $L = L_0/\gamma$ is the length contracted distance in her frame.

SIMPLIFY:

(a) The speed of the spaceship is given by

$$u' = \frac{u-v}{1-uv/c^2} \Rightarrow u' - \frac{uvu'}{c^2} = u - v \Rightarrow u - u' = v - \frac{uvu'}{c^2} \Rightarrow v = \frac{u-u'}{1-uu'/c^2}.$$

(b) The time for Alice's return flight is $t = \frac{L_0}{\gamma v} = \frac{L_0}{v}\sqrt{1-(v/c)^2}$.

CALCULATE:

(a) To Alice, the Earth is moving toward her with a speed of $u' = -0.650c$, so

$$v = \frac{(0.650c)-(-0.650c)}{1-\frac{(0.650c)(-0.650c)}{c^2}} = 0.91388c.$$

(b) The time duration of the flight as measured by Alice is

$$t = \frac{(3.25 \text{ ly})}{(0.91388c)}\sqrt{1-(0.91388c/c)^2} = 1.4438 \text{ years}.$$

ROUND: The answers should be given to three significant figures.

(a) As required, the velocity of the ship relative to the Earth is $v = 0.914c$.

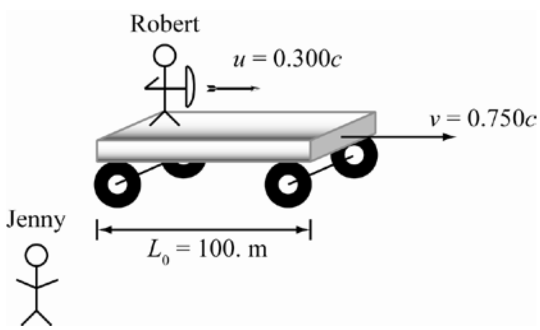
(b) The duration of Alice's return flight as measured by her is $t = 1.44$ years.

DOUBLE-CHECK: The speed $v = 0.914c$ gives

$$u' = \frac{(0.650c)-(0.914c)}{1-(0.650c)(0.914c)/c^2} = -0.650c.$$

35.51. **THINK:** The arrow has a velocity of $u' = 0.300c$ in Robert's reference frame. The railroad car has a length of $L_0 = 100.$ m and travels at a speed of $v = 0.750c$. The velocity transformation equations and the equation for length contraction can be used to determine the values observed by Jenny.

SKETCH:



RESEARCH: As observed by Jenny,

(a) the railroad car is length contracted: $L = \frac{L_0}{\gamma}$,

(b) the velocity of the arrow is given by the inverse relativistic velocity transformation: $u = \frac{u'+v}{1+vu'/c^2}$,

(c) the time of the arrow's flight is given by the inverse Lorentz transformation: $t = \gamma\left(t' + \frac{vx'}{c^2}\right)$, and

(d) the distance traveled by the arrow is given by the inverse Lorentz transformation: $x = \gamma(x' + vt')$.

SIMPLIFY: Here $x' = L_0$ is the length of the railroad car and $t' = L_0/u'$ is the time of the arrow`s flight in Robert`s frame of reference. As observed by Jenny,

(a) $L = L_0\sqrt{1-(v/c)^2}$,

(c) the time taken by the arrow to cover the length of the car is $t=\frac{L_0}{\sqrt{1-(v/c)^2}}\left(\frac{1}{u'}+\frac{v}{c^2}\right)$, , so if we take v

as $v=kc$ and $u'=jc$, we have $t=\frac{L_0}{\sqrt{1-(v/c)^2}}\left(\frac{1}{jc}+\frac{kc}{c^2}\right)=\frac{L_0}{c}\cdot\frac{1}{\sqrt{1-(v/c)^2}}\left(\frac{1}{j}+k\right)$, and

(d) the distance covered by the arrow is $x=\frac{L_0}{\sqrt{1-(v/c)^2}}\left(1+\frac{v}{u'}\right)$.

CALCULATE:

(a) $L=(100.\text{ m})\sqrt{1-(0.750c/c)^2}=66.14\text{ m}$

(b) $u=\frac{(0.300c)+(0.750c)}{1+(0.750c)(0.300c)/c^2}=0.85714c$

(c) $t=\frac{(100.\text{ m})}{(2.9979\cdot 10^8\text{ m/s})\sqrt{1-(0.750c/c)^2}}\left(\frac{1}{(0.300)}+(0.750)\right)=2.059\cdot 10^{-6}\text{ s}$

(d) $x=\frac{(100.\text{ m})}{\sqrt{1-(0.750c/c)^2}}\left(1+\frac{(0.750c)}{(0.300c)}\right)=529.2\text{ m}$

ROUND: The answers should be given to three significant figures. As observed by Jenny,
(a) the railroad car is $L=66.1\text{ m}$ long,
(b) the velocity of the arrow is $u=0.857c$,
(c) the time it takes the arrow to cover the length of the railroad car is $t=2.06\ \mu\text{s}$, and
(d) the arrow covers a distance of $x=529\text{ m}$.

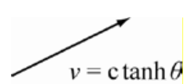
DOUBLE-CHECK: The railroad car length is contracted from Jenny's viewpoint, as expected. Multiplying the answer to part (b) by the answer to part (c):

$$x=(0.8571)(2.9979\cdot 10^8\text{ m/s})(2.059\cdot 10^{-6}\text{ s})=529\text{ m},$$

as found in part (d). So, the answers are consistent.

35.52. **THINK:** The speed of an object can be described by the relation $v=c\tanh\theta$ where θ is known as the rapidity. The question asks to prove that two velocities adding via the Lorentzian rule, corresponds to adding the rapidity of the two velocities. The question also asks for the Lorentz transformation of two coordinate systems using the rapidity. The Lorentz transformation equations can be used to solve this problem.

SKETCH:



RESEARCH:

(a) The Lorentzian rule for adding two velocities is $v=\frac{u_1+u_2}{1+u_1u_2/c^2}$. The Lorentz transformation between two frames with relative velocity v in the x direction is given by the equations

$$x'=\gamma(x-\beta ct),\quad y'=y,\quad z'=z,\quad \text{and}\quad t'=\gamma(t-\beta x/c).$$

Velocities that add according to the Lorentzian rule correspond to adding the rapidity of each:

$$v=c\tanh(\theta_1+\theta_2)=\frac{u_1+u_2}{1+u_1u_2/c^2}=\frac{c\tanh\theta_1+c\tanh\theta_2}{1+\tanh\theta_1\tanh\theta_2}$$

(b) For the derivation it is useful to know that the hyperbolic tangent is related to exponentials by

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

The following relations are also useful:

$$1 = \operatorname{sech}^2\theta + \tanh^2\theta \text{ and } \tanh\theta = \frac{\sinh\theta}{\cosh\theta}.$$

SIMPLIFY:

(a) According to the Lorentzian rule,

$$v = \frac{c\tanh\theta_1 + c\tanh\theta_2}{1 + c\tanh\theta_1 c\tanh\theta_2 / c^2} = c\frac{\left(\tanh\theta_1 + \tanh\theta_2\right)}{1 + \tanh\theta_1\tanh\theta_2} = c\frac{\dfrac{e^{\theta_1} - e^{-\theta_1}}{e^{\theta_1} + e^{-\theta_1}} + \dfrac{e^{\theta_2} - e^{-\theta_2}}{e^{\theta_2} + e^{-\theta_2}}}{1 + \left(\dfrac{e^{\theta_1} - e^{-\theta_1}}{e^{\theta_1} + e^{-\theta_1}}\right)\left(\dfrac{e^{\theta_2} - e^{-\theta_2}}{e^{\theta_2} + e^{-\theta_2}}\right)}$$

$$= c\frac{\left(e^{\theta_1} - e^{-\theta_1}\right)\left(e^{\theta_2} + e^{-\theta_2}\right) + \left(e^{\theta_1} + e^{-\theta_1}\right)\left(e^{\theta_2} - e^{-\theta_2}\right)}{\left(e^{\theta_1} + e^{-\theta_1}\right)\left(e^{\theta_2} + e^{-\theta_2}\right) + \left(e^{\theta_1} - e^{-\theta_1}\right)\left(e^{\theta_2} - e^{-\theta_2}\right)} = c\frac{2e^{\theta_1+\theta_2} - 2e^{-(\theta_1+\theta_2)}}{2e^{\theta_1+\theta_2} + 2e^{-(\theta_1+\theta_2)}} = c\tanh\left(\theta_1 + \theta_2\right),$$

as required.

(b) If $v = c\tanh\theta$ then,

$$\gamma = \left(1 - \left(v/c\right)^2\right)^{-\frac{1}{2}} = \left(1 - \left(c\tanh\theta / c\right)^2\right)^{-\frac{1}{2}} = \left(1 - \tanh^2\theta\right)^{-\frac{1}{2}} = \left(\operatorname{sech}^2\theta\right)^{-\frac{1}{2}} = \cosh\theta \text{ and}$$

$$\beta = \frac{v}{c} = \frac{c\tanh\theta}{c} = \tanh\theta.$$

The Lorentz transformation becomes

$x' = \gamma\left(x - \beta ct\right) = \cosh\theta x - \cosh\theta\tanh\theta ct = x\cosh\theta - ct\sinh\theta,$

$y' = y,$

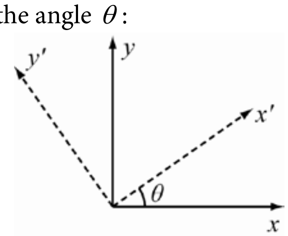
$z' = z,$ and

$t' = \gamma\left(t - \beta x / c\right) = t\cosh\theta - \frac{x}{c}\cosh\theta\tanh\theta = t\cosh\theta - \frac{x}{c}\sinh\theta$

CALCULATE: Not required.

ROUND: Not required.

DOUBLE-CHECK: Note that the transformation is similar to a transformation from one coordinate system to another where they differ by the angle θ:



$$x' = x\cos\theta + y\sin\theta \text{ and } y' = -x\sin\theta + y\cos\theta.$$

35.53. The relativistic momentum is $p = \gamma mv$. If the momentum is equal to $p = mc$ then $\gamma mv = mc$ or

$$\gamma = \frac{1}{\sqrt{1 - \left(v/c\right)^2}} = \frac{c}{v} \Rightarrow \frac{1}{1 - \left(v/c\right)^2} = \left(\frac{c}{v}\right)^2 \Rightarrow 1 = \left(\frac{c}{v}\right)^2 - 1 \Rightarrow \left(\frac{c}{v}\right)^2 = 2 \Rightarrow v = \frac{c}{\sqrt{2}}.$$

This can be left in exact form, or written as $v \approx 0.707c$.

35.54. (a) The energy of the electron is $E = \gamma mc^2$. For the energy to be 10 times greater than its rest energy of $E_0 = mc^2$,

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = 10 \Rightarrow 1-(v/c)^2 = \frac{1}{100} \Rightarrow v = \sqrt{\frac{99}{100}}c = 0.995c$$

(b) The momentum is $p = \gamma mv = 10\left(0.511 \text{ MeV}/c^2\right)\left(\sqrt{99/100}c\right) = 5.08 \text{ MeV}/c.$

35.55. The kinetic energy of the colliding beams in the center-of-mass reference frame is related to the fixed-target equivalent, or lab reference frame by

$$K^{\text{lab}} = 4K^{\text{cm}} + \frac{2\left(K^{\text{cm}}\right)^2}{m_p c^2} = 4(197)(100.\text{ GeV}) + \frac{2\left((197)(100.\text{ GeV})\right)^2}{(197)(1.00\text{ GeV})} = 4.02 \cdot 10^6 \text{ GeV}.$$

This is an incredibly large energy.

35.56. The work done on the proton is equal to the change in kinetic energy of the proton.

$$W = \Delta K = \gamma m_p c^2 - m_p c^2 = (\gamma - 1) m_p c^2 = \left(\frac{1}{\sqrt{1-(v/c)^2}} - 1\right) m_p c^2$$

$$= \left(\frac{1}{\sqrt{1-(0.997c/c)^2}} - 1\right)\left(1.672 \cdot 10^{-27} \text{ kg}\right)\left(2.9979 \cdot 10^8 \text{ m/s}\right)^2$$

$$= 11.1179 \text{ GeV} \approx 11.1 \text{ GeV}.$$

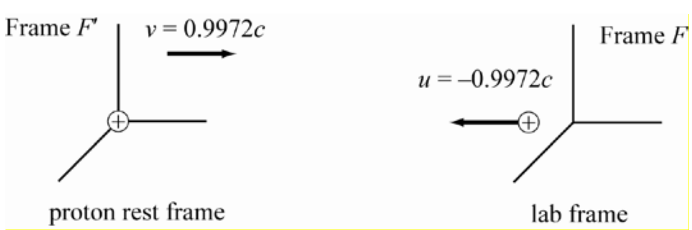
35.57. The energy of the proton is

$$E = \gamma m_p c^2 = \frac{1}{\sqrt{1-(0.61c/c)^2}}\left(1.672 \cdot 10^{-27} \text{ kg}\right)\left(2.9979 \cdot 10^8 \frac{\text{m}}{\text{s}}\right)^2 \left(\frac{6.241 \cdot 10^{18} \text{ eV}}{1 \text{ J}}\right)\left(\frac{1 \text{ MeV}}{1 \cdot 10^6 \text{ eV}}\right)$$

$$= \frac{1}{\sqrt{1-(0.61c/c)^2}}(938 \text{ MeV}) = 1183.53 \text{ MeV} \approx 1200 \text{ MeV}.$$

35.58. **THINK:** Two protons in an accelerator are on a head-on collision course. In the lab reference frame (frame *F*) the protons reach a speed of $v = 0.9972c$. The relativistic velocity transformation and the relativistic formula for kinetic energy can be used to solve the problem.

SKETCH:



RESEARCH: The speed of the proton in the other proton's rest frame (frame F') is given by $u' = \frac{u - v}{1 - vu/c^2}$. The kinetic energy of a relativistic particle is $K = (\gamma - 1)mc^2$. The mass of a proton is $m_p = 938.27 \text{ MeV}/c^2$.

SIMPLIFY: Let u denote the speed of the proton in the lab frame. In the proton reference frame, the speed of the other proton is

$$u' = \frac{v-(-v)}{1-(-v)v/c^2} = \frac{2v}{1+(v/c)^2}.$$

The kinetic energy K of the protons in the lab reference frame is the sum of the kinetic energy of each proton:

$$K = K_1 + K_2 = (\gamma-1)m_\text{p}c^2 + (\gamma-1)m_\text{p}c^2 = 2(\gamma-1)m_\text{p}c^2 = 2\left(\frac{1}{\sqrt{1-(v/c)^2}}-1\right)m_\text{p}c^2.$$

The kinetic energy K' in the proton reference frame is $K' = (\gamma-1)m_\text{p}c^2 = \left(\frac{1}{\sqrt{1-(u/c)^2}}-1\right)m_\text{p}c^2.$

CALCULATE:

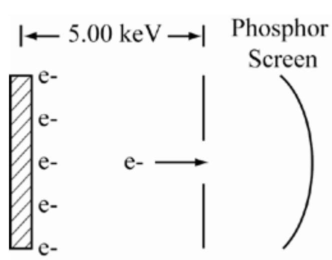
(a) $u' = \dfrac{2(0.9972c)}{1+(0.9972c/c)^2} = 0.999996c$

(b) $K = 2\left(\dfrac{1}{\sqrt{1-(0.9972c/c)^2}}-1\right)(938.27 \text{ MeV}/c^2)c^2 = 23217.35 \text{ MeV}$

(c) $K' = \left(\dfrac{1}{\sqrt{1-(0.999996c/c)^2}}-1\right)(938.27 \text{ MeV}/c^2)c^2 = 333689.6 \text{ MeV}$

ROUND: (a) To six significant figures, the speed of one proton with respect to another is $u' = 0.999996c$.
(b) To four significant figures, in the lab reference frame, the particles have a kinetic energy of $K = 23{,}220$ MeV.
(c) To four significant figures, in the proton's reference frame, the other proton has a kinetic energy of 333,700 MeV.
DOUBLE-CHECK: These are typical speeds and energies for protons to have in proton accelerators.

35.59. **THINK:** Electrons acquire kinetic energy as they accelerate through the potential difference. The speed acquired by the electron after moving through this potential can be found and then the appropriate classical and relativistic formulae can be used to find the total energy and momentum. Many of the answers only make sense if they are given to three significant figures, so rounding will be nonstandard.
SKETCH:



RESEARCH:

(a) The kinetic energy gained by the electron in moving through the potential difference V is equal to the work done by the potential difference: $W = K = qV$.

(b) The kinetic energy of a relativistic particle is $K = (\gamma-1)E_0$.

(c) The relativistic values for the total energy and momentum are $E_R = \gamma E_0$ and $p_R = \gamma mv$. Classically, these values are given by $E_C = K = \frac{1}{2}mv_C^2$ and $p_C = mv_C$.

The rest mass energy of an electron is $E_0 = 511$ keV.

SIMPLIFY:

(a) $K = eV$

(b) The speed of the particle is found using the relativistic formula $K = (\gamma - 1)E_0$:

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{K+E_0}{E_0} \Rightarrow v = \sqrt{1-\left(\frac{E_0}{K+E_0}\right)^2}\,c = \sqrt{\frac{(K+E_0)^2 - E_0^2}{(K+E_0)^2}}\,c = \frac{\sqrt{K^2+2KE_0}}{K+E_0}c.$$

(c) The relativistic values for the total energy and momentum are

$$E_R = \gamma E_0 = K + E_0, \text{ and}$$

$$p_R = \gamma mv = \left(\frac{K+E_0}{E_0}\right)\left(\frac{E_0}{c^2}\right)\frac{\sqrt{K^2+2KE_0}}{K+E_0}c = \sqrt{K^2+2KE_0}\,/\,c.$$

Classically, the total energy and momentum are

$$E_C = K, \text{ and}$$

$$p_C = mv_C = m\sqrt{2K/m} = \sqrt{2Km} = \sqrt{2KE_0}\,/\,c.$$

CALCULATE:

(a) $K = e(5.00 \text{ kV}) = 5.00 \text{ keV}$

(b) $v = \dfrac{\sqrt{(5.00 \text{ keV})^2 + 2(5.00 \text{ keV})(511 \text{ keV})}}{(5.00 \text{ keV}) + (511 \text{ keV})}c = 0.1389c$

(c) $E_R = (5.00 \text{ keV}) + (511 \text{ keV}) = 516 \text{ keV}$

$p_R = \sqrt{(5.00 \text{ keV})^2 + 2(5.00 \text{ keV})(511 \text{ keV})}\,/\,c = 71.659 \text{ keV}/c$

$E_C = 5.00 \text{ keV}$

$p_C = \sqrt{2(5.00 \text{ keV})(511 \text{ keV})}\,/\,c = 71.484 \text{ keV}/c$

ROUND:

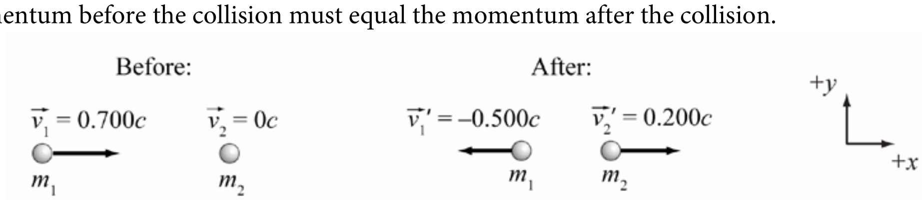
(a) The kinetic energy that the electron acquires is $K = 5.00$ keV.

(b) The electron has a speed of $v = 0.139c$, thus the electron will have only a small difference between its classical and relativistic values, but this can still be considered a relativistic speed.

(c) The relativistic and classical energies are $E_R = 516$ keV and 5.00 keV, respectively. (The difference is due to the fact that the relativistic energy includes the rest energy). The relativistic and classical momenta are $p_R = 71.7 \text{ keV}/c$ and $p_C = 71.5 \text{ keV}/c$, respectively.

DOUBLE-CHECK: The classical and relativistic momenta are similar, as expected for such a low speed.

35.60. The momentum before the collision must equal the momentum after the collision.



$$p_1 + p_2 = p_1' + p_2'$$

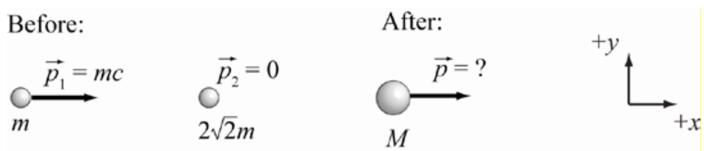
$$\gamma_1 m_1 v_1 = \gamma_1' m_1 v_1' + \gamma_2' m_2 v_2'.$$

The ratio is

$$\frac{m_2}{m_1} = \frac{\gamma_1 v_1 - \gamma_1' v_1'}{\gamma_2' v_2'} = \frac{\dfrac{(0.700c)}{\sqrt{1-(0.700c/c)^2}} - \dfrac{(-0.500c)}{\sqrt{1-(0.500c/c)^2}}}{\dfrac{(0.200c)}{\sqrt{1-(0.200c/c)^2}}} = 7.63.$$

35.61. **THINK:** Two particles collide inelastically. One particle has a mass of $m_1 = m$ and momentum $p_1 = mc.$ The second particle has a mass of $m_2 = 2\sqrt{2}m.$ Conservation of energy and momentum can be used with the relativistic energy equation to determine the speed and mass of the new particle.

SKETCH:



RESEARCH: The relativistic momentum is $p = \gamma m v.$ The energy of the particles is $E^2 = p^2c^2 + m^2c^4$ after the collision.

SIMPLIFY:

(a) The speed of the projectile with momentum $p_1 = mc$ before the collision is given by

$$v_1 = \frac{mc}{m\gamma} = c\sqrt{1-(v_1/c)^2} \Rightarrow (v_1/c)^2 = 1-(v_1/c)^2 \Rightarrow 2v_1^2 = c^2 \Rightarrow v_1 = \frac{c}{\sqrt{2}}.$$

(b) The total energy is conserved before and after the collision. Therefore,

$$E_f = E_i$$
$$\sqrt{p^2c^2 + M^2c^4} = \sqrt{p_1^2c^2 + m_1^2c^4} + \sqrt{p_2^2c^2 + m_2^2c^4}$$
$$\sqrt{p^2c^2 + M^2c^4} = \sqrt{(mc)^2c^2 + m^2c^4} + \sqrt{0 + (2\sqrt{2}m)^2c^4}$$
$$\sqrt{p^2c^2 + M^2c^4} = \sqrt{2}mc^2 + 2\sqrt{2}mc^2$$
$$\sqrt{p^2c^2 + M^2c^4} = 3\sqrt{2}mc^2$$
$$p^2 + M^2c^2 = 18m^2c^2$$

From conservation of momentum, $p = p_1 = mc.$ Therefore, the above equation becomes:

$$(mc)^2 + M^2c^2 = 18m^2c^2$$
$$M^2c^2 = 17m^2c^2$$
$$M = \sqrt{17}m$$

Note that there is more mass than there was before the collision. Some kinetic energy has become mass energy.

(c) Using the conservation of momentum

$$p = p_1 \Rightarrow \gamma M v = mc \Rightarrow Mv = mc\sqrt{1-(v/c)^2} \Rightarrow M^2v^2 = m^2c^2 - m^2v^2$$

$$v = \frac{mc}{\sqrt{M^2 + m^2}} = \frac{mc}{\sqrt{(\sqrt{17}m)^2 + m^2}} = \frac{c}{\sqrt{18}}$$

CALCULATE: Not required.

ROUND: Not required.

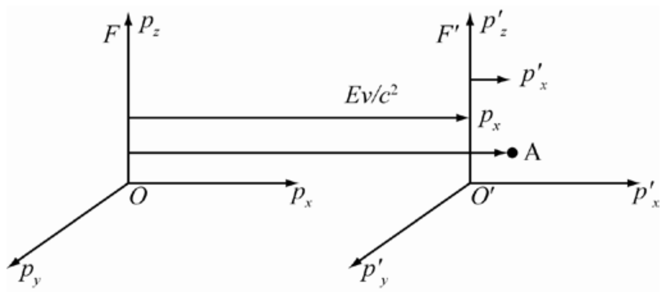
DOUBLE-CHECK: It is reasonable that the speed of the new particle is smaller than the speed of the projectile. The momentum of the new particle is given by

$$p=\gamma Mv=\frac{Mv}{\sqrt{1-(v/c)^2}}=\frac{\left(\sqrt{17}m\right)\left(c/\sqrt{18}\right)}{\sqrt{1-\left(\left(c/\sqrt{18}\right)/c\right)^2}}=\frac{\sqrt{17}}{\sqrt{18}\sqrt{1-(1/18)}}mc=\frac{\sqrt{17}}{\sqrt{18-1}}mc=mc.$$

This is the initial momentum of the projectile, as expected by conservation of momentum.

35.62. **THINK:** To derive the Lorentz transformation for momentum, follow Derivation 35.3. In this case, the momentum is similar to the position coordinates and the energy is analogous to the time.

SKETCH:



RESEARCH: The energy is given by $E=\gamma mc^2$ and the momentum is given by $p=\gamma mv$. In order to use the energy as a momentum, it must be of the form $p=E\dfrac{v}{c^2}=\beta\dfrac{E}{c}$.

SIMPLIFY: In frame F, the vectors are

$$\overrightarrow{OA}=p_x,\ \overrightarrow{O'A}=\frac{p'_x}{\gamma}\ \text{and}\ \overrightarrow{OO'}=\frac{Ev}{c^2}.$$

Using the equation $\overrightarrow{OA}=\overrightarrow{O'A}+\overrightarrow{OO'}$ gives

$$p_x=\frac{p'_x}{\gamma}+\frac{Ev}{c^2}\Rightarrow p'_x=\gamma\left(p_x-\frac{Ev}{c^2}\right). \qquad (1)$$

For frame F', the vectors are

$$\overrightarrow{OA}=\frac{p_x}{\gamma},\ \overrightarrow{O'A}=p'_x,\ \text{and}\ \overrightarrow{OO'}=\frac{E'v}{c^2}.$$

Using the equation $\overrightarrow{OA}=\overrightarrow{O'A}+\overrightarrow{OO'}$ gives

$$\frac{p_x}{\gamma}=p'_x+\frac{E'v}{c^2}. \qquad (2)$$

Substituting from equation (1) for p'_x into equation (2) gives $\dfrac{p_x}{\gamma}=\gamma\left(p_x-\dfrac{Ev}{c^2}\right)+\dfrac{E'v}{c^2}$. Solving for E':

$$E'=\frac{c^2}{v}\frac{p_x}{\gamma}-\frac{c^2}{v}\gamma p_x+\gamma E=\frac{c^2}{v}\left(\frac{1}{\gamma}-\gamma\right)p_x+\gamma E=\gamma E-\gamma vp_x\frac{1}{\beta^2}\left(1-\frac{1}{\gamma^2}\right).$$

From $\gamma=\dfrac{1}{\sqrt{1-\beta^2}}$, it is easy to show that $\dfrac{1}{\beta^2}\left(1-\dfrac{1}{\gamma^2}\right)=1$. Therefore, $E'=\gamma\left(E-vp_x\right)$. Of course, for motion in one dimension (the x-direction), $p'_y=p_y$ and $p'_z=p_z$. Thus the Lorentz transformation for momentum and energy is established.

CALCULATE: Not required.

ROUND: Not required.

DOUBLE-CHECK: This result matches with the required expressions.

35.63. **THINK:** The Lorentz transformations for energy and momentum in the frame F' can be used to write the quantity $E'^2 - p'^2c^2$ in terms of the values in the unprimed frame *F*.
SKETCH: Not required.
RESEARCH: The Lorentz transformations are

$$E' = \gamma\left(E - vp_x\right), \quad p'_x = \gamma\left(p_x - vE/c^2\right), \quad p'_y = p_y \text{ and } p'_z = p_z.$$

SIMPLIFY: Apply the transformations:

$$\begin{aligned} E'^2 - p'^2c^2 &= E'^2 - p'^2_x c^2 - p'^2_y c^2 - p'^2_z c^2 = \gamma^2\left(E - vp_x\right)^2 - \gamma^2\left(p_x - vE/c^2\right)^2 c^2 - p_y^2c^2 - p_z^2c^2 \\ &= \gamma^2E^2 - 2\gamma^2 Evp_x + \gamma^2v^2p_x^2 - \gamma^2p_x^2c^2 + 2\gamma^2p_x vE - v^2E^2\gamma^2/c^2 - p_y^2c^2 - p_z^2c^2 \\ &= \gamma^2\left(1-(v/c)^2\right)E^2 + \gamma^2\left(v^2 - c^2\right)p_x^2 - p_y^2c^2 - p_z^2c^2 \\ &= \gamma^2\left(1-(v/c)^2\right)E^2 - \gamma^2\left(1-(v/c)^2\right)p_x^2c^2 - p_y^2c^2 - p_z^2c^2 = E^2 - p^2c^2. \end{aligned}$$

CALCULATE: Not required.
ROUND: Not required.
DOUBLE-CHECK: The statement in the problem has been proved using only the Lorentz transformation equations. One could check the result for special and limiting cases. For example, if $v = 0$ then $\gamma = 1$ and the Lorentz transformations reduce to $E' = E$ and $p'_x = p_x$, so the result holds. When $p = 0$ in the frame *F*,

$$\begin{aligned} E'^2 - p'^2c^2 &= E'^2 - p'^2_x c^2 - p'^2_y c^2 - p'^2_z c^2 = \gamma^2E^2 - \gamma^2\left(-vE/c^2\right)^2 c^2 \\ &= \gamma^2E^2 - v^2E^2\gamma^2/c^2 = \gamma^2\left(1-(v/c)^2\right)E^2 = E^2. \end{aligned}$$

35.64. The gravitational potential at the surface of the Earth – taking the potential to be zero at infinity – is the same as would be produced by a point mass $m_\oplus$ at the center of the Earth. Hence, the desired ratio is:

$$\frac{\Phi}{c^2} = -\frac{Gm_\oplus}{c^2 r_\oplus} = -\frac{\left(6.674\cdot 10^{-11}\text{ m}^3/\left(\text{kg s}^2\right)\right)\left(5.9736\cdot 10^{24}\text{ kg}\right)}{\left(2.998\cdot 10^8\text{ m/s}\right)^2\left(6.371\cdot 10^6\text{ m}\right)} = -6.962\cdot 10^{-10},$$

a dimensionless quantity. The deviation from flat space-time geometry produced by the Earth's gravitation is rather small.

35.65. (a) Using the formula for the Schwarzschild radius, the Schwarzschild radius corresponding to the mass of the Sun is

$$r_\text{S} = \frac{2GM_\text{S}}{c^2} = \frac{2\left(6.674\cdot 10^{-11}\text{ m}^3/\left(\text{kg s}^2\right)\right)\left(1.989\cdot 10^{30}\text{ kg}\right)}{\left(2.998\cdot 10^8\text{ m/s}\right)^2} = 2.954\text{ km},$$

a characteristic size scale for stellar-mass black holes.
(b) The Schwarzschild radius corresponding to a proton mass is

$$r_\text{S} = \frac{2Gm_\text{P}}{c^2} = \frac{2\left(6.674\cdot 10^{-11}\text{ m}^3/\left(\text{kg s}^2\right)\right)\left(1.673\cdot 10^{-27}\text{ kg}\right)}{\left(2.998\cdot 10^8\text{ m/s}\right)^2} = 2.485\cdot 10^{-54}\text{ m}.$$

This is much smaller than the femtometer size scale usually associated with protons: it is orders of magnitude smaller than the Planck scale (see Chapter 39), generally considered the smallest scale on which our basic notions of length make sense. Hence, it is unlikely that a proton could usefully be described via a classical black-hole geometry.

35.66. The time dilation between the Earth and the satellite is $\Delta t = \gamma \Delta t_0 \approx \left(1+\frac{1}{2}\beta^2\right)\Delta t_0$. The difference per Earth second is: $\Delta t = \gamma \Delta t_0 \approx \left(1+\frac{1}{2}\beta^2\right)\Delta t_0 \Rightarrow \left(\Delta t - \Delta t_0\right) = \frac{1}{2}\beta^2 \Delta t_0 \Rightarrow \frac{\Delta t - \Delta t_0}{\Delta t_0} = \frac{1}{2}\beta^2.$

$$\frac{1}{2}\beta^2 = \frac{1}{2}\left(\frac{\left(4.00\cdot 10^3 \text{ m/s}\right)}{\left(2.9979\cdot 10^8 \text{ m/s}\right)}\right)^2 = 8.90\cdot 10^{-11} \text{ s/}\left(\text{Earth second}\right) = 89.0 \text{ ps/}\left(\text{Earth second}\right)$$

$$\Rightarrow \frac{89.0 \text{ ps}}{\text{Earth second}}\cdot\frac{1 \text{ s}}{10^{12} \text{ ps}}\cdot\frac{86,400 \text{ Earth seconds}}{1 \text{ Earth Day}} = 7.69\cdot 10^{-6} \text{ s}.$$

This corresponds to a difference of $7.69\cdot 10^{-6}$ s/day μs/day.

35.67. The Schwarzschild radius of a black hole is $R_\text{S} = \frac{2GM}{c^2}$. The black hole at the center of the Milky Way in Example 12.4 was found to be $3.72\cdot 10^6$ solar masses. The mass of the Sun is $1.989\cdot 10^{30}$ kg. The Schwarzschild radius of this black hole is

$$R_\text{S} = \frac{2\left(6.674\cdot 10^{-11} \text{ N m}^2/\text{kg}^2\right)\left(3.72\cdot 10^6\right)\left(1.989\cdot 10^{30} \text{ kg}\right)}{\left(2.9979\cdot 10^8 \text{ m/s}\right)^2} = 10.99\cdot 10^9 \text{ m}\left(\frac{\text{AU}}{149.60\cdot 10^9 \text{ m}}\right)$$
$$= 0.0735 \text{ AU}.$$

35.68. In the garage's reference frame, the limousine is length contracted. The speed required for it to fit into the garage is

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1-\left(v/c\right)^2} \Rightarrow \left(L/L_0\right)^2 = 1-\left(v/c\right)^2$$
$$v = \sqrt{1-\left(L/L_0\right)^2}\,c = c\sqrt{1-\left(\left(35.0 \text{ ft}\right)/\left(50.0 \text{ ft}\right)\right)^2} = 0.71c.$$

In the limousine's reference frame, the length of the garage is length contracted by a factor of

$$L = \frac{L_0}{\gamma} \Rightarrow \gamma = \frac{L_0}{L} = \frac{50.0 \text{ ft}}{35.0 \text{ ft}} = 1.43.$$

35.69. The relativistic momentum of an electron is given by $p_\text{R} = \gamma m_e v = \frac{m_e v}{\sqrt{1-\left(v/c\right)^2}}$ where m_e is the mass of the electron. The classical momentum is $p_\text{C} = mv$. Therefore, the percentage difference between the classical and relativistic momenta is

$$\Delta p = \frac{p_\text{R} - p_\text{C}}{p_\text{C}}\left(100\,\%\right) = \frac{\gamma m_e v - m_e v}{\left(m_e v\right)}\left(100\,\%\right) = \left(\gamma - 1\right)\left(100\,\%\right)$$

For an electron moving at $v = 2.00\cdot 10^8 \text{ m/s} = \left(2.00/3.\right)00c$,

$$\gamma = 1/\sqrt{1-\left(v/c\right)^2} = 1/\sqrt{1-\left(2.00/3.00\right)^2} = 1.342.$$

Its relativistic momentum is

$$p_\text{R} = \gamma m_e v = \left(1.342\right)m_e(2c/3) = 0.8944\left(9.109\cdot 10^{-31} \text{ kg}\right)\left(\frac{2}{3}\left(3.00\cdot 10^8 \text{ m/s}\right)\right) = 2.44\cdot 10^{-22} \text{ kg m/s},$$

which differs from its classical value by $\Delta p = \left(\gamma - 1\right)\left(100\,\%\right) = 34\,\%$. For an electron moving at $v = 2.00\cdot 10^3 \text{ m/s} = \left(2.00\cdot 10^{-5}/3.00\right)c$,

$$\gamma = 1/\sqrt{1-(v/c)^2} = 1/\sqrt{1-\left(2.00\cdot 10^{-5}/3.00\right)^2} = 1.000$$

Its relativistic momentum is

$$p_{\text{R}} = \gamma m_e v = (1.000)m_e\left(2.00\cdot 10^{-5}/3\right)c = \left(9.109\cdot 10^{-31}\right)\left(2.00\cdot 10^{-5}/3\right)\left(3.00\cdot 10^{8}\text{ m/s}\right) = 1.82\cdot 10^{-27}\text{ kg m/s.}$$

This does not differ appreciably from its classical value since $(\gamma-1)(100\,\%)=0$ to many decimal places. For small velocities, the classical momentum of the electron is a good approximation.

35.70. Let the Earth be frame *F* and rocket A be the moving frame F'. The speed of rocket B in frame *F* is then $u = 0.95c$. The speed of frame F' with respect to frame *F* is $v = 0.75c$. The speed of rocket B relative to rocket A is then

$$u' = \frac{u-v}{1-uv/c^2} = \frac{(0.95c)-(0.75c)}{1-(0.95c)(0.75c)/c^2} = 0.70c.$$

35.71. The Newtonian and relativistic kinetic energies of a particle are $K_{\text{N}} = (1/2)mv^2$ and $K_{\text{R}} = (\gamma-1)mc^2$, respectively. In Newtonian mechanics, the difference in their kinetic energy is

$$\Delta K_{\text{N}} = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 = \frac{1}{2}m\left(v_1^2 - v_2^2\right) = \frac{1}{2}\left(0.9999^2 - 0.9900^2\right)mc^2$$
$$= \frac{1}{2}\left(0.9999^2 - 0.9900^2\right)\left(0.511\text{ MeV}/c^2\right)c^2 = 5.03\text{ keV.}$$

The difference using special relativity is

$$\Delta K_{\text{R}} = (\gamma_1 - 1)mc^2 - (\gamma_2 - 1)mc^2 = (\gamma_1 - \gamma_2)mc^2 = \left(\frac{1}{\sqrt{1-(v_1/c)^2}} - \frac{1}{\sqrt{1-(v_2/c)^2}}\right)mc^2$$
$$= \left(\frac{1}{\sqrt{1-0.9999^2}} - \frac{1}{\sqrt{1-0.9900^2}}\right)(0.511\text{ MeV}) = 32.5\text{ MeV.}$$

Therefore, we see that at velocities near *c*, the Newtonian approximation of Kinetic energy diverges from the relativistic Kinetic energy by several orders of magnitude.

35.72. (a) The clock of the friend waiting in B will show a longer time interval due to time dilation. The person traveling experiences time "slowing down" relative to a stationary observer.

(b) The time dilation is given by $\Delta t = \gamma \Delta t_0$. Since the velocity of the airplane is small compared to the speed of light, γ can be approximated as $\gamma \approx 1 + \frac{1}{2}\beta^2$. The difference in time between the two clocks is

$$\Delta t - \Delta t_0 = \gamma\Delta t_0 - \Delta t_0 = \frac{1}{2}\beta^2\Delta t_0 = \frac{1}{2}\left(\frac{240\text{ m/s}}{3.00\cdot 10^{8}\text{ m/s}}\right)^2(3.00\text{ h})(3600\text{ s/h}) = 3.5\text{ ns.}$$

35.73. The mass can be found from the energy:

$$E = mc^2 \Rightarrow m = \frac{E}{c^2} = \frac{(15.0)\left(4.18\cdot 10^{12}\text{ J}\right)}{\left(3.00\cdot 10^{8}\text{ m/s}\right)^2} = 6.9667\cdot 10^{-4}\text{ kg} = 0.697\text{ g.}$$

35.74. The speed can be found using the equation for length contraction:

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1-(v/c)^2} \Rightarrow (v/c)^2 = 1-(L/L_0)^2 \Rightarrow v = \sqrt{1-(L/L_0)^2}\,c = \sqrt{1-\left(\frac{90.0\text{ cm}}{100.\text{ cm}}\right)^2}\,c = 0.436c.$$

35.75. Using the relativistic velocity transformation, the speed of object A relative to object B as measured by an observer on object B is

$$u' = \frac{v_A - v_B}{1 - v_A v_B / c^2} = \frac{(0.600c) - (-0.600c)}{1 - (0.600c)(-0.600c)/c^2} = 0.882c.$$

35.76. The length contraction factor is one-third so $\gamma = 3$. Therefore, the relative velocity is

$$\frac{1}{\sqrt{1-(v/c)^2}} = 3 \Rightarrow 1-(v/c)^2 = \frac{1}{9} \Rightarrow 1 - \frac{1}{9} = (v/c)^2 \Rightarrow v = \frac{2\sqrt{2}}{3}c$$

35.77. The average speed on the ~~trip, which~~ took 40.0 hours to travel 2200.0 miles, was 55.0 mph. Since the velocity of the vehicle is small compared to the speed of light, γ can be approximated as $\gamma \approx 1 + \frac{1}{2}\beta^2$. Therefore, the difference in time between your watch and your professor's watch (your watch runs slow) is

$$\Delta t - \Delta t_0 = \gamma \Delta t_0 - \Delta t_0 = \frac{1}{2}\beta^2 \Delta t_0 = \frac{1}{2}\left(\frac{55.0 \text{ mph}\left(\frac{1 \text{ h}}{3600 \text{ s}}\right)\left(\frac{1609.3 \text{ m}}{1 \text{ mi}}\right)}{3.00 \cdot 10^8 \text{ m/s}} \right)^2 (40.0 \text{ h})(3600. \text{ s/h}) = 0.484 \text{ ns}$$

This amount of time is very tiny and could not be a reason for being late.

35.78. Because of the second postulate of relativity, both observers measure the speed of light to be the same.

(a) The speed of light measured on the spaceship is *c*.

(b) The speed of light measured on the asteroid is also *c*.

35.79. The distance of 100. ly was measured by someone on one of the space stations. Someone on the spaceship will measure a different distance, one that is shorter according to the formula for length contraction, $L = L_0 / \gamma$. The time it takes to travel from one space station to the next as measured by someone on the spaceship is

$$t_1 = \frac{L}{v} = \frac{L_0}{\gamma v} = \frac{L_0}{v}\sqrt{1-(v/c)^2} = \frac{(100. \text{ ly})}{(0.950c)}\sqrt{1-(0.950c/c)^2} = 32.8684 \text{ years} \approx 33 \text{ years}.$$

As seen by someone on the space station, the time will be

$$t_2 = \frac{L}{v} = \frac{(100. \text{ ly})}{(0.950c)} = 105 \text{ years}.$$

35.80. The electron gains kinetic energy from the potential: $K = (\gamma - 1)mc^2 = qV$. Solving for the velocity v:

$$(\gamma-1)mc^2 = qV \Rightarrow \gamma - 1 = \frac{qV}{mc^2} \Rightarrow \frac{1}{\sqrt{1-(v/c)^2}} = \frac{qV}{mc^2} + 1 \Rightarrow \frac{1}{1-(v/c)^2} = \left(\frac{qV}{mc^2}+1\right)^2$$

$$\Rightarrow 1-(v/c)^2 = \frac{1}{\left(1+\frac{qV}{mc^2}\right)^2} \Rightarrow v = \sqrt{1-\left(1+\frac{qV}{mc^2}\right)^{-2}}\; c.$$

The rest mass energy of the electron is $mc^2 = 0.511 \text{ MeV}$ and the potential energy is $qV = e(1.0 \cdot 10^6 \text{ V}) = 1.0 \text{ MeV}$. Thus the electron attains a speed of

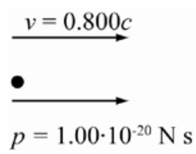
$$v = \sqrt{1-\left(1+\frac{(1.0 \text{ MeV})}{(0.511 \text{ MeV})}\right)^{-2}}\; c = 0.94c.$$

35.81. As seen by those on the ship, the round trip distance is length contracted to $L = \frac{L_0}{\gamma}$, where $L_0 / c = 4000.0 \text{ yr}$ constitutes the time for round-trip travel. If the speed of the ship is v and the journey must take only $t = 40.000 \text{ yr}$ then the required speed is

$$v = \frac{L_0}{\gamma t} = \frac{L_0}{t}\sqrt{1-(v/c)^2} \Rightarrow (ct/L_0)^2 v^2 = c^2 - v^2 \Rightarrow v = \frac{c}{\sqrt{1+(ct/L_0)^2}}$$

$$v = \frac{c}{\sqrt{1+\left((40.000 \text{ yr})/(4000.0 \text{ yr})\right)^2}} = 0.99995c.$$

35.82. **THINK:** The particle is moving at a speed of $v = 0.800c$. The mass of the particle is unknown, but the momentum of the particle is $p = 1.00 \cdot 10^{-20}$ N s. This is all that is required to find the energy of the particle.
SKETCH:



RESEARCH: The energy and momentum of a relativistic particle are $E = \gamma mc^2$ and $p = \gamma mv$ respectively.

SIMPLIFY: $E = \gamma mc^2 = \frac{p}{mv} mc^2 = \frac{pc^2}{v}$

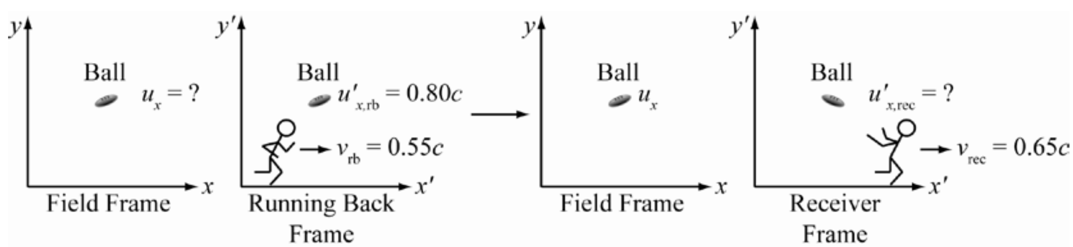
CALCULATE: $E = \frac{pc^2}{v} = \frac{(1.00 \cdot 10^{-20} \text{ N}\cdot\text{s})(2.9979 \cdot 10^8 \text{ m/s})c}{(0.800c)} = 3.747 \cdot 10^{-12} \text{ J} = 23.392 \text{ MeV}$

ROUND: To three significant figures, the energy of the particle is $3.75 \cdot 10^{-12}$ J or 23.4 MeV.
DOUBLE-CHECK: This is a typical energy for a high energy particle. For $v = 0.800c$, the value of γ is found to be 5/3. Hence the mass of the particle is $m = p/\gamma v = 0.25 \cdot 10^{-28}$ kg , which is a reasonable mass for an atomic particle. Using this mass, the energy of the particle is

$$E = \gamma mc^2 = (5/3)(0.25 \cdot 10^{-28} \text{ kg})(2.9979 \cdot 10^8 \text{ m/s})^2 = 3.747 \cdot 10^{-12} \text{J},$$

which agrees with the calculated value.

35.83. **THINK:** The running back is travelling at 55.0% the speed of light relative to the field. He throws the ball to a receiver running at 65.0% the speed of light relative to the field in the same direction. The speed of the ball relative to the running back is 80.0% the speed of light. The relativistic velocity transformation can be used to find the speed that the receiver perceives the ball to be travelling at. Recall that the speed of light is the same in all reference frames.
SKETCH:



RESEARCH: The velocity of the ball with respect to the running back is $u'_x = 0.800c$. The velocity of the running back with respect to the field is $v_{\text{rb}} = 0.550c$. The inverse Lorentz transformation can be used to find the velocity u_x of the ball in the field frame:

$$u_x = \frac{u'_{x,\text{rb}} + v_{\text{rb}}}{1 + u'_{x,\text{rb}} v_{\text{rb}} / c^2}.$$

Using a Lorentz transform gives the speed of the ball relative to the receiver:

$$u'_{x,\text{rec}} = \frac{u_x - v_{\text{rec}}}{1 - u_x v_{\text{rec}} / c^2},$$

where $v_{\text{rec}} = 0.650c$ is the velocity of the receiver relative to the field.
SIMPLIFY: Not required.
CALCULATE:

(a) $u_x = \dfrac{(0.800c) + (0.550c)}{1 + (0.800c)(0.550c)/c^2} = 0.9375c \Rightarrow u'_{x,\text{rec}} = \dfrac{(0.9375c) - (0.650c)}{1 - (0.9375c)(0.650c)/c^2} = 0.7360c$

(b) Photons travel at the speed of light and the speed of light is the same in any reference frame; therefore, the photons would appear to be travelling at the speed of light to the receiver.
ROUND:
(a) To three significant figures, the speed of the ball perceived by the receiver is $u'_{x,\text{rec}} = 0.736c = 2.21 \cdot 10^8$ m/s.
DOUBLE-CHECK: The calculated value of the football's relative speed was less than the speed of light as it must be, since no massive object can travel at the speed of light.

35.84. **THINK:** The ^{14}C electrons have kinetic energy $K = 0.305E_0$, where E_0 is the rest energy. The baseline between the detectors is $\Delta x = 2.0$ m. Find the necessary timing accuracy needed by the detectors to show that the expression for the relativistic momentum, and not the expression for the non-relativistic momentum, is correct.
SKETCH: Not required.
RESEARCH: The rest energy is $E_0 = mc^2$. The non-relativistic momentum is $p_{\text{nr}} = m_{\text{e}} v_{\text{nr}}$, and the non-relativistic kinetic energy is $K_{\text{nr}} = (1/2) m_{\text{e}} v_{\text{nr}}^2$, where m_{e} is the electron's mass. The non-relativistic velocity v_{nr} can be determined from these equations. The relativistic momentum is $p_{\text{r}} = \gamma m_{\text{e}} v_{\text{r}}$, where $\gamma = 1/\sqrt{1-(v_{\text{r}}/c)^2}$. The relativistic kinetic energy is $K = (\gamma - 1)E_0$. The relativistic velocity v_{r} can be determined from these equations. Finally, the time needed to travel a distance Δx is $t = \Delta x / v$.
SIMPLIFY: Non-relativistic case:

$$K_{\text{nr}} = \frac{1}{2} m_{\text{e}} v_{\text{nr}}^2 = 0.305E_0 \Rightarrow v_{\text{nr}} = \sqrt{2(0.305E_0)/m_{\text{e}}}$$

Substituting $E_0 = m_{\text{e}} c^2$ into the equation gives

$$v_{\text{nr}} = \sqrt{2(0.305 m_{\text{e}} c^2)/m_{\text{e}}} = \sqrt{0.610}c \Rightarrow t_{\text{nr}} = \frac{\Delta x}{v_{\text{nr}}} = \frac{\Delta x}{\sqrt{0.610}c}$$

Relativistic case:

$$K = (\gamma - 1)E_0 = 0.305E_0 \Rightarrow \gamma = 1.305$$

$$\gamma = \frac{1}{\sqrt{1-(v_{\text{r}}/c)^2}} = 1.305 \Rightarrow (1.305)^2 = \frac{c^2}{c^2 - v_r^2} \Rightarrow (1.305)^2 (c^2 - v_r^2) = c^2$$

$$\Rightarrow (1.305)^2 v_{\text{r}}^2 = \left((1.305)^2 - 1\right)c^2 \Rightarrow v_{\text{r}} = \sqrt{\frac{(1.305)^2 - 1}{(1.305)^2}}c = 0.6425c \Rightarrow t_{\text{r}} = \frac{\Delta x}{v_{\text{r}}} = \frac{\Delta x}{c}\sqrt{\frac{(1.305)^2}{(1.305)^2 - 1}}$$

CALCULATE: $t_{nr} = \dfrac{(2.0\text{ m})}{\sqrt{0.610}\left(3.00\cdot 10^{8}\text{ m/s}\right)} = 8.5358\text{ ns}, \quad t_{r} = \dfrac{(2.0\text{ m})}{\left(3.00\cdot 10^{8}\text{ m/s}\right)}\sqrt{\dfrac{(1.305)^2}{(1.305)^2 - 1}} = 10.376\text{ ns}$

ROUND: To two significant figures, $t_{nr} = 8.5\text{ ns}$ and $t_r = 10.\text{ ns}$. By comparison of the calculated values t_{nr} and t_r, the necessary timing accuracy is on the order of 1 ns.

DOUBLE-CHECK: The calculated values for t_{nr} and t_r had the correct units.

35.85. **THINK:** The spacecraft travels a distance of $d = 1.00\cdot 10^{-3}$ ly in a time of $\Delta t = 20.0$ hrs as measured by an observer stationed on Earth. The length of the journey, Δt_0, as measured by the captain of the spacecraft will be shorter due to time dilation.

SKETCH: Not required.

RESEARCH: The speed of the spacecraft is given by $v = d/\Delta t$. The expression for time dilation is given by $\Delta t = \gamma \Delta t_0$.

SIMPLIFY: $\Delta t = \dfrac{\Delta t_0}{\sqrt{1-(v/c)^2}} \Rightarrow \Delta t_0 = \Delta t\sqrt{1-(v/c)^2} \Rightarrow \Delta t_0 = \Delta t\sqrt{1-\left(d/c\Delta t\right)^2}$

CALCULATE: Since $d/c = 1.00\cdot 10^{-3}$ yr,

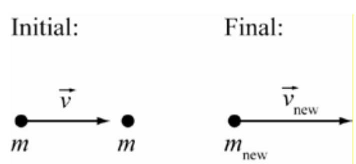
$$\Delta t_0 = (20.0\text{ hr})\sqrt{1-\left(\frac{\left(1.00\cdot 10^{-3}\text{ yr}\right)\left(8.766\cdot 10^{3}\text{ hr/yr}\right)}{(20.0\text{ hr})}\right)^2} = 17.977\text{ hr}.$$

ROUND: To three significant figures, $\Delta t_0 = 18.0$ hr.

DOUBLE-CHECK: The time measured by the captain is shorter than the time measured by the observer on the Earth. This makes sense because the captain is traveling at the same speed as the spacecraft (e.g. the captain is at rest with respect to the spacecraft). According to the time dilation theory, a moving clock runs slower than a clock at rest.

35.86. **THINK:** A hypothetical particle with rest mass $m = 1.000\text{ GeV}/c^2$ and kinetic energy $K = 1.000$ GeV collides with an identical particle at rest. The two particles fuse to form a single new particle. Total energy and momentum are both conserved in the collision. Find (a) the momentum p and speed v of the first particle and (b) the rest mass m_{new} and speed v_{new} of the new particle.

SKETCH:



RESEARCH: The total energy is $E = \gamma mc^2 = E_0 + K$, where $E_0 = mc^2 = 1.000\text{ GeV}$ and $K = (\gamma - 1)E_0$. The relationship between energy and momentum is given by $E^2 = p^2c^2 + m^2c^4$.

SIMPLIFY:

(a) The momentum of the first particle is given by

$$E^2 = p^2c^2 + m^2c^4 = p^2c^2 + E_0^2 \Rightarrow p^2 = \frac{E^2 - E_0^2}{c^2}$$

$$p = \sqrt{\left(E_0 + K\right)^2 - E_0^2}\,/\,c = \sqrt{2E_0K + K^2}\,/\,c.$$

The speed of the first particle is given by

$$K=\left(\frac{1}{\sqrt{1-(v/c)^2}}-1\right)E_0 \Rightarrow K+E_0=\frac{E_0}{\sqrt{1-(v/c)^2}} \Rightarrow v=\sqrt{\left(1-\frac{E_0^2}{(K+E_0)^2}\right)}c.$$

(b) The rest mass m_{new} of the new particle can be found by using the relationship between energy and momentum:

$$E_{\text{new}}^2=p_{\text{new}}^2c^2+m_{\text{new}}^2c^4 \Rightarrow m_{\text{new}}^2=\frac{E_{\text{new}}^2-p_{\text{new}}^2c^2}{c^4} \Rightarrow m_{\text{new}}=\sqrt{E_{\text{new}}^2-p_{\text{new}}^2c^2}\,/\,c^2.$$

By energy and momentum conservation, the newly formed particle has the same total energy and momentum as the two original particles did prior to the collision, so

$$E_{\text{new}}=2E_0+K=3.000\text{ GeV},$$

and $p_{\text{new}}=p,$ which was found in part (a). The speed of the new particle is given by:

$$p_{\text{new}}=\gamma m_{\text{new}}v_{\text{new}}=\frac{m_{\text{new}}v_{\text{new}}}{\sqrt{1-(v_{\text{new}}/c)^2}} \Rightarrow p_{\text{new}}^2=\frac{m_{\text{new}}^2v_{\text{new}}^2}{1-(v_{\text{new}}/c)^2}=\frac{m_{\text{new}}^2c^2v_{\text{new}}^2}{c^2-v_{\text{new}}^2}$$

$$p_{\text{new}}^2c^2-p_{\text{new}}^2v_{\text{new}}^2=m_{\text{new}}^2c^2v_{\text{new}}^2 \Rightarrow v_{\text{new}}=\frac{p_{\text{new}}c}{\sqrt{m_{\text{new}}^2c^2+p_{\text{new}}^2}}.$$

CALCULATE:

(a) $p=\sqrt{2(1.000\text{ GeV})(1.000\text{ GeV})+(1.000\text{ GeV})^2}\,/\,c=1.73205\text{ GeV}/c$

$$v=\sqrt{\left(1-\frac{(1.000\text{ GeV})^2}{((1.000\text{ GeV})+(1.000\text{ GeV}))^2}\right)}c=0.86603c$$

(b) $m_{\text{new}}=\sqrt{(3.000\text{ GeV})^2-(1.73205\text{ GeV}/c)^2c^2}\,/\,c^2=2.44949\text{ GeV}/c^2$

$$v_{\text{new}}=\frac{(1.73205\text{ GeV}/c)c}{\sqrt{(2.44949\text{ GeV}/c^2)^2c^2+(1.73205\text{ GeV}/c)^2}}=0.57735c$$

ROUND: To four significant figures,

(a) $p=1.732\text{ GeV}/c,\quad v=0.8660c$

(b) $m_{\text{new}}=2.449\text{ GeV}/c^2,\quad v_{\text{new}}=0.5774c$

DOUBLE-CHECK: The mass of the new particle is on the same order as the mass of a proton, $m_{\text{p}}=0.938\text{ GeV}/c^2,$ so it is reasonable. The calculated speeds are large, but are realistic for small masses.

35.87. **THINK:** In considering accelerating bodies with special relativity, the acceleration experienced by the moving body is constant; that is, in each increment of the body's own proper time, $d\tau,$ the body acquires velocity increment $dv=gd\tau$ as measured in the body's frame (the inertial frame in which the body is momentarily at rest). Given this interpretation,

(a) Write a differential equation for the velocity v of the body, moving in one spatial dimension, as measured in the inertial frame in which the body was initially at rest (the "ground frame").

(b) Solve this equation for $v(t),$ where both v and t are measured in the ground frame.

(c) Verify that the solution behaves appropriately for small and large values of t.

(d) Calculate the position of the body $x(t),$ as measured in the ground frame.

(e) Identify the trajectory of the body on a Minkowski diagram with coordinates x and $ct,$ as measured in the ground frame.

(f) For $g = 9.81\text{ m/s}^2$, calculate how much time t it takes the body to accelerate from rest to 70.7% of c, as measured in the ground frame, and how much ground-frame distance, Δx, the body covers in this time.

SKETCH: Not required.

RESEARCH: In moving from the ground frame to the next frame, the body's velocity was incremented by dv. Since we are interested in a differential equation for the velocity as measured in the ground frame, an inverse Lorentz transformation from the next frame to the ground frame is necessary:

$$u_{\text{ground}} = \frac{u_{\text{next}} + v}{1 + u_{\text{next}} v / c^2} \Rightarrow v + dv = \frac{v + dv}{1 + v dv / c^2}.$$

The increment of the body's proper time $d\tau$ is related to the increment of ground-frame time dt by time dilation, $d\tau = \left(1 - (v/c)^2\right)^{1/2} dt$. The trajectory of the body in a space-time diagram will be determined by examining the position as a function of time, which is determined in part (d).

SIMPLIFY:

(a) Ignoring squares and higher powers of differentials,

$$v + dv = \frac{v + g d\tau}{1 + v g d\tau / c^2} = (v + g d\tau)\left(1 - \frac{v g d\tau}{c^2} + \ldots\right) = v + g\left(1 - (v/c)^2\right) d\tau + \ldots, \text{ or } dv = g\left(1 - (v/c)^2\right) d\tau.$$

But the increment of proper time $d\tau$ is related to the increment of ground-frame time dt by time dilation so the differential equation, in terms of ground frame quantities, becomes

$$dv = g\left(1 - (v/c)^2\right) d\tau = g\left(1 - (v/c)^2\right)\left(1 - (v/c)^2\right)^{1/2} dt$$

$$\frac{dv}{dt} = g\left(1 - (v/c)^2\right)^{3/2}$$

(b) The above differential equation separates, yielding

$$g\int_0^t dt' = \int_0^{v(t)} \frac{dv'}{\left(1 - (v'/c)^2\right)^{3/2}} \Rightarrow gt = \frac{v(t)}{\left(1 - \left(v(t)/c\right)^2\right)^{1/2}}.$$

This is readily solved, giving $v(t) = \dfrac{gt}{\left(1 + (gt/c)^2\right)^{1/2}}$ for the ground-frame velocity of the accelerating body as a function of ground-frame time.

(c) For $gt \ll c$, i.e., the Newtonian limit, the above result takes the form $v(t) \cong gt$, exactly as expected. The relativistic limit, as time approaches infinity is:

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{gt}{\sqrt{1 + (gt/c)^2}} = \frac{gt}{\sqrt{(gt/c)^2}} = c.$$

That is, the velocity of the accelerating body asymptotically approaches c, as expected.

(d) The position follows from the velocity through integration:

$$x(t) = \frac{c^2}{g} + \int_0^t v(t')dt' = \frac{c^2}{g} + \int_0^t \frac{gt'dt'}{\left[1 + (gt'/c)^2\right]^{1/2}} = \frac{c^2}{g} + \frac{c^2}{g}\left[1 + \left(\frac{gt'}{c}\right)^2\right]^{1/2}\Bigg|_0^t = \frac{c^2}{g}\left[1 + \left(\frac{gt}{c}\right)^2\right]^{1/2}$$

(e) The above result implies the simple relation $x^2 - (ct)^2 = c^4 / g^2$. The right-hand side is constant. Hence, the trajectory is a branch of a hyperbola on a Minkowski diagram.

(f) Consider the ground-frame speed as a function of ground-frame time from part (b),

$$v(t)=\frac{gt}{\sqrt{1+(gt/c)^2}}.$$

The time t required for the body to accelerate from rest to $v=0.707c$ is given by:

$$v=\frac{gt}{\sqrt{1+(gt/c)^2}} \Rightarrow v^2\left(1+(gt/c)^2\right)=(gt)^2 \Rightarrow t=\frac{v}{g\sqrt{1-(v/c)^2}}.$$

The ground-frame distance travelled in this time is $\Delta x = x(t)-x(0)$. As stated in the problem, the ground-frame position at ground-frame time $t=0$ is $x(0)=c^2/g$. Then

$$\Delta x=\frac{c^2}{g}\left[1+\left(\frac{gt}{c}\right)^2\right]^{1/2}-\frac{c^2}{g}=\frac{c^2}{g}\left(\sqrt{1+\frac{(v/c)^2}{1-(v/c)^2}}-1\right).$$

CALCULATE:

(f) $t=\dfrac{(0.707)(2.998\cdot 10^8\text{ m/s})}{(9.81\text{ m/s}^2)\sqrt{1-((0.707c)/c)^2}}=3.055\cdot 10^7\text{ s}=353.6\text{ days},$

$$\Delta x=\frac{(2.998\cdot 10^8\text{ m/s})^2}{(9.81\text{ m/s}^2)}\left(\sqrt{1+\frac{((0.707c)/c)^2}{1-((0.707c)/c)^2}}-1\right)=3.793\cdot 10^{15}\text{ m}=0.4009\text{ ly}$$

ROUND: The answers should be quoted to three significant figures:

(f) $t=354$ days, and $\Delta x=0.401$ ly.

DOUBLE-CHECK: The motion of an object with constant proper acceleration in special relativity should be described by a hyperbola, as found in parts (d) and (e). The values found in part (f) are reasonable considering the relatively slow acceleration of 9.81 m/s^2.

Multi-Version Exercises

35.88. $W=\Delta E=(\gamma_2-\gamma_1)mc^2=\left(\dfrac{1}{\sqrt{1-(v_2/c)^2}}-\dfrac{1}{\sqrt{1-(v_1/c)^2}}\right)mc^2$

$$=\left(\frac{1}{\sqrt{1-0.8433^2}}-\frac{1}{\sqrt{1-0.5785^2}}\right)(183.473\text{ GeV})=116.4493263\text{ GeV}=116\text{ GeV}$$

Note that because of the subtraction rule the answer has only three significant figures.

35.89. $W = \Delta E = (\gamma_2 - \gamma_1)mc^2 = \left(\dfrac{1}{\sqrt{1-\beta_2^2}} - \dfrac{1}{\sqrt{1-\beta_1^2}} \right) mc^2$

$$\Rightarrow \frac{W}{mc^2} + \frac{1}{\sqrt{1-\beta_1^2}} = \frac{1}{\sqrt{1-\beta_2^2}}$$

$$\Rightarrow \frac{1}{\left(\dfrac{W}{mc^2} + \dfrac{1}{\sqrt{1-\beta_1^2}} \right)^2} = 1 - \beta_2^2$$

$$\Rightarrow \beta_2 = \sqrt{1 - \frac{1}{\left(\dfrac{W}{mc^2} + \dfrac{1}{\sqrt{1-\beta_1^2}} \right)^2}} = \sqrt{1 - \frac{1}{\left(\dfrac{140.779}{183.473} + \dfrac{1}{\sqrt{1-0.4243^2}} \right)^2}} = 0.8453$$

35.90. $W = \Delta E = (\gamma_2 - \gamma_1)mc^2 = \left(\dfrac{1}{\sqrt{1-\beta_2^2}} - \dfrac{1}{\sqrt{1-\beta_1^2}} \right) mc^2$

$$\Rightarrow \frac{1}{\sqrt{1-\beta_1^2}} = \frac{1}{\sqrt{1-\beta_2^2}} - \frac{W}{mc^2}$$

$$\Rightarrow \frac{1}{\left(\dfrac{1}{\sqrt{1-\beta_2^2}} - \dfrac{W}{mc^2} \right)^2} = 1 - \beta_1^2$$

$$\Rightarrow \beta_1 = \sqrt{1 - \frac{1}{\left(\dfrac{1}{\sqrt{1-\beta_2^2}} - \dfrac{W}{mc^2} \right)^2}} = \sqrt{1 - \frac{1}{\left(\dfrac{1}{\sqrt{1-0.8475^2}} - \dfrac{137.782}{183.473} \right)^2}} = 0.4701$$

35.91. $K_{\text{lab}} = 4K_{\text{cm}} + 2K^2_{\text{cm}} / mc^2 = 4(503.01 \text{ GeV}) + 2(503.01 \text{ GeV})^2 / (50.30 \text{ GeV}) = 12.072 \text{ TeV}$

35.92. $K_{\text{lab}} = 4K_{\text{cm}} + 2K^2_{\text{cm}} / mc^2$

$\Rightarrow mc^2 = 2K^2_{\text{cm}} / (K_{\text{lab}} - 4K_{\text{cm}})$

$= 2(621.38 \text{ GeV})^2 / (15161.70 \text{ GeV} - 4(621.38 \text{ GeV})) = 60.92 \text{ GeV}$

35.93. $K_{\text{lab}} = 4K_{\text{cm}} + 2K^2_{\text{cm}} / mc^2$

$\Rightarrow K^2_{\text{cm}} + 2K_{\text{cm}} mc^2 - \tfrac{1}{2} K_{\text{lab}} mc^2 = 0$

$\Rightarrow K_{\text{cm}} = \sqrt{\tfrac{1}{2} K_{\text{lab}} mc^2 + m^2c^4} - mc^2$

$= \sqrt{\tfrac{1}{2}(10868.96 \text{ GeV})(23.94 \text{ GeV}) + (23.94 \text{ GeV})^2} - (23.94 \text{ GeV})$

$= 337.6 \text{ GeV}$

Chapter 36: Quantum Physics

Concept Checks

36.1. a **36.2.** b **36.3.** d **36.4.** d **36.5.** b **36.6.** d

Multiple-Choice Questions

36.1. b **36.2.** e **36.3.** a and c **36.4.** c **36.5.** c **36.6.** c **36.7.** c **36.8.** b

Conceptual Questions

36.9. The spectral emittance as a function of wavelength is given by equation (36.13): $\varepsilon(\lambda)=\dfrac{2\pi hc^2}{\lambda^5\left(e^{hc/(\lambda K_B T)}-1\right)}.$

For a given temperature the emittance depends on the wavelength of the light. Red light corresponds to a certain wavelength, but white light is made up of a distribution of visible light with different wavelengths. This is shown in Figure 36.4 of the textbook. The temperature of an object is inversely proportional to its minimum wavelength, as ε is explained by Wien's displacement law. In the spectrum shown in Figure 36.4, you can see that the minimum wavelength of the red light corresponds to a longer wavelength than that of the white light, therefore the white-hot object is hotter than the red-hot object.

36.10. It is generally accepted that electrons exhibit both wave and particle like behavior. The wave like nature of electrons was postulated by de Broglie and experimental evidence was provided later. Details of the experimental evidence of the wave like properties of elections are discussed in the double-slit experiment for particles in the textbook. Einstein's analysis of the photoelectric effect provides evidence of the particle like behavior of electrons.

36.11. The formula for Compton scattering is given by equation (36.20) in the textbook: $\lambda'=\lambda+\dfrac{h}{m_e c}(1-\cos\theta).$

Blue light corresponds to photons with wavelength ranging from $450\cdot 10^{-9}$ m to $495\cdot 10^{-9}$ m. If you choose a value of $\theta=45°$ and $\lambda=450\cdot 10^{-9}$ m, then the final wavelength of the photon after scattering is:

$$\lambda'=450\cdot 10^{-9}\text{ m}+\frac{\left(6.626\cdot 10^{-34}\text{ Js}\right)}{\left(9.109\cdot 10^{-31}\text{ kg}\right)\left(3.00\cdot 10^{8}\text{ m/s}\right)}\left(1-\cos 45°\right)=450.00071\cdot 10^{-9}\text{ m},$$

which is still well within the range of wavelengths corresponding to the blue light. The increase in wavelength after Compton Scattering is on the order of 10^{-12} m, which is too small to affect wavelengths in the visible spectrum.

36.12. The Heisenberg uncertainty relation for energy and time is given by equation (36.27) in the textbook: $\Delta E\cdot\Delta t\geq\hbar/2.$ Rearranging this equation to solve for Δt gives: $\Delta t=\hbar/\left(2\Delta E\right).$ For a proton-antiproton pair $\Delta E=2\left(m_p c^2\right)=2\left(1.672\cdot 10^{-27}\text{ kg}\right)\left(3.00\cdot 10^{8}\text{ m/s}\right)^2\dfrac{1\text{ eV}}{1.602\cdot 10^{-19}\text{ J}}\approx 1879\cdot 10^{6}\text{ eV}.$ Substituting this into the equation for Δt gives: $\Delta t=\dfrac{6.5821\cdot 10^{-16}\text{ eV s}}{2\left(1879\cdot 10^{6}\text{ eV}\right)}\approx 1.75\cdot 10^{-25}\text{ s}.$ This is the maximum lifetime of the proton-antiproton pair.

36.13. If Planck's constant was 5 J s the wavelength of the tennis ball, as well as other objects, would be very large relative to our universe. Consider a tennis ball with a mass, $m=0.050$ kg that is travelling at a speed $v=20$ m/s. In the universe where $h=5$ J s the tennis ball would have a wavelength of

$\lambda' = \dfrac{5 \text{ J s}}{(0.050 \text{ kg})(20 \text{ m/s})} = 5 \text{ m}.$ By comparison in our universe $h = 6.626 \cdot 10^{-34}$ J s, so the wavelength of the tennis ball is $\lambda = \dfrac{6.626 \cdot 10^{-34} \text{ J s}}{(0.050 \text{ kg})(20 \text{ m/s})} = 6.626 \cdot 10^{-34}$ m. The tennis ball with the 5 m wavelength would appear very fuzzy and could possibly diffract through the spacing in the tennis rackets or the net.

36.14. In classical mechanics force is defined as $\vec{F} = m\vec{a}$, where m is the mass of the object and $\vec{a}$ is its acceleration. If $\vec{F} = 0$ for a massive particle it means that $\vec{a} = 0$ which implies the particle is travelling at a constant velocity. If you know the particles position and velocity at one point you can predict where the particle will be at some later time. In quantum mechanics the Heisenberg uncertainty relation states that you cannot instantaneously know, with absolute certainty an object's position and momentum. Since momentum depends on the object's mass and velocity ($p = mv$), you cannot predict with certainty the object's trajectory from this information.

36.15. The classical physicist would expect that increasing the intensity of the UV light shining on the metal surface should increase the maximum kinetic energy of the electrons ejected from that surface. This was not observed in experiments. The experimental observations showed that increasing the intensity of the light increased the number of elections ejected from the metal surface but it did not increase their kinetic energy.

36.16. The power of the visible light source is $P_\text{v} = 60 \text{ W} = 60 \text{ J/s}$. The power of the X-ray source is $P_\text{x} = 0.002 \text{ W} = 0.002 \text{ J/s}$. The wavelength of visible light, λ_v is on the order of $\sim 1 \cdot 10^{-7}$ m to $1 \cdot 10^{-6}$ m. The wavelength of X-rays are on the order of $\sim 1 \cdot 10^{-13}$ m (hard X-rays) to 10^{-8} m (soft X-rays). The electromagnetic spectrum is shown in Figure 31.10 of the textbook. If you consider a one second time interval, the energy of the visible light is $E_\text{v} = 60$ J and the energy of the X-rays is $E_\text{x} = 0.002$ J, so in terms of the total energy from the visible light source is $E_\text{v} = 30000 E_\text{x}$ in one second. In terms of energy per photon, the X-ray photons will have higher energy and be much more damaging to the skin than a photon of visible light. For example taking values of $\lambda_\text{v} = 1 \cdot 10^{-6}$ m and $\lambda_\text{x} = 1 \cdot 10^{-10}$ m. Using the equation $E = hc/\lambda$ you can obtain the ratio $\dfrac{E'_\text{v}}{E'_\text{x}} = \dfrac{\lambda_\text{x}}{\lambda_\text{v}} = 1 \cdot 10^{-4}$ or $E'_\text{v} = 1 \cdot 10^{-4} E'_\text{x}$, where the prime denote that we are discussing energy per photon. It should be noted that hard X-rays ($\lambda \sim 10^{-12}$ m) are very energetic and can cause immediate damage to human cells.

36.17. Neutrons in the neutron beam that have spins that are aligned with the spin of the neutrons in the nucleus of the polarized ^{3}He will not be absorbed because of the Pauli Exclusion Principle (The Pauli exclusion principle is discussed in the textbook). Fewer neutrons in the unpolarized beam will be absorbed to form ^{4}He nuclei. The aligned neutrons in the beam that are not absorbed by the ^{3}He can be reused on a neutron beam to polarize the neutron beam.

36.18. The photocathode is made of Cesium which has a work function $\phi = 2.1$ eV, this corresponds to a maximum wavelength of $\lambda_\text{max} = 590 \text{ nm} = 590 \cdot 10^{-9}$ m (see Table 36.1 in textbook). Using green laser light of wavelength $\lambda = 514.5$ nm corresponds to an energy of $E = hc/\lambda \approx 2.4$ eV which is enough energy to cause electrons to be emitted from the Cesium cathode. Doubling the power of the green laser results in an increase in intensity of the light hitting the cathode, this will increase the number of ejected electrons, but the energy per electron will remain the same.

Exercises

36.19. Assuming the surface temperature of the Sun is $T_{\text{Sun}} = 5800.\text{ K}$ and the surface temperature of the Earth is $T_{\text{earth}} = 300.\text{ K}$, Wien's displacement law can be used to calculate the respective peak wavelengths.

(a) $\lambda_{\text{m,Sun}} = \dfrac{2.90\cdot 10^{-3}\text{ K m}}{T_{\text{Sun}}} = \dfrac{2.90\cdot 10^{-3}\text{ K m}}{5800.\text{ K}} = 5.00\cdot 10^{-7}\text{ m}$

(b) $\lambda_{\text{m,Earth}} = \dfrac{2.90\cdot 10^{-3}\text{ K m}}{T_{\text{Earth}}} = \dfrac{2.90\cdot 10^{-3}\text{ K m}}{300.\text{ K}} \approx 9.67\cdot 10^{-6}\text{ m}$

36.20. The peak emission temperature can be calculated using Wien's displacement law: $T = \dfrac{2.90\cdot 10^{-3}\text{ K m}}{\lambda_{\text{m}}}$. The total intensity of the radiation from the filament at a given temperature can be found using equation (36.1) from the textbook: $I = \sigma T^4$, where σ is the Stefan-Boltzmann constant. At the short end of the visible spectrum $\lambda_{\text{m}_1} = 380\text{ nm} = 380\cdot 10^{-9}\text{ m}$ and $T_1 = \dfrac{2.90\cdot 10^{-3}\text{ K m}}{380\cdot 10^{-9}\text{ m}} = 7631.6\text{ K}$. The intensity at this temperature is $I_1 = 5.6704\cdot 10^{-8}\text{ W/}\left(\text{m}^2\text{ K}^4\right)\cdot\left(7631.6\text{ K}\right)^4 = 1.9\cdot 10^8\text{ W/m}^2$. At the long end of the visible spectrum $\lambda_{\text{m}_2} = 780\text{ nm} = 780\cdot 10^{-9}\text{ m}$ and $T_2 = \dfrac{2.90\cdot 10^{-3}\text{ K m}}{780\cdot 10^{-9}\text{ m}} = 3717.9\text{ K}$.

The intensity at this temperature is:

$$I_2 = 5.6704\cdot 10^{-8}\text{ W/}\left(\text{m}^2\text{ K}^4\right)\cdot\left(3717.9\text{ K}\right)^4 = 1.1\cdot 10^7\text{ W/m}^2.$$

36.21. A gamma ray with energy $E = 3.5\cdot 10^{12}\text{ eV}$ would have a very short wavelength. The wavelength of a gamma ray with this energy is $\lambda_\gamma = \dfrac{hc}{E} = \dfrac{\left(4.136\cdot 10^{-15}\text{ eV s}\right)\left(3.00\cdot 10^8\text{ m/s}\right)}{3.5\cdot 10^{12}\text{ eV}} = 3.545\cdot 10^{-19}\text{ m} \approx 3.5\cdot 10^{-19}\text{ m}$.

The rest mass energy of a proton is $E_{o,\text{p}} = m_{\text{p}}c^2$, where $m_{\text{p}} = 938.3\cdot 10^6\text{ eV/c}^2$, so $E_{o,\text{p}} = 938.3\cdot 10^6\text{ eV}$.

The energy of the gamma ray is $\dfrac{3.5\cdot 10^{12}\text{ eV}}{938.3\cdot 10^6\text{ eV}} \approx 3700$ times greater than the rest mass energy of a proton.

36.22. The temperature of the object is $T = 20.\text{ °C} = 293\text{ K}$. Consider the radiation the object emits at the peak of the spectral energy density.

(a) The peak wavelength can be calculated using Wien's displacement law.

$$\lambda_{\text{m}} = \frac{2.90\cdot 10^{-3}\text{ m K}}{293\text{ K}} = 9.8976\cdot 10^{-6}\text{ m} \approx 9.90\cdot 10^{-6}\text{ m}$$

(b) The frequency at this wavelength is $f = \dfrac{c}{\lambda_{\text{m}}} = \dfrac{3.00\cdot 10^8\text{ m/s}}{9.8976\cdot 10^{-6}\text{ m}} = 3.0310\cdot 10^{13}\text{ Hz} \approx 3.03\cdot 10^{13}\text{ Hz}$.

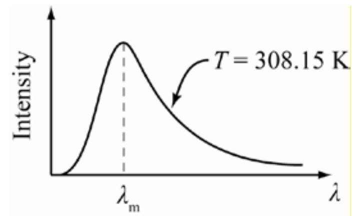
(c) The energy of one photon of light at this frequency is given by:

$$E = hf = \left(6.626\cdot 10^{-34}\text{ J s}\right)\left(3.0310\cdot 10^{13}\text{ s}^{-1}\right) = 2.008366\cdot 10^{-20}\text{ J} \approx 2.01\cdot 10^{-20}\text{ J}.$$

Expressed in eV, $E = 2.008366\cdot 10^{-20}\text{ J}\cdot\dfrac{1\text{ eV}}{1.602\cdot 10^{-19}\text{ J}} = 0.125366\text{ eV} \approx 0.125\text{ eV}$.

36.23. **THINK:** The temperature of your skin is approximately $T = 35.0\text{ °C} = 308.15\text{ K}$. Assume that it is a blackbody. Consider a total surface area of $A = 2.00\text{ m}^2$. (a) The Wien displacement law can be used to determine the peak wavelength λ_{m} of the radiation emitted by the skin, (b) the Stefan-Boltzmann radiation law can be used to determine the total power P emitted by your skin, and (c) the wavelength of the radiation needs to be considered.

SKETCH:



RESEARCH:

(a) The Wien displacement law is $\lambda_m T = 2.90 \cdot 10^{-3}$ K m.

(b) The total power is $P = IA$, where I, the intensity, is given by the Stefan-Boltzmann radiation law:

$$I = \sigma T^4, \text{ using } \sigma = 5.6704 \cdot 10^{-8} \text{ W}/\left(\text{m}^2 \text{ K}^4\right).$$

(c) Power is energy per unit time. The relationship between energy and wavelength for photons is: $E = hc / \lambda$.

SIMPLIFY:

(a) $\lambda_m = \dfrac{2.90 \cdot 10^{-3} \text{ K m}}{T}$

(b) $P = IA$, substituting $P = \sigma T^4$ gives: $P = \sigma T^4 A$.

CALCULATE:

(a) $\lambda_m = \dfrac{2.90 \cdot 10^{-3} \text{ K m}}{308.15 \text{ K}} = 9.4110 \cdot 10^{-6}$ m

(b) $P = \left(5.6704 \cdot 10^{-8} \text{ W}/\left(\text{m}^2 \text{ K}^4\right)\right)\left(308.15 \text{ K}\right)^4 \left(2.00 \text{ m}^2\right) = 1022.57$ W

(c) Considering that a typical light bulb has a power of 100 W, why is it that a person does not glow with a power output of about 1000 W? The reason is because in order for you to "glow" your wavelength must be in the visible spectrum. However, the peak wavelength calculated in part (a) is $\lambda_m = 9.4111$ μm. This wavelength is in the infrared part of the spectrum, not the visible part. Your wavelength is not in the visible spectrum.

ROUND:

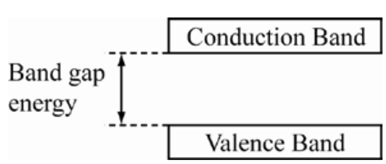
(a) To three significant figures, the peak wavelength is $\lambda_m = 9.41$ μm.

(b) To three significant figures, the total power emitted by your skin is $P = 1.02$ kW.

DOUBLE-CHECK: The calculated values seem reasonable considering the given values. Comparing the calculated peak wavelength to the values in Figure 31.10 of the textbook shows that the wavelength of the emitted radiation is out of the visible spectrum.

36.24. **THINK:** The known room-temperature band-gap energies for germanium, silicon and gallium-arsenide are $E_{Ge} = 0.66$ eV, $E_{Si} = 1.12$ eV, and $E_{Ga\text{-}As} = 1.42$ eV, respectively. The wavelength of photons can be calculated from the photon energy to (a) find the room-temperature transparency range of these three semiconductors, and (b) to explain the yellow color observed for ZnSe crystals, which have a band-gap of 2.67 eV. Semiconductors are only transparent if the energy (wavelength) of the photon is lower (higher) than the band-gap energy. (c) For a material to be used as a light detector, it must be able to absorb the incident light. This means the detecting material must have a band-gap corresponding to a longer wavelength than the incident light.

SKETCH:



RESEARCH: The wavelengths associated with the given band-gap energies can be determined using the equation: $\lambda = hc/E$.
SIMPLIFY: Not required.
CALCULATE:

(a) $$\lambda_{\text{Ge}} = \frac{\left(4.136\cdot 10^{-15}\text{ eV s}\right)\left(3.00\cdot 10^{8}\text{ m/s}\right)}{\left(0.66\text{ eV}\right)} = 1.880\cdot 10^{-6}\text{ m}$$

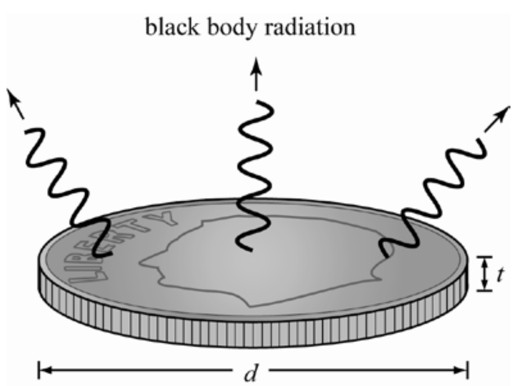
$$\lambda_{\text{Si}} = \frac{\left(4.136\cdot 10^{-15}\text{ eV s}\right)\left(3.00\cdot 10^{8}\text{ m/s}\right)}{\left(1.12\text{ eV}\right)} = 1.108\cdot 10^{-6}\text{ m}$$

$$\lambda_{\text{Ga-As}} = \frac{\left(4.136\cdot 10^{-15}\text{ eV s}\right)\left(3.00\cdot 10^{8}\text{ m/s}\right)}{\left(1.42\text{ eV}\right)} = 8.738\cdot 10^{-7}\text{ m}$$

(b) $$\lambda_{\text{ZnSe}} = \frac{\left(4.136\cdot 10^{-15}\text{ eV s}\right)\left(3.00\cdot 10^{8}\text{ m/s}\right)}{\left(2.67\text{ eV}\right)} = 4.647\cdot 10^{-7}\text{ m}$$

ROUND: The transparency range for these semiconductors is:
(a) $\lambda_{\text{Ge}} > 1900$ nm, $\lambda_{\text{Si}} > 1110$ nm, and $\lambda_{\text{Ga-As}} > 874$ nm.
(b) The photon wavelength corresponding to the band-gap energy for ZnSe is $\lambda_{\text{ZnSe}} = 465$ nm. Therefore, only the blue end of the visible spectrum will be absorbed by the ZnSe. This results in the yellow color that is observed for ZnSe crystals.
(c) The only material that had a wavelength greater than 1550 nm was germanium $\left(\lambda_{\text{Ge}} = 1880\text{ nm}\right)$. This means that germanium is not transparent to the 1550 nm light and would be useful as a detector for this optical communications wavelength.
DOUBLE-CHECK: The calculated wavelengths are reasonable and all had the correct units. It is expected that a material with a low band-gap energy will be able to absorb radiation with a large wavelength.

36.25. **THINK:** The mass of a dime is $m = 2.268\cdot 10^{-3}$ kg, its diameter is $d = 17.91\cdot 10^{-3}$ m, and its thickness is $t = 1.350\cdot 10^{-3}$ m. (a) The Stefan-Boltzmann radiation law can be used to determine the total radiant energy coming from the dime. (b) Wien's displacement law can be used to determine the wavelength of peak emission of each photon. Since each photon carries the same amount of energy, the number of photons can be determined. (c) With the temperature known, the thermal energy of air can be calculated. The Ideal Gas Law can be used to determine the volume of air required for it to have the same energy as the energy radiated from the dime in 1 second. Take room temperature to be $T = 20.0\text{ °C} = 293.15\text{ K}$.
SKETCH:



RESEARCH:

(a) The radiant energy per second can be found using the equation $P = IA$, where I is the intensity and A_{t} is the total surface area of the dime, given by: $A_{\text{t}} = \pi d\left(\left(d/2\right)+t\right)$. By assuming the dime is an ideal radiator, it is valid to use the Stefan-Boltzmann radiation law: $I = \sigma T^4$.

(b) The energy of one photon is given by $E = hc/\lambda$. The wavelength that corresponds to peak emission can be found using Wien's displacement law: $\lambda_{\text{m}} T = 2.90\cdot 10^{-3}$ K m.

(c) If it is assumed that the air is made up of diatomic molecules, the energy per molecule is: $E_{\text{air}} = \left(3/2\right)k_{\text{B}}T$. Note that at at room temperature and standard pressure one mole ($6.022\cdot 10^{23}$ molecules) of air occupies a volume of $V_1 = 22.4\cdot 10^{-3}\text{ m}^3$.

SIMPLIFY:

(a) $P = IA$, substituting $I = \sigma T^4$ gives: $P = \sigma T^4 \pi d\left(\left(d/2\right)+t\right)$. The radiant energy per second is $P\cdot 1$ second.

(b) $E = nhf$, where $f = c/\lambda$ therefore the energy is: $E = nhc/\lambda$. The wavelength can be found from Wien's displacement law $\lambda_{\text{m}} T = 2.90\cdot 10^{-3}$ K m, substitution into the energy equation gives:

$$E = nhcT/\left(2.90\cdot 10^{-3}\text{ K m}\right)$$

$$n = \left(2.90\cdot 10^{-3}\text{ K m}\right)E/\left(hcT\right)$$

(c) $E_{\text{air}} = \frac{3}{2}k_{\text{B}}T$. The number of molecules of air corresponding to the radiant energy emitted from the dime in one second is: $N = E/E_{\text{air}}$. The volume of air with energy equal to one second of radiation from the dime is given by:

$$V_{\text{T}} = \nu N = \frac{(22.4\text{ L})N}{6.022\cdot 10^{23}}$$

CALCULATE:

(a) $P = \left(5.6704\cdot 10^{-8}\text{ W}/\left(\text{m}^2\text{K}^4\right)\right)\left(293.15\text{ K}\right)^4 \pi\left(17.91\cdot 10^{-3}\text{ m}\right)\left(\frac{17.91}{2}\cdot 10^{-3}\text{ m}+1.350\cdot 10^{-3}\text{ m}\right)$

$= 0.2428$ W

$E = \left(0.2428\text{ J/s}\right)\left(1\text{ sec}\right) = 0.2428$ J

(b) $n = \dfrac{\left(2.90\cdot 10^{-3}\text{ K m}\right)\left(0.2428\text{ J}\right)}{\left(6.626\cdot 10^{-34}\text{ Js}\right)\left(3.00\cdot 10^{8}\text{ m/s}\right)\left(293.15\text{ K}\right)}$

$= 1.208\cdot 10^{19}$ photons/second

(c) $N = \dfrac{0.2428\text{ J}}{\frac{3}{2}\left(1.381\cdot 10^{-23}\text{ J K}^{-1}\right)\left(293\text{ K}\right)} = 3.998\cdot 10^{19}$ molecules of air

$$V_{\text{T}} = \frac{\left(22.4\cdot 10^{-3}\text{ m}^3\right)\left(3.998\cdot 10^{19}\text{ molecules}\right)}{\left(6.022\cdot 10^{23}\text{ molecules}\right)} = 1.487\cdot 10^{-6}\text{ m}^3$$

ROUND: The calculated values should be reported to three significant figures, therefore:

(a) $E = 0.243$ J

(b) $n = 1.21\cdot 10^{19}$ photons per second

(c) $V_{\text{T}} = 1.49 \cdot 10^{-6}\text{ m}^3$.

DOUBLE-CHECK: The calculated values all had the correct units. It is reasonable that only a very small amount of energy is radiated from the dime at room temperature. It also seems reasonable that the volume of air that has energy equal to one second of radiation from the dime is small.

36.26. The given work function is $\phi = 5.8\text{ eV}$. The minimum light frequency necessary for the photoelectric effect to occur is given by equation (36.15) in the textbook.

$$f_{\text{min}} = \frac{\phi}{h} = \frac{5.8\text{ eV}}{4.136 \cdot 10^{-15}\text{ eV s}} = 1.402321 \cdot 10^{15}\text{ s}^{-1} \approx 1.4 \cdot 10^{15}\text{ s}^{-1}$$

36.27. The light that is incident on the sodium surface is $\lambda = 470\text{ nm} = 470 \cdot 10^{-9}\text{ nm}$. The work function for sodium is $\phi = 2.3\text{ eV}$ (see Table 36.1 in textbook). The maximum kinetic energy of the electrons ejected from the sodium surface is $K_{\text{max}} = eV_0 = hf - \phi$. For photons $f = c/\lambda$,

$$f = \frac{3.00 \cdot 10^{8}\text{ m/s}}{470 \cdot 10^{-9}\text{ m}} = 6.38 \cdot 10^{14}\text{ s}^{-1}.$$

Inserting this value into the equation for K_{max} gives:

$$K_{\text{max}} = \left(4.136 \cdot 10^{-15}\text{ eV s}\right)\left(6.38 \cdot 10^{14}\text{ s}^{-1}\right) - 2.3\text{ eV} = 0.34\text{ eV}.$$

36.28. The threshold wavelength is given as $\lambda = 400.\text{ nm} = 400. \cdot 10^{-9}\text{ m}$. Frequency and wavelength for photons are related by the equation $f = c/\lambda$. The work function, ϕ of the alloy can be determined using equation (36.15) from the textbook:

$$\phi = f_{\text{min}} h = \frac{hc}{\lambda_{\text{min}}} = \frac{\left(3.00 \cdot 10^{8}\text{ m/s}\right)\left(4.136 \cdot 10^{-15}\text{ eV s}\right)}{400. \cdot 10^{-9}\text{ m}} = 3.10\text{ eV}$$

36.29. The work function of Cesium is $\phi = 2.100\text{ eV}$. The stopping potential for this material is $V_0 = 0.310\text{ V}$. When the laser is shined on cathode made of an unknown material the stopping potential is found to be $V_0' = 0.110\text{ V}$.

(a) The wavelength of the laser light is found using equation (36.16): $eV_0 = hf - \phi = \frac{hc}{\lambda} - \phi$.

Where $\lambda = \frac{hc}{\left(eV_0 + \phi\right)} = \frac{\left(4.136 \cdot 10^{-15}\text{ eV s}\right)\left(3.00 \cdot 10^{8}\text{ m/s}\right)}{\left(0.310\text{ eV} + 2.100\text{ eV}\right)} = 5.15 \cdot 10^{-7}\text{ m}$, this wavelength can be used to find the work function of the unknown material, ϕ_{u}.

$$\phi_{\text{u}} = \frac{hc}{\lambda} - eV_0 = \frac{\left(4.136 \cdot 10^{-15}\text{ eV s}\right)\left(3.00 \cdot 10^{8}\text{ m/s}\right)}{5.1485 \cdot 10^{-7}\text{ m}} - 0.110\text{ eV} = 2.30\text{ eV}$$

(b) Work function for a number of common elements are listed in Table 36.1 in the textbook. Possible candidate materials for the unknown cathode would be potassium or sodium. They both have work functions of 2.3 eV.

36.30. The incident light has a wavelength of $\lambda = 550\text{ nm} = 550 \cdot 10^{-9}\text{ m}$. The work function of zinc is $\phi = 4.3\text{ eV}$. (See table 36.1 in text) In order for the photoelectric effect to occur the energy of the incident light must be equal to or greater than the work function of zinc. The energy of a photon of light with $\lambda = 550 \cdot 10^{-9}\text{ m}$ is given by: $E = \frac{hc}{\lambda} = \frac{\left(4.136 \cdot 10^{-15}\text{ eV s}\right)\left(3.00 \cdot 10^{8}\text{ m/s}\right)}{550 \cdot 10^{-9}\text{ m}} = 2.3\text{ eV}$. The energy of the incident light is not sufficient to eject any electrons from the zinc surface so there will not be any photoelectric current and therefore no stopping voltage is required.

36.31. White light is made up of photons with wavelengths ranging from $\lambda = 4.00 \cdot 10^2$ nm to $7.50 \cdot 10^2$ nm ($4.00 \cdot 10^{-7}$ m to $7.50 \cdot 10^{-7}$ m). The work function of barium is given as $\phi = 2.48$ eV.

(a) The maximum kinetic energy of an electron ejected from the barium surface will correspond to a photon with the minimum wavelength.

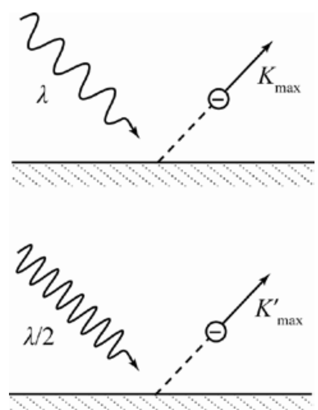
$$K_{\text{max}} = \frac{hc}{\lambda_{\text{min}}} - \phi = \frac{\left(4.136 \cdot 10^{-15} \text{ eV s}\right)\left(3.00 \cdot 10^8 \text{ m/s}\right)}{4.00 \cdot 10^{-7} \text{ m}} - 2.48 \text{ eV} = 0.622 \text{ eV}$$

(b) The longest wavelength of light that could eject electrons is given by $\lambda = \frac{hc}{\phi} = \frac{\left(4.136 \cdot 10^{-15} \text{ eV s}\right)\left(3.00 \cdot 10^8 \text{ m/s}\right)}{2.48 \text{ eV}} = 5.00 \cdot 10^{-7} \text{ m} = 5.00 \cdot 10^2 \text{ nm}$. This means that the $7.50 \cdot 10^2$ nm wavelength light would not eject electrons from the barium surface.

(c) The wavelength of light that would eject electrons with zero kinetic energy is given by: $\lambda = hc / \phi$ which was solved in part (b). The wavelength was $\lambda = 5.00 \cdot 10^2$ nm.

36.32. **THINK:** The maximum kinetic energy measured is $K_{\text{max}} = 1.50$ eV when the wavelength is λ. When the wavelength is decreased to $\lambda / 2$, the maximum kinetic energy measured is $K'_{\text{max}} = 3.80$ eV. By considering the photoelectric effect, (a) the work function of the material and (b) the original wavelength can be determined.

SKETCH:



RESEARCH: Combining equation (36.14) and (36.16) from the textbook gives: $K_{\text{max}} = hf - \phi$, where the frequency is $f = c / \lambda$.

SIMPLIFY:

(a) $K_{\text{max}} = \frac{hc}{\lambda} - \phi \Rightarrow \frac{hc}{\lambda} = K_{\text{max}} + \phi$, $K'_{\text{max}} = \frac{hc}{\lambda / 2} - \phi \Rightarrow \frac{K'_{\text{max}}}{2} = \frac{hc}{\lambda} - \frac{\phi}{2} = \left(K_{\text{max}} + \phi\right) - \frac{\phi}{2}$,

$\frac{K'_{\text{max}}}{2} = K_{\text{max}} + \frac{\phi}{2}$, so $\phi = K'_{\text{max}} - 2K_{\text{max}}$

(b) $K_{\text{max}} = \frac{hc}{\lambda} - \phi \Rightarrow \lambda = \frac{hc}{K_{\text{max}} + \phi} = \frac{hc}{K_{\text{max}} + \left(K'_{\text{max}} - 2K_{\text{max}}\right)}$, and therefore $\lambda = \frac{hc}{K'_{\text{max}} - K_{\text{max}}}$.

CALCULATE:

(a) $\phi = \left(3.80 \text{ eV}\right) - 2\left(1.50 \text{ eV}\right) = 0.800 \text{ eV}$

(b) $$\lambda = \frac{\left(4.136 \cdot 10^{-15} \text{ eV s}\right)\left(3.00 \cdot 10^8 \text{ m/s}\right)}{\left(3.80 \text{ eV}\right) - \left(1.50 \text{ eV}\right)} = 5.39478 \cdot 10^{-7} \text{ m}$$

ROUND:

(a) To three significant figures, the work function is $\phi = 0.800$ eV.

(b) To three significant figures, the original wavelength is $\lambda = 539$ nm.
DOUBLE-CHECK: The calculated values have the correct units.

36.33. The X-rays have wavelength, $\lambda = 0.120 \text{ nm} = 0.120 \cdot 10^{-9}$ m. They are scattered by the carbon. The angle between the incoming and outgoing photon is $\theta = 90.0°$. The formula for Compton scattering is given by Equation (36.20) in the textbook: $\lambda' = \lambda + \frac{h}{m_e c}(1-\cos\theta)$. $\Rightarrow$ $\lambda' - \lambda = \Delta\lambda = \frac{h}{m_e c}(1-\cos\theta)$. Inserting the proper values gives:

$$\Delta\lambda = \frac{(6.626 \cdot 10^{-34} \text{ J s})(1-\cos 90.0°)}{(9.109 \cdot 10^{-31} \text{ kg})(3.00 \cdot 10^{8} \text{ m/s})} = 2.42 \cdot 10^{-12} \text{ m}$$

This is the Compton wavelength shift.

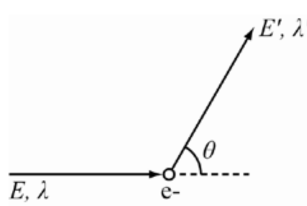
36.34. The wavelength of the incoming photon is: $\lambda = \frac{hc}{E} = \frac{1240 \text{ eV nm}}{2.0 \cdot 10^{6} \text{ eV}} = 6.20 \cdot 10^{-4} \text{ nm} = 6.20 \cdot 10^{-13}$ m. The outgoing photon's wavelength can be found using the Compton scattering formula $\lambda' = \lambda + \frac{h}{m_e c}(1-\cos\theta) = (6.20 \cdot 10^{-13} \text{ m}) + (2.426 \cdot 10^{-12} \text{ m})(1-\cos 53°) = 1.6 \cdot 10^{-12} \text{ m} = 1.6 \cdot 10^{-3}$ nm.

36.35. The wavelength of the incoming photon is $\lambda = 0.30$ nm; its original energy was: $E = hf = \frac{hc}{\lambda} = \frac{(4.13567 \cdot 10^{-15} \text{ eV s})(2.998 \cdot 10^{8} \text{ m/s})}{3.0 \cdot 10^{-10} \text{ m}} = 4133$ eV. It rebounds at angle of $\theta = 160°$. Its new wavelength can be found using the Compton scattering formula.

$$\lambda' = \lambda + \frac{h}{m_e c}(1-\cos\theta) = (3.0 \cdot 10^{-10} \text{ m}) + \frac{(6.626 \cdot 10^{-34} \text{ J s})(1-\cos 160°)}{(9.109 \cdot 10^{-31} \text{ kg})(2.998 \cdot 10^{8} \text{ m/s})} = 3.047 \cdot 10^{-10} \text{ m}$$

Its new energy is: $E' = \frac{hc}{\lambda'} = \frac{(4.13567 \cdot 10^{-15} \text{ eV s})(2.998 \cdot 10^{8} \text{ m/s})}{3.047 \cdot 10^{-10} \text{ m}} = 4069$ eV. The amount of energy lost is $\Delta E = E - E' = 4133 \text{ eV} - 4069 \text{ eV} = 64$ eV.

36.36. **THINK:** The X-rays have an initial energy $E = 4.000 \cdot 10^{5}$ eV. They undergo Compton scattering from a target, and the scattered rays are detected at $\theta = 25.0°$ relative to the incident rays. (a) The formula for Compton scattering can be used to find the energy of the scattered X-ray, E', and (b) conservation of energy can be used to find the energy of the recoiling electron, E_e.
SKETCH:



RESEARCH:
(a) The energy of a photon is $E = hc/\lambda$. The wavelength of the scattered X-ray is given by the Compton scattering formula:

$$\lambda' = \lambda + \frac{h}{m_e c}(1-\cos\theta).$$

(b) Due to energy conservation in Compton scattering, the energy lost by the scattered photon is imparted onto the electron, that is, $K_e = E - E'$.

SIMPLIFY:
(a) The energy of the scattered X-ray is:

$$E' = \frac{hc}{\lambda'} = \frac{hc}{\dfrac{hc}{E} + \dfrac{h}{m_e c}(1-\cos\theta)} = \left(\frac{1}{E} + \frac{(1-\cos\theta)}{m_e c^2}\right)^{-1}.$$

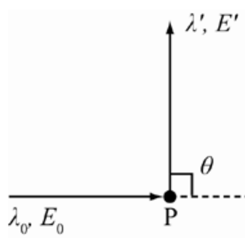
(b) No simplification is required.
CALCULATE:

(a) $E' = \left(\dfrac{1}{(4.000\cdot 10^5 \text{ eV})} + \dfrac{(1-\cos(25.0°))}{(0.51100\cdot 10^6 \text{ eV})}\right)^{-1} = 3.7267\cdot 10^5 \text{ eV}$

(b) $K_e = (4.000\cdot 10^5 \text{ eV}) - (3.7267\cdot 10^5 \text{ eV}) = 2.7332\cdot 10^4 \text{ eV}$

ROUND:
(a) The energy of the scattered X-ray is $E' = 373$ keV.
(b) The kinetic energy of the recoiling electron is $K_e = 27.3$ keV.
DOUBLE-CHECK: The X-ray should lose energy after scattering off of the electron. As expected, this energy loss is equal to the kinetic energy of the electron: $E' + K_e = 372.7 \text{ keV} + 27.33 \text{ keV} = 400.0 \text{ keV} = E.$

36.37. **THINK:**
(a) X-rays of energy $E_0 = 140. \text{ keV} = 2.243\cdot 10^{-14}$ J bounce off of a proton at $\theta = 90.0°$. The Compton scattering formula can be used to find their fractional change in energy, $f = (E_0 - E)/E_0$.
(b) The equation derived in part (a) can be used to find the energy of a photon that would be necessary to cause a 1.00% change in energy at $\theta = 90.0°$ scattering.
SKETCH:



RESEARCH: The energy of a photon is $E = hc/\lambda$. The wavelength of the scattered photon is found from the Compton scattering formula, but with the mass of a proton substituted for the mass of an electron:

$$\lambda' = \lambda + \frac{h}{m_p c}(1-\cos\theta)$$

The mass of a proton is $m_p = 1.673\cdot 10^{-27}$ kg.
SIMPLIFY:

(a) $f = \dfrac{E_0 - E}{E_0} = \dfrac{E_0 - \dfrac{hc}{\lambda'}}{E_0} = 1 - \dfrac{hc}{E_0\lambda'} = 1 - \dfrac{hc}{E_0\left(\dfrac{hc}{E_0} + \dfrac{h}{m_p c}(1-\cos\theta)\right)} = \dfrac{E_0(1-\cos\theta)}{m_p c^2 + E_0(1-\cos\theta)}$

(b) Using the equation from part (a),

$$f = \frac{E_0(1-\cos\theta)}{m_p c^2 + E_0(1-\cos\theta)} \Rightarrow f\left(m_p c^2 + E_0(1-\cos\theta)\right) = E_0(1-\cos\theta) \Rightarrow E_0 = \frac{f m_p c^2}{(1-f)(1-\cos\theta)}$$

CALCULATE:

(a) $$f=\frac{\left(2.243\cdot 10^{-14}\text{ J}\right)\left(1-\cos 90.0°\right)}{(1.673\cdot 10^{-27}\text{ kg})(2.998\cdot 10^{8}\text{ m/s})^2+\left(2.243\cdot 10^{-14}\text{ J}\right)\left(1-\cos 90.0°\right)}=1.491\cdot 10^{-4}$$

(b) For a 1.00% change in energy, $f=0.0100$:

$$E_0=\frac{\left(0.0100\right)\left(1.673\cdot 10^{-27}\text{ kg}\right)\left(2.998\cdot 10^{8}\text{ m/s}\right)^2}{\left(1-(0.0100)\right)\left(1-\cos 90.0°\right)}=1.519\cdot 10^{-12}\text{ J}=9.481\text{ MeV}$$

ROUND:

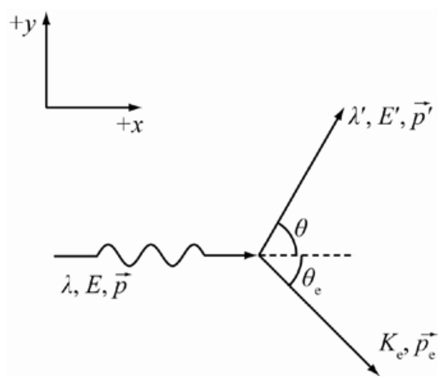
(a) $f=1.49\cdot 10^{-4}$

(b) $E_0=9.48$ MeV

DOUBLE-CHECK: To get a 1.00% fractional change in energy, gamma-rays would be required. These are extremely high energy photons. This is one reason why electrons are used for scattering experiments (the photons do not have to be as energetic).

36.38. **THINK:** The X-ray photon has an energy of $E=5.00\cdot 10^4$ eV. It strikes an electron which is initially at rest inside a metal and is scattered at an angle of $\theta=45°$. The Compton scattering formula can be used to find the kinetic energy K_e and momentum p_e (magnitude and direction) of the electron after the collision. Conservation of energy and momentum can also be used to solve the problem.

SKETCH:



RESEARCH: The energy of a photon is $E=hc/\lambda$. The momentum of a photon is $p=h/\lambda$. The wavelength of the scattered photon, according to the Compton scattering formula, is

$$\lambda'=\lambda+\frac{h}{m_e c}\left(1-\cos\theta\right).$$

In Compton scattering, energy is conserved. The energy that is lost by the photon is imparted to the electron, that is, $K_e=E-E'$. Momentum is also conserved in this collision, that is, $\vec{p}=\vec{p}'+\vec{p}_e$. For scattering in two dimensions, this becomes $p_x=p_x'+p_{ex}$ and $p_y=p_y'+p_{ey}$. The magnitude of the electron's momentum is $p_e=\sqrt{p_{ex}^{\ 2}+p_{ey}^{\ 2}}$, and the direction is $\theta_e=\tan^{-1}\left(p_{ey}/p_{ex}\right)$.

SIMPLIFY: The kinetic energy of the electron is given by:

$$\lambda'=\lambda+\frac{h}{m_e c}\left(1-\cos\theta\right)=\frac{hc}{E}+\frac{h}{m_e c}\left(1-\cos\theta\right)$$

$$K_e=E-E'=E-\frac{hc}{\lambda'}$$

The momentum of the electron is given by:

$$p_{ex} = p_x - p'_x = p - p'\cos\theta = \frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta = \frac{E}{c} - \frac{h\cos\theta}{\lambda'}$$
$$p_{ey} = p_y - p'_y = 0 - p'\sin\theta = -\frac{h}{\lambda'}\sin\theta$$
$$p_e = \sqrt{p_{ex}^{\ 2} + p_{ey}^{\ 2}}$$
$$\theta_e = \tan^{-1}\left(\frac{p_{ey}}{p_{ex}}\right)$$

CALCULATE:

$$\lambda' = \frac{\left(4.13567\cdot 10^{-15}\text{ eV s}\right)\left(2.998\cdot 10^{8}\text{ m/s}\right)}{\left(5.00\cdot 10^{4}\text{ eV}\right)} + \frac{\left(6.626\cdot 10^{-34}\text{ J s}\right)\left(1-\cos\left(45°\right)\right)}{\left(9.109\cdot 10^{-31}\text{ kg}\right)\left(2.998\cdot 10^{8}\text{ m/s}\right)} = 2.5508\cdot 10^{-11}\text{ m}$$

$$K_e = \left(5.00\cdot 10^{4}\text{ eV}\right) - \frac{\left(4.13567\cdot 10^{-15}\text{ eV s}\right)\left(2.998\cdot 10^{8}\text{ m/s}\right)}{\left(2.5508\cdot 10^{-11}\text{ m}\right)} = 50.0\text{ keV} - 48.607\text{ keV} = 1.393\text{ keV}$$

$$p_{ex} = \frac{\left(5.00\cdot 10^{4}\text{ eV}\right)\left(1.602\cdot 10^{-19}\text{ J/eV}\right)}{\left(2.998\cdot 10^{8}\text{ m/s}\right)} - \frac{\left(6.626\cdot 10^{-34}\text{ J s}\right)\cos\left(45°\right)}{\left(2.5508\cdot 10^{-11}\text{ m}\right)} = 8.350\cdot 10^{-24}\text{ kg m/s}$$

$$p_{ey} = -\frac{\left(6.626\cdot 10^{-34}\text{ J s}\right)\sin\left(45°\right)}{\left(2.5508\cdot 10^{-11}\text{ m}\right)} = -1.837\cdot 10^{-23}\text{ kg m/s}$$

$$p_e = \sqrt{\left(8.350\cdot 10^{-24}\text{ kg m/s}\right)^2 + \left(-1.837\cdot 10^{-23}\text{ kg m/s}\right)^2} = 2.018\cdot 10^{-23}\text{ kg m/s}$$

$$\theta_e = \tan^{-1}\left(\frac{\left(-1.837\cdot 10^{-23}\text{ kg m/s}\right)}{\left(8.350\cdot 10^{-24}\text{ kg m/s}\right)}\right) = -65.6°$$

(The negative means the angle is made below the positive x-axis.)

ROUND: To two significant figures: $K_e = 1.4$ keV, $p_e = 2.0\cdot 10^{-23}$ kg m/s, and $\theta_e = -66°$.

DOUBLE-CHECK: Using the non relativistic equation, the momentum of the electron is $p_e = m_e v_e$, when $v_e = \sqrt{\frac{2K_e}{m_e}}$. Then,

$$p_e = \sqrt{2K_e m_e} = \sqrt{2\left(1.393\cdot 10^{3}\text{ eV}\right)\left(1.602\cdot 10^{-19}\text{ J/eV}\right)\left(9.109\cdot 10^{-31}\text{ kg}\right)} \approx 2.0\cdot 10^{-23}\text{ kg m/s.}$$

This is in agreement with the value in the solution.

36.39. (a) The wavelength of a photon is $\lambda = hc/E$. For a photon of energy $E = 2.00$ eV, the wavelength is: $\lambda = \left(4.13567\cdot 10^{-15}\text{ eV s}\right)\left(2.998\cdot 10^{8}\text{ m/s}\right)/\left(2.00\text{ eV}\right) = 6.1994\cdot 10^{-7}\text{ m} \approx 620.\text{ nm}$.

(b) The wavelength of an electron is $\lambda = h/p = h/\left(m_e v\right)$, and its kinetic energy is $K = m_e v^2/2$. In terms of K, the velocity v is $v = \sqrt{\frac{2K}{m_e}}$. Then the wavelength of the electron is $\lambda = \frac{h}{m_e}\sqrt{\frac{m_e}{2K}} = \sqrt{\frac{h^2}{2Km_e}}$. For an electron of kinetic energy $K = \left(2.00\text{ eV}\right)\cdot\left(1.602\cdot 10^{-19}\text{ J}\right)/\left(1\text{ eV}\right) = 3.204\cdot 10^{-19}$ J, the wavelength is:

$$\lambda = \sqrt{\frac{\left(6.626\cdot 10^{-34}\text{ J s}\right)^2}{2\left(3.204\cdot 10^{-19}\text{ J}\right)\left(9.109\cdot 10^{-31}\text{ kg}\right)}} = 8.673\cdot 10^{-10}\text{ m} \approx 0.867\text{ nm.}$$

36.40. The car has a mass of $m = 2.000 \cdot 10^3$ kg and a speed

$$v = (100.0 \text{ km/h})(1000 \text{ m/km})(1 \text{ hr/3600 s}) \approx 27.78 \text{ m/s}.$$

The de Broglie Wavelength is $\lambda = h / p = h / (mv)$. The wavelength of this car is therefore

$$\lambda = (6.626 \cdot 10^{-34} \text{ J s}) / \left[(2.000 \cdot 10^3 \text{ kg})(27.78 \text{ m/s})\right] = 1.193 \cdot 10^{-38} \text{ m}$$

36.41. The nitrogen molecule has a mass of $m = 4.648 \cdot 10^{-26}$ kg and a speed $v = 300.0$ m/s.

(a) The de Broglie wavelength is $\lambda = h / p = h / (mv)$. The wavelength of this nitrogen molecule is therefore $\lambda = (6.626 \cdot 10^{-34} \text{ J s}) / \left[(4.648 \cdot 10^{-26} \text{ kg})(300.0 \text{ m/s})\right] = 4.752 \cdot 10^{-11} \text{ m} \approx 47.52$ pm.

(b) For a double slit experiment, the fringes are $\Delta x = 0.30$ cm apart and the screen is $L = 70.0$ cm in front of the slits. In a double slit experiment with particles the distance between the fringes is $\Delta x = \lambda L / d$. In that case, the distance d between the slits is $d = \lambda L / \Delta x = (4.752 \cdot 10^{-11} \text{ m})(70.0 \text{ cm}) / (0.30 \text{ cm}) = 11$ nm.

36.42. The alpha particles are accelerated through a potential difference of magnitude $V = 20000$ V. Alpha particles are composed of 2 protons and 2 neutrons, and therefore have a charge of $q = 2e$, where $e = 1.602 \cdot 10^{-19}$ C. Assuming the alpha particles are accelerated from rest, the final kinetic energy of each alpha particle is $K = |\Delta U| = q|V| = (2 \text{ e}) \cdot (20000 \text{ V}) = 40000$ eV, or $6.408 \cdot 10^{-15}$ J. The de Broglie wavelength is $\lambda = h / p = h / (mv)$. Kinetic energy is $K = mv^2 / 2$. In terms of kinetic energy, the speed is $v = \sqrt{\dfrac{2K}{m_e}}$. Substituting, the de Broglie wavelength becomes $\lambda = \dfrac{h}{m_e}\sqrt{\dfrac{m_e}{2K}} = \dfrac{h}{\sqrt{2Km_e}}$. Note the mass of an alpha particle is $m = 6.645 \cdot 10^{-27}$ kg. The de Broglie wavelength of the alpha particle is

$$\lambda = \frac{6.626 \cdot 10^{-34} \text{ J s}}{\sqrt{2(6.408 \cdot 10^{-15} \text{ J})(6.645 \cdot 10^{-27} \text{ kg})}} = 7.18 \cdot 10^{-14} \text{ m}.$$

36.43. The electron has a de Broglie wavelength of $\lambda = 550$ nm.

(a) The de Broglie wavelength is $\lambda = h / p = h / (mv)$. The speed of the electron is

$$v = \frac{h}{m_e \lambda} = \frac{6.626 \cdot 10^{-34} \text{ J s}}{(9.109 \cdot 10^{-31} \text{ kg})(5.5 \cdot 10^{-7} \text{ m})} = 1323 \text{ m/s} \approx 1300 \text{ m/s}.$$

(b) This speed is much less than the speed of light, so the non-relativistic approximation is sufficient.

(c) In non-relativistic terms, the electron's kinetic energy is

$$K = \frac{mv^2}{2} = \frac{(9.109 \cdot 10^{-31} \text{ kg})(1323 \text{ m/s})^2}{2} = 7.967 \cdot 10^{-25} \text{ J}.$$

In eV, this becomes $K = (7.967 \cdot 10^{-25} \text{ J})(1 \text{ eV}) / (1.602 \cdot 10^{-19} \text{ J}) = 4.973 \cdot 10^{-6} \text{ eV} \approx 5.0 \ \mu\text{eV}$.

36.44. **THINK:** The roommate wants to know if he could be diffracted when passing through a doorway. His mass is $m = 60.0$ kg. The width of the doorway is $d = 0.900$ m. The de Broglie wavelength can be used to find (a) the maximum speed v_{max} at which the roommate can pass through the doorway in order to be significantly diffracted and (b) the time Δt it would take the roommate to make a step of length $\Delta x = 0.75$ m in order to be significantly diffracted. Assume that significant diffraction occurs when the width of the diffraction aperture is less than 10.0 times the wavelength of the wave being diffracted, that is, $d < 10.0\lambda_{rm}$, where λ_{rm} is the de Broglie wavelength of the roommate.

SKETCH: Not applicable.

RESEARCH:

(a) The de Broglie wavelength is $\lambda = \dfrac{h}{p} = \dfrac{h}{mv}$.

(b) Speed is $v = \dfrac{\Delta x}{\Delta t}$.

SIMPLIFY:

(a) For significant diffraction take $\lambda_{\text{rm}} > \dfrac{d}{10.0}$. The speed is given by: $v = \dfrac{h}{m\lambda}$. Since v and λ are inversely proportional, the minimum λ will yield a maximum v. Then:

$$v_{\text{max}} = \frac{h}{m\lambda_{\text{rm}}} = \frac{10.0\,h}{md}$$

(b) $\Delta t = \dfrac{\Delta x}{v_{\text{max}}}$

CALCULATE:

(a) $v_{\text{max}} = \dfrac{10.0\left(6.626 \cdot 10^{-34}\text{ J s}\right)}{\left(60.0\text{ kg}\right)\left(0.900\text{ m}\right)} = 1.2270 \cdot 10^{-34}\text{ m/s}$

(b) $\Delta t = \dfrac{\left(0.75\text{ m}\right)}{\left(1.2270 \cdot 10^{-34}\text{ m/s}\right)} = 6.1123 \cdot 10^{33}\text{ s}$

ROUND:

(a) $v_{\text{max}} = 1.23 \cdot 10^{-34}\text{ m/s}$

(b) $\Delta t = 6.1 \cdot 10^{33}\text{ s}$

(c) To achieve a de Broglie wavelength capable of diffracting through the doorway the roommate must move at a speed of $v = 1.23 \cdot 10^{-34}$ m/s. This would take him $6.11 \cdot 10^{33}$ s, or $1.94 \cdot 10^{26}$ years! This is more than 10^{16} times the age of the universe. The roommate does not need to worry about diffracting through the doorway. Particles like electrons and protons can diffract because they are many orders of magnitude smaller in mass than a person.

DOUBLE-CHECK: It is reasonable that the roommate would have to move extremely slow in order for him to be diffracted since his mass is so large and the doorway is so large.

36.45. **THINK:** The de Broglie waves have a wavelength $\lambda = h/p$ and a frequency $f = E/h$. A Newtonian particle of mass m, has momentum $p = mv$, and energy $E = p^2/\left(2m\right)$. (a) To calculate the dispersion relation for the de Broglie waves of a Newtonian particle, the angular frequency ω needs to be found as a function of wave number, κ. (b) The phase velocity v_{p} and group velocity v_{g} can be determined by using the dispersion relation.

SKETCH: Not applicable.

RESEARCH:

(a) The dispersion relation is an expression for the angular frequency, $\omega = 2\pi f$, as a function of wave number $\kappa = 2\pi/\lambda$; that is, $\omega = \omega\left(\kappa\right)$.

(b) The phase velocity of a wave is $v_{\text{p}} = \omega/\kappa$, while the group velocity of a wave is $v_{\text{g}} = \partial\omega/\partial\kappa$.

SIMPLIFY:

(a) For a Newtonian particle, the dispersion relation is

$$\omega = 2\pi f \;\Rightarrow\; \omega = 2\pi\frac{E}{h} = \frac{2\pi}{h}\left(\frac{p^2}{2m}\right) = \frac{2\pi}{h}\left(\frac{h^2}{2m\lambda^2}\right) = \frac{2\pi}{h}\left(\frac{h^2\kappa^2}{2m\left(2\pi\right)^2}\right) \;\Rightarrow\; \omega\left(\kappa\right) = \frac{h\kappa^2}{4\pi m}.$$

(b) For the same particle, the phase velocity is,

$$v_{\mathrm{p}} = \frac{\omega}{\kappa} = \frac{h\kappa}{4\pi m} = \frac{h}{2\lambda m} = \frac{p}{2m},$$

while the group velocity is

$$v_{\mathrm{g}} = \frac{d\omega}{dk} = \frac{h\kappa}{2\pi m} = \frac{h}{\lambda m} = \frac{p}{m}.$$

Note: since the momentum of a Newtonian particle is $p = mv$, it is the group velocity that corresponds to the classical velocity of the particle.
CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: When several waves are superimposed to result in a single wave shape (the envelope of the wave) the speed of the overall wave shape is described by the group velocity. The phase velocity describes the velocity at which the peaks, or phases, of the waves propagate. The classical velocity should correspond to the group velocity of the particle.

36.46. **THINK:** The de Broglie waves have a wavelength $\lambda = h/p$ and a frequency $f = E/h$. A relativistic particle of mass m has momentum $p = \gamma mv$ and energy $E = mc^2\gamma$, where $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$. (a) To calculate the dispersion relation for the de Broglie waves of a relativistic particle, the angular frequency ω needs to be found as a function of wave number, κ. (b) The phase velocity v_{p} and group velocity v_{g} can be determined by using the dispersion relation.
SKETCH: Not applicable.
RESEARCH:
(a) The dispersion relation is an expression for the angular frequency $\omega = 2\pi f$, as a function of wave number $\kappa = 2\pi/\lambda$; that is, $\omega = \omega(\kappa)$.
(b) The phase velocity of a wave is $v_{\mathrm{p}} = \omega/\kappa$, while the group velocity of a wave is $v_{\mathrm{g}} = \partial\omega/\partial\kappa$.
SIMPLIFY:
(a) For a relativistic particle, the dispersion relation is

$$\omega = 2\pi f = 2\pi\frac{E}{h} = \frac{2\pi}{h}\left[(pc)^2 + m^2c^4\right]^{1/2} = \frac{2\pi}{h}\left[\left(\frac{hc}{\lambda}\right)^2 + m^2c^4\right]^{1/2} = \frac{2\pi}{h}\left[\left(\frac{h\kappa c}{2\pi}\right)^2 + m^2c^4\right]^{1/2}$$

$$\Rightarrow \omega(\kappa) = \left(\kappa^2c^2 + \frac{4\pi^2m^2c^4}{h^2}\right)^{1/2}$$

(b) For the same particle, the phase velocity is,

$$v_{\mathrm{p}} = \frac{\omega}{\kappa} = \left(c^2 + \frac{4\pi^2m^2c^4}{h^2\kappa^2}\right)^{1/2} = c\left(1 + \frac{4\pi^2m^2c^2}{h^2\kappa^2}\right)^{1/2},$$

while the group velocity is,

$$v_{\mathrm{g}} = \frac{d\omega}{d\kappa} = \frac{\kappa c^2}{\left(\kappa^2c^2 + \dfrac{4\pi^2m^2c^4}{h^2}\right)^{1/2}} = \frac{c}{\left(1 + \dfrac{4\pi^2m^2c^2}{h^2\kappa^2}\right)^{1/2}}.$$

Using the relation, $h^2\kappa^2 = 4\pi^2p^2$, the group velocity can be written as:

$$v_{\mathrm{g}}^2 = \frac{c^2}{1 + \dfrac{4\pi^2m^2c^2}{h^2\kappa^2}} = \frac{c^2}{1 + \dfrac{4\pi^2m^2c^2}{4\pi^2p^2}} = \frac{c^2}{1 + \dfrac{m^2c^2}{p^2}} = \frac{p^2c^2}{p^2 + m^2c^2} = \frac{p^2c^4}{p^2c^2 + m^2c^4}$$

$$v_{\mathrm{g}} = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}} = \frac{pc^2}{E} = \frac{(mv\gamma)c^2}{(mc^2\gamma)} = v.$$

Therefore, (c) the group velocity is the classical velocity of the particle.

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: Note that the phase velocity can exceed the speed of light (this is not unusual, or worrisome, as the phase velocity does not transmit any energy or information), while the group velocity cannot. This further enforces that it must be the group velocity, and not the phase velocity, which corresponds to the classical velocity of the particle.

36.47. The mass of the particle is $m = 50.0$ kg. It has a de Broglie wavelength of $\lambda = 20.0$ cm.
(a) The de Broglie wavelength is $\lambda = h/p = h/(mv)$. The speed is therefore

$$v = \frac{h}{m\lambda} = \frac{6.626 \cdot 10^{-34} \text{ J s}}{(50.0 \text{ kg})(0.200 \text{ m})} = 6.626 \cdot 10^{-35} \text{ m/s} \approx 6.63 \cdot 10^{-35} \text{ m/s}.$$

(b) From the uncertainty relation $\Delta x \cdot \Delta p_x \geq (1/2)\hbar \Rightarrow \Delta x \cdot m\Delta v_x \geq (1/2)\hbar$, the uncertainty in the speed must be $\Delta v_x \geq (1/2)\hbar / (\Delta x \cdot m)$. The minimum uncertainty is

$$\Delta v_x = \frac{\hbar}{2 \cdot \Delta x \cdot m} = \frac{1.0546 \cdot 10^{-34} \text{ J s}}{2(0.200 \text{ m})(50.0 \text{ kg})} = 5.273 \cdot 10^{-36} \text{ m/s} \approx 5.27 \cdot 10^{-36} \text{ m/s}.$$

36.48. The distance through a hydrogen atom of radius $r = 0.53 \cdot 10^{-10}$ m is $d = 2r = 1.06 \cdot 10^{-10}$ m. The time required for the light to travel through it is

$$t = \frac{d}{v} = \frac{d}{c} = \frac{1.06 \cdot 10^{-10} \text{ m}}{2.998 \cdot 10^{8} \text{ m/s}} = 3.54 \cdot 10^{-19} \text{ s}.$$

The largest time uncertainty cannot be greater than the actual travel time, that is $\Delta t_{max} = 3.54 \cdot 10^{-19}$ s. The uncertainty relation between time and energy is $\Delta E \cdot \Delta t \geq (1/2)\hbar$. The uncertainty in the energy is therefore

$$\Delta E \cdot \Delta t \geq (1/2)\hbar \Rightarrow \Delta E \geq \frac{\hbar}{2 \cdot \Delta t} = \frac{1.0546 \cdot 10^{-34} \text{ J s}}{2(3.54 \cdot 10^{-19} \text{ s})} = 1.4895 \cdot 10^{-16} \text{ J}.$$

In terms of eV, this is

$$\Delta E \geq (1.4895 \cdot 10^{-16} J)(1 \text{ eV})/(1.602 \cdot 10^{-19} \text{ J}) = 929.8 \text{ eV} \approx 0.930. \text{ keV}.$$

The smallest ΔE can be is 0.930 keV. As Δt decreases from its maximum value, ΔE must increase according to the uncertainty relation.

36.49. The uncertainty relation between time and energy is $\Delta E \cdot \Delta t \geq (1/2)\hbar$. In terms of mass, $\Delta E = \Delta mc^2$. The neutron's mass is $m = 1.67 \cdot 10^{-27}$ kg. It has an average lifetime of $t = 882$ s. The largest time uncertainty cannot be greater than the actual lifetime of the particle, which is $\Delta t_{max} = 882$ s. The uncertainty in the mass of the neutron is therefore $\Delta E = \Delta mc^2 \geq (1/2)\hbar / \Delta t$.

$$\Delta m \geq \frac{\hbar}{2\Delta tc^2} = \frac{1.0546 \cdot 10^{-34} \text{ J s}}{2(882 \text{ s})(3.00 \cdot 10^{8} \text{ m/s})^2} = 6.643 \cdot 10^{-55} \text{ kg} \approx 6.64 \cdot 10^{-55} \text{ kg}.$$

As the uncertainty in the time Δt decreases from its maximum value, the uncertainty in the mass increases, according to the uncertainty relation.

36.50. Fuzzy lives in a universe where $\hbar = 1.00$ J s. Fuzzy's mass is $m = 0.500$ kg and he lives somewhere within a 0.750 m wide pond. The uncertainty relation between position and momentum (in one dimension) is $\Delta x \cdot \Delta p \geq (1/2)\hbar$. In terms of velocity, $\Delta p = m\Delta v$, and so the uncertainty relation becomes

$\Delta x \cdot \Delta v \geq (1/2)\hbar / m.$ Since the largest uncertainty in Fuzzy's position is the width of the pond, $\Delta x_{\text{max}} = 0.750$ m, the minimum uncertainty in his speed is

$$\Delta v_{\text{min}} = (1/2)\hbar / (m\Delta x_{\text{max}}) = (1/2)(1.00 \text{ J s}) / [(0.500 \text{ kg})(0.750 \text{ m})] = 1.3333 \text{ m/s} \approx 1.33 \text{ m/s}.$$

As the uncertainty in his position Δx decreases from its maximum value, the uncertainty in his velocity increases, according to the uncertainty relation. If the uncertainty prevails for $t = 5.00$ s, Fuzzy could move $\Delta x = \Delta v t = (1.3333 \text{ m/s})(5.00 \text{ s}) = 6.6666 \text{ m} \approx 6.67 \text{ m}$ away from his pond.

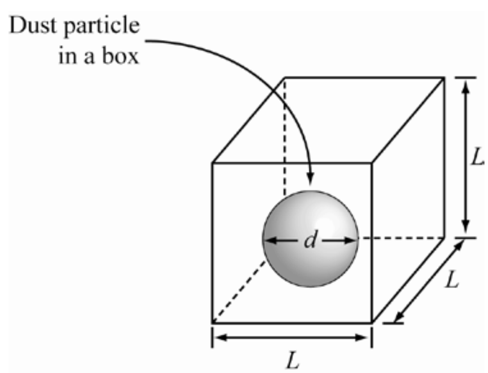
36.51. The uncertainty relation between position and momentum (in one dimension) is $\Delta x \cdot \Delta p \geq (1/2)\hbar.$ In terms of speed, $\Delta p = m\Delta v,$ and so the uncertainty relation becomes $\Delta x \cdot \Delta v \geq (1/2)\hbar / m.$ The electron is confined to a box of dimensions $L = 20.0\ \mu$m. The maximum uncertainty in the (one-dimensional) position of the electron is the dimension of the box, that is $\Delta x_{\text{max}} = L = 20.0\ \mu$m. The minimum uncertainty in the speed of the electron is

$$\Delta v_{\text{min}} = (1/2)\hbar / (m\Delta x_{\text{max}}) = (1/2)(1.05457 \cdot 10^{-34} \text{ J s}) / [(9.109 \cdot 10^{-31} \text{ kg})(20.0\ \mu\text{m})] = 2.894 \text{ m/s}.$$

The minimum speed the electron can have is 2.89 m/s.

36.52. **THINK:** The dust particle has mass $m = 1.00 \cdot 10^{-16}$ kg and diameter $d = 5.00$ μm. It is confined to a box of length $L = 15.0$ μm. The Heisenberg uncertainty relation can be used to determine (a) if the particle can be at rest, (b) the range of its velocity, and (c) how long it will take for it to move a distance of $x = 1.00 \cdot 10^{-3}$ m at the lower range of the velocity.

SKETCH:



RESEARCH: The uncertainty relation between position and momentum (in one dimension) is

$$\Delta x \Delta p_x \geq \frac{1}{2}\hbar.$$

In terms of velocity, $\Delta p = m\Delta v$, and so the uncertainty relation becomes

$$\Delta x (m\Delta v_x) \geq \frac{1}{2}\hbar.$$

The equation for velocity is $v_x = x / t.$

SIMPLIFY:

(a) If the particle is at rest, then there is no uncertainty in the momentum, $\Delta p = 0.$ Then Heisenberg's uncertainty relation,

$$\Delta x \Delta p_x \geq \frac{1}{2}\hbar \Rightarrow \Delta x \geq \frac{1}{2}\frac{\hbar}{\Delta p_x},$$

would required that $\Delta x = \infty.$ However, the particle is known to be contained in the box, so $\Delta x = L$ (the length of the box). Therefore, due to Heisenberg's uncertainty relation, we cannot know if the particle is at rest.

(b) With the particle confined to the box, the uncertainty in position is $\Delta x = L - d$. The uncertainty in the speed is:

$$\Delta v_x \geq \frac{\hbar}{2m\Delta x}$$

Therefore, the particle's velocity must be somewhere in the range

$$-\frac{\Delta v_x}{2} \leq v_x \leq \frac{\Delta v_x}{2}$$

$$-\frac{\hbar}{4m\Delta x} \leq v_x \leq \frac{\hbar}{4m\Delta x}.$$

(c) $t = x / v_x$

CALCULATE:

(b) $\dfrac{\Delta v_x}{2} = \dfrac{\left(1.0546 \cdot 10^{-34} \text{ J s}\right)}{4\left(1.00 \cdot 10^{-16} \text{ kg}\right)\left(10.0 \cdot 10^{-6} \text{ m}\right)} = 2.6365 \cdot 10^{-14} \text{ m/s}$

(c) $t = \dfrac{\left(1.00 \cdot 10^{-3} \text{ m}\right)}{\left(2.6365 \cdot 10^{-14} \text{ m/s}\right)} = 3.79 \cdot 10^{10} \text{ s}$

ROUND: To three significant figures:

(b) $-2.64 \cdot 10^{-14} \text{ m/s} \leq v_x \leq 2.64 \cdot 10^{-14} \text{ m/s}$

(c) $t = 3.79 \cdot 10^{10} \text{ s} \approx 1.20 \cdot 10^{3}$ years

DOUBLE-CHECK: For all intent, the dust particle is at rest since it would take it 2400 years to move just 1 mm. However, by the Heisenberg uncertainty principle, one cannot be sure that at any given time the particle is truly at rest.

36.53. **THINK:** A quantum state of energy E can be occupied by any number n of bosonic particles (including $n = 0$). At absolute temperature T, the probability of finding n particles in this state is $P_n = N\exp\left(-\dfrac{nE}{k_{\text{B}}T}\right)$, where k_{B} is Boltzmann's constant and N is the normalization factor. Calculate the mean or expected value of n, $\langle n \rangle$, i.e. the occupancy of this state, given this probability distribution.

SKETCH: Not applicable.

RESEARCH: The expectation value of n is $\langle n \rangle = \sum_{n=0}^{\infty} nP_n$. The value of the constant N is determined by the requirement that all the probabilities sum to one, that is $\sum_{n=0}^{\infty} P_n = 1$. To simplify the notation, let $z = \exp\left(-\dfrac{E}{k_{\text{B}}T}\right)$. With this, $P_n = Nz^n$.

SIMPLIFY: In order to evaluate the normalization factor N: $1 = \sum_{n=0}^{\infty} P_n = N\sum_{n=0}^{\infty} z^n = \dfrac{N}{1-z} \Rightarrow N = 1 - z$. Then simplify the expected value to

$$\langle n \rangle = \sum_{n=0}^{\infty} nP_n = 0 + N\sum_{n=1}^{\infty} n\exp\left(-\frac{nE}{k_B T}\right) = N\sum_{n=1}^{\infty} nz^n = (1-z)\sum_{n=1}^{\infty} nz^n.$$

There are several ways to evaluate the sum in this expression. If $|z| < 1$, then the original series $\sum_{n=1}^{\infty} nz^n$ is absolutely convergent, and it is okay to interchange the order of the sums.

One way to evaluate it is as a sequence of sequences: $\sum_{n=1}^{\infty} nz^n = \sum_{n=1}^{\infty} z^n + \sum_{n=2}^{\infty} z^n + \sum_{n=3}^{\infty} z^n + \ldots = \sum_{k=0}^{\infty}\left(z^k \sum_{n=k}^{\infty} z^n \right)$.

Next, $\sum_{n=k}^{\infty} z^n = z^k \sum_{n=k}^{\infty} z^{n-k} = z^k \sum_{j=0}^{\infty} z^j = z^k \cdot \frac{1}{1-z}$. Substituting this into the sequence of sequences formula:

$$\sum_{n=1}^{\infty} nz^n = \sum_{k=0}^{\infty}\left(z^k \sum_{n=k}^{\infty} z^n \right) = \sum_{k=0}^{\infty}\left(z^k \left(\frac{1}{1-z} \right)\right) = \frac{z}{1-z}\sum_{k=0}^{\infty} z^k = \frac{z}{1-z}\cdot\frac{1}{1-z} = \frac{z}{(1-z)^2}.$$

Altogether, this makes the occupancy:

$$\langle n \rangle = N\sum_{n=1}^{\infty} nz^n = (1-z)\frac{z}{(1-z)^2} = \frac{z}{1-z} = \frac{\exp\left(\frac{-E}{k_B T}\right)}{1-\exp\left(\frac{-E}{k_B T}\right)}.$$

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: The expectation value of n calculated above is an expected result as it is the Bose-Einstein distribution, which describes the distribution of identical (and therefore indistinguishable) bosons in an energy state E at thermal equilibrium.

36.54. **THINK:** The quantum state of energy E and temperature T has a probability distribution

$$P_n = N\exp\left(-\frac{nE}{k_B T}\right),$$

(as the preceding problem), but with fermionic particles. Here k_B is Boltzmann's constant and N is the normalization factor. Due to the Pauli exclusion principle, the only possible occupation numbers are $n=0$ and $n=1$. Calculate the mean occupancy $\langle n \rangle$ of the state in this case.
SKETCH: Not applicable.
RESEARCH: The expectation value of n is

$$\langle n \rangle = \sum_{n=0}^{1} nP_n.$$

The normalization factor N is determined by the requirement that all the probabilities sum to unity:

$$1 = \sum_{n=0}^{1} P_n.$$

SIMPLIFY: $\langle n \rangle = \sum_{n=0}^{1} nP_n = (0)P_0 + (1)P_1 = P_1$. The normalization factor N is determined from

$$1 = \sum_{n=0}^{1} P_n = P_0 + P_1.$$

From the probability distribution:

$$P_0 = N\exp(0) = N \text{ and } P_1 = N\exp\left(-\frac{E}{k_B T}\right).$$

Therefore,

$$1 = N + N\exp\left(-\frac{E}{k_B T}\right) = N\left[1+\exp\left(-\frac{E}{k_B T}\right)\right] \Rightarrow N = \frac{1}{1+\exp\left(-\frac{E}{k_B T}\right)}.$$

The occupancy of the state is

$$\langle n\rangle = P_1 = \frac{\exp\left(-\frac{E}{k_{\mathrm{B}}T}\right)}{1+\exp\left(-\frac{E}{k_{\mathrm{B}}T}\right)} = \frac{1}{\exp\left(\frac{E}{k_{\mathrm{B}}T}\right)+1}.$$

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: The expectation value of n calculated above for fermions is an expected result as it is the Fermi-Dirac distribution, which describes the distribution of identical (and therefore indistinguishable) fermions in an energy state E at thermal equilibrium.

36.55. **THINK:** The system is made up of N particles. The average energy per particle is given by

$$\langle E\rangle = \frac{\sum E_i e^{-E_i/k_{\mathrm{B}}T}}{Z},$$

where Z is the partition function,

$$Z = \sum_i g_i e^{-E_i/k_{\mathrm{B}}T},$$

and g_i is the degeneracy of the state with energy E_i. This system is a 2-state system with $E_1 = 0$ and $E_2 = E$ and $g_1 = g_2 = 1$. Calculate the heat capacity of the system, $C = N\left(d\langle E\rangle / dT\right)$, and approximate its behavior at very high and very low temperatures (i.e. $k_{\mathrm{B}}T \gg 1$ and $k_{\mathrm{B}}T \ll 1$).
SKETCH: Not applicable.
RESEARCH: Not applicable as the necessary equations were all given in the problem.
SIMPLIFY: The average energy per particle is for $E_1 = 0$ and $E_2 = E$ is:

$$\langle E\rangle = \frac{(0)\exp\left(-\frac{(0)}{k_{\mathrm{B}}T}\right)+(E)\exp\left(-\left(\frac{(E)}{k_{\mathrm{B}}T}\right)\right)}{(1)\exp\left(-\frac{(0)}{k_{\mathrm{B}}T}\right)+(1)\exp\left(-\frac{(E)}{k_{\mathrm{B}}T}\right)} = \frac{E\exp\left(-\frac{E}{k_{\mathrm{B}}T}\right)}{1+\exp\left(-\frac{E}{k_{\mathrm{B}}T}\right)}$$

Therefore,

$$\langle E\rangle = \frac{E}{1+\exp\left(\frac{E}{k_{\mathrm{B}}T}\right)} \Rightarrow N\langle E\rangle = \frac{NE}{1+\exp\left(\frac{E}{k_{\mathrm{B}}T}\right)}.$$

The heat capacity of the system is,

$$C = N\frac{d\langle E\rangle}{dT} = \frac{d(N\langle E\rangle)}{dT} = Nk_{\mathrm{B}}\left(\frac{E}{k_{\mathrm{B}}T}\right)^2 \frac{\exp\left(\frac{E}{k_{\mathrm{B}}T}\right)}{\left(\exp\left(\frac{E}{k_{\mathrm{B}}T}\right)+1\right)^2}.$$

For $k_{\mathrm{B}}T >> 1$, $\exp\left(\frac{E}{k_{\mathrm{B}}T}\right) \approx 1$:

$$C \approx \frac{Nk_{\mathrm{B}}}{4}\left(\frac{E}{k_{\mathrm{B}}T}\right)^2$$

For $0 < k_\text{B}T << 1$, $\exp\left(E/k_\text{B}T\right) >> 1$:

$$C \approx Nk_\text{B}\left(\frac{E}{k_\text{B}T}\right)^2 \frac{\exp\left(\dfrac{E}{k_\text{B}T}\right)}{\left(\exp\left(\dfrac{E}{k_\text{B}T}\right)\right)^2} \Rightarrow C \approx Nk_\text{B}\left(\frac{E}{k_\text{B}T}\right)^2 \exp\left(-\frac{E}{k_\text{B}T}\right)$$

For each temperature extreme, the heat capacity approaches zero.

CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: In general, in the extremely low temperature limit, the heat capacity must approach zero to be consistent with the third law of thermodynamics.

36.56. The work function of tungsten is $\phi = 4.55$ eV. For a photon of wavelength $\lambda = 360$ nm, its energy is

$$E_\text{ph} = hf = \frac{hc}{\lambda} = \frac{\left(4.13567\cdot 10^{-15} \text{ eV s}\right)\left(2.998\cdot 10^{8} \text{ m/s}\right)}{3.6\cdot 10^{-7} \text{ m}} = 3.44409 \text{ eV} \approx 3.4 \text{ eV}$$

These photons are not energetic enough to overcome the work function of tungsten, and so no electrons are ejected from the tungsten cathodes. No stopping potential is required ($v_0 = 0$).

36.57. The de Broglie wavelength is $\lambda = h/p$. The proton and the electron have the same kinetic energy. In terms of kinetic energy, momentum p can be written as: $K = \dfrac{p^2}{2m} \Rightarrow p = \sqrt{2mK}$. Then, the de Broglie wavelength becomes: $\lambda = \dfrac{h}{p} = \dfrac{h}{\sqrt{2mK}}$. The ratio of the de Broglie wavelengths of a proton and an electron of the same kinetic energy K is:

$$\frac{\lambda_\text{p}}{\lambda_\text{e}} = \frac{\dfrac{h}{\sqrt{2m_\text{p}K}}}{\dfrac{h}{\sqrt{2m_\text{e}K}}} = \sqrt{\frac{m_\text{e}}{m_\text{p}}} = \sqrt{\frac{9.109\cdot 10^{-31} \text{ kg}}{1.673\cdot 10^{-27} \text{ kg}}} = 0.0233.$$

36.58. In one einstein of light there are $N = 6.02\cdot 10^{23}$ photons. If these photons have a wavelength of $\lambda = 400$ nm, the energy contained in one Einstein of photons is

$$E_\text{tot} = NE_\text{ph} = N\frac{hc}{\lambda} = \frac{\left(6.02\cdot 10^{23}\right)\left(6.626\cdot 10^{-34} \text{ J s}\right)\left(2.998\cdot 10^{8} \text{ m/s}\right)}{4.00\cdot 10^{-7} \text{ m}} = 2.99\cdot 10^{5} \text{ J}.$$

36.59. The de Broglie wavelength is $\lambda = h/p$. The momentum of the baseball is:

$$p = mv = \left(0.100 \text{ kg}\right)\left(100. \text{ mi/h}\right)\left(1609 \text{ m/mi}\right)\left(1 \text{ h/3600 s}\right) = 4.469 \text{ kg m/s}.$$

The de Broglie wavelength of the baseball is:

$$\lambda = h/p = \left(6.626\cdot 10^{-34} \text{ J s}\right)/\left(4.469 \text{ kg m/s}\right) = 1.48\cdot 10^{-34} \text{ m}.$$

The momentum of the spacecraft is:

$$p = mv = \left(250. \text{ kg}\right)\left(125000 \text{ km/h}\right)\left(1000 \text{ m/km}\right)\left(1 \text{ h/3600 s}\right) = 8.681\cdot 10^{6} \text{ kg m/s}.$$

The de Broglie wavelength of the spacecraft is:

$$\lambda = h/p = \left(6.626\cdot 10^{-34} \text{ J s}\right)/\left(8.681\cdot 10^{6} \text{ kg m/s}\right) = 7.63\cdot 10^{-41} \text{ m}.$$

36.60. The Heisenberg uncertainty relation can be used to find the uncertainty in the velocity. In one dimension it is stated as: $\Delta x\Delta p \geq \hbar/2$. Writing Δp as $\Delta p = m\Delta v$, the uncertainty relationship becomes $\Delta x m\Delta v \geq \hbar/2$. The uncertainty in the velocity is therefore $\Delta v \geq \hbar/\left(2\cdot\Delta x\cdot m\right)$. The minimum uncertainty in the velocity corresponds to the maximum uncertainty in the position. In this case, if the particle of mass $m = 1.0 \text{ ng} = 1.0 \text{ pkg} = 1.0 \cdot 10^{-12}$ kg is restricted to be somewhere on the pinhead, the maximum uncertainty in its position is the width of the pinhead, $\Delta x_{\text{max}} = 1.0 \text{ mm} = 0.0010 \text{ m}$. The minimum uncertainty in the velocity of the particle is

$$\Delta v_{\text{min}} = \frac{\hbar}{2\Delta x_{\text{max}} m} = \frac{1.0546 \cdot 10^{-34} \text{ J s}}{2(0.0010 \text{ m})(1.0 \cdot 10^{-12} \text{ kg})} = 5.273 \cdot 10^{-20} \text{ m/s} \approx 5.3 \cdot 10^{-20} \text{ m/s}.$$

36.61. The wavelength of light is $\lambda = 700. \text{ nm} = 7.00 \cdot 10^{-7}$ m. The energy of each photon is therefore

$$E_{\text{ph}} = \frac{hc}{\lambda} = \frac{(6.626 \cdot 10^{-34} \text{ J s})(2.998 \cdot 10^{8} \text{ m/s})}{7.00 \cdot 10^{-7} \text{ m}} = 2.8378 \cdot 10^{-19} \text{ J}.$$

The light intensity on the surface of area $A = 10.0 \text{ cm}^2$ is $I = 0.300 \text{ W/cm}^2$. The total power incident on the surface is therefore $P = IA = (0.300 \text{ W/cm}^2)(10.0 \text{ cm}^2) = 3.00 \text{ W} = 3.00 \text{ J/s}$. The photon flux Φ, or number of photons per unit time, through the surface A is

$$\Phi = \frac{P}{E_{\text{ph}}} = \frac{3.00 \text{ J/s}}{2.8378 \cdot 10^{-19} \text{ J}} = 1.06 \cdot 10^{19} \text{ s}^{-1}.$$

36.62. The intensity of the Sun is measured to be about $I = 1400. \text{ W/m}^2$. The peak of the wavelength spectrum emitted by the Sun is at $\lambda = 500.$ nm.

(a) The corresponding photon frequency is

$$f = \frac{c}{\lambda} = \frac{2.998 \cdot 10^{8} \text{ m/s}}{5.00 \cdot 10^{-7} \text{ m}} = 5.996 \cdot 10^{14} \text{ Hz} \approx 6.00 \cdot 10^{14} \text{ Hz}.$$

(b) The corresponding energy per photon is

$$E_{\text{ph}} = hf = (6.626 \cdot 10^{-34} \text{ J s})(5.996 \cdot 10^{14} \text{ Hz}) = 3.973 \cdot 10^{-19} \text{ J} \approx 3.97 \cdot 10^{-19} \text{ J}.$$

(c) The number flux of photons Φ arriving at the Earth (assuming all light emitted by the Sun has the same peak wavelength) is

$$\Phi = \frac{I}{E_{\text{ph}}} = \frac{1400. \text{ W/m}^2}{3.973 \cdot 10^{-19} \text{ J}} = 3.52 \cdot 10^{21} \text{ m}^{-2} \text{ s}^{-1}.$$

That is, about $3.52 \cdot 10^{21}$ photons hit one meter-squared of surface area of the Earth per second.

36.63. The plates have a potential difference of $V = 5.0$ V between them. The magnitude of the stopping potential is therefore $V_0 = 5.0$ V. The work function of silver is $\phi = 4.7$ eV. The largest wavelength (lowest frequency and energy) of light λ_{max} that can be shined on the cathode to produce a current through the anode is found from the equation, $eV_0 = hf - \phi$. The wavelength of light is

$$f_{\text{min}} = \frac{(eV_0 + \phi)}{h} \Rightarrow \frac{c}{\lambda_{\text{max}}} = \frac{(eV_0 + \phi)}{h} \Rightarrow \lambda_{\text{max}} = \frac{hc}{(eV_0 + \phi)}$$

$$\lambda_{\text{max}} = \frac{(4.136 \cdot 10^{-15} \text{ eV s})(3.00 \cdot 10^{8} \text{ m/s})}{e(5.0 \text{ V}) + (4.7 \text{ eV})} = \frac{(4.136 \cdot 10^{-15} \text{ eV s})(3.00 \cdot 10^{8} \text{ m/s})}{(5.0 \text{ eV}) + (4.7 \text{ eV})} = 1.279 \cdot 10^{-7} \text{ m}$$

$$\lambda_{\text{max}} \approx 130 \text{ nm}.$$

36.64. The surface has an area $A = 10.0 \text{ m}^2$. A force of $F = 0.100$ N is exerted on the surface by photons of wavelength $\lambda = 600.$ nm. In general, force is the rate of change of momentum (by Newton's Second Law). By the conservation of momentum, the momentum supplied to the plate must be the momentum of the incoming photons. The momentum of a single photon is $p_{\text{ph}} = E / c$.

Since the photon energy is given by $E = hf$, the momentum of each photon is:

$$p_{\text{ph}} = hf/c = h/\lambda = \left(6.626 \cdot 10^{-34} \text{ J s}\right)/\left(6.00 \cdot 10^{-7} \text{ m}\right) = 1.1043 \cdot 10^{-27} \text{ kg m/s}.$$

The total momentum transferred by the photons is $p_{\text{total}} = np_{\text{ph}}$, where n is the total number of photons. The total momentum per second transferred to the surface must be equal to the force F exerted on the surface, that is $p_{\text{total}}/s = F$. The number of photons required per second is

$$\frac{np_{\text{ph}}}{s} = F \;\Rightarrow\; \frac{n}{s} = \frac{F}{p_{\text{ph}}} = \frac{0.100 \text{ N}}{1.1043 \cdot 10^{-27} \text{ kg m/s}} = 9.05551 \cdot 10^{25} \text{ s}^{-1} \approx 9.06 \cdot 10^{25} \text{ s}^{-1}.$$

About $9.06 \cdot 10^{25}$ photons per second must strike the surface to exert a force of $F = 0.100$ N.

36.65. The wave function describing an electron predicts a statistical spread of $\Delta v = 1.00 \cdot 10^{-4}$ m/s in the electron's velocity. The corresponding statistical spread in its position Δx is found from the Heisenberg uncertainty principle, $\Delta x \Delta p \geq \hbar/2$. In terms of velocity and electron mass, this is $\Delta x \cdot m_{\text{e}} \Delta v \geq \hbar/2$. Solving for Δx gives:

$$\Delta x \geq \hbar/\left(2m_{\text{e}}\Delta v\right)$$
$$\Delta x \geq \left(1.0546 \cdot 10^{-34} \text{ J s}\right)/\left[2\left(9.109 \cdot 10^{-31} \text{ kg}\right)\left(1.00 \cdot 10^{-4} \text{ m/s}\right)\right]$$
$$\Delta x \geq 0.579 \text{ m}$$

The uncertainty in the electron's position is at least $\Delta x = 0.579$ m.

36.66. Wien's displacement law states $\lambda T = 2.90 \cdot 10^{-3}$ K m. For a blackbody whose peak emitted wavelength is in the X-ray portion of the spectrum, that is, $10^{-13} \text{ m} < \lambda < 10^{-8} \text{ m}$, the temperature of the blackbody ranges from:

$$\left(2.90 \cdot 10^{-3} \text{ K m}\right)/\lambda_{\text{max}} < T < \left(2.90 \cdot 10^{-3} \text{ K m}\right)/\lambda_{\text{min}}$$
$$\left(2.90 \cdot 10^{-3} \text{ K m}\right)/\left(10^{-8} \text{ m}\right) < T < \left(2.90 \cdot 10^{-3} \text{ K m}\right)/\left(10^{-13} \text{ m}\right)$$
$$2.90 \cdot 10^{5} \text{ K} < T < 2.90 \cdot 10^{10} \text{ K}$$

or, approximately, $10^{5} \text{ K} < T < 10^{10} \text{ K}$, depending on the exact wavelength of the emitted light.

36.67. A nocturnal bird's eye can detect monochromatic light of frequency $f = 5.8 \cdot 10^{14}$ Hz with a power as small as $P = 2.333 \cdot 10^{-17}$ W. The energy of each detected photon is

$$E_{\text{ph}} = hf = \left(6.626 \cdot 10^{-34} \text{ J s}\right)\left(5.8 \cdot 10^{14} \text{ Hz}\right) = 3.843 \cdot 10^{-19} \text{ J}.$$

The number of photons, n, detected by the bird per second is:

$$n/s = P/E_{\text{ph}}$$
$$n/s = \left(2.333 \cdot 10^{-17} \text{ W}\right)/\left(3.843 \cdot 10^{-19} \text{ J}\right) \approx 61 \text{ photons/s}.$$

That is, the minimum number of photons that this bird can detect in one second is about 61 photons.

36.68. The UV light wavelength is $\lambda = 355$ nm. The work function of calcium is $\phi = 2.9$ eV. The stopping potential is found from $\text{eV}_0 = hf - \phi$. The stopping potential in this case is

$$V_0 = \left(hf - \phi\right)/e = \left(\left(hc/\lambda\right) - \phi\right)/e$$
$$V_0 = \left[\left(4.13567 \cdot 10^{-15} \text{ eV s}\right)\left(2.998 \cdot 10^{8} \text{ m/s}\right)/\left(3.55 \cdot 10^{-7} \text{ m}\right) - 2.9 \text{ eV}\right]/e$$
$$V_0 = \left(3.493 \text{ eV} - 2.9 \text{ eV}\right)/e = 0.593 \text{ V} \approx 0.59 \text{ V}.$$

36.69. The electron is accelerated from rest through a potential difference of $V = 1.00 \cdot 10^{-5}$ V. From energy conservation, $\Delta U = -\Delta K \;\Rightarrow\; e\Delta V = (1/2)m_{\text{e}}v_{\text{f}}^{2}$. The electron's final velocity is then

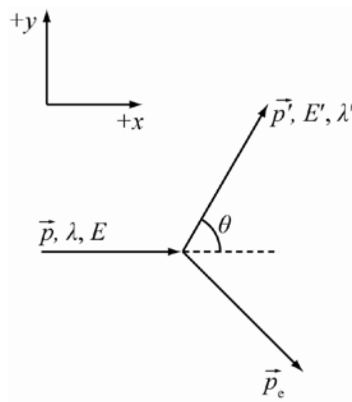
$$v_{\text{f}} = \left(2e\Delta V/m_{\text{e}}\right)^{1/2} = \left[2\left(1.60 \cdot 10^{-19} \text{ C}\right)\left(1.00 \cdot 10^{-5} \text{ V}\right)/\left(9.109 \cdot 10^{-31} \text{ kg}\right)\right]^{1/2} = 1874.30 \text{ m/s}.$$

From the de Broglie wavelength formula, the wavelength of the electron is

$$\lambda = h/p = h/(mv)$$
$$\lambda = \left(6.626\cdot 10^{-34}\text{ J s}\right)/\left[\left(9.109\cdot 10^{-31}\text{ kg}\right)\left(1874.30\text{ m/s}\right)\right]$$
$$\lambda = 3.88098\cdot 10^{-7}\text{ m} \approx 388\text{ nm}.$$

36.70. **THINK:** Compton used photons of wavelength $\lambda = 0.0711$ nm. The formula for Compton scattering can be used to find (a) the wavelength λ_e' of the photons scattered at $\theta = 180.^\circ$ from an electron, (b) the energy of these photons, and (c) the wavelength λ_p' of the photons scattered at $\theta = 180.^\circ$ from a proton.

SKETCH:



RESEARCH: For an electron, the formula for Compton scattering is

$$\lambda_e' = \lambda + \frac{h}{m_e c}(1-\cos\theta).$$

The energy of a photon is $E = hc/\lambda$. If the target were a proton and not an electron, the electron mass m_e in the Compton scattering formula would need to be replaced with the mass of a proton, m_p.

SIMPLIFY:

(b) $E' = \dfrac{hc}{\lambda_e'}$

(c) $\lambda_p' = \lambda + \dfrac{h}{m_p c}(1-\cos\theta)$

CALCULATE:

(a) $$\lambda_e' = (0.0711\text{ nm}) + \frac{\left(6.626\cdot 10^{-34}\text{ J s}\right)\left(1-\cos\left(180.^\circ\right)\right)}{\left(9.109\cdot 10^{-31}\text{ kg}\right)\left(2.998\cdot 10^{8}\text{ m/s}\right)}$$
$$= 0.0711\text{ nm} + 0.00485\text{ nm}$$
$$= 0.07595\text{ nm}$$

(b) $$E' = \frac{\left(6.626\cdot 10^{-34}\text{ J s}\right)\left(2.998\cdot 10^{8}\text{ m/s}\right)}{\left(0.07595\cdot 10^{-9}\text{ m}\right)} = 2.6155\cdot 10^{-15}\text{ J}$$

(c) For a proton target,

$$\lambda_p' = (0.0711\text{ nm}) + \frac{\left(6.626\cdot 10^{-34}\text{ J s}\right)\left(1-\cos\left(180.^\circ\right)\right)}{\left(1.673\cdot 10^{-27}\text{ kg}\right)\left(2.998\cdot 10^{8}\text{ m/s}\right)}$$
$$= 0.0711\text{ nm} + 2.64\cdot 10^{-6}\text{ nm}$$
$$= 0.07110264\text{ nm}.$$

ROUND:

(a) To four decimal places, $\lambda_e' = 0.0760$ nm.

(b) To three significant figures, $E' = 2.62 \cdot 10^{-15}$ J.

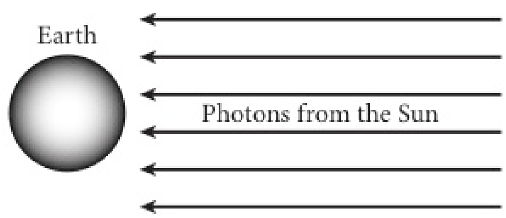
(c) To four decimal places, $\lambda_p' = 0.0711$ nm. Therefore, the wavelength of the photon will be smaller if the target electron is replaced by a proton.

DOUBLE-CHECK: Since some of the initial photon's energy is imparted on the electron upon scattering, it is expected that the wavelength of the photon will increase. Since a proton is about 1000 times more massive than an electron, it is expected that the wavelength of the photon will change very little (in this case, by a negligible amount).

36.71. **THINK:** To estimate the number of photons that impact the Earth, it is useful to know that the intensity of the Sun's radiation on the Earth is $I = 1370 \text{ W/m}^2$. Use the peak wavelength of the light emitted by the Sun, $\lambda = 500.$ nm, as stated in section 36.2. Note that the Earth's upper atmosphere, the ionosphere, is $d = 300.$ km above the Earth's surface. The radius of the Earth is $R = 6378$ km. Finally, keep in mind that only half of the Earth's surface can face the Sun at any given time. Note that one year has approximately

$$t = 1 \text{ year}(365.24 \text{ days/yr})(24 \text{ hr/day})(3600 \text{ s/hr}) = 31{,}556{,}736 \text{ s}.$$

SKETCH:



RESEARCH: The energy of a photon is $E_{ph} = hf = hc/\lambda$. The photon flux rate Φ (the number of photons per unit area per unit time) is found from $\Phi = I/E_{ph}$. The number of photons N that strike the Earth's upper atmosphere per year is $N = \Phi \cdot \frac{A_{atm}}{2} \cdot t_{year}$. The area that the Earth presents to the flux of photons from the Sun is $A = \pi r^2$.

SIMPLIFY: $\Phi = \dfrac{I}{E_{ph}} = \dfrac{I\lambda}{hc}$, $N = \dfrac{1}{2}\Phi A_{atm} \cdot t_{year} = \dfrac{1}{2}\left(\dfrac{I\lambda}{hc}\right)\left(\pi (R+d)^2\right) t_{year} = \dfrac{\pi I \lambda (R+d)^2 t_{year}}{2hc}$.

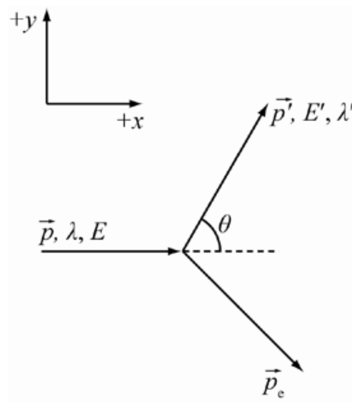
CALCULATE: $N = \dfrac{\pi\left(1370 \text{ W/m}^2\right)\left(5.00 \cdot 10^{-7} \text{ m}\right)\left(6.678 \cdot 10^6 \text{ m}\right)^2 (1 \text{ yr})(31556736 \text{ s/yr})}{2\left(6.626 \cdot 10^{-34} \text{ J s}\right)\left(3.00 \cdot 10^8 \text{ m/s}\right)} = 7.618 \cdot 10^{42}.$

ROUND: To three significant figures, the number of photons received by Earth's upper atmosphere in one year is $N = 7.62 \cdot 10^{42}$.

DOUBLE-CHECK: This is a huge number, but is expected for the number of photons from the Sun to hit the Earth in one full year. Dimensional analysis confirms that the calculation yields a dimensionless result.

36.72. **THINK:** The energy of the backscatter peak corresponds to the energy of the gamma-ray after Compton scattering at an angle of $\theta = 180.^\circ$. The Compton edge energy is the energy cut off or the maximum energy that can be transferred to an electron. The Compton scattering formula can be used to determine the energies of the Compton edge and the back scatter peak for a gamma-ray photon of energy $E = 511$ keV. The mass of an electron is $m_e = 511 \text{ keV}/c^2$.

SKETCH:



RESEARCH: The Compton scattering formula is given by

$$\lambda' = \lambda + \frac{h(1-\cos\theta)}{mc}.$$

Using $\lambda = \frac{hc}{E}$, it becomes $\frac{hc}{E'} = \frac{hc}{E} + \frac{h(1-\cos\theta)}{mc}$ or $\frac{1}{E'} = \frac{1}{E} + \frac{(1-\cos\theta)}{mc^2}$.

SIMPLIFY: For the backscatter peak energy, substituting $\theta = 180°$ yields

$$\frac{1}{E_{\text{bs}}} = \frac{1}{E} + \frac{2}{m_e c^2} \Rightarrow E_{\text{bs}} = \frac{E m_e c^2}{2E + m_e c^2}.$$

The maximum energy transferred to an electron occurs when the scattered photon energy is a minimum, which occurs when $\theta = 180.°$, or when $E' = E_{\text{bs}}$.

$$E_{\text{edge}} = E - E' = E - E_{\text{bs}} = E - \frac{E m_e c^2}{2E + m_e c^2} \Rightarrow E_{\text{edge}} = \frac{2E^2}{2E + m_e c^2}.$$

CALCULATE: $E_{\text{bs}} = \frac{(511\text{ keV})(511\text{ keV}/c^2)c^2}{2(511\text{ keV}) + (511\text{ keV}/c^2)c^2} = 170.333\text{ keV}$

$$E_{\text{edge}} = \frac{2(511\text{ keV})^2}{2(511\text{ keV}) + (511\text{ keV}/c^2)c^2} = 340.667\text{ keV}$$

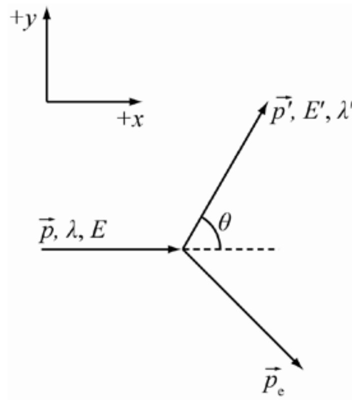
ROUND: Rounding to three significant figures, $E_{\text{bs}} = 170.\text{ keV}$ and $E_{\text{edge}} = 341\text{ keV}$.

DOUBLE-CHECK: Since the energy of the incident gamma-ray is the same as the rest energy of an electron, it is reasonable that the energy of the Compton edge is exactly twice the energy of the backscatter peak.

Multi-Version Exercises

36.73. **THINK:** An X-ray has an initial wavelength of $\lambda = 6.37\text{ nm}$. Its wavelength is increased by $\Delta\lambda = 1.13\text{ pm}$ in a collision with an electron. Some of the energy of the photon will be imparted to the electron, giving it a velocity. To solve this problem, the conservation of energy is used. It is assumed that the electron is initially at rest.

SKETCH:



RESEARCH: Since energy is conserved in this scattering event, the kinetic energy that the electron receives is simply equal to the photon's energy loss:

$$K_e = E - E'.$$

Before the collision, the energy of the photon is $E = hc/\lambda$. After the collision, the energy of the X-ray is

$$E' = \frac{hc}{\lambda'} = \frac{hc}{\lambda + \Delta\lambda}.$$

SIMPLIFY: The kinetic energy of the electron after the collision is:

$$K_e = \frac{hc}{\lambda} - \frac{hc}{\lambda + \Delta\lambda} = hc\left(\frac{(\lambda + \Delta\lambda) - \lambda}{\lambda^2 + \lambda\Delta\lambda}\right) = \frac{hc\Delta\lambda}{\lambda^2 + \lambda\Delta\lambda} = \frac{1}{2}m_e v^2$$

$$v = \sqrt{\frac{2hc\Delta\lambda}{m_e\left(\lambda^2 + \lambda\Delta\lambda\right)}}$$

CALCULATE: $v = \sqrt{\dfrac{2\left(6.626\cdot 10^{-34}\text{ J s}\right)\left(2.998\cdot 10^{8}\text{ m/s}\right)\left(1.13\cdot 10^{-12}\text{ m}\right)}{\left(9.109\cdot 10^{-31}\text{ kg}\right)\left[\left(6.37\cdot 10^{-9}\text{ m}\right)^2 + \left(6.37\cdot 10^{-9}\text{ m}\right)\left(1.13\cdot 10^{-12}\text{ m}\right)\right]}} = 110{,}200\text{ m/s}$

ROUND: Rounding the result to two significant figures gives $v = 110.2$ km/s.

DOUBLE-CHECK: Momentum must also be conserved: $p = p' + p_e$. The initial momentum of the photon is $p = h/\lambda$. Since the x-direction is chosen to be the initial direction of the photon,

$$p = p_x \text{ and } p_y = 0.$$

The final direction of the photon is given by the Compton scattering formula,

$$\lambda' = \lambda + \frac{h}{m_e c}\left[1 - \cos\theta\right] \Rightarrow \theta = \cos^{-1}\left(1 - \frac{m_e c\Delta\lambda}{h}\right) = 57.70^\circ.$$

The components of the final momentum of the photon are

$$p'_x = \frac{h}{\lambda'}\cos\theta = \frac{h}{\lambda + \Delta\lambda}\cos\theta \text{ and } p'_y = \frac{h}{\lambda'}\sin\theta = \frac{h}{\lambda + \Delta\lambda}\sin\theta.$$

The difference between the final and initial momentum of the photon must be equal to the final momentum of the electron.

$$p_{e,x} = p_x - p'_x = \frac{h}{\lambda} - \frac{h}{\lambda + \Delta\lambda}\cos\theta = \frac{6.626\cdot 10^{-34}\text{ J s}}{6.37\cdot 10^{-9}\text{ m}} - \frac{6.626\cdot 10^{-34}\text{ J s}}{6.37113\cdot 10^{-9}\text{ m}}\cos\left(57.70^\circ\right) = 4.8452\cdot 10^{-26}\text{ kg m/s}$$

$$p_{e,y} = p_y - p'_y = 0 - \frac{h}{\lambda + \Delta\lambda}\sin\theta = -\frac{6.626\cdot 10^{-34}\text{ J s}}{6.37113\cdot 10^{-9}\text{ m}}\sin\left(57.70^\circ\right) = -8.7912\cdot 10^{-26}\text{ kg m/s}$$

$$p_e = \sqrt{\left(p_{e,x}\right)^2 + \left(p_{e,y}\right)^2} = \sqrt{\left(4.8452\cdot 10^{-26}\text{ J s/m}\right)^2 + \left(-8.7912\cdot 10^{-26}\text{ J s/m}\right)^2} = 1.0038\cdot 10^{-25}\text{ kg m/s}$$

The momentum of the electron from the original calculation is

$$p_{\text{e}} = m_{\text{e}} v = \left(9.109 \cdot 10^{-31}\text{ kg}\right)\left(110{,}200\text{ m/s}\right) = 1.0038 \cdot 10^{-25}\text{ kg m/s.}$$

Since the calculated momentum using two methods is the same, the speed of the electron found is correct.

36.74. $\Delta E = K = \tfrac{1}{2}mv^2$

$$\Rightarrow v = \sqrt{2\Delta E / m} = \sqrt{2\left(28.52\text{ meV}\right)\left(1.602 \cdot 10^{-22}\text{ J/meV}\right) / \left(9.109 \cdot 10^{-31}\text{ kg}\right)} = 100.2\text{ km/s}$$

36.75. $\Delta E = K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}\left(9.109 \cdot 10^{-31}\text{ kg}\right)\left(92170\text{ m/s}\right)^2 / \left(1.602 \cdot 10^{-22}\text{ J/meV}\right) = 24.15\text{ meV}$

36.76. $K = \tfrac{1}{2}mv^2 = E_{\text{i}} - E_{\text{f}} = \dfrac{hc}{\lambda_i} - \dfrac{hc}{\lambda_i + \Delta\lambda}$

$$\Rightarrow \Delta\lambda = \frac{hc}{\dfrac{hc}{\lambda_i} - \tfrac{1}{2}mv^2} - \lambda_i$$

$$= \frac{\left(6.626 \cdot 10^{-34}\text{ J s}\right)\left(2.998 \cdot 10^{8}\text{ m/s}\right)}{\dfrac{\left(6.626 \cdot 10^{-34}\text{ J s}\right)\left(2.998 \cdot 10^{8}\text{ m/s}\right)}{5.43 \cdot 10^{-9}\text{ m}} - \dfrac{1}{2}\left(9.109 \cdot 10^{-31}\text{ kg}\right)\left(132700\text{ m/s}\right)^2} - 5.43 \cdot 10^{-9}\text{ m}$$

$$= 1.19\text{ pm}$$

36.77. $K = \tfrac{1}{2}mv^2 = E_{\text{i}} - E_{\text{f}} = h(f_{\text{i}} - f_{\text{f}}) \Rightarrow \Delta f = -\tfrac{1}{2}mv^2 / h = -9.65 \cdot 10^{12}\text{ Hz}$

36.78. $p = \dfrac{h}{\lambda}; \quad E = \sqrt{p^2c^2 + m^2c^4} = \sqrt{\left(\dfrac{hc}{\lambda}\right)^2 + m^2c^4}$

$$= \sqrt{\left[\frac{\left(4.136 \cdot 10^{-21}\text{ MeV s}\right)\left(2.998 \cdot 10^{8}\text{ m/s}\right)}{3.63 \cdot 10^{-15}\text{ m}}\right]^2 + \left(938.272\text{ MeV}\right)^2} = 998.5\text{ MeV}$$

36.79. $p = \dfrac{h}{\lambda}; \quad E = \sqrt{p^2c^2 + m^2c^4}$

$$K = E - mc^2 = \sqrt{\left(\frac{hc}{\lambda}\right)^2 + m^2c^4} - mc^2$$

$$\sqrt{\left[\frac{\left(4.136 \cdot 10^{-21}\text{ MeV s}\right)\left(2.998 \cdot 10^{8}\text{ m/s}\right)}{4.43 \cdot 10^{-15}\text{ m}}\right]^2 + \left(938.272\text{ MeV}\right)^2} - \left(938.272\text{ MeV}\right) = 40.9\text{ MeV}$$

36.80. $p = \dfrac{h}{\lambda} = \dfrac{6.626 \cdot 10^{-34}\text{ J s}}{1.71 \cdot 10^{-15}\text{ m}} = 3.8749 \cdot 10^{-19}\text{ kg m/s}$

$$E = \sqrt{\left(3.8749 \cdot 10^{-19}\text{ kg m/s}\right)^2\left(2.9979 \cdot 10^{8}\text{ m/s}\right)^2 + \left(1.6726 \cdot 10^{-27}\text{ kg}\right)^2\left(2.9979 \cdot 10^{8}\text{ m/s}\right)^4} = 1.8998 \cdot 10^{-10}\text{ J}$$

$$v = \frac{pc^2}{E} = \frac{\left(3.8749 \cdot 10^{-19}\text{ kg m/s}\right)\left(2.9979 \cdot 10^{8}\text{ m/s}\right)^2}{1.8998 \cdot 10^{-10}\text{ J}} = 1.833 \cdot 10^{5}\text{ km/s}$$

Chapter 37: Quantum Mechanics

Concept Checks

37.1. e **37.2.** b **37.3.** d **37.4.** b

Multiple-Choice Questions

37.1. c **37.2.** b **37.3.** d **37.4.** b **37.5.** b **37.6.** e **37.7.** d **37.8.** a, e **37.9.** a **37.10.** a

Conceptual Questions

37.11. The answer can be true or false depending on the system. Let us consider the case of a particle in an infinite potential well. The wave function for this potential is given by $\psi_n(x)=\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$ where a is the width of the infinite potential well. The kinetic energy is given by $E_n=\frac{\hbar^2\pi^2n^2}{2ma^2}$. It can be seen that if the amplitude of ψ_n, $\sqrt{2/a}$, is larger, a must be smaller. As a consequence, the kinetic energy is larger as long as n is the same. Therefore, the statement is true. However, if n is not the same, the kinetic energy cannot be compared from the amplitude of wave functions.

37.12. Since the wave functions of a particle in an square infinite potential well have symmetric property for odd n and an antisymmetric property for an even n, where n is the quantum number. Therefore, the probability of the particle is symmetric about $c=L/2$. This means that the probability of finding the particle in the interval between 0 and $L/2$ is the same as for the interval between $L/2$ and L. This does not depend on the energy of the particle. Therefore, the probability of finding the particle between 0 and $L/2$ stays the same regardless the value of the energy of the particle.

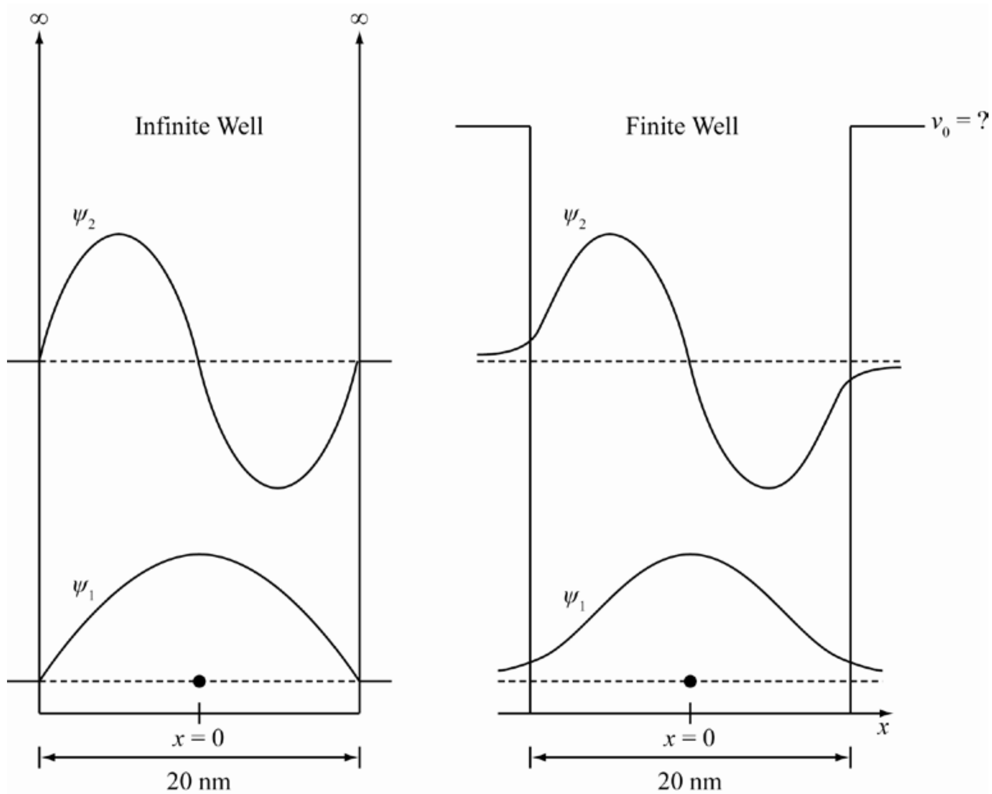
37.13. The wave function for a particle in an infinite square well is given by $\psi_{\text{n}}=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$, where L is the width of the potential well. The probability of finding the particle is $\Pi(x)=\frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right)$. As n increases, the probability density fluctuates around an average probability density given by $\langle\Pi(x)\rangle=\frac{2}{L}\left\langle\sin^2\left(\frac{n\pi x}{L}\right)\right\rangle$. Since $\left\langle\sin^2\left(\frac{n\pi x}{L}\right)\right\rangle=\frac{1}{2}$, it becomes $\langle\Pi(x)\rangle=\frac{1}{L}$. This is exactly the classical probability distribution. Therefore, it does obey the correspondence principle.

37.14. It is known that the wave functions for a particle in a one dimensional harmonic oscillator are symmetric for even-n states. It can be shown that the first derivative of the wave functions with respect to the spatial variable is antisymmetric for even-n states. Since the expectation value of the momentum is defined as $\langle\hat{P}\rangle=\int_{-\infty}^{\infty}\psi_n^*(x)\hat{P}\psi_n(x)dx=\int_{-\infty}^{\infty}-i\hbar\psi_n^*(x)\frac{d}{dx}\psi_n(x)dx$ and $\psi_n^*(x)$ is symmetric and $\frac{d\psi_n}{dx}$ is antisymmetric. Therefore, the above integral is zero, and thus $\langle\hat{P}\rangle=0$.

37.15. The expectation value of the position is defined as $\langle x\rangle=\int|\psi|^2xdx$. Since x is an odd function (antisymmetric), the integral becomes zero where $|\psi|^2$ is symmetric (or even function). The probability $\Pi(x)=|\psi|^2$ is symmetric when the wave function is a symmetric function or an antisymmetric function.

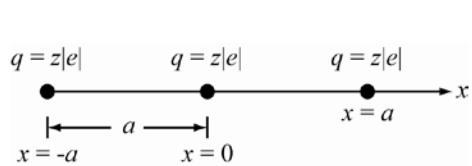
For an antisymmetric wave function, the probability at $x=0$ is zero. Therefore, $\langle x \rangle = 0$ and $\Pi(0)=0$ for an antisymmetric or odd wave function. As an example, the first excited state of a particle in a harmonic oscillator.

37.16. The two lowest energies for an electron in an infinite potential wall are $E_1 = \frac{\hbar^2\pi^2}{2ma^2} = \frac{(1.0546\cdot 10^{-34}\text{ J s})^2\pi^2}{2(9.109\cdot 10^{-31}\text{ kg})(20\cdot 10^{-9}\text{ m})^2} = 1.506\cdot 10^{-22}\text{ J} \approx 0.00094\text{ eV}$ and $E_2 = 2^2 E_1 = 0.0038\text{ eV}$.



Since v_0 is much larger than E_1 or E_2, the two lowest energies for the particle in a finite well is approximately the same as E_1 and E_2 for the infinite well. However, since the electron in the finite well can penetrate into the classically forbidden region, the effective wavelength for the finite well is larger than the wavelength for the infinite well. Since energy is proportional to κ^2 or $1/\lambda^2$, the energy of the particle in the finite well is lower than the energy of the particle in the infinite well.

37.17.



The Coulomb potential energy of the central nucleus due to only two adjacent nuclei is $U(x) = \frac{1}{4\pi\varepsilon_0} z^2 e^2 \left[\frac{1}{a-x} + \frac{1}{a+x}\right]$. For small oscillation about an equilibrium point $(x=0)$, the potential energy can be approximated by a simply harmonic oscillator potential. Expanding the potential energy in

Taylor series about $x=0$ and keeping only up to the term x^2 yield $U(x)\approx U(0)+U'(0)x+(1/2)U''(0)x^2$, where

$$U(0)=\left(\frac{1}{2\pi\varepsilon_0}\right)\frac{z^2e^2}{a},\ U'(0)=0 \text{ and}$$

$$U''(0)=-\frac{1}{2\pi\varepsilon_0}z^2e^2\left[\frac{1}{(a-x)^3}+\frac{1}{(a+x)^3}\right]\Bigg|_{x=0}=\frac{z^2e^2}{\pi\varepsilon_0 a^3}.$$

Thus $U(x)-U(0)=(1/2)U''(0)x^2=-(1/2)kx^2$ with $k=\dfrac{z^2e^2}{\pi\varepsilon_0 a^3}$. The energy $U(0)$ is just a shift in energy and it can be neglected. The energy states of this harmonic oscillator is

$$E_n=\left(n+\frac{1}{2}\right)\left(1.054\cdot 10^{-34}\text{ J s}\right)\sqrt{\frac{6^2\left(1.6\cdot 10^{-19}\text{ C}\right)^2}{12\left(1.66\cdot 10^{-27}\text{ kg}\right)\pi\left(8.85\cdot 10^{-12}\text{ C}^2/(\text{N m}^2)\right)\left(20\cdot 10^{-15}\text{ m}\right)^3}}$$
$$=4.8\cdot 10^{-14}\text{ J}\left(n+\frac{1}{2}\right).$$

The maximum energy allowed is given by $E_{\max}=\dfrac{3}{2}k_{\text{B}}T=\dfrac{3}{2}\left(1.38\cdot 10^{-23}\text{ J/K}\right)\cdot 10^4\text{ K}=2\cdot 10^{-19}\text{ J}$. Therefore, the central nucleus is in its ground state $n=0$.

37.18. The wave functions for a finite square well is in the form of

$$\psi=\begin{cases}A\exp(-\alpha x)+B\exp(+\alpha x) & \text{if } E<U_0\\ C\sin(kx)+D\cos(kx) & \text{if } E>U_0\end{cases}$$

where $k=\sqrt{\dfrac{2m(E-U_0)}{\hbar^2}}$ and $\alpha=\sqrt{\dfrac{2m(U_0-E)}{\hbar^2}}$. If $U=U_0$, both solutions are equal, $\psi=$ a constant. This corresponds to a wavelength $\lambda=\infty$. This is impossible.

37.19. Since the potential $U(x)=\infty$ for $x\le 0$, the condition for the solution is $\psi(x=0)=0$. For $x\le 0$, the solution is known to be $\psi(x)=0$. For $x>0$, the wave function must satisfy a harmonic oscillation potential. Therefore, the solution of the potential should be the wave functions of the harmonic oscillator with the requirement $\psi(x=0)=0$. This requirement is satisfied by all wave functions with odd n. Thus the energies of the states are $E_n=(n+(1/2))\hbar w_0$ where n is an odd number.

37.20. The probability density is given by $P=\psi^*(x)\psi(x)$. The new probability is $P_{\text{new}}=\psi_{\text{new}}{}^*(x)\psi_{\text{new}}(x)=\psi^*(x)e^{-i\phi}e^{+i\phi}\psi(x)=\psi^*(x)\psi(x)=P$. The probability is the same. An additional phase does not change the probability.

37.21. The ground state is approximated by approximating the potential of a harmonic oscillator potential about the equilibrium position. The equilibrium position of the potential in Taylor series up to x^2 yields

$$U(x)\approx U(0)+U'(0)x+(1/2)U''(0)x^2,\text{ where } U(0)=U_0,$$

$$U'(0)=0\text{ and } U''(0)=\frac{U_0}{a^2}\cos h\left(\frac{x}{a}\right)=\frac{U_0}{a^2}.$$

Since $k = U''(0) = U_0 / a^2$, $\omega_0 = \sqrt{k/m}$ and the ground state energy for the harmonic oscillator is $E_{\text{osc}} = (1/2)\hbar\omega_0$, the ground state energy of the particle in $U(x)$ is therefore,

$$E = U(0) + E_{\text{osc}} = U_0 + \left(\frac{1}{2}\right)\hbar\left(\frac{U_0}{ma^2}\right)^{\frac{1}{2}}.$$

37.22. The operator for energy is $i\hbar\partial/(\partial t)$ and the operator for momentum is $-i\hbar\nabla$. Replacing the energy and the momentum in the relativistic energy-momentum relation, $E^2 - p^2c^2 = m^2c^4$ yields

$$\left[(i\hbar)^2 \frac{\partial^2}{\partial t^2} - (-i\hbar)^2 c^2\nabla^2\right]\psi(r) = m^2c^4\psi(r)$$

$$\hbar^2c^2\left[\frac{-1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right]\psi(r) = m^2c^4\psi(r)$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\right]\psi(r) = 0.$$

This is known as the Klein-Gordon equation.

Exercises

37.23. The kinetic energy of a neutron is $10.0\text{ MeV} = 1.60\cdot 10^{-12}$ J. The size of an object that is necessary to observe diffraction effects is on the order of the de Broglie wavelength of the neutron. The (relativistic) de Broglie wavelength is given by

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K^2 + 2Kmc^2}}.$$

$$\lambda = \frac{\left(6.63\cdot 10^{-34}\text{ J s}\right)\left(3.00\cdot 10^{8}\text{ m/s}\right)}{\sqrt{\left(1.60\cdot 10^{-12}\text{ J}\right)^2 + 2\left(1.60\cdot 10^{-12}\text{ J}\right)\left(1.67\cdot 10^{-27}\text{ kg}\right)\left(3.00\cdot 10^{8}\text{ m/s}\right)^2}} = 9.0454\cdot 10^{-15}\text{ m} = 9.05\text{ fm}.$$

Since protons and neutrons have a diameter of about 1.00 fm, they would be useful targets to demonstrate the wave nature of 10.0-MeV neutrons.

37.24. Given $f(x) = (8+3i) + (7-2i)x = (8+7x) + (3-2x)i$,

$$|f(x)|^2 = f(x)^* f(x) = [(8+7x) - (3-2x)i][(8+7x) + (3-2x)i]$$
$$= (8+7x)^2 + (8+7x)(3-2x)i - (8+7x)(3-2x)i - (3-2x)^2 i^2$$
$$= (8+7x)^2 - (3-2x)^2\left(\sqrt{-1}\right)^2$$
$$= (8+7x)^2 + (3-2x)^2$$
$$= 64 + 112x + 49x^2 + 9 - 12x + 4x^2$$
$$= 53x^2 + 100x + 73.$$

37.25. The energies of an electron in a box are given by $E_n = \dfrac{\hbar^2\pi^2}{2ma^2}n^2$. The two lowest energies are:

$$E_1 = \frac{\left(1.055\cdot 10^{-34}\text{ J s}\right)^2\pi^2}{2\left(9.11\cdot 10^{-31}\text{ kg}\right)\left(2.0\cdot 10^{-9}\text{ m}\right)^2}(1)^2 = 1.5\cdot 10^{-20}\text{ J} = 0.094\text{ eV} \text{ and}$$

$$E_2 = (2)^2 E_1 = 6.0\cdot 10^{-20}\text{ J} = 0.38\text{ eV}.$$

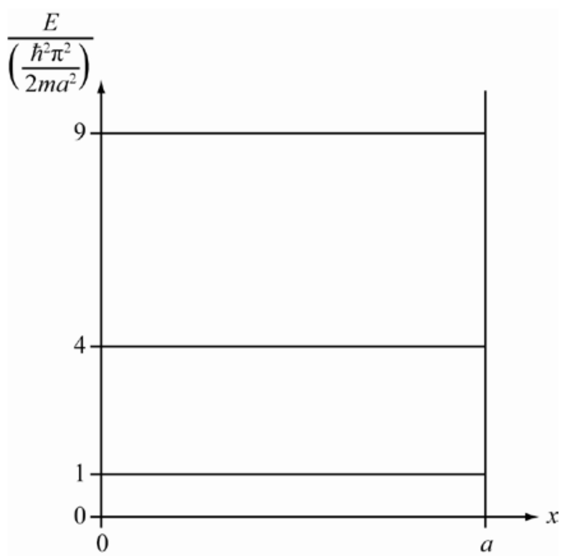
37.26. The energies of a proton in a box are given by $E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2$. The three lowest energies are

$$E_1 = \frac{\left(1.055\cdot 10^{-34}\text{ J s}\right)^2\pi^2}{2\left(1.67\cdot 10^{-27}\text{ kg}\right)\left(1.0\cdot 10^{-10}\text{ m}\right)^2} = 3.3\cdot 10^{-21}\text{ J} = 0.021\text{ eV},\ E_2 = (2)^2 E_1 = 1.3\cdot 10^{-20}\text{ J} = 0.082\text{ eV} \text{ and}$$

$E_3 = (3)^2 E_1 = 3.0\cdot 10^{-20}\text{ J} = 0.18\text{ eV}.$

37.27. The energies for a particle in an infinite square well are $E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2$ for a square well of length L and $E_n = \frac{\hbar^2\pi^2}{2m(2L)^2}n^2$ for a square well of length $2L$. Therefore, $\frac{(E_2 - E_1)L}{(E_2 - E_1)2L} = \frac{\hbar^2\pi^2\left(2^2-1^2\right)/\left(2mL^2\right)}{\hbar^2\pi^2\left(2^2-1^2\right)/\left(8mL^2\right)} = 4.$

37.28. **THINK:** The second excited state is the state with $n = 3$.
SKETCH:



RESEARCH: The energy state of an electron in a one-dimensional infinite well is given by $E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2$. The wavelength of light emitted by the transition from the second excited state to the ground state is found by $\Delta E = \frac{hc}{\lambda}$.

SIMPLIFY:

(a) The energy difference between the second excited state and the ground state is:

$$\Delta E = E_3 - E_1 = \frac{\hbar^2\pi^2}{2ma^2}\left(3^2 - 1^2\right) = 8\frac{\hbar^2\pi^2}{ma^2}.$$

(b) The wavelength is $\lambda = \frac{hc}{\Delta E}$.

CALCULATE:

(a) $\Delta E = 8\frac{\left(1.055\cdot 10^{-34}\text{ J s}\right)^2\pi^2}{2\left(9.11\cdot 10^{-31}\text{ kg}\right)\left(1.0\cdot 10^{-9}\text{ m}\right)^2} = 4.823\cdot 10^{-19}\text{ J} = 3.0\text{ eV}$

(b) $\lambda = \dfrac{\left(6.63\cdot 10^{-34}\text{ J s}\right)\left(3.00\cdot 10^{8}\text{ m/s}\right)}{\left(4.823\cdot 10^{-19}\text{ J}\right)} = 4.12\cdot 10^{-7}\text{ m}$

ROUND: Round to two significant figures.

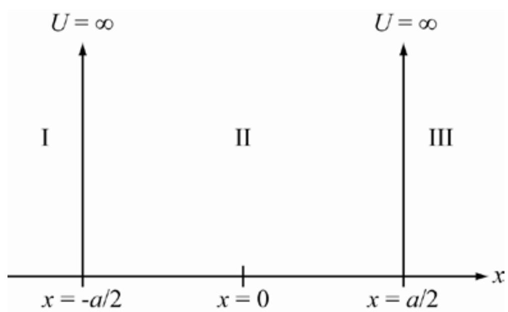
(a) The energy difference between the second excited state and the ground state is $\Delta E = 3.0$ eV.

(b) The wavelength of light emitted is $\lambda = 410$ nm.

DOUBLE-CHECK: The wavelength of falls in the visible part of the electromagnetic spectrum. The energy and wavelength are reasonable results.

37.29. **THINK:** In order to get the solution of the Schrödinger equation for a given potential, the continuity conditions need to be satisfied.

SKETCH:



RESEARCH: In regions I and III the potential energy is infinite, so the wave function is $\psi(x)=0$ for these regions. In region II, the potential energy is zero. Therefore, the potential energy is given by:

$$U(x)=\begin{cases}\infty & \text{for } x<-a/2\\ 0 & \text{for } -a/2\le x\le a/2\\ \infty & \text{for } x>a/2\end{cases}$$

The wave function must satisfy the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}=E\psi(x).$$

The solution of this equation has the form of $\psi(x)=A\sin(\kappa x)+B\cos(\kappa x)$, where $\kappa=\sqrt{2mE/\hbar^2}$ for an infinite square well. Since the wave function must be continuous at the boundaries, the wave function must satisfy $\psi(a/2)=\psi(-a/2)=0$.

SIMPLIFY: Continuity at the boundaries gives:

$$\psi(a/2)=A\sin(\kappa a/2)+B\cos(\kappa a/2)=0 \qquad (1)$$

$$\psi(-a/2)=A\sin(\kappa(-a/2))+B\cos(\kappa(-a/2))=0 \Rightarrow -A\sin(\kappa a/2)+B\cos(\kappa a/2)=0 \qquad (2)$$

Subtracting (1) with (2) yields $2A\sin(\kappa a/2)=0$. This implies that $\kappa a/2=n\pi/2 \Rightarrow \kappa=n\pi/a$, with n even. Adding equations (1) and (2) yields $2B\cos(\kappa a/2)=0$. This implies that $\kappa a/2=n\pi/2 \Rightarrow \kappa=n\pi/a$, with n odd. Therefore, there are two sets of solutions:

$$\psi(x)=\begin{cases}A\sin\left(\dfrac{n\pi x}{a}\right), & \text{with } n \text{ even}\\[2ex] B\cos\left(\dfrac{n\pi x}{a}\right), & \text{with } n \text{ odd}\end{cases}$$

The normalization condition can be used to determine the constants A and B. The result is the same as that shown in the text: $A = B = \sqrt{2/a}$. Therefore, the solution to the Schrödinger equation for this potential is:

$$\psi(x) = \begin{cases} 0 & \text{for } x < -a/2 \text{ and } x > a/2 \\ \sqrt{\dfrac{2}{a}}\sin\left(\dfrac{n\pi x}{a}\right) & \text{for } -a/2 \le x \le a/2 \text{ with even } n \\ \sqrt{\dfrac{2}{a}}\cos\left(\dfrac{n\pi x}{a}\right) & \text{for } -a/2 \le x \le a/2 \text{ with odd } n \end{cases}$$

CALCULATE: Not required.
ROUND: Not required.
DOUBLE-CHECK: The above solutions can be found from the solutions for the infinite square well with interval $(0,a)$ by replacing the variable x with $x+a/2$. Doing so yields:

$$\psi(x+a/2) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}\left(x+\frac{a}{2}\right)\right).$$

Using the trigonometric identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, the above equation becomes:

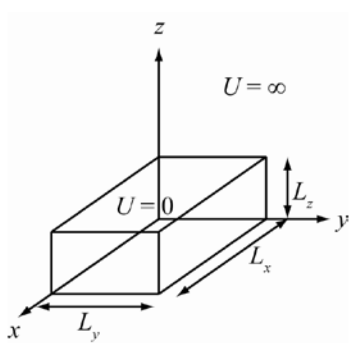
$$\psi(x+a/2) = \sqrt{\frac{2}{a}}\left[\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi}{2}\right) + \cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi}{2}\right)\right].$$

Here if *n* is odd then $\cos(n\pi/2) = 0, \sin(n\pi/2) = \pm 1$ and if *n* is even then $\sin(n\pi/2) = 0, \cos(n\pi/2) = \pm 1$. Therefore, the wave function is given by:

$$\psi(x) = \begin{cases} 0 & \text{for } x < -a/2 \text{ or } x > a/2 \\ \pm\sqrt{\dfrac{2}{a}}\sin\left(\dfrac{n\pi x}{a}\right) & \text{for } -a/2 \le x \le a/2 \text{ with even } n \\ \pm\sqrt{\dfrac{2}{a}}\cos\left(\dfrac{n\pi x}{a}\right) & \text{for } -a/2 \le x \le a/2 \text{ with odd } n \end{cases}$$

This matches the solution above to within a minus sign, which is physically insignificant.

37.30. **THINK:** The three dimensional Schrödinger equation can be used and separation of variables can be assumed in order to solve the problem.
SKETCH:



RESEARCH: Separation of variables allows us to write the potential as $U(x,y,z) = U_1(x)\cdot U_2(y)\cdot U_3(z)$ with:

$$U_1(x) = \begin{cases} \infty & \text{for } x < 0 \\ 0 & \text{for } 0 \le x \le L_x \\ \infty & \text{for } x > L_x \end{cases}, \quad U_2(y) = \begin{cases} \infty & \text{for } y < 0 \\ 0 & \text{for } 0 \le y \le L_y \\ \infty & \text{for } y > L_y \end{cases}, \quad U_3(z) = \begin{cases} \infty & \text{for } z < 0 \\ 0 & \text{for } 0 \le z \le L_z \\ \infty & \text{for } z > L_z \end{cases}$$

The three dimensional Schrödinger equation is given by

$$\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi(x,y,z)+U(x,y,z)\psi(x,y,z)=E\psi(x,y,z).$$

SIMPLIFY: The wave function is also a product of three separable functions, $\psi(x,y,z)=\psi_1(x)\cdot\psi_2(y)\cdot\psi_3(z)$ with:

$$\psi_1(x)=\begin{cases}0 & \text{for } x<0\\ \sqrt{\frac{2}{L_x}}\sin\left(\frac{n_x\pi x}{L_x}\right)\text{ with } n_x=1,2,3... & \text{for } 0\le x\le L_x\\ 0 & \text{for } x>L_x\end{cases}$$

$$\psi_2(y)=\begin{cases}0 & \text{for } y<0\\ \sqrt{\frac{2}{L_y}}\sin\left(\frac{n_y\pi y}{L_y}\right)\text{ with } n_y=1,2,3... & \text{for } 0\le y\le L_y\\ 0 & \text{for } y>L_y\end{cases}$$

$$\psi_3(z)=\begin{cases}0 & \text{for } z<0\\ \sqrt{\frac{2}{L_z}}\sin\left(\frac{n_z\pi z}{L_z}\right)\text{ with } n_z=1,2,3... & \text{for } 0\le z\le L_z\\ 0 & \text{for } z>L_z\end{cases}$$

(a) Therefore, the solution of the wave function of an electron in a potential rectangle is:

$$\psi(x,\,y,\,z)=\psi_1(x)\cdot\psi_2(y)\cdot\psi_3(z)=\sqrt{\frac{8}{L_xL_yL_z}}\sin\left(\frac{n_x\pi x}{L_x}\right)\sin\left(\frac{n_y\pi y}{L_y}\right)\sin\left(\frac{n_z\pi z}{L_z}\right).$$

In the same fashion that the allowed energies were derived in the text for the one-dimensional infinite potential well, the allowed energies are:

$$E_{n_x,n_y,n_z}=\frac{\hbar^2\pi^2}{2mL_x^2}n_x^2+\frac{\hbar^2\pi^2}{2mL_y^2}n_y^2+\frac{\hbar^2\pi^2}{2mL_z^2}n_z^2.$$

(b) The lowest energy for a potential cube with side L occurs when $n_x=n_y=n_z=1,$ and is given by:

$$E_{1,1,1}=\frac{3\hbar^2\pi^2}{2mL^2}.$$

CALCULATE:

(b) For a potential cube with side $1.00\cdot10^{-10}$ m, the lowest allowed energy for the electron is:

$$E_{1,1,1}=\frac{3\left(1.055\cdot10^{-34}\text{ J s}\right)^2\pi^2}{2\left(9.11\cdot10^{-31}\text{ kg}\right)\left(1.00\cdot10^{-10}\text{ m}\right)^2}=1.81\cdot10^{-17}\text{ J}=113.0\text{ eV}.$$

ROUND:

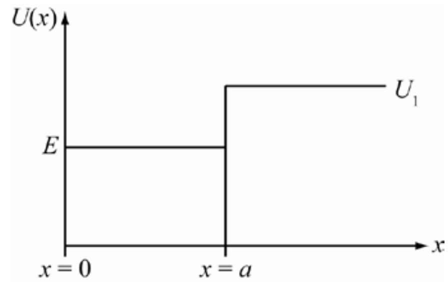
(b) To 3 significant figures, the lowest energy is $E_{1,1,1}=113$ eV.

DOUBLE-CHECK: This is a reasonable amount of energy for an electron to have in such a small volume.

37.31. The potential energy for the well is given by:

$$U(x)=\begin{cases}\infty & \text{for } x<0\\ 0 & \text{for } 0\le x\le a\\ U_1 & \text{for } x>a\end{cases}$$

This is illustrated in the diagram:



Since the question states that the electron is confined to the potential well, $E < U_1$. As shown in the text, the wave function for this finite potential well can be written as:

$$\psi(x) = \begin{cases} 0 & \text{for } x < 0 \\ A\sin(\kappa x) & \text{for } 0 \le x \le a \\ Be^{-\gamma x} & \text{for } x > a \end{cases}$$

Where $\kappa = \sqrt{\frac{2mE}{\hbar^2}}$ and $\gamma = \sqrt{\frac{2m(U_1 - E)}{\hbar^2}}$. The wave function $\psi(x)$ must satisfy the boundary conditions at $x = a$:

$$(1)\quad A\sin(\kappa a) = Be^{-\gamma a}$$
$$(2)\quad \kappa A\cos(\kappa a) = -\gamma Be^{-\gamma a}.$$

Dividing (1) and (2) yields $\frac{\tan(\kappa a)}{\kappa} = -\frac{1}{\gamma}$ or $\tan(\kappa a) = -\frac{\kappa}{\gamma}$. Since κ and γ are positive, $\tan(\kappa a)$ must be negative. This is satisfied when

$$(2n-1)\frac{\pi}{2} < \kappa a < n\pi,\ n = 1,2,3...$$

For the third state $(n = 3)$:

$$\frac{5\pi}{2} < \kappa a < 3\pi \ \Rightarrow\ \frac{25\pi^2}{4} < \kappa^2 a^2 < 9\pi^2 \ \Rightarrow\ \frac{25\pi^2}{4} < \frac{2mEa^2}{\hbar^2} < 9\pi^2.$$

Therefore,

$$\frac{25}{4}\frac{\hbar^2\pi^2}{2ma^2} < E_3 < 9\frac{\hbar^2\pi^2}{2ma^2} \ \Rightarrow\ \frac{25}{4}E_1 < E_3 < 9E_1,$$

where E_1 is the ground state energy for the infinite square well:

$$E_1 = \frac{\left(1.055\cdot 10^{-34}\text{ J s}\right)^2 \pi^2}{2\left(9.11\cdot 10^{-31}\text{ kg}\right)\left(1.0\cdot 10^{-9}\text{ m}\right)^2} = 6.03\cdot 10^{-20}\text{ J} = 0.376\text{ eV}.$$

Therefore,

$$\frac{25}{4}(0.376\text{ eV}) < E_3 < 9(0.376\text{ eV}) \ \Rightarrow\ \ 2.4\text{ eV} < E_3 < 3.4\text{ eV}.$$

Since $U_1 = 2.0\text{ eV} < E_3$, the third state is not a bound state.

37.32. The tunneling probability or transmission coefficient is given by:

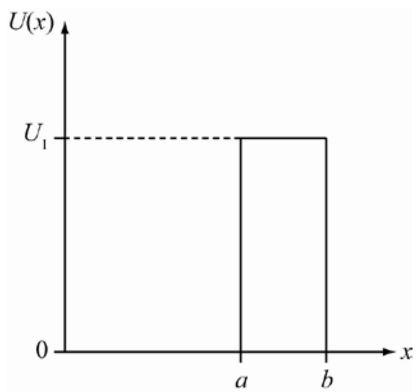
$$T = e^{-2\gamma(b-a)} \text{ where } \gamma = \sqrt{\frac{2m(U_1 - E)}{\hbar^2}}.$$

$$\gamma = \sqrt{\frac{2\left(1.67 \cdot 10^{-27} \text{ kg}\right)\left(29.8 \text{ MeV} - 18.0 \text{ MeV}\right)\left(1.602 \cdot 10^{-13} \text{ J/MeV}\right)}{\left(1.055 \cdot 10^{-34} \text{ J s}\right)^2}} = 7.53 \cdot 10^{14} \text{ m}^{-1}$$

The tunneling probability is $T = e^{-2\left(7.53 \cdot 10^{14} \text{ m}^{-1}\right)\left(1.00 \cdot 10^{-15} \text{ m}\right)} = 0.222$. Therefore, there is a 22.2% chance that the proton will tunnel through the barrier.

37.33. **THINK:** The equation for the transmission coefficient can be used to calculate the tunneling probability. The factor that the neutron's probability of tunneling through the barrier increases by can be found by taking a ratio of the tunneling probabilities. The potential barrier is $b - a = 8.40$ fm wide and $U_1 = 36.2$ MeV high. Originally, the neutron has a kinetic energy of $E_1 = 22.4$ MeV and this is increased to $E_2 = 1.15E_1$.

SKETCH:



RESEARCH: The tunneling probability for a square barrier is given by

$$T = e^{-2\gamma(b-a)} \text{ where } \gamma = \sqrt{\frac{2m(U_1 - E)}{\hbar^2}}.$$

SIMPLIFY: The ratio of the two tunneling probabilities for the two energies E_2 and E_1 is

$$\frac{T_2}{T_1} = \frac{e^{-2\gamma_2(b-a)}}{e^{-2\gamma_1(b-a)}} = e^{-2(\gamma_2 - \gamma_1)(b-a)}.$$

CALCULATE: Since $E_2 = 1.15E_1$:

$$\gamma_2 - \gamma_1 = \sqrt{\frac{2\left(1.67 \cdot 10^{-27} \text{ kg}\right)\left(36.2 \text{ MeV} - (1.15)(22.4 \text{ MeV})\right)\left(1.6 \cdot 10^{-13} \text{ J/MeV}\right)}{\left(1.055 \cdot 10^{-34} \text{ J} \cdot \text{s}\right)^2}} - \sqrt{\frac{2\left(1.67 \cdot 10^{-27} \text{ kg}\right)\left(36.2 \text{ MeV} - 22.4 \text{ MeV}\right)\left(1.6 \cdot 10^{-13} \text{ J/MeV}\right)}{\left(1.055 \cdot 10^{-34} \text{ J} \cdot \text{s}\right)^2}}$$

$$= -1.060 \cdot 10^{14} \text{ m}^{-1}.$$

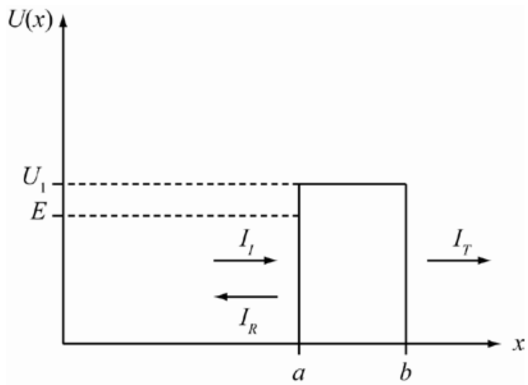
Therefore, ratio is $\frac{T_2}{T_1} = e^{-2\left(-1.060 \cdot 10^{14} \text{ m}^{-1}\right)\left(8.4 \cdot 10^{-15} \text{ m}\right)} = 5.935.$

ROUND: To two significant figures, the neutron's probability of tunneling through the barrier increases by 5.9 times.

DOUBLE-CHECK: Due to the exponential equation it is reasonable that a small increase in energy leads to a large increase in the probability of tunneling.

37.34. **THINK:** The rate of tunneling I_T is proportional to the tunneling probability and the rate of incidence $I_I = 1000.$ electrons/s, and the rate of reflection I_R is the rate of incidence minus the rate of tunneling. The width and height of the potential barrier are $b-a = 1.00$ nm and $U_1 = 2.51$ eV, respectively. Each electron has kinetic energy $E = 2.50$ eV.

SKETCH:



RESEARCH: The tunneling probability is given by:

$$T = e^{-2\gamma(b-a)} \text{ where } \gamma = \sqrt{\frac{2m(U_1 - E)}{\hbar^2}}.$$

The reflection probability is given by $R = 1 - T = 1 - e^{-2\gamma(b-a)}$. The wavelength of the electron is calculated using $\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K^2 + 2Kmc^2}}$.

SIMPLIFY: The rate of tunneling is given by $I_T = I_I T$ and the rate of reflection is $I_R = I_I R = I_I - I_T$.

CALCULATE: $\gamma = \sqrt{\frac{2\left(9.11\cdot 10^{-31} \text{ kg}\right)\left(2.51 \text{ eV} - 2.50 \text{ eV}\right)\left(1.602\cdot 10^{-19} \text{ J/eV}\right)}{\left(1.055\cdot 10^{-34} \text{ J s}\right)^2}} = 5.121\cdot 10^{8} \text{ m}^{-1}$,

$I_T = \left(1000. \text{ electrons/s}\right)e^{-2\left(5.121\cdot 10^{8} \text{ m}^{-1}\right)\left(1.00\cdot 10^{-9} \text{ m}\right)} = 359.1$ electrons/s, and

$I_R = \left(1000. - 359.1\right)$ electrons/s $= 640.9$ electrons/s. The wavelength of an electron is

$$\lambda = \frac{\left(6.63\cdot 10^{-34} \text{ J s}\right)\left(3.00\cdot 10^{8} \text{ m/s}\right)}{\sqrt{\left(\left(2.50 \text{ eV}\right)\left(1.602\cdot 10^{-19} \text{ J/eV}\right)\right)^2 + 2\left(2.50 \text{ eV}\right)\left(1.602\cdot 10^{-19} \text{ J/eV}\right)\left(9.11\cdot 10^{-31} \text{ kg}\right)\left(3.00\cdot 10^{8} \text{ m/s}\right)^2}}$$

$= 7.761\cdot 10^{-10}$ m.

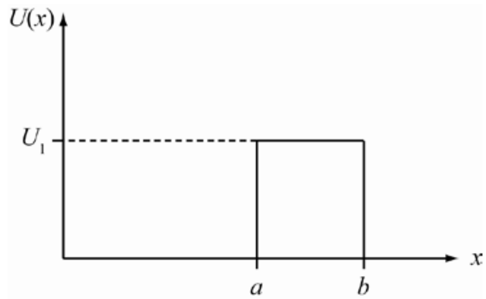
The wavelength of an electron before and after passing the barrier is the same because $U(x) = 0$ on either side of the barrier.

ROUND: To three significant figures, the rate at which electrons pass through the barrier is $I_T = 359$ electrons/s, the rate at which electrons reflect back from the barrier is $I_R = 641$ electrons/s, and the wavelength of the electrons before and after they pass through the barrier is $\lambda = 0.776$ nm.

DOUBLE-CHECK: Since there is such a small difference in energy between the energy of the incident electrons and the potential energy of the barrier, it is reasonable that a large portion of the electrons tunnel through the barrier.

37.35. **THINK:** Given that the tunneling probability is $T = 0.100,$ the equation for the transmission coefficient can be used to calculate the energy of the electron. The potential barrier is $b - a = 2.00\text{ nm}$ wide and $U_1 = 7.00\text{ eV}$ high.

SKETCH:



RESEARCH: The tunneling probability of the electron is given by:

$$T = e^{-2\gamma(b-a)} \text{ where } \gamma = \sqrt{\frac{2m(U_1 - E)}{\hbar^2}}.$$

SIMPLIFY: Solving for E gives:

$$\ln(T) = -2\sqrt{\frac{2m(U_1 - E)}{\hbar^2}}(b-a)$$

$$\frac{2m(U_1 - E)}{\hbar^2} = \left(-\frac{\ln(T)}{2(b-a)}\right)^2$$

$$U_1 - E = \frac{\hbar^2}{2m}\left(-\frac{\ln(T)}{2(b-a)}\right)^2$$

$$E = U_1 - \frac{\hbar^2}{2m}\left(-\frac{\ln(T)}{2(b-a)}\right)^2.$$

CALCULATE:

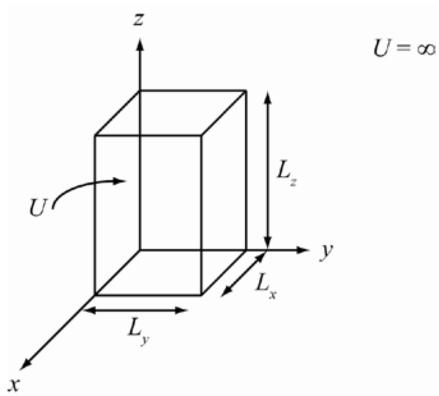
$$E = (7.00\text{ eV})(1.602 \cdot 10^{-19}\text{ J/eV}) - \frac{(1.055 \cdot 10^{-34}\text{ J s})^2}{2(9.11 \cdot 10^{-31}\text{ kg})}\left(-\frac{\ln(0.100)}{2(2.00 \cdot 10^{-9}\text{ m})}\right)^2 = 1.119 \cdot 10^{-18}\text{ J} = 6.987\text{ eV}$$

ROUND: To three significant figures, the energy of the electron is $E = 6.99\text{ eV}.$

DOUBLE-CHECK: The electron energy comes out as less than the potential barrier, as expected.

37.36. **THINK:** The three dimensional Schrödinger equation can be used and separation of variables can be assumed in order to solve the problem. The infinite potential box has dimensions $L_x = 1.00\text{ nm},$ $L_y = 2.00\text{ nm}$ and $L_z = 3.00\text{ nm}.$

SKETCH:



RESEARCH: Separation of variables allows us to write the potential as $U(x,y,z) = U_1(x)\cdot U_2(y)\cdot U_3(z)$ with:

$$U_1(x) = \begin{cases} \infty & \text{for } x<0 \\ 0 & \text{for } 0 \le x \le L_x \\ \infty & \text{for } x > L_x \end{cases},\ U_2(y) = \begin{cases} \infty & \text{for } y<0 \\ 0 & \text{for } 0 \le y \le L_y \\ \infty & \text{for } y > L_y \end{cases},\ U_3(z) = \begin{cases} \infty & \text{for } z<0 \\ 0 & \text{for } 0 \le z \le L_z \\ \infty & \text{for } z > L_z \end{cases}$$

The three dimensional Schrödinger equation is given by

$$\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi(x,y,z)+U(x,y,z)\psi(x,y,z)=E\psi(x,y,z).$$

SIMPLIFY: The wave function is also a product of three separable functions, $\psi(x,y,z) = \psi_1(x)\cdot\psi_2(y)\cdot\psi_3(z)$ with:

$$\psi_1(x) = \begin{cases} 0 & \text{for } x<0 \\ \sqrt{\frac{2}{L_x}}\sin\left(\frac{n_x \pi x}{L_x}\right) \text{ with } n_x = 1,2,3... & \text{for } 0 \le x \le L_x \\ 0 & \text{for } x > L_x \end{cases}$$

$$\psi_2(y) = \begin{cases} 0 & \text{for } y<0 \\ \sqrt{\frac{2}{L_y}}\sin\left(\frac{n_y \pi y}{L_y}\right) \text{ with } n_y = 1,2,3... & \text{for } 0 \le y \le L_y \\ 0 & \text{for } y > L_y \end{cases}$$

$$\psi_3(z) = \begin{cases} 0 & \text{for } z<0 \\ \sqrt{\frac{2}{L_z}}\sin\left(\frac{n_z \pi z}{L_z}\right) \text{ with } n_z = 1,2,3... & \text{for } 0 \le z \le L_z \\ 0 & \text{for } z > L_z \end{cases}$$

Therefore, the solution of the wave function of an electron in a potential rectangle is:

$$\psi(x,\ y,\ z) = \psi_1(x)\psi_2(y)\psi_3(z) = \sqrt{\frac{8}{L_x L_y L_z}}\sin\left(\frac{n_x \pi x}{L_x}\right)\sin\left(\frac{n_y \pi y}{L_y}\right)\sin\left(\frac{n_z \pi z}{L_z}\right).$$

In the same fashion that the allowed energies were derived in the text for the one-dimensional infinite potential well, the allowed energies are:

$$E_{n_x,n_y,n_z} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{L_x^2}+\frac{n_y^2}{L_y^2}+\frac{n_z^2}{L_z^2}\right).$$

CALCULATE: By trial and error one finds from the term

$$\frac{n_x^2}{L_x^2}+\frac{n_y^2}{L_y^2}+\frac{n_z^2}{L_z^2}=\frac{n_x^2}{1.00\text{ nm}^2}+\frac{n_y^2}{4.00\text{ nm}^2}+\frac{n_z^2}{9.00\text{ nm}^2},$$

that the six lowest energy levels correspond to:

$$\left(n_x,n_y,n_z\right)=(1,1,1),\ (1,1,2),\ (1,2,1),\ (1,1,3),\ (1,2,2),\ (1,2,3).$$

The energy is given by:

$$\begin{aligned}E_{n_x,\,n_y,\,n_z}&=\frac{\left(1.055\cdot10^{-34}\text{ J s}\right)^2\pi^2}{2\left(9.11\cdot10^{-31}\text{ kg}\right)}\left(\frac{n_x^2}{1.00\text{ m}^2}+\frac{n_y^2}{4.00\text{ m}^2}+\frac{n_z^2}{9.00\text{ m}^2}\right)\left(10^9\right)^2\\&=\left(6.02915\cdot10^{-20}\text{ J m}^2\right)\left(\frac{n_x^2}{1.00\text{ m}^2}+\frac{n_y^2}{4.00\text{ m}^2}+\frac{n_z^2}{9.00\text{ m}^2}\right)\\&=\left(0.37635\text{ eV m}^2\right)\left(\frac{n_x^2}{1.00\text{ m}^2}+\frac{n_y^2}{4.00\text{ m}^2}+\frac{n_z^2}{9.00\text{ m}^2}\right)\end{aligned}$$

The six lowest energy states are given by

$n_x,\ n_y,\ n_z$	$E_{n_x,\,n_y,\,n_z}$ (eV)
(1,1,1)	0.51225
(1,1,2)	0.63770
(1,2,1)	0.79452
(1,1,3)	0.84679
(1,2,2)	0.91997
(1,2,3)	1.12905

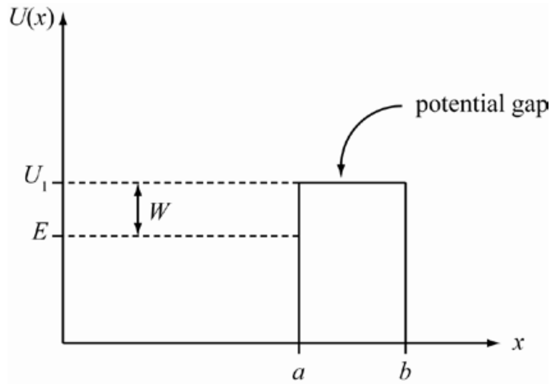
Since none of the quantum states have the same energy, none of the levels are degenerate.
ROUND: The answers should be rounded to three significant figures:

$n_x,\ n_y,\ n_z$	$E_{n_x,\,n_y,\,n_z}$ (eV)
(1,1,1)	0.512
(1,1,2)	0.638
(1,2,1)	0.795
(1,1,3)	0.847
(1,2,2)	0.920
(1,2,3)	1.13

DOUBLE-CHECK: These are reasonable energy values for an electron confined to a small infinite potential box. Any other combination of $\left(n_x,n_y,n_z\right)$ leads to a larger energy, so these are the six lowest energy states.

37.37. **THINK:** The work function is given by $W=U_1-E.$ The equation for the transmission coefficient can be used to find the work function of the probe given that the width of the barrier is $b-a=0.100$ nm and the tunneling probability is 0.100% or $T=0.00100.$ Use the conversion factor: $1.000\text{ J}=6.242\cdot10^{18}\text{ eV}.$

SKETCH:



RESEARCH: The tunneling probability of the electron is given by:

$$T = e^{-2\gamma(b-a)} \text{ where } \gamma = \sqrt{\frac{2m(U_1 - E)}{\hbar^2}}.$$

SIMPLIFY: Solving for the work function W gives:

$$\ln(T) = -2\sqrt{\frac{2mW}{\hbar^2}}(b-a)$$

$$\frac{2mW}{\hbar^2} = \left(-\frac{\ln(T)}{2(b-a)}\right)^2$$

$$W = \frac{\hbar^2}{2m}\left(-\frac{\ln(T)}{2(b-a)}\right)^2.$$

CALCULATE: $W = \dfrac{\left(1.055\cdot 10^{-34}\text{ J s}\right)^2}{2\left(9.11\cdot 10^{-31}\text{ kg}\right)}\left(-\dfrac{\ln(0.00100)}{2\left(0.100\cdot 10^{-9}\text{ m}\right)}\right)^2 = 7.287\cdot 10^{-18}\text{ J}\left(6.242\cdot 10^{18}\text{ eV/J}\right) = 45.5\text{ eV}$

ROUND: To 3 significant figures, the work function of the probe of the scanning tunneling microscope is $W = 45.5$ eV.

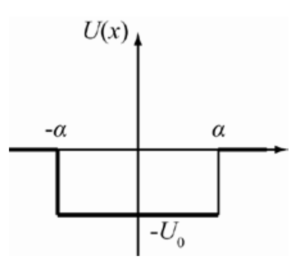
DOUBLE-CHECK: The unit of the work function is electron volts, as expected.

37.38. **THINK:** The attractive square well potential is given by the function:

$$U(x) = \begin{cases} 0 & \text{for } x < -\alpha \\ -U_0 & \text{for } -\alpha < x < \alpha \\ 0 & \text{for } x > \alpha \end{cases}$$

The one-dimensional Schrödinger equation and the boundary conditions can be used to determine the reflection amplitude, R.

SKETCH:



RESEARCH: The solution to the Schrödinger equation for each region is given by:

$$\psi(x)=\begin{cases} e^{i\kappa x}+Re^{-i\kappa x} & \text{for } x<-\alpha \\ Ae^{i\kappa' x}+Be^{-i\kappa' x} & \text{for } -\alpha<x<\alpha \\ Te^{i\kappa x} & \text{for } x>\alpha \end{cases}$$

where R is the amplitude of the reflected wave, T is the amplitude of the transmitted, and

$$\kappa^2=\frac{2mE}{\hbar^2},\ (\kappa')^2=\frac{2m(E+U_0)}{\hbar^2}$$

As is suggested in the question, boundary conditions at $x=-\alpha$ and $x=\alpha$ are required in order to find an expression for R. Boundary conditions require that the wave function and its derivative are continuous at $x=-\alpha$:

$$e^{-i\kappa\alpha}+Re^{i\kappa\alpha}=Ae^{-i\kappa'\alpha}+Be^{i\kappa'\alpha} \tag{1}$$

$$i\kappa\left(e^{-i\kappa\alpha}-Re^{i\kappa\alpha}\right)=i\kappa'\left(Ae^{-i\kappa'\alpha}-Be^{i\kappa'\alpha}\right) \tag{2}$$

At $x=\alpha$:

$$Ae^{i\kappa'\alpha}+Be^{-i\kappa'\alpha}=Te^{i\kappa\alpha} \tag{3}$$

$$i\kappa'\left(Ae^{i\kappa'\alpha}-Be^{-i\kappa'\alpha}\right)=i\kappa Te^{i\kappa\alpha} \tag{4}$$

SIMPLIFY: There are four equations and four unknown coefficients, so an expression for R can be found. Equations (3) and (4) can be used to eliminate T:

$$Ae^{i\kappa'\alpha}+Be^{-i\kappa'\alpha}=\frac{\kappa'}{\kappa}Ae^{i\kappa'\alpha}-\frac{\kappa'}{\kappa}Be^{-i\kappa'\alpha}$$

$$\kappa Ae^{i\kappa'\alpha}+\kappa Be^{-i\kappa'\alpha}=\kappa' Ae^{i\kappa'\alpha}-\kappa' Be^{-i\kappa'\alpha}$$

$$A=\frac{(\kappa'+\kappa)}{(\kappa'-\kappa)}Be^{-2i\kappa'\alpha} \tag{5}$$

Substituting (5) into (1) and solving for B gives: $e^{-i\kappa\alpha}+Re^{i\kappa\alpha}=B\frac{(\kappa'+\kappa)}{(\kappa'-\kappa)}e^{-3i\kappa'\alpha}+Be^{i\kappa'\alpha}$, which implies:

$$B=\frac{e^{-i\kappa\alpha}+Re^{i\kappa\alpha}}{\frac{(\kappa'+\kappa)}{(\kappa'-\kappa)}e^{-3i\kappa'\alpha}+e^{i\kappa'\alpha}}. \tag{6}$$

Substituting (5) into (2) and solving for B gives: $\kappa e^{-i\kappa\alpha}-\kappa Re^{i\kappa\alpha}=B\kappa'\frac{(\kappa'+\kappa)}{(\kappa'-\kappa)}e^{-3i\kappa'\alpha}-\kappa' Be^{i\kappa'\alpha}$, which implies:

$$B=\frac{\kappa e^{-i\kappa\alpha}-\kappa Re^{i\kappa\alpha}}{\kappa'\frac{(\kappa'+\kappa)}{(\kappa'-\kappa)}e^{-3i\kappa'\alpha}-\kappa' e^{i\kappa'\alpha}} \tag{7}$$

Setting (6) and (7) equal and solving for R gives: $\dfrac{e^{-i\kappa\alpha}+Re^{i\kappa\alpha}}{\frac{(\kappa'+\kappa)}{(\kappa'-\kappa)}e^{-3i\kappa'\alpha}+e^{i\kappa'\alpha}}=\dfrac{\kappa e^{-i\kappa\alpha}-\kappa Re^{i\kappa\alpha}}{\kappa'\frac{(\kappa'+\kappa)}{(\kappa'-\kappa)}e^{-3i\kappa'\alpha}-\kappa' e^{i\kappa'\alpha}}$.

$$\kappa'\frac{(\kappa'+\kappa)}{(\kappa'-\kappa)}e^{-i(\kappa+3\kappa')\alpha}-\kappa' e^{i(\kappa'-\kappa)\alpha}+R\kappa'\frac{(\kappa'+\kappa)}{(\kappa'-\kappa)}e^{i(\kappa-3\kappa')\alpha}-R\kappa' e^{i(\kappa+\kappa')\alpha}=\kappa\frac{(\kappa'+\kappa)}{(\kappa'-\kappa)}e^{-i(\kappa+3\kappa')\alpha}+\kappa e^{i(\kappa'-\kappa)\alpha}-$$

$$\ldots-R\kappa\frac{(\kappa'+\kappa)}{(\kappa'-\kappa)}e^{i(\kappa-3\kappa')\alpha}-R\kappa e^{i(\kappa+\kappa')\alpha}$$

Gathering like terms and simplifying gives:

$$R\left[(\kappa'+\kappa)\frac{(\kappa'+\kappa)}{(\kappa'-\kappa)}e^{i(\kappa-3\kappa')\alpha}-(\kappa'-\kappa)e^{i(\kappa+\kappa')\alpha}\right]=(\kappa'+\kappa)e^{i(\kappa'-\kappa)\alpha}-(\kappa'-\kappa)\frac{(\kappa'+\kappa)}{(\kappa'-\kappa)}e^{-i(\kappa+3\kappa')\alpha}$$

$$R\left[(\kappa'+\kappa)^2e^{i(\kappa-3\kappa')\alpha}-(\kappa'-\kappa)^2e^{i(\kappa+\kappa')\alpha}\right]=(\kappa'+\kappa)(\kappa'-\kappa)\left[e^{i(\kappa'-\kappa)\alpha}-e^{-i(\kappa+3\kappa')\alpha}\right]$$

$$R\left[(\kappa'+\kappa)^2e^{i(\kappa-3\kappa')\alpha}-(\kappa'-\kappa)^2e^{i(\kappa+\kappa')\alpha}\right]=(\kappa'^2-\kappa^2)\left[e^{i(\kappa'-\kappa)\alpha}-e^{-i(\kappa+3\kappa')\alpha}\right]$$

At this point it is convenient to multiply both sides by $e^{i(\kappa'-\kappa)\alpha}$. Doing so and solving for R gives:

$$R\left[(\kappa'+\kappa)^2e^{-2i\kappa'\alpha}-(\kappa'-\kappa)^2e^{2i\kappa'\alpha}\right]=(\kappa'^2-\kappa^2)\left[e^{2i(\kappa'-\kappa)\alpha}-e^{-2i(\kappa+\kappa')\alpha}\right]$$

$$=e^{-2i\kappa\alpha}(\kappa'^2-\kappa^2)\left[e^{2i\kappa'\alpha}-e^{-2i\kappa'\alpha}\right]$$

$$R=\frac{e^{-2i\kappa\alpha}(\kappa'^2-\kappa^2)\left[e^{2i\kappa'\alpha}-e^{-2i\kappa'\alpha}\right]}{(\kappa'+\kappa)^2e^{-2i\kappa'\alpha}-(\kappa'-\kappa)^2e^{2i\kappa'\alpha}}. \quad (8)$$

Using Euler`s formula, $e^{i\vartheta}=\cos\vartheta+i\sin\vartheta$, the exponential terms become:

$$e^{2i\kappa'\alpha}=\cos(2\kappa'\alpha)+i\sin(2\kappa'\alpha)$$

$$e^{-2i\kappa'\alpha}=\cos(-2\kappa'\alpha)+i\sin(-2\kappa'\alpha)=\cos(2\kappa'\alpha)-i\sin(2\kappa'\alpha)$$

Substitution of these expressions into (8) and further simplification gives:

$$R=\frac{e^{-2i\kappa\alpha}(\kappa'^2-\kappa^2)\left[(\cos(2\kappa'\alpha)+i\sin(2\kappa'\alpha))-(\cos(2\kappa'\alpha)-i\sin(2\kappa'\alpha))\right]}{(\kappa'+\kappa)^2(\cos(2\kappa'\alpha)-i\sin(2\kappa'\alpha))-(\kappa'-\kappa)^2(\cos(2\kappa'\alpha)+i\sin(2\kappa'\alpha))}$$

$$=\frac{2ie^{-2i\kappa\alpha}(\kappa'^2-\kappa^2)\sin(2\kappa'\alpha)}{(\kappa'^2+2\kappa\kappa'+\kappa^2)(\cos(2\kappa'\alpha)-i\sin(2\kappa'\alpha))-(\kappa'^2-2\kappa\kappa'+\kappa^2)(\cos(2\kappa'\alpha)+i\sin(2\kappa'\alpha))}$$

$$=\frac{ie^{-2i\kappa\alpha}(\kappa'^2-\kappa^2)\sin(2\kappa'\alpha)}{2\kappa\kappa'\cos(2\kappa'\alpha)-i(\kappa'^2+\kappa^2)\sin(2\kappa'\alpha)}$$

No reflected wave, $R=0$, occurs when:

$$\sin(2\kappa'\alpha)=0\Rightarrow 2\kappa'\alpha=n\pi\Rightarrow 2^2\kappa'^2\alpha^2=(n\pi)^2\Rightarrow\frac{2m(E+U_0)}{\hbar^2}\alpha^2=\frac{n^2\pi^2}{2^2}\Rightarrow E_n+U_0=\frac{\hbar^2\pi^2}{2m(2\alpha)^2}n^2.$$

CALCULATE: Not required.
ROUND: Not required.
DOUBLE-CHECK: The energies,

$$E_n+U_0=\frac{\hbar^2\pi^2}{2m(2\alpha)^2}n^2,$$

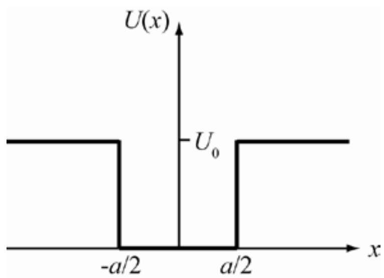
are the allowed energies for the infinite square well of width 2α. Remarkably, perfect transmission occurs when the energy of the particle plus the potential of the well is equal to the allowed energies of an infinite square well.

37.39. **THINK:**
(a) The Schrödinger equation and the relevant boundary conditions can be used to find the wave function and the energy levels.
(b) The solution to the Schrödinger equation can be used to find the penetration distance η for a decrease in the wave function by a factor of $1/e$.

(c) This particular quantum well has width 1 nm and depth 0.300 eV with energy 0.125 eV.
The finite well potential is given by the function:

$$U(x)=\begin{cases}U_0 & \text{for } x\le -a/2\\ 0 & \text{for } -a/2\le x\le a/2\\ U_0 & \text{for } x\ge a/2\end{cases}$$

SKETCH:



RESEARCH:
(a) The solution to the Schrödinger equation for each region is given by:

$$\psi(x)=\begin{cases}Ae^{\gamma x}+Be^{-\gamma x} & \text{for } x\le -a/2\\ C\cos(\kappa x)+D\sin(\kappa x) & \text{for } -a/2\le x\le a/2\\ Ge^{\gamma x}+Fe^{-\gamma x} & \text{for } x\ge a/2\end{cases}$$

where,

$$\kappa^2=\frac{2mE}{\hbar^2},\ \gamma^2=\frac{2m(U_0-E)}{\hbar^2}.$$

Combining these expressions gives

$$\gamma^2=\frac{2m(U_0-E)}{\hbar^2}=\frac{2mU_0}{\hbar^2}-\kappa^2 \Rightarrow \gamma^2+\kappa^2=\frac{2mU_0}{\hbar^2},$$

which represents circles in the $\kappa\gamma$-plane of radius $\frac{2mU_0}{\hbar^2}$. However, in the region $x<-a/2$, as $x\to-\infty$ the B term blows up and in the region $x>a/2$, as $x\to\infty$ the G term blows up. Therefore, the physical solution is given by:

$$\psi(x)=\begin{cases}Ae^{\gamma x} & \text{for } x\le -a/2\\ C\cos(\kappa x)+D\sin(\kappa x) & \text{for } -a/2\le x\le a/2\\ Fe^{-\gamma x} & \text{for } x\ge a/2\end{cases}$$

b) For $x\ge a/2$ the solution requires that

$$\psi(x)\big|_{x=a/2+\eta}=\frac{1}{e}\psi(x)\big|_{x=a/2}$$

SIMPLIFY:
(a) At $x=-a/2$, continuity of the function and its derivative requires:

$$Ae^{-\gamma a/2}=C\cos(-\kappa a/2)+D\sin(-\kappa a/2)=C\cos(\kappa a/2)-D\sin(\kappa a/2) \quad (1)$$

$$-A\gamma e^{-\gamma a/2}=-C\kappa\sin(\kappa a/2)-D\kappa\cos(\kappa a/2)=-\left(C\kappa\sin(\kappa a/2)+D\kappa\cos(\kappa a/2)\right)$$

$$A\gamma e^{-\gamma a/2}=C\kappa\sin(\kappa a/2)+D\kappa\cos(\kappa a/2) \quad (2)$$

At $x=a/2$, continuity of the function and its derivative requires:

$$Fe^{-\gamma a/2}=C\cos(\kappa a/2)+D\sin(\kappa a/2) \quad (3)$$

$$-F\gamma e^{-\gamma a/2}=-C\kappa\sin(\kappa a/2)+D\kappa\cos(\kappa a/2) \quad (4)$$

These four equations can be simplified:
Adding (1) and (3):

$$(A+F)e^{-\gamma a/2} = 2C\cos(\kappa a/2) \quad (5)$$

Subtracting (4) from (2):

$$(A+F)\gamma e^{-\gamma a/2} = 2C\kappa\sin(\kappa a/2) \quad (6)$$

Adding (2) and (4):

$$(A-F)\gamma e^{-\gamma a/2} = 2D\kappa\cos(\kappa a/2) \quad (7)$$

Subtracting (1) from (3):

$$(F-A)e^{-\gamma a/2} = 2D\sin(\kappa a/2) \quad (8)$$

If $C \neq 0$ and $A \neq -F$, dividing (6) by (5) yields:

$$\gamma = \kappa\tan(\kappa a/2) \;\Rightarrow\; \tan(\kappa a/2) = \gamma/\kappa$$

If $D \neq 0$ and $A \neq F$, dividing (7) by (8) yields:

$$-\gamma = \kappa\cot(\kappa a/2) \;\Rightarrow\; \tan(\kappa a/2) = -\kappa/\gamma$$

If these two equations are simultaneously valid then they imply that $\tan^2(\kappa a/2) = -1$ which cannot be true for real values of the energy (i.e. κ must be real). This means that solutions can be divided into two separate classes. The wave functions split into even and odd parity solutions are given by:

(i) For even parity solutions where $\Psi(x) = C\cos(\kappa x)$ in the well, $D = 0$ and $A = F$. The wave function is given by:

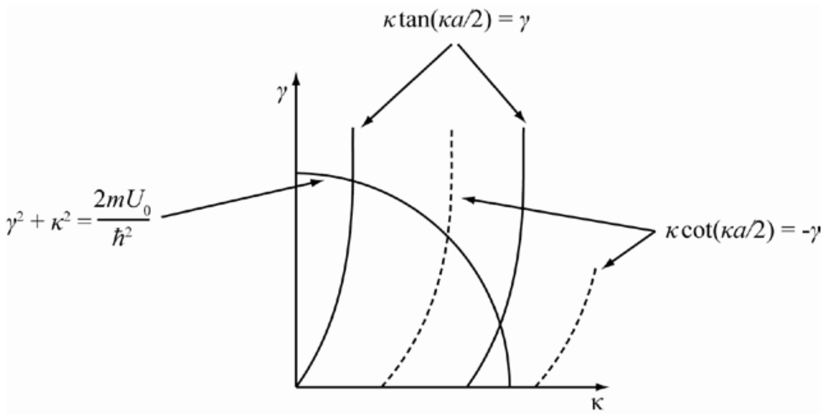
$$\psi(x) = \begin{cases} Ae^{\gamma x} & \text{for } x \leq -a/2 \\ C\cos(\kappa x) & \text{for } -a/2 \leq x \leq a/2 \\ Ae^{-\gamma x} & \text{for } x \geq a/2 \end{cases}$$

This leads to the solution $\kappa\tan(\kappa a/2) = \gamma$.

(ii) For odd parity solutions where $\Psi(x) = D\sin(\kappa x)$ in the well, $C = 0$ and $A = -F$. The wave function is given by:

$$\psi(x) = \begin{cases} Ae^{\gamma x} & \text{for } x \leq -a/2 \\ D\sin(\kappa x) & \text{for } -a/2 \leq x \leq a/2 \\ -Ae^{-\gamma x} & \text{for } x \geq a/2 \end{cases}$$

This leads to the solution $\kappa\cot(\kappa a/2) = -\gamma$. The energy levels can be found by solving numerically or graphically each of these solutions with the required relation between κ and γ: $\gamma^2 = \frac{2mU_0}{\hbar^2} - \kappa^2$. Solving $\kappa\tan(\kappa a/2) = \gamma$ and $\kappa\cot(\kappa a/2) = -\gamma$ graphically (intersection points) gives discrete values for κ and γ and hence the allowed energy levels are obtained from the κ values at the intersection points and $E = \frac{\hbar^2\kappa^2}{2m}$. A sketch of such a graph is shown:



(b) $Fe^{-\gamma(a/2+\eta)} = \frac{1}{e}\left(Fe^{-\gamma(a/2)}\right) \Rightarrow e^{-\gamma\eta} = e^{-1} \Rightarrow \gamma\eta = 1$

The penetration distance is given by $\eta = \frac{1}{\gamma} = \sqrt{\frac{\hbar^2}{2m(U_0 - E)}} = \frac{\hbar}{\sqrt{2m(U_0 - E)}}$.

CALCULATE:

(c) $\eta_{\text{GaAs-GaAlAs}} = \frac{(1.055 \cdot 10^{-34} \text{ J s})}{\sqrt{2(9.109 \cdot 10^{-31} \text{ kg})(0.300 \text{ eV} - 0.125 \text{ eV})(1.602 \cdot 10^{-19} \text{ J/eV})}} = 4.668 \cdot 10^{-10} \text{ m}$

ROUND:

(c) To three significant figures, the penetration distance is $\eta_{\text{GaAs-GaAlAs}} = 467$ pm.

DOUBLE-CHECK: It is reasonable that the penetration depth is independent of the width of the well. A unit analysis of the units for the penetration depth provides the correct unit of length:

$$\frac{\text{J s}}{\sqrt{\text{J kg}}} = \sqrt{\frac{\text{J s}^2}{\text{kg}}} = \sqrt{\frac{(\text{kg m}^2/\text{s}^2)\text{ s}^2}{\text{kg}}} = \sqrt{\text{m}^2} = \text{m}.$$

37.40. The energy states of a harmonic oscillator are given by: $E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0$, $\omega_0 = 2.99 \cdot 10^{14}$ rad/s. The energy of the ground state and the first two excited states are:

$$E_0 = \frac{1}{2}\hbar\omega_0 = \frac{1}{2}(1.055 \cdot 10^{-34} \text{ J s})(2.99 \cdot 10^{14} \text{ rad/s}) = 1.58 \cdot 10^{-20} \text{ J} = 0.0985 \text{ eV},$$
$$E_1 = \frac{3}{2}\hbar\omega_0 = \frac{3}{2}(1.055 \cdot 10^{-34} \text{ J s})(2.99 \cdot 10^{14} \text{ rad/s}) = 4.73 \cdot 10^{-20} \text{ J} = 0.295 \text{ eV},$$
$$E_2 = \frac{5}{2}\hbar\omega_0 = \frac{5}{2}(1.055 \cdot 10^{-34} \text{ J s})(2.99 \cdot 10^{14} \text{ rad/s}) = 7.89 \cdot 10^{-20} \text{ J} = 0.492 \text{ eV}.$$

37.41. The energy levels of a harmonic oscillator are given by: $E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0$. The energy of a photon is given by $E = hc/\lambda$. The energy of the photon with wavelength $\lambda_{3\to 1}$ is given by:

$$\Delta E_{3\to 1} = E_3 - E_1 = \left(\frac{7}{2} - \frac{3}{2}\right)\hbar\omega_0 = 2\hbar\omega_0.$$

The energy of a photon with wavelength $\lambda_{3\to 2}$ is given by:

$$\Delta E_{3\to 2} = E_3 - E_2 = \left(\frac{7}{2} - \frac{5}{2}\right)\hbar\omega_0 = \hbar\omega_0.$$

Then

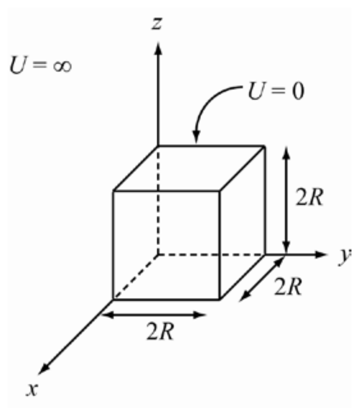
$$\frac{\Delta E_{3\to 1}}{\Delta E_{3\to 2}} = \frac{hc/\lambda_{3\to 1}}{hc/\lambda_{3\to 2}} = \frac{\lambda_{3\to 2}}{\lambda_{3\to 1}} \Rightarrow \lambda_{3\to 2} = \frac{\Delta E_{3\to 1}}{\Delta E_{3\to 2}}\lambda_{3\to 1} = \frac{(2\hbar\omega_0)}{(\hbar\omega_0)}(360 \text{ nm}) = 720 \text{ nm}.$$

37.42. The spacing of two energy levels in a harmonic oscillator is given by: $\Delta E = E_{n+1} - E_n = \hbar\omega_0 = 9\cdot 10^{-20}$ J. For a spring, the frequency is given by: $\omega_0 = \sqrt{k/m}$ where $m = 2\cdot 1.67\cdot 10^{-27} = 3.34\cdot 10^{-27}$ kg is the mass of a diatomic hydrogen molecule. Therefore,

$$\Delta E = \hbar\sqrt{\frac{k}{m}} \Rightarrow k = \frac{(\Delta E)^2 m}{\hbar^2} = \frac{\left(9\cdot 10^{-20}\text{ J}\right)^2\left(3.34\cdot 10^{-27}\text{ kg}\right)}{\left(1.055\cdot 10^{-34}\text{ J s}\right)^2} = 2431 \text{ N/m} \approx 2000 \text{ N/m}.$$

37.43. **THINK:** Since the electron is confined to a cube, the electron can be treated as if it was inside a three-dimensional infinite potential well. In the text, the equation for the energy states for a two dimensional infinite potential is derived. An analogous form for the three dimensional case can be used to determine the ground state energy of the electron in the cube of side length $2R$, where $R = 0.0529$ nm. The spring constant can be found by setting the ground state energy for a potential well equal to the ground state energy for a harmonic oscillator.

SKETCH:



3D infinite potential cube

RESEARCH: The three dimensional energy states (analogous to equation (37.16)) for the electron are:

$$E_{n_x,n_y,n_z} = \frac{\hbar^2\pi^2}{2m_e a^2}n_x^2 + \frac{\hbar^2\pi^2}{2m_e a^2}n_y^2 + \frac{\hbar^2\pi^2}{2m_e a^2}n_z^2,$$

where a is the side length of the cube and m_e is the mass of an electron. The ground state of a harmonic oscillator is given by:

$$E_0 = \left(0 + \frac{1}{2}\right)\hbar\omega_0 = \frac{\hbar}{2}\sqrt{\frac{k}{m_e}},$$

where k is the spring constant.

SIMPLIFY: The ground state, $(n_x, n_y, n_z) = (1,1,1)$, energy for a three dimensional infinite potential well of side length $a = 2R$ is:

$$E_{1,1,1} = \frac{3\hbar^2\pi^2}{8m_e R^2}.$$

For the case $E_{1,1,1} = E_0$: $E_{1,1,1} = \frac{3\hbar^2\pi^2}{8m_e R^2} = \frac{\hbar}{2}\sqrt{\frac{k}{m_e}} \Rightarrow \frac{3\hbar\pi^2}{4m_e R^2} = \sqrt{\frac{k}{m_e}} \Rightarrow k = \frac{9\hbar^2\pi^4}{16m_e R^4}.$

CALCULATE: $E_{1,1,1}=\dfrac{3\hbar^2\pi^2}{8m_eR^2}=\dfrac{3\left(1.055\cdot 10^{-34}\text{ J s}\right)^2\pi^2}{8\left(9.11\cdot 10^{-31}\text{ kg}\right)\left(0.0529\cdot 10^{-9}\text{ m}\right)^2}=1.6159\cdot 10^{-17}\text{ J}=100.87\text{ eV}$

$$k=\frac{9\left(1.055\cdot 10^{-34}\text{ J s}\right)^2\pi^4}{16\left(9.11\cdot 10^{-31}\text{ kg}\right)\left(0.0529\cdot 10^{-9}\text{ m}\right)^4}=8.5484\cdot 10^4\text{ N/m}$$

ROUND: To three significant figures, the ground state energy for an electron confined to a cube of twice the Bohr radius is $E=101\text{ eV}$ and the spring constant that would give the same ground state energy for a harmonic oscillator is $k=85.5\text{ kN/m}$.

DOUBLE-CHECK: The ionization energy of an electron in a hydrogen atom is 13.6 eV and is comparable to the energy calculated.

37.44. **THINK:** The normalization condition,

$$\int_{-\infty}^{\infty}\left|\psi(x)\right|^2 dx=1,$$

can be used to normalize the given wave function, $\psi(x,0)=A\left[\psi_0(x)+\psi_1(x)\right]$.

SKETCH: Not required.

RESEARCH: The oscillator wave functions are given by: $\psi_0(x)=\dfrac{1}{\sqrt{\sigma\pi^{1/4}}}e^{-x^2/2\sigma^2}$, and $\psi_1(x)=\dfrac{1}{\sqrt{\sigma\pi^{1/4}}}\dfrac{1}{\sqrt{2}}\left(2\dfrac{x}{\sigma}\right)e^{-x^2/2\sigma^2}$, where $\sigma=\sqrt{\dfrac{\hbar}{mw_0}}$. Normalization of the wave function requires that,

$$\int_{-\infty}^{\infty}A^2\left|\psi_0(x)+\psi_1(x)\right|^2dx=1.$$

SIMPLIFY:

$$\begin{aligned}
1&=\int_{-\infty}^{\infty}A^2\left|\frac{1}{\sqrt{\sigma\pi^{1/4}}}e^{-x^2/2\sigma^2}+\frac{1}{\sqrt{\sigma\pi^{1/4}}}\frac{1}{\sqrt{2}}\left(2\frac{x}{\sigma}\right)e^{-x^2/2\sigma^2}\right|^2dx\\
&=A^2\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{\sigma\pi^{1/4}}}e^{-x^2/2\sigma^2}\right)^2\left(1+\frac{1}{\sqrt{2}}\left(2\frac{x}{\sigma}\right)\right)^2dx\\
&=\frac{A^2}{\sigma\pi^{1/2}}\int_{-\infty}^{\infty}e^{-x^2/\sigma^2}\left(1+\frac{\sqrt{2}x}{\sigma}\right)^2dx\\
&=\frac{A^2}{\sigma\pi^{1/2}}\int_{-\infty}^{\infty}e^{-x^2/\sigma^2}\left(1+\frac{2\sqrt{2}x}{\sigma}+\frac{2x^2}{\sigma^2}\right)dx\\
&=\frac{A^2}{\sigma\pi^{1/2}}\int_{-\infty}^{\infty}e^{-x^2/\sigma^2}\left(1+\frac{x^2}{\sigma^2}\right)dx
\end{aligned}$$

The $\left(e^{-x^2/\sigma^2}\dfrac{2\sqrt{2}x}{\sigma}\right)$ term vanishes because this is an odd function, so the result will be zero when integrating from $-\infty$ to ∞. Using integral tables, the Gaussian integrals are evaluated:

$$1=A^2\frac{1}{\sigma\pi^{1/2}}\left[\sigma\sqrt{\pi}+\frac{1}{2}\frac{\sigma^3\sqrt{\pi}}{\sigma^2}\right]=A^2\left(1+\frac{1}{2}\right)\Rightarrow A=\sqrt{\frac{2}{3}}.$$

CALCULATE: Not required.

ROUND: Not required.

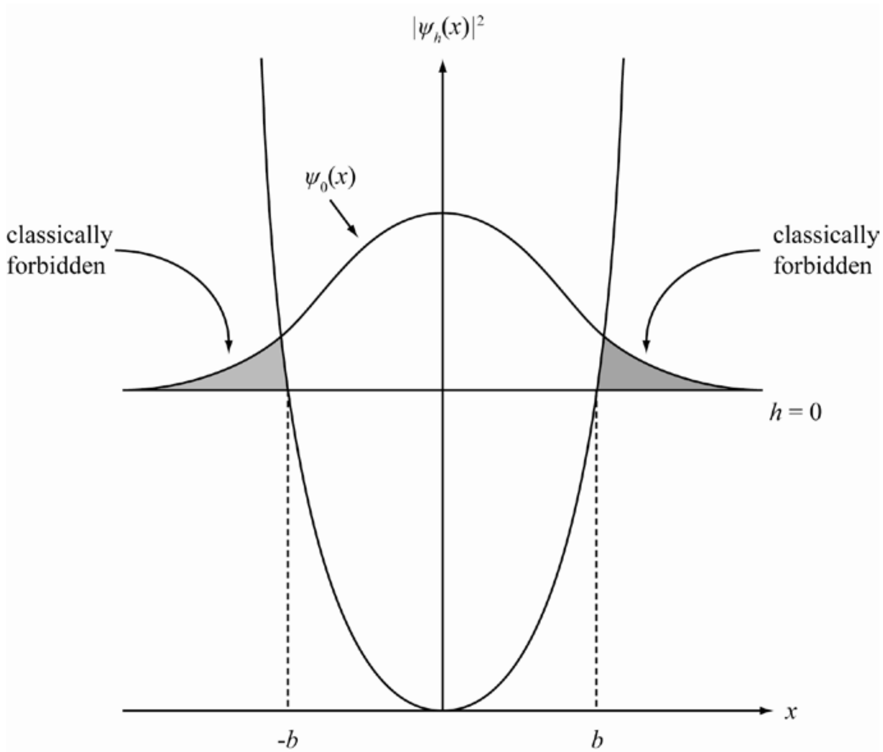
DOUBLE-CHECK: As expected, the coefficient A does not depend on σ, so it is unitless.

37.45. **THINK:** The normalization condition,

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1,$$

can be used to normalize the given wave function, $\psi_0(x) = A_2 e^{-x^2/2b^2}$.

SKETCH:



RESEARCH:

(a) The oscillator wave function is given by:

$$\psi_0(x) = A_2 e^{-x^2/2b^2}.$$

Normalization of the wave function requires that,

$$\int_{-\infty}^{\infty} |\psi_0(x)|^2 dx = 1.$$

(b) As seen from the sketch, the probability that the quantum harmonic oscillator will be found in the classically forbidden region is given by:

$$\Pi = \int_b^{\infty} |\psi_0(x)|^2 \, dx + \int_{-\infty}^{-b} |\psi_0(x)|^2 \, dx = 2\int_b^{\infty} |\psi_0(x)|^2 \, dx.$$

SIMPLIFY:

(a) $1 = \int_{-\infty}^{\infty} |\psi_0(x)|^2 dx = \int_{-\infty}^{\infty} \left| A_2 e^{-x^2/2b^2} \right|^2 dx = A_2{}^2 \int_{-\infty}^{\infty} e^{-x^2/b^2} dx = A_2{}^2 b\sqrt{\pi} \Rightarrow A_2 = \dfrac{1}{\sqrt[4]{\pi}\sqrt{b}}$

(b) Consider the equation: $\Pi = 2\int_b^{\infty} \left| A_2 e^{-x^2/2b^2} \right|^2 dx = \dfrac{2}{\sqrt{\pi} b} \int_b^{\infty} e^{-x^2/b^2} dx.$ With the substitution $u = x/b, dx = b\,du$ the expression becomes:

$$\Pi = \frac{2}{\sqrt{\pi}} \int_1^{\infty} e^{-u^2} du.$$

CALCULATE:

(b) An integration table provides $\int_1^\infty e^{-u^2}du = 0.139,$ so $\Pi = 2\frac{1}{\sqrt{\pi}}(0.139) = 0.157$

ROUND: No rounding is required.

DOUBLE-CHECK: The ratio is less than one, as it must be.

37.46. The wave function for an infinite square well is derived in the text. For a well of width L and for the $n = 3$ state, the wave function inside the well is given by:

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{3\pi x}{L}\right)$$

The probability that the particle is found in the rightmost 10.0% of the well is given by:

$$\Pi = \int_{0.9L}^{L}|\psi(x)|^2 dx = \frac{2}{L}\int_{0.9L}^{L}\sin^2\left(\frac{3\pi x}{L}\right)dx$$

The identity $2\sin^2\theta = 1-\cos 2\theta$ can be used to simplify the integrand:

$$\begin{aligned}\Pi &= \frac{1}{L}\int_{0.9L}^{L}\left[1-\cos\left(\frac{6\pi x}{L}\right)\right]dx \\ &= \frac{1}{L}\left[x - \frac{L}{6\pi}\sin\left(\frac{6\pi x}{L}\right)\right]_{0.9L}^{L} \\ &= \left(0.100 - \frac{1}{6\pi}(\sin 6\pi - \sin 5.4\pi)\right) = 0.0495 = 4.95\%.\end{aligned}$$

37.47. (a) Normalization requires that $\int_{-\infty}^{\infty}|\psi(x)|^2 dx = 1.$ Given that the wave function of the electron in the region $0 < x < L$ is $\psi(x) = A\sin(2\pi x/L),$

$$\int_0^L A^2\sin^2\left(\frac{2\pi x}{L}\right)dx = 1$$

The identity $2\sin^2\theta = 1-\cos 2\theta$ can be used to simplify the integrand:

$$\begin{aligned}1 &= \frac{A^2}{2}\int_0^L\left[1-\cos\left(\frac{4\pi x}{L}\right)\right]dx \\ &= \frac{A^2}{2}\left[x - \frac{L}{4\pi}\sin\left(\frac{4\pi x}{L}\right)\right]_0^L \\ &= \frac{A^2}{2}L \Rightarrow A = \sqrt{\frac{2}{L}}.\end{aligned}$$

(b) The probability of finding the electron in the region $0 < x < L/3$ is:

$$\begin{aligned}\Pi &= \int_0^{L/3}|\psi(x)|^2 dx = \int_0^{L/3}\left|\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)\right|^2 dx = \frac{2}{L}\int_0^{L/3}\sin^2\left(\frac{2\pi x}{L}\right)dx \\ &= \frac{1}{L}\left[x - \frac{L}{4\pi}\sin\left(\frac{4\pi x}{L}\right)\right]_0^{L/3} = \frac{1}{3} - \frac{1}{4\pi}\sin\left(\frac{4\pi}{3}\right) = 0.402.\end{aligned}$$

37.48. The wave function for an infinite square well is derived in the text. For a well of width $L = 2.00$ nm and for the $n = 2$ state, the wave function inside the well is given by:

$$\psi(x) = \sqrt{\frac{2}{2.00\text{ nm}}}\sin\left(\frac{2\pi x}{2.00\text{ nm}}\right).$$

The probability that the particle is found between $x = 0.800 \text{ nm}$ and $x = 0.900 \text{ nm}$ is given by:

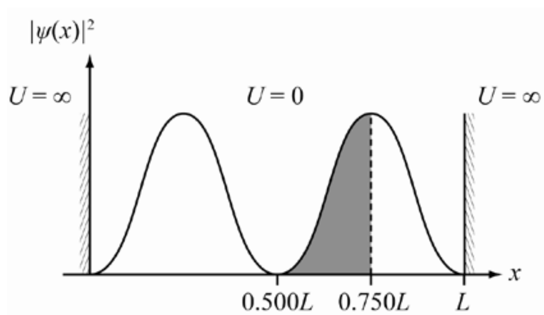
$$\Pi = \int_{0.800 \text{ nm}}^{0.900 \text{ nm}} |\psi(x)|^2 \, dx = \int_{0.800 \text{ nm}}^{0.900 \text{ nm}} \left| \sqrt{\frac{2}{2.00 \text{ nm}}} \sin\left(\frac{2\pi x}{2.00 \text{ nm}}\right) \right|^2 dx = \frac{2}{2.00 \text{ nm}} \int_{0.800 \text{ nm}}^{0.900 \text{ nm}} \sin^2\left(\frac{2\pi x}{2.00 \text{ nm}}\right) dx.$$

The identity $2\sin^2\theta = 1 - \cos 2\theta$ can be used to simplify the integrand:

$$\begin{aligned}
\Pi &= \frac{1}{2.00 \text{ nm}} \int_{0.800 \text{ nm}}^{0.900 \text{ nm}} \left[1 - \cos\left(\frac{4\pi x}{2.00 \text{ nm}}\right)\right] dx \\
&= \frac{1}{2.00 \text{ nm}} \left[x - \frac{2.00 \text{ nm}}{4\pi} \sin\left(\frac{4\pi x}{2.00 \text{ nm}}\right)\right]_{0.800 \text{ nm}}^{0.900 \text{ nm}} \\
&= \frac{0.100 \text{ nm}}{2.00 \text{ nm}} - \frac{1}{4\pi}\left(\sin\left(4\pi \frac{0.900 \text{ nm}}{2.00 \text{ nm}}\right) - \sin\left(4\pi \frac{0.800 \text{ nm}}{2.00 \text{ nm}}\right)\right) \\
&= 0.02109 = 2.11\%.
\end{aligned}$$

37.49. **THINK:** An electron is trapped in a one dimensional infinite potential well of width $L = 300.\text{ pm}$. The wave function for a particle in an infinite potential well can be integrated over the range $(0.500 \text{ L}, 0.750 \text{ L})$ to find the probability that the electron in its first excited state is within this range.

SKETCH:



RESEARCH: The wave function for an infinite square well is derived in the text. For a well of width L and for the first excited state $(n = 2)$, the wave function inside the well is given by:

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right).$$

SIMPLIFY: The probability that the particle is found in the range $0.500L < x < 0.750L$ is given by:

$$\Pi = \int_{0.500L}^{0.750L} |\psi(x)|^2 \, dx = \int_{0.500L}^{0.750L} \left| \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) \right|^2 dx = \frac{2}{L} \int_{0.500L}^{0.750L} \sin^2\left(\frac{2\pi x}{L}\right) dx.$$

The identity $2\sin^2\theta = 1 - \cos 2\theta$ can be used to simplify the integrand:

$$\begin{aligned}
\Pi &= \frac{1}{L} \int_{0.500L}^{0.750L} \left[1 - \cos\left(\frac{4\pi x}{L}\right)\right] dx \\
&= \frac{1}{L}\left[x - \frac{L}{4\pi}\sin\left(\frac{4\pi x}{L}\right)\right]_{0.500L}^{0.750L}
\end{aligned}$$

CALCULATE: $\Pi = 0.250 - \frac{1}{4\pi}\left(\sin(4\pi \cdot 0.750) - \sin(4\pi \cdot 0.500)\right) = 0.250$

ROUND: Therefore, the probability that the electron in the first excited state is found in the range $0.500L < x < 0.750L$ is 0.250.

DOUBLE-CHECK: It is reasonable that the actual length, $L = 300.\text{ pm},$ is irrelevant in finding the probability since the range was given in terms of L. The diagram agrees with this probability that was found.

37.50. **THINK:** The relationship shown in the text for the uncertainty in position can be used for the wave function,

$$\Psi(x,t) = Ae^{-\lambda x^2}e^{-i\omega t}.$$

Leave the normalization constant A as a variable, and do not attempt to determine it numerically.
SKETCH: Not required.

RESEARCH: The uncertainty is given by $(\Delta x)^2 = \left\langle \left(x - \langle x \rangle\right)^2 \right\rangle$. Since $\Psi(x,t)$ is symmetric about $x = 0,$ $\langle x \rangle = 0,$ and so

$$(\Delta x)^2 = \langle x^2 \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)x^2\Psi(x,t)dx.$$

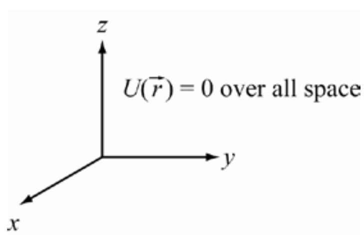
SIMPLIFY: $(\Delta x)^2 = \int_{-\infty}^{\infty} Ae^{-\lambda x^2}e^{i\omega t}x^2 Ae^{-\lambda x^2}e^{-i\omega t}dx = A^2\int_{-\infty}^{\infty} e^{-\lambda x^2}x^2e^{-\lambda x^2}dx = A^2\int_{-\infty}^{\infty} x^2e^{-2\lambda x^2}dx$

CALCULATE: Using integral tables, the uncertainty of x for the given wave function is:

$$\Delta x = A\sqrt[4]{\frac{\pi}{32\lambda^3}}.$$

ROUND: Not required.
DOUBLE-CHECK: The expression of the uncertainty in x states that the larger λ is, the smaller the uncertainty is. This is logical since $\Psi(x,t)$ decays more rapidly for larger λ making $\Psi(x,t)$ more localized.

37.51. **THINK:** A one dimensional plane-wave wave function can be generalized for three dimensions to find $\Psi(\vec{r},\, t)$ for a non relativistic particle of mass m and momentum p. For a free particle, $U(\vec{r}) = 0$ identically. It is constantly zero.
SKETCH:



RESEARCH: A plane-wave wave function in one dimension is given by:

$$\Psi(\vec{x},t) = Ae^{i\kappa x}e^{-i\omega t}, \text{ where } \kappa = p/\hbar \text{ and } \omega = E/\hbar.$$

The wave function can be assumed separable into spatial and time dependent parts. Here p is the momentum of the particle and E is the energy. The probability density function is $\left|\Psi(\vec{r},t)\right|^2$.

SIMPLIFY: The spatial wave function for such a particle can be written as the product of three plane waves. Hence, the wave function takes the form

$$\Psi(\vec{r},t) = Ae^{i(\vec{p}\cdot\vec{r})/\hbar}e^{-iEt/\hbar},$$

where $\vec{r} = \vec{x} + \vec{y} + \vec{z}$. κ and ω have been rewritten as $\kappa = p/\hbar$ and $\omega = E/\hbar$. Since $E = p^2/(2m)$ is the energy of a non relativistic particle, the full wave function can also be written as

$$\Psi(\vec{r},t) = Ae^{i(\vec{p}\cdot\vec{r})/\hbar}e^{-ip^2t/2m\hbar}.$$

The probability density is $\left|\Psi\left(\vec{r},t\right)\right|^2 = \left(Ae^{-i(\vec{p}\cdot\vec{r})/\hbar}e^{ip^2t/2m\hbar}\right)\left(Ae^{i(\vec{p}\cdot\vec{r})/\hbar}e^{-ip^2t/2m\hbar}\right) = A^2.$

CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: The spatial part of this wave function clearly represents a plane wave as $\vec{k}\cdot\vec{r} = c$, where c is a constant, is the general form of a plane perpendicular to $\vec{k}$. The wave function can also be substituted into the time dependent Schrödinger Equation satisfying the equation:

$$(-\hbar^2/2m)\partial^2\left(\Psi(\vec{r},t)\right)/\partial\vec{r}^2 + U(r)\Psi(\vec{r},t) = i\hbar\partial\left(\Psi(\vec{r},t)\right)/\partial t$$

Since U = 0 for a free particle Schrödinger's Equation becomes:

$$(-\hbar^2/2m)\partial^2\left(\Psi(\vec{r},t)\right)/\partial\vec{r}^2 = i\hbar\partial\left(\Psi(\vec{r},t)\right)/\partial t$$

Substituting for $\Psi\left(\vec{r},t\right) = \Psi\left(\vec{r}\right)\Psi\left(t\right) = Ae^{i(\vec{p}\cdot\vec{r})/\hbar}e^{-ip^2t/2m\hbar}$ and differentiating, the left side is:

$$\frac{-\hbar^2(-p^2)Ae^{i(\vec{p}\cdot\vec{r})/\hbar}e^{-ip^2t/2m\hbar}}{2\hbar^2 m}$$

and the right side is:

$$\frac{-i\hbar Aip^2e^{i(\vec{p}\cdot\vec{r})/\hbar}e^{-ip^2t/2m\hbar}}{2m\hbar}$$

After cancelling like terms and recalling that $i^2 = -1$ these are equal and so the wave function does satisfy the time dependent Schrödinger Equation.

37.52. **THINK:** Separation of variables can be used to write the wave function as a product of two functions that depend on only one variable. The equation for the expectation value of x is given in the text. The derivative of this expression provides $d\langle x\rangle/dt$.

SKETCH: Not required.

RESEARCH: The expectation value of the particle's position is given by: $\langle x\rangle = \int_{-\infty}^{\infty}\Psi^*\left(x,t\right)x\Psi\left(x,t\right)dx.$

As shown in the text, the wave function can be written as:

$$\Psi\left(x,t\right) = \psi\left(x\right)\chi\left(t\right), \text{ where } \chi\left(t\right) = Ae^{-iEt/\hbar}$$

using separation of variables. Therefore, the expectation value of x is:

$$\langle x\rangle = \int_{-\infty}^{\infty}\psi^*\left(x\right)\chi^*\left(t\right)x\psi\left(x\right)\chi\left(t\right)dx.$$

SIMPLIFY: The expectation value of x can be simplified as:

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty}\left[\psi^*\left(x\right)x\psi\left(x\right)\right]\chi^*\left(t\right)\chi\left(t\right)dx\\ &= A^2\int_{-\infty}^{\infty}\left[\psi^*\left(x\right)x\psi\left(x\right)\right]e^{iEt/\hbar}e^{-iEt/\hbar}dx\\ &= A^2\int_{-\infty}^{\infty}\left[\psi^*\left(x\right)x\psi\left(x\right)\right]dx\end{aligned}$$

Since the time dependence vanishes,

$$\frac{d\langle x\rangle}{dt} = \frac{d}{dt}\left(A^2\int_{-\infty}^{\infty}\psi^*\left(x\right)x\psi\left(x\right)dx\right) = 0$$

CALCULATE: Not required.

ROUND: Not required.

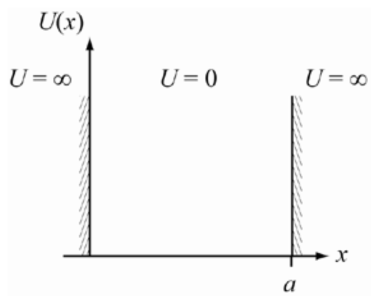
DOUBLE-CHECK: It is reasonable that for a stationary state, the expectation value of the position of the particle does not depend on time (i.e. it remains stationary).

37.53. **THINK:** A quantum particle of mass m is in an infinite one dimensional potential well and has a wave function given by: $\Psi(x,t)=\frac{1}{\sqrt{2}}\left[\Psi_1(x,t)+\Psi_2(x,t)\right]$. The time-independent wave function for an infinite potential well is derived in the text. Since the wave function is separable,

$$\Psi(x,t)=\psi(x)\chi(t), \text{ with } \chi(t)=e^{-iEt/\hbar}.$$

The probability density distribution is just $\left|\Psi(x,t)\right|^2$.

SKETCH:



RESEARCH: The probability density distribution is given by:

$$\left|\Psi(x,t)\right|^2=\frac{1}{2}\left[\Psi_1^*(x,t)+\Psi_2^*(x,t)\right]\left[\Psi_1(x,t)+\Psi_2(x,t)\right]$$

The wave functions Ψ_1 and Ψ_2 are:

$$\Psi_1(x,t)=\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)e^{-iE_1t/\hbar}$$

$$\Psi_2(x,t)=\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)e^{-iE_2t/\hbar}$$

with $E_1=\frac{\hbar^2\pi^2}{2ma^2}$ and $E_2=\frac{2\hbar^2\pi^2}{ma^2}$.

SIMPLIFY: The probability density distribution is:

$$\begin{aligned}
\left|\Psi(x,t)\right|^2 &=\frac{1}{2}\left[\left|\Psi_1(x,t)\right|^2+\left|\Psi_2(x,t)\right|^2+\Psi_1^*(x,t)\Psi_2(x,t)+\Psi_1(x,t)\Psi_2^*(x,t)\right]\\
&=\frac{1}{2}\left[\begin{array}{l}\frac{2}{a}\sin^2\left(\frac{\pi x}{a}\right)+\frac{2}{a}\sin^2\left(\frac{2\pi x}{a}\right)+\frac{2}{a}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)e^{i(E_1-E_2)t/\hbar}+\\ \qquad +\frac{2}{a}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)e^{-i(E_1-E_2)t/\hbar}\end{array}\right]\\
&=\frac{1}{a}\left[\sin^2\left(\frac{\pi x}{a}\right)+\sin^2\left(\frac{2\pi x}{a}\right)+\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\left[e^{i(E_1-E_2)t/\hbar}+e^{-i(E_1-E_2)t/\hbar}\right]\right]
\end{aligned}$$

Using Euler`s formula, $e^{i\theta}=\cos\theta+i\sin\theta$, the exponential terms become:

$$e^{i(E_1-E_2)t/\hbar}=\cos\left(\frac{(E_1-E_2)t}{\hbar}\right)+i\sin\left(\frac{(E_1-E_2)t}{\hbar}\right)$$

$$e^{-i(E_1-E_2)t/\hbar}=\cos\left(-\frac{(E_1-E_2)t}{\hbar}\right)+i\sin\left(-\frac{(E_1-E_2)t}{\hbar}\right)=\cos\left(\frac{(E_1-E_2)t}{\hbar}\right)-i\sin\left(\frac{(E_1-E_2)t}{\hbar}\right)$$

Therefore, the imaginary terms cancel out to give:

$$\begin{aligned}\left|\Psi(x,t)\right|^2 &= \frac{1}{a}\left[\sin^2\left(\frac{\pi x}{a}\right)+\sin^2\left(\frac{2\pi x}{a}\right)+2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{(E_1-E_2)t}{\hbar}\right)\right]\\ &= \frac{1}{a}\left[\sin^2\left(\frac{\pi x}{a}\right)+\sin^2\left(\frac{2\pi x}{a}\right)+2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos\left(-\frac{3\hbar^2\pi^2}{2ma^2}\frac{t}{\hbar}\right)\right]\\ &= \frac{1}{a}\left[\sin^2\left(\frac{\pi x}{a}\right)+\sin^2\left(\frac{2\pi x}{a}\right)+2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{3\hbar\pi^2 t}{2ma^2}\right)\right]\end{aligned}$$

Using trig identities, this reduces to:

$$\left|\Psi(x,t)\right|^2 = \frac{1}{a}\sin^2\left(\frac{\pi x}{a}\right)\left[1+4\cos^2\left(\frac{\pi x}{a}\right)+4\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{3\hbar\pi^2 t}{2ma^2}\right)\right].$$

CALCULATE: Not required.
ROUND: Not required.
DOUBLE-CHECK: The probability density function is real and has units of inverse length, as expected.

37.54. The energy released by the annihilation of a proton and an antiproton is

$$E = mc^2 = 2m_\text{p}c^2 = 2\left(1.6726\cdot 10^{-27}\text{ kg}\right)\left(3.00\cdot 10^8\text{ m/s}\right)^2 = 3.01\cdot 10^{-10}\text{ J} = 1.88\cdot 10^9\text{ eV} \approx 1.9\cdot 10^9\text{ eV}.$$

The energy released from the annihilation is about 4500 times greater than that for a nuclear-fusion reaction.

37.55. The energy time uncertainty relation is given by:

$$\Delta E\Delta t \geq \frac{\hbar}{2}.$$

A violation of classical energy conservation by an amount ΔE is then possible as long as the time interval over which this happens is at most Δt, the value of which is given by the minimal value obtained from the uncertainty relation. ΔE in this case is at least the sum of the rest energies of the particle and the antiparticle. This means that the maximum lifetime of the particle-antiparticle pair is given by:

(a) For an electron/positron pair:

$$\begin{aligned}\Delta t &= \frac{\hbar}{2\left(2m_e c^2\right)}\\ &= \frac{\left(1.055\cdot 10^{-34}\text{ J s}\right)}{2\left(2\left(9.11\cdot 10^{-31}\text{ kg}\right)\left(3.00\cdot 10^8\text{ m/s}\right)^2\right)}\\ &= 3.22\cdot 10^{-22}\text{ s.}\end{aligned}$$

(b) For a proton/anti-proton pair:

$$\Delta t = \frac{\hbar}{2\left(2m_p c^2\right)} = \frac{\left(1.055\cdot 10^{-34}\text{ J s}\right)}{2\left(2\left(1.67\cdot 10^{-27}\text{ kg}\right)\left(3.00\cdot 10^8\text{ m/s}\right)^2\right)} = 1.75\cdot 10^{-25}\text{ s.}$$

37.56. The positron-electron annihilation releases two 2.0 MeV gamma rays or a total of $E_\text{tot} = 4.0\text{ MeV}$. Since energy must be conserved, the kinetic energy of the two particles and the energy created due to the annihilation must be equal to 4.0 MeV. The energy released when the positron and electron annihilate is:

$$E = mc^2 = 2m_\text{e}c^2 = 2\left(9.11\cdot 10^{-31}\text{ kg}\right)\left(3.00\cdot 10^8\text{ m/s}\right)^2 = 1.64\cdot 10^{-13}\text{ J} = 1.02\text{ MeV}.$$

Therefore, the total kinetic energy of the particles is $K_\text{p}+K_\text{e} = E_\text{tot}-E.$ Since $K_\text{e} = K_\text{p}/2,$ the kinetic energy of the electron is:

$$K_p + \frac{K_p}{2} = E_{tot} - E \Rightarrow \frac{3}{2}K_p = E_{tot} - E \Rightarrow K_p = \frac{2}{3}\left(E_{tot} - E\right)$$

$$K_p = \frac{2}{3}\left(4.0 \text{ MeV} - 1.02 \text{ MeV}\right) = 2.0 \text{ MeV}$$

Finally, $K_e = K_p / 2 = 2.0 \text{ MeV} / 2 = 1.0 \text{ MeV}$.

37.57. The allowed energies for a proton in a one dimensional infinite potential well of width α are $E_n = \frac{\hbar^2\pi^2}{2m\alpha^2}n^2$. For the first excited state, $n = 2$. Therefore, the energy of the first excited state of a proton is:

$$E_2 = \frac{4\left(1.055\cdot 10^{-34} \text{ J s}\right)^2 \pi^2}{2\left(1.67\cdot 10^{-27} \text{ kg}\right)\left(1.00\cdot 10^{-9} \text{ m}\right)^2} = 1.31558\cdot 10^{-22} \text{ J} = 8.21\cdot 10^{-4} \text{ eV}.$$

37.58. The probability of tunneling is given by:

$$T = e^{-2\gamma(b-a)}, \text{ where } \gamma = \sqrt{\frac{2m_e\left(U-E\right)}{\hbar^2}}$$

The factor by which the tunneling current changes is:

$$\begin{aligned}\frac{T_i}{T_f} &= \frac{e^{-2\gamma(b-a)}}{e^{-2\gamma(b-a+0.10 \text{ nm})}} \\ &= \exp\left[2\gamma\left(0.10 \text{ nm}\right)\right] \\ &= \exp\left[2\sqrt{\frac{2\left(9.11\cdot 10^{-31} \text{ kg}\right)\left(4.0 \text{ eV}\right)\left(1.602\cdot 10^{-19} \text{ J/eV}\right)}{\left(1.055\cdot 10^{-34} \text{ J s}\right)^2}}\left(0.10\cdot 10^{-9} \text{ m}\right)\right] \\ &= 7.8.\end{aligned}$$

Therefore, the tunneling current decreases by a factor of 7.8 when the tip moves 0.10 nm farther from the surface.

37.59. The normalized solution of the wave function in the ground state $\left(n=1\right)$ for an electron in an infinite cubic potential well of side length L is given by:

(a) $\psi = \psi_x\left(x\right)\psi_y\left(y\right)\psi_z\left(z\right) = \left(\sqrt{\frac{2}{L}}\right)^3 \sin\frac{\pi x}{L}\sin\frac{\pi y}{L}\sin\frac{\pi z}{L};\ 0 < x,y,z < L$

(b) Since the energies are given by

$$E = \frac{\hbar^2\pi^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right),$$

the different energies depend on the energy state, $\left(n_x^2 + n_y^2 + n_z^2\right)$. The ground state is for $\left(n_x, n_y, n_z\right) = \left(1,1,1\right)$, the first excited state is for $\left(n_x, n_y, n_z\right) = \left(1,2,1\right)$, $\left(2,1,1\right)$, $\left(1,1,2\right)$, and the second excited state is for $\left(n_x, n_y, n_z\right) = \left(1,2,2\right)$, $\left(2,1,2\right)$, $\left(2,2,1\right)$. Since an electron has two spin states (up or down), there are a total of 14 possible energy states.

37.60. The energy of a harmonic oscillator is given by $E_n = \hbar\omega_0\left(n+\frac{1}{2}\right)$. The quantum excitation number is then

$$n = \frac{E_n}{\hbar\omega_0} - \frac{1}{2} = \frac{1.00\text{ J}}{\left(1.055\cdot 10^{-34}\text{ J s}\right)\left(4.45\text{ s}^{-1}\right)} - \frac{1}{2} = 2.13\cdot 10^{33}.$$

37.61. The distance between fringes (central maximum and first order peak) for a double slit setup is given by $\Delta y = \frac{\lambda L}{d}$. The wavelength is given by:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mK}} = \frac{\left(6.626\cdot 10^{-34}\text{ J s}\right)}{\sqrt{2\left(1.67\cdot 10^{-27}\text{ kg}\right)\left(4.005\cdot 10^{-21}\text{ J}\right)}} = 1.812\cdot 10^{-10}\text{ m}$$

The distance between interference peaks is:

$$\Delta y = \frac{\left(1.812\cdot 10^{-10}\text{ m}\right)\left(1.5\text{ m}\right)}{\left(0.50\cdot 10^{-3}\text{ m}\right)} = 5.44\cdot 10^{-7}\text{ m} = 0.54\ \mu\text{m}.$$

37.62. The ground state $(n=1)$ energy of an electron in a one dimensional quantum box (infinite well) of length $L = 0.100\text{ nm}$ is:

$$E_1 = \frac{\hbar^2\pi^2}{2m_e L^2} = \frac{\left(1.055\cdot 10^{-34}\text{ J s}\right)^2\pi^2}{2\left(9.11\cdot 10^{-31}\text{ kg}\right)\left(0.100\cdot 10^{-9}\text{ nm}\right)^2} = 6.02915\cdot 10^{-18}\text{ J} = 37.6\text{ eV}.$$

37.63. The ground state $(n=1)$ energy of an electron in a one dimensional infinite well of length L is:

$$E_1 = \frac{\hbar^2\pi^2}{2mL^2}$$

(a) For 2 GaAs layers, $L = 0.56\text{ nm}$, so the energy is:

$$E_1 = \frac{\left(1.055\cdot 10^{-34}\text{ J s}\right)^2\pi^2}{2\left(9.11\cdot 10^{-31}\text{ kg}\right)\left(0.56\cdot 10^{-9}\text{ m}\right)^2} = 1.9\cdot 10^{-19}\text{ J} = 1.2\text{ eV}.$$

(b) For 5 GaAs layers, $L = 1.4\text{ nm}$, so the energy is:

$$E_1 = \frac{\left(1.055\cdot 10^{-34}\text{ J s}\right)^2\pi^2}{2\left(9.11\cdot 10^{-31}\text{ kg}\right)\left(1.4\cdot 10^{-9}\text{ m}\right)^2} = 3.1\cdot 10^{-20}\text{ J} = 0.19\text{ eV}.$$

37.64. (a) The ground state $(n=1)$ energy of a water vapor molecule in a room (an infinite potential well) is:

$$E_1 = \frac{\hbar^2\pi^2}{2m}\left(\frac{1}{(10.0\text{ m})^2} + \frac{1}{(10.0\text{ m})^2} + \frac{1}{(4.00\text{ m})^2}\right)$$

$$= \frac{\left(1.055\cdot 10^{-34}\text{ J s}\right)^2\pi^2}{2\left(2.992\cdot 10^{-26}\text{ kg}\right)}\left(\frac{1}{(10.0\text{ m})^2} + \frac{1}{(10.0\text{ m})^2} + \frac{1}{(4.00\text{ m})^2}\right)$$

$$= 1.5145\cdot 10^{-43}\text{ J} = 9.45\cdot 10^{-25}\text{ eV}.$$

(b) The average kinetic energy of a molecule is given by:

$$K_{\text{avg}} = \frac{3}{2}kT,$$

where *k* is the Boltzmann constant and T is the temperature.

Therefore,

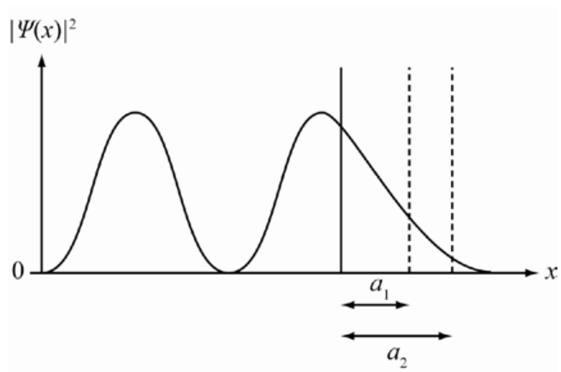
$$K_{\text{avg}} = \frac{3}{2}\left(1.38\cdot 10^{-23}\text{ J/K}\right)\left(300.\text{ K}\right) = 6.21\cdot 10^{-21}\text{ J} = 0.0388\text{ eV}.$$

(c) Since $K_{\text{avg}} >> E,$ thermal energies are so great on a macroscopic scale that quantum effects cannot be observed.

37.65. The fundamental state $(n=1)$ energy of a neutron between rigid walls (a one dimensional infinite potential well) $L = 8.4\text{ fm}$ apart is:

$$E_1 = \frac{\hbar^2\pi^2}{2mL^2} = \frac{\left(1.055\cdot 10^{-34}\text{ J s}\right)^2\pi^2}{2\left(1.67\cdot 10^{-27}\text{ kg}\right)\left(8.4\cdot 10^{-15}\text{ m}\right)^2} = 4.7\cdot 10^{-13}\text{ J} = 2.9\text{ MeV}.$$

37.66. **THINK:** Since the tunneling current is proportional to the tunneling probability, the ratio of the current is found by using the given wave function dependence and the two working gap distances.
SKETCH:



RESEARCH: The electron wave function falls off exponentially as:

$$|\psi| = e^{-\left(10.0\text{ nm}^{-1}\right)a}.$$

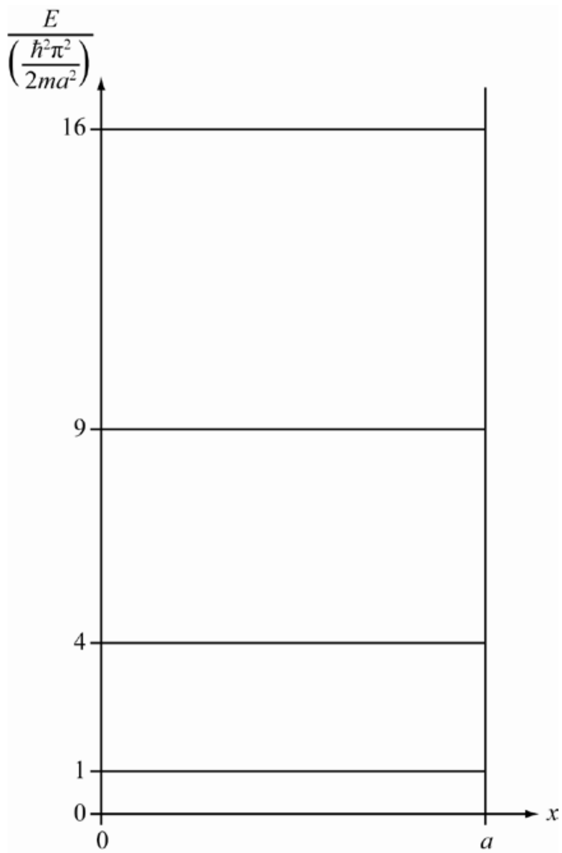
SIMPLIFY: Equation 37.23 shows that the ratio of tunneling currents is: $\dfrac{|\psi_2|^2}{|\psi_1|^2} = \dfrac{e^{-2\left(10.0\text{ nm}^{-1}\right)(0.400\text{ nm})}}{e^{-2\left(10.0\text{ nm}^{-1}\right)(0.420\text{ nm})}}.$

CALCULATE: $\dfrac{|\psi_2|^2}{|\psi_1|^2} = e^{2\left(10.0\text{ nm}^{-1}\right)(0.020\text{ nm})} = 1.49$

ROUND: To three significant figures, the ratio of the current when the STM tip is 0.400 nm above a surface feature to the current when the tip is 0.420 nm above the surface is 1.49.
DOUBLE-CHECK: It is expected that the tunneling current is greater when the STM is closer to the surface since tunneling probability is greater.

37.67. **THINK:** The equation for the allowed energy states of a particle in an infinite square well can be found in the text. The energy difference between the $n=4$ state and the $n=2$ state is the energy of the resulting radiation. The wavelength of the radiation can be found from this energy.

SKETCH:



RESEARCH: The energy of a particle in a one dimensional infinite potential well of width L is given by:

$$E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2.$$

The wavelength of a photon with energy E is given by: $\lambda = \dfrac{hc}{E}.$

SIMPLIFY: For an electron transition from the $n = 4$ state to the $n = 2$ state the change in energy is

$$E_{4\to2} = \frac{\hbar^2\pi^2}{2m_eL^2}(16-4) = \frac{6\hbar^2\pi^2}{m_eL^2}.$$

Therefore, the corresponding wavelength of the radiation is given by:

$$\lambda_{4\to2} = \frac{hc}{E_{4\to2}} = \frac{cm_eL^2}{3\pi\hbar}.$$

CALCULATE: The wavelength of the radiation for a transition from the $n = 4$ state to the $n = 2$ state is:

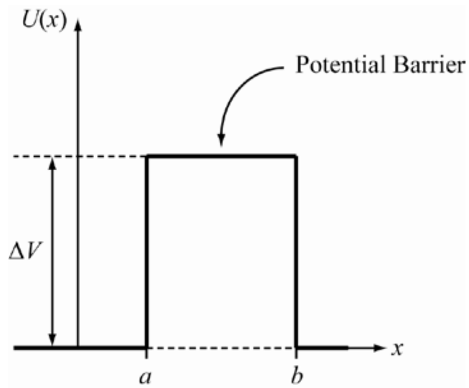
$$\lambda_{4\to2} = \frac{\left(3.00\cdot10^{8}\text{ m/s}\right)\left(9.11\cdot10^{-31}\text{ kg}\right)\left(2.00\cdot10^{-9}\text{ m}\right)^2}{3\pi\left(1.055\cdot10^{-34}\text{ J s}\right)} = 1.099\cdot10^{-6}\text{ m} = 1099\text{ nm}.$$

ROUND: To 3 significant figures, the wavelength is $\lambda_{4\to2} = 1.10\cdot10^{3}$ nm.

DOUBLE-CHECK: The units work out to get a length for the wavelength, as it should.

37.68. **THINK:** This scenario can be modeled as a tunneling problem with a potential barrier height of $\Delta U = 1.00 \text{ eV}$ and a width of $b - a = 2.00 \text{ nm}$.

SKETCH:



RESEARCH: The tunneling probability for an electron is given by:

$$T = e^{-2\gamma(b-a)}, \text{ where } \gamma = \sqrt{\frac{2m_e \Delta U}{\hbar^2}}.$$

SIMPLIFY: $T = \exp\left[-2\sqrt{\frac{2m_e\Delta U}{\hbar^2}}(b-a)\right].$

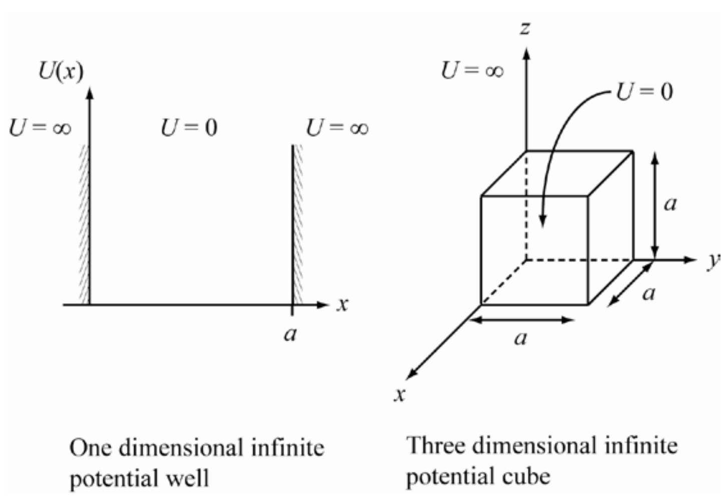
CALCULATE: $T = \exp\left[-2\sqrt{\frac{2(9.11\cdot 10^{-31} \text{ kg})(1.00 \text{ eV})(1.602\cdot 10^{-19} \text{ J/eV})}{(1.055\cdot 10^{-34} \text{ J s})^2}}(2.00\cdot 10^{-9} \text{ m})\right] = 1.270\cdot 10^{-9}.$

ROUND: To 3 significant figures, the probability that a conduction electron in one wire will be found in the other wire after arriving at the gap is $T = 1.27\cdot 10^{-9}$.

DOUBLE-CHECK: Classically, the probability that an electron in one wire can be found in the other wire is zero. However, quantum mechanically it is expected that there is a small probability that this can happen.

37.69. **THINK:** In the text, the equations for the energy states for a one and two dimensional infinite potential are derived. An analogous form for the three dimensional case can be used to determine the ground state energy of the electron in the potential cube of side length $a = 0.10 \text{ nm}$.

SKETCH:



RESEARCH: The allowed energies for the one dimensional infinite potential well are given by:

$$E_{n,\,1\mathrm{D}} = \frac{\hbar^2\pi^2}{2ma^2}n^2.$$

The allowed energies for the three dimensional infinite potential cube in its ground state are given by:

$$E_{n,\,3\mathrm{D}} = \frac{\hbar^2\pi^2}{2ma^2}\left(n_x^2 + n_y^2 + n_z^2\right).$$

SIMPLIFY: The electron confined to the cube is in its ground state, so:

$$E_{1,\,3\mathrm{D}} = \frac{3\hbar^2\pi^2}{2ma^2}.$$

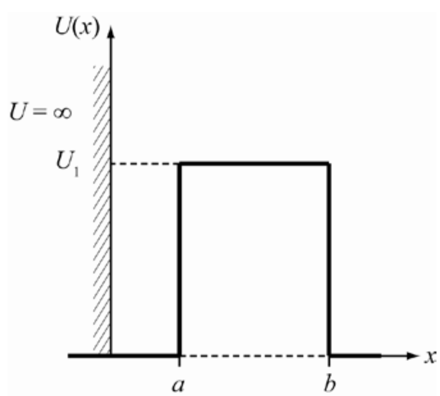
$E_{n,\,1\mathrm{D}}$ is closest to $E_{1,\,3\mathrm{D}}$ for $n = 2$ (the first excited state), so the smallest energy difference is given by:

$$E_{\text{min}} = E_{2,\,1\mathrm{D}} - E_{1,3\mathrm{D}} = \frac{4\hbar^2\pi^2}{2ma^2} - \frac{3\hbar^2\pi^2}{2ma^2} = \frac{\hbar^2\pi^2}{2ma^2}.$$

CALCULATE: $E_{\text{min}} = \dfrac{\left(1.055\cdot 10^{-34}\text{ J s}\right)^2\pi^2}{2\left(9.11\cdot 10^{-31}\text{ kg}\right)\left(0.10\cdot 10^{-9}\text{ m}\right)^2} = 6.029\cdot 10^{-18}\text{ J} = 37.6\text{ eV}$

ROUND: To two significant figures, the minimum energy difference is 38 eV.
DOUBLE-CHECK: The energy is of the same order of magnitude with the ionization energy of an electron (13.6 eV) in a hydrogen atom. Therefore, the answer is reasonable.

37.70. **THINK:** This scenario can be modeled as a tunneling problem with a potential barrier height of U_1 and width $b - a = 2116.8\text{ fm} - 529.2\text{ fm}$ for an electron with energy $E = 129\text{ keV}$. Given that the probability of tunneling is 10%, the equation for the tunneling probability can be used to determine the height of the potential barrier U_1.
SKETCH:



RESEARCH: The probability of tunneling is given by:

$$T = e^{-2\gamma(b-a)}, \text{ where } \gamma = \sqrt{\frac{2m_{\mathrm{e}}\left(U_1 - E\right)}{\hbar^2}}$$

SIMPLIFY: $$T = \exp\left[-2\sqrt{\frac{2m_e\left(U_1 - E\right)}{\hbar^2}}\left(b-a\right)\right]$$

$$\ln\left(T\right) = -2\sqrt{\frac{2m_\text{e}\left(U_1 - E\right)}{\hbar^2}}\left(b-a\right)$$

$$\frac{1}{4}\left(\frac{\ln\left(T\right)}{\left(b-a\right)}\right)^2 = \frac{2m_\text{e}\left(U_1 - E\right)}{\hbar^2}$$

$$U_1 = \frac{\hbar^2}{8m_\text{e}}\left(\frac{\ln\left(T\right)}{\left(b-a\right)}\right)^2 + E$$

CALCULATE:

$$U_1 = \frac{\left(1.055\cdot 10^{-34}\text{ J s}\right)^2}{2\left(9.11\cdot 10^{-31}\text{ kg}\right)}\left(\frac{\ln\left(0.100\right)}{2\left(2116.8\text{ fm} - 529.2\text{ fm}\right)\left(10^{-15}\text{ m/fm}\right)}\right)^2 + \left(129\cdot 10^3\text{ eV}\right)\left(1.602\cdot 10^{-19}\text{ J/eV}\right)$$

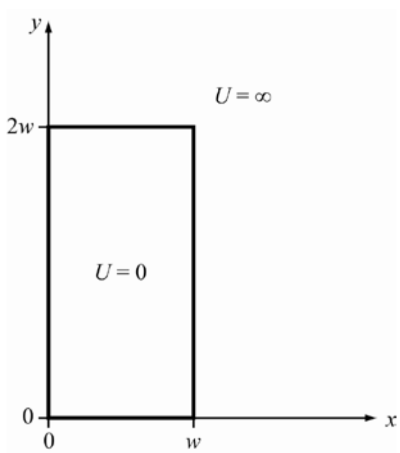
$$= 2.388\cdot 10^{-14}\text{ J} = 149.1\text{ keV}$$

ROUND: To three significant figures, the height of the potential barrier is $U_1 = 149$ keV.

DOUBLE-CHECK: It is expected that the potential barrier is larger than the energy of the particle in order to allow for tunneling. Since the tunneling probability is 10.0% it is reasonable that the potential barrier is comparable to the kinetic energy of the particle.

37.71. **THINK:** The equation for the allowed energies of a two dimensional infinite potential well is given in the text.

SKETCH:



RESEARCH: The allowed energies for an electron in an infinite potential rectangle of dimensions L_x and L_y are given by:

$$E_{n_x,n_y} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{L_x{}^2} + \frac{n_y^2}{L_y{}^2}\right).$$

SIMPLIFY: For $L_x = w$ and $L_y = 2w$,

$$E_{n_x,n_y} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{w^2} + \frac{n_y^2}{4w^2}\right) = \frac{\hbar^2\pi^2}{8mw^2}\left(4n_x^2 + n_y^2\right).$$

CALCULATE: The lowest energy for which degeneracy occurs is for:

$$\left(n_x,n_y\right) = \left(2,2\right)\text{ and }\left(n_x,n_y\right) = \left(1,4\right).$$

ROUND: Not required.

DOUBLE-CHECK: $E_{2,\,2}=\dfrac{\hbar^2\pi^2}{8mw^2}\left(4\left(2^2\right)+\left(2^2\right)\right)=\dfrac{5\hbar^2\pi^2}{2mw^2}$ and $E_{1,4}=\dfrac{\hbar^2\pi^2}{8mw^2}\left(4\left(1^2\right)+\left(4^2\right)\right)=\dfrac{5\hbar^2\pi^2}{2mw^2}$. These values are the same, as required.

Multi-Version Exercises

37.72. $T=e^{-2\gamma w}$; width $=w$

decay constant $\gamma=\sqrt{2(mc^2)(U-E)/(\hbar c)^2}$

$=\sqrt{2(3727.4\text{ MeV})(15.5\text{ MeV}-5.15\text{ MeV})/(197.327\text{ MeV fm})^2}=1.40767\text{ fm}^{-1}$

$T=e^{-2(1.40767)(11.7)}=4.95\cdot 10^{-15}$

37.73. decay constant $\gamma=\sqrt{2(mc^2)(U-E)/(\hbar c)^2}$

$=\sqrt{2(3727.4\text{ MeV})(15.7\text{ MeV}-6.31\text{ MeV})/(197.327\text{ MeV fm})^2}=1.3\text{ fm}^{-1}$

37.74. $\gamma=\sqrt{2(mc^2)(U-E)/(\hbar c)^2}$

$\Rightarrow E=U-\gamma^2(\hbar c)^2/(2mc^2)$

$=15.7\text{ MeV}-(1.257\text{ fm}^{-1})^2(197.327\text{ MeV fm})^2/(2\cdot 3727.4\text{ MeV})=7.4\text{ MeV}$

37.75. $T=e^{-2\gamma w}$; width $=w=-(\ln T)/(2\gamma)$

decay constant $\gamma=\sqrt{2(mc^2)(U-E)/(\hbar c)^2}$

$=\sqrt{2(3727.4\text{ MeV})(15.9\text{ MeV}-8.59\text{ MeV})/(197.327\text{ MeV fm})^2}=1.183\text{ fm}^{-1}$

$w=-\ln(1.042\cdot 10^{-18})/(2\cdot 1.183\text{ fm}^{-1})=17\text{ fm}$

The value of w comes out very close to 17.5 fm, but the subtraction of the two energies in the calculation of the decay constant limits the final result to two significant figures.

37.76. $\Delta E=E_\text{f}-E_\text{i}=(n_\text{f}^2-n_\text{i}^2)\dfrac{\hbar^2\pi^2}{2ma^2}=(25-1)\dfrac{(1.0546\cdot 10^{-34}\text{ J s})^2\pi^2}{2(9.109\cdot 10^{-31}\text{ kg})(13.5\cdot 10^{-9}\text{ m})^2}=7.93\cdot 10^{-21}\text{ J}$

37.77. $\Delta E=E_\text{f}-E_\text{i}=(n_\text{f}^2-n_\text{i}^2)\dfrac{\hbar^2\pi^2}{2ma^2}$

$\Rightarrow n_\text{f}^2-n_\text{i}^2=\dfrac{2ma^2\Delta E}{\hbar^2\pi^2}$

$=\dfrac{2(1.673\cdot 10^{-27}\text{ kg})(23.9\cdot 10^{-9}\text{ m})^2(1.08\cdot 10^{-6}\text{ eV})(1.602\cdot 10^{-19}\text{ J/eV})}{(1.0546\cdot 10^{-34}\text{ J s})^2\pi^2}$

$=3$ (allowing for rounding error)

With $n_\text{f}=2$, this means $n_\text{i}=1$, which is our answer.

37.78. $\Delta E=E_\text{f}-E_\text{i}=(n_\text{f}^2-n_\text{i}^2)\dfrac{\hbar^2\pi^2}{2ma^2}$

$\Rightarrow m=(n_\text{f}^2-n_\text{i}^2)\dfrac{\hbar^2\pi^2}{2a^2\Delta E}$

$=(4-1)\dfrac{(1.0546\cdot 10^{-34}\text{ J s})^2\pi^2}{2(19.3\cdot 10^{-9}\text{ m})^2(2.639\cdot 10^{-25}\text{ J})}=1.675\cdot 10^{-27}\text{ kg}$

It looks like our particle is a neutron.

Chapter 38: Atomic Physics

Concept Checks

38.1. c **38.2.** e **38.3.** c **38.4.** d **38.5.** a **38.6.** c

Multiple-Choice Questions

38.1. d **38.2.** c **38.3.** d **38.4.** a **38.5.** b **38.6.** c **38.7.** b **38.8.** a and c

Conceptual Questions

38.9. Electrons are not solid particles, so the idea of an electron orbiting the nucleus (analogous to a planet orbiting the Sun) is not an accurate description. As described by quantum mechanics, electrons are point particles whose locations are described by probability density distributions filling three-dimensional orbitals.

38.10. A hydrogen atom cannot absorb just any wavelength of light because of the quantization of orbital angular momentum and energy predicted by the Bohr model. Only light of a specific wavelength, which allows the hydrogen atom to transition to an excited quantum state, can be absorbed. In addition, a photon of more than 13.6 eV will just ionize the H atom, so there is a minimum absorbable wavelength.

38.11. As shown in Chapter 37, the energy levels for an infinite square well are given by,

$$E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2,$$

and the energy levels for a quantum harmonic oscillator are given by,

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0.$$

For a hydrogen atom, the energy levels are given by:

$$E_n = -\frac{\mu k^2 e^4}{2\hbar^2}\frac{1}{n^2}.$$

Therefore, for increasing n, the difference in energy levels increases for an infinite square well, remains constant for a quantum harmonic oscillator, and decreases for a hydrogen atom.

38.12. The energy levels for a hydrogen atom are given by

$$E_n = -\frac{\mu k^2 e^4}{2\hbar^2}\frac{1}{n^2},$$

and the orbital radius is given by

$$r = \frac{\hbar^2}{\mu k e^2}n^2.$$

Since the Coulomb force is proportional to k, doubling the Coulomb force effectively changes k to $2k$. Therefore, the energy levels would increase by a factor of four and the size of the atoms would decrease by a factor of two.

38.13. The Bohr model predicts that the electron orbits the nucleus at a certain distance. In the ground state, the electron orbits at the Bohr radius. The quantum mechanical model predicts that the location of the electron is governed by the probability density distribution of its wave function. Therefore, the quantum mechanical model predicts that the electron spends more time near to the nucleus than the Bohr model does.

38.14. The associated Legendre functions, $P_l^m(\cos\theta)$, for $l<4$ are given on page 1266:

	$m=0$	$m=1$	$m=2$	$m=3$
$l=0$	1			
$l=1$	$\cos\theta$	$-\sin\theta$		
$l=2$	$\frac{1}{2}\left(3\cos^2\theta-1\right)$	$-3\cos\theta\sin\theta$	$3\sin^2\theta$	
$l=3$	$\frac{1}{2}\left(5\cos^3\theta-3\cos\theta\right)$	$-\frac{3}{2}\sin\theta\left(5\cos^2\theta-1\right)$	$15\cos\theta\sin^2\theta$	$-15\sin^3\theta$

The xy-plane occurs at $\theta=\pi/2$. Since $\cos(\pi/2)=0$ and $\sin(\pi/2)=1$, the wave functions that have a maximum probability in the xy-plane are: $l=1$, $m=1$ and $l=2$, $m=2$ and $l=3$, $m=3$.

38.15. Electrons are fermions and by the Pauli exclusion principle, two identical fermions cannot exist together. Therefore, within a subshell, electrons must have opposite spins (one electron must be spin up and one must be spin down). By Hund's rule, the orbitals are each occupied by electrons of parallel spin before double occupation of electrons with opposite spin takes place. This occurs because electrons having parallel spins (symmetric wave functions) are, on average, further apart than electrons having opposite spins (antisymmetric wave functions). It is commonly thought that this larger separation between electrons reduces the electron-electron repulsion energy and puts the atom in a lower energy state. However, it is now known that parallel-spin electrons of larger separation are less effectively shielded by one another from the nucleus. This results in orbitals that are more tightly bound to the nucleus, yielding a lower energy state.

38.16. The nonradiative transitions have to be faster to achieve the population inversion needed for laser operation. The fast nonradiative transition from level 4 to level 3 feeds level 3 at a rate higher than the laser transition rate, allowing for population build-up on this level. At the same time, the fast nonradiative transition from level 2 to level 1 depletes the lower level of the laser transition (level 2) fast enough for a population inversion to exist between levels 2 and 3 at all times.

38.17. When the wave function is zero it means that the particle will never be found at that point, since the probability density is zero. There is a difference between the two given descriptions. Electrons cannot be observed passing through a point because that would violate the Heisenberg uncertainty principle.

38.18. The electron in the ground state $(n=1)$ of the hydrogen atom has an energy of -13.6 eV, and the electron in the first excited state $(n=2)$ has an energy of -3.4 eV. The difference in energy between these two states is 10.2 eV. This is the minimum energy required to excite the hydrogen atom. The 10 eV electron does not have enough energy to excite the hydrogen atom, so it will remain in the ground state. Therefore, no emission of photons will occur.

Exercises

38.19. The shortest wavelength of light that a hydrogen atom can emit occurs when an electron is captured by the hydrogen atom and jumps directly to the ground state, i.e. a transition from $n_2=\infty$ to $n_1=1$. The Rydberg formula gives a wavelength of:

$$\lambda=\left[R_{\mathrm{H}}\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)\right]^{-1}=\left[\left(1.097\cdot10^7\text{ m}^{-1}\right)\left(\frac{1}{1}-\frac{1}{\infty}\right)\right]^{-1}=91.16\text{ nm}$$

or 91.2 nm to the usual three significant figures.

38.20. The second line in the Paschen series corresponds to the transition of an electron from the $n_2 = 5$ state to the $n_1 = 3$ state. The Rydberg formula gives a wavelength of:

$$\lambda = \left[R_{\text{H}} \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \right]^{-1} = \left[\left(1.097 \cdot 10^7 \text{ m}^{-1}\right) \left(\frac{1}{(3)^2} - \frac{1}{(5)^2} \right) \right]^{-1} = 1282 \text{ nm}.$$

38.21. The shortest wavelength photons emitted in the Pfund series occurs when an electron is captured by the hydrogen atom and jumps directly to the fourth excited state, i.e. a transition from $n_2 = \infty$ to $n_1 = 5$. The Rydberg formula gives the shortest wavelength:

$$\lambda = \left[R_{\text{H}} \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \right]^{-1} = \left[\left(1.097 \cdot 10^7 \text{ m}^{-1}\right) \left(\frac{1}{(5)^2} - \frac{1}{\infty} \right) \right]^{-1} = 2279 \text{ nm}.$$

The longest wavelength photons emitted in the Pfund series corresponds to the transition of an electron from the fifth excited state to the fourth excited state, i.e. a transition from $n_2 = 6$ to $n_1 = 5$. The Rydberg formula gives the longest wavelength:

$$\lambda = \left[R_{\text{H}} \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \right]^{-1} = \left[\left(1.097 \cdot 10^7 \text{ m}^{-1}\right) \left(\frac{1}{(5)^2} - \frac{1}{(6)^2} \right) \right]^{-1} = 7458 \text{ nm}.$$

Both of these wavelengths lie in the near-infrared range, so they are not visible.

38.22. The electron in the second excited state has $n = 3$. There are two different paths the electron can take to get to the ground state:

$$n = 3 \rightarrow n = 2 \rightarrow n = 1 \text{ or } n = 3 \rightarrow n = 1.$$

Two photons are emitted for the first path. The wavelength of the first photon is

$$\lambda_{3\rightarrow 2} = \left[R_{\text{H}} \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \right]^{-1} = \left[\left(1.097 \cdot 10^7 \text{ m}^{-1}\right) \left(\frac{1}{(2)^2} - \frac{1}{(3)^2} \right) \right]^{-1} = 656.3 \text{ nm},$$

and the wavelength of the second photon is

$$\lambda_{2\rightarrow 1} = \left[R_{\text{H}} \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \right]^{-1} = \left[\left(1.097 \cdot 10^7 \text{ m}^{-1}\right) \left(\frac{1}{(1)^2} - \frac{1}{(2)^2} \right) \right]^{-1} = 121.5 \text{ nm}.$$

One photon is emitted for the second path. The wavelength of this photon is

$$\lambda_{3\rightarrow 1} = \left[R_{\text{H}} \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \right]^{-1} = \left[\left(1.097 \cdot 10^7 \text{ m}^{-1}\right) \left(\frac{1}{(1)^2} - \frac{1}{(3)^2} \right) \right]^{-1} = 102.6 \text{ nm}.$$

The photon of wavelength 656.3 nm is red. The other two photons are in the UV part of the spectrum so they are not a visible color.

38.23. The quantum number of the fifth excited state is $n = 6$. The energy of the fifth excited state is:

$$E_n = -E_0 \frac{1}{n^2} \Rightarrow E_6 = -\left(13.6 \text{ eV}\right) \frac{1}{(6)^2} = -0.378 \text{ eV}.$$

38.24. The allowed energies for the hydrogen atom are given by:

$$E_n = -E_0 \frac{1}{n^2}.$$

The bombarding electron has an energy of 13.1 eV, which is less than the ionization energy of the hydrogen atom, so there is specific state that the hydrogen atom cannot be excited beyond. The shortest wavelength that can be emitted occurs for a transition from the highest possible state to the ground state.

To determine the highest possible state, the difference in energy between the excited states and the ground state is required:

$$\Delta E_{2\to 1} = -(13.6\text{ eV})\left(\frac{1}{(2)^2}-1\right) = 10.20\text{ eV}$$

$$\Delta E_{3\to 1} = -(13.6\text{ eV})\left(\frac{1}{(3)^2}-1\right) = 12.09\text{ eV}$$

$$\Delta E_{4\to 1} = -(13.6\text{ eV})\left(\frac{1}{(4)^2}-1\right) = 12.75\text{ eV}$$

$$\Delta E_{5\to 1} = -(13.6\text{ eV})\left(\frac{1}{(5)^2}-1\right) = 13.06\text{ eV}$$

$$\Delta E_{6\to 1} = -(13.6\text{ eV})\left(\frac{1}{(6)^2}-1\right) = 13.22\text{ eV}$$

Since the bombarding electron has an energy of 13.1 eV, the highest possible state that the bombarding electron can excite the hydrogen atom into is the $n=5$ state. Therefore, the shortest wavelength line that the atom will emit is

$$\lambda = \frac{hc}{E} = \frac{(4.136\cdot 10^{-15}\text{ eV s})(2.998\cdot 10^{8}\text{ m/s})}{(13.06\text{ eV})} = 9.4973\cdot 10^{-8}\text{ m} \approx 94.97\text{ nm}.$$

38.25. (a) In a hydrogen atom, the nucleus is composed of a single proton. The modified Rydberg constant for hydrogen is:

$$R_{\text{modified}} = \frac{R_{\text{H}}}{1+m/M} = \frac{(1.09737\cdot 10^{7}\text{ m}^{-1})}{1+(9.10938\cdot 10^{-31}\text{ kg})/(1.67262\cdot 10^{-27}\text{ kg})} = 1.0968\cdot 10^{7}\text{ m}^{-1}.$$

(b) In a positronium atom, the nucleus is composed of a single positron. The mass of the positron is the same as the mass of an electron, so $m/M=1.$ The modified Rydberg constant for positronium is:

$$R_{\text{modified}} = \frac{R_{\text{H}}}{1+(m/M)} = \frac{(1.09737\cdot 10^{7}\text{ m}^{-1})}{2} = 5.4869\cdot 10^{6}\text{ m}^{-1}.$$

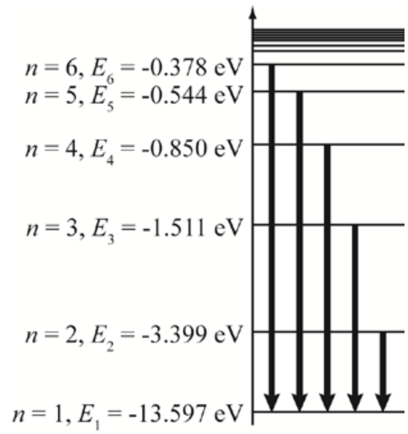
38.26. (a) The reduced mass of a hydrogen-like muonic atom is:

$$\mu = \frac{m_{\text{muon}}m_{\text{p}}}{m_{\text{muon}}+m_{\text{p}}} = \frac{(1.88\cdot 10^{-28}\text{ kg})(1.67\cdot 10^{-27}\text{ kg})}{(1.88\cdot 10^{-28}\text{ kg})+(1.67\cdot 10^{-27}\text{ kg})} = 1.69\cdot 10^{-28}\text{ kg}.$$

(b) The reduced mass of the muonic atom is about 186 times that of the hydrogen atom. Since the ionization energy of the hydrogen is directly proportional to the reduced mass, the ionization energy of the muonic atom is about 186 times that of the hydrogen atom. Since it takes 13.6 eV to ionize the hydrogen atom, it would take $E=186(13.6\text{ eV})=2520\text{ eV}$ to ionize the hydrogen-like muonic atom.

38.27. **THINK:** An excited hydrogen atom emits a photon with an energy of $E_{\text{ph}}=1.133$ eV. When emitting a photon, the hydrogen atom loses the energy of the photon. To determine the initial and final states of the hydrogen atom, the energy levels must be analyzed to determine which set of two can be separated by the given photon energy. Start with the final energy level being the ground state, and then progress to higher states as the final state.

SKETCH:



RESEARCH: The energy of the n^{th} energy level in a hydrogen atom is $E_n = -13.6 \text{ eV}/n^2$. Since the hydrogen atom loses the energy that is gained by the emitted photon, $E_{\text{ph}} = -\Delta E = E_{\text{i}} - E_{\text{f}}$.

SIMPLIFY: $E_{\text{i}} = E_{\text{f}} + E_{\text{ph}}$

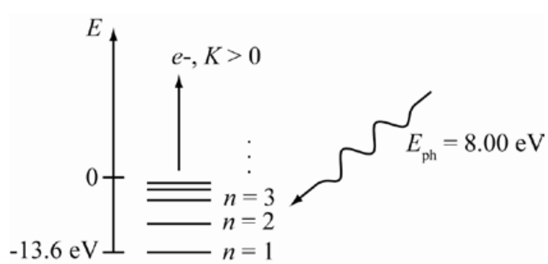
CALCULATE: The lowest energy levels of the hydrogen atom are: $E_1 = -13.597$ eV, $E_2 = -3.399$ eV, $E_3 = -1.511$ eV, $E_4 = -0.850$ eV, $E_5 = -0.544$ eV, $E_6 = -0.378$ eV. For $E_{\text{f}} = E_1$, $E_{\text{i}} = (-13.597 \text{ eV}) + (1.133 \text{ eV}) = -12.464 \text{ eV}$, which is not an allowed level. For $E_{\text{f}} = E_2$, $E_{\text{i}} = (-3.399 \text{ eV}) + (1.133 \text{ eV}) = -2.266 \text{ eV}$, which is not an allowed level. For $E_{\text{f}} = E_3$, $E_{\text{i}} = (-1.511 \text{ eV}) + (1.133 \text{ eV}) = -0.378 \text{ eV}$, which is the energy level E_6. The photon was emitted when an excited atom in the $n = 6$ state transitioned to the $n = 3$ state. This is the only allowed transition.

ROUND: Not applicable.

DOUBLE-CHECK: For a transition from the $n = 6$ state to the $n = 3$ state, the emitted photon has energy $E = E_6 - E_3 = (-0.378 \text{ eV}) - (-1.511 \text{ eV}) = 1.133 \text{ eV}$, as required.

38.28. **THINK:** The incident photon has an energy of $E_{\text{ph}} = 8.00$ eV. The electron is in an initial state with quantum number $n_{\text{i}} = 2$. To calculate the final speed, v, of the electron, the final kinetic energy of the electron must be determined. To determine the kinetic energy of the electron, the amount of energy required to remove the electron from the atom needs to be determined.

SKETCH:



RESEARCH: The energy of the n^{th} electron state is $E_n = -13.6 \text{ eV}/n^2$. Since the photon is absorbed, the final energy of the electron is $E_{\text{f}} = E_{\text{i}} + E_{\text{ph}}$. The non-relativistic speed of the electron is found from considering its kinetic energy, $K = mv^2/2$.

SIMPLIFY: The initial energy of the electron is $E_{\text{i}} = -13.6 \text{ eV}/n_{\text{i}}^2$. The final energy of the electron is $E_{\text{f}} = E_{\text{ph}} - 13.6 \text{ eV}/n_{\text{i}}^2$. The final energy is equal to the kinetic energy of the electron:

$$E_f = K = E_{ph} - 13.6 \text{ eV}\frac{1}{n_i^2} \Rightarrow \frac{1}{2}m_e v^2 = E_{ph} - 13.6 \text{ eV}\frac{1}{n_i^2}$$

$$v = \sqrt{\frac{2\left(E_{ph} - 13.6 \text{ eV}\frac{1}{n_i^2}\right)}{m_e}}$$

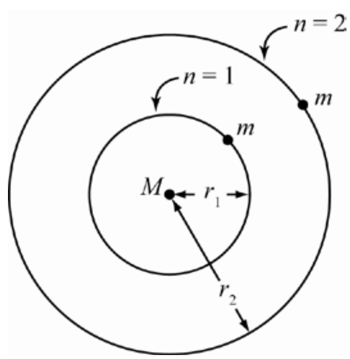
CALCULATE: $v = \sqrt{\frac{2\left((8.00 \text{ eV}) - 13.6 \text{ eV}\frac{1}{(2)^2}\right)(1.602 \cdot 10^{-19} \text{ J/eV})}{(9.109 \cdot 10^{-31} \text{ kg})}} = 1.272 \cdot 10^6 \text{ m/s}$

ROUND: Rounding to three significant figures, $v = 1.27 \cdot 10^6$ m/s.

DOUBLE-CHECK: The final speed of the ejected electron is only about 0.4% the speed of light. This justifies using a non-relativistic method to determine the final speed.

38.29. **THINK:** Assume that Bohr's quantized energy levels apply to planetary orbits; that is, there are discrete allowed radial distances from the center of rotation for which a planet can orbit. Consider Newton's law of universal gravitation, the equation for centripetal acceleration, and the quantization of orbital angular momentum to derive an equation similar to 38.6 and to determine the value of the principal quantum number for the Earth's orbit.

SKETCH:



RESEARCH: Newton's law of gravitational attraction is $F = GMm / r^2$. The equation for centripetal acceleration is $a_c = v^2 / r$. Orbital angular momentum has discrete values, and is characterized by a principal quantum number, n: $L = |\vec{r} \times \vec{p}| = n\hbar$, with $n = 1, 2, 3, \ldots$. For circular motion, $\vec{r}$ and $\vec{p}$ are perpendicular, so $L = rp = rmv = n\hbar$. The following values are useful for the problem: Earth's mass, $m_E = 5.974 \cdot 10^{24}$ kg; the Sun's mass, $m_S = 1.989 \cdot 10^{30}$ kg; and the center-to-center distance between the Earth and the Sun, $R = 1.4960 \cdot 10^{11}$ m.

SIMPLIFY: Using Newton's second law, $F = ma$:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow v = \sqrt{\frac{GM}{r}},$$

$$L = rmv = rm\sqrt{\frac{GM}{r}} = m\sqrt{GMr} = n\hbar \Rightarrow m^2 GMr = n^2\hbar^2$$

Therefore, the allowed radius of the orbits is given by

$$r = \frac{\hbar^2}{GMm^2}n^2,$$

where M is the mass at the center of the orbit and m is the mass orbiting M.

The principal quantum number is then

$$n = \frac{m\sqrt{GMr}}{\hbar}.$$

CALCULATE: With $M = m_S$, $m = m_E$ and $r = R$:

$$n = \frac{\left(5.974 \cdot 10^{24}\text{ kg}\right)\sqrt{\left(6.674 \cdot 10^{-11}\text{ m}^3/\text{kg s}^2\right)\left(1.989 \cdot 10^{30}\text{ kg}\right)\left(1.4960 \cdot 10^{11}\text{ m}\right)}}{\left(1.055 \cdot 10^{-34}\text{ J s}\right)} = 2.52342 \cdot 10^{74}.$$

ROUND: Rounding to four significant figures, the principal quantum number is $n = 2.523 \cdot 10^{74}$.
DOUBLE-CHECK: For macroscopic objects such as the Earth and the Sun, the principal quantum number of the Earth's orbit should be very large. Note the derived expression for r has the correct units:

$$[r] = \frac{[\text{J s}]^2}{\left[\text{m}^3/\text{kg s}^2\right][\text{kg}]^3} = \frac{\left[\text{kg m}^2/\text{s}^2\right]^2[\text{s}]^2}{\left[\text{m}^3\text{ kg}^2/\text{s}^2\right]} = \frac{\text{kg}^2\text{ m}^4/\text{s}^2}{\text{kg}^2\text{ m}^3/\text{s}^2} = \text{m}.$$

38.30. **THINK:** The period of rotation of an electron on the n^{th} Bohr orbit is given by:

$$T = \frac{n^3}{2cR_H}, \text{ with } n = 1, 2, 3, \ldots.$$

To prove this, define the period in terms of the radius of the Bohr orbit and the speed of an electron in the Bohr orbit, and use the corresponding equations to simplify the period into the desired result. It is useful to know that the Rydberg constant can be written as $R_H = \mu k^2 e^4 / \left(4\pi c\hbar^3\right)$.
SKETCH: Not applicable.
RESEARCH: The period for any circular motion is $T = 2\pi r / v$. The radius of the n^{th} Bohr orbit is:

$$r = \frac{\hbar^2}{\mu k e^2} n^2 \equiv a_0 n^2.$$

The speed of the electron in the Bohr orbit is:

$$v = \sqrt{\frac{ke^2}{\mu a_0 n^2}} = \frac{e}{n}\sqrt{\frac{k}{\mu a_0}}.$$

SIMPLIFY: $T = \frac{2\pi r}{v} = 2\pi a_0 n^2\left(\frac{n}{e}\sqrt{\frac{\mu a_0}{k}}\right) = \frac{2\pi}{e}n^3\sqrt{\frac{\mu a_0^3}{k}} = \frac{2\pi}{e}n^3\sqrt{\frac{\mu\left(\hbar^2/\mu k e^2\right)^3}{k}} = \frac{2\pi}{e}n^3\sqrt{\frac{\hbar^6}{\mu^2 k^4 e^6}}$

$$= \frac{2\pi\hbar^3}{\mu k^2 e^4}n^3 = \frac{2\pi\hbar^3}{\mu k^2 e^4}n^3\left(\frac{2c}{2c}\right) = \frac{1}{2c}\left(\frac{4\pi c\hbar^3}{\mu k^2 e^4}\right)\left(n^3\right) = \frac{1}{2c}\left(\frac{1}{R_H}\right)\left(n^3\right) = \frac{n^3}{2cR_H}, \text{ with } n = 1, 2, 3, \ldots.$$

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: The units of R_H are m^{-1}. By dimensional analysis, the units of the derived equation are:

$$\left[\frac{n^3}{2cR_H}\right] = \frac{1}{(\text{m/s})\text{m}^{-1}} = \frac{1}{1/\text{s}} = \text{s}.$$

This result is a unit of time, which is appropriate for a period.

38.31. For an electron in the $n = 5$ shell, the orbital angular momentum quantum number can range from 0 to $n-1$; that is, $\ell = 0, 1, 2, 3, 4$. The magnitude of the total orbital angular momentum is $L = \sqrt{\ell(\ell+1)}\hbar$. The largest possible value for the angular momentum occurs with $\ell = 4$: $L = \sqrt{4(4+1)}\hbar = \sqrt{20}\hbar \approx 4.716 \cdot 10^{-34}$ J s. The smallest possible value for the orbital angular momentum occurs with $\ell = 0$: $L = \sqrt{0(0+1)}\hbar = 0$.

38.32. (a) The K shell has quantum number $n=1$. The electron configuration of a full K shell is $1s^2$. Therefore, the maximum allowed number of electrons is 2.

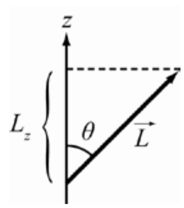
(b) The L shell has quantum number $n=2$. The electron configuration of a full L shell is $2s^2 2p^6$. Therefore, the maximum allowed number of electrons is 8.

(c) The M shell has quantum number $n=3$. The electron configuration of a full M shell is $3s^2 3p^6 3d^{10}$. Therefore, the maximum allowed number of electrons is 18.

In general, the maximum number of electrons in a shell n is $2n^2$.

38.33. **THINK:** The hydrogen atom is in the stationary state $(n,\ell,m)=(3,2,1)$. Determine the angle, θ, between the total angular momentum vector and the z-axis for this state.

SKETCH: The classical picture of angular momentum is shown below.



RESEARCH: The magnitude of the total orbital angular momentum is $L=\sqrt{\ell(\ell+1)}\hbar$. The z-projection of the total orbital angular momentum is $L_z = m\hbar$.

SIMPLIFY: From the classical picture above,

$$\cos\theta = \frac{L_z}{L} = \frac{m}{\sqrt{\ell(\ell+1)}}.$$

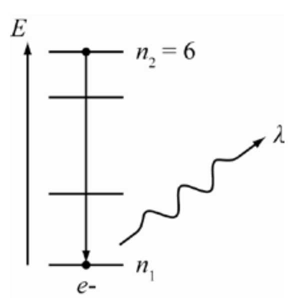
CALCULATE: $\cos\theta = \dfrac{1}{\sqrt{2(2+1)}} = \dfrac{1}{\sqrt{6}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{\sqrt{6}}\right) = 65.905°$

ROUND: The value can be given to arbitrary precision. It is reasonable to use three significant figures, so $\theta = 65.9°$.

DOUBLE-CHECK: The angle should be between $0°$ and $180°$. Note that the value of L_z can never exceed the magnitude of the total orbit angular momentum, L.

38.34. **THINK:** A hydrogen atom is in its fifth excited state, with principal quantum number $n=6$. The atom emits a photon with a wavelength of $\lambda = 410$ nm. This photon is emitted when the electron falls from the $n=6$ energy level to a less energetic energy level. By energy conservation, the energy of the emitted photon must match the change in energy of the fallen electron. Determine the maximum possible orbital angular momentum, L, of the electron after emission.

SKETCH:



RESEARCH: The maximum possible orbital angular momentum of the electron is related to the maximum possible orbital angular momentum quantum number, ℓ, of the final state. Specifically,

$L=\sqrt{\ell(\ell+1)}\hbar,$ where the maximum ℓ is $\ell_{\max}=n-1.$ The energy of the n^{th} level in a hydrogen atom is $E_n=-13.6\text{ eV}/n^2.$ The energy of a photon is $E=hc/\lambda.$

SIMPLIFY: When the photon is emitted, the electron loses energy, $E_{\text{ph}}.$ The final state of the electron must have an energy that is accessible via this photon emission; that is, $E_{n_1}=E_{n_2}-E_{\text{ph}}.$ Substitution gives

$$E_{n_1}=-\frac{13.6\text{ eV}}{n_2^2}-\frac{hc}{\lambda}\Rightarrow -\frac{13.6\text{ eV}}{n_1^2}=-\frac{13.6\text{ eV}}{n_2^2}-\frac{hc}{\lambda}.$$

The quantum number of the final state is

$$n_1=\sqrt{\frac{-13.6\text{ eV}}{\left(-\frac{13.6\text{ eV}}{n_2^2}-\frac{hc}{\lambda}\right)}}.$$

Then, $\ell_{\max}=n_1-1$ and $L_{\max}=\sqrt{\ell_{\max}(\ell_{\max}+1)}\hbar=\sqrt{n_1(n_1-1)}\hbar.$

CALCULATE: $n_1=\sqrt{\dfrac{-13.6\text{ eV}}{\left(-\dfrac{13.6\text{ eV}}{(6)^2}-\dfrac{(4.136\cdot 10^{-15}\text{ eV s})(2.998\cdot 10^{8}\text{ m/s})}{(410\cdot 10^{-9}\text{ m/s})}\right)}}=1.999\approx 2$

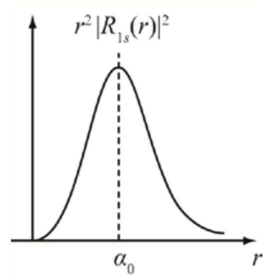
$\Rightarrow L_{\max}=\sqrt{2(2-1)}\hbar=\sqrt{2}\hbar$

ROUND: Since n_1 is an integer, it was rounded to 2 above.

DOUBLE-CHECK: The electron transitioned from the $n=6$ state to the $n=2$ state by emitting a photon of wavelength $\lambda=410$ nm. This matches the case shown in Figure 38.6 for the $n=6$ to $n=2$ transition, in which a photon is emitted of wavelength $\lambda=410$ nm.

38.35. **THINK:** The radial wave function for hydrogen in the $n=1$ state is given by $\psi_1(r)=A_1e^{-r/a_0},$ where the normalization constant is calculated in Example 38.2 to be $A_1=1/\left(\sqrt{\pi}a_0^{3/2}\right).$ The radial part of the hydrogen wave function must normalize to 1.

SKETCH:



RESEARCH: The probability density is given by

$$P_n(r)=r^2\left|\psi_n(r)\right|^2.$$

The radial wave function for $n=1$ is

$$\psi_1(r)=\frac{1}{\sqrt{\pi}a_0^{3/2}}e^{-r/a_0}.$$

SIMPLIFY: (a) The probability density is given for $n=1$ by

$$P_{n=1}(r)=r^2\left|\psi_1(r)\right|^2=r^2\left|\frac{1}{\sqrt{\pi}a_0^{3/2}}e^{-r/a_0}\right|^2=\frac{r^2}{\pi a_0^3}e^{-2r/a_0}.$$

For $r = a_0/2$,

$$P_{n=1}(a_0/2) = \frac{(a_0/2)^2}{\pi a_0^3} e^{-2(a_0/2)/a_0} = \frac{1}{4\pi a_0} e^{-1} = \frac{1}{4\pi a_0 e}.$$

(b) The radial function, $\psi_1(r)$ has a maximum at the origin $(r=0)$. To find the maximum in the probability density we need to take its derivative with respect to r, set it equal to zero, and solve for r.

$$\frac{dP_1(r)}{dr} = \frac{d}{dr}\left(\frac{r^2}{\pi a_0^3} e^{-2r/a_0}\right) = \frac{1}{\pi a_0^3}\left(2re^{-2r/a_0} - \frac{2r^2}{a_0} e^{-2r/a_0}\right) = 0.$$

$$2re^{-2r/a_0} = \frac{2r^2}{a_0} e^{-2r/a_0} \;\Rightarrow\; 1 = \frac{r}{a_0} \;\Rightarrow\; r = a_0.$$

Thus, because the probability density contains an r^2 factor instead of the r in the wavefunction, it has a maximum at a_0 rather than the origin.

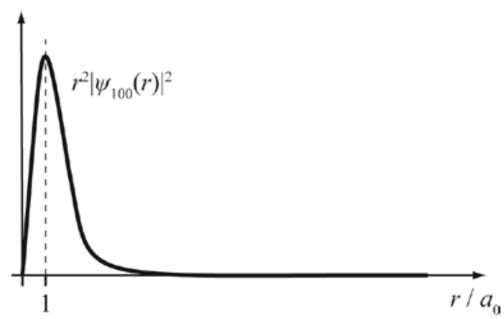
CALCULATE:
Not applicable

ROUND: Not applicable.

DOUBLE-CHECK: The probability density for $n=1$ peaks at $r=a_0$. We can see that this is true in Figure 38.9b. However, the probability density does not peak at $r=4a_0$ for $n=2$ and it does not peak at $r=9a_0$ for $n=3$.

38.36. **THINK:** The value of r for which its probability is a maximum is determined by taking the derivative of the probability function with respect to r, then setting it to zero and solving for r.

SKETCH:



RESEARCH: The given wave function is

$$\psi_{100}(r) = \frac{1}{\sqrt{\pi} a_0^{3/2}} e^{-r/a_0}.$$

The probability function is $P(r) = 4\pi r^2 \left|\psi_{100}(r)\right|^2$.

SIMPLIFY: Taking the derivative of $P(r)$ with respect to r gives:

$$\frac{dP(r)}{dr} = \frac{d}{dr}\left(4\pi r^2 \left|\psi_{100}(r)\right|^2\right) = \frac{d}{dr}\left(4\pi r^2 \left|\frac{e^{-r/a_0}}{\sqrt{\pi} a_0^{3/2}}\right|^2\right) = \frac{4}{a_0^3}\frac{d}{dr}\left(r^2 e^{-2r/a_0}\right)$$

$$= \frac{4}{a_0^3}\left(2re^{-2r/a_0} - \frac{2r^2}{a_0} e^{-2r/a_0}\right) = \frac{8re^{-2r/a_0}}{a_0^3}\left(1 - \frac{r}{a_0}\right)$$

Setting $\dfrac{dP(r)}{dr} = 0$:

$$\frac{8re^{-2r/a_0}}{a_0^3}\left(1 - \frac{r}{a_0}\right) = 0 \;\Rightarrow\; 1 - \frac{r}{a_0} = 0 \text{ and } r = 0,$$

which corresponds to $r=0$ and $r=a_0$. The probability is zero when $r = 0$, so this solution is a minimum. Therefore, the value for r for which the probability function is a maximum is $r=a_0$.
CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: The second derivative test can be used to verify that $r=a_0$ corresponds to a maximum. The second derivative is:

$$\frac{d^2P(r)}{dr^2}=\frac{8e^{-2r/a_0}}{a_0^3}\left(\frac{2r^2}{a_0^2}-\frac{4r}{a_0}+1\right).$$

In order for $r=a_0$ to be a local maximum point for the function P, it must be the case that

$$\left.\frac{d^2P(r)}{dr^2}\right|_{r=a_0}<0.$$

Check:

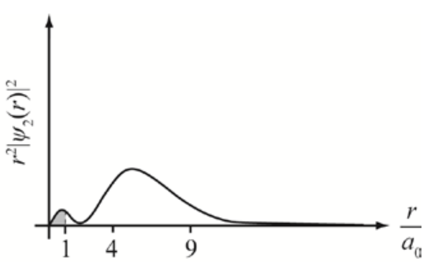
$$\left.\frac{d^2P(r)}{dr^2}\right|_{r=a_0}=\left.\frac{8e^{-2r/a_0}}{a_0^3}\left(\frac{2r^2}{a_0^2}-\frac{4r}{a_0}+1\right)\right|_{r=a_0}=\frac{8e^{-2}}{a_0^3}(2-4+1)=-\frac{8e^{-2}}{a_0^3}<0.$$

Hence, $r=a_0$ is a local maximum for $P(r)$.

38.37. **THINK:** To calculate the probability that the electron is found within the Bohr radius, $a_0=0.05295$ nm, the function,

$$\psi_{2s}(r)=\frac{1}{4\sqrt{2\pi a_0^3}}\left(2-\frac{r}{a_0}\right)e^{-\frac{r}{2a_0}},$$

must be squared and integrated from a radius of $r=0$ to $r=a_0$.
SKETCH:



RESEARCH: The probability that the electron is between $r=0$ and $r=a_0$ is given by:

$$P(r\le a_0)=\int_0^{a_0}\left|\psi_{2s}(r)\right|d^3r.$$

SIMPLIFY: $P(r\le a_0)=4\pi\int_0^{a_0}r^2\left|\psi_{2s}(r)\right|dr$
CALCULATE:

$$P(r\le a_0)=4\pi\left(\int_0^{a_0}\frac{r^2}{32\pi a_0^3}\left(2-\frac{r}{a_0}\right)^2e^{-\frac{r}{a_0}}dr\right)=\frac{1}{8a_0^3}\int_0^{a_0}\left(4r^2-\frac{4r^3}{a_0}+\frac{r^4}{a_0^2}\right)e^{-\frac{r}{a_0}}dr$$

$$=\frac{1}{2a_0^3}\int_0^{a_0}r^2e^{-\frac{r}{a_0}}dr-\frac{1}{2a_0^4}\int_0^{a_0}r^3e^{-\frac{r}{a_0}}dr+\frac{1}{8a_0^5}\int_0^{a_0}r^4e^{-\frac{r}{a_0}}dr$$

After evaluating the integrals, the probability is:

$$P(r\le a_0)=\left[\frac{-e^{-\frac{r}{a_0}}\left(8a_0^4+8a_0^3r+4a_0^2r^2+r^4\right)}{8a_0^4}\right]_0^{a_0}=-e^{-1}\left(\frac{21}{8}\right)+e^0=1-\frac{21}{8e}=0.0343165.$$

ROUND: Rounding to four decimal places is sufficient. The probability of finding the electron within the Bohr radius is $P(r \le a_0) = 0.03432$.

DOUBLE-CHECK: Figure 38.8(b) shows the probability density function for $n=2$ (blue curve). Here, for $r/a_0 \le 1$, the area under the curve looks to be about small, so an answer of about 3% is reasonable.

38.38. The energy of the electron in He^+ is given by:

$$E_1 = \frac{-\mu k^2 Z^2 e^4}{2\hbar^2} = -Z^2\left(\frac{ke^2}{2a_0}\right) = -Z^2 E_0, \text{ where } E_0 = 13.6 \text{ eV and } Z = 2 \text{ for helium.}$$

Therefore, the energy required to convert He^+ to He^{2+} is $|E_1| = \left|-(2)^2(13.6 \text{ eV})\right| = 54.4$ eV. This is four times the energy needed to ionize the hydrogen atom.

38.39. By following the derivation of the Bohr radius for hydrogen from the textbook, but using 2*e* for the charge of the nucleus, one obtains:

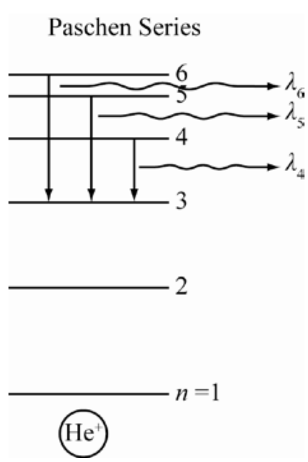
$$\frac{k(2e)e}{r^2} = \frac{\mu v^2}{r} \Rightarrow r\mu k 2e^2 = \mu^2 v^2 r^2 = n^2\hbar^2 \Rightarrow$$

$$r = \frac{n^2\hbar^2}{2\mu ke^2} = a_0' n^2 \Rightarrow a_0' = \frac{\hbar^2}{2\mu ke^2} = \frac{1}{2}\left(\frac{\hbar^2}{\mu ke^2}\right) = \frac{a_0}{2} = \frac{0.05295 \text{ nm}}{2} = 0.0265 \text{ nm.}$$

The Bohr radius of He^+ is half that of hydrogen.

38.40. **THINK:** The Paschen series occurs for transitions to the $n=3$ state, so the three lowest energies correspond to transitions from the $n=4$, $n=5$, and $n=6$ states. The Paschen series for He^+ is determined in the same manner as that for hydrogen. The main difference is that the Rydberg constant, R_H, will be different.

SKETCH:



RESEARCH: The Rydberg constant for hydrogen is given by $R_H = \mu k^2 e^4 / (4\pi c\hbar^3)$. The charge of a helium nucleus is 2*e*. The Paschen series is given by:

$$\frac{1}{\lambda_n} = R_H\left(\frac{1}{9} - \frac{1}{n^2}\right), \text{ with } n = 4, 5, 6\ldots$$

SIMPLIFY: Since the charge of a helium nucleus is 2*e*, the Rydberg constant for He is:

$$R_H' = \frac{\mu k^2 (2e^2)^2}{4\pi c\hbar^3} = 4\frac{\mu k^2 e^4}{4\pi c\hbar^3} = 4R_H.$$

The wavelength for the Paschen series is then:

$$\lambda_n = \frac{1}{R'_{\rm H}}\left(\frac{1}{9}-\frac{1}{n^2}\right)^{-1} = \frac{1}{4R_{\rm H}}\left(\frac{1}{9}-\frac{1}{n^2}\right)^{-1}.$$

CALCULATE: $\lambda_4 = \frac{1}{4\left(1.097\cdot 10^7\text{ m}^{-1}\right)}\left(\frac{1}{9}-\frac{1}{16}\right)^{-1} = 468.81\cdot 10^{-9}\text{ m}$

$$\lambda_5 = \frac{1}{4\left(1.097\cdot 10^7\text{ m}^{-1}\right)}\left(\frac{1}{9}-\frac{1}{25}\right)^{-1} = 320.48\cdot 10^{-9}\text{ m}$$

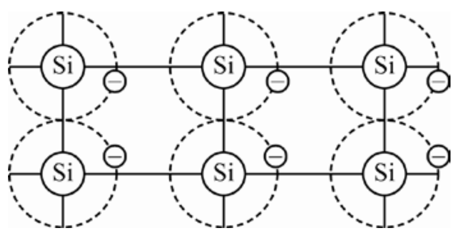
$$\lambda_6 = \frac{1}{4\left(1.097\cdot 10^7\text{ m}^{-1}\right)}\left(\frac{1}{9}-\frac{1}{36}\right)^{-1} = 273.47\cdot 10^{-9}\text{ m}$$

ROUND: Rounding to three significant figures is sufficient: $\lambda_4 = 469\text{ nm}$, $\lambda_5 = 320.\text{ nm}$ and $\lambda_6 = 273\text{ nm}$.

DOUBLE-CHECK: While λ_5 and λ_6 are outside the visible spectrum, λ_4 does coincide with a wavelength in the visible spectrum, so the answers are reasonable.

38.41. **THINK:** The Coulomb constant is given by $k = 1/\left(4\pi\varepsilon_0\right)$. We can derive the effective Coulomb constant by replacing ε_0 with $\kappa\varepsilon_0$. So $k_{\rm eff} = 1/\left(4\pi\kappa\varepsilon_0\right) = k/\kappa$. Since the ground state energy of the hydrogen-like atom is proportional to k, it will then be affected by κ. In addition, we replace the reduced mass of the electron with the effective mass of the electron $m_{\rm eff}$ in the crystal. The reduced mass in this case is very close to the mass of the electron.

SKETCH:



RESEARCH: The ground state energy of a hydrogen atom is $E_0 = \mu k^2 e^4/\left(2\hbar^2\right)$.

SIMPLIFY: The ground energy of a hydrogen-like atom is in the silicon crystal is

$$E_0^{\rm eff} = \frac{m_{\rm eff}\left(k_{\rm eff}\right)^2 e^4}{2\hbar^2} = \frac{0.200 m_{\rm e} k^2 e^4}{2\hbar^2\kappa^2} = \frac{0.200}{\kappa^2}\left(\frac{m_{\rm e}k^2e^4}{2\hbar^2}\right) = \frac{0.200E_0}{\kappa^2}.$$

CALCULATE:

(a) Not necessary

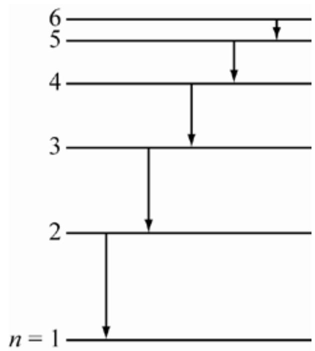
(b) $E_0^{\rm eff} = \frac{0.200}{(10.0)^2}\left(13.6\text{ eV}\right) = 0.0272\text{ eV}.$

ROUND: To three significant figures, $E_0^{\rm eff} = 27.2\text{ meV}$.

DOUBLE-CHECK: Since silicon is considered a semiconductor, it should have a relatively low binding energy so that at room temperature, $T = 300\text{ K}$, the thermal energy $\left(E \approx k_{\rm B}T = 26\text{ meV}\right)$ is enough to free the electrons for conduction. The result is reasonable.

38.42. **THINK:** The spacing between energy levels in Li^{2+} will be proportional to $1/n^2$. The energy level spacing will then be inversely proportional to the wavelength of light for that transition from level n to n'. The lowest end of the visible spectrum is $\lambda_{\min} = 380\text{ nm}$. The wavelength increases with decreasing energy, so if the transition ends on level n', the lowest wavelength associated with that final position is when $n = n' + 1$. The series can be examined starting with $n' = 1$ until the desired wavelength is obtained.

SKETCH:



RESEARCH: Change in energy is given by $\Delta E = hc/\lambda$. The energy of each level is given by $E_n = E_0'/n^2$. For lithium, $E_0' = Z^2 E_0$, where $Z = 3$.

SIMPLIFY: The energy difference between level n' and n $(n > n')$ is,

$$\Delta E = \frac{E_0'}{n'^2} - \frac{E_0'}{n^2} = 9E_0\left(\frac{1}{n'^2} - \frac{1}{n^2}\right) = \frac{hc}{\lambda}.$$

Therefore, the wavelength of the emitted photon is,

$$\lambda_n = \frac{hc}{9E_0}\left(\frac{1}{n'^2} - \frac{1}{n^2}\right)^{-1}.$$

CALCULATE: $n = 2,\ n' = 1:\ \lambda_1 = \dfrac{(4.136 \cdot 10^{-15}\ \text{eV s})(3.00 \cdot 10^8\ \text{m/s})}{9(13.6\ \text{eV})}\left(1 - \dfrac{1}{4}\right)^{-1} = 13.51\ \text{nm}$

$n = 3,\ n' = 2:\ \lambda_2 = \dfrac{(4.136 \cdot 10^{-15}\ \text{eV s})(3.00 \cdot 10^8\ \text{m/s})}{9(13.6\ \text{eV})}\left(\dfrac{1}{4} - \dfrac{1}{9}\right)^{-1} = 72.99\ \text{nm}$

$n = 4,\ n' = 3:\ \lambda_3 = \dfrac{(4.136 \cdot 10^{-15}\ \text{eV s})(3.00 \cdot 10^8\ \text{m/s})}{9(13.6\ \text{eV})}\left(\dfrac{1}{9} - \dfrac{1}{16}\right)^{-1} = 208.54\ \text{nm}$

$n = 5,\ n' = 4:\ \lambda_4 = \dfrac{(4.136 \cdot 10^{-15}\ \text{eV s})(3.00 \cdot 10^8\ \text{m/s})}{9(13.6\ \text{eV})}\left(\dfrac{1}{16} - \dfrac{1}{25}\right)^{-1} = 450.55\ \text{nm}\ (> \lambda_{\text{min}})$

ROUND: Rounding to three significant figures is sufficient. For doubly ionized lithium, the transition from state $n = 5$ to state $n' = 4$ is the first visible, with wavelength 451 nm.

DOUBLE-CHECK: The next two closest possible transitions (going down two levels) would be $n = 5$, $n' = 3$ and $n = 6$, $n' = 4$, which give wavelengths of 143 nm and 292 nm, respectively. Since neither of these is in the visible spectrum, the result found must be the first transition to cross the threshold.

38.43. **THINK:** The derivations for the desired expressions are the same as the derivation in the textbook for hydrogen, except instead of the Coulomb force being proportional to e^2, it is now proportional to Ze^2.

SKETCH: Not applicable.

RESEARCH: The Coulomb force and the centripetal forces on the electron are equal:

$$k\frac{Ze^2}{r'^2} = \mu\frac{v'^2}{r'}.$$

The ground state energy is given by: $E_0' = \dfrac{1}{2}\mu v'^2 - k\dfrac{Ze^2}{r'}.$

SIMPLIFY:

(a) Solving for r' in terms of Z gives:

$$k\frac{Ze^2}{r'^2}=\mu\frac{v'^2}{r'} \Rightarrow \mu r'kZe^2=\mu^2v'^2r'^2.$$

Recall $\mu^2v'^2r'^2=n^2\hbar^2$, so

$$\mu r'kZe^2=n^2\hbar^2 \Rightarrow r'=\left(\frac{\hbar^2}{\mu ke^2}\right)\frac{n^2}{Z}=\frac{a_0n^2}{Z}.$$

(b) Solving for v' in terms of Z gives:

$$k\frac{Ze^2}{r'^2}=\mu\frac{v'^2}{r'} \Rightarrow v'^2=\frac{Zke^2}{\mu r'}=\frac{Zke^2}{\mu\left(a_0n^2/Z\right)}=\frac{Z^2ke^2}{\mu a_0n^2} \Rightarrow v'=\frac{Z}{n}\sqrt{\frac{ke^2}{\mu a_0}}.$$

(c) Solving for E' in terms of Z gives:

$$E'=\frac{1}{2}\mu v'^2-k\frac{Ze^2}{r'}=\frac{1}{2}\left(k\frac{Ze^2}{r'}\right)-k\frac{Ze^2}{r'}=-k\frac{Ze^2}{2r'}=-\frac{kZe^2}{2\left(a_0n^2/Z\right)}=-Z^2\left(\frac{ke^2}{2a_0n^2}\right)=\frac{-Z^2}{n^2}\left(\frac{ke^2}{2a_0}\right)=Z^2\left(-\frac{E_0}{n^2}\right).$$

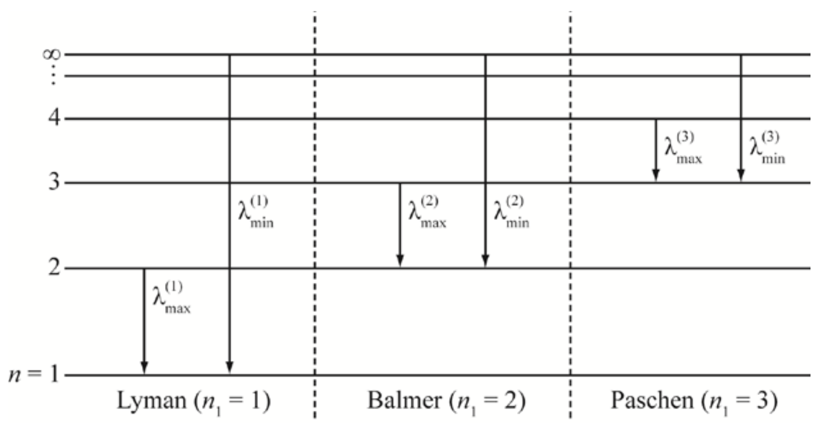
CALCULATE: There is nothing to calculate.

ROUND: Not applicable.

DOUBLE-CHECK: As Z increases, there are more protons attracting the electron. The radius decreases and the speed increases with Z, which is expected for a stronger attractive force. Likewise, as the electron gets closer and Z gets larger it will be more bound to the nucleus, so $E'\propto Z^2$ is also reasonable.

38.44. **THINK:** The solution from Problem 38.43 indicates that for an atom with atomic number Z, the energy of the electron in the orbit is directly proportional to Z^2. This means the Rydberg constant for helium will be different than that for hydrogen. For each of the series, the maximum wavelength occurs between the two lowest levels, while the minimum wavelength occurs between the lowest energy level and the ionization energy. For He, $Z=2$.

SKETCH:



RESEARCH: From the previous question, the energy of the electron is:

$$E'=\frac{-Z^2}{n^2}E_0.$$

The energy is given by $E=hc/\lambda$.

SIMPLIFY: For an electron transition from n_1 to n_2, the energy of the emitted photon is:

$$E=E_2-E_1=\frac{-Z^2}{n_2^2}E_0-\frac{-Z^2}{n_1^2}E_0=Z^2\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)E_0.$$

Therefore, the wavelength of the emitted photon is:

$$\frac{hc}{\lambda^{(n_1)}} = Z^2\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)E_0 \Rightarrow \lambda^{(n_1)} = \frac{hc}{Z^2E_0}\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)^{-1}.$$

The maximum and minimum wavelengths are given by:

$$\lambda_{\text{max}}^{(n_1)} = \frac{hc}{Z^2E_0}\left(\frac{1}{n_1^2}-\frac{1}{(n_1+1)^2}\right)^{-1} \text{ and } \lambda_{\text{min}}^{(n_1)} = \frac{hc}{Z^2E_0}\left(\frac{1}{n_1^2}-\frac{1}{\infty}\right)^{-1} = \frac{hcn_1^2}{Z^2E_0}.$$

CALCULATE: For the Lyman series:

$$\lambda_{\text{max}}^{(1)} = \frac{(4.136\cdot 10^{-15}\text{ eV s})(3.00\cdot 10^8\text{ m/s})}{(2)^2(13.6\text{ eV})}\left(\frac{1}{(1)^2}-\frac{1}{(1+1)^2}\right)^{-1} = 30.41\cdot 10^{-9}\text{ m}$$

$$\lambda_{\text{min}}^{(1)} = \frac{(4.136\cdot 10^{-15}\text{ eV s})(3.00\cdot 10^8\text{ m/s})(1)^2}{(2)^2(13.6\text{ eV})} = 22.81\cdot 10^{-9}\text{ m}$$

For the Balmer series:

$$\lambda_{\text{max}}^{(2)} = \frac{(4.136\cdot 10^{-15}\text{ eV s})(3.00\cdot 10^8\text{ m/s})}{(2)^2(13.6\text{ eV})}\left(\frac{1}{(2)^2}-\frac{1}{(2+1)^2}\right)^{-1} = 164.2\cdot 10^{-9}\text{ m},$$

$$\lambda_{\text{min}}^{(2)} = \frac{(4.136\cdot 10^{-15}\text{ eV s})(3.00\cdot 10^8\text{ m/s})(2)^2}{(2)^2(13.6\text{ eV})} = 91.24\cdot 10^{-9}\text{ m}.$$

For the Paschen series:

$$\lambda_{\text{max}}^{(3)} = \frac{(4.136\cdot 10^{-15}\text{ eV s})(3.00\cdot 10^8\text{ m/s})}{(2)^2(13.6\text{ eV})}\left(\frac{1}{(3)^2}-\frac{1}{(3+1)^2}\right)^{-1} = 469.2\cdot 10^{-9}\text{ m},$$

$$\lambda_{\text{min}}^{(3)} = \frac{(4.136\cdot 10^{-15}\text{ eV s})(3.00\cdot 10^8\text{ m/s})(3)^2}{(2)^2(13.6\text{ eV})} = 205.3\cdot 10^{-9}\text{ m}.$$

ROUND: Rounding to three significant figures is sufficient:

$\lambda_{\text{max}}^{(1)} = 30.4$ nm, $\lambda_{\text{min}}^{(1)} = 22.8$ nm, $\lambda_{\text{max}}^{(2)} = 164$ nm, $\lambda_{\text{min}}^{(2)} = 91.2$ nm, $\lambda_{\text{max}}^{(3)} = 469$ nm, $\lambda_{\text{min}}^{(3)} = 205$ nm

DOUBLE-CHECK: As the energy level, *n*, increases, the difference in the energy between the maximum and minimum transition decreases. Therefore, the wavelength difference should increase. This trend is observed in the results.

38.45. For hydrogen, the wavelength to go from $n_1 = 1$ to $n_2 = 2$ is:

$$\lambda_1 = \frac{1}{R_{\text{H}}}\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)^{-1} = \frac{1}{(1.097\cdot 10^7\text{ m}^{-1})}\left(\frac{1}{(1)^2}-\frac{1}{(2)^2}\right)^{-1} = 121.5\text{ nm}.$$

The wavelength to go from $n_1 = 2$ to $n_2 = 3$ is:

$$\lambda_2 = \frac{1}{R_{\text{H}}}\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)^{-1} = \frac{1}{(1.097\cdot 10^7\text{ m}^{-1})}\left(\frac{1}{(2)^2}-\frac{1}{(3)^2}\right)^{-1} = 656.3\text{ nm}.$$

Therefore, a laser with a wavelength 5.4 times larger is needed.

38.46. (a) The intensity of the laser is its power, $P_{\text{L}} = 0.50 \text{ mW},$ divided by its cross-sectional area, which is a circle of diameter $d = 3.0 \text{ mm}.$ Therefore, the intensity is:

$$I_{\text{L}} = \frac{P_{\text{L}}}{A} = \frac{P_{\text{L}}}{\pi\left(d/2\right)^2} = \frac{\left(0.50 \text{ mW}\right)}{\pi\left(1.5 \text{ mm}\right)^2} = 71 \text{ W/m}^2.$$

(b) The intensity of the light bulb is its power, $P_{\text{B}} = 100. \text{ W},$ divided by the area of the sphere of radius $r = 2.0 \text{ m}.$ Therefore, the intensity is:

$$I_{\text{B}} = \frac{P_{\text{B}}}{A} = \frac{P_{\text{B}}}{4\pi r^2} = \frac{\left(100. \text{ W}\right)}{4\pi\left(2.0 \text{ m}\right)^2} = 2.0 \text{ W/m}^2.$$

The ratio of intensities is:

$$\frac{I_{\text{L}}}{I_{\text{B}}} = \frac{\left(71 \text{ W/m}^2\right)}{\left(2.0 \text{ W/m}^2\right)} = 35.$$

Therefore, the laser is 35 times more intense than the light bulb.

38.47. The laser has a power of $P = 3.00 \text{ kW}$ and a wavelength of $\lambda = 694 \text{ nm}.$ It emits a light pulse of duration $\Delta t = 10.0 \text{ ns}.$

(a) The energy of each of the photons in the pulse is

$$E_0 = hf = h\frac{c}{\lambda} = \left(6.626 \cdot 10^{-34} \text{ J s}\right)\frac{3.00 \cdot 10^8 \text{ m/s}}{694 \cdot 10^{-9}} = 2.86 \cdot 10^{-19} \text{ J}.$$

(b) The total energy in each laser pulse is

$$E = P\Delta t = \left(3.00 \cdot 10^3 \text{ W}\right)\left(10.0 \cdot 10^{-9} \text{ s}\right) = 30.0 \ \mu\text{J}.$$

The number of chromium atoms undergoing stimulated emission is:

$$N = \frac{E}{E_0} = \frac{30.0 \cdot 10^{-6} \text{ J}}{2.86 \cdot 10^{-19} \text{ J}} = 1.05 \cdot 10^{14}.$$

38.48. The green laser has wavelength $\lambda_{\text{G}} = 543 \text{ nm}$ and power $P_{\text{G}} = 5.00 \text{ mW}.$ The red laser has wavelength $\lambda_{\text{R}} = 633 \text{ nm}$ and power $P_{\text{R}} = 4.00 \text{ mW}.$ The energy of each photon is given by $E_0 = hc/\lambda.$ For a time duration $\Delta t,$ the total energy of the photons is given by $E = P\Delta t.$ The number of photons per unit time is:

$$E = P\Delta t = NE_0 = \frac{Nhc}{\lambda} \ \Rightarrow \ \frac{N}{\Delta t} = \frac{P\lambda}{hc}.$$

The number of photons per second produced by the green laser is:

$$\frac{N_{\text{G}}}{\Delta t} = \frac{P_{\text{G}}\lambda_{\text{G}}}{hc} \frac{\left(5.00 \cdot 10^{-3} \text{ W}\right)\left(543 \cdot 10^{-9} \text{ m}\right)}{\left(6.626 \cdot 10^{-34} \text{ J s}\right)\left(3.00 \cdot 10^8 \text{ m/s}\right)} = 1.37 \cdot 10^{16} \text{ s}^{-1}.$$

The number of photons per second produced by the red laser is:

$$\frac{N_{\text{R}}}{\Delta t} = \frac{P_{\text{R}}\lambda_{\text{R}}}{hc} \frac{\left(4.00 \cdot 10^{-3} \text{ W}\right)\left(633 \cdot 10^{-9} \text{ m}\right)}{\left(6.626 \cdot 10^{-34} \text{ J s}\right)\left(3.00 \cdot 10^8 \text{ m/s}\right)} = 1.27 \cdot 10^{16} \text{ s}^{-1}.$$

The green laser produces more photons per second.

38.49. The Lyman series has the electron falling to $n_1 = 1.$ The shortest possible wavelength occurs when $n_2 = \infty.$ Therefore, the shortest possible wavelength of the Lyman series in hydrogen is:

$$\lambda_{\text{min}} = \frac{1}{R_{\text{H}}}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)^{-1} = \frac{1}{\left(1.097 \cdot 10^{-7} \text{ m}^{-1}\right)}\left(\frac{1}{\left(1\right)^2} - \frac{1}{\infty}\right)^{-1} = 91.16 \text{ nm}.$$

38.50. The binding energy of the electron is given by: $E = -E_0 \dfrac{1}{n^2}$. Therefore, the energy required in ionize hydrogen when the electron is in the nth level is $|E| = E_0 \dfrac{1}{n^2}$.

38.51. The mass of an electron is $m = 9.109 \cdot 10^{-31}$ kg. The mass of a proton is $M = 1.673 \cdot 10^{-27}$ kg. By using the reduced mass, $\mu = mM/(m+M)$, the percent change in mass of the electron is:

$$\left(\frac{m-\mu}{m}\right) = \left(1 - \frac{M}{m+M}\right)100\% = \left(1 - \frac{\left(1.673 \cdot 10^{-27}\text{ kg}\right)}{\left(9.109 \cdot 10^{-31}\text{ kg}\right) + \left(1.673 \cdot 10^{-27}\text{ kg}\right)}\right)100\% = 0.05442\%.$$

If $M = m$, the reduced mass would be:

$$\mu = \frac{mM}{m+M} = \frac{m^2}{2m} = \frac{m}{2} = \frac{\left(9.109 \cdot 10^{-31}\text{ kg}\right)}{2} = 4.555 \cdot 10^{-31}\text{ kg}.$$

38.52. For a given n, ℓ lies between $0 \le \ell \le n-1$. For each ℓ, the number of possible orbitals (characterized by the quantum number, m) is $2\ell + 1$. By the Pauli exclusion principle, only one fermion can exist in a certain state at a time. This means that up to two electrons can exist in a certain orbital at the same time, where the two electrons have opposite spin angular momenta. Then, in the case of electrons, for each ℓ, the number of possible electron states is $2(2\ell+1)$. The number of possible states for a given n is therefore:

$$\sum_{\ell=0}^{n-1} 2(2\ell+1) = 4\sum_{\ell=0}^{n-1} \ell + \sum_{\ell=0}^{n-1} 2 = 4\frac{n(n-1)}{2} + 2n = 2n^2.$$

38.53. In Section 38.2, the speed of the electron in a hydrogen atom was determined to be

$$v = \sqrt{\frac{ke^2}{\mu a_0 n^2}}.$$

To scale with increasing atomic charge, replace e^2 with Ze^2. Therefore, the radius is given by:

$$r = \frac{\hbar^2}{\mu k Z e^2} n^2 = \frac{a_0}{Z} n^2.$$

Changing a_0 to a_0 / Z and e^2 to Ze^2 in the equation for speed gives:

$$v = Z\sqrt{\frac{ke^2}{\mu a_0 n^2}} \Rightarrow Z = v\sqrt{\frac{\mu a_0 n^2}{ke^2}}$$

$$Z = (0.500)\left(2.998 \cdot 10^8\text{ m/s}\right)\sqrt{\frac{\left(9.104 \cdot 10^{-31}\text{ kg}\right)\left(5.295 \cdot 10^{-11}\text{ m}\right)(1)^2}{\left(8.99 \cdot 10^9\text{ N m}^2/\text{C}^2\right)\left(1.602 \cdot 10^{-19}\text{ C}\right)^2}} = 68.5.$$

This is either the element erbium, Er, $Z = 68$, or the element thulium, Tm, $Z = 69$.

38.54. There are various paths from the third excited state to the ground state. These paths are:
(1) $n = 4 \to n = 3 \to n = 2 \to n = 1$,
(2) $n = 4 \to n = 3 \to n = 1$,
(3) $n = 4 \to n = 2 \to n = 1$, and
(4) $n = 4 \to n = 1$.
The wavelength is given by:

$$\lambda = \frac{1}{R_\text{H}}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)^{-1}.$$

The possible photon wavelengths are calculated below.
(1) In the first case, three photons are emitted. For $n=4 \rightarrow n=3$,

$$\lambda_{4\rightarrow 3} = \frac{1}{\left(1.097 \cdot 10^7 \text{ m}^{-1}\right)}\left(\frac{1}{(3)^2}-\frac{1}{(4)^2}\right)^{-1} = 1875 \text{ nm}.$$

For $n=3 \rightarrow n=2$,

$$\lambda_{3\rightarrow 2} = \frac{1}{\left(1.097 \cdot 10^7 \text{ m}^{-1}\right)}\left(\frac{1}{(2)^2}-\frac{1}{(3)^2}\right)^{-1} = 656.3 \text{ nm}.$$

For $n=2 \rightarrow n=1$,

$$\lambda_{2\rightarrow 1} = \frac{1}{\left(1.097 \cdot 10^7 \text{ m}^{-1}\right)}\left(\frac{1}{(1)^2}-\frac{1}{(2)^2}\right)^{-1} = 121.5 \text{ nm}.$$

(2) For $n=3 \rightarrow n=1$,

$$\lambda_{3\rightarrow 1} = \frac{1}{\left(1.097 \cdot 10^7 \text{ m}^{-1}\right)}\left(\frac{1}{(1)^2}-\frac{1}{(3)^2}\right)^{-1} = 102.6 \text{ nm}.$$

(3) For $n=4 \rightarrow n=2$,

$$\lambda_{4\rightarrow 2} = \frac{1}{\left(1.097 \cdot 10^7 \text{ m}^{-1}\right)}\left(\frac{1}{(2)^2}-\frac{1}{(4)^2}\right)^{-1} = 486.2 \text{ nm}.$$

(4) For $n=4 \rightarrow n=1$,

$$\lambda_{4\rightarrow 1} = \frac{1}{\left(1.097 \cdot 10^7 \text{ m}^{-1}\right)}\left(\frac{1}{(1)^2}-\frac{1}{(4)^2}\right)^{-1} = 97.23 \text{ nm}.$$

38.55. For the muonic hydrogen atom the energy is calculated in the same way as that of the usual hydrogen atom with an electron, but with a new reduced mass, μ_μ. The energy is

$$E_{n,\mu} = -\frac{1}{n^2}\frac{\mu_\mu k^2 e^4}{2\hbar^2} = -\frac{\mu_\mu}{\mu}\left(\frac{1}{n^2}\frac{\mu k^2 e^4}{2\hbar^2}\right) = -\frac{\mu_\mu}{\mu}\frac{1}{n^2}E_0,$$

where E_0 is the ionization energy of hydrogen. The ratio of reduced masses is

$$\begin{aligned}\frac{\mu_\mu}{\mu} &= \left(\frac{m_\mu m_\text{p}}{m_\mu + m_\text{p}}\right)\left(\frac{m_\text{e}+m_\text{p}}{m_\text{e} m_\text{p}}\right) = \frac{m_\mu}{m_\text{e}}\left(\frac{m_\text{e}+m_\text{p}}{m_\mu + m_\text{p}}\right)\\ &= \frac{\left(105.66 \text{ MeV}/c^2\right)}{\left(0.511 \text{ MeV}/c^2\right)}\left(\frac{\left(0.511 \text{ MeV}/c^2\right)+\left(938.27 \text{ MeV}/c^2\right)}{\left(105.66 \text{ MeV}/c^2\right)+\left(938.27 \text{ MeV}/c^2\right)}\right) = 185.94 \text{ MeV}/c^2.\end{aligned}$$

Therefore, the first three energy levels are:

$$E_{1,\mu} = -(185.94)\frac{1}{(1)^2}(13.6 \text{ eV}) = -2530 \text{ eV}$$

$$E_{2,\mu} = -(185.94)\frac{1}{(2)^2}(13.6 \text{ eV}) = -632 \text{ eV}$$

$$E_{3,\mu} = -(185.94)\frac{1}{(3)^2}(13.6 \text{ eV}) = -281 \text{ eV}$$

38.56. The energy required to ionize an electron from the $n=2$ state is the absolute value of the binding energy:

$$E=\left|-\left(\frac{1}{n^2}\right)E_0\right|=\frac{1}{(2)^2}(13.6\text{ eV})=3.40\text{ eV}.$$

38.57. For helium, the atomic number is $Z=2$ since there are two protons in the nucleus. Using equation 38.32, the energy levels are given by:

$$E_n=-\frac{\mu' k^2 Z^2 e^4}{2\hbar^2 n^2}=-\frac{\mu'}{\mu}\left(\frac{Z}{n}\right)^2 E_0,$$

where E_0 is the ionization energy of the hydrogen atom. The ratio of the reduced masses is:

$$\frac{\mu'}{\mu}=\left(\frac{m_e(2m_p)}{m_e+2m_p}\right)\left(\frac{m_e+m_p}{m_e m_p}\right)=2\left(\frac{m_e+m_p}{m_e+2m_p}\right)=2\frac{(9.10938\cdot 10^{-31}\text{ kg})+(1.67262\cdot 10^{-27}\text{ kg})}{(9.10938\cdot 10^{-31}\text{ kg})+2(1.67262\cdot 10^{-27}\text{ kg})}=1.0002722.$$

The first three energy levels of the He^+ are:

$$E_1=-(1.0002722)\left(\frac{2}{1}\right)^2(13.6\text{ eV})=-54.4\text{ eV}$$

$$E_2=-(1.0002722)\left(\frac{2}{2}\right)^2(13.6\text{ eV})=-13.6\text{ eV}$$

$$E_3=-(1.0002722)\left(\frac{2}{3}\right)^2(13.6\text{ eV})=-6.05\text{ eV}$$

38.58. The energy of a transition capable of producing light of wavelength $10.6\ \mu\text{m}$ is:

$$E=\frac{hc}{\lambda}=\frac{(4.136\cdot 10^{-15}\text{ eV s})(2.998\cdot 10^{8}\text{ m/s})}{(10.6\cdot 10^{-6}\text{ m})}=0.117\text{ eV}.$$

38.59. The energy of an electron in the *n*th orbital of a hydrogen atoms is given by:

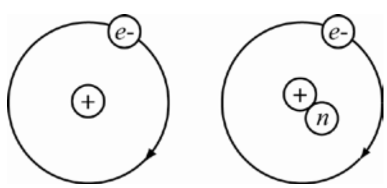
$$E_n=-\frac{1}{n^2}E_0.$$

The energy of the orbiting electron in a hydrogen atom with quantum number $n=45$ is:

$$E_{45}=-\frac{1}{(45)^2}(13.6\text{ eV})=-6.72\text{ meV}.$$

38.60. **THINK:** One first must determine the expression of the ground state energy in terms of the electron mass, *m*, and the mass of the proton, *M*.

SKETCH:



RESEARCH: The ground state energy for hydrogen is given by $E_0=\mu k^2 e^4/(2\hbar^2)$, where the reduced mass is $\mu=mM/(M+m)$. The reduced mass of deuterium is given by $\mu'=2mM/(m+2M)$, since the mass of the proton is about the same as the mass of a neutron.

SIMPLIFY: The energy difference is given by:

$$\Delta E=\frac{k^2e^4}{2\hbar^2}(\mu'-\mu)=\frac{\mu k^2 e^4}{2\hbar^2}\left(\frac{\mu'}{\mu}-1\right)=\left(\frac{\mu'}{\mu}-1\right)E_0.$$

The term $\left(\dfrac{\mu'}{\mu}-1\right)$ is given by:

$$\frac{\mu'}{\mu}-1=\frac{2mM/(m+2M)}{mM/(m+M)}-1=\frac{2(m+M)}{m+2M}-1=\frac{2m+2M-m-2M}{m+2M}=\frac{m}{m+2M}.$$

Therefore, the energy difference between the ground state of hydrogen and deuterium is:

$$\Delta E=\left(\frac{m}{m+2M}\right)E_0.$$

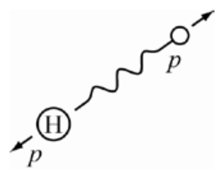
CALCULATE: $\Delta E=\left(\dfrac{(9.109\cdot 10^{-31}\text{ kg})}{(9.109\cdot 10^{-31}\text{ kg})+2(1.672\cdot 10^{-27}\text{ kg})}\right)(13.6\text{ eV})=3.704\cdot 10^{-3}\text{ eV}$

ROUND: To three significant figures, $\Delta E=3.70$ meV.

DOUBLE-CHECK: Since the neutron is not charged, adding it to the system will not affect the energy significantly, so the answer is reasonable.

38.61. **THINK:** The electron emits a photon in going from the $n=4$ state to the ground state, so the atom will recoil since the photon carries momentum.

SKETCH:



RESEARCH: The momentum of the photon is given by $p=E/c$. The energy of the electron is:

$$E_n=-\frac{1}{n^2}E_0$$

The momentum that the hydrogen atom gains is p in the opposite direction. The speed is given by $v=p/m_{\text{H}}$. The ground state energy for a hydrogen atom is $E_0=13.6$ eV.

SIMPLIFY: For a transition from the $n=4$ state to the $n=1$ state, the energy of the emitted photon is:

$$E=-E_0\left(\frac{1}{(4)^2}-\frac{1}{(1)^2}\right)=\frac{15}{16}E_0$$

Therefore, the speed of the hydrogen atom is

$$v=\frac{p}{m_{\text{H}}}=\frac{E}{m_{\text{H}}c}=\frac{15}{16}\frac{E_0}{m_{\text{H}}c}.$$

CALCULATE: $v=\dfrac{15}{16}\dfrac{(13.6\text{ eV})(1.602\cdot 10^{-19}\text{ J/eV})}{(1.674\cdot 10^{-27}\text{ kg})(3.00\cdot 10^{8}\text{ m/s})}=4.067\text{ m/s}$

ROUND: To three significant figures, the speed of the recoiling hydrogen atom is $v=4.07$ m/s.

DOUBLE-CHECK: This recoil speed is reasonable for a low energy photon emitted from a small mass.

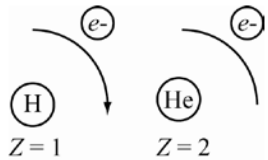
38.62. **THINK:**

(a) Using the equation for the allowed radii in the Bohr model, an expression for the radius as a function of n can be determined.

(b) The Rydberg formula provides the wavelength of the emitted radiation.

(c) Helium has a different atomic number and reduced mass.

SKETCH:



RESEARCH:

(a) The allowed radii in a hydrogen atom is $r = a_0 n^2$.

(b) The Rydberg formula for the wavelength in the hydrogen atom is $\frac{1}{\lambda} = R_{\text{H}}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$. The frequency is given by $f = c/\lambda$.

(c) The reduced mass for He^+ is

$$\mu' = \frac{2mM}{m+2M}.$$

The Bohr radius for He^+ is given by

$$a_0' = \frac{\hbar^2}{\mu' k e^2}.$$

For larger atoms, the e^2 term is replaced with Ze^2.

SIMPLIFY:

(b) $\lambda = \frac{1}{R_{\text{H}}}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)^{-1}$, $f = cR_{\text{H}}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$.

(c) $a_0' = \frac{\hbar^2}{\mu' k e^2} = \frac{\mu}{\mu'}\frac{\hbar^2}{\mu k e^2} = \frac{\mu}{Z\mu'} a_0$

$$r_{6,\text{ He}} = a_0' n^2 = \frac{\mu}{Z\mu'} a_0 n^2 = \frac{\left(\frac{mM}{m+M}\right)}{Z\left(\frac{2mM}{m+2M}\right)} a_0 n^2 = \frac{m+2M}{2Z(m+M)} a_0 n^2.$$

CALCULATE:

(a) $r_6 = (0.053\text{ nm})(6)^2 = 1.908\text{ nm}$

$$\frac{r_6}{r_1} = \frac{a_0(6)^2}{a_0(1)^2} = 36$$

(b) $\lambda = \frac{1}{(1.097\cdot 10^7\text{ m}^{-1})}\left(\frac{1}{(1)^2} - \frac{1}{(6)^2}\right)^{-1} = 9.376\cdot 10^{-8}\text{ m}$

$$f = (2.998\cdot 10^8\text{ m/s})(1.097\cdot 10^7\text{ m}^{-1})\left(\frac{1}{(1)^2} - \frac{1}{(6)^2}\right) = 3.197\cdot 10^{15}\text{ Hz}$$

Therefore, ultra-violet radiation was emitted.

(c) $r_{6,\text{ He}} = \frac{(9.109\cdot 10^{-31}\text{ kg}) + 2(1.673\cdot 10^{-27}\text{ kg})}{2(2)((9.109\cdot 10^{-31}\text{ kg}) + (1.673\cdot 10^{-27}\text{ kg}))}(0.053\text{ nm})(6)^2 = 0.9537\text{ nm}$

The ratio of the radii would be the same: $\frac{r_{6,\text{ He}}}{r_{1,\text{ He}}} = 36.$

ROUND:

(a) To two significant figures, $r_6 = 1.9 \text{ nm}$ and $r_6 / r_1 = 36.$

(b) Rounding to three significant figures is sufficient: $\lambda = 93.8$ nm, $f = 3.20 \cdot 10^{15}$ Hz.

(c) To two significant figures, $r_{6,\text{He}} = 0.95 \text{ nm}$ and $r_{6,\text{He}} / r_{1,\text{He}} = 36.$

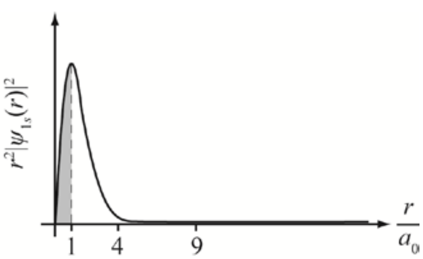
DOUBLE-CHECK: UV radiation is reasonable for a transition from the $n = 6$ state to the ground state.

38.63. **THINK:** The probability of finding the electron within the Bohr radius is determined by integrating the wave function,

$$\psi_{1s}(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0},$$

from 0 to $a_0 = 0.05295$ nm.

SKETCH:



RESEARCH: The probability is given by: $P(r \le a_0) = \int_0^{a_0} |\psi_{1s}(r)|^2 \, d^3r$

SIMPLIFY: Using an integration table:

$$P(r \le a_0) = \frac{4\pi}{\pi a_0^3} \int_0^{a_0} r^2 e^{-2r/a_0} dr = \frac{4}{a_0^3}\left[-\frac{a_0}{2} e^{-2r/a_0}\left(\frac{a_0^2}{2} + ra_0 + r^2\right)\right]_0^{a_0}$$

$$= \frac{4}{a_0^3}\left[-\frac{a_0}{2} e^{-2}\left(\frac{5}{2}a_0^2\right) + \frac{a_0}{2}\left(\frac{1}{2}a_0^2\right)\right]$$

$$= 1 - 5e^{-2}.$$

CALCULATE: $P(r \le a_0) = 1 - 5e^{-2} = 0.32332$

ROUND:. Rounding the answer four significant figures gives as $P(r \le a_0) = 0.3233.$

DOUBLE-CHECK: Figure 38.8(b) shows the probability density function for $n = 1$ (red curve). Here, for $r / a_0 \le 1,$ the area under the curve looks to be about 30%, so the answer is reasonable.

Multi-Version Exercises

38.64. **THINK:** The kinetic energy of the electrons must provide enough energy for an electron in the hydrogen atom to move from the $n_1 = 1$ state to the $n_2 = 2$ state. Then the emission of light from the $n_2 = 2$ to $n_1 = 1$ can occur. Note that in this collision between the electron and the atom the total momentum is also conserved, and the hydrogen atom will recoil from the collision. However, the effect of this is a very small correction to the overall energy, because the mass of the hydrogen atom is ~2,000 that of the electron. This is why the problem statement instructs us to neglect the recoil.

SKETCH: No sketch is needed.

RESEARCH: The wavelength is given by: $\frac{1}{\lambda} = R_\text{H}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$. The kinetic energy must be equal to the energy of the photon $K = \frac{1}{2} m_\text{e} v^2 = \frac{hc}{\lambda}.$

SIMPLIFY: $\frac{1}{2}m_e v^2 = hcR_H\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right) \Rightarrow v = \sqrt{\frac{2hcR_H}{m_e}\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)}$

CALCULATE: $v = \sqrt{\frac{2\left(6.626\cdot 10^{-34}\text{ J s}\right)\left(2.998\cdot 10^{8}\text{ m/s}\right)\left(1.097\cdot 10^{7}\text{ m}^{-1}\right)}{9.109\cdot 10^{-31}\text{ kg}}\left(\frac{1}{1}-\frac{1}{4}\right)} = 1.894327754\cdot 10^{6}\text{ m/s}$

ROUND: Rounding to four significant figures, $v = 1.894\cdot 10^6$ m/s. If we wanted to be conservative, due to our neglect of the recoil correction, we could round to three figures and give the result as $v = 1.89\cdot 10^6$ m/s.

DOUBLE-CHECK: In the final expression, for larger values of m, v is smaller. This is reasonable. Also comforting is that the speed required increases with higher value of n_2 and decreases with higher value of n_1, both of which are expected.

38.65. **THINK:** Assume that some of the excited atoms are in the $n = 3$ state. The kinetic energy of the electrons must provide enough energy for an electron in the hydrogen atom to move from the $n_1 = 3$ state to the higher state. Note that in this collision between the electron and the atom the total momentum is also conserved, and the hydrogen atom will recoil from the collision. However, the effect of this is a very small correction to the overall energy, because the mass of the hydrogen atom is ~2,000 that of the electron. This is why the problem statement instructs us to neglect the recoil.

SKETCH: We can use the same basic sketch as in the previous problem.

RESEARCH: The energy levels of the hydrogen atom are given by: $E_0 = -13.6$ eV, with $\Delta E = E_0\left(\frac{1}{n_2^2}-\frac{1}{n_1^2}\right)$ The kinetic energy of the electron is $K = \frac{1}{2}m_e v^2$.

SIMPLIFY: We set the kinetic energy of the incoming electrons equal to the energy difference between the two levels:

$$\Delta E = E_0\left(\frac{1}{n_2^2}-\frac{1}{n_1^2}\right) = \tfrac{1}{2}m_e v^2 \Rightarrow \frac{1}{n_2^2} = \frac{1}{n_1^2}+\frac{m_e v^2}{2E_0} = \frac{E_0+\tfrac{1}{2}m_e v^2 n_1^2}{n_1^2 E_0} \Rightarrow$$

$$n_2 = n_1\sqrt{\frac{E_0}{E_0+\tfrac{1}{2}m_e v^2 n_1^2}}$$

CALCULATE: It is perhaps easiest to compute the kinetic energy of the electron in eV and insert this value. We can easily obtain it from using that $m_e c^2 = 511$ keV. So

$$\tfrac{1}{2}m_e v^2 = \tfrac{1}{2}m_e c^2\left(v/c\right)^2 = \tfrac{1}{2}(511\text{ keV})\left(6.7601\cdot 10^5 / 2.998\cdot 10^8\right)^2 = 1.2991\text{ eV}$$

Now we can insert this into our expression for the final quantum number and obtain

$$n_2 = 3\sqrt{\frac{-13.6\text{ eV}}{-13.6\text{ eV}+(1.2991\text{ eV})9}} = 8.009$$

ROUND: Obviously, our answer needs to be an integer, so we round our final result to $n_2 = 8$.

DOUBLE-CHECK: Just the fact that all of our units cancel out and that we are left with a dimensionless number is comforting by itself. That we find a value for $n_2 > 3 = n_1$ is also as expected.

38.66. **THINK:** The kinetic energy of the electrons must provide enough energy for an electron in the hydrogen atom to move from some initial state with quantum number n_1 state to the higher state with quantum number $n_2 = 10$. Note that in this collision between the electron and the atom the total momentum is also conserved, and the hydrogen atom will recoil from the collision. However, the effect of this is a very small correction to the overall energy, because the mass of the hydrogen atom is ~2,000 that of the electron. This is why the problem statement instructs us to neglect the recoil.

SKETCH: We can use the same basic sketch as in the previous problem.

RESEARCH: The energy levels of the hydrogen atom are given by: $E_0 = -13.6 \text{ eV}$, with $\Delta E = E_0\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right)$. The kinetic energy of the electron is $K = \frac{1}{2}m_e v^2$.

SIMPLIFY: We set the kinetic energy of the incoming electrons equal to the energy difference between the two levels:

$$\Delta E = E_0\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right) = \tfrac{1}{2}m_e v^2 \Rightarrow \frac{1}{n_1^2} = \frac{1}{n_2^2} - \frac{m_e v^2}{2E_0} = \frac{E_0 - \frac{1}{2}m_e v^2 n_2^2}{n_2^2 E_0} \Rightarrow$$

$$n_1 = n_2\sqrt{\frac{E_0}{E_0 - \frac{1}{2}m_e v^2 n_2^2}}$$

CALCULATE: It is perhaps easiest to compute the kinetic energy of the electron in eV and insert this value. We can easily obtain it from using that $m_e c^2 = 511 \text{ keV}$. So

$$\tfrac{1}{2}m_e v^2 = \tfrac{1}{2}m_e c^2 (v/c)^2 = \tfrac{1}{2}(511 \text{ keV})\left(3.7892\cdot 10^5 / 2.998\cdot 10^8\right)^2 = 0.40815 \text{ eV}$$

Now we can insert this into our expression for the final quantum number and obtain

$$n_1 = 10\sqrt{\frac{-13.6 \text{ eV}}{-13.6 \text{ eV} - (0.40815 \text{ eV})100}} = 4.9993$$

ROUND: Obviously, our answer needs to be an integer, so we round our final result to $n_1 = 5$.

DOUBLE-CHECK: Just the fact that all of our units cancel out and that we are left with a dimensionless number is comforting by itself. That we find a value for $n_1 < 10 = n_2$ is also as expected.

38.67. $$\frac{n_{\text{lower}}}{n_{\text{higher}}} = \exp\left[(-13.6 \text{ eV})\left(1/n_2^2 - 1/n_1^2\right)/\left(k_B T\right)\right]$$

$$\frac{n_{\text{lower}}}{n_{\text{higher}}} = \exp\left[(-13.6 \text{ eV})\left(1/7^2 - 1/3^2\right)/\left((8.61733\cdot 10^{-5} \text{ eV/K})(528.3 \text{ K})\right)\right] = 5.85722\cdot 10^{11} = 5.86\cdot 10^{11}$$

38.68. $$\frac{n_{\text{lower}}}{n_{\text{higher}}} = \exp\left[(-13.6 \text{ eV})\left(1/n_2^2 - 1/n_1^2\right)/\left(k_B T\right)\right]$$

$$\Rightarrow T = \frac{(-13.6 \text{ eV})\left(1/n_2^2 - 1/n_1^2\right)}{k_B \ln(n_{\text{lower}}/n_{\text{higher}})}$$

$$T = \frac{(-13.6 \text{ eV})\left(1/64 - 1/9\right)}{(8.61733\cdot 10^{-5} \text{ eV/K})\ln(5.1383\cdot 10^5)} = 1146.02 \text{ K} = 1150 \text{ K}$$

Chapter 39: Elementary Particle Physics

Concept Checks

39.1. b **39.2.** c **39.3.** b

Multiple-Choice Questions

39.1. a **39.2.** c **39.3.** c **39.4.** c **39.5.** d **39.6.** b **39.7.** c

Conceptual Questions

39.8. (a) The Baryon number is not conserved so this reaction cannot occur.
(b) The Lepton number is not conserved so this reaction cannot occur.
(c) The Baryon number is conserved. The masses can be considered:

$$\Lambda^0(1116) \to p + K^- + \pi^+$$

$$1116 \text{ MeV/c}^2 < 938.272 \text{ MeV/c}^2 + 493.68 \text{ MeV/c}^2 + 139.570 \text{ MeV/c}^2 = 1572 \text{ MeV/c}^2$$

This decay is not possible since conservation of energy is violated.
(d) The Baryon number is conserved. The masses can be considered:

$$\Lambda^0(1450) \to p + K^- + \pi^+$$

$$1450 \text{ MeV/c}^2 < 938.272 \text{ MeV/c}^2 + 493.68 \text{ MeV/c}^2 + 139.570 \text{ MeV/c}^2 = 1572 \text{ MeV/c}^2$$

This decay is not possible since conservation of energy is violated.

39.9. Electromagnetic waves would be more appropriate for investigating the scattering cross-section of the atom since electromagnetic waves are scattered by the electrons surrounding the nucleus. On the other hand, since neutrons are electrically neutral particles they do not interact with the electron cloud. This allows neutrons to penetrate to the nucleus of an atom. Atoms with a larger atomic number Z have larger electron clouds, so electromagnetic waves are more affected by the atomic number. The atomic number will not affect neutrons.

39.10. The Heisenberg uncertainty relation allows a violation of energy conservation on the order of ΔE over a time $\Delta t < \hbar / \Delta E < \hbar / \left(m_p c^2\right)$. Since the maximum speed of the boson is the speed of light, the range is $\Delta x \le c\Delta t$. Therefore, the maximum range of such a force would be:

$$\Delta x < \frac{\hbar}{m_p c} = \frac{1.055 \cdot 10^{-34} \text{ J s}}{\left(1.673 \cdot 10^{-27} \text{ kg}\right)\left(3.00 \cdot 10^8 \text{ m/s}\right)} = 2.10 \cdot 10^{-16} \text{ m.}$$

This distance is overestimated since the particle would require a great deal of additional energy to reach a speed close to *c*. In this case, the maximum range of the force would be significantly smaller.

39.11. No, the spin and the mass also determine the type of meson. π^- and ρ^-, for example, have the same quark constituents ($\overline{u}d$) but different spin and mass.

39.12. Composite particles are defined by their constituent particles. Both protons and neutrons are baryons, and both are composed of three quarks. This is the basis of grouping particles together, not their time scales of decay. A neutron is not considered to be a proton-electron composite because the fundamental make-up of a neutron is three quarks, as for the proton. For this reason a neutron and a proton are both considered to be baryons.

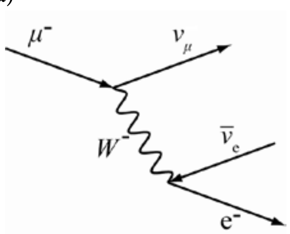
39.13. A positron will collide with an electron in the metal and result in two gamma rays. Since conservation of momentum must be obeyed, measuring the momentum of the two gamma rays allows one to infer the momentum of the electron.

39.14. The Heisenberg uncertainty relation allows a violation of energy conservation on the order of E over a time $\Delta t < \hbar / E$. Since the speed of the virtual photon is the speed of light, the range is $\Delta x = c\Delta t$. Therefore, the range of the electromagnetic interaction is given by: $\Delta x < \hbar c / E$.

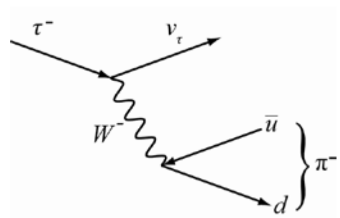
39.15. (a) Absorption of a photon by a positron.
(b) Electron-positron annihilation into a photon.
(c) Electron-positron scattering.

39.16.

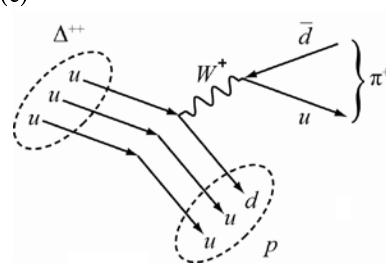
(a)



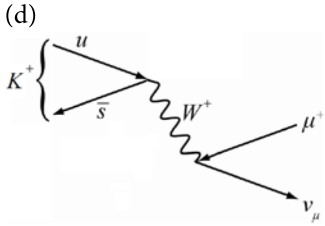
(b)



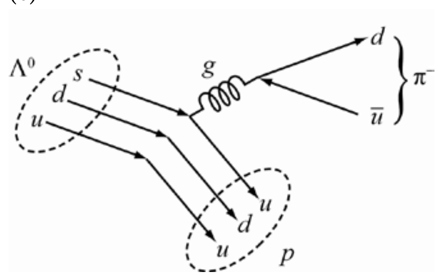
(c)



(d)



(e)



39.17. The neutron and the proton are both baryons so the baryon number is conserved. Checking energy conservation:

$$n \to p + \pi^-$$

$$939.6 \text{ MeV/c}^2 < 938.3 \text{ MeV/c}^2 + 139.6 \text{ MeV/c}^2 = 1078 \text{ MeV/c}^2$$

This decay is not possible since conservation of energy is violated.

39.18. In the decay $\pi^+ \to \mu^+ + \nu_\mu + \nu_e$, the left side has a lepton number of zero and the right side has a lepton number of $-1+1+1=1$. Since the lepton number is not conserved, this decay cannot occur.

39.19. Baryon number is conserved. Strangeness is conserved. Checking energy conservation:

$$\pi^0 + n \to K^- + \Sigma^+$$

$$135.0 \text{ MeV/c}^2 + 939.6 \text{ MeV/c}^2 \to 493.7 \text{ MeV/c}^2 + 1189.4 \text{ MeV/c}^2$$

$$1074.6 \text{ MeV/c}^2 < 1683.1 \text{ MeV/c}^2$$

This reaction cannot occur since conservation of energy is violated.

39.20. The scattering process $e^+ + \nu_\mu \to e^+ + \nu_\mu$ cannot proceed through a charged W boson because it would have to be a W^+ boson and this would decay into a positron and an electron neutrino (just as W^- a W^- boson decays into an electron and an anti-electron-neutrino, as mentioned in the text). Therefore, the scattering process must proceed through the Z boson. The scattering process $e^+ + \nu_e \to e^+ + \nu_e$ is completely permissible, because emission of a W^+ boson by a positron yields an electron neutrino and absorption of a W^+ boson by a neutrino yields a positron. The neutral exchange, elastic scattering by means of a Z boson is perfectly permissible as well for this interaction.

39.21. The baryons with quarks *uds* are: Λ^0: 1115.683 MeV/c^2; Σ^0: 1192.642 MeV/c^2.

39.22. $n\begin{Bmatrix} u\to u \\ d\to d \\ d\to u \end{Bmatrix}p \xrightarrow{\bar{u}d\left(\pi^-\text{ meson}\right)} p\begin{Bmatrix} u\to d \\ u\to u \\ d\to d \end{Bmatrix}n$ The virtual particle is a π^- meson.

Exercises

39.23. r_{min} occurs when the initial kinetic energy is equal to the potential energy at r_{min}.

$$K = 4.50\text{ MeV} = U(r) = \frac{k\left(Z_{\text{p}}e\right)\left(Z_{\text{t}}e\right)}{r_{\text{min}}}$$

$$r_{\text{min}} = \frac{k\left(Z_{\text{p}}e\right)\left(Z_{\text{t}}e\right)}{K} = \frac{\left(ke^2\right)Z_{\text{p}}Z_{\text{t}}}{K} = \frac{\left(1.44\text{ MeV fm}\right)\left(2\right)\left(78\right)}{4.50\text{ MeV}} = 49.9\text{ fm}$$

39.24. The alpha particle has energy, $E_\alpha = 6.50$ MeV. It is incident on a lead nucleus. The alpha particle has a charge number of $Z_{\text{p}} = 2$ and the lead nucleus has a charge number of $Z_{\text{t}} = 82,$ where *e* is the elementary unit of charge.

(a) r_{min} can be determined using the equation: $U(r) = kZ_{\text{p}}Z_{\text{t}}e^2 / r_{\text{min}},$ where k is the Coulomb constant. From conservation of energy it is reasoned that at the point of closest approach all of the alpha particles energy has been converted from kinetic to potential so $U(r) = 6.50$ MeV. Solving for r_{min} gives: $r_{\text{min}} = kZ_{\text{p}}Z_{\text{t}}e^2 / U(r) = \left(ke^2\right)Z_{\text{p}}Z_{\text{t}} / U(r).$ It is common to use $ke^2 = 1.44$ MeV fm, as found in Example 39.1.

This gives: $r_{\text{min}} = \dfrac{\left(1.44\text{ MeV fm}\right)\left(2\right)\left(82\right)}{6.50\text{ MeV}} = 36.3\text{ fm}.$

(b) If the kinetic energy is increased, the potential energy of the alpha particle at the point of closest approach will increase due to conservation of energy. The potential $U(r)$ is inversely proportional to r_{min}, which implies that if the potential energy is higher, then r_{min} will be smaller. Therefore, if the kinetic energy of the alpha particle is increased, the particle's distance of approach will decrease. It should be noted that at higher values of kinetic energy this model breaks down because at high enough values of kinetic energy the model would predict the two nuclei would "touch". This is discussed further in the text.

39.25. Assuming Rutherford scattering, the equation for the differential scattering cross section is given by equation 39.3 in the text: $\dfrac{d\sigma}{d\Omega} = \left(\dfrac{kZ_{\text{p}}Z_{\text{t}}e^2}{4K}\right)^2 \dfrac{1}{\sin^4\left(\frac{1}{2}\theta\right)},$ where k is the Coulomb constant and K is the kinetic energy of the projectile. The energy of the alpha particle is $K = 6.50\text{ MeV} = 6.50\cdot 10^6\text{ eV} = \left(6.50\cdot 10^6\text{ eV}\right)\cdot\left(1.602\cdot 10^{-19}\text{ J/eV}\right) = 1.0413\cdot 10^{-12}\text{ J}.$ The angle is $\theta = 60.0°.$ The charge number for the alpha particle is $Z_{\text{p}} = 2,$ the charge number for the lead nucleus is $Z_{\text{t}} = 82.$ Inserting the known values into equation (39.3) gives:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\left(8.9876\cdot 10^9\text{ J m/C}^2\right)\left(2\right)\left(82\right)\left(1.602\cdot 10^{-19}\text{ C}\right)^2}{4\left(1.0413\cdot 10^{-12}\text{ J}\right)}\right)^2 \frac{1}{\sin^4\left(30.0°\right)}$$

$$\frac{d\sigma}{d\Omega} = 1.32\cdot 10^{-27}\text{ m}^2/\text{sr} = 13.2\text{ b/sr}.$$

39.26. The protons have kinetic energy of $K = 2.00\text{ MeV} = \left(2.00\cdot 10^6\text{ eV}\right)\cdot\left(1.602\cdot 10^{-19}\text{ J/eV}\right) = 3.204\cdot 10^{-13}\text{ J}.$ Each proton has a charge number of $Z_{\text{p}} = 1$ and the gold nucleus has a charge number of $Z_{\text{t}} = 79.$ The equation for the differential scattering cross section is given by equation 39.3 in the text:

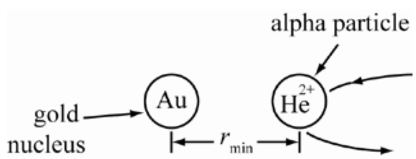
$$\frac{d\sigma}{d\Omega}=\left(\frac{kZ_{\text{p}}Z_{\text{t}}e^2}{4K}\right)^2\frac{1}{\sin^4\left(\frac{1}{2}\theta\right)},$$

where $\theta=30.0°$ and k is the Coulomb constant. Inserting the known values into equation (39.3) gives:

$$\frac{d\sigma}{d\Omega}=\left(\frac{\left(8.9876\cdot10^9\text{ J m/C}^2\right)(1)(79)\left(1.602\cdot10^{-19}\text{ C}\right)^2}{4\left(3.204\cdot10^{-13}\text{ J}\right)}\right)^2\frac{1}{\sin^4\left(15.0°\right)}=4.51\cdot10^{-26}\text{ m}^2/\text{sr}=451\text{ b/sr}.$$

39.27. **THINK:** The alpha particle has a de Broglie wavelength of $\lambda=6.40\text{ fm}=6.40\cdot10^{-15}\text{ m}$ and kinetic energy $K=5.00\text{ MeV}=5.00\cdot10^6\text{ eV}$. The closest distance this alpha particle can get to the gold nucleus is $r_{\text{min}}=45.5\text{ fm}=45.5\cdot10^{-15}\text{ m}$. Determine how the ratio r_{min}/λ varies with the kinetic energy of the alpha particle. Note that r_{min}/λ is unitless.

SKETCH:



RESEARCH: At the point of closest approach all of the alpha particle's kinetic energy has been converted into potential energy: $U(r_{\text{min}})=K$, where K is the initial kinetic energy of the alpha particle and $U(r_{\text{min}})$ is given by the Coulomb potential: $U(r_{\text{min}})=kZ_\alpha Z_{\text{Au}}e^2/r_{\text{min}}$. The de Broglie wavelength of the alpha particle is $\lambda=h/p=h/(m_\alpha v_\alpha)$. The equation for K is $K=(1/2)m_\alpha v_\alpha^2$. The charge number for an alpha particle is $Z_\alpha=2$, the charge number for gold is $Z_{\text{Au}}=79$.

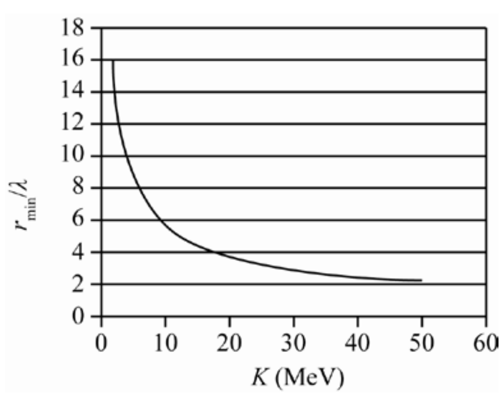
SIMPLIFY: $U(r_{\text{min}})=K=\dfrac{kZ_\alpha Z_{\text{Au}}e^2}{r_{\text{min}}}\;\Rightarrow\; r_{\text{min}}=\dfrac{kZ_\alpha Z_{\text{Au}}e^2}{K}\;\Rightarrow\;\dfrac{r_{\text{min}}}{\lambda}=\dfrac{kZ_\alpha Z_{\text{Au}}e^2p}{Kh}=\dfrac{kZ_\alpha Z_{\text{Au}}e^2m_\alpha v_\alpha}{Kh},$

but $m_\alpha v_\alpha=\sqrt{2m_\alpha K}$. Substituting this into the equation gives:

$$\begin{aligned}\frac{r_{\text{min}}}{\lambda}&=\frac{kZ_\alpha Z_{\text{Au}}e^2\sqrt{2m_\alpha K}}{Kh}\\&=\frac{kZ_\alpha Z_{\text{Au}}e^2\sqrt{2m_\alpha}}{h\sqrt{K}}=\frac{Z_\alpha Z_{\text{Au}}ke^2\sqrt{2m_\alpha}}{h\sqrt{K}}=\frac{158ke^2\sqrt{2m_\alpha}}{h\sqrt{K}}.\end{aligned}$$

Therefore, the ratio of r_{min}/λ is proportional to $1/\sqrt{K}$.

CALCULATE:

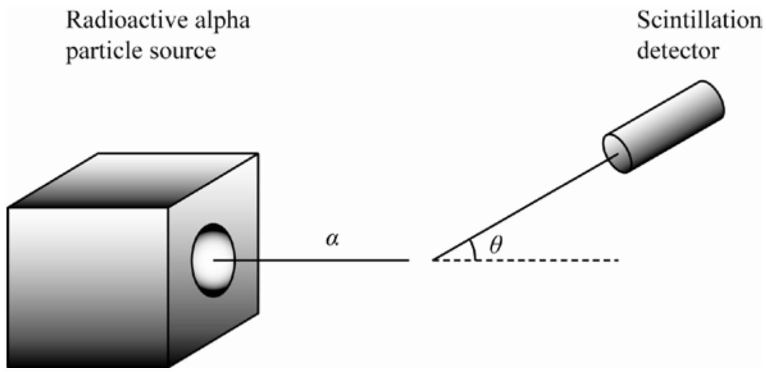


ROUND: Not applicable.

DOUBLE-CHECK: It makes sense that the ratio r_{min}/λ is affected by kinetic energy but not rapidly; that is, it is not inversely proportional to the square or cube of kinetic energy. Greater kinetic energy means one can probe smaller distances, and at the same time it means smaller wavelength. So r_{min} and λ both get smaller, but r_{min} does so a little faster.

39.28. **THINK:** Gold foil of thickness $t = 1.00\ \mu\text{m} = 1.00\cdot 10^{-6}$ m is bombarded with alpha rays of energy $K_{\text{E}} = 8.00\text{ MeV} = 8.00\cdot 10^{6}$ eV. Find the fraction of particles scattered to an angle: (a) between 5.00° and 6.00° and (b) between 30.0° and 31.0°. The atomic mass of gold is $m_{\text{Au}} = 197$ g/mol and its density is $\rho = 19.3\text{ g/cm}^3$.

SKETCH:



RESEARCH: The equation for the differential scattering cross section is given by equation 39.3 in the text: $\dfrac{d\sigma}{d\Omega} = \left(\dfrac{kZ_{\text{p}}Z_{\text{t}}e^2}{4K}\right)^2 \dfrac{1}{\sin^4\left(\frac{1}{2}\theta\right)} = \dfrac{d\sigma}{d\theta}\dfrac{d\theta}{d\Omega}$ where $\dfrac{d\theta}{d\Omega} = \dfrac{1}{2\pi\sin\theta}$. The charge number for an alpha particle is $Z_{\text{p}} = 2$ and the charge number for gold is $Z_{\text{t}} = 79$. The number of scattering centers per unit volume is $n_{\text{v}} = \rho N_{\text{A}}/m_{\text{Au}}$ where 1 mole has Avogadro's number of particles ($N_A = 6.022\cdot 10^{23}\text{ mol}^{-1}$). The number of scattering centers per unit area is $n_{\text{t}} = n_{\text{v}}t$. The cross section is defined as: $\sigma = \dfrac{\text{Number of reactions per scattering center/s}}{\text{Number of impinging particles/s/m}^2}$. Therefore the fraction of particles scattered into the angle range $\Delta\theta$ is given by $n_{\text{t}}\Delta\sigma = n_{\text{t}}\left(d\sigma/d\theta\right)\Delta\theta$. Note that $1° = 0.01745$ radians.

SIMPLIFY: $\dfrac{d\sigma}{d\Omega} = \dfrac{d\sigma}{d\theta}\dfrac{d\theta}{d\Omega} \Rightarrow \dfrac{d\sigma}{d\Omega} = \dfrac{d\sigma}{d\theta}\dfrac{1}{2\pi\sin\theta} \Rightarrow \dfrac{d\sigma}{d\theta} = \dfrac{2\pi\sin\theta}{\sin^4(\theta/2)}\left(\dfrac{kZ_{\text{p}}Z_{\text{t}}e^2}{4K_{\text{E}}}\right)^2$, since $\dfrac{d\theta}{d\Omega} = \dfrac{1}{2\pi\sin\theta}$

and $\dfrac{d\sigma}{d\Omega} = \left(\dfrac{kZ_{\text{p}}Z_{\text{t}}e^2}{4K}\right)^2 \dfrac{1}{\sin^4\left(\frac{1}{2}\theta\right)} = \dfrac{d\sigma}{d\theta}\dfrac{d\theta}{d\Omega}$. $n_{\text{t}} = n_{\text{v}}t$, substituting $n_{\text{v}} = \dfrac{\rho N_{\text{A}}}{m_{\text{Au}}}$ gives $n_{\text{t}} = \dfrac{\rho N_{\text{A}}t}{m_{\text{Au}}}$.

Substituting the above expressions into the equation for the number of particles scattered through an angle gives:

$$n_{\text{t}}\Delta\sigma = n_{\text{t}}\frac{d\sigma}{d\theta}\Delta\theta \Rightarrow n_{\text{t}}\Delta\sigma = \frac{\rho N_{\text{A}}t}{m_{\text{Au}}}\frac{2\pi\sin\theta}{\sin^4(\theta/2)}\left(\frac{kZ_{\text{p}}Z_{\text{t}}e^2}{4K_{\text{E}}}\right)^2\Delta\theta.$$

CALCULATE:

(a) $n_{\text{t}}\Delta\sigma = \left(5.8997\cdot 10^{22}\text{ m}^{-2}\right)\left(3.177\cdot 10^{-28}\text{ m}^2/\text{sr}\right)\dfrac{\sin 5.00°}{\sin^4 2.50°}\left(0.01745\text{ rad}\right) = 7.8748\cdot 10^{-3}$

(b) $n_{\text{t}}\Delta\sigma = \left(5.8997\cdot 10^{22}\text{ m}^{-2}\right)\left(3.177\cdot 10^{-28}\text{ m}^2/\text{sr}\right)\dfrac{\sin 30.0°}{\sin^4 15.0°}\left(0.01745\text{ rad}\right) = 3.6445\cdot 10^{-5}$

ROUND: The answers should be reported to three significant figures so the fraction of particles scattered to an angle: (a) between $5.00°$ and $6.00°$ is $7.87 \cdot 10^{-3}$ and (b) between $30.0°$ and $31.0°$ is $3.64 \cdot 10^{-5}$.
DOUBLE-CHECK: By inspection of the θ dependence in our derived equation it is reasonable that the fraction particles scattered between $5.00°$ and $6.00°$ was higher than the fraction scattered between $30.0°$ and $31.0°$. If you look at the geometry of the experiment that is shown in Figure 39.10 of the text the finding that a large fraction of particles are scattered at lower values of θ appears to make physical sense as well.

39.29. **THINK:** The differential scattering cross section for particles to scatter by $\theta = 55.0°$ off a target is $d\sigma / d\Omega = 4.00 \cdot 10^{-18}\ \text{m}^2/\text{sr}$. The detector has an area of $A = 1.00\ \text{cm}^2 = 1.00 \cdot 10^{-4}\ \text{m}^2$ and is placed a distance $d = 1.00$ m away from the target. The target has an area of $A_T = 1.00\ \text{mm}^2 = 1.00 \cdot 10^{-6}\ \text{m}^2$. Assume that $N = 3.00 \cdot 10^{17}$ particles hit the target area every second. Find the number of particles n_D that hit the detector per second. Note: $N = 3.00 \cdot 10^{17}\ \text{particles/mm}^2 = 3.00 \cdot 10^{23}\ \text{particles/m}^2$
SKETCH: A suitable sketch is provided with the question.
RESEARCH: Equation 39.1 in the text states that the differential cross section is:

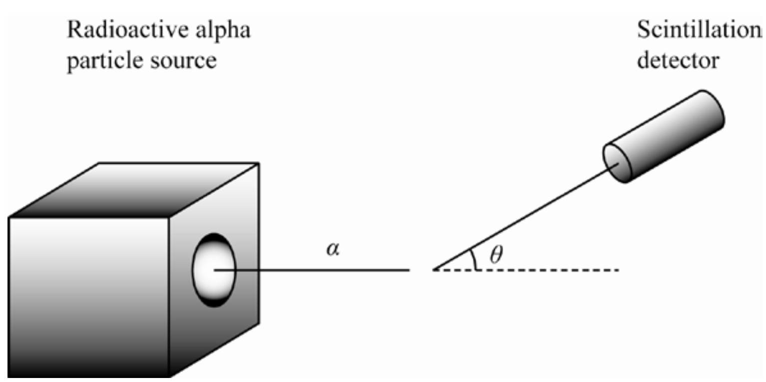
$$\frac{d\sigma}{d\Omega} = \frac{\text{Number of scatterings into solid angle } d\Omega \text{ per scattering center/s}}{\text{Number of impinging particles / s / m}^2}.$$

The total number of particles that hit the detector in one second is n_D and is given by $n_D = \Delta\sigma I_{\text{beam}}$, where I_{beam} is the number of particles per m^2 per second and $\Delta\sigma = (d\sigma / d\Omega)\Delta\Omega$. The solid angle difference $\Delta\Omega = 4\pi A / (\text{Cross sectional area of sphere})$, where the cross sectional area of the sphere that the detector lies on is $4\pi d^2$, therefore $\Delta\Omega = A / d^2$ sr.

SIMPLIFY: $n_D = \Delta\sigma I_{\text{beam}} = \dfrac{d\sigma}{d\Omega}\dfrac{A}{d^2}N$

CALCULATE:
$n_D = (4.00 \cdot 10^{-18}\ \text{m}^2/\text{sr})(1.00 \cdot 10^{-4}\ \text{m}^2/\text{sr} / 1.00\ \text{m}^2)(3.00 \cdot 10^{23}\ \text{particles/m}^2\ \text{s}) = 120.0\ \text{particles/s}.$
ROUND: The answer should be reported to three significant figures, so $n_D = 120.$ particles/s.
DOUBLE-CHECK: It seems reasonable that the detector will only be struck by a relatively small number of particles per second because the detector occupies only a small portion of the scattering sphere.

39.30. **THINK:** Assume symmetry about the axis of the incoming particle beam. Manipulate the Rutherford scattering formula to obtain the total number of particles, N, detected within angle range $d\theta$ as a function of the scattering angle θ, off a target hit by n particles where the density of the scattering centers is ρ.
SKETCH: Figure 39.10 in text.



RESEARCH: The Rutherford formula is given by equation (39.3) in the text: $\frac{d\sigma}{d\Omega}=\left(\frac{kZ_{\mathrm{p}}Z_{\mathrm{t}}e^2}{4K}\right)^2\frac{1}{\sin^4\left(\frac{1}{2}\theta\right)}$.

The differential cross section is defined by equation (39.1) in the text:

$$\frac{d\sigma}{d\Omega}=\frac{\text{Number of scatterings into solid angle } d\Omega \text{ per scattering center / s}}{\text{Number of impinging particles / s / m}^2}$$

By the chain rule, $\frac{d\sigma}{d\Omega}=\frac{d\sigma}{d\theta}\frac{d\theta}{d\Omega}$ and therefore $\frac{d\sigma}{d\theta}=\frac{d\sigma/d\Omega}{d\theta/d\Omega}$. Having found $\frac{d\sigma}{d\theta}$, we can write

$$\frac{d\sigma}{d\theta}=\frac{\text{Number of scatterings into angle } d\theta \text{ per scattering center / s}}{\text{Number of impinging particles / s / m}^2}$$

$$=\frac{\text{Number of scatterings into } d\theta}{\text{(Number of impinging particles)(scattering centers / m}^2\text{)}}$$

So then

$$d\sigma=\frac{d\sigma}{d\theta}d\theta=\frac{N}{n\rho}\Rightarrow N=n\rho d\sigma=n\rho\frac{d\sigma}{d\theta}d\theta.$$

SIMPLIFY: $\frac{d\sigma}{d\Omega}=\frac{d\sigma}{d\theta}\frac{d\theta}{d\Omega}$, and since the solid angle of a sphere is $\Omega=\iint_S\sin\theta d\theta d\phi,\quad d\Omega=\sin\theta\, d\theta\, d\phi$. If there is symmetry about the axis of the incoming particle beam, this expression can be simplified to $d\Omega=\sin\theta\, d\theta(2\pi)$ or $\frac{d\theta}{d\Omega}=\frac{1}{2\pi\sin\theta}$. Substituting this into the equation gives: $\frac{d\sigma}{d\Omega}=\frac{d\sigma}{d\theta}\frac{1}{2\pi\sin\theta}$, but $\frac{d\sigma}{d\Omega}=\left(\frac{kZ_{\mathrm{p}}Z_{\mathrm{t}}e^2}{4K}\right)^2\frac{1}{\sin^4\left(\frac{1}{2}\theta\right)}$ so the equation becomes: $\left(\frac{kZ_{\mathrm{p}}Z_{\mathrm{t}}e^2}{4K}\right)^2\frac{1}{\sin^4\left(\frac{1}{2}\theta\right)}=\frac{d\sigma}{d\theta}\frac{1}{2\pi\sin\theta}$, solving for $d\sigma$ gives: $d\sigma=\frac{2\pi\sin\theta}{\sin^4\left(\frac{1}{2}\theta\right)}\left(\frac{kZ_{\mathrm{p}}Z_{\mathrm{t}}e^2}{4K}\right)^2d\theta$. The total number of particles scattered as a function of the scattering angle θ can be written as $N=n\rho d\sigma=\frac{n\rho 2\pi\sin\theta}{\sin^4\left(\frac{1}{2}\theta\right)}\left(\frac{kZ_{\mathrm{p}}Z_{\mathrm{t}}e^2}{4K}\right)^2d\theta$.

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: From inspection of Figure 39.10 in the text it can be seen that the number of scattered particles that are detected should depend on θ. The functional form of the derived equation implies that more particles will be detected while moving through small values of θ.

39.31. **THINK:** The form factor is $F^2(\Delta p)$ and the Coulomb scattering differential cross section is $d\sigma/d\Omega$. Evaluate these quantities for an electron beam that is scattering off a uniform density sphere. The sphere has total charge Ze and radius R. Describe the scattering pattern.
SKETCH: See Figure 39.14 in the text.

RESEARCH: The differential cross section is given by equation 39.8 in the text: $\frac{d\sigma}{d\Omega}=\left(\frac{d\sigma}{d\Omega}\right)_{\text{point}}F^2\left(\Delta p\right)$, where $\left(\frac{d\sigma}{d\Omega}\right)_{\text{point}}=\frac{\left(2kZ_{\mathrm{p}}Z_{\mathrm{t}}e^2m_{\mathrm{p}}\right)^2}{\left(\Delta p\right)^4}$ (equation 39.4 in the text) and the form factor is given by equation 39.7: $F^2(\Delta p)=\left|\frac{1}{Ze}\int\rho(\vec{r})e^{i\Delta\vec{p}\cdot\vec{r}/\hbar}dV\right|^2$. Note that in equation 39.4, m_{p} is the mass of the projectile. The total charge of the sphere is $Ze = \rho V = \rho \frac{4}{3}\pi R^3$. Because of the symmetry of the problem it will be most convenient to use spherical coordinates when evaluating the form factor. Note that by the definition of the dot product $\Delta\vec{p}\bullet\vec{r} = \Delta pr\cos\theta$.

SIMPLIFY: First determine the form factor. In spherical coordinates

$$F^2\left(\Delta p\right) = \left|\frac{\rho}{Ze}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} e^{i\Delta pr\cos\theta/\hbar} r^2 \sin\theta drd\theta d\phi\right|^2 = \left|\frac{2\pi\rho}{Ze}\int_0^{\pi}\int_0^{R} e^{i\Delta pr\cos\theta/\hbar} r^2 \sin\theta drd\theta\right|^2.$$

Now, consider just the integral $\int_0^{\pi} e^{i\Delta pr\cos\theta/\hbar}\sin\theta d\theta$. This integral can be evaluated using substitution: let $i\Delta pr/\hbar = \alpha$ and let $\cos\theta = u$, then $du = -\sin\theta d\theta$. Substituting these values into the integral gives:

$$-\int e^{\alpha u}\frac{\sin\theta du}{\sin\theta} = -\int e^{\alpha u}du = -\frac{1}{\alpha}e^{\alpha u}.$$

Evaluating gives: $-\frac{\hbar e^{i\Delta pr\cos\theta/\hbar}}{i\Delta pr}\Bigg|_0^{\pi} = \frac{-\hbar}{i\Delta pr}\left(e^{-i\Delta pr/\hbar} - e^{i\Delta pr/\hbar}\right)$, note that $\frac{e^{i\Delta pr/\hbar} - e^{-i\Delta pr/\hbar}}{2i} = \sin\left(\frac{\Delta pr}{\hbar}\right)$, so the evaluated value is $\frac{2\hbar}{\Delta pr}\sin\left(\frac{\Delta pr}{\hbar}\right)$. Substituting this back into the integral equation for $F^2\left(\Delta p\right)$ gives:

$F^2\left(\Delta p\right) = \left|\frac{4\pi\rho\hbar}{Ze\Delta p}\int_0^R r\sin\left(\frac{\Delta pr}{\hbar}\right)dr\right|^2$. This integral can be solved using integration by parts. Let $\Delta p/\hbar = \beta$, let $u = r$, $du = dr$, $dv = \sin(\beta r)$ and $v = (-1/\beta)\cos(\beta r)$.

$$\frac{4\pi\rho\hbar}{Ze\Delta p}\int_0^R r\sin\left(\beta r\right)dr = \frac{4\pi\rho\hbar}{Ze\Delta p}\left(\frac{-r\cos\left(\beta r\right)}{\beta}\Bigg|_0^R + \int_0^R \frac{1}{\beta}\cos\left(\beta r\right)dr\right) = \frac{4\pi\rho\hbar}{Ze\Delta p}\left(\frac{-R\cos\left(\beta R\right)}{\beta} + \frac{1}{\beta^2}\sin\left(\beta R\right)\right)$$

Substituting for β gives the desired version of the form factor:

$$F^2(\Delta p) = \left|\frac{4\pi\rho\hbar^3}{Ze(\Delta p)^3}\left(\sin\left(\frac{\Delta pR}{\hbar}\right) - \left(\frac{\Delta pR}{\hbar}\right)\cos\left(\frac{\Delta pR}{\hbar}\right)\right)\right|^2.$$

Substituting $F^2(\Delta p)$ and $\left(\frac{d\sigma}{d\Omega}\right)_{\text{point}}$ into the differential cross section gives:

$$\frac{d\sigma}{d\Omega} = \frac{(2kZ_{\text{p}}Z_{\text{t}}e^2m_{\text{p}})^2}{(\Delta p)^4}\left(\frac{4\pi\rho\hbar^3}{Ze(\Delta p)^3}\left[\sin\left(\frac{\Delta pR}{\hbar}\right) - \left(\frac{\Delta pR}{\hbar}\right)\cos\left(\frac{\Delta pR}{\hbar}\right)\right]\right)^2.$$

Recall that $Ze = \rho V = \rho\frac{4}{3}\pi R^3$ so this can be substituted into the equation to get:

$$\frac{d\sigma}{d\Omega} = \frac{(2kZ_{\text{p}}Z_{\text{t}}e^2m_{\text{p}})^2}{(\Delta p)^4}\left(\frac{3\hbar^3}{\left(\Delta p\right)^3 R^3}\left[\sin\left(\frac{\Delta pR}{\hbar}\right) - \left(\frac{\Delta pR}{\hbar}\right)\cos\left(\frac{\Delta pR}{\hbar}\right)\right]\right)^2.$$

For $\frac{\Delta pR}{\hbar} \approx 1$ this equation gives the expected result for Rutherford scattering, but at large values of momentum transfer the differential cross section decreases much faster.

CALCULATE: Does not apply.

ROUND: Does not apply.

DOUBLE-CHECK: The derived equation implies that the differential cross section drops off quickly for larger values of Δp, this is consistent with the discussion of quantum limitations in the text and explains why at high kinetic energies the Rutherford model breaks down.

39.32. Table 39.1 in the text provides a list of elementary fermions with their symbols, charges, masses and antiparticles. The proton is made up of 2 up quarks and 1 down quark. The charge of an up quark is $+\frac{2}{3}e$ and the charge of a down quark is $-\frac{1}{3}e.$ The proton's charge is: $q = 2\left(\frac{2}{3}e\right) - \frac{1}{3}e = e = 1.602 \cdot 10^{-19}$ C.

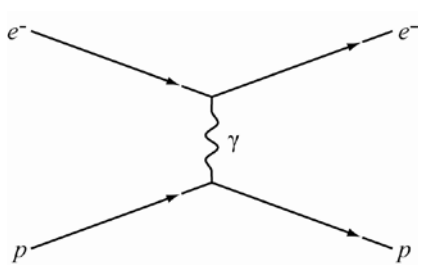
39.33. The observed magnetic moment of the proton is $\mu = 1.4 \cdot 10^{-26}$ A m^2. Estimate the speed of the quarks. Assume that the quarks move in circular orbits of radius $r = 0.80$ fm $= 0.80 \cdot 10^{-15}$ m and that they all move at the same speed and direction. The relationship between magnetic moment and orbital angular momentum can be used solve this problem. Using Equation 28.15 gives: $\vec{\mu}_{\text{orb}} = \frac{q}{2m}\vec{L}_{\text{orb}}$, where the magnitude of $\vec{L}_{\text{orb}}$ is $L_{\text{orb}} = rmv$. Substituting for L_{orb} in the equation gives: $\mu_{\text{orb}} = \frac{qrmv}{2m} = \frac{qrv}{2}$. Note that the total magnetic moment is the sum of the moments from the three quarks in the proton. The proton has two up quarks, each with charge $q = +\frac{2}{3}e$ and one down quark with charge $q = -\frac{1}{3}e$. Inserting these values into the equation for μ_{orb} gives:

$$\mu_{\text{orb}} = \left(\frac{2}{3}e\right)\frac{rv}{2} + \left(\frac{2}{3}e\right)\frac{rv}{2} - \left(\frac{1}{3}e\right)\frac{rv}{2} = \frac{erv}{2}.$$

Solving for v gives: $v = \frac{2\mu_{\text{orb}}}{er} = \frac{2(1.4 \cdot 10^{-26}\text{ A m}^2)}{(1.602 \cdot 10^{-19}\text{ C})(0.80 \cdot 10^{-15}\text{ m})} = 2.2 \cdot 10^8$ m/s.

39.34. The energy of the photon is $E_{\min} = 2.0$ keV. The minimum particle energy to probe a spatial size Δx is given by the equation: $E_{\min} = \sqrt{\frac{\hbar^2 c^2}{\Delta x^2} + m^2c^4}$, but for a photon $m = 0$ so the equation reduces to $E_{\min} = \frac{\hbar c}{\Delta x}$. Solving for Δx gives: $\Delta x = \frac{\hbar c}{E_{\min}} = \frac{(6.582 \cdot 10^{-16}\text{ eVs})(3.00 \cdot 10^8\text{ m/s})}{2.0 \cdot 10^3\text{ eV}} = 9.9 \cdot 10^{-11}$ m.

39.35. The Feynman diagram for an election–proton scattering event, $e^- + p \rightarrow e^- + p$, that is mediated by photon (γ) exchange can be drawn as follows:

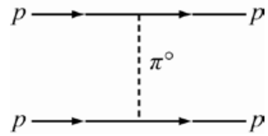


39.36. The mass of the Higgs boson is listed as $M = 125$ GeV/c^2 in Table 39.2 of the text. The range of reactions mediated by bosons is discussed in the text. The upper bound on the range of a reaction that is mediated by a Higgs boson can be found using the following equation:

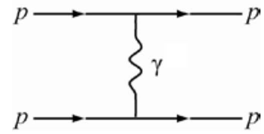
$$\Delta x < \frac{\hbar c}{m_b c^2} = \frac{(6.582 \cdot 10^{-16}\text{ eVs})(3.00 \cdot 10^8\text{ m/s})}{\left(125 \cdot 10^9\text{ eV/c}^2\right)c^2} = 1.58 \cdot 10^{-18}\text{ m}.$$

39.37. a) For the purposes of this question, the main scenario is interaction via the strong force. However, the electromagnetic force is another possibility.

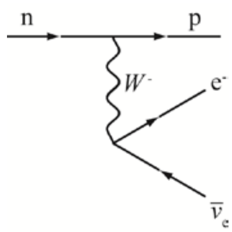
Proton-proton scattering, strong interaction:



Proton-proton scattering, EM interaction:



b) Neutron beta decays to a proton: $n \to p + e^- + \bar{\nu}_e$:



39.38. THINK:

(a) Perform a rough estimate of the mass, m, of the meson. Use the uncertainty principle and the known dimensions of a nucleus, $d \approx 10^{-15}$ m. Assume that the meson travels at relativistic speed.

(b) Follow the method of part (a) to prove that the theoretically expected rest mass of the photon is zero.

SKETCH: Not applicable.

RESEARCH: The Heisenberg uncertainty relations for energy and time, and position and momentum, are respectively $\Delta E \cdot \Delta t \geq \hbar / 2$ and $\Delta x \cdot \Delta p \geq \hbar / 2$. For relativistic energies, $E = K + mc^2$, where $K \cong mv^2 / 2$ for relativistic particles with mass and $E^2 = p^2c^2 + \left(mc^2\right)^2$. Recall that $\Delta p \cong m\Delta v$ for relativistic particles with mass, and generally $\Delta v = \Delta x / \Delta t$ (or equally useful, $\Delta t = \Delta x / \Delta v$). Recall for photons $v = c$. In this problem, the minimum uncertainty in the energy corresponds to the maximum uncertainty in time; the minimum uncertainty in energy is therefore $\Delta E = \hbar / \left(2\Delta t\right)$. Similarly, the minimum uncertainty in the momentum is $\Delta p = \hbar / \left(2\Delta x\right) = \hbar / \left(2d\right)$, where the maximum uncertainty in the position is restricted to the size of the nucleus, that is the maximum uncertainty in the position is $\Delta x = d$. For particles with mass the minimum uncertainty in the velocity corresponds to the maximum uncertainty in the position, that is the minimum uncertainty in the velocity is $\Delta v = \hbar / \left(2m\Delta x\right) = \hbar / \left(2md\right)$.

SIMPLIFY:

(a) $\Delta E = \hbar / \left(2\Delta t\right) = \hbar \Delta v / \left(2\Delta x\right) = \hbar \Delta v / \left(2d\right) = \hbar / \left(2d\right) \cdot \Delta v = \hbar / \left(2d\right) \cdot \hbar / \left(2md\right) = \hbar^2 / \left(4md^2\right)$

Also $\Delta E = \Delta K + mc^2 = m\left(\Delta v\right)^2 / 2 + mc^2 = m\left(\hbar / \left(2md\right)\right)^2 / 2 + mc^2 = \hbar^2 / \left(8md^2\right) + mc^2$.

Equating both expressions, $\hbar^2 / \left(8md^2\right) + mc^2 = \hbar^2 / \left(4md^2\right)$

$$mc^2 = \hbar^2 / \left(4md^2\right) - \hbar^2 / \left(8md^2\right)$$

$$mc^2 = \hbar^2 / \left(8md^2\right)$$

$$m^2 = \hbar^2 / \left(8c^2d^2\right)$$

$$m = \hbar / \left(\sqrt{8}cd\right) = \frac{\hbar c}{\sqrt{8}d} \cdot \frac{1}{c^2}$$

(b) For the photon, assume $v = c$: $\Delta E = \hbar/(2\Delta t) = \hbar c/(2\Delta x) = \hbar c/(2d)$. Also $(\Delta E)^2 = (\Delta p)^2 c^2 + (mc^2)^2$.

Then equating both expressions:

$$(\Delta p)^2 c^2 + (mc^2)^2 = \hbar^2 c^2/(4d^2)$$
$$(\hbar/(2d))^2 c^2 + (mc^2)^2 = \hbar^2 c^2/(4d^2)$$
$$\hbar^2 c^2/(4d^2) + (mc^2)^2 = \hbar^2 c^2/(4d^2)$$
$$(mc^2)^2 = \hbar^2 c^2/(4d^2) - \hbar^2 c^2/(4d^2) = 0$$

Therefore the rest mass is zero for the photon.

CALCULATE:

(a) $m = \dfrac{(6.582 \cdot 10^{-16} \text{ eV s})(3 \cdot 10^8 \text{ m/s})}{\sqrt{8}(1 \cdot 10^{-15} \text{ m})c^2} = 69.812652 \cdot 10^6 \text{ eV/c}^2$

(b) Not applicable.

ROUND: The answer should be reported to one significant figure, therefore a) $m = 70 \cdot 10^6 \text{ eV/c}^2 = 70 \text{ MeV/c}^2$.

DOUBLE-CHECK: Comparing the calculated mass to the values listed in Table 39.3 of the textbook shows that the calculated mass is of the right order of magnitude.

39.39. The Carbon dioxide molecule (CO_2) is made up of 1 carbon atom and 2 oxygen atoms. The atomic number for carbon is $Z = 6$, so there are 6 protons, 6 neutrons and 6 electrons in this atom. Each proton is made of 3 quarks (uud) and each neutron is made of 3 quarks (udd), so the carbon atom has 6 electrons + 3(6) quarks + 3(6) quarks = 42 fermions. The atomic number for oxygen is $Z = 8$, so there are 8 electrons + 3(8) quarks + 3(8) quarks = 56 fermions in each oxygen atom. The total number of fermions in the carbon dioxide molecule = 42+56 +56 = 154 fermions.

39.40. (a) The equation describing a neutral pion decaying into photons is given by equation 39.13 in the text: $\pi^\circ \to 2\gamma$. The mass of the neutral pion is listed in Table 39.3: $m_{\text{pion}} = 134.977 \cdot 10^6 \text{ eV/c}^2$. Using the mass-energy equivalence equation gives $E = m_{\text{pion}} c^2$. By conservation of energy the energy of each of the two identical photons is $\frac{1}{2}E$. The energy of each photon is therefore given by:

$$E_{\text{photon}} = \frac{1}{2}E = \frac{m_{\text{pion}} c^2}{2} = \frac{(134.977 \cdot 10^6 \text{ eV/c}^2)(c^2)}{2} = 67.4885 \cdot 10^6 \text{ eV}$$

(b) The energy of a photon can also be written as $E_{\text{photon}} = hf$, solving for f gives

$$f = \frac{E_{\text{photon}}}{h} = \frac{67.4885 \cdot 10^6 \text{ eV}}{4.136 \cdot 10^{-15} \text{ eVs}} = 1.6317336 \cdot 10^{22} \text{ s}^{-1} \approx 1.632 \cdot 10^{22} \text{ s}^{-1}.$$

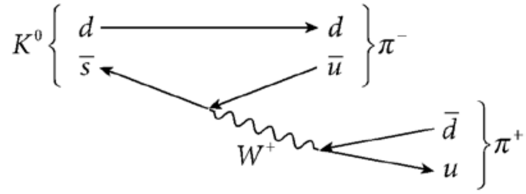
(c) A photon with this high of frequency would be in the gamma ray portion of the electromagnetic spectrum. This is shown in Figure 31.10 of the text.

39.41. **THINK:** The neutral kaon decays into two charged pions via the reaction: $K^0 \to \pi^+ + \pi^-$. Draw the quark level Feynman diagram for this process.

SKETCH: The diagram will be developed during the question.

RESEARCH: Figure 39.31 in the text shows that the neutral kaon decays via the weak interaction. Table 39.3 tells us that the neutral kaon (K^0) contains a down quark (d) of charge $-\frac{1}{3}e$ and an anti-strange quark ($\bar{s}$) of charge $+\frac{1}{3}e$. The positively charged pion (π^+) has an up quark (u) of charge $+\frac{2}{3}e$ and an anti-down quark ($\bar{d}$) of charge $+\frac{1}{3}e$. The negatively charged pion (π^-) has an anti-up quark ($\bar{u}$) of charge $-\frac{2}{3}e$ and a down quark (d) of charge $-\frac{1}{3}e$. The anti-strange quark beta-decays to an anti-up quark, charge $-\frac{2}{3}e$, emitting a W^+ boson. The W^+ boson then decays into an up quark of charge $+\frac{2}{3}e$ and an anti-down quark of charge $+\frac{1}{3}e$, which form a positively charged pion.

SIMPLIFY:



CALCULATE: Not applicable.

ROUND: Not applicable.

DOUBLE-CHECK: The quark level Feynman diagram is consistent with charge conservation. The mass of the K^0 is 497.61 MeV/c^2. The sum of the masses of the two pions is $2 \cdot 139.57$ MeV/c^2 = 279.14 MeV/c^2. Thus, the decay is energetically allowed because the mass of the neutral kaon is larger than the summed mass of the two pions.

39.42. During the radiation-dominated era the temperature was gradually decreasing with time. This relationship is given by equation 39.17 in the text: $T(t) = 1.5 \cdot 10^{10}\text{ K s}^{1/2}/\sqrt{t}$. Stefan's law describes the power radiated per unit area and is given by equation 36.1 in the text: $I = \sigma T^4$, where σ is the Stefan-Boltzmann constant ($\sigma = 5.670400 \cdot 10^{-8}$ W m^{-2} K^{-4}). Substituting equation 39.17 into equation 36.1 gives:

$$I = \sigma(1.5 \cdot 10^{10}\text{ K s}^{1/2})^4 / t^2 = \left(5.670400 \cdot 10^{-8}\text{ W m}^{-2}\text{ K}^{-4}\right) \cdot \left(1.5 \cdot 10^{10}\text{ K s}^{1/2}\right)^4 / t^2$$

$$I = 2.870640 \cdot 10^{33} t^{-2}\text{ W m}^{-2} \approx 2.9 \cdot 10^{33} t^{-2}\text{ W m}^{-2}.$$

The background radiation intensity during the radiation dominated era $\propto t^{-2}$.

39.43. Equation 39.17 from the text gives the time dependence of the temperature during the radiation-dominated era. $T(t) = 1.5 \cdot 10^{10}\text{ K s}^{1/2}/\sqrt{t}$. In the text, in the section discussing Quark-Gluon plasma, it is mentioned that color singlets began to form at a temperature of approximately $2.1 \cdot 10^{12}$ K. This can be used as the temperature when proton and neutrons began to form. Solving equation 39.17 in terms of time gives:

$$t = \frac{\left(1.5 \cdot 10^{10}\text{ K s}^{1/2}\right)^2}{T^2} = \frac{\left(1.5 \cdot 10^{10}\text{ K s}^{1/2}\right)^2}{\left(2.1 \cdot 10^{12}\text{ K}\right)^2} = 5.1 \cdot 10^{-5}\text{ s}.$$

This would be the estimated age of the universe when protons and neutrons began to form.

39.44. The average temperature of the universe three hundred thousand years after the Big Bang was $T \approx 3000$ K.

(a) Wien's displacement law can be used to find the peak wavelength (λ_{max}) of the blackbody spectrum at this temperature.

$$\lambda_{\text{max}} T = 2.90 \cdot 10^{-3}\text{ K m} \Rightarrow \lambda_{\text{max}} = 2.90 \cdot 10^{-3}\text{ K m} / 3000\text{ K} = 1 \cdot 10^{-6}\text{ m}$$

(b) Radiation of this wavelength would be in the infrared portion of the electromagnetic spectrum.

39.45. **THINK:** At a time of about $t = 10^{-6}$ s after the Big Bang, the universe had cooled to a temperature of $T \approx 10^{13}$ K. The electron volt can be used as a unit of temperature.

SKETCH: Not applicable.

RESEARCH: The thermal energy is $E = k_B T$, where $k_B = 1.381 \cdot 10^{-23}$ J/K is the Boltzmann constant. The electron volt can be used as a unit of temperature by the following conversion: $1 \text{ eV} = 1 \text{ eV} \cdot \left(1.602 \cdot 10^{-19} \text{ J/eV}\right) \cdot 1/k_B$. Einstein's mass energy equivalence theorem can be stated as: $E = mc^2$. The masses of the most important baryons are listed in Table 39.4 of the text. The masses of the elementary fermions are listed in Table 39.1.

SIMPLIFY:

(a) In units of eV, the equation $E = k_B T$ can be stated as $E = k_B T / \left(1.602 \cdot 10^{-19} \text{ J/eV}\right)$.

CALCULATE:

(a) $E = \left(1.381 \cdot 10^{-23} \text{ J/K}\right)\left(1 \cdot 10^{13} \text{ K}\right) / \left(1.602 \cdot 10^{-19} \text{ J/eV}\right) = 8.620 \cdot 10^{8} \text{ eV}$

(b) At this temperature there were a small number of protons and neutrons present. At this thermal energy (≈ 1 GeV) the protons and neutrons would be constantly converted into one another. This was possible because energy was much greater than the mass difference between the proton and neutron. This implies that the protons and neutrons were in equilibrium at this temperature.

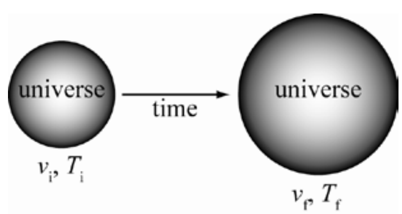
(c) The energy required to form an electron-position pair is given by the equation $E = 2m_e c^2 = 2\left(0.511 \cdot 10^{6} \text{ eV}/c^2\right)c^2 = 1.02 \cdot 10^{6}$ eV. A temperature of $\approx 10^{13}$ K would facilitate this easily. At this temperature there were a large number of electrons and positrons present. The temperature at which there was not sufficient thermal energy for electron-position pairs to form was approximately 10^{10} K, at this temperature the thermal energy was approximately $9 \cdot 10^{5}$ eV. From Figure 39.42 in the text, the temperature of the universe dropped below this value ~1 s after the Big Bang.

ROUND: The answer should be reported to one significant figure: (a) $E = 9 \cdot 10^{8} \text{ eV} = 9 \cdot 10^{5} \text{ keV}$.

DOUBLE-CHECK: The thermal energy that was calculated for a temperature of $\approx 10^{13}$ K is consistent with the values presented in Chapter 39 of the text. The discussion of protons, neutrons, electrons and positrons follows the reasoning given in the text.

39.46. **THINK:** The initial temperature to consider is $T_i = 3000$ K, and the final temperature to consider is $T_f = 2.75$ K. Model the universe as an ideal gas and assume that the expansion of the universe is adiabatic. Determine how much the volume of the universe has changed. Next assume that the process is irreversible and determine the change in the entropy of the universe, ΔS, based on the change in volume.

SKETCH:



RESEARCH: For a reversible adiabatic process $TV^{\gamma-1}$ = constant as given by Equation 19.27 in the text. Modeling the universe as a monatomic ideal gas, $\gamma = 5/3$. The change in entropy with respect to the change in volume is given by: $\Delta S = k_B \ln(V_f / V_i)$, where k_B is the Boltzmann constant. Let V_i be the volume of the universe at temperature T_i and V_f be the volume at temperature T_f.

SIMPLIFY: $TV^{\gamma-1}$ = constant, so $T_iV_i^{\gamma-1}=T_fV_f^{\gamma-1} \Rightarrow \frac{T_i}{T_f}=\left(\frac{V_f}{V_i}\right)^{\gamma-1} \Rightarrow \frac{V_f}{V_i}=\left(\frac{T_i}{T_f}\right)^{(\gamma-1)^{-1}}$

CALCULATE: $\frac{V_f}{V_i}=\left(\frac{3000\text{ K}}{2.75\text{ K}}\right)^{\left(\frac{5}{3}-1\right)^{-1}}=\left(\frac{3000\text{ K}}{2.75\text{ K}}\right)^{3/2}=36031.5,$

$\Delta S=\left(1.381\cdot 10^{-23}\text{ J/K}\right)\ln\left(36031.5\right)=1.449\cdot 10^{-22}\text{ J/K}$

ROUND: The answers should be reported to one significant figure, therefore $V_f/V_i=40{,}000$ and $\Delta S=1\cdot 10^{-22}$ J/K.

DOUBLE-CHECK: It seems reasonable that there would be a large increase in the volume of the universe because there was a large decrease in its temperature. It is also reasonable that the entropy increased (by a small amount) because the entropy of a closed system always stays the same or increases.

39.47. **THINK:** The ratio of the wavelength of light received, λ_{rec} from a galaxy to its wavelength at emission, λ_{emit} is equal to the ratio of the scale factor (e.g., radius of curvature) of the Universe at reception to its value at emission ($a_{\text{rec}}/a_{\text{emit}}$). The redshift, z of the light is defined by $1+z=\lambda_{\text{rec}}/\lambda_{\text{emit}}=a_{\text{rec}}/a_{\text{emit}}$.

(a) Hubble's Law states that the redshift z of light from a galaxy is proportional to the galaxy's distance from us (for reasonably nearby galaxies). Derive this law from the first relationships above, and determine the Hubble constant in terms of the scale-factor function $a(t)$.

(b) If the present Hubble constant has the value $H_0=72$ (km/s)/Mpc determine the distance Δs from us to a galaxy which has light with a redshift of $z=0.10$.

SKETCH: Not applicable.

RESEARCH:

(a) Hubble's law states $z\approx c^{-1}H\Delta s$, where c is the speed of light in vacuum, H is the Hubble constant, and Δs is the distance to the galaxy. The scale factor function $a_{\text{emit}}(t)$ can be expanded backwards in time from the present using a Taylor expansion. For a reasonably close source, expanding the series to first order should be a good approximation: $a_{\text{emit}}(t)\approx a_{\text{rec}}-(da/dt)_{\text{rec}}\Delta t$

(b) Although the numerical value is not required in the calculation, the megaparsec (Mpc) is a unit of length equal to $3.26\cdot 10^6$ light years. Hubble's law can be used to calculate the distance.

SIMPLIFY:

(a) $a_{\text{emit}}\cong a_{\text{rec}}-(da/dt)_{\text{rec}}\Delta t$, substituting this into the equation that defines the redshift gives:

$$1+z\cong\frac{a_{\text{rec}}}{a_{\text{rec}}-\left(\frac{da}{dt}\right)_{\text{rec}}\Delta t}=\frac{a_{\text{rec}}}{a_{\text{rec}}\left(1-\frac{1}{a_{\text{rec}}}\left(\frac{da}{dt}\right)_{\text{rec}}\Delta t\right)}=\left(1-\left(\frac{1}{a}\frac{da}{dt}\right)_{\text{rec}}\Delta t\right)^{-1}\Rightarrow 1+z\cong 1+\left(\frac{1}{a}\frac{da}{dt}\right)_{\text{rec}}\Delta t$$

. We know that $\Delta t=\Delta s/c$. Therefore, $1+z\cong 1+\frac{1}{c}\left(\frac{1}{a}\frac{da}{dt}\right)_{\text{rec}}\Delta s \Rightarrow z\cong\frac{1}{c}\left(\frac{1}{a}\frac{da}{dt}\right)_{\text{rec}}\Delta s$. Comparison of this equation and the equation for Hubble's law shows that $H=\left(\frac{1}{a}\frac{da}{dt}\right)_{\text{rec}}$.

(b) $\Delta s\approx zc/H_0$

CALCULATE:

(a) Not applicable.

(b) $\Delta s\cong\frac{(0.10)(3.00\cdot 10^5\text{ km/s})}{72\text{ (km/s)/Mpc}}=416.66\text{ Mpc}$

ROUND: The answer should be reported to two significant figures, therefore (b) $\Delta s=420$ Mpc.

DOUBLE-CHECK: The calculated distance to the galaxy has units of length, which is expected. This distance is approximately 1.4 billion light years. This distance corresponds to a time that falls within the age of the universe.

39.48. To produce an electron-positron pair, the photon must have at least enough energy to create an electron and a position at rest. The rest mass of both the electron and the positron is $m = 0.511\text{ MeV/c}^2$. The minimum energy required to produce this pair is therefore:

$$E_{\text{ph}} = m_{\text{e-}}c^2 + m_{\text{e+}}c^2 = 2(0.511\text{ MeV/c}^2)c^2 = 1.022\text{ MeV} = 1.022\cdot 10^6\text{ eV}.$$

The wavelength of this photon is found from $E_{\text{ph}} = hc/\lambda.$ Rearranging

$$\lambda = hc/E_{\text{ph}} = \left(4.13567\cdot 10^{-15}\text{ eV s}\right)\left(2.998\cdot 10^8\text{ m/s}\right)/\left(1.022\ \cdot 10^6\text{ eV}\right)) = 1.21\text{ pm}.$$

39.49. (a) The kinetic energy K of the neutron is found from $K = p^2/2m.$ The de Broglie wavelength formula is $\lambda = h/p.$ Solving for momentum p in terms of the neutron's wavelength gives: $p = h/\lambda = \left(6.626\cdot 10^{-34}\text{ J s}\right)/\left(0.15\cdot 10^{-9}\text{ m}\right) = 4.417\cdot 10^{-24}\text{ kg m/s}.$ Since the mass of a neutron is $1.675\cdot 10^{-27}\text{ kg},$ the kinetic energy of the neutron is:

$$K = \left(4.417\cdot 10^{-24}\text{ kg m/s}\right)^2/\left(2\cdot 1.675\cdot 10^{-27}\text{ kg}\right) \approx 5.825\cdot 10^{-21}\text{ J} \approx 0.036\text{ eV}.$$

The energy of a photon with the same wavelength is:

$$E_{\text{ph}} = hc/\lambda = \left(6.626\cdot 10^{-34}\text{ J s}\right)\left(2.998\cdot 10^8\text{ m/s}\right)/\left(0.15\cdot 10^{-9}\text{ m}\right) = 1.3\cdot 10^{-15}\text{ J} = 8.3\text{ keV}.$$

The energy of the X-ray photon is more than 200,000 times greater than that of the neutron having the same wavelength.

(b) While the wavelength is the same, the X-rays are orders of magnitude more energetic, and bound to create more damage (induce serious molecular degradation). For reference, the C-C bond energy is only about 4 eV. Neutrons, on the other hand, have very little kinetic energy are less likely to damage the biological samples investigated.

39.50. To produce a proton-antiproton pair, the photon must have at least enough energy to create a proton and antiproton at rest. The rest mass of each of the particles is $m = 938\text{ MeV/c}^2$. The minimum energy required of the photon is therefore: $E_{\text{ph}} = m_{\text{p}}c^2 + m_{\bar{\text{p}}}c^2 = 2(938\text{ MeV/c}^2)c^2 = 1880\text{ MeV}.$

39.51. The resolution provided by the electrons with kinetic energy $K = p^2/2m = 100.\text{ eV}$ is dependent on their de Broglie wavelength ($\lambda = h/p$), which is inversely proportional to their momentum. With the rest mass of an electron being $m_{\text{e}} = 0.511\text{ MeV/c}^2,$ the momentum of the electrons is:

$$p = \left(2m_{\text{e}}K\right)^{1/2} = \left(2\cdot\left(0.511\text{ MeV/c}^2\right)\left(100.\text{ eV}\right)\right)^{1/2} = 10.1\text{ keV/c} = 5.40\cdot 10^{-24}\text{ kg m/s}$$

To obtain the same resolution, the neutrons must have the same momentum (wavelength) as the electrons. Since the mass of the neutron is $m_{\text{n}} = 1.675\cdot 10^{-27}\text{ kg},$ this corresponds to a kinetic energy of:

$$K = p^2/2m_{\text{n}} = \left(5.40\cdot 10^{-24}\text{ kg m/s}\right)^2/\left(2\cdot 1.675\cdot 10^{-27}\text{ kg}\right) = 8.71\cdot 10^{-21}\text{ J} = 0.0544\text{ eV}.$$

The neutrons would require a kinetic energy of 54.4 meV.

39.52. The rest mass of the α-particle is $m_\alpha = 3.73\text{ GeV/c}^2 = 3.73\cdot 10^3\text{ MeV/c}^2.$ The kinetic energy of the α-article is given as $K = p^2/2m_\alpha = 100.\text{ MeV}.$ Solving for p gives:

$$p = \left(2m_\alpha K\right)^{1/2} = \left(2\cdot\left(3.73\cdot 10^3\text{ MeV/c}^2\right)\left(100.\text{ MeV}\right)\right)^{1/2} = 863.7\text{ MeV/c} = 8.637\cdot 10^8\text{ eV/c}$$

Using the de Broglie formula $\lambda = h/p,$

$$\lambda = \left(4.13567\cdot 10^{-15}\text{ eV s}\right)/\left(8.637\cdot 10^8\text{ eV/c}\right) = 1.44\cdot 10^{-15}\text{ m}$$

The de Broglie wavelength of this α-particle is $\lambda = 1.44$ fm. Figure 39.13 shows a length scale between 10^{-15} m and 10^{-16} m for a 100.-MeV α-particle. This is comparable to the de Broglie wavelength.

39.53. The Heisenberg uncertainty relation allows a violation of energy conservation on the order of ΔE over a time $\Delta t < \hbar / \Delta E$. Since the maximum speed of the boson is the speed of light, the range of interaction is $\Delta x \le c\Delta t$. Therefore $\Delta x < \frac{c\hbar}{\Delta E} = \frac{\left(2.998 \cdot 10^8 \text{ m/s}\right)\left(1.0546 \cdot 10^{-34} \text{ J s}\right)}{\left(91.1876 \cdot 10^9 \text{ eV}\right)\left(1.6022 \cdot 10^{-19} \text{ J/eV}\right)} = 2.164 \cdot 10^{-18} \text{ m}$. The order of magnitude of the range of the weak interaction is 10^{-18} m.

39.54. For the momentum to be conserved, the energies of the two photons must be equal and must sum to the combined rest mass energy of the proton and antiproton.

$$2m_0c^2 = 2hc / \lambda \Rightarrow \lambda = \frac{h}{m_0 c} = \frac{6.63 \cdot 10^{-34} \text{ J s}}{\left(1.6726 \cdot 10^{-27} \text{ kg}\right)\left(2.9979 \cdot 10^8 \text{ m/s}\right)} = 1.32 \cdot 10^{-15} \text{ m}.$$

39.55. The cross section of the interaction is $\sigma_\Lambda = (\Delta x)^2$ where Δx is the interaction range. The range is related to the decay time by $\Delta x \approx \tau c$. Therefore $\sigma_\Lambda = (\Delta x)^2 \approx (\tau c)^2 = \left(\left(10^{-10} \text{ s}\right)\cdot\left(3.00 \cdot 10^8 \text{ m/s}\right)\right)^2 = 9 \cdot 10^{-4} \text{ m}^2$.

39.56. The classical Rutherford differential scattering is described by $\frac{d\sigma}{d\Omega} = \left(\frac{kZ_\text{p}Z_\text{t}e^2}{4K}\right)^2 \frac{1}{\sin^4(\theta/2)}$. The alpha particle has a charge number $Z_\text{p} = 2$ and uranium has a charge number $Z_\text{t} = 92$. The differential scattering cross section is then

$$\frac{d\sigma}{d\Omega} = \left(\frac{\left(8.9876 \cdot 10^9 \text{ N m}^2/\text{C}^2\right)(2)(92)\left(1.602 \cdot 10^{-19} \text{ C}\right)^2}{4\left(5.00 \cdot 10^6 \text{ eV}\right)\left(1.602 \cdot 10^{-19} \text{ J/eV}\right)}\right)^2 \frac{1}{\sin^4\left(35.0°/2\right)}$$
$$= 2.15 \cdot 10^{-26} \text{ (N m}^2\text{/J)}^2 = 2.15 \cdot 10^{-26} \text{ m}^2.$$

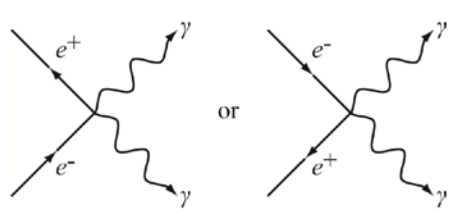
39.57. (a) The speed of the alpha particle is only 5.00% of the speed of light. This speed allows for a classical approach. The kinetic energy must equal the Coulomb potential between the particles at closest approach.

$$\frac{1}{2}mv^2 = \frac{kZ_\text{p}Z_\text{t}e^2}{r} \text{ or } r = \frac{2kZ_pZ_te^2}{mv^2} = \frac{2Z_pZ_tke^2}{mv^2} = \frac{2(2)(79)\left(ke^2\right)}{mv^2} = \frac{316ke^2}{mv^2}$$

(b) The mass of the alpha particle is 3.73 GeV/c^2. $r_\text{min} = \frac{316\left(1.44 \text{ MeV fm}\right)}{\left(3.73 \cdot 10^3 \text{ MeV/c}^2\right)\left(0.0500c\right)^2} = 48.8$ fm.

39.58. **THINK:** The two photons created by the annihilation of the electron-positron pair must have the same energy as the combined energy of the particles. To conserve momentum, the photons must have equal and opposite momentum. The electron and positron have the same rest mass $E_0 = 0.511$ MeV and are each traveling at $v = 0.99c$ with respect to their center of mass.

SKETCH:



RESEARCH: The energy of a relativistic particle is $E = \gamma mc^2 = \gamma E_0 = E_0 / \sqrt{1-(v/c)^2}$. The wavelength of a photon of energy E_γ is $\lambda = hc / E_\gamma$.

SIMPLIFY: The energy of one photon is half that of the total energy of the electron-positron pair:

$$E_\gamma = \frac{1}{2}(E_{e^-} + E_{e^+}) = \frac{1}{2}(\gamma E_0 + \gamma E_0) = \gamma E_0 = \frac{E_0}{\sqrt{1-(v/c)^2}}.$$

The wavelength of the photon is then $\lambda = \dfrac{hc}{E_\gamma} = \dfrac{hc\sqrt{1-(v/c)^2}}{E_0}$.

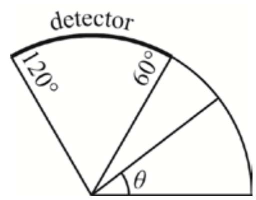
CALCULATE: $\lambda = \dfrac{1240 \text{ eV nm}}{0.511 \cdot 10^6 \text{ eV}} \sqrt{1-0.99^2} = 342.3 \text{ fm}$

ROUND: The speed is given to two significant figures, so the wavelength is accurate to two significant figures. The annihilation produces photons of wavelength $\lambda = 3.4 \cdot 10^{-13}$ m = 340 fm.

DOUBLE-CHECK: Because the electrons are traveling at a speed close to that of light, the annihilation should produce very energetic photons. Energetic photons have a high frequency, but a small wavelength as these photons do.

39.59. **THINK:** The solid-angular distribution of the tau-leptons varies as $(1+\cos^2\theta)$. The fraction of particles detected is proportional to the integral over the solid angle that the detector covers. In this case, the detector covers $\theta_1 = 60°$ to $\theta_2 = 120°$.

SKETCH:



RESEARCH: The intensity varies: $I \propto \int_{\theta_1}^{\theta_2} (1+\cos^2\theta)\sin\theta d\theta.$

SIMPLIFY: Let $x = \cos\theta$ and $dx = -\sin\theta d\theta$. Then the fraction is:

$$\frac{I}{I_{\text{tot}}} = \frac{-\int_{x_1}^{x_2}(1+x^2)dx}{-\int_{-1}^{1}(1+x^2)dx} = \frac{x+\frac{1}{3}x^3 \Big|_{x_1}^{x_2}}{x+\frac{1}{3}x^3 \Big|_{-1}^{1}} = \frac{x_2 + \frac{1}{3}x_2^3 - x_1 - \frac{1}{3}x_1^3}{\frac{8}{3}}$$

CALCULATE: $x_1 = \cos 60° = 1/2$ and $x_2 = \cos 120° = -1/2$.

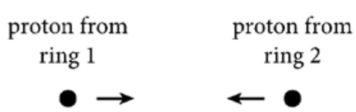
$$\left|\frac{I}{I_{\text{tot}}}\right| = \left|\frac{\left(-\frac{1}{2}\right)+\left(\frac{1}{3}\right)\left(-\frac{1}{8}\right)-\left(\frac{1}{2}\right)-\left(\frac{1}{3}\right)\left(\frac{1}{8}\right)}{\frac{8}{3}}\right| = \frac{13}{32} = 0.40625.$$

ROUND: There is no need to round. The fraction of the tau-leptons captured is $13/32$.

DOUBLE-CHECK: With a distribution of $(1+\cos^2\theta)$, the majority of the leptons are captured at angles close to $0°$ and $180°$. The detector is not located at these regions, so a value of less than half is reasonable.

39.60. **THINK:** The beam has a luminosity of $L = 4.00 \cdot 10^{33}$ cm^{-2} s^{-1} and the cross section for the creation of a Higgs boson is $\sigma = 1.00$ pb. How many Higgs events can be expected at the LHC in 1.00 year?

SKETCH:



RESEARCH: The number of Higgs created per second is $R=\sigma L.$ The total number of Higgs produced in a time period *t* is $N=Rt.$ Use

$$t=1\text{ yr}=\left(1\text{ yr}\cdot\frac{3600\text{ s}}{1\text{ hr}}\cdot\frac{24\text{ hr}}{1\text{ day}}\cdot\frac{365\text{ day}}{1\text{ yr}}\right)=3.1536\cdot 10^{7}\text{ s.}$$

SIMPLIFY: $N_{\text{Higgs/yr}}=Rt=\sigma Lt.$

CALCULATE: $N_{\text{Higgs/yr}}=\left(1\cdot 10^{-12}\text{ b}\right)\left(10^{-28}\text{ m}^2/\text{b}\right)\left[4.00\cdot 10^{33}\text{ cm}^{-2}\text{ s}^{-1}\cdot\left(\frac{100\text{ cm}}{1\text{ m}}\right)^2\right]\cdot\left(3.1536\cdot 10^{7}\text{ s/yr}\right)$

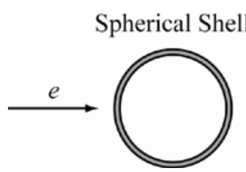
$=126,252$ Higgs/yr.

ROUND: The value is accurate to three significant figures. The number of Higgs produced in a year at LHC is 126,000.

DOUBLE-CHECK: The Higgs boson is an extremely hard particle to produce. The rate of Higgs boson production in this case is 126,000 Higgs / $3.15\cdot 10^{7}$ s = 14 Higgs bosons per hour.

39.61. **THINK:** What is the differential cross section for a beam of electrons Coulomb-scattering off a thin spherical shell of total charge *Ze* and radius *a*? Can this experiment distinguish between the thin-shell and a solid-sphere charge distribution?

SKETCH:



RESEARCH: The differential cross section is given by $\frac{d\sigma}{d\Omega}=\left(\frac{2Ze^2m_e}{4\pi\varepsilon_0}\right)^2\frac{1}{(\Delta p)^4}F^2(\Delta p),$ with form factor $F^2(\Delta p)=\left|\frac{1}{Ze}\int\rho(\vec{r})\exp\left(\frac{i}{\hbar}\Delta\vec{p}\bullet\vec{r}\right)dV\right|^2$. But in this case the target charge density ρ is concentrated in a thin spherical shell of radius *a*. The integral over *r* extends only over the thickness of the shell, with all the functions in the integrand evaluated at $r=a$. The charge density takes the form $\rho(r)=\frac{Ze}{4\pi a^2}\delta(r-a),$ using the Dirac delta function. Equivalently, the integral can be taken to be a surface integral, with the charge density ρ replaced by a surface charge density $Ze/(4\pi a^2).$ The cross section for a solid sphere of radius *a* is (see Solution 39.31)

$$\frac{d\sigma}{d\Omega}=\left(\frac{2Ze^2m_e}{4\pi\varepsilon_0}\right)^2\frac{1}{(\Delta p)^4}\left\{\frac{3\hbar}{(\Delta p)^3a^3}\left[\sin\left(\frac{(\Delta p)a}{\hbar}\right)-\frac{(\Delta p)a}{\hbar}\cos\left(\frac{(\Delta p)a}{\hbar}\right)\right]\right\}^2$$

SIMPLIFY: The form factor is given by

$$F(\Delta p)=\frac{1}{4\pi a^2}\int_0^{2\pi}\int_0^{\pi}\exp\left(\frac{i(\Delta p)a}{\hbar}\cos\theta\right)a^2\sin\theta d\theta d\phi$$
$$=\frac{1}{4\pi a^2}a^2\int_0^{\pi}\exp\left(\frac{i(\Delta p)a}{\hbar}\cos\theta\right)\sin\theta\, d\theta\int_0^{2\pi}d\phi$$
$$=\frac{1}{2}\int_0^{\pi}\exp\left(\frac{i(\Delta p)a}{\hbar}\cos\theta\right)\sin\theta\, d\theta$$
$$=\frac{\hbar}{2(\Delta p)ai}\left[-\exp\left(\frac{i(\Delta p)a}{\hbar}\cos\theta\right)\right]\Bigg|_0^{\pi}$$
$$=\frac{\hbar}{2(\Delta p)ai}\left(\exp[i(\Delta p)a/\hbar]-\exp[-i(\Delta p)a/\hbar]\right)$$
$$=\frac{\sin[(\Delta p)a/\hbar]}{(\Delta p)a/\hbar}$$

The cross section is therefore $\dfrac{d\sigma}{d\Omega}=\left(\dfrac{2Ze^2m_e}{4\pi\varepsilon_0}\right)^2\dfrac{1}{(\Delta p)^4}\left(\dfrac{\sin[(\Delta p)a/\hbar]}{(\Delta p)a/\hbar}\right)^2.$

CALCULATE: There is no need to calculate.

ROUND: There is no need to round. Like the cross section for the solid sphere, this matches the point-target result in the $\Delta p\to 0$ limit, but falls off much more rapidly for large momentum transfer. It also has zero for (in this case, periodic) values of the momentum transfer, so this scattering pattern too can show a central maximum surrounded by bright and dark rings. However, this pattern is distinguishable from that for the solid sphere target. It falls off more quickly with increasing Δp for small values, but for large values it falls off much more slowly: as $(\Delta p)^{-6}$, rather than $(\Delta p)^{-8}$ though still faster than the $(\Delta p)^{-4}$ fall off of the point-target cross section. So, yes, a scattering experiment with sufficient data could distinguish between a solid and a hollow spherical target.

DOUBLE-CHECK: The technique should be able to distinguish between the two charge distributions. One might expect that both spheres can be approximated by a point charge at the center of the sphere. The charge distribution, however, that the electron 'sees' is a function of its momentum. The faster the electron goes the closer it gets to the real particles and the more it can resolve the real charge distribution.

Multi-Version Exercises

39.62. $\Pi=\rho N_A t\sigma/M$

$$=\frac{(2.77\text{ g/cm}^3)(6.022\cdot 10^{23}\text{/mole})(68.5\text{ cm})(0.68\cdot 10^{-38}\text{ cm}^2\text{/GeV})(337\text{ GeV})}{27.0\text{ g/mole}}$$
$$=9.7\cdot 10^{-12}$$

39.63. $\Pi=\rho N_A t\sigma/M\Rightarrow$

$$t=M\Pi/(\rho N_A\sigma)$$
$$=\frac{(27.0\text{ g/mole})(4.19\cdot 10^{-12})}{(2.77\text{ g/cm}^3)(6.022\cdot 10^{23}\text{/mole})(0.68\cdot 10^{-38}\text{ cm}^2\text{/GeV})(143\text{ GeV})}$$
$$=70.\text{ cm}$$

39.64. $\Pi = \rho N_A t\sigma / M = \rho N_A t\left(0.68\cdot 10^{-38}\text{cm}^2/\text{GeV}\right)E/M \Rightarrow$

$E = M\Pi / [\rho N_A t\left(0.68\cdot 10^{-38}\text{cm}^2/\text{GeV}\right)]$

$$= \frac{(27.0\text{ g/mole})(6.00\cdot 10^{-12})}{\left(2.77\text{ g/cm}^3\right)\left(6.022\cdot 10^{23}/\text{mole}\right)\left(71.1\text{ cm}\right)\left(0.68\cdot 10^{-38}\text{ cm}^2/\text{GeV}\right)}$$

$= 2.0\cdot 10^2\text{ GeV}$

39.65. $I(\theta) \propto \sin^{-4}(\tfrac{1}{2}\theta) \Rightarrow$

$I(\theta_2) = I(\theta_1)\sin^4(\tfrac{1}{2}\theta_1)/\sin^4(\tfrac{1}{2}\theta_2)$

$= (853\text{ /s})\sin^4(47.45°)/\sin^4(30.25°) = 3900.671\text{ /s} = 3.90\cdot 10^3\text{ /s}$

39.66. $I(\theta) \propto \sin^{-4}(\tfrac{1}{2}\theta) \Rightarrow$

$I(\theta_2)\sin^4(\tfrac{1}{2}\theta_2) = I(\theta_1)\sin^4(\tfrac{1}{2}\theta_1) \Rightarrow$

$\theta_2 = 2\sin^{-1}[\sqrt[4]{I(\theta_1)/I(\theta_2)}\sin\tfrac{1}{2}\theta_1]$

$= 2\sin^{-1}[\sqrt[4]{1129/4840}\sin(47.55°)] = 61.7°$

39.67. $I(\theta) \propto \sin^{-4}(\tfrac{1}{2}\theta) \Rightarrow$

$I(\theta_1)/I(\theta_2) = \sin^4(\tfrac{1}{2}\theta_2)/\sin^4(\tfrac{1}{2}\theta_1)$

$= [\sin(31.45)/\sin(42.55)]^4 = 0.354$

Chapter 40: Nuclear Physics

Concept Checks

40.1. a **40.2.** b **40.3.** b, e **40.4.** d **40.5.** d

Multiple-Choice Questions

40.1. b **40.2.** c **40.3.** b **40.4.** e **40.5.** d **40.6.** b **40.7.** d

Conceptual Questions

40.8. If the half-life is long then a given species of radioactive nuclei will decay very slowly and if the half-life is short the species will decay very quickly. Since radioactive particles are dangerous, a radioactive material with a short half-life is more dangerous.

40.9. There are many high energy particles that travel to Earth from cosmic ray sources. As these particles collide with the atoms, they turn into non-harmful particles. Thus, very few of these particles reach the surface of the Earth due to the atmosphere. However, since pilots climb to an altitude of 10 km above sea-level or higher there are more harmful particles since they have interacted with a smaller amount of the atmosphere. Health concerns are one reason why flight times must be limited.

40.10. The magic numbers: 2, 8, 20, 28, 50, 82, 126, etc., are the number of nucleons of each type that are needed to form closed shells. When either the protons or the neutrons form a closed shell, a nucleus becomes more stable.

40.11. The binding energy difference between ${}^{3}_{2}\text{He}$ and ${}^{3}_{1}\text{H}$ is 0.764 MeV. The Coulomb interaction energy between two protons that are separated by a distance of the nuclear radius of ${}^{3}_{2}\text{He}$ is $U = ke^2 / R,$ where $R = R_0 A^{1/3} = 1.12 \text{ fm}(3)^{1/3} = 1.615 \text{ fm}.$ Then, $U = 1.429 \cdot 10^{-13} \text{ J} = 0.892 \text{ MeV}.$ This accounts for the difference in binding energy between the two atoms and why the binding energy of ${}^{3}_{2}\text{He}$ is lower than that of ${}^{3}_{1}\text{H},$ where there is no repulsive Coulomb interaction.

40.12. All the quantities are conserved during a nuclear reaction, not for the individual nuclei involved in the reaction, but for the ensemble of nuclei in the reaction.

40.13. Gamma radiation is an electromagnetic wave and hence does not remain in one location. When it is emitted from the radioactive source, it either continues traveling or is absorbed by matter. When it is absorbed, its energy is transformed into other types of energy and hence the original gamma radiation ceases to exist. Thus, the food will not contain any gamma radiation when it is sold unless the radiation treatment caused nuclei in the food to be transformed into radioactive isotopes.

40.14. The confinement of the particles and energy are kept together under high pressure and temperature by the enormous gravity of the Sun. Since this is not available on Earth, other methods must be found.

40.15. The nuclear strong force must balance the Coulomb repulsion, as well as the asymmetry term in the binding energy equation. Having too many protons is prohibited by the Coulomb repulsion, while the asymmetry term keeps the neutrons and protons more or less in balance (Pauli exclusion principle). On the other hand, the force that holds the nuclei together is comparatively short-ranged. Thus, when too many protons get close enough together, the Coulomb repulsion is more prevalent (over the longer range) than the strong nuclear force. This limits the size of nuclei.

40.16. (a) Helium-3 has "picked up" a neutron from the target nucleus. With this characteristic, this kind of reaction is traditionally called a pickup reaction.
(b) From the conservation of mass number, which started at 15, the final product should also have the same mass number of 15. The alpha particle has 4 nucleons, so the X nucleus should have 11 nucleons. From the conservation of protons, the X nucleus must be ${}^{11}_{6}\text{C}$.
(c)
$$\begin{aligned} Q &= \left[m(1,2)+m(6,6)-m(5,6)-m(2,2)\right]c^2 \\ &= \left[(3.016029\text{ u})+(12.000000\text{ u})-(11.0114336\text{ u})-(4.0026033\text{ u})\right]c^2\left(931.4940\text{ MeV}/(c^2\text{u})\right) \\ &= 1.856\text{ MeV} \end{aligned}$$
(d) Since the Q-value is positive, the reaction is exothermic.

40.17. (a) Four states can be constructed with two $t = 1/2$ objects: $(1/2, 1/2)$, $(1/2, -1/2)$, $(-1/2, 1/2)$, and $(-1/2, -1/2)$. The state $(1/2, 1/2)$, a symmetric combination of $(1/2, -1/2)$, $(-1/2, 1/2)$ and $(-1/2, -1/2)$, is grouped together in an iso-triplet or $t = 1$ set of states. The state $(-1/2, -1/2)$, an anti-symmetric combination of $(1/2, -1/2)$ and $(-1/2, 1/2)$, forms an "iso-singlet" or $t = 0$ object. The iso-triplet states would correspond to the di-proton or ${}^2\text{He}$ nucleus, a variant of the deuteron or ${}^2\text{H}$ nucleus, and the di-neutron, but these do not exist as bound states. The iso-singlet is the deuteron, or ${}^2\text{H}$ nucleus.
(b) Eight states can be formed from three $t = 1/2$ objects. Four of these can be grouped into a $t = 3/2$ set of states: $(1/2, 1/2, 1/2)$; symmetric combinations of $(1/2, 1/2, -1/2)$, $(1/2, -1/2, 1/2)$, and $(-1/2, 1/2, 1/2)$; symmetric combinations of $(1/2, -1/2, -1/2)$, $(-1/2, -1/2, 1/2)$, and $(-1/2, 1/2, -1/2)$; and $(-1/2, -1/2, -1/2)$. These would correspond to ${}^3\text{Li}$, a variant of ${}^3\text{He}$, a variant of ${}^3\text{H}$ (tritium), and the tri-neutron, respectively, but again, they do not exist as bound states. The other four states can be grouped into two iso-doublet, $t = 1/2$ sets. They consist of combinations of $(1/2, 1/2, -1/2)$, $(1/2, -1/2, 1/2)$, and $(-1/2, 1/2, 1/2)$; and $(1/2, -1/2, -1/2)$, $(-1/2, -1/2, 1/2)$, and $(-1/2, 1/2, -1/2)$. These correspond to the nuclei ${}^3\text{He}$ and ${}^3\text{H}$.

40.18. The deuteron is a spin-1 object. The combination of spin-1/2 objects to produce spin-1 is symmetric: "up-up", the symmetric combination of "up-down" and "down-up", and "down-down". As fermions, the two nucleons making up the deuteron must have anti-symmetric total wave function. The iso-spin portion of the wave function is anti-symmetric; the neutron and proton have opposite iso-spins, with no companion parallel iso-spin states (no di-proton or di-neutron). With the spins in a symmetric combination, the spatial portion of the wave function is forced to be asymmetric. This enhances the probability density at the origin, i.e. zero separation. It puts the nucleons closer together, on average, than the alternative. This enables the nuclear force between the nucleons to produce a bound state. A spin-zero deuteron, with an anti-symmetric spin wave function, must have an anti-symmetric spatial wave function as well. This would keep the nucleons farther apart on average (see Problem 38.4). The nuclear force between them is not strong enough to produce a bound state in this case. As such, the spin-1 deuteron is only weakly bound.

40.19. ${}^{39}\text{Ar}$ has $Z = 18$. In β^- decay, a neutron gets converted into a proton, an electron and an antineutrino:
$${}^{39}_{18}\text{Ar} \rightarrow {}^{39}_{19}\text{X} + e^- + \overline{\nu}_e$$
Therefore, the new isotope is ${}^{39}\text{K}$.

40.20. A neutron star is the core of a star that has undergone a supernova explosion, leaving only the core collapsed to nuclear density. An ordinary star exists in a hydrostatic equilibrium between gas pressure and its own gravity. A neutron star exists in an equilibrium between "degeneracy pressure" (pressure arising from the Pauli exclusion principle) and its own gravity. Neutrons fill every available quantum state from the ground state up to their Fermi surface. The neutron star contains just enough protons so that the Fermi surface for protons is high enough to make the decay of a neutron into a proton (which must go into the next available energy level) and an electron energetically unfavorable. Hence, it is energetically favorable for surplus protons to combine with electrons to form neutrons. The number of electrons is just that required for overall charge neutrality. This is, of course, a highly simplified description. To understand the structure of neutron stars in detail, one must take into account nuclear physics, elementary particle physics, general relativity, fluid dynamics and even superfluidity.

40.21. Alpha decay emits a ${}^{4}_{2}\text{He}$ atom. Thus, the decay equation is ${}^{157}_{72}\text{Hf} \to {}^{4}_{2}\text{He} + {}^{153}_{70}\text{X}.$ The daughter nucleus is $\text{Yb}\left(A = 153, Z = 70\right).$

Exercises

40.22. For ${}^{235}\text{U},$ the mass number is A = 235. Since $R(A) = R_0 A^{1/3},$ the volume is:

$$V = \frac{4\pi R(A)^3}{3} = \frac{4\pi R_0^3 A}{3} = \frac{4\pi}{3}\left(1.12 \cdot 10^{-15} \text{ m}\right)^3 \left(235\right) = 1.38 \cdot 10^{-42} \text{ m}^3.$$

40.23. In general, binding energy is given by

$$B\left(N,Z\right) = \left[Zm(0,1) + Nm_{\text{n}} - m\left(N,Z\right)\right]c^2,$$

where $m(0,1) = 1.007825032$ u, $m_{\text{n}} = 1.008664916$ u and $1\text{ u} = 931.4940 \text{ MeV}/c^2.$ The choice of how to round the values is arbitrary. Round to the nearest integer.

(a) ${}^{7}\text{Li}$ has N = 4, Z = 3 and m(4,3) = 7.0160045 u, so:

$$\begin{aligned} B\left(4,3\right) &= \left[3\left(1.007825032\right) + 4\left(1.008664916\right) - \left(7.0160045\right)\right]c^2\left(931.4940 \text{ MeV}/c^2\right) \\ &= 39 \text{ MeV}. \end{aligned}$$

(b) ${}^{12}\text{C}$ has N = 6, Z = 6 and m(6,6) = 12.000000 u, so:

$$\begin{aligned} B\left(6,6\right) &= \left[6\left(1.007825032\right) + 6\left(1.008664916\right) - \left(12.000000\right)\right]c^2\left(931.4940 \text{ MeV}/c^2\right) \\ &= 92 \text{ MeV}. \end{aligned}$$

(c) ${}^{56}\text{Fe}$ has N = 30, Z = 26 and m(30,26) = 55.93493748 u, so:

$$\begin{aligned} B\left(30,26\right) &= \left[30\left(1.007825032\right) + 26\left(1.008664916\right) - \left(55.93493748\right)\right]c^2\left(931.4940 \text{ MeV}/c^2\right) \\ &= 489 \text{ MeV}. \end{aligned}$$

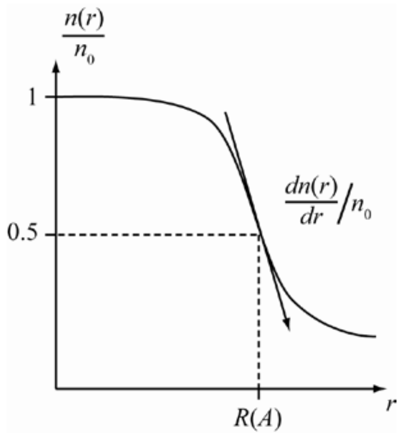
(d) ${}^{85}\text{Rb}$ has N = 48, Z = 37 and m(48,37)= 84.91178974 u, so:

$$\begin{aligned} B\left(48,37\right) &= \left[48\left(1.007825032\right) + 37\left(1.008664916\right) - \left(84.91178974\right)\right]c^2\left(931.4940 \text{ MeV}/c^2\right) \\ &= 731 \text{ MeV}. \end{aligned}$$

40.24. When in standard notation, ${}^{134}_{54}\text{Xe}$ is the same as ${}^{A}_{Z}\text{X},$ where A is the number of nucleons, Z is the number of protons and electrons, and $A - Z$ is the number of neutrons. Therefore, there are 134 nucleons, 54 protons and electrons, and $134 - 54 = 80$ neutrons.

40.25. **THINK:** Since the derivative is performed when $r = R(A)$, the term in the exponential will disappear, meaning the result should be independent of the number of nucleons, *A*.

SKETCH:



RESEARCH: The Fermi function is given by: $n(r) = \dfrac{n_0}{1+e^{(r-R(A))/a}}$.

SIMPLIFY: The derivative with respect to *r*, when $r = R_A$ is:

$$\left[\frac{dn(r)/n_0}{dr}\right]_{r=R(A)} = \left[\frac{d}{dr}\left(1+e^{(r-R(A))/a}\right)^{-1}\right]_{r=R(A)} = \left[-\left(1+e^{(r-R(A))/a}\right)^{-2}\frac{1}{a}e^{(r-R(A))/a}\right]_{r=R(A)}$$

$$= \left[\frac{-e^{(r-R(A))/a}}{a\left(1+e^{(r-R(A))/a}\right)^2}\right]_{r=R(A)} = \frac{-e^0}{a\left(1+e^0\right)^2} = -\frac{1}{4a}.$$

CALCULATE: $\left[\dfrac{dn(r)/n_0}{dr}\right]_{r=R(A)} = -\dfrac{1}{4(0.54\text{ fm})} = -0.46296\text{ fm}^{-1}$

ROUND: To two significant figures, the relative change in density at the nuclear surface is

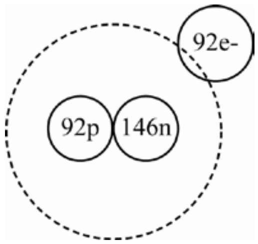
$$\left[\frac{dn(r)/n_0}{dr}\right]_{r=R(A)} = -0.46\text{ fm}^{-1}.$$

DOUBLE-CHECK: The number should decrease with increasing *r*, so a negative value for the rate change is appropriate.

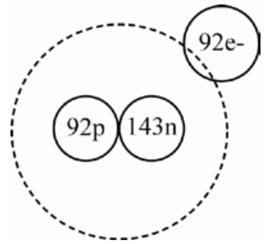
40.26. **THINK:** ${}^{238}_{92}\text{U}$ has 92 protons and electrons, each of which have a mass of $m(0,1) = 1.007825032\text{ u}$, and 146 neutrons each with a mass of $m_\text{n} = 1.008664916\text{ u}$. The overall atomic mass is $m(146,92) = 238.0507826\text{ u}$. ${}^{235}_{92}\text{U}$ has 92 protons and electrons as well, but has only 143 neutrons and has a total atomic mass of $m(143,92) = 235.0439299\text{ u}$.

SKETCH:

(a)



(b)



RESEARCH: The binding energy is given by $B(N,Z)=\left[Zm(0,1)+Nm_{\text{n}}-m(N,Z)\right]c^2$.

SIMPLIFY: Not applicable.

CALCULATE:

(a) $B(146,92)=\left[92(1.007825032)+146(1.008664916)-(238.0507826)\right]c^2\left(931.4940\text{ MeV}/c^2\right)$
$=1.80169\text{ GeV}$

(b) $B(143,92)=\left[92(1.007825032)+143(1.008664916)-(235.0439299)\right]c^2\left(931.4940\text{ MeV}/c^2\right)$
$=1.78386\text{ GeV}$

ROUND: Rounding to the nearest hundredth is sufficient to contrast the results.

(a) $B(146,92)=1.80\text{ GeV}$

(b) $B(143,92)=1.78\text{ GeV}$

The binding energy of ^{238}U is larger, so it is more stable.

DOUBLE-CHECK: The half-life of ^{238}U is 4.5 billion years, and for ^{235}U it is 700 million years. Since the half-life of ^{238}U is greater, it is more stable than ^{235}U.

40.27. In β^- decay, the atomic number, *Z*, increases by one, and an electron and electron anti-neutrino is emitted.

(a) $^{60}_{27}\text{Co}\rightarrow{}^{60}_{28}\text{Ni}+e^-+\overline{\nu}_e$

(b) $^{3}_{1}\text{H}\rightarrow{}^{3}_{2}\text{He}+e^-+\overline{\nu}_e$

(c) $^{14}_{6}\text{C}\rightarrow{}^{14}_{7}\text{N}+e^-+\overline{\nu}_e$

40.28. Each α decay emits a $^{4}_{2}\text{He}$ atom, so the atomic number, *Z*, decreases by two and the daughter nucleus has four fewer nucleons.

(a) $^{212}_{86}\text{Rn}\rightarrow{}^{208}_{84}\text{Po}+{}^{4}_{2}\text{He}$

(b) $^{241}_{95}\text{Am}\rightarrow{}^{237}_{93}\text{Np}+{}^{4}_{2}\text{He}$

40.29. The decay of $^{14}_{6}\text{C}$ via β^- decay is given by: $^{14}_{6}\text{C}\rightarrow{}^{14}_{7}\text{N}+e^-+\overline{\nu}_e$. The Q-value of this reaction is $Q=m(8,6)-m(7,7)-m_{\text{e}}$. The energy released is then $E=|Q|c^2$. Substituting the expression for Q gives:

$$E=\left|(14.0032420)-(14.003074)-\left(5.485799\cdot10^{-4}\right)\right|\left(931.4940\text{ MeV}/c^2\right)c^2=0.352\text{ MeV}.$$

40.30. (a) When the isotope goes through three half-lives, there is $(1/2)^3=1/8$ of the original amount remaining. Therefore, in t = 5.0 h, three half-lives have occurred. Therefore,

$$t=3t_{1/2}\ \Rightarrow\ t_{1/2}=\frac{t}{3}=\frac{(5.0\text{ h})}{3}=1.667\text{ h}=1.7\text{ h}.$$

(b) The mean lifetime, τ, is related to the half-life by:

$$t_{1/2} = \tau \ln 2 \Rightarrow \tau = \frac{t_{1/2}}{\ln 2} = \frac{(1.667 \text{ h})}{\ln 2} = 2.4 \text{ h}.$$

40.31. Since the isotope decays to 1/8 its original amount in $t' = 5.00$ h, it goes through three half-lives. Therefore the equation describing the amount of isotope remaining after *t* hours is

$$N(t) = N_0 e^{-\lambda t} = N_0 e^{-\ln(2)t/t_{1/2}} = N_0 \left(\frac{1}{2}\right)^{t/t_{1/2}}$$

$$N(t) = 0.900 N_0 = N_0 \left(\frac{1}{2}\right)^{t/t_{1/2}} \Rightarrow 0.900 = \left(\frac{1}{2}\right)^{t/t_{1/2}} \Rightarrow t = \frac{\ln(0.900)}{\ln(1/2)} t_{1/2} = \frac{\ln(0.900)}{\ln(1/2)} (5.00 \text{ h}/3) = 0.253 \text{ h}$$

40.32. The decay constant is:

$$\lambda = \frac{\ln 2}{t_{1/2}} = \frac{\ln 2}{(1600 \text{ yr})} = 4.3 \cdot 10^{-4} \text{ yr}^{-1}.$$

40.33. The mass of ^{228}Th is $m = 1.00$ g, while the molar mass is $m(138, 90) = 228.0287411$ g/mol. Therefore, the number of ^{228}Th atoms present is:

$$N = \frac{m}{m(138,90)} N_A = \left(\frac{1.00 \text{ g}}{228.0287411 \text{ g/mol}}\right) 6.022 \cdot 10^{23} \text{ atoms/mol} = 2.6409 \cdot 10^{21} \text{ atoms}.$$

The measured activity is $A' = 75$ counts/day. Since only 10.0% of all decays are picked up, the actual activity is ten times greater, therefore, $A = 10A' = 750$ decays/day. The activity is also given by:

$$A = \lambda N \Rightarrow \lambda = \frac{A}{N} = \frac{(750 \text{ decays/day})}{(2.6409 \cdot 10^{21} \text{ atoms})} = 2.8399 \cdot 10^{-19} \text{ day}^{-1}.$$

Therefore, the lifetime is:

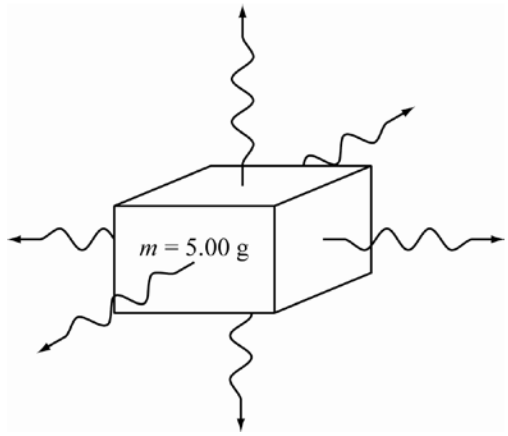
$$\tau = \frac{1}{\lambda} = \frac{1}{(2.8399 \cdot 10^{-19} \text{ day}^{-1})} = 3.5212 \cdot 10^{18} \text{ days} \left(\frac{1 \text{ yr}}{365 \text{ days}}\right) = 9.65 \cdot 10^{15} \text{ yr}.$$

40.34. The half-life is $t_{1/2} = 10$ min, and there are initially $N_0 = 10^{11}$ atoms. In 100 min, 10 half-lives have passed and in 200 min, 20 half-lives have passed. Therefore, the number of alpha particles emitted over this time frame is:

$$\Delta N = N(t = 10t_{1/2}) - N(t = 20t_{1/2}) = N_0 (1/2)^{10} - N_0 (1/2)^{20} = 10^{11} \left[(1/2)^{10} - (1/2)^{20}\right] = 10^8 \text{ particles}.$$

40.35. **THINK:** Carbon-14 has a half-life of $t_{1/2} = 5730$ yr. For a piece of wood, the radioactive decay of ^{14}C follows an exponential decay law, while the number of ^{12}C isotopes stays constant in time because this isotope is stable. Since the ratio of the number of ^{14}C atoms to the number of ^{12}C stays constant until the intake of ^{14}C ceases (when the tree died), the initial amount of ^{14}C can be found. It can be assumed that ^{12}C and ^{14}C comprise all of the mass of the wood, $m = 5.00$ g. Even though the wood was cut on January 1, 1700, the actual date today is not important to determine the activity today. It can be easily approximated by simply using the year associated with the date of the measurement (i.e. 2010). This is because the half-life is large (on a year scale), so being off by a few months (or even a full year) would only result in an error of less than 1%.

SKETCH:



RESEARCH: The number of atoms in a given mass is $N = mN_A / M$, where M is the molar mass. The activity of a material is $A = \lambda N$, where the decay constant is $\lambda = \ln 2 / t_{1/2}$. The specific activity is $S_A = A / m\left({}^{14}\text{C}\right)$. The radioactive decay law is given by $N(t) = N_0 e^{-\lambda t}$. As stated in the text, the initial ratio of ${}^{14}\text{C}$ to ${}^{12}\text{C}$ atoms is $r = N_0\left({}^{14}\text{C}\right) / N\left({}^{12}\text{C}\right) = 1.20 \cdot 10^{-12}$.

SIMPLIFY:

(a) The number of ${}^{14}\text{C}$ atoms per gram is given by:

$$N\left({}^{14}\text{C}\right) = \frac{m\left({}^{14}\text{C}\right)N_A}{M\left({}^{14}\text{C}\right)} \Rightarrow \frac{N\left({}^{14}\text{C}\right)}{m\left({}^{14}\text{C}\right)} = \frac{N_A}{M\left({}^{14}\text{C}\right)}.$$

The specific activity of ${}^{14}\text{C}$ is then:

$$S_A = \frac{A}{m\left({}^{14}\text{C}\right)} = \frac{\lambda N\left({}^{14}\text{C}\right)}{m\left({}^{14}\text{C}\right)} = \frac{\ln 2}{t_{1/2}} \frac{N\left({}^{14}\text{C}\right)}{m\left({}^{14}\text{C}\right)} = \frac{\ln 2 N_A}{t_{1/2} M\left({}^{14}\text{C}\right)}.$$

(b) To find the initial activity of a piece of wood with $m = 5.00$ g, the mass of ${}^{14}\text{C}$ present in the piece of wood needs to be found:

$$m\left({}^{12}\text{C}\right) = \left(\frac{N\left({}^{12}\text{C}\right)M\left({}^{12}\text{C}\right)}{N_0\left({}^{14}\text{C}\right)M\left({}^{14}\text{C}\right)}\right) m\left({}^{14}\text{C}\right) \text{ and } m\left({}^{12}\text{C}\right) + m\left({}^{14}\text{C}\right) = m$$

$$m\left({}^{14}\text{C}\right) = \frac{m}{\left(\frac{N\left({}^{12}\text{C}\right)M\left({}^{12}\text{C}\right)}{N_0\left({}^{14}\text{C}\right)M\left({}^{14}\text{C}\right)}\right) + 1}$$

Then, the initial activity is given by:

$$A = S_A m\left({}^{14}\text{C}\right).$$

(c) The time passed in years is $\Delta t = 2010 \text{ yr} - 1700 \text{ yr} = 310. \text{ yr}$. Treating the year 1700 as $t = 0$, the change in the number of atoms, i.e. disintegrations is:

$$\Delta N\left({}^{14}\text{C}\right) = N_0\left({}^{14}\text{C}\right) - N\left({}^{14}\text{C}\right) = N_0\left({}^{14}\text{C}\right)\left(1 - e^{-\lambda t}\right) = \frac{m\left({}^{14}\text{C}\right)N_A}{M\left({}^{14}\text{C}\right)}\left(1 - e^{-t \ln 2 / t_{1/2}}\right).$$

CALCULATE:

(a) $$S_A = \frac{\ln 2\left(6.022\cdot 10^{23}\text{ atoms/mol}\right)}{\left(5730\text{ yr}\right)\left(365.25\text{ days/yr}\right)\left(24\text{ hr/day}\right)\left(3600\text{ s/hr}\right)\left(14.0032420\text{ g/mol}\right)}$$
$$= 1.64846\cdot 10^{11}\text{ disint/}\left(\text{g s}\right)$$
$$= 1.64846\cdot 10^{11}\text{ Bq/g}$$
$$= 4.4553\text{ Ci/g}$$

(b) $$m\left({}^{14}\text{C}\right) = \frac{\left(5.00\text{ g}\right)}{\left(\dfrac{\left(12.0000000\text{ g}\right)}{\left(1.20\cdot 10^{-12}\right)\left(14.0032420\text{ g}\right)}\right)+1} = 7.001621\cdot 10^{-12}\text{ g}$$

$$A = \left(1.64846\cdot 10^{11}\text{ Bq/g}\right)\left(7.001621\cdot 10^{-12}\text{ g}\right) = 1.15419\text{ Bq}$$

(c) $$\Delta N\left({}^{14}\text{C}\right) = \frac{\left(7.001621\cdot 10^{-12}\text{ g}\right)\left(6.022\cdot 10^{23}\text{ atoms/mol}\right)}{\left(14.0032420\text{ g/mol}\right)}\left(1-e^{-\left(310.\text{ yr}\right)\ln 2/5730\text{ yr}}\right) = 1.108219\cdot 10^{10}\text{ disint}$$

ROUND:

(a) $S_A = 1.65\cdot 10^{11}\text{ disint/}\left(\text{g s}\right) = 1.65\cdot 10^{11}\text{ Bq/g} = 4.46\text{ Ci/g}$

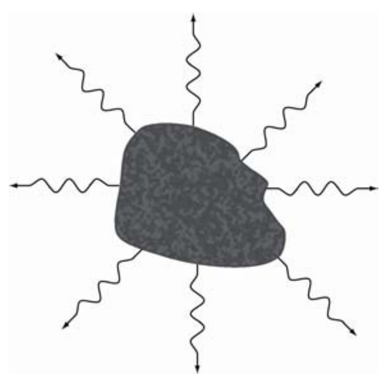
(b) $A = 1.15\text{ Bq}$

(c) $\Delta N\left({}^{14}\text{C}\right) = 1.11\cdot 10^{10}\text{ disint}$

DOUBLE-CHECK: These are reasonable results for radioactive decay over a long time period.

40.36. **THINK:** The radioactive decay of ${}^{14}\text{C}$ follows an exponential decay law, while the number of ${}^{12}\text{C}$ isotopes stays constant in time because this isotope is stable. Since the ratio of the number of ${}^{14}\text{C}$ atoms to the number of ${}^{12}\text{C}$ stays constant until the intake of ${}^{14}\text{C}$ ceases (when the tree died), the initial amount of ${}^{14}\text{C}$ can be found. It can be assumed that ${}^{12}\text{C}$ comprises all of the mass of the charcoal; that is, $m = m\left({}^{12}\text{C}\right) = 7.2\text{ g}$. The activity, $A = 0.42\text{ Bq}$, can be used along with the half-life, $t_{1/2} = 1.81\cdot 10^{3}\text{ s}$, to determine the current number of ${}^{14}\text{C}$ atoms. Using all of this information will provide an approximate age for the site.

SKETCH:



RESEARCH: The exponential decay law for the number of atoms remaining as a function of time is given by

$$N(t) = N_0 e^{-\lambda t},$$

where $\lambda = \ln 2 / t_{1/2}$. The activity of ${}^{14}\text{C}$ is given by $A = \lambda N\left({}^{14}\text{C}\right)$. As stated in the text, the initial number ratio of ${}^{14}\text{C}$ to ${}^{12}\text{C}$ atoms is $r = N_0\left({}^{14}\text{C}\right) / N\left({}^{12}\text{C}\right) = 1.20 \cdot 10^{-12}$. Therefore, the number of initial ${}^{14}\text{C}$ atoms is $N_0\left({}^{14}\text{C}\right) = rN\left({}^{12}\text{C}\right)$. The number of ${}^{12}\text{C}$ atoms is given by

$$N\left({}^{12}\text{C}\right) = \frac{m\left({}^{12}\text{C}\right)}{M\left({}^{12}\text{C}\right)} N_\text{A},$$

where $M\left({}^{12}\text{C}\right)$ is the molar mass of ${}^{12}\text{C}$.

SIMPLIFY: The decay for ${}^{14}\text{C}$ is

$$N\left({}^{14}\text{C}\right) = N_0\left({}^{14}\text{C}\right)e^{-\lambda t} = rN\left({}^{12}\text{C}\right)e^{-\lambda t}.$$

Simplifying and solving for *t* gives:

$$\frac{A}{\lambda} = \frac{rm\left({}^{12}\text{C}\right)N_\text{A}e^{-\lambda t}}{M\left({}^{12}\text{C}\right)} \Rightarrow \frac{At_{1/2}}{\ln 2} = \frac{rm\left({}^{12}\text{C}\right)N_\text{A}e^{-\ln 2t/t_{1/2}}}{M\left({}^{12}\text{C}\right)} \Rightarrow e^{-\ln 2t/t_{1/2}} = \frac{AM\left({}^{12}\text{C}\right)t_{1/2}}{rm\left({}^{12}\text{C}\right)N_\text{A}\ln 2}$$

$$t = -\frac{t_{1/2}}{\ln 2}\ln\left(\frac{AM\left({}^{12}\text{C}\right)t_{1/2}}{rm\left({}^{12}\text{C}\right)N_\text{A}\ln 2}\right).$$

CALCULATE:

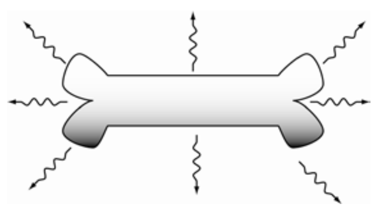
$$t = -\frac{\left(1.81 \cdot 10^{11}\text{ s}\right)}{\ln 2}\ln\left[\frac{\left(0.42\text{ Bq}\right)\left(12.0000\text{ g/mol}\right)\left(1.81 \cdot 10^{11}\text{ s}\right)}{\left(1.20 \cdot 10^{-12}\right)\left(7.2\text{ g}\right)\left(6.022 \cdot 10^{23}\text{ atoms/mol}\right)\ln 2}\right] = 3.5894 \cdot 10^{11}\text{ s} = 11382\text{ yr}$$

ROUND: To two significant figures, $t = 11000$ yr.

DOUBLE-CHECK: 11000 years is well within the history of mankind and is still old enough to warrant an excavation site. Therefore, the result makes sense.

40.37. **THINK:** The specific activity decays exponentially. The initial specific activity is $S_{A,0} = 0.270$ Bq/g, and the current specific activity is $S_A = 0.268$ Bq/g. The half-life of ${}^{14}\text{C}$ is $t_{1/2} = 5.73 \cdot 10^3$ yr.

SKETCH:



RESEARCH: The specific activity decays exponentially as $S_A(t) = S_{A,0}e^{-\lambda t}$, the decay constant is given by $\lambda = \ln 2 / t_{1/2}$.

SIMPLIFY: Solving for time yields:

$$S_A = S_{A,0}e^{-\lambda t} \Rightarrow \ln\left(\frac{S_A}{S_{A,0}}\right) = -\frac{t\ln 2}{t_{1/2}} \Rightarrow t = \frac{t_{1/2}}{\ln 2}\ln\left(\frac{S_{A,0}}{S_A}\right).$$

CALCULATE: $t = \dfrac{\left(5.73 \cdot 10^3\text{ yr}\right)}{\ln 2}\ln\left(\dfrac{0.270\text{ Bq/g}}{0.268\text{ Bq/g}}\right) = 61.462\text{ yr}$

ROUND: To three significant figures, $t = 61.5$ yr. The victim was murdered in 1946 or 1947.

DOUBLE-CHECK: While this is an old crime, it still warrants an investigation; therefore, the result is reasonable. As the question does not indicate when in 2008 the remains were found, the answer might be 1946 or 1947.

40.38. **THINK:** For part (a), the energy required for the combustion is the heat of combustion of hydrogen, $\Delta E = 285.83$ kJ/mol. This is the change in energy for this reaction, which is related to its change in mass. For the remaining processes, the difference in their atomic masses is all that needs to be considered.

SKETCH: Not applicable.

RESEARCH: The change in mass is converted to energy via $\Delta m = \Delta E / c^2$. The fractional change in mass that is converted to energy is $\Delta m / m$.

SIMPLIFY:

(a) For every mole of H_2, the combustion uses a half mole of O_2. The initial molar mass of the reaction is then $M = M(H_2) + M(O_2)/2$. Therefore,

$$\frac{\Delta m}{m} = \frac{\Delta E / c^2}{M(H_2) + M(O_2)/2}.$$

(b) The fractional change in mass for this reaction is given by:

$$\frac{\Delta m}{m} = 1 - \frac{m(53,36) + m(86,56) + 5m_{\text{n}}}{m(143,92) + m_{\text{n}}}.$$

(c) The fractional change in mass for this reaction is given by:

$$\frac{\Delta m}{m} = 1 - \frac{m(3,4) + m_{\text{n}}}{m(3,3) + m(1,1)}.$$

(d) The fractional change in mass for this reaction is given by:

$$\frac{\Delta m}{m} = 1 - \frac{m_{\text{p}} + m_{\text{e}} + m_{\bar{\nu}_e}}{m_{\text{n}}} \approx 1 - \frac{m_{\text{p}} + m_{\text{e}}}{m_{\text{n}}}.$$

The electron antineutrino $\bar{\nu}_e$ has virtually no mass.

(e) The fractional change in mass for this reaction is given by:

$$\frac{\Delta m}{m} = 1 - \frac{m_{\text{e}} + m_{\nu_\mu} + m_{\bar{\nu}\text{e}}}{m_\mu} \approx 1 - \frac{m_{\text{e}} + m_{\nu_\mu}}{m_\mu}.$$

The electron antineutrino, $\bar{\nu}_e$, has virtually no mass, but the muon neutrino, m_{ν_μ}, has a non-negligible mass.

(f) Since $\gamma -$rays have no mass, all the mass is converted to energy. Therefore,

$$\frac{\Delta m}{m} = \frac{m_{e^-} + m_{e^+}}{m_{e^-} + m_{e^+}} = 1.$$

CALCULATE:

(a) $$\frac{\Delta m}{m} = \frac{(285.83 \text{ kJ/mol})/(3.00 \cdot 10^8 \text{ m/s})^2}{2.015894 \text{ g/mol} + (31.9988 \text{ g/mol})/2} = 1.76288 \cdot 10^{-13}$$

(b) $$\frac{\Delta m}{m} = 1 - \frac{(88.917631 \text{ u}) + (141.916454 \text{ u}) + 5(1.008664916 \text{ u})}{(235.0439299 \text{ u}) + (1.008664916 \text{ u})} = 7.421 \cdot 10^{-4}$$

(c) $$\frac{\Delta m}{m} = 1 - \frac{(7.0169292 \text{ u}) + (1.008664916 \text{ u})}{(6.0151223 \text{ u}) + (2.014101778 \text{ u})} = 4.521 \cdot 10^{-4}$$

(d) $$\frac{\Delta m}{m} = 1 - \frac{(1.007276467 \text{ u}) + (5.485799 \cdot 10^{-4} \text{ u})}{(1.008664916 \text{ u})} = 8.327 \cdot 10^{-4}$$

(e) $$\frac{\Delta m}{m} = 1 - \frac{(5.485799 \cdot 10^{-4} \text{ u}) + (1.825025 \cdot 10^{-7} \text{ u})}{(0.1134290 \text{ u})} = 0.99516$$

(f) $\dfrac{\Delta m}{m}=1$

ROUND: Rounding to three significant figures is sufficient to compare the results:

(a) $\dfrac{\Delta m}{m}=1.76\cdot 10^{-13}$

(b) $\dfrac{\Delta m}{m}=7.42\cdot 10^{-4}$

(c) $\dfrac{\Delta m}{m}=4.52\cdot 10^{-4}$

(d) $\dfrac{\Delta m}{m}=8.33\cdot 10^{-4}$

(e) $\dfrac{\Delta m}{m}=0.995$

(f) $\dfrac{\Delta m}{m}=1$ (exact)

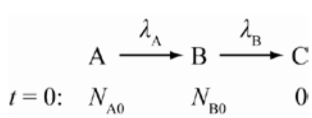
DOUBLE-CHECK: Since reaction (a) does not split up any atoms and simply recombines them, very little mass energy is lost. For reactions $(\text{b})\rightarrow(\text{d}),$ the atoms are split up quite considerably, so binding energies would change by an appreciable amount, thus releasing energy or causing a noticeable change in mass. For reaction (e), the upper limit value of m_{ν_μ} was used. Anything smaller and the value of $\Delta m/m$ gets closer to one, which makes sense since the mass of an electron is about 200 times smaller than m_{ν_μ}, meaning:

$$\frac{\Delta m}{m}=\frac{m_{\nu_\mu}-\left(m_{\nu_\mu}/200\right)}{m_\mu}\approx 0.995.$$

Obviously for reaction (f), all of the mass is converted so it is the most efficient. For these reasons, the results are reasonable.

40.39. **THINK:** $N_\text{B}(t)$ decreases via decay $\text{B}\rightarrow\text{C},$ but is then replenished via $\text{A}\rightarrow\text{B}.$ Thus, the total activity of B is the difference between the activity of A becoming B, and the activity of B becoming C. Initially, both A and B have some nuclei.

SKETCH:



RESEARCH: The activity of each nuclei at any time is $\left|dN_i(t)/dt\right|=-\lambda_i N_i(t).$ The number of atoms present at any time is $N_i(t)=N_{i0}e^{-\lambda_i t}$, where subscript *i* denotes either A, B or C.

SIMPLIFY: The total activity of atom B is:

$$\frac{dN_\text{B}(t)}{dt}=-\lambda_\text{B}N_\text{B}(t)+\lambda_\text{A}N_\text{A}(t)=-\lambda_\text{B}N_\text{B}(t)+\lambda_\text{A}N_{\text{A}0}e^{-\lambda_\text{A}t}\;\Rightarrow\; dN_\text{B}(t)=\lambda_\text{A}N_{\text{A}0}e^{-\lambda_\text{A}t}dt-\lambda_\text{B}N_\text{B}(t)dt$$

$$\Rightarrow\; dN_\text{B}(t)e^{\lambda_\text{B}t}+\lambda_\text{B}N_\text{B}(t)e^{\lambda_\text{B}t}dt=\lambda_\text{A}N_{\text{A}0}e^{(\lambda_\text{B}-\lambda_\text{A})t}dt.$$

Consider $N_\text{B}(t)e^{\lambda_\text{B}t}$:

$$\frac{d}{dt}\left(N_\text{B}(t)e^{\lambda_\text{B}t}\right)=\frac{dN_\text{B}(t)}{dt}e^{\lambda_\text{B}t}+N_\text{B}(t)\lambda_\text{B}e^{\lambda_\text{B}t}\;\Rightarrow\; d\left(N_\text{B}(t)e^{\lambda_\text{B}t}\right)=dN_\text{B}(t)e^{\lambda_\text{B}t}+\lambda_\text{B}N_\text{B}(t)e^{\lambda_\text{B}t}dt.$$

Therefore,

$$\int_{N_{\text{B},0}}^{N_{\text{B}}(t)} d\left(N_{\text{B}}(t)e^{\lambda_{\text{B}}t}\right)=\int_0^t \lambda_{\text{A}}N_{\text{A0}}e^{(\lambda_{\text{B}}-\lambda_{\text{A}})t}dt$$

$$N_{\text{B}}(t)e^{\lambda_{\text{B}}t}-N_{\text{B0}}=\frac{\lambda_{\text{A}}}{\lambda_{\text{B}}-\lambda_{\text{A}}}N_{\text{A0}}e^{(\lambda_{\text{B}}-\lambda_{\text{A}})t}-\frac{\lambda_{\text{A}}}{\lambda_{\text{B}}-\lambda_{\text{A}}}N_{\text{A0}}$$

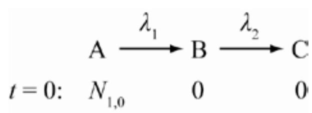
$$N_{\text{B}}(t)=\frac{\lambda_{\text{A}}}{\lambda_{\text{B}}-\lambda_{\text{A}}}N_{\text{A0}}e^{-\lambda_{\text{A}}t}+\left(N_{\text{B0}}-\frac{\lambda_{\text{A}}}{\lambda_{\text{B}}-\lambda_{\text{A}}}N_{\text{A0}}\right)e^{-\lambda_{\text{B}}t}.$$

CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: In general, the population of atom B would decrease. However, if $\lambda_{\text{A}} >> \lambda_{\text{B}}$, then B is replenished by A faster than B can decay to C. The solution when $\lambda_{\text{A}} >> \lambda_{\text{B}}$, simplifies to:

$$N_{\text{B}}(t)\approx -N_{\text{A0}}e^{-\lambda_{\text{A}}t}+N_{\text{B0}}e^{-\lambda_{\text{B}}t}+N_{\text{A0}}e^{-\lambda_{\text{B}}t}\approx\left(N_{\text{B0}}+N_{\text{A0}}\right)e^{-\lambda_{\text{B}}t}-N_{\text{A0}}e^{-\lambda_{\text{A}}t}.$$

The first term is the regular decay of B if all of A was instantly converted to B (which is true when $\lambda_{\text{A}} >> \lambda_{\text{B}}$). The second term is the number subtracting the actual number of atoms still in A. Since this approximation validates what is known should happen in the limit, the solution is reasonable.

40.40. **THINK:** $N_{\text{B}}(t)$ decreases via decay $B\rightarrow \text{C}$, but is then replenished via $\text{A}\rightarrow\text{B}$. Thus, the total activity of B is the difference between the activity of A becoming B, and the activity of B becoming C. Initially, only A nuclei are present and the daughter B nuclei are not present.
SKETCH:



RESEARCH: The activity of each nuclei at any time is $dN_i(t)/dt=-\lambda_iN_i(t)$. The number of nuclei at any time is $N_i(t)=N_{i0}e^{-\lambda_it}$, where subscript *i* denotes either 1 or 2.
SIMPLIFY:
(a) Since the nuclei of A have to decay through both λ_1 and λ_2 to get to C, and C increases as A decreases, then $N_3(t)=N_{1,0}\left[1-e^{-(\lambda_1+\lambda_2)t}\right]$. The A nuclei decrease as $N_1(t)=N_{1,0}e^{-\lambda_1t}$. The total activity of nuclei B is:

$$\frac{dN_2(t)}{dt}=-\lambda_2N_2(t)+\lambda_1N_1(t)=-\lambda_2N_2(t)+\lambda_1N_{1,0}e^{-\lambda_1t}\ \Rightarrow\ dN_2(t)+\lambda_2N_2(t)dt=\lambda_1N_{1,0}e^{-\lambda_1t}dt$$

$$\Rightarrow\ dN_2(t)e^{\lambda_2t}+\lambda_2N_2(t)e^{\lambda_2t}dt=\lambda_1N_{1,0}e^{(\lambda_2-\lambda_1)t}dt.$$

Consider $N_2(t)e^{\lambda_2t}$:

$$\frac{d}{dt}\left(N_2(t)e^{\lambda_2t}\right)=\frac{dN_2(t)}{dt}e^{\lambda_2t}+N_2(t)\lambda_2e^{\lambda_2t}\ \Rightarrow\ d\left(N_2(t)e^{\lambda_2t}\right)=dN_2(t)e^{\lambda_2t}+\lambda_2N_2(t)e^{\lambda_2t}dt.$$

Therefore,

$$\int_0^{N_2(t)}d\left(N_2(t)e^{\lambda_2t}\right)=\int_0^t\lambda_1N_{1,0}e^{(\lambda_2-\lambda_1)t}dt\ \Rightarrow\ N_2(t)e^{\lambda_2t}=\frac{\lambda_1}{\lambda_2-\lambda_1}N_{1,0}e^{(\lambda_2-\lambda_1)t}-\frac{\lambda_1}{\lambda_2-\lambda_1}N_{1,0}$$

$$\Rightarrow\ N_2(t)=\frac{\lambda_1}{\lambda_2-\lambda_1}N_{1,0}\left(e^{-\lambda_1t}-e^{-\lambda_2t}\right).$$

The activity of B is then: $A_2(t)=\frac{\lambda_2\lambda_1}{\lambda_2-\lambda_1}N_{1,0}\left(e^{-\lambda_1t}-e^{-\lambda_2t}\right).$

(b) When $\lambda_2 \approx 10\lambda_1$, $e^{-\lambda_2 t}$ dies off faster than $e^{-\lambda_1 t}$, therefore,

$$N_2(t) \approx \frac{\lambda_1}{\lambda_2 - \lambda_1} N_{1,0} e^{-\lambda_1 t} = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(t) \Rightarrow \frac{N_1(t)}{N_2(t)} = \frac{\lambda_2 - \lambda_1}{\lambda_1}.$$

At any time, the ratio of atoms is constant. When $\lambda_2 \approx 100\lambda_1$, the result is the same as before, except also $\lambda_2 - \lambda_1 \approx \lambda_2$. Therefore,

$$\frac{N_1(t)}{N_2(t)} = \frac{\lambda_2}{\lambda_1} \Rightarrow \lambda_1 N_1(t) = \lambda_2 N_2(t) \Rightarrow \mathrm{A}_1(t) = \mathrm{A}_2(t).$$

The activities are the same.
CALCULATE: Not applicable.
ROUND: Not applicable.
DOUBLE-CHECK: At $t = 0$, $N_3(0) = 0$, $N_2(0) = 0$ and $N_1(0) = N_{1,0}$, which is expected. As $t \to \infty$, $N_3(t) = N_{1,0}$, $N_2(t) = N_1(t) = 0$, which is also expected. When $\lambda_2 >> \lambda_1$, as soon as an atom in A decays to B, it decays to C before another A can decay. Therefore, for every A decay, there is one B decay, in some time interval, so the activities are expected to be equal.

40.41. **THINK:** To determine the binding energy, multiply both sides of the Bethe-Weizsäcker formula by A. Since $A = Z + N$, if A is odd then either Z is odd and N is even or Z is even and N is odd. The most stable isotope for when $A = 117$ is for a Z where the binding energy is at a minimum. Therefore, once the quadratic equation is obtained, differentiate with respect to Z and set it equal to solve for Z.
SKETCH: Not applicable.
RESEARCH: The Bethe-Weizsäcker formula multiplied by A is:

$$B(N,Z) = a_{\mathrm{v}} A - a_{\mathrm{s}} A^{2/3} - a_{\mathrm{c}} \frac{Z^2}{A^{1/3}} - a_{\mathrm{a}} A \left(\frac{Z}{A} - \frac{1}{2} \right)^2 + a_{\mathrm{p}} \left[(-1)^N + (-1)^Z \right] A^{-1/2}.$$

SIMPLIFY: When Z is odd and N is even (or vice-versa), the last term becomes:

$$a_{\mathrm{p}} \left[(-1)^{\text{even}} + (-1)^{\text{odd}} \right] A^{-1/2} = a_{\mathrm{p}} (1-1) A^{-1/2} = 0.$$

Therefore, the equation is:

$$\begin{aligned} B(N,Z) &= a_{\mathrm{v}} A - a_{\mathrm{s}} A^{2/3} - a_{\mathrm{c}} Z^2 A^{-1/3} - a_{\mathrm{a}} A \left(\frac{Z^2}{A^2} - \frac{Z}{A} + \frac{1}{4} \right) \\ &= \left(a_{\mathrm{v}} A - a_{\mathrm{s}} A^{2/3} - \frac{a_{\mathrm{a}} A}{4} \right) + a_{\mathrm{a}} Z + \left(-\frac{a_{\mathrm{c}}}{A^{1/3}} - \frac{a_{\mathrm{a}}}{A} \right) Z^2 \equiv a + bZ + cZ^2. \end{aligned}$$

The minimum in $B(N,Z)$ is when:

$$\left[\frac{dB(N,Z)}{dZ} \right]_{A=117} = \left[\frac{d}{dZ} \left(a + bZ + cZ^2 \right) \right]_{A=117} = \left[b + 2cZ \right]_{A=117} = 0$$

$$\Rightarrow Z = \left[-\frac{b}{2c} \right]_{A=117} = \left[\frac{-a_{\mathrm{a}}}{2\left(-a_{\mathrm{c}} / A^{1/3} - a_{\mathrm{a}} / A \right)} \right]_{A=117}.$$

CALCULATE: The values for the constants are given in the text:

$$Z = \frac{-(92.86\text{ MeV})}{2\left(-(0.71\text{ MeV})/(117)^{1/3} - (92.86\text{ MeV})/(117) \right)} = 49.455$$

ROUND: Since the result must be an integer and rounded up, $Z = 50$.

DOUBLE-CHECK: This atom ($Z = 50$, $A = 117$) corresponds to $^{117}_{50}\mathrm{Sn}$, which does exist and is stable.

40.42. **THINK:** Plot S_n and S_{2n} for element Sn (Z = 50) versus neutron number, N. Since N is an integer, simply examining the plot will be sufficient to determine when each plot crosses into the negative. The following constants are given on page 13.47: $a_V = 15.85$ MeV, $a_s = 18.34$ MeV, $a_c = 0.71$ MeV, $a_a = 92.86$ MeV, $a_p = 11.46$ MeV. Use the fact that $A = N + Z$ to write the formulas that will be obtained in the RESEARCH step in terms of N and Z.

SKETCH: The plots will be in the CALCULATE step.

RESEARCH: The Bethe-Weizsäcker formula is given in Equation 40.37:

$$\frac{B(N,Z)}{A} = a_v - a_s A^{-1/3} - a_c \frac{Z^2}{A^{4/3}} - a_a \left(\frac{Z}{A} - \frac{1}{2}\right)^2 + a_p \frac{(-1)^Z + (-1)^N}{A^{3/2}}.$$

Page 1333 discusses the separation energy, S, required to separate some part of an isotope away from the remainder of the nucleus. It gives the formula $S = B(N_1 + N_2, Z_1 + Z_2) - B(N_1, Z_1) - B(N_2, Z_2)$. To compute S_n, let $N_2 = 1$, and $Z_2 = 0$. To compute S_{2n}, let $N_2 = 2$, and $Z_2 = 0$. As mentioned in Example 40.2, the binding energy of two neutrons is zero. Also, the binding energy for a single neutron is zero, as there is nothing to bind. This means $B(1,0) = B(2,0) = 0$.

SIMPLIFY: $S_n = B(N,Z) - B(N-1,Z) - B(1,0) = B(N,Z) - B(N-1,Z)$, and $S_{2n} = B(N,Z) - B(N-2,Z) - B(2,0) = B(N,Z) - B(N-2,Z)$. Now find expressions for $B(N,Z)$, $B(N-1,Z)$, and $B(N-2,Z)$.

$$\begin{aligned} B(N,Z) &= a_v A - a_s A^{2/3} - a_c \frac{Z^2}{A^{1/3}} - a_a \left(Z - \frac{A}{2}\right)^2 + a_p \frac{(-1)^Z + (-1)^N}{A^{1/2}} \\ &= a_v (Z+N) - a_s (Z+N)^{2/3} - a_c \frac{Z^2}{(Z+N)^{1/3}} - a_a \left(Z - \frac{(Z+N)}{2}\right)^2 + a_p \frac{(-1)^Z + (-1)^N}{(Z+N)^{1/2}} \end{aligned}$$

$$B(N-1,Z) = a_v (Z+N-1) - a_s (Z+N-1)^{2/3} - a_c \frac{Z^2}{(Z+N-1)^{1/3}} - a_a \left(Z - \frac{(Z+N-1)}{2}\right)^2 + a_p \frac{(-1)^Z + (-1)^{N-1}}{(Z+N-1)^{1/2}}$$

$$B(N-2,Z) = a_v (Z+N-2) - a_s (Z+N-2)^{2/3} - a_c \frac{Z^2}{(Z+N-2)^{1/3}} - a_a \left(Z - \frac{(Z+N-2)}{2}\right)^2 + a_p \frac{(-1)^Z + (-1)^N}{(Z+N-2)^{1/2}}$$

$$\begin{aligned} S_n(N,Z) &= B(N,Z) - B(N-1,Z) \\ &= \left[a_v (Z+N) - a_s (Z+N)^{2/3} - a_c \frac{Z^2}{(Z+N)^{1/3}} - a_a \left(Z - \frac{(Z+N)}{2}\right)^2 + a_p \frac{(-1)^Z + (-1)^N}{(Z+N)^{1/2}} \right] \\ &\quad - \left[a_v (Z+N-1) - a_s (Z+N-1)^{2/3} - a_c \frac{Z^2}{(Z+N-1)^{1/3}} - a_a \left(Z - \frac{(Z+N-1)}{2}\right)^2 + a_p \frac{(-1)^Z + (-1)^{N-1}}{(Z+N-1)^{1/2}} \right] \end{aligned}$$

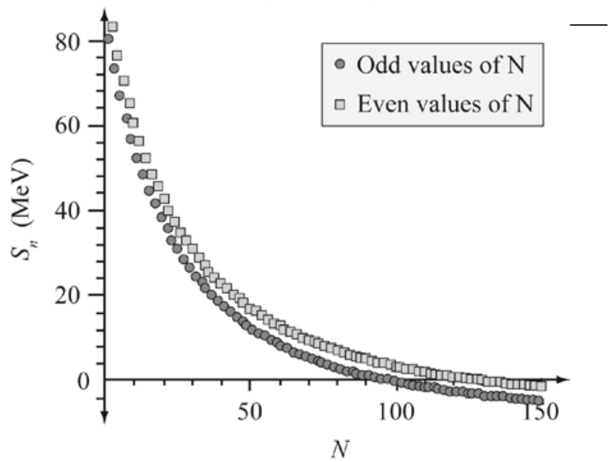
$$\begin{aligned} S_{2n}(N,Z) &= B(N,Z) - B(N-2,Z) \\ &= \left[a_v (Z+N) - a_s (Z+N)^{2/3} - a_c \frac{Z^2}{(Z+N)^{1/3}} - a_a \left(Z - \frac{(Z+N)}{2}\right)^2 + a_p \frac{(-1)^Z + (-1)^N}{(Z+N)^{1/2}} \right] \\ &\quad - \left[a_v (Z+N-2) - a_s (Z+N-2)^{2/3} - a_c \frac{Z^2}{(Z+N-2)^{1/3}} - a_a \left(Z - \frac{(Z+N-2)}{2}\right)^2 + a_p \frac{(-1)^Z + (-1)^N}{(Z+N-2)^{1/2}} \right] \end{aligned}$$

At this point, further algebraic simplification will only make the functions messier. Move on to the following step where the substitution of some values will give the explicit function.

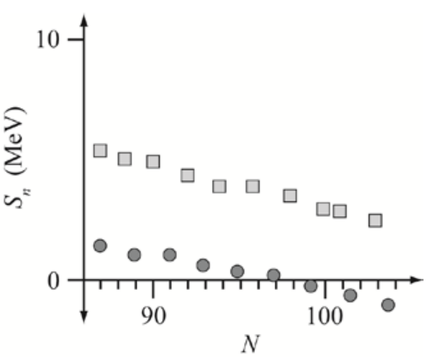
CALCULATE: These formulas are unpleasant, and clearly impossible to graph without the aid of a computer graphing utility.

$$S_n(N,50) = 15.85 - 18.34(50+N)^{2/3} - \frac{1775}{(50+N)^{1/3}} - 92.86\left(\frac{25-\frac{N}{2}}{50+N}\right)^2 + 11.46\frac{1+(-1)^N}{\sqrt{50+N}}$$

$$+18.34(49+N)^{2/3} + \frac{1775}{(49+N)^{1/3}} + 92.86\left(\frac{25.5-\frac{N}{2}}{49+N}\right)^2 - 11.46\frac{1+(-1)^{N+1}}{\sqrt{49+N}}$$

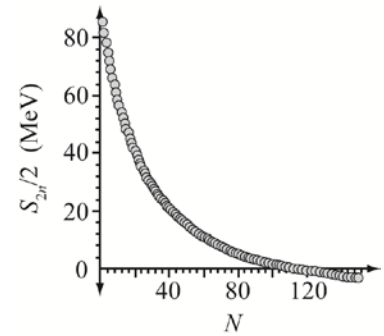


Here is the same function plotted again zoomed in on where it crosses the *x*-axis:

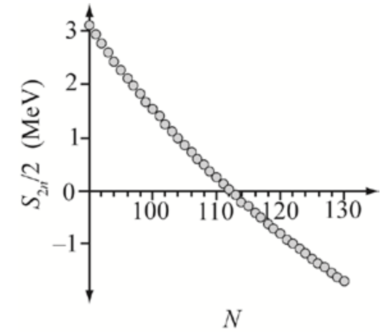


$$S_{2n}(N,50) = 31.70 - 18.34(50+N)^{2/3} - \frac{1775}{(50+N)^{1/3}} - 92.86\left(\frac{25-\frac{N}{2}}{50+N}\right)^2 + 11.46\frac{1+(-1)^N}{\sqrt{50+N}}$$

$$+18.34(48+N)^{2/3} + \frac{1775}{(48+N)^{1/3}} + 92.86\left(\frac{26-\frac{N}{2}}{48+N}\right)^2 - 11.46\frac{1+(-1)^N}{\sqrt{48+N}}$$



Again, the function is replotted, zoomed in on where it crosses the x-axis:



ROUND: S_n becomes negative for the first time at $N = 99$. S_{2n} becomes negative at $N = 113$.

DOUBLE-CHECK: As the number of neutrons in the atom increases, they help keep the atom together so that the binding energy decreases, which is evident in the plots. Also, when the separation energy is negative, energy is necessary to separate the atom.

40.43. Given that the power plant with $P = 1.50$ GW has an efficiency of $\varepsilon = 0.350$, the total power that is created is $P_0 = P/\varepsilon$. The energy that the plant produces in $\Delta t = 1$ day is given by $E = P_0\Delta t$. Since each ^{235}U reaction is $\Delta E = 200.$ MeV, the number of ^{235}U consumed per day is given by $N = E/\Delta E$. Since ^{235}U has a molar mass of $m_{\text{M}} = 235.0439299$ g/mol, the mass of ^{235}U consumed in one day is:

$$m = \frac{N}{N_{\text{A}}}m_{\text{M}} = \frac{Em_{\text{M}}}{\Delta EN_{\text{A}}} = \frac{P\Delta tm_{\text{M}}}{\varepsilon\Delta EN_{\text{A}}} = \frac{\left(1.50\cdot 10^{9}\text{ W}\right)\left(86400\text{ s/day}\right)\left(235.0439299\text{ g/mol}\right)}{\left(0.350\right)\left(200.\cdot 10^{6}\text{ eV}\right)\left(1.602\cdot 10^{-19}\text{ J/eV}\right)\left(6.022\cdot 10^{23}\text{ atoms/mol}\right)}$$
$$= 4.51\text{ kg}.$$

40.44. The reaction is ${}^{2}_{1}\text{H}+{}^{2}_{1}\text{H} \rightarrow {}^{4}_{2}\text{He}+Q$. The atomic masses of ${}^{2}_{1}\text{H}$ and ${}^{4}_{2}\text{He}$ are $m(1,1) = 2.014101778$ u and $m(2,2) = 4.002603254$ u.

(a) The energy released is $\Delta E = Qc^2$, where Q is the change in mass; therefore,

$$\Delta E = \left(2m(1,1) - m(2,2)\right)c^2 = \left[2(2.014101778) - (4.002603254)\right]c^2\left(931.4940\text{ MeV}/c^2\right) = 23.8465\text{ MeV}.$$

(b) The total mass of water in the ocean is approximately $M = 1.50\cdot 10^{16}$ kg. Since every reaction needs two ${}^{2}_{1}\text{H}$ atoms, and 0.0300% of the mass is ${}^{2}_{1}\text{H}$ atoms, then the number of reactions that could occur is:

$$N = \frac{M(0.000300)}{2m(1,1)} = \frac{\left(1.50\cdot 10^{16}\text{ kg}\right)\left(0.000300\right)}{2\left(2.014101778\text{ u}\right)\left(1.661\cdot 10^{-27}\text{ kg/u}\right)} = 6.7256\cdot 10^{38}\text{ reactions}.$$

Therefore, the total energy is:

$$E = N\Delta E = \left(23.8465\text{ MeV}\right)\left(1.602\cdot 10^{-19}\text{ J/eV}\right)\left(6.7256\cdot 10^{38}\right) = 2.57\cdot 10^{27}\text{ J}.$$

(c) If the world uses $P = 1.00 \cdot 10^{13}$ W, then the energy would last for:

$$\Delta t = \frac{E}{P} = \frac{\left(2.569 \cdot 10^{27} \text{ J}\right)}{\left(1.00 \cdot 10^{13} \text{ J/s}\right)} = 2.57 \cdot 10^{14} \text{ s}\left(\frac{1 \text{ hr}}{3600 \text{ s}}\right)\left(\frac{1 \text{ day}}{24 \text{ hr}}\right)\left(\frac{1 \text{ yr}}{365 \text{ day}}\right) = 8.15 \cdot 10^{6} \text{ yr}.$$

40.45. (a) The Sun radiates energy at a rate of $P = 3.85 \cdot 10^{26}$ W, giving off energy ΔE in time Δt: $P = \Delta E / \Delta t$. The change in energy is related to the change in mass by $\Delta E = \Delta mc^2$. Therefore, the mass loss rate is:

$$P = \frac{\Delta E}{\Delta t} = \frac{\Delta mc^2}{\Delta t} \Rightarrow \frac{\Delta m}{\Delta t} = \frac{P}{c^2} = \frac{\left(3.85 \cdot 10^{26} \text{ W}\right)}{\left(3.00 \cdot 10^{8} \text{ m/s}\right)^2} = 4.2778 \cdot 10^{9} \text{ kg/s} \approx 4.28 \cdot 10^{9} \text{ kg/s}.$$

(b) This result is the rate at which mass itself is converted into energy. In Example 40.6, the result was the rate of proton mass required as protons fuse to form helium-4 (only a fraction of the proton's mass gets converted into energy).

(c) If a star loses mass at a constant rate, $\Delta m / \Delta t$, for $T = 4.50 \cdot 10^{9}$ yr, and its current mass is $M = 1.99 \cdot 10^{30}$ kg, then the percent change in mass over the star's lifetime, T, is:

$$\begin{aligned}\%\Delta m &= \frac{\left(\Delta m / \Delta t\right)T}{M + \left(\Delta m / \Delta t\right)T}\left(100\%\right) \\ &= \frac{\left(4.2778 \cdot 10^{9} \text{ kg/s}\right)\left(4.50 \cdot 10^{9} \text{ yr}\right)\left(365 \text{ days/yr}\right)\left(24 \text{ hr/day}\right)\left(3600 \text{ s/hr}\right)}{\left(1.99 \cdot 10^{30} \text{ kg}\right) + \left(4.2778 \cdot 10^{9} \text{ kg/s}\right)\left(4.50 \cdot 10^{9} \text{ yr}\right)\left(365 \text{ days/yr}\right)\left(24 \text{ hr/day}\right)\left(3600 \text{ s/hr}\right)}\left(100\%\right) \\ &= 0.0305\%.\end{aligned}$$

40.46. The reaction is:

$${}^{3}_{2}\text{He} + {}^{4}_{2}\text{He} \rightarrow {}^{7}_{4}\text{Be} + \gamma,$$

where $m(2,1) = 3.016029$ u, $m(2,2) = 4.002603$ u, $m(4,3) = 7.0169298$ u, and the energy of the photon is given by $E = \Delta mc^2$. Therefore, the minimum possible energy of the photon that is emitted is:

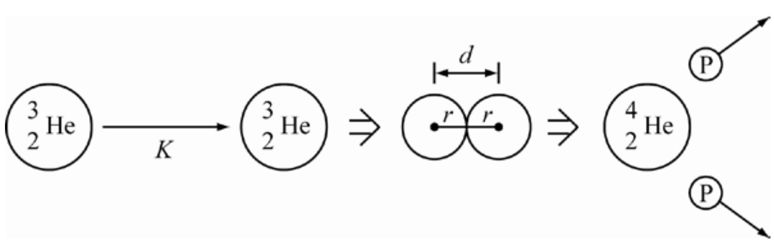
$$\begin{aligned}E_{\text{min}} &= \left(m(2,1) + m(2,2) - m(4,3)\right)c^2 \\ &= \left(3.016029 + 4.002603 - 7.0169298\right)c^2\left(931.4940 \text{ MeV}/c^2\right) = 1.58559 \text{ MeV}.\end{aligned}$$

The maximum possible wavelength of this photon is:

$$\lambda_{\text{max}} = \frac{hc}{E_{\text{min}}} = \frac{\left(6.626 \cdot 10^{-34} \text{ J s}\right)\left(3.00 \cdot 10^{8} \text{ m/s}\right)}{\left(1.58558 \cdot 10^{6} \text{ eV}\right)\left(1.602 \cdot 10^{-19} \text{ J/eV}\right)} = 783 \text{ fm}.$$

40.47. **THINK:** In order for the two ${}^{3}_{2}\text{He}$ atoms to bind, they must come close enough for the strong force to overcome the Coulomb repulsion. The closest the two can get is when their centers are separated by the sum of the radii of the atoms, i.e. the diameter of one atom. Assuming that one is at rest, the kinetic energy of the other atom must be greater than the potential barrier due to the repulsion force. The kinetic energy is directly proportional to the temperature of the surroundings.

SKETCH:



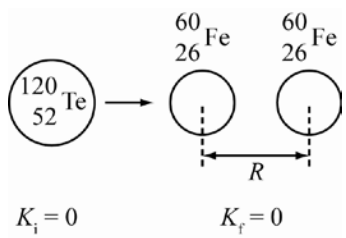
RESEARCH: Thermal energy of a particle is given by $K=\frac{3}{2}k_{\text{B}}T$. The Coulomb potential is given by $U_{\text{C}}=kq^2/d$, where $q=2e$ for ${}_2^3\text{He}$. The diameter of ${}_2^3\text{He}$ is $d = 2\,R(A)$.

SIMPLIFY: The temperature is given by: $K=U_{\text{C}} \Rightarrow \frac{3}{2}k_{\text{B}}T=\frac{k(2e)^2}{2R(A)} \Rightarrow T=\frac{4ke^2}{3k_{\text{B}}R_0A^{1/3}}$.

CALCULATE: $T=\dfrac{4\left(8.99\cdot 10^9\text{ N m}^2/\text{C}^2\right)\left(1.602\cdot 10^{-19}\text{ C}\right)^2}{3\left(1.12\cdot 10^{-15}\text{ m}\right)(3)^{1/3}\left(1.381\cdot 10^{-23}\text{ J/K}\right)}=13.790\cdot 10^9\text{ K}$

ROUND: To three significant figures, the temperature required to make the fusion occur is T = 13.8 GK.
DOUBLE-CHECK: This result is about 1000 times hotter than the core of the Sun. However, the temperature really only needs to be a fraction of this because there will be nuclei in the high energy "tail" of the energy distribution of a lower temperature.

40.48. **THINK:** From the principle of conservation of energy, the net Coulomb repulsion can be determined.
SKETCH:



RESEARCH: The conservation of energy is given by

$$E_{\text{i}}=E_{\text{f}} \Rightarrow K_{\text{i}}+m(68,52)c^2=2m(34,26)c^2+U+K_{\text{f}}.$$

The Coulomb potential energy is given by:

$$U_{\text{C}}=\frac{kZ^2e^2}{R}.$$

SIMPLIFY: Letting $K_{\text{i}}=K_{\text{f}}=0,$ the potential energy U of the two Fe nuclei is given by

$$U=\left(m(68,52)-2m(34,26)\right)c^2.$$

This energy must be equal to the Coulomb potential energy, $U=U_{\text{C}}$, so the separation between the two iron nuclei is given by $R=\dfrac{kZ^2e^2}{\left(m(68,52)-2m(34,26)\right)c^2}$.

CALCULATE: Substituting the values for the masses and $Z=26$ gives:

$$R=\frac{\left(8.99\cdot 10^9\text{ N m}^2/\text{C}^2\right)(26)^2\left(1.602\cdot 10^{-19}\text{ C}\right)^2}{\left[(119.904040)-2(59.934078)\right]c^2\left(931.494\ \cdot 10^6\text{ eV}/c^2\right)\left(1.602\cdot 10^{-19}\text{ J/eV}\right)}=2.9126\cdot 10^{-14}\text{ m}.$$

ROUND: Rounding the result to three significant figures gives $R=29.1$ fm.

DOUBLE-CHECK: A typical distance between nucleons is of the order of 10 fm, so the answer is reasonable.

40.49. **THINK:** From the definition of the mass excess, the atomic mass can be determined. For this problem, use the energy conversion, $1\text{ u}=931.49\text{ MeV}/c^2$.
SKETCH: A sketch is not necessary.
RESEARCH:
(a) The definition of the mass excess is $\Delta m=m(N,Z)-A\text{ u}$.

(b) Using the atomic masses, the mass-energy difference between the initial and final states for a fission reaction is $\Delta E = \left(m_{\text{i}} - \sum m_{\text{f}}\right)c^2 = \left(\Delta m_{\text{i}} - \sum \Delta m_{\text{f}}\right)c^2$. Since mass number is conserved, the mass excess can be used.

SIMPLIFY:

(a) The atomic mass is $m(N,Z) = A\text{ u} + \Delta m.$

(b) The mass-energy difference between the initial and the final states of two fission reactions are:

$$\Delta E_{\text{Cf}} = \left(\Delta m(154,98) - \Delta m(84,56) - \Delta m(67,42) - 3\Delta m_{\text{n}}\right)c^2$$

$$\Delta E_{\text{Fm}} = \left(\Delta m(156,100) - \Delta m(86,54) - \Delta m(46,66) - 4\Delta m_{\text{n}}\right)c^2.$$

(c) Since ΔE_{Cf} and ΔE_{Fm} are larger than zero, the reactions can occur spontaneously. Therefore, energy is released in both reactions.

CALCULATE:

(a) Using the energy conservation and $1\text{ u} = 931.49\text{ MeV}/c^2$, the atomic masses are determined and given in the following table (rounding to five significant figures).

No.	**Nuclide**	**Mass number, *A***	**Mass excess, $\Delta m\ (\text{keV}/c^2)$**	**Atomic mass (u)**
1	${}_0^1n$	1	8071.3	1.0087
2	${}_{98}^{252}\text{Cf}$	252	76034	252.08
3	${}_{100}^{256}\text{Fm}$	256	85496	256.09
4	${}_{56}^{140}\text{Ba}$	140	-83271	139.91
5	${}_{54}^{140}\text{Xe}$	140	-72990	139.92
6	${}_{46}^{112}\text{Pd}$	112	-86336	111.91
7	${}_{42}^{109}\text{Mo}$	109	-67250	108.93

(b) $\Delta E_{\text{Cf}} = \left[76034 + 83271 + 67250 - 3(8071.3)\right]\text{ keV} = 202.3411\text{ MeV}$

$\Delta E_{\text{Fm}} = \left[85496 + 72990 + 86336 - 4(8071.3)\right]\text{ keV} = 212.5368\text{ MeV}$

(c) Not necessary.

ROUND:

(a) Not necessary.

(b) $\Delta E_{\text{Cf}} = 202.34\text{ MeV}$ and $\Delta E_{\text{Fm}} = 212.54\text{ MeV}.$

(c) Not necessary.

DOUBLE-CHECK: As a comparison, the mass-energy difference is $\Delta E_{\text{U}} = 173.3\text{ MeV}$ for the reaction:

$$n + {}^{235}\text{U} \rightarrow {}^{141}\text{Ba} + {}^{92}\text{Kr} + 3n,$$

so the values obtained here are of a reasonable magnitude.

40.50. It is known that the nuclear radius is proportional to $A^{1/3}$, as given by $R = R_0 A^{1/3}$. So, $A = (R/R_0)^3$, where $R_0 = 1.12\text{ fm}$ and *A* is the number of nucleons in a nucleus. Therefore, the number of nucleons in a 10.0 km diameter star is:

$$A = \left(\frac{5.00 \cdot 10^3\text{ m}}{1.12 \cdot 10^{-15}\text{ m}}\right)^3 = 8.90 \cdot 10^{55}\text{ nucleons.}$$

40.51. The average kinetic energy is related to the temperature by $K_{\text{ave}} = 3kT/2$. Substituting the numerical values gives:

$$K_{\text{ave}} = \frac{3}{2}\left(1.38\cdot 10^{-23}\text{ J/K}\right)\left(1.00\cdot 10^{7}\text{ K}\right) = 2.07\cdot 10^{-16}\text{ J}.$$

This corresponds to an average velocity of:

$$K_{\text{ave}} = \frac{1}{2}mv_{\text{ave}}^2 \Rightarrow v_{\text{ave}} = \sqrt{\frac{2K_{\text{ave}}}{m}} = \sqrt{\frac{2\left(2.07\cdot 10^{-16}\text{ J}\right)}{\left(1.67\cdot 10^{-27}\text{ kg}\right)}} = 4.98\cdot 10^{5}\text{ m/s}.$$

40.52. **THINK:** It is possible to make a rough estimate of the age of the solar system by comparing the current ratio of ^{235}U to ^{238}U isotopes to the ratio which is believed to have been present at the time of the solar system's formation.

SKETCH: Not applicable.

RESEARCH: It is known that the half-life of ^{235}U and ^{238}U are $2.22\cdot 10^{16}$ s and $1.41\cdot 10^{17}$ s, respectively. Using the exponential decay law, $N = N_0 e^{-t\ln 2/t_{1/2}}$, the ratio of the numbers of the two isotopes yields:

$$\frac{N_{238}(t)}{N_{235}(t)} = \frac{N_{0,238}e^{-t\ln 2/t_{1/2,238}}}{N_{0,235}e^{-t\ln 2/t_{1/2,235}}}.$$

SIMPLIFY: Since initially, $N_{0,238} = N_{0,235}$, the ratio becomes:

$$\frac{N_{238}(t)}{N_{235}(t)} = \exp\left[-t\ln 2\left(\frac{1}{t_{1/2,238}} - \frac{1}{t_{1/2,235}}\right)\right] \Rightarrow t = -\frac{\ln\left(\frac{N_{238}(t)}{N_{235}(t)}\right)}{\ln 2\left(\left(1/t_{1/2,238}\right) - \left(1/t_{1/2,235}\right)\right)}.$$

CALCULATE: $t = -\dfrac{\ln\left(\dfrac{0.9928}{0.0072}\right)}{\ln 2\left(\left(1/1.41\cdot 10^{17}\text{ s}\right) - \left(1/2.22\cdot 10^{16}\text{ s}\right)\right)} = 1.873\cdot 10^{17}\text{ s} = 5.938\cdot 10^{9}\text{ yr}$

ROUND: The answer should be rounded to two significant figures. The explosion was about 5.9 billion years ago.

DOUBLE CHECK: The formation of our solar system is believed to have occurred approximately 4.5 billion years ago. The calculated answer is of the correct order of magnitude.

40.53. The activity at time, *t*, is related to the initial activity by $A(t) = A_0(1/2)^{t/t_{1/2}}$. The activity after 2.50 h is $A(t = 2.50\text{ h}) = A_0(1/2)^{(2.50\text{ h})/t_{1/2}} = 1.50$ μCi. Thus, the initial activity should be:

$$A_0 = \left(1.50\ \mu\text{Ci}\right)2^{(2.50\text{ h})/(6.05\text{ h})} = 2.00\ \mu\text{Ci}.$$

40.54. The frequency of a photon is given to be $f = 40.58\cdot 10^{6}$ Hz. The wavelength of the photon is:

$$\lambda = \frac{c}{f} = \frac{\left(2.998\cdot 10^{8}\text{ m/s}\right)}{\left(40.58\cdot 10^{6}\text{ s}^{-1}\right)} = 7.388\text{ m}.$$

The energy of the photon is $E = hf = \left(6.626\cdot 10^{-34}\text{ J s}\right)\left(40.58\cdot 10^{6}\text{ s}^{-1}\right) = 2.689\cdot 10^{-26}$ J. This photon lies in the radio wave spectrum. Since the energy is much less than the energy to ionize electrons from an atom, it is not harmful to the human body. Radio waves are constantly around us.

40.55. The exponential decay law can be expressed as $N = N_0 (1/2)^{t/t_{1/2}}$. It is given that $N/N_0 = 0.1000$. The time required is determined using:

$$\ln\left(\frac{N}{N_0}\right) = \ln\left(\frac{1}{2}\right)\frac{t}{t_{1/2}} \Rightarrow t = t_{1/2}\frac{\ln(N/N_0)}{\ln(1/2)} = (3.825 \text{ days})\frac{\ln(0.1000)}{\ln(1/2)} = 12.71 \text{ days.}$$

40.56. The absorbed dose is given by:

$$\text{dose} = \frac{\text{amount of radiation energy}}{\text{mass of absorbing material}} = \frac{0.180 \text{ J}}{0.0500 \text{ kg}} = 3.60 \text{ J/kg} = 3.60 \text{ Gy} = 360 \text{ rd.}$$

40.57. Using the mass-energy relation, $E = mc^2$, the energy required to take apart the nucleus into its constituent pieces is:

$$E = E_f - E_i = N_p m_p c^2 + N_n m_n c^2 - m_i c^2$$
$$= \left[11(1.007276) + 11(1.008665) - (21.994435)\right]c^2\left(931.494 \text{ MeV}/c^2\right) = 168.522 \text{ MeV.}$$

40.58. The initial measurement is $N_0 = 7210$ count/s. After 45 min, the measurement is N = 4585 counts/s. Using the exponential decay law,

$$N = N_0 e^{-t/\tau} \text{ or } N = N_0\left(\frac{1}{2}\right)^{t/t_{1/2}},$$

it is found that the half-life of the material is:

$$\ln\left(\frac{N}{N_0}\right) = \ln\left(\frac{1}{2}\right)\frac{t}{t_{1/2}} \Rightarrow t_{1/2} = t\left[\frac{\ln(1/2)}{\ln(N/N_0)}\right]$$
$$t_{1/2} = (45 \text{ min})\left[\frac{\ln(1/2)}{\ln(4585/7210)}\right] = 69 \text{ min.}$$

40.59. Initially, all of the energy is in the form of kinetic energy (assume the potential is zero at a large distance). When the particles are at their closest approach, all of the energy is potential energy. Therefore,

$$K_i = U_f \Rightarrow K_i = \frac{kZ_1Z_2q_p^2}{r} \Rightarrow r = \frac{kZ_1Z_2q_p^2}{K_i}$$
$$r = \frac{\left(8.99\cdot 10^9 \text{ N m}^2/\text{C}^2\right)(2)(92)\left(1.602\cdot 10^{-19} \text{ C}\right)^2}{\left(5.00\cdot 10^6 \text{ eV}\right)\left(1.602\cdot 10^{-19} \text{ J}\right)} = 53.0 \text{ fm.}$$

40.60. The activity of a radioactive material is given by $A = \lambda N$. The number of nuclei in the sample of mass, m, is $N = mN_A / M$, where M is the molar mass of the material. Therefore, the activity is:

$$A = \lambda\frac{m}{M}N_A = \frac{mN_A \ln 2}{t_{1/2}M} = \frac{(1.00 \text{ kg})\left(6.02\cdot 10^{23} \text{ mol}^{-1}\right)\ln 2}{(24100 \text{ yr})\left(3.1536\cdot 10^7 \text{ s/yr}\right)(0.239052 \text{ kg/mol})} = 2.30\cdot 10^{12} \text{ Bq.}$$

40.61. The activity of a sample is determined using the exponential decay law, $A = A_0 e^{-\lambda t}$ or $A = A_0 (1/2)^{t/t_{1/2}}$. After one year, the activity is

$$A = (1.000 \text{ μCi})\left(\frac{1}{2}\right)^{(365 \text{ days})/(5.01 \text{ days})} = 1.17\cdot 10^{-22} \text{ μCi.}$$

40.62. Denote the fraction of mass of carbon atoms in a human body by $f_C = 0.14$. For a living object, it is known that the abundance of ^{14}C to ^{12}C is $r = N(^{14}C)/N(^{12}C) = 1.20 \cdot 10^{-12}$. The activity of ^{14}C is given by $A = \lambda N(^{14}C)$. The decay constant is given by $\lambda = \ln 2 / t_{1/2}$ and the number of ^{14}C nucleons is

$$N(^{14}C) = \frac{m(^{14}C)N_A}{M(^{14}C)},$$

where $M(^{14}C)$ is the molar mass of ^{14}C. For this person of mass $m = 75$ kg, the mass of ^{14}C present is given by:

$$m(^{12}C) = \left(\frac{N(^{12}C)M(^{12}C)}{N(^{14}C)M(^{14}C)}\right)m(^{14}C) \text{ and } m(^{12}C) + m(^{14}C) = f_C m$$

$$m(^{14}C) = \frac{f_C m}{\left(\frac{N(^{12}C)M(^{12}C)}{N(^{14}C)M(^{14}C)}\right)+1}.$$

Combining the above equations and simplifying gives the activity:

$$\begin{aligned} A &= \frac{N_A f_C r m \ln 2}{t_{1/2}\left(M(^{12}C) + rM(^{14}C)\right)} \\ &= \frac{(6.02 \cdot 10^{23} \text{ mol}^{-1})(0.14)(1.20 \cdot 10^{-12})(75 \cdot 10^3 \text{ g})\ln 2}{(1.81 \cdot 10^{11} \text{ s})\left((12.000000 \text{ g/mol}) + (1.20 \cdot 10^{-12})(14.003242 \text{ g/mol})\right)} \\ &= 2.4 \cdot 10^3 \text{ Bq} \end{aligned}$$

40.63. The binding energy of a nucleus is given by

$$B(N,Z) = \left[Zm(0,1) + Nm_n - m(N,Z)\right]c^2.$$

Substituting $Z = 3$, $N = 5$, $m(0,1) = 1.007825032$ u, $m_n = 1.008664916$ u, $m(5,3) = 8.022485$ u and $\text{u} = 931.494 \text{ MeV}/c^2$ gives:

$$B(5,3) = \left[3(1.007825032) + 5(1.008664916) - (8.022485)\right]c^2(931.494 \text{ MeV}/c^2) = 41.279 \text{ MeV}.$$

40.64. The energy released in decay $n \to p + e^- + \bar{\nu}_e$, is equal to the difference between the initial and final energies, $E = E_i - E_f$. Since the mass of a neutrino is negligible, the total energy released is:

$$\begin{aligned} E &= m_n c^2 - (m_p c^2 + m_e c^2) = (m_n - m_p - m_e)c^2 \\ &= \left((1.008664916) - (1.007276467) - (5.4858 \cdot 10^{-4})\right)c^2(931.494 \text{ MeV}/c^2) = 0.782 \text{ MeV}. \end{aligned}$$

40.65. The rest mass energy is defined by the mass-energy relation, $E = mc^2$. Therefore, the rest mass energy is

$$E = \rho V c^2 = (737 \text{ kg/m}^3)(3.785 \text{ L})(10^{-3} \text{ m}^3/\text{L})(3.00 \cdot 10^8 \text{ m/s})^2 = 2.51 \cdot 10^{11} \text{ MJ}.$$

40.66. The exponential decay law is given by $N(t) = N_0 e^{-\lambda t}$ or $N(t) = N_0(1/2)^{t/t_{1/2}}$. If $t = 10t_{1/2}$, the number of remaining atoms is

$$N(t = 10t_{1/2}) = N_0(1/2)^{10}.$$

If $t = 20t_{1/2}$, the number of remaining atoms is

$$N(t = 20t_{1/2}) = N_0(1/2)^{20} = \left[N_0(1/2)^{10}\right](1/2)^{10} = N(t = 10t_{1/2})(1/2)^{10}.$$

Thus,

$$N\left(t = 20t_{1/2}\right) = \left(10^{30}\right)\left(1/2\right)^{10} = 10^{27} \text{ atoms.}$$

40.67. The binding energy per nucleon is:

$$\frac{B(N,Z)}{A} = \frac{1}{A}\left[Zm(0,1) + Nm_\text{n} - m(N,Z)\right]c^2.$$

(a) For ${}^4_2\text{He}$,

$$\frac{B(2,2)}{4} = \frac{1}{4}\left[2(1.007825032) + 2(1.008664916) - (4.002603)\right]c^2\left(931.494 \text{ MeV}/c^2\right)$$
$$= 7.074 \text{ MeV}$$

(b) For ${}^3_2\text{He}$,

$$\frac{B(1,2)}{3} = \frac{1}{3}\left[2(1.007825032) + (1.008664916) - (3.016030)\right]c^2\left(931.494 \text{ MeV}/c^2\right)$$
$$= 2.572 \text{ MeV}$$

(c) For ${}^3_1\text{H}$,

$$\frac{B(2,1)}{3} = \frac{1}{3}\left[(1.007825032) + 2(1.008664916) - (3.016050)\right]c^2\left(931.494 \text{ MeV}/c^2\right)$$
$$= 2.827 \text{ MeV}$$

(d) For ${}^2_1\text{H}$,

$$\frac{B(1,1)}{2} = \frac{1}{2}\left[(1.007825032) + (1.008664916) - (2.014102)\right]c^2\left(931.494 \text{ MeV}/c^2\right)$$
$$= 1.112 \text{ MeV}$$

40.68. The mean lifetime is related to the half-life by $t_{1/2} = \tau \ln 2$. Therefore, the half-life is:

$$t_{1/2} = \left(4300 \text{ s}\right)\ln 2 = 3.0 \cdot 10^3 \text{ s.}$$

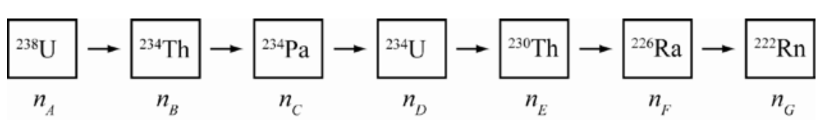
40.69. The exponential decay law is given by $N = N_0 e^{-\lambda t}$ or $N = N_0\left(1/2\right)^{t/t_{1/2}}$. Since 90% of the sample has decayed, there is 10.0% of the sample remaining. Therefore, $N/N_0 = 0.100$. The time required to reach $N/N_0 = 0.100$. is:

$$\frac{N}{N_0} = \left(\frac{1}{2}\right)^{t/t_{1/2}} \Rightarrow \frac{t}{t_{1/2}}\ln(1/2) = \ln\left(N/N_0\right) \Rightarrow t = t_{1/2}\frac{\ln\left(N/N_0\right)}{\ln(1/2)}$$

$$t = \left(26.8 \text{ min}\right)\frac{\ln(0.100)}{\ln(1/2)} = 89.0 \text{ min.}$$

40.70. **THINK:** The concentrations of ${}^{238}\text{U}$ and its first five daughters are in equilibrium, that is each daughter is produced as fast as it decays. This means the rates of decay (or activities) of ${}^{238}\text{U}$ and its daughters are the same.

SKETCH:



RESEARCH: The rate of decay or activity is defined as $A = -dN/dt = \lambda N = \lambda n N_\text{A}$, where n is the number of moles. For simplicity, denote ${}^{238}\text{U}$ and its daughters by letters *A*, *B*, *C*, *D*, *E*, *F*, and *G*. Since the mixture is in equilibrium, the rates of decay are the same for all nuclei, except for Rn that is:

$A_A = A_B = A_C = A_D = A_E = A_F \equiv A$ or $\lambda_A n_A = \lambda_B n_B = \lambda_C n_C = \lambda_D n_D = \lambda_E n_E = \lambda_F n_F$. Therefore, the amount of each species is $n_i = \lambda_A n_A / \lambda_i$, where *i* = *B*, *C*, *D*, *E*, and *F*. The masses of all the species must add up to the total mass, m_{tot}. This means: $m_{tot} = \sum_i n_i M_i$, where M_i is the molar mass of i^{th} species.

SIMPLIFY:

(a) $m_{tot} = \sum_{i=A}^{F} \lambda_A n_A \frac{M_i}{\lambda_i} = \lambda_A n_A \sum_{i=A}^{F} \frac{M_i}{\lambda_i} \Rightarrow \lambda_A n_A = \frac{m_{tot}}{\sum_{i=A}^{F} M_i / \lambda_i}$

Using $\lambda = \ln 2 / t_{1/2}$, the above equation becomes:

$$\lambda_A n_A = \frac{m_{tot} \ln 2}{\sum_{i=A}^{F} M_i t_{1/2,i}} \text{ or } A = \frac{m_{tot} N_A \ln 2}{\sum_{i=A}^{F} M_i t_{1/2,i}}.$$

Then, the rate, in mass per unit time, that ^{222}Rn is produced is:

$$r = \frac{A M_G}{N_A}$$

(b) The rate of activity of radon is:

$$\frac{dA_{Rn}}{dt} = \lambda_{Rn} \left(\frac{dN_{Rn}}{dt} \right) = \lambda_{Rn} A = \frac{A \ln 2}{t_{1/2,G}}.$$

CALCULATE:

(a) The following values are found in Appendix B:

^{238}U: $M_A = 238.0508$ g/mol and $t_{1/2,A} = 1.41 \cdot 10^{17}$ s,

^{234}Th: $M_B = 234.0436$ g/mol and $t_{1/2,B} = 2.08 \cdot 10^{6}$ s,

^{234}Pa: $M_C = 234.0433$ g/mol and $t_{1/2,C} = 2.41 \cdot 10^{4}$ s,

^{234}U: $M_D = 234.0410$ g/mol and $t_{1/2,D} = 7.74 \cdot 10^{12}$ s,

^{230}Th: $M_E = 230.0331$ g/mol and $t_{1/2,E} = 2.38 \cdot 10^{12}$ s,

^{226}Ra: $M_F = 226.0254$ g/mol and $t_{1/2,F} = 5.05 \cdot 10^{10}$ s, and

^{222}Rn: $M_G = 222.0176$ g/mol and $t_{1/2,G} = 3.30 \cdot 10^{5}$ s.

Therefore, the sum is: $\sum_{i=A}^{F} M_i t_{1/2,i} = 3.35675 \cdot 10^{19}$ g s/mol $= 3.35675 \cdot 10^{16}$ kg s/mol. The activity is given by:

$$A = \frac{(1.00 \text{ kg}) \ln(2) (6.02 \cdot 10^{23} \text{ mol}^{-1})}{(3.35675 \cdot 10^{16} \text{ kg s/mol})} = 1.2431 \cdot 10^{7} \text{ decay/s}.$$

The rate, in mass per unit time, that ^{222}Rn is produced is:

$$r = \frac{(1.2431 \cdot 10^{7} \text{ decay/s})(222.0176 \text{ g/mol})}{(6.02 \cdot 10^{23} \text{ mol}^{-1})} = 4.5845 \cdot 10^{-15} \text{ g/s} = 0.1446\ \mu\text{g/yr}.$$

(b) $\frac{dA_{Rn}}{dt} = \frac{(1.2431 \cdot 10^{7} \text{ decay/s}) \ln 2}{(3.30 \cdot 10^{5} \text{ s})} = 26.11 \text{ decays/s}^2 = 26.11 \text{ Bq/s} = 8.234 \cdot 10^{8} \text{ Bq/yr} = 22.25 \text{ mCi/yr}.$

ROUND: Round the results to three significant figures.

(a) $r = 0.145\ \mu\text{g/yr}$

(b) $\frac{dA_{Rn}}{dt} = 22.3$ mCi/yr

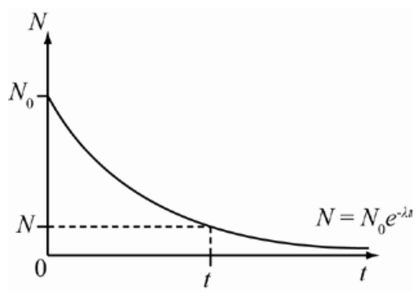
DOUBLE-CHECK: The activity of radon can be approximated by assuming that the sample is composed of only ^{238}U. That is,

$$A = \lambda N = \frac{mN_\text{A}\ln 2}{t_{1/2,\text{A}}M_\text{A}} = \frac{(1.00 \text{ kg})(6.02\cdot 10^{23} \text{ mol}^{-1})\ln 2}{(1.41\cdot 10^{17} \text{ s})(0.2380508 \text{ kg/mol})} = 1.24\cdot 10^{7} \text{ decays/s}.$$

This is in agreement with the activity calculated above.

40.71. **THINK:** The radioactive decay of ^{14}C follows an exponential decay law, while the number of ^{12}C isotopes stays constant in time because this isotope is stable. It can be assumed that ^{12}C comprises all of the mass of the ash; that is, $m = m\left(^{12}\text{C}\right) = 50.0 \text{ g}$. The activity, $A = 20.0 \text{ decays/hr}$, can be used along with the half-life, $t_{1/2} = 5730 \text{ yr}$, to determine the current number of ^{14}C atoms. Using all of this information will provide an approximate age for the tree.

SKETCH:



RESEARCH: The exponential decay law for the number of atoms remaining as a function of time is given by

$$N(t) = N_0 e^{-\lambda t},$$

where $\lambda = \ln 2 / t_{1/2}$. The activity of ^{14}C is given by $A = \lambda N\left(^{14}\text{C}\right)$. As stated in the question, the initial ratio of ^{14}C to ^{12}C atoms is $r = N_0\left(^{14}\text{C}\right)/N\left(^{12}\text{C}\right) = 1.300\cdot 10^{-12}$. Therefore, the number of initial ^{14}C atoms is $N_0\left(^{14}\text{C}\right) = rN\left(^{12}\text{C}\right)$. The number of ^{12}C atoms is given by

$$N\left(^{12}\text{C}\right) = \frac{m\left(^{12}\text{C}\right)}{M\left(^{12}\text{C}\right)}N_\text{A},$$

where $M\left(^{12}\text{C}\right)$ is the molar mass of ^{12}C.

SIMPLIFY: The decay law for ^{14}C is

$$N\left(^{14}\text{C}\right) = N_0\left(^{14}\text{C}\right)e^{-\lambda t} = rN\left(^{12}\text{C}\right)e^{-\lambda t}.$$

Simplifying and solving for *t* gives:

$$\frac{A}{\lambda} = \frac{rm\left(^{12}\text{C}\right)N_\text{A}e^{-\lambda t}}{M\left(^{12}\text{C}\right)} \Rightarrow \frac{At_{1/2}}{\ln 2} = \frac{rm\left(^{12}\text{C}\right)N_\text{A}e^{-\ln 2t/t_{1/2}}}{M\left(^{12}\text{C}\right)} \Rightarrow e^{-\ln 2t/t_{1/2}} = \frac{AM\left(^{12}\text{C}\right)t_{1/2}}{rm\left(^{12}\text{C}\right)N_\text{A}\ln 2}$$

$$t = -\frac{t_{1/2}}{\ln 2}\ln\left(\frac{AM\left(^{12}\text{C}\right)t_{1/2}}{rm\left(^{12}\text{C}\right)N_\text{A}\ln 2}\right).$$

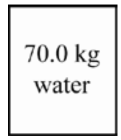
CALCULATE: $t = -\frac{(5730 \text{ yr})}{\ln 2}\ln\left(\frac{(20.0 \text{ decays/hr})(8760 \text{ hr/yr})(12.000000 \text{ g/mol})(5730 \text{ yr})}{(1.300\cdot 10^{-12})(50.0 \text{ g})(6.02\cdot 10^{23} \text{ mol}^{-1})\ln 2}\right) = 63813 \text{ yr}.$

ROUND: Rounding to three significant figure yields $t = 63800 \text{ yr}$.

DOUBLE-CHECK: This is a reasonable age for a campfire, considering that fossil evidence indicates that modern humans originated in Africa 200,000 years ago.

40.72. **THINK:** It is known that protons decay according to an exponential decay law. To determine the number of protons that would decay during 70.0 years, the initial number of protons inside a human body is needed. It is assumed that the mass of the human body is 70.0 kg and the human body is made entirely of water.

SKETCH:



RESEARCH: The exponential decay law is given by $N = N_0 e^{-\lambda t}$ or $N = N_0 (1/2)^{t/t_{1/2}}$. The initial number of protons, N_0, is determined by assuming the human body is composed entirely of water. The number of water molecules in the human body is $N_w = mN_A / M_w$, where *m* is the mass and M_w is the molar mass. Each water molecule contains ten protons, so the initial number of protons is $N_0 = 10N_w = 10mN_A / M_w$.

SIMPLIFY: The number of protons that decay during an interval of time, *t*, is:

$$N_{\text{decay}} = N_0 - N = N_0 \left(1 - \left(\frac{1}{2} \right)^{t/t_{1/2}} \right) = \left(\frac{10mN_A}{M_w} \right) \left[1 - \left(\frac{1}{2} \right)^{t/t_{1/2}} \right].$$

CALCULATE: Substituting the numerical values gives:

$$N_{\text{decay}} = \frac{10\left(70.0 \cdot 10^3 \text{ g}\right)\left(6.022 \cdot 10^{23} \text{ mol}^{-1}\right)}{\left(18.015 \text{ g/mol}\right)} \left(1 - \left(\frac{1}{2} \right)^{(70.0 \text{ yr})/\left(1.00 \cdot 10^{30} \text{ yr}\right)} \right)$$

$$= \left(2.33993994 \cdot 10^{28}\right) \left(1 - \left(\frac{1}{2} \right)^{7.00 \cdot 10^{-29} \text{ yr}} \right).$$

Evaluating this expression is tricky on a handheld calculator, because the first factor is very large, while the second factor is indistinguishable from zero when evaluated. The product is not easily determined. To deal with this complication, we rewrite the exponential term using the approximation $e^x \approx 1 + x$, which is good for small *x* (as is certainly the case here):

$$\begin{aligned} N_{\text{decay}} &= \left(2.33993894 \cdot 10^{28}\right)\left(1 - \exp\left[-7.00 \ln(2) \cdot 10^{-29}\right]\right) \\ &\approx \left(2.33993894 \cdot 10^{28}\right)\left(1 - \left[1 - 7.00 \ln(2) \cdot 10^{-29}\right]\right) \\ &= \left(2.33993894 \cdot 10^{28}\right)\left(7.00 \ln(2) \cdot 10^{-29}\right) \\ &= \left(2.33993894\right)\left(7.00 \ln(2)\right)\left(10^{-1}\right) \\ &= 1.1353 \end{aligned}$$

ROUND: Rounding to three significant figure gives $N_{\text{decay}} = 1.14$ decays.

DOUBLE-CHECK: The activity of the protons is:

$$A = \lambda N \approx \frac{N_0 \ln 2}{t_{1/2}} = \frac{\left(2.34 \cdot 10^{28}\right) \ln 2}{\left(1.00 \cdot 10^{30} \text{ yr}\right)} = 0.01622 \text{ decays/yr}.$$

Therefore, during 70.0 years, the number of decays is approximately (since $N \approx N_0$):

$$N_{\text{decay}} \approx A\Delta t = \left(0.01622 \text{ decays/yr}\right)\left(70.0 \text{ yr}\right) = 1.14 \text{ decays}.$$

This agrees with the above result. This is expected since the half-life of the proton is very large.

40.73. **THINK:** Since the theory predicts that protons never get any older it can be assumed that the activity is constant. The half-life of the proton, $t_{1/2} = 1.80 \cdot 10^{29}$ yr, can be used to determine how many protons, in a tank with $1.00 \cdot 10^4$ tons of water, will decay over two years.

SKETCH: Not necessary.

RESEARCH: Assuming that the activity is constant, the number of decays over a given time period is $N_{\text{decay}} \approx \lambda N \Delta t$, where $\lambda = \ln 2 / t_{1/2}$. Since there are ten protons in a water molecule, the number of protons, *N*, is given by $N = 10mN_{\text{A}} / M_{\text{w}}$, where $M_{\text{w}} = 18.015$ g/mol is the molar mass of water.

SIMPLIFY: The number of proton decays is

$$N_{\text{decay}} = \frac{10mN_{\text{A}}\Delta t \ln 2}{M_{\text{w}} t_{1/2}}.$$

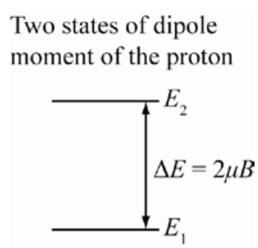
CALCULATE: $N_{\text{decay}} = \dfrac{10\left(1.00^{10}\text{ g}\right)\left(6.02 \cdot 10^{23}\text{ mol}^{-1}\right)\left(2\text{ yr}\right)\ln 2}{\left(18.015\text{ g/mol}\right)\left(1.80 \cdot 10^{29}\text{ yr}\right)} = 2.5736 \cdot 10^4$ decays

ROUND: Rounding to three significant figures, $N_{\text{decay}} = 2.57 \cdot 10^4$ decays.

DOUBLE-CHECK: Although the half-life of the proton is very large, a large number decay over two years because there are so many $\left(N = 3 \cdot 10^{33}\right)$ in the tank of water.

40.74. **THINK:** When a proton is placed in a magnetic field, the magnetic dipole moment of the proton can have only two directions: parallel or anti-parallel to the external field. By introducing a time-varying electric field at a proper frequency, it can induce a proton to flip its magnetic dipole moment.

SKETCH:



RESEARCH: The energy required to flip the dipole moment of the proton is equal to $\Delta E = 2\mu B$. This corresponds to a frequency given by $f = \Delta E / h$.

SIMPLIFY: The magnetic field is given by: $B = \dfrac{hf}{2\mu}$.

CALCULATE: $B = \dfrac{\left(6.626075 \cdot 10^{-34}\text{ J s}\right)\left(15.35850 \cdot 10^6\text{ s}^{-1}\right)}{2\left(1.410608 \cdot 10^{-26}\text{ J/T}\right)} = 0.360718828$ T

ROUND: To seven significant figures, $B = 0.3607188$ T.

DOUBLE-CHECK: This result is a typical value used in NMR spectroscopy of a proton.

40.75. **THINK:** Radioactive decay follows an exponential decay law. In this problem, two species of radioactive nuclei, A and B, are compared. After a time of 100. s, it is observed that $N_{\text{A}} = 100N_{\text{B}}$ with $\tau_{\text{A}} = 2\tau_{\text{B}}$. The initial number of nuclei for both species is N_0.

SKETCH: A sketch is not necessary.

RESEARCH: The exponential decay law is given by $N = N_0 e^{-t/\tau}$. After an interval of time, *t*, the number of nuclei A and B are $N_{\text{A}} = N_0 e^{-t/\tau_{\text{A}}}$ and $N_{\text{B}} = N_0 e^{-t/\tau_{\text{B}}}$.

SIMPLIFY: Taking the ratio of N_A to N_B and using $\tau_\text{A} = 2\tau_\text{B}$ gives

$$\frac{N_\text{A}}{N_\text{B}} = e^{-t\left(1/\tau_\text{A} - 1/\tau_\text{B}\right)} = e^{-\left(t/\tau_\text{B}\right)\left[(1/2)-1\right]} \Rightarrow \ln\left(\frac{N_\text{A}}{N_\text{B}}\right) = \frac{t}{2\tau_\text{B}} \Rightarrow \tau_\text{B} = \frac{t}{2\ln\left(N_\text{A}/N_\text{B}\right)}.$$

CALCULATE: $\tau_\text{B} = \dfrac{(100.\text{ s})}{2\ln(100)} = 10.86 \text{ s}$

ROUND: Rounding to three significant figure, $\tau_\text{B} = 10.9$ s.

DOUBLE-CHECK: If $\tau_\text{B} = 10.86$ s, then $\tau_\text{A} = 21.72$ s. Inserting these results into the equation $N(t) = N_0 e^{-t/\tau}$ gives

$$N_\text{A}\left(t = 100.\text{ s}\right) = 0.01N_0 \text{ and } N_\text{B}\left(t = 100.\text{ s}\right) = 0.0001N_0.$$

$N_\text{A}\left(t = 100.\text{ s}\right)$ is larger than $N_\text{B}\left(t = 100.\text{ s}\right)$ by a factor of 100, as required.

Multi-Version Exercises

40.76. $\lambda = \dfrac{\ln 2}{t_{1/2}} = \dfrac{\ln 2}{5730 \text{ y}} = 3.833 \cdot 10^{-12} \text{ s}^{-1}$

$$t = -\ln\left(\frac{A \cdot M\left({}^{12}\text{C}\right)}{r \cdot m\left({}^{12}\text{C}\right) N_\text{A} \lambda}\right) / \lambda$$

$$= -\ln\left[\frac{(105.\text{ decays/min})(1 \text{ min/60 s})(12 \text{ g/mol})}{\left(1.20 \cdot 10^{-12}\right)(12.43 \text{ g})\left(6.022 \cdot 10^{23} \text{ mol}^{-1}\right)(3.833 \cdot 10^{-12} \text{ s}^{-1})}\right] / (3.833 \cdot 10^{-12} \text{ s}^{-1})$$

$= 4100$ yr.

40.77. $\lambda = \dfrac{\ln 2}{t_{1/2}} = \dfrac{\ln 2}{5730 \text{ y}} = 3.833 \cdot 10^{-12} \text{ s}^{-1}$

$$t = -\ln\left(\frac{A \cdot M\left({}^{12}\text{C}\right)}{r \cdot m\left({}^{12}\text{C}\right) N_\text{A} \lambda}\right) / \lambda \Rightarrow e^{-\lambda t} = \frac{A \cdot M\left({}^{12}\text{C}\right)}{r \cdot m\left({}^{12}\text{C}\right) N_\text{A} \lambda} \Rightarrow$$

$$m\left({}^{12}\text{C}\right) = \frac{A \cdot M\left({}^{12}\text{C}\right) e^{\lambda t}}{r \cdot N_\text{A} \lambda}$$

$$= \frac{(107.\text{ decays/min})(1 \text{ min/60 s})(12 \text{ g/mol})}{\left(1.20 \cdot 10^{-12}\right)\left(6.022 \cdot 10^{23} \text{ mol}^{-1}\right)(3.833 \cdot 10^{-12} \text{ s}^{-1})} \exp\left[(3.833 \cdot 10^{-12} \text{ s}^{-1})(4384 \text{ y})\right]$$

$= 13.1$ g

40.78. $\lambda = \dfrac{\ln 2}{t_{1/2}} = \dfrac{\ln 2}{5730 \text{ y}} = 3.833 \cdot 10^{-12} \text{ s}^{-1}$

$$t = -\ln\left(\frac{A \cdot M\left({}^{12}\text{C}\right)}{r \cdot m\left({}^{12}\text{C}\right) N_\text{A} \lambda}\right) / \lambda \Rightarrow e^{-\lambda t} = \frac{A \cdot M\left({}^{12}\text{C}\right)}{r \cdot m\left({}^{12}\text{C}\right) N_\text{A} \lambda} \Rightarrow$$

$$A = \frac{r \cdot N_\text{A} m\left({}^{12}\text{C}\right) \lambda e^{-\lambda t}}{M\left({}^{12}\text{C}\right)}$$

$$= \frac{\left(1.20 \cdot 10^{-12}\right)\left(6.022 \cdot 10^{23} \text{ mol}^{-1}\right)(13.83 \text{ g})(3.833 \cdot 10^{-12} \text{ s}^{-1}) \exp\left[-(3.833 \cdot 10^{-12} \text{ s}^{-1})(4814 \text{ y})\right]}{(12 \text{ g/mol})}$$

$= 107$ /min