Appendix C

THE SCATTERING MATRIX

Let us consider an arbitrary network with N ports and the corresponding reference planes (Fig. C.1). This network can be characterized by means of the impedance (**Z**) or the admittance (**Y**) matrix, where $\mathbf{V} = \mathbf{Z} \cdot \mathbf{I}$ (**V** and **I** being column vectors composed of the voltages and currents, respectively, at the ports of the network) and $\mathbf{Y} = \mathbf{Z}^{-1}$. At microwave frequencies, a variation (displacement) of the reference plane of the ports modifies the elements of the Z- and Y-matrix in a so complex form, that it might be difficult (not to say impossible) to identify two identical networks with the ports located at different planes.

Microwave networks are usually (although not exclusively) described by means of the scattering matrix (S-matrix). Let us consider that the incident and reflected waves[1] at the reference plane of port i are characterized by the voltages and currents (V_i^+, I_i^+) and (V_i^-, I_i^-), respectively, and that the characteristic impedance of port i is Z_{oi}. We can define the normalized voltages as follows:

$$a_i = \frac{V_i^+}{\sqrt{Z_{oi}}}; \; b_i = \frac{V_i^-}{\sqrt{Z_{oi}}} \tag{C.1}$$

[1] Notice that the meaning of incident and reflected waves simply refers to waves impinging externally to the port (i.e., entering the network) and waves coming from the network (i.e., traveling to the external region of the network). Thus, the incident waves are not necessarily generated by an external source (they can be generated by internal port reflection caused by a mismatched load). Similarly, the reflected waves are not necessarily caused by external reflection of a source connected to the port (they can be generated by external sources connected to other ports of the network).

Artificial Transmission Lines for RF and Microwave Applications, First Edition. Ferran Martín.

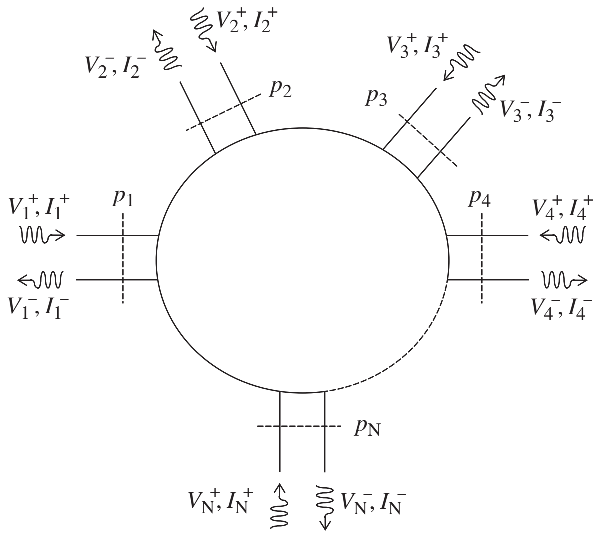

FIGURE C.1 Arbitrary microwave *N*-port network. The reference planes of the ports are labeled p_i.

so that the total voltage and current at port *i* can be expressed as follows:

$$V_i = V_i^+ + V_i^- = \sqrt{Z_{oi}}(a_i + b_i) \tag{C.2a}$$

$$I_i = I_i^+ - I_i^- = \frac{1}{\sqrt{Z_{oi}}}(a_i - b_i) \tag{C.2b}$$

and, from (1.17), the power delivered to the *i*-th port is

$$P_i = \frac{1}{2}\left(|a_i|^2 - |b_i|^2\right) \tag{C.3}$$

The *S*-matrix relates the normalized voltages of the incident and reflected waves, that is, $\mathbf{b} = \mathbf{S} \cdot \mathbf{a}$, where **a** and **b** are column vectors composed of the normalized voltages of the incident and reflected waves, respectively, and the elements of the *S*-matrix are given by

$$S_{ij} = \left.\frac{b_i}{a_j}\right|_{a_k = 0,\ \text{for } k \neq j} \tag{C.4}$$

Notice that if all the port impedances are identical, the *S*-matrix elements can be written in terms of the incident and reflected port voltages as follows:

$$S_{ij} = \left.\frac{V_i^-}{V_j^+}\right|_{V_k^+ = 0,\ \text{for } k \neq j} \tag{C.5}$$

The diagonal elements of the S-matrix are the reflection coefficients at the different ports looking into the network, when no incident waves are present on the other ports. To guarantee that no incident waves impinge on a port, that port must be terminated with a matched load. For networks with identical port impedances, the elements S_{ij} with $i \neq j$ can be interpreted as the transmission coefficients between port j and port i when all the ports except the port j are terminated in matched loads. In two-port networks, such as microwave filters, S_{11} and S_{21} are usually expressed in decibel and referred to as return and insertion loss, respectively, rather than reflection and transmission coefficients.[2]

For a given network, the calculation of the S-matrix may be tedious, as compared to the Z- or Y-matrix. However, since these matrices completely characterize the network, they must be related. In other words, the S-matrix can be expressed in terms of the Z- or Y-matrix, and vice versa. Let us consider that the impedances of the ports are all identical (Z_o), which is the most usual situation. Introducing the incident and reflected wave variables in the matrix equation $\mathbf{V} = \mathbf{Z} \cdot \mathbf{I}$, the following expression results:

$$\mathbf{V}^+ + \mathbf{V}^- = \mathbf{Z}(\mathbf{I}^+ - \mathbf{I}^-) \tag{C.6}$$

or

$$\mathbf{V}^+ + \mathbf{S}\mathbf{V}^+ = \mathbf{Z}\left(\frac{\mathbf{V}^+}{Z_o} - \frac{\mathbf{V}^-}{Z_o}\right) \tag{C.7}$$

$$Z_o(\mathbf{1} + \mathbf{S})\mathbf{V}^+ = \mathbf{Z}(\mathbf{1} - \mathbf{S})\mathbf{V}^+ \tag{C.8}$$

where $\mathbf{1}$ is the identity matrix. Therefore, the normalized impedance matrix ($\bar{\mathbf{Z}} = \mathbf{Z}/Z_o$) can be expressed in terms of the S-matrix as follows:

$$\bar{\mathbf{Z}} = (\mathbf{1} + \mathbf{S})(\mathbf{1} - \mathbf{S})^{-1} \tag{C.9}$$

From (C.8), we can isolate the S-matrix:

$$\mathbf{S} = (\bar{\mathbf{Z}} + \mathbf{1})^{-1}(\bar{\mathbf{Z}} - \mathbf{1}) \tag{C.10}$$

Since $\mathbf{Y} = \mathbf{Z}^{-1}$, the relations between the Y- and the S-matrix can be easily inferred.

For two-port networks, the transmission $ABCD$ matrix is a very useful matrix to describe the structure. In reference to the two-port network of Figure C.2a, and the total voltages and currents in the ports, the $ABCD$ matrix is given by

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} \tag{C.11}$$

[2] Normally, port 1 and 2 are the input and output ports, respectively.

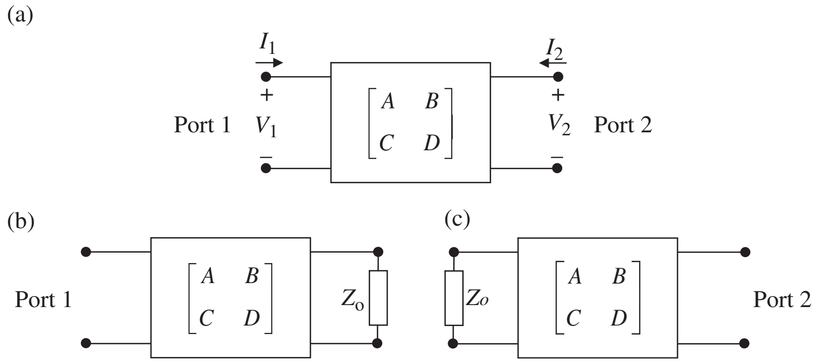

FIGURE C.2 Arbitrary two-port network (a). The loaded networks used to express S_{11}, S_{21}, and S_{22}, S_{12} as a function of the *ABCD* parameters are shown in (b) and (c), respectively.

The main advantage of the transmission *ABCD* matrix over the *S*-, *Z*-, or *Y*-matrix is that if several two-port networks are cascaded, the *ABCD* matrix of the resulting two-port network is given by the product of the *ABCD* matrices of the individual networks. Thus, if several two-port networks are cascaded, finding first the *ABCD* matrix and then transforming to the *S*-, *Z*-, or *Y*-matrices is the convenient way to obtain these matrices for the complete network.

Let us show how to express the *S*-matrix in terms of the *ABCD* matrix. $S_{11}$is the reflection coefficient from port 1 when port 2 is terminated with the impedance Z_o (which is assumed to be the impedance of both ports). Such reflection coefficient can be calculated from the impedance, Z_{in}, seen from port 1, given by

$$Z_{in} = \frac{V_1}{I_1} = \frac{A + B/Z_o}{C + D/Z_o} \tag{C.12}$$

as results from the following expressions, given by the *ABCD* matrix for the structure of Figure C.2b:

$$V_1 = AV_2 + B\frac{V_2}{Z_o} \tag{C.13a}$$

$$I_1 = CV_2 + D\frac{V_2}{Z_o} \tag{C.13b}$$

From (C.12) and (1.20) (with $Z_L = Z_{in}$), S_{11} is found to be

$$S_{11} = \frac{A + B/Z_o - CZ_o - D}{A + B/Z_o + CZ_o + D} \tag{C.14}$$

The transmission coefficient S_{21} can be obtained from the circuit of Figure C.2b, taking into account that $V_2^- = V_2$ and $V_1 = (1 + S_{11})V_1^+$:

$$S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^+=0} = \frac{V_2}{V_1}(1 + S_{11}) \tag{C.15}$$

From (C.13a) and (C.14),

$$S_{21} = \frac{2}{A + B/Z_o + CZ_o + D} \tag{C.16}$$

To determine S_{22}, port 1 is loaded with Z_o (Fig. C.2c). The impedance seen from port 2 can be expressed in terms of the *ABCD* matrix as follows:

$$Z_{out} = \frac{V_2}{I_2} = \frac{D + B/Z_o}{C + A/Z_o} \tag{C.17}$$

and the reflection coefficient is

$$S_{22} = \frac{-A + B/Z_o - CZ_o + D}{A + B/Z_o + CZ_o + D} \tag{C.18}$$

Finally, the transmission coefficient from port 2 to port 1 is given by

$$S_{12} = \left.\frac{V_1^-}{V_2^+}\right|_{V_1^+=0} = \frac{V_1}{V_2}(1 + S_{22}) \tag{C.19}$$

From the first algebraic equation inferred from the *ABCD* matrix applied to the circuit of Figure C.2c, that is

$$V_1 = AV_2 - B\frac{V_2}{Z_{out}} \tag{C.20}$$

and (C.17) and (C.18), we obtain

$$S_{12} = \frac{2(AD - BC)}{A + B/Z_o + CZ_o + D} \tag{C.21}$$

Since for reciprocal networks[3] $S_{21} = S_{12}$, comparison of (C.16) and (C.21) gives $AD - BC = 1$.

[3] Reciprocal networks, defined in Chapter 1, exhibit a symmetric *S*-, *Y*-, and *Z*-matrix.

The dependence of the *ABCD* matrix elements with the *S*-parameters can be inferred from (C.14), (C.16), (C.18), and (C.21):

$$A = \frac{(1+S_{11})(1-S_{22}) + S_{12}S_{21}}{2S_{21}} \tag{C.22a}$$

$$B = Z_o \frac{(1+S_{11})(1+S_{22}) - S_{12}S_{21}}{2S_{21}} \tag{C.22b}$$

$$C = \frac{1}{Z_o} \frac{(1-S_{11})(1-S_{22}) - S_{12}S_{21}}{2S_{21}} \tag{C.22c}$$

$$D = \frac{(1-S_{11})(1+S_{22}) + S_{12}S_{21}}{2S_{21}} \tag{C.22d}$$

The conversion between those pairs of matrices not given above can be indirectly inferred. Nevertheless, a complete conversion table for two-port network parameters can be found in Ref. [1].

REFERENCE

1. D. M. Pozar, *Microwave Engineering*, Addison Wesley, New York, 1990.