36 Layout of Orthogonal Arrays Using Linear Graphs

36.1. Introduction

Linear graphs have been developed by Taguchi for easy assignment of experiments. These graphs can be used to assign interactions between factors in an array to calculate interactions in the design of experiments. But in quality engineering, linear graphs are not used for assigning interactions but for special cases such as multilevel assignment. This chapter is based on Genichi Taguchi, *Design of Experiments*. Tokyo: Japanese Standards Association, 1973.

36.2. Linear Graphs of Orthogonal Array L_8

Before explaining the modifications of orthogonal arrays for experimental layout, it is necessary to know the column or columns where an interaction between columns is confounded. To simplify explanation, orthogonal array L_8 is used. Two linear graphs can be drawn for the L_8 table as shown in Figure 36.1.

Linear Graph (1)

Linear graph (1) in Figure 36.1 means that the interaction between columns 1 and 2 comes out to column 3, and the interaction between columns 1 and 4 comes out to column 5. Column 7 is shown as an independent point apart from the triangle.

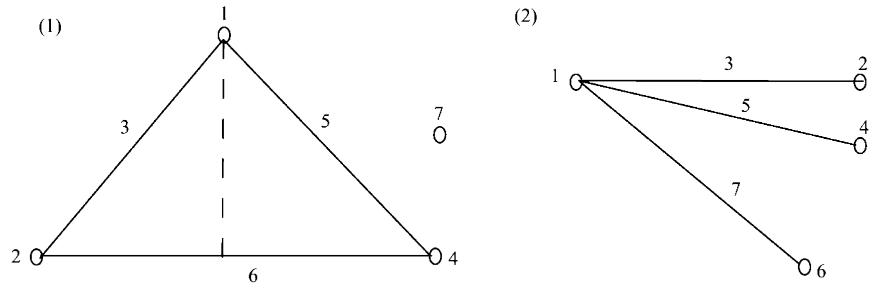

Figure 36.1
Linear graphs of orthogonal array L_8

If there are four two-level factors, *A*, *B*, *C*, and *D*, and if *A* is assigned to column 1, *B* to column 2, *C* to column 4, and *D* to column 7, then interaction $A \times B$ is obtained from column 3, $A \times C$ from column 5, and $B \times C$ from column 6. Such assignment problems are easily solved using linear graph (1).

Linear Graph (2)

Linear graph (2) is used for an experiment where the interactions between one particular factor and some other factors are important. This is illustrated in the following experiment, with two types of raw materials, A_1 and A_2; two annealing methods, B_1 and B_2; two temperatures, C_1 and C_2; and two treating times, D_1 and D_2.

It is expected that since B_1 and B_2 use different types of furnaces, their operations are quite different and the optimum temperature or optimum treating time might not be the same for B_1 as for B_2. On the other hand, it is expected that a better raw material must always be better for any annealing method. Accordingly, only the main effect is cited for factor *A*. Thus, we need to get the information of main effects *A*, *B*, *C*, *D* and interactions $B \times C$ and $B \times D$.

These requirements are shown in Figure 36.2. Such a layout is easily arranged using two lines out of the three radial lines of linear graph (2) shown in Figure 36.1.

The remaining columns, 6 and 7, are removed from the form, as shown in Figure 36.3, with the independent points as indicated. An independent point is used for the assignment of a main effect; factor *A* is assigned to any one of the remaining two points, since only the main effect is required for *A*. The layout is shown in Table 36.1.

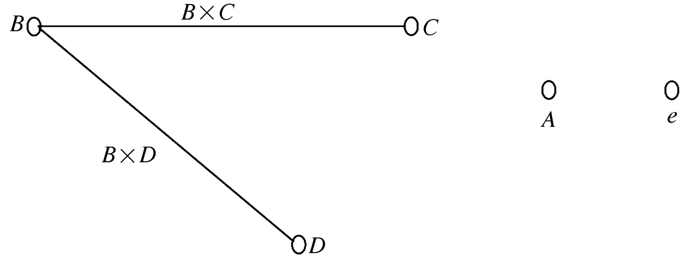

Figure 36.2
Information required for a linear graph

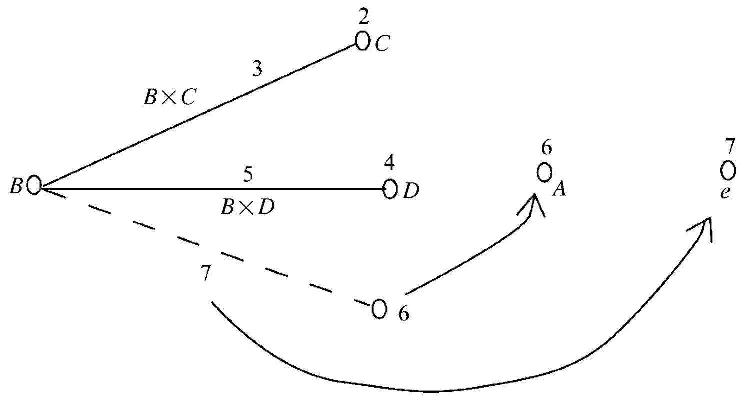

Figure 36.3
Layout of the experiment

Such a layout can also be arranged from linear graph (1). Remove column 6, which is the line connected by columns 2 and 4, indicate it as an independent point like column 7, then assign factor *A*, as shown in Figure 36.4.

36.3. Multilevel Arrangement

At the planning stage of an experiment, we sometimes want to include some multilevel factors. That is to arrange a four- or eight-level column in two-level series orthogonal arrays, or a nine-level column in three-level series orthogonal arrays. Using linear graphs, a four- or eight-level factor can be assigned from a two-level series orthogonal array.

Next, the method of arranging a four-level factor from orthogonal array L_8 is explained. Assume that *A* is a four-level factor, and *B*, *C* ,and *D* are two-level factors.

Table 36.1
Layout of experiment

Column	*B* 1	*C* 2	*BC* 3	*D* 4	*BD* 5	*A* 6	*e* 7
1	1	1	1	1	1	1	1
2	1	1	1	2	2	2	2
3	1	2	2	1	1	2	2
4	1	2	2	2	2	1	1
5	2	1	2	1	2	1	2
6	2	1	2	2	1	2	1
7	2	2	1	1	2	2	1
8	2	2	1	2	1	1	2

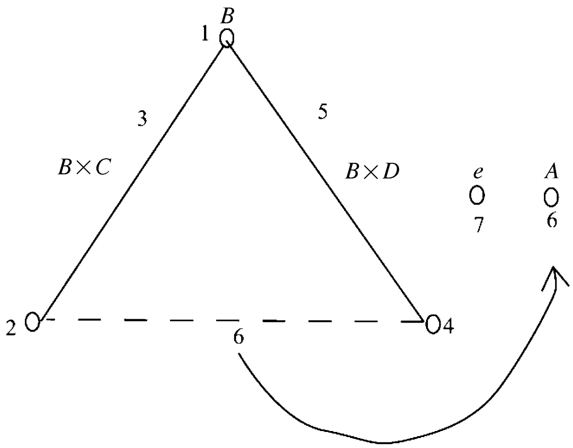

Figure 36.4
Layout using linear graph (1)

To obtain these main effects from an L_8 array, it is necessary to prepare a four-level column in the array. For instance, line 3 with two points 1 and 2 in linear graph (1) is selected. Then three columns are removed from the linear graph and rearranged using a four-level column, as shown in Table 36.2. That is, from the two columns shown as the two points at both ends of the line (or any two of the three columns), four combinations (11, 12, 21, and 22) are obtained; these combinations are then substituted for 1, 2, 3, and 4, respectively, to form a four-level column. Then columns 1, 2, and 3 are removed from the table, as shown on the right side of Table 36.2.

Table 36.2
Preparation of orthogonal array $L_8(4 \times 2^4)$ from L_8 (2^7)

	$L_8(2^7)$							$L_8(4 \times 2^4)$				
No.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**123**	**4**	**5**	**6**	**7**
1	1	1	1(1)	1	1	1	1	1	1	1	1	1
2	1	1	1(1)	2	2	2	2	1	2	2	2	2
3	1	2	2(2)	1	1	2	2	2	1	1	2	2
4	1	2	2(2)	2	2	1	1	2	2	2	1	1
5	2	1	2(3)	1	2	1	2	3	1	2	1	2
6	2	1	2(3)	2	1	2	1	3	2	1	2	1
7	2	2	1(4)	1	2	2	1	4	1	2	2	1
8	2	2	1(4)	2	1	1	2	4	2	1	1	2

It is seen from the example above that for a four-level column, a line with points at each end is necessary, as shown in Figure 36.5. The figure shows a four-level column of three degrees of freedom replaced by three columns of one degree of freedom.

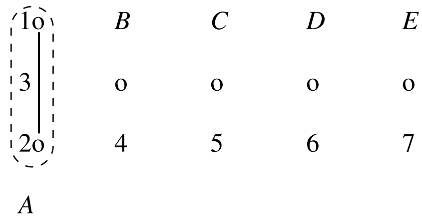

Figure 36.5
Layout with a four-level factor

When a four-level column is arranged in a two-level series orthogonal array, the four-level column is substituted for the three columns of the array; the three columns consist of any two columns and the column of interaction between the two columns. These three columns are similar to those illustrated in the linear graph in Figure 36.5. Notice the graph has a line with two points at each end. To do this, pick up a line and the two points at the ends of the line, select any two points (normally select the two points at the ends) out of the three columns to form a new four-level column. Letting the four combinations of the two columns, 11, 12, 21, and 22, correspond to 1, 2, 3, and 4, respectively, a four-level column is then formed. Delete the three columns described above and then put the four-level column in the table, as shown in Table 36.2. To determine the degrees of freedom, the three columns with one degree of freedom each are replaced by a column with three degrees of freedom.

Next, the formation of an eight-level column is illustrated. For this purpose, a closed triangle (plus the line to represent the interaction between a point at the top and the baseline in the triangle) from a two-level series linear graph is used. These seven columns (i.e., three apexes, three bases, and one perpendicular line) with seven degrees of freedom in total are replaced by a eight-level column with seven degrees of freedom.

Figure 36.6 shows a triangle chosen from a linear graph of the L_{16} array. From apex 1, draw a perpendicular line to base 6; the line is found to be column 7 from Appendix A. There are eight combinations of 1 and 2 formed by the three columns (three apexes 1, 2, and 4). The combinations are 111, 112, 121, 122, 211, 212, 221, and 222. As in the case of the four-level arrangement, it is necessary to erase column 1 through 7 from the orthogonal array after inserting the eight-level column. Thus, the $L_{16}(2^{15})$ array is rearranged to be an $L_{16}(8 \times 2^8)$ table. The new orthogonal array is shown in Table 36.3.

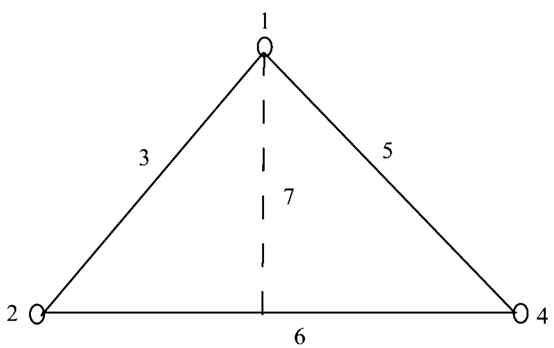

Figure 36.6
Linear graph from an L_{16} array

Table 36.3
Orthogonal array $L_{16}(8 \times 2^8)$ from $L_{16}(2^{15})$

Column	1	2	4	*A*	1	2	4	*A*
	1	1	1	1	2	1	1	5
	1	1	2	2	2	1	2	6
	1	2	1	3	2	2	1	7
	1	2	2	4	2	2	2	8

Col.	**1–7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
No.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2
3	2	1	1	1	1	2	2	2	2
4	2	2	2	2	2	1	1	1	1
5	3	1	1	2	2	1	1	2	2
6	3	2	2	1	1	2	2	1	1
7	4	1	1	2	2	2	2	1	1
8	4	2	2	1	1	1	1	2	2
9	5	1	2	1	2	1	2	1	2
10	5	2	1	2	1	2	1	2	1
11	6	1	2	1	2	2	1	2	1
12	6	2	1	2	1	1	2	1	2
13	7	1	2	2	1	1	2	2	1
14	7	2	1	1	2	2	1	1	2
15	8	1	2	2	1	2	1	1	2
16	8	2	1	1	2	1	2	2	1

❒ Example

To obtain information concerning the tires for passenger cars, five two-level factors from the manufacturing process, *A*, *B*, *C*, *D,* and *E,* are cited. It is judged that factor A is most important, followed by *B* and *C*. Accordingly, it is desirable to obtain the interactions $A \times B$ and $A \times C$. For the life test of tires, four passenger cars, R_1, R_2, R_3, and R_4, are available and there are no driving restrictions during the testing.

In other words, once the tires for the test are put on the cars, there is no restriction on the roads driven or the distance traveled, and the cars are allowed to run as usual. As a result, the driving conditions among these four cars must differ considerably.

On the other hand, it is known that the different positions of tires of a car cause different conditions; the condition of the rear tires is worse than the front and there is also a difference between the two front tires since a car is not symmetrical, a driver does not sit in the middle, the road conditions for both sides are not the same, and so on.

For these reasons, we must consider the difference among the four positions in a car: V_1: front right; V_2: front left; V_3: rear right; and V_4: rear left. Such influences should not interfere with the effects of the five factors and interactions $A \times B$ and $A \times C$.

The real purpose of the experiment is to obtain the information concerning factors A, B, C, D, and E, and not for R and V. The character of factors A, B, C, D, and E is different from that of factors R and V.

The purpose of factors A, B, C, D, and E, called *control factors,* is the selection of the optimum levels. There are four cars, R_1, R_2, R_3, and R_4. For each level of R, the number of tests is limited to four (four wheels). The factor that has restricted the number of tests for each level, in this case four, is called a *block factor.* From a technical viewpoint, the significance of the difference between levels in a block factor is not important. Therefore, the effect of such a factor is not useful for adjustment of the levels of other factors. The purpose of citing a block factor is merely to avoid its effect being mixed with control factors.

The factor, like the position of tire, V, is called an *indicative factor*. It is meaningless to find the best level among the levels of an indicative factor.

From the explanation of block factor or indicative factor, the information of R (cars) and V (positions of tires) are not needed; these factors are cited so to avoid mixing the control factors (A, B, C, D, and E).

This experiment is assigned as follows. Factors and degrees of freedom are listed in Table 36.4. All factorial effects belong to either two levels or four levels; the total degrees of freedom is 13; hence L_{16} (with 15 total degrees of freedom) is probably an appropriate one. (The selection of either a two- or three-level series orthogonal array depends on the majority of the number of levels of factors.)

R and V are both four-level factors. As described in Chapter 35, a four-level factor has to replace three two-level columns, which consist of a line and two points at the ends. Accordingly, the linear graph required in this experiment is shown in Figure 36.7.

There are six standard types of linear graphs given for an L_{16} array [1]. But normally, we cannot find an exactly identical linear graph to the one that is required in a particular experiment. What we must do is to examine the standard types of linear graphs, compare them with the linear graph required for our purpose, and select the one that might be most easily modified. Here, type 3 is selected, for example, and modified to get the linear graph we require. Figure 36.8 shows type 3.

Table 36.4
Distribution of degrees of freedom

Factorial Effect	Degrees of Freedom	Factorial Effect	Degrees of Freedom
A	1	$A \times B$	1
B	1	$A \times C$	1
C	1	R	3
D	1	V	3
E	1		

First, factorial effects A, B, C, $A \times B$, and $A \times C$ are assigned to the figure on the right side. Next, R and V are assigned as a line to the figure on the left side. To the remaining columns, factors D, E, and error are assigned. The layout is shown in Figure 36.9 and Table 36.5.

From the table, it is known that in experiment 10, the tire is manufactured under condition $A_1B_1C_2D_2E_1$; the tire thus manufactured is put on V_2, the front left wheel of car R_3. This is the case when only one tire is manufactured for each experiment number.

When four cars are used, two replications are performed for each experiment. When eight cars are used, the half portion, or 16 tires, are tested by cars R_1, R_2, R_3, and R_4, as shown in Figure 36.9. Then another half portion is tested using cars R_5, R_6, R_7, and R_8, which replace R_1, R_2, R_3, and R_4 as follows: R_5 for R_1, R_7 for R_3, R_6 for R_2, and R_8 for R_4.

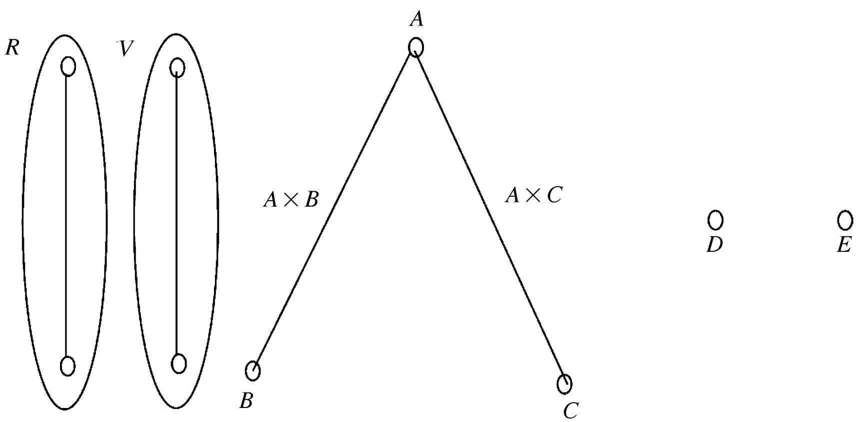

Figure 36.7
Linear graph for experiment

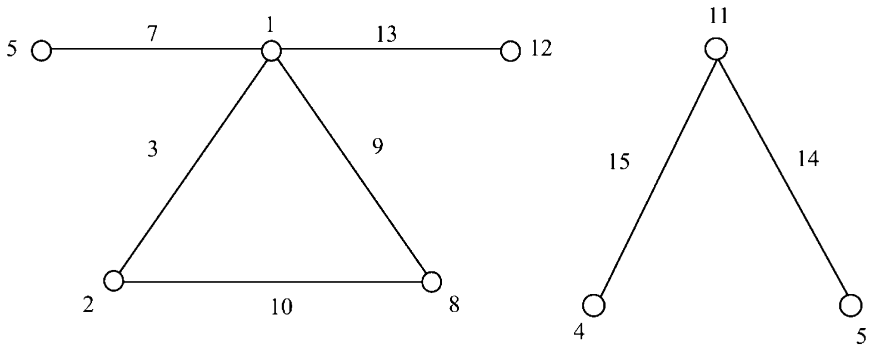

Figure 36.8
Type 3 linear graph

36.4. Dummy Treatment

To put a two-level factor A in a three-level series table, A is formally treated as a three-level factor: that is, to provide three levels for A and run one of the two actual levels twice. Usually, we select one level that is probably more important than the other, and let this level replicate. For instance, $A_1 = A_1$, $A_2 = A_2$, and $A_3 = A_1$. A_3 is formally treated as the third level of A, but actually it is A_1.

When a factor such as A is assigned to column 3 of an $L_9(3^4)$ array, the layout would look like the one shown in Table 36.6. In the table, 3 in the third column of the original L_9 array is rewritten as $1'$ to show that it is the dummy level of A.

The main effect A is calculated as

$$S_A = \frac{(A_1 + A_{1'})^2}{6r} + \frac{A_2^2}{3r} - \frac{(A_1 + A_1' + A_2)^2}{9r} \qquad (36.1)$$

or

$$S_A = \frac{(A_1 + A_1' - 2)^2}{18r} \qquad (36.2)$$

where r is the number of replication in orthogonal array L_9.

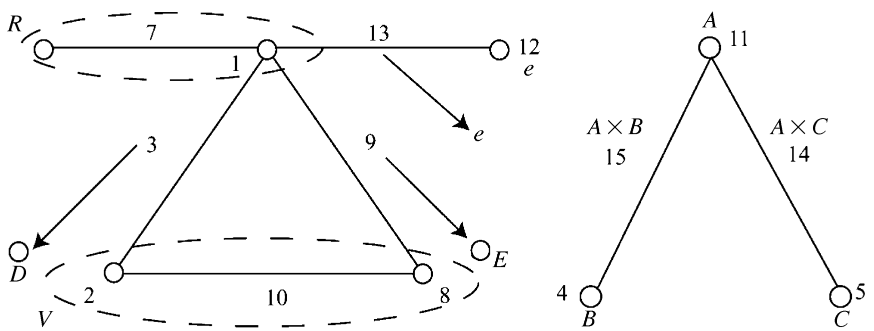

Figure 36.9
Layout of experiment

Table 36.5
Layout of experiment

No.	A	B	C	D	E	$A \times C$	$A \times B$	Car	Position	e	e
1	1	1	1	1	1	1	1	R_1	V_1	1	1
2	2	1	1	1	2	2	2	R_1	V_2	2	2
3	1	2	2	1	1	2	2	R_2	V_2	1	1
4	2	2	2	1	2	1	1	R_2	V_2	1	1
5	2	1	1	2	1	2	2	R_2	V_3	1	1
6	1	1	1	2	2	1	1	R_2	V_4	2	2
7	2	2	2	2	1	1	1	R_1	V_3	2	2
8	1	2	2	2	2	2	2	R_1	V_4	1	1
9	2	1	2	2	2	1	2	R_3	V_1	1	2
10	1	1	2	2	1	2	1	R_3	V_2	2	1
11	2	2	1	2	2	2	1	R_4	V_1	2	1
12	1	2	1	2	1	1	2	R_4	V_2	1	2
13	1	1	2	1	2	2	1	R_4	V_3	1	2
14	2	1	2	1	1	1	2	R_4	V_4	2	1
15	1	2	1	1	2	1	2	R_3	V_3	2	1
16	2	2	1	1	1	2	1	R_3	V_4	1	2
Column	11	4	5	3	9	14	15	1	2	1	1
								6	8	2	3
								7	10		

36.5. Combination Design

There is a technique to put two-level factors in a three-level series orthogonal array. When there are two factors with two levels, and if their interaction is not required, the combination design is used instead of the dummy treatment.

A combined factor (AB) with three levels is formed as follows:

$$(AB)_1 = A_1B_1$$

$$(AB)_2 = A_2B_1$$

$$(AB)_3 = A_1B_2$$

The main effect, A, is obtained from the difference of $(AB)_1$ and $(AB)_2$. Similarly, the main effect, B, is obtained from $(AB)_1$ and $(AB)_3$.

When there are 20 two-level factors, 10 three-level factors are formed by each of the two factors; these combined factors are assigned to an L_{27} array. Because

Table 36.6
Assignment of *A* to column 3

No.	Level 1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	3	1′	3
4	2	1	2	3
5	2	2	1′	1
6	2	3	1	2
7	3	1	1′	2
8	3	2	1	3
9	3	3	2	1

there are so many two-level factors, it is better to assign these factors to an $L_{32}(2^{31})$ array. But for an experiment such as the $2^{10} \times 3^6$ type, it is more practical to combine each two for the 10 two-level factors to get five three-level combined factors and assign the other six three-level factors to an L_{27} array.

Also, if it is necessary to obtain the interactions between combined factor *AB* and the three-level factor *C*, *AB* and *C* are assigned to the two dots of a line to obtain interactions $A \times C$ and $B \times C$.

A three-level factor (*A*) and a two-level factor (*B*) are combined to form a combined four-level factor:

$$(AB)_1 = A_1B_1$$

$$(AB)_2 = A_2B_1$$

$$(AB)_3 = A_3B_1$$

$$(AB)_4 = A_1B_2$$

The variation of a combined three-level factor (*AB*) by two factors (*A*, *B*) of two levels is calculated as

$$S_{(AB)} = \frac{1}{r}\,[(AB)_1^2 + (AB)_2^2 + (AB)_3^2] - \text{CF} \qquad (36.3)$$

where r is the number of units of A_1B_1, A_2B_1, and A_1B_2.

However, the following relationship does not hold, since *A* and *B* are not orthogonal:

$$S_{(AB)} = S_A + S_B \qquad (36.4)$$

If we want to separate A and B, calculate

$$S_A = \frac{[(AB)_1 - (AB)_2]^2}{2r} \tag{36.5}$$

$$S_B = \frac{[(AB)_1 - (AB)_3]^2}{2r} \tag{36.6}$$

and decompose $S_{(AB)}$ into

$$S_{(AB)} = S_A + S_{\text{res}} \tag{36.7}$$

or

$$S_{(AB)} = S_B + S'_{\text{res}} \tag{36.8}$$

In the analysis of variance table, $S_{(AB)}$ is listed inside the table, while S_A and S_B are listed outside.

Reference

1. ASI, 1987. *Orthogonal Arrays and Linear Graphs*. Livonia, Michigan: American Supplier Institute.