Appendix B

THE SMITH CHART

The Smith chart is a graphical means to simultaneously visualize the reflection coefficient and impedance of a certain load (i.e., a lumped element, a terminated transmission line, or even a complex one-port system). It is very useful for the analysis and synthesis of circuits based on transmission lines and stubs, and most microwave commercial CAD tools and test equipment allow for the visualization of simulation and measurement results on a Smith chart. The Smith chart (see Fig. B.1) is essentially the representation of the reflection coefficient in polar coordinates, $\rho = |\rho|e^{j\theta}$, where $-\pi \le \theta \le \pi$, and the origin of θ is the right-hand side of the horizontal axis. For any passive load ($|\rho| \le 1$), the reflection coefficient is given by a single point within the circle of unit radius, which is the considered region for the representation of the reflection coefficient in the chart.

However, the main relevant use of the Smith chart is the direct conversion from reflection coefficients to normalized impedances, or admittances, and vice versa. Let us consider for the moment the conversion to normalized impedances, defined by the quotient between a given impedance Z and a reference impedance Z_o, which is typically the characteristic impedance of a transmission line:

$$\bar{Z} = \frac{Z}{Z_o} \tag{B.1}$$

Considering Z the load impedance of a transmission line, the reflection coefficient can be expressed as follows:

$$\rho = \frac{\bar{Z} - 1}{\bar{Z} + 1} \tag{B.2}$$

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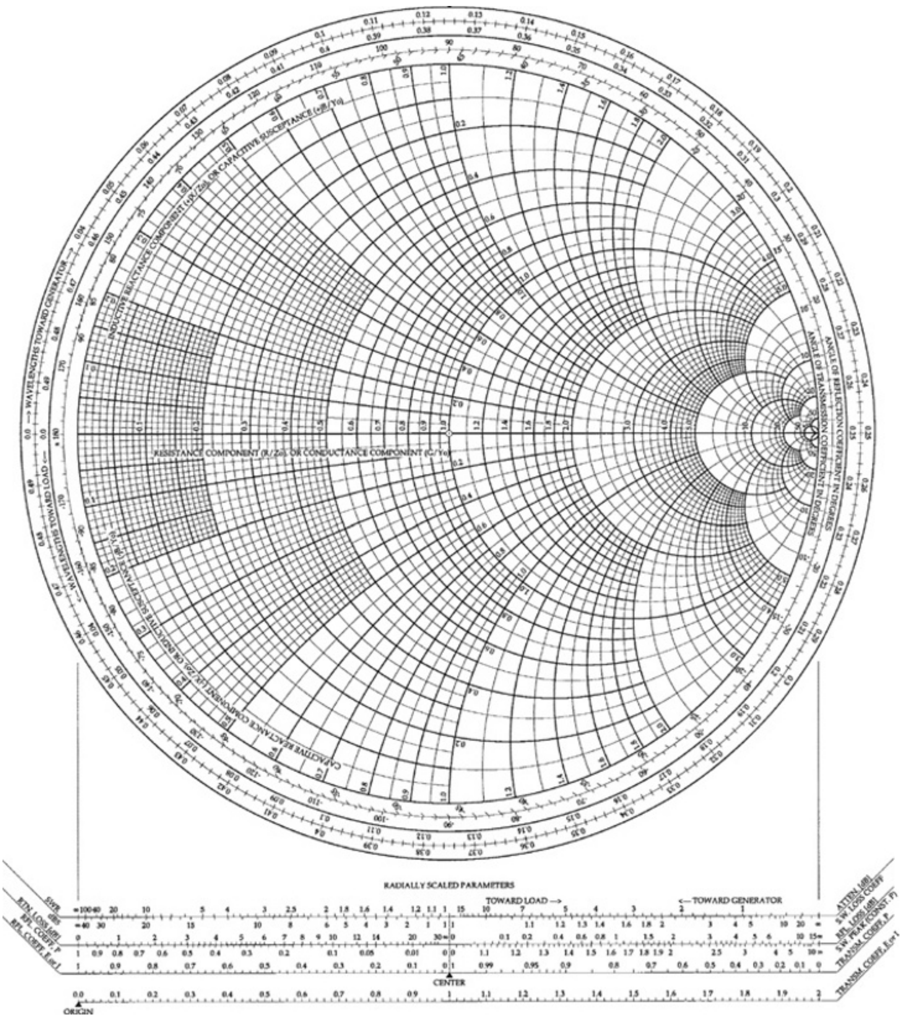

FIGURE B.1 Smith chart.

and there is a univocal correspondence between ρ and $\bar{Z}$. From B.2, the normalized impedance can be isolated and expressed as follows:

$$\bar{Z} = \frac{1+\rho}{1-\rho} \tag{B.3}$$

or

$$\bar{R} + j\bar{\chi} = \frac{1+\rho_{\mathrm{r}}+j\rho_{\mathrm{i}}}{1-\rho_{\mathrm{r}}-j\rho_{\mathrm{i}}} \tag{B.4}$$

where $\bar{Z}$ and ρ have been decomposed into the real and imaginary parts. In the previous complex equation, both the real and the imaginary parts must be equal. This gives

$$\left(\rho_r - \frac{\bar{R}}{1+\bar{R}}\right)^2 + \rho_i^2 = \left(\frac{1}{1+\bar{R}}\right)^2 \tag{B.5a}$$

$$(\rho_r - 1)^2 + \left(\rho_i - \frac{1}{\bar{\chi}}\right)^2 = \left(\frac{1}{\bar{\chi}}\right)^2 \tag{B.5b}$$

Considering $\bar{R}$ constant, (B.5a) is the equation of a circumference in the ρ_r–ρ_i plane. Thus, (B.5a) is a family of circumferences parameterized by $\bar{R}$, and each circumference has its center in the $\rho_i = 0$ axis. These circumferences, all contained within the Smith chart, are called constant resistance circumferences, and the value of $\bar{R}$ is indicated in the horizontal axis of the Smith chart. Notice that the $\bar{R} = 0$ circumference coincides with the unit radius circumference, as one expects for a purely reactive load, where $|\rho| = 1$. As $\bar{R}$ increases, the radius of the constant resistance circumferences decrease, and the circumference degenerates in a single point $\rho_r = 1$, $\rho_i = 0$ (or $|\rho| = 1$, $\theta = 0$) for $\bar{R} \to \infty$.

By contrast, (B.5b) is the equation of a family of circumferences parameterized by the normalized reactance (constant reactance circumferences). However, for any constant reactance circumference, only a portion of it lies within the Smith chart (the centers of these circumferences lie in the vertical line $\rho_r = 1$ of the chart). For increasing values of $\bar{\chi}$, the radius of the reactance circumferences decreases and $\rho_r = 1$, $\rho_i = 0$ for $\bar{\chi} \to \infty$. For $\bar{\chi} = 0$, the constant reactance curve degenerates in the straight line $\rho_i = 0$, as expected on account of the real reflection coefficient for a purely resistive load. The normalized reactance values that label the different reactance circumferences are indicated in the Smith chart (along the whole external circumference).

Notice that the constant resistance and reactance circumferences are orthogonal. Some relevant curves and normalized impedances are indicated in the Smith chart of Figure B.2.

Given a reflection coefficient, represented by a unique point in the Smith chart, the normalized impedance can be immediately visualized by directly reading in the chart. The Smith chart has many applications, but one relevant use of the Smith chart concerns the graphical solution of the input impedance, Z_{in}, of a terminated transmission line (expression 1.31). Since the reflection coefficient at the input of the terminated line is given by (1.27), it is clear that increasing the line length is equivalent to a clockwise rotation of an angle $\theta = 2\beta l$ (l being the line length) from the point of the load (ρ_L) with center on the center of the Smith chart. Notice that for a $\lambda/2$ line, $\theta = 2\pi$, corresponding to a complete rotation and $Z_{in} = Z_L$ (the input impedance does not experience any change for transmission line lengths that are multiple of half wavelength). To facilitate the solution of this type of transmission line problems, the Smith chart has scales around its periphery calibrated in terms of the wavelength toward (clockwise) or away from (counterclockwise) the generator. Figure B.3 plots the location of

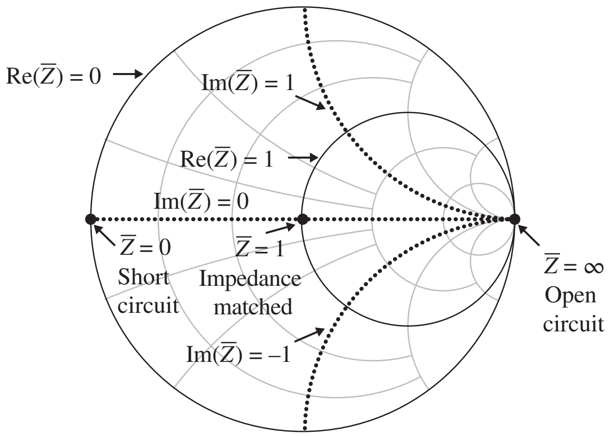

FIGURE B.2 Some relevant constant resistance/reactance curves and normalized impedances plotted in the Smith chart. The constant resistance circumference that crosses the center of the Smith chart (Re($\bar{Z}$) = 1) is called unit resistance circumference.

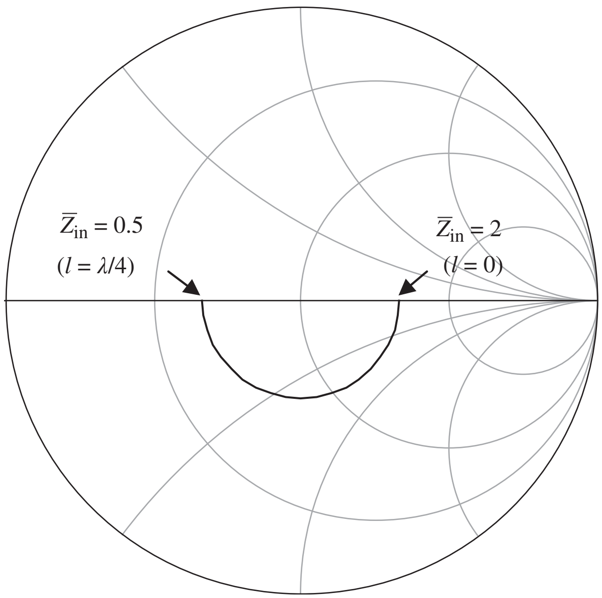

FIGURE B.3 Location of the reflection coefficient of the indicated terminated line for line length increasing from 0 to $\lambda/4$. Notice that the left extreme of the plot gives the normalized admittance of the load ($\bar{Y} = 0.5$), which coincides with the normalized input impedance of the line.

the reflection coefficient at the input port of a transmission line terminated with a purely resistive load ($\bar{Z} = 2$) when the line length is increased from $l = 0$ to $l = \lambda/4$.

Following a similar procedure, a Smith chart for the normalized admittance can be constructed. However, the normalized admittance of a given load can be directly visualized in the impedance Smith chart. The reason is that the normalized input impedance of a $\lambda/4$ terminated line (impedance inverter) is

$$\overline{Z_{\text{in}}} = \frac{1}{\overline{Z_{\text{L}}}} \tag{B.6}$$

which is the normalized admittance of the load. Since a $\lambda/4$ transformation is equivalent to a 180° rotation in the Smith chart, the normalized admittance of the load can be simply inferred by imaging the impedance point across the center of the Smith chart. Thus, the impedance Smith chart can be used to deal with normalized impedances or admittances, indistinctly. Figure B.3, indicates that the normalized admittance of the load considered in the example is $\bar{Y} = 0.5$.

Since the reflection coefficient of a matched load is null, that is, it is located in the centre of the Smith chart, the Smith chart is a useful tool to evaluate the level of matching of a given load and its dependence with frequency (strong deviations from the center of the chart indicate significant mismatch). In reactive loads, departure from the zero-resistance circumference must be attributed to the presence of a non-negligible-resistive component in the load impedance, and gives an indication of the deviation from the purely reactive nature of the load.

To finalize this appendix, let us consider an illustrative example to show how the Smith chart can be used to obtain information on reactive two-port circuits loaded with resistive impedances. Let us consider the lumped-element circuit of Figure B.4a (this circuit describes the unit cell of a particular type of artificial transmission line, as discussed in Chapter 3), and let us assume that the series inductance L and the elements of the shunt branch (C, C_{c}, and L_{c}) are unknown. Let us represent the dependence of the input impedance (or reflection coefficient) with frequency in the Smith chart, considering that the output port is loaded with the reference impedance Z_{o} (Fig. B.4b). There are two singular frequencies in the circuit: (1) the resonance frequency of the L_{c}–C_{c} tank, f_{o} and (2) the frequency that nulls the shunt reactance, f_{z}. At f_{z}, given by

$$f_{\text{z}} = \frac{1}{2\pi\sqrt{L_{\text{c}}(C + C_{\text{c}})}} \tag{B.7}$$

the impedance seen from the input port is simply $j\omega L$ since the shunt branch is short-circuited to ground. This means that the input impedance must be tangent to the zero-resistance circle of the Smith chart at f_{z}. Thus, from the value of the reactance at this point, the inductance L can be inferred, and from the value of f_{z}, a condition for the parameters of the shunt branch is inferred. A second condition comes from f_{o},

$$f_{\text{o}} = \frac{1}{2\pi\sqrt{L_{\text{c}}C_{\text{c}}}} \tag{B.8}$$

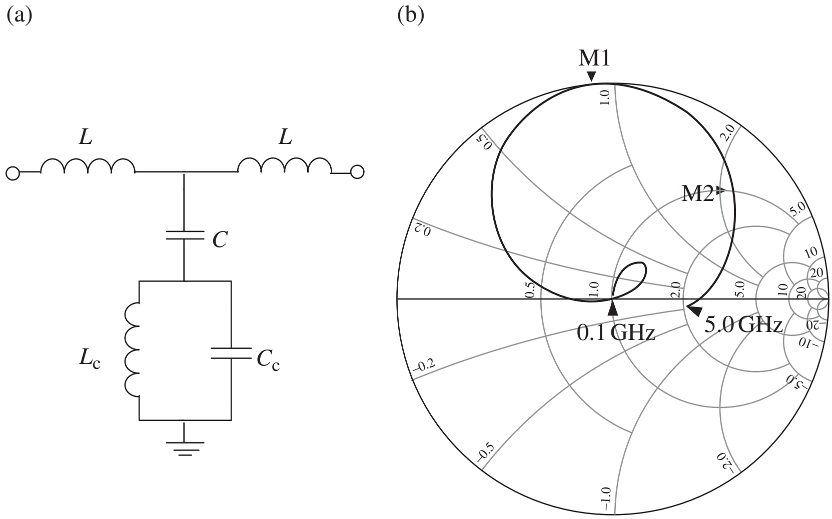

FIGURE B.4 Lumped-element-reactive two-port network (a), and its reflection coefficient considering that the output port is terminated with the reference impedance Z_o (b). The relevant frequencies are $f_o = 3.56\,\text{GHz}$ (M2) and $f_z = 3\,\text{GHz}$ (M1). The normalized impedance at f_z is $\bar{Z} = j0.946$, giving a value for the series inductance of $L = 2.5\,\text{nH}$. The other element values are $C = 0.8\,\text{pF}$, $C_c = 2\,\text{pF}$, and $L_c = 1\,\text{nH}$. The frequency swept covers the range 0.1–5.0 GHz.

This frequency can be easily identified because at this frequency the shunt branch opens and the normalized input impedance can be expressed as $\overline{Z_{in}} = 1 + j\bar{\chi}$. In other words, f_o is that frequency where the input impedance crosses the so-called unit resistance circumference, and the second condition is given by expression (B.8). To univocally determine the parameters of the shunt branch an additional condition is necessary, for instance, the reactance value at a given frequency. However, a more elegant procedure to extract the parameters of structures described by this and other similar circuits is detailed in Appendix G.