38 Youden Squares

38.1. Introduction 617
38.2. Objective of Using Youden Squares 617
38.3. Calculations 620
38.4. Derivations 623
Reference 625

38.1. Introduction

Youden squares are a type of layout used in design of experiments. A *Youden square* is a fraction of a *Latin square*. They are also called *incomplete Latin squares*. Table 38.1 shows a Latin square commonly seen in experimental design texts. Table 38.2 is another way of showing the contents of a Latin square using an L_9 orthogonal array, where only the first three columns of the array are shown.

Table 38.3 shows a Youden square. In the table, a portion of the layout from Table 38.1 is missing. Table 38.4 is another way of showing the contents of a Youden square.

From Tables 38.1 and 38.2, it is seen that all three factors, A, B, and C, are balanced (orthogonal), whereas in Tables 38.3 and 38.4, factors B and C are unbalanced. Although factors B and C are unbalanced in a Youden square, their effects can be calculated by solving the equations generated from the relationship shown in this layout.

38.2. Objective of Using Youden Squares

The objective of using Youden squares is to utilize an L_{18} orthogonal array efficiently to obtain more information from the same number of runs of the array. Normally, an L_{18} array is used to assign one two-level factor and seven three-level factors. By putting a Youden square in an L_{18} array, one more three-level factor can be assigned without adding extra experimental runs.

Taguchi's Quality Engineering Handbook. Genichi Taguchi, Subir Chowdhury and Yuin Wu

Table 38.1
Common Latin square

	B_1	B_2	B_3
A_1	C_1	C_2	C_3
A_2	C_2	C_3	C_1
A_3	C_3	C_1	C_2

Table 38.5 is a Youden square where an L_{18} array is assigned with an extra column; the ninth factor is denoted by C, as column 9. In the table, y'_1, y'_2, ... , y'_{18} are the outputs of 18 runs. Comparing Tables 38.4 and 38.5, the first two columns of the two arrays correspond to each other, with the exception that there are three repetitions in Table 38.5. For example, combination A_1B_1 appears once in Table 38.4, while the combination appears three times (numbers 1, 2, and 3) in Table 38.5. It is the same for other combinations.

It is known that the number of degrees of freedom means the amount of information available. There are 18 runs in an L_{18} array; its number of degrees of freedom is 17. When one two-level factor and seven three-level factors are assigned, the total number of degrees of freedom from this layout is 15. The difference, two degrees of freedom, is hidden inside. One of the ways to utilize these two degrees of freedom to obtain more information without having more runs is by substituting columns 1 and 2 with a six-level column to create a column that can assign a factor of up to six levels. This is called *multilevel assignment.* Another way of utilizing the two degrees of freedom is to assign a three-level column using a Youden square. Thus, 17 degrees of freedom are all utilized.

Table 38.2
Latin square using an orthogonal array

No.	A 1	B 2	C 3	y
1	1	1	1	y_1
2	1	2	2	y_2
3	1	3	3	y_3
4	2	1	2	y_4
5	2	2	3	y_5
6	2	3	1	y_6
7	3	1	3	y_7
8	3	2	1	y_8
9	3	3	2	y_9

Table 38.3
Youden square

	B_1	B_2	B_3
A_1	$C_1(y_1)$	$C_2(y_2)$	$C_3(y_3)$
A_2	$C_2(y_4)$	$C_3(y_5)$	$C_1(y_6)$

Table 38.4
Alternative Youden square depiction

No.	A 1	B 2	C 3	y
1	1	1	1	y_1
2	1	2	2	y_2
3	1	3	3	y_3
4	2	1	2	y_4
5	2	2	3	y_5
6	2	3	1	y_6

Table 38.5
Youden square

No.	A 1	B 2	C 9	D_3 3	… …	I 8	Results	Subtotal
1	1	1	1				y'_1	
2	1	1	1				y'_2	y_1
3	1	1	1				y'_3	
4	1	2	2				y'_4	
5	1	2	2				y'_5	y_2
6	1	2	2				y'_6	
7	1	3	3				y'_7	
8	1	3	3				y'_8	y_3
9	1	3	3				y'_9	
10	2	1	2				y'_{10}	
11	2	1	2				y'_{11}	y_4
12	2	1	2				y'_{12}	
13	2	2	3				y'_{13}	
14	2	2	3				y'_{14}	y_5
15	2	2	3				y'_{15}	
16	2	3	1				y'_{16}	
17	2	3	1				y'_{17}	y_6
18	2	3	1				y'_{18}	

38.3. Calculations

In Table 38.5, one two-level factor, A, and eight three-level factors, B, C, D, E, F, G, H, and I, were assigned. Eighteen outputs, $y_1, y_2, \ldots, y_{18}$, were obtained, and results, $y'_1, y'_2, \ldots, y'_{18}$, were obtained. Calculation of main effects A, D, E, F, G, H, and I are exactly identical to regular calculation. Main effects B and C are calculated after calculating their level averages as follows:

$$
\begin{aligned}
y_1 &= y'_1 + y'_2 + y'_3 \\
y_2 &= y'_4 + y'_5 + y'_6 \\
&\vdots \\
y_6 &= y'_{16} + y'_{17} + y'_{18}
\end{aligned}
\qquad (38.1)
$$

Let the effect of each factor level, A_1, A_2, B_1, B_2, B_3, C_1, C_2, and C_3, be denoted by a_1, a_2, b_1, b_2, b_3, c_1, c_2, and c_3, respectively, as shown in Figure 38.1.

For example, a_1 is the deviation of A_1 from the grand average, $\overline{T}$. As described previously, a_1 and a_2 can be determined as usual without considering the layout of Youden squares. The effects of factor levels of B and C are calculated as follows:

$$
\begin{aligned}
b_1 &= \tfrac{1}{9}(y_1 - y_2 + y_4 - y_6) \\
b_2 &= \tfrac{1}{9}(y_2 - y_3 - y_4 + y_5) \\
b_3 &= \tfrac{1}{9}(-y_1 + y_3 - y_5 + y_6)
\end{aligned}
\qquad (38.2)
$$

$$
\begin{aligned}
c_1 &= \tfrac{1}{9}(y_1 - y_3 - y_4 + y_6) \\
c_2 &= \tfrac{1}{9}(-y_1 + y_2 + y_4 - y_5) \\
c_3 &= \tfrac{1}{9}(-y_2 + y_3 + y_5 - y_6)
\end{aligned}
\qquad (38.3)
$$

❒ Example [1]

In a study of air turbulence in a conical air induction system (conical AIS) for automobile engines, the following control factors were studied:

A: honeycomb, two levels

B: large screen, three levels

C: filter medium, three levels

D: element part support, three levels

E: inlet geometry, three levels

F: gasket, three levels

G: inlet cover geometry, three levels

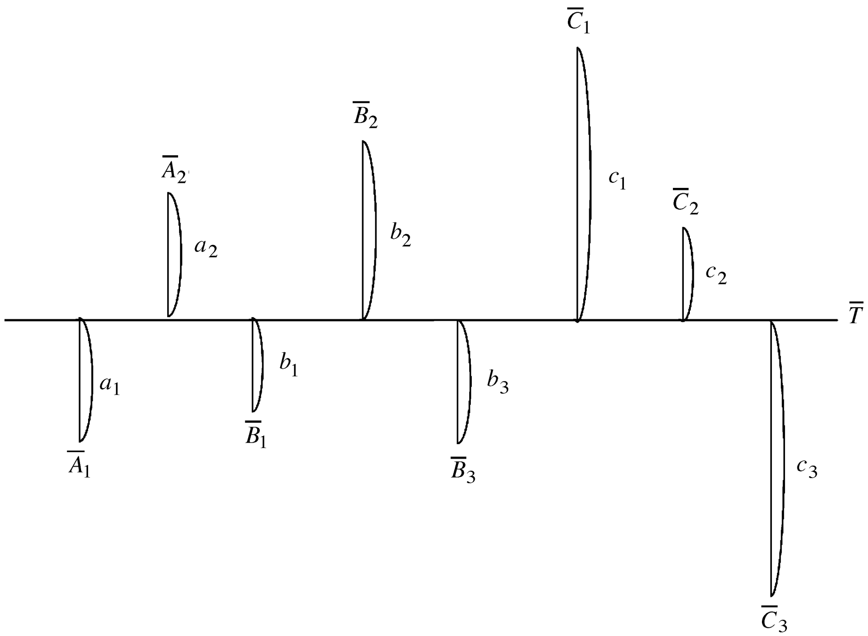

Figure 38.1
Main effects of A, B, and C

H: bypass screen, three levels

I: MAFS orientation to housing, three levels

A Youden square was used for assigning the factors above. The dynamic SN ratio and sensitivity were calculated from the experiment. Table 38.6 shows the layout and SN ratio.

Comparing Tables 38.5 and 38.6, it is seen that in Table 38.5, the combinations of columns 1, 2, and 9 are based on a Youden square, and factors A, B, and C are assigned. But in Table 38.6, the combinations of columns 1, 2, and 9 are still based on a Youden square, where factors A, B, and I are assigned.

The effect of each factor level of B and I is calculated as follows:

$$b_1 = \frac{y_1 - y_2 + y_4 - y_6}{9}$$

$$= \frac{9.05 - 8.77 + 8.88 - 8.23}{9} = 0.1033 \qquad (38.4)$$

$$b_2 = \frac{y_2 - y_3 - y_4 + y_5}{9}$$

$$= \frac{8.77 - 8.31 - 8.88 + 8.58}{9} = 0.0177 \qquad (38.5)$$

Table 38.6
Layout and results

No.	1 *A*	2 *B*	3 *C*	4 *D*	5 *E*	6 *F*	7 *G*	8 *H*	9 *I*	SN	Subtotal
1	1	1	1	1	1	1	1	1	1	4.24	$y_1 = 9.05$
2	1	1	2	2	2	2	2	2	1	2.21	
3	1	1	3	3	3	3	3	3	1	2.60	
4	1	2	1	1	2	2	3	3	2	2.20	$y_2 = 8.77$
5	1	2	2	2	3	3	1	1	2	4.01	
6	1	2	3	3	1	1	2	2	2	2.56	
7	1	3	1	2	1	3	2	3	3	2.20	$y_3 = 8.31$
8	1	3	2	3	2	1	3	1	3	3.87	
9	1	3	3	1	3	2	1	2	3	2.24	
10	2	1	1	3	3	2	2	1	2	2.66	$y_4 = 8.88$
11	2	1	2	1	1	3	3	2	2	2.66	
12	2	1	3	2	2	1	1	3	2	2.31	
13	2	2	1	2	3	1	3	2	3	2.20	$y_5 = 8.58$
14	2	2	2	3	1	2	1	3	3	2.35	
15	2	2	3	1	2	3	2	1	3	4.03	
16	2	3	1	3	2	3	1	2	1	2.11	$y_6 = 8.23$
17	2	3	2	1	3	1	2	3	1	2.16	
18	2	3	3	2	1	2	3	1	1	3.96	

$$b_3 = \frac{-y_1 + y_3 - y_5 + y_6}{9}$$

$$= \frac{-9.05 + 8.31 - 8.58 + 8.23}{9} = -0.1211 \qquad (38.6)$$

$$i_1 = \frac{y_1 - y_3 - y_4 + y_6}{9}$$

$$= \frac{9.05 - 8.31 - 8.88 + 8.23}{9} = 0.01 \qquad (38.7)$$

$$i_2 = \frac{-y_1 + y_2 + y_4 - y_5}{9}$$

$$= \frac{-9.05 + 8.77 + 8.88 - 8.58}{9} = 0.0022 \qquad (38.8)$$

$$i_3 = \frac{-y_2 + y_3 + y_5 - y_6}{9}$$

$$= \frac{-8.77 + 8.31 + 8.58 - 8.23}{9} = -0.0122 \qquad (38.9)$$

The grand average, $\bar{T}$, is

$$\bar{T} = \frac{4.24 + 2.21 + \cdots + 3.96}{18} = 2.88 \qquad (38.10)$$

Therefore, the level averages of B and I are calculated as

$$\bar{B}_1 = \bar{T} + b_1 = 2.88 + 0.1033 = 2.98 \qquad (38.11)$$

$$\bar{B}_2 = \bar{T} + b_2 = 2.88 + 0.0177 = 2.90 \qquad (38.12)$$

$$\bar{B}_3 = \bar{T} + b_3 = 2.88 - 0.121 = 2.76 \qquad (38.13)$$

$$\bar{I}_1 = \bar{T} + i_1 = 2.88 + 0.01 = 2.89 \qquad (38.14)$$

$$\bar{I}_2 = \bar{T} + i_2 = 2.88 - 0.0022 = 2.88 \qquad (38.15)$$

$$\bar{I}_3 = \bar{T} + i_3 = 2.88 - 0.0122 = 2.87 \qquad (38.16)$$

Other level totals are calculated in the normal way. Table 38.7 shows the results of calculation.

38.4. Derivations

The derivations of equations (38.2) and (38.3) are illustrated. Table 38.8 is constructed from Table 38.4 using symbols a_1, a_2, b_1, b_2, b_3, c_1, c_2, and c_3, as described in Section 38.3. It is seen from the table that y_1 is the result of a_1, b_1, and c_1; y_2 is the result of a_1, b_2, and c_2; and so on. Therefore, the following equations can be written:

Table 38.7
Response table for SN ratio

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1	2.90	2.98	2.81	2.92	3.00	2.89	2.88	4.00	2.89
2	2.86	2.90	2.88	2.82	2.79	2.81	2.84	2.33	2.88
3		2.76	2.95	2.90	2.85	2.94	2.92	2.30	2.87

Table 38.8
Youden square

No.	*A* 1	*B* 2	*C* 3	*y*
1	a_1	b_1	c_1	y_1
2	a_1	b_2	c_2	y_2
3	a_1	b_3	c_3	y_3
4	a_2	b_1	c_2	y_4
5	a_2	b_2	c_3	y_5
6	a_2	b_3	c_1	y_6

$$y_1 = a_1 + b_1 + c_1 \tag{38.17}$$

$$y_2 = a_1 + b_2 + c_2 \tag{38.18}$$

$$y_3 = a_1 + b_3 + c_3 \tag{38.19}$$

$$y_4 = a_2 + b_1 + c_2 \tag{38.20}$$

$$y_5 = a_2 + b_2 + c_3 \tag{38.21}$$

$$y_6 = a_2 + b_3 + c_1 \tag{38.22}$$

In addition, the following three equations exist based on the definition of a_1, a_2, ... , c_3:

$$a_1 + a_2 = 0 \tag{38.23}$$

$$b_1 + b_2 + b_3 = 0 \tag{38.24}$$

$$c_1 + c_2 + c_3 = 0 \tag{38.25}$$

By solving these equations, b_1, b_2, b_3, c_1, c_2, and c_3 are calculated, as shown in equations (38.2) and (38.3).

For example, the effect of C_2 is derived as follows.

equation (38.17) − equation (38.18):

$$y_1 - y_2 = b_1 - b_2 + c_1 - c_2 \tag{38.26}$$

equation (38.20) − equation (38.21):

$$y_4 - y_5 = b_1 - b_2 + c_2 - c_3 \tag{38.27}$$

equation (38.26) − equation (38.27):

$$y_1 - y_2 - (y_4 - y_5) = c_1 - c_2 - (c_2 - c_3) \tag{38.28}$$

$$y_1 - y_2 - y_4 + y_5 = c_1 + c_3 - 2c_2 \tag{38.29}$$

From equation (38.25),

$$c_1 + c_2 + c_3 = 0 \tag{38.30}$$

Therefore, equation (38.29) can be written as

$$y_1 - y_2 - y_4 + y_5 = c_1 + c_3 + c_2 - 3c_2 \tag{38.31}$$

$$c_2 = \tfrac{1}{3}(-y_1 + y_2 + y_4 - y_5) \tag{38.32}$$

This is shown in equation (38.3).

The Youden square shown in Table 38.4 has six runs; each run (combination) has one repetition. When a Youden square is assigned in L_{18} arrays, as shown in Table 38.5, each combination of the Youden square has three repetitions. Therefore, the effect of each factor level of B and C must be divided by 9, as shown in equations (38.2) and (38.3), instead of 3 as in equation (38.31).

Reference

1. R. Khami, et al., 1994. Optimization of the conical air induction system's mass airflow sensor performance. Presented at the Taguchi Symposium.