34 Two-Way Layout with Repetition

34.1 Introduction

In this chapter we describe experiments with repetitions. This chapter is based on Genichi Taguchi et al., *Design of Experiments*. Tokyo: Japanese Standards Association, 1973.

34.2. Same Number of Repetitions

The data in Table 34.1 show the height of bounding for two brands of golf balls: A_1 (Dunlop) and A_2 (Eagle). The bounding of two balls from each brand was measured at four temperatures: $B_1 = 0°\text{C}$, $B_2 = 10°\text{C}$, $B_3 = 20°\text{C}$, and $B_4 = 30°\text{C}$. After subtracting a working mean of 105.0 cm (Table 34.2*a*), the sums of repetitions, the sums of rows, and sums of columns are then calculated to get a supplementary table (Table 34.2*b*).

$$\text{CF} = \frac{27.0^2}{16}$$

$$= 45.56 \qquad (f = 1) \tag{34.1}$$

$$S_T = (-6.0)^2 + 0.1^2 + \cdots + 11.0^2 - \text{CF}$$

$$= 833.50 \qquad (f = 15) \tag{34.2}$$

Table 34.1
Bounding height of golf balls (cm)

	B_1	B_2	B_3	B_4
A_1	99.0 98.2	105.1 104.6	110.3 112.8	114.5 116.1
A_2	96.1 95.2	101.6 102.4	109.8 108.2	117.1 116.0

Next, the *variation between the combinations* of A and B, say S_{T_1}, is obtained. S_{T_1} can also be called either the *variation between experiments* (there are eight experimental combinations with different conditions), or the *variation between the primary units.*

$$S_{T_1} = \frac{1}{2}[(-12.8)^2 + (-0.3)^2 + \cdots + 23.1^2] - \text{CF}$$

$$= 826.04 \qquad (f = 7) \tag{34.3}$$

The variation between experiments is decomposed into the following variations:

S_A: main effect of A

S_{B_l}: linear effect of B

S_{B_q}: quadratic effect of B

S_{B_c}: cubic effect of B

Table 34.2
Supplementary table

	B_1	B_2	B_3	B_4
A_1	−6.0 −6.8	0.1 −0.4	5.3 7.8	9.5 11.1
A_2	−8.9 −9.8	−3.4 −2.6	4.8 3.2	12.1 11.0

(*a*)

	B_1	B_2	B_3	B_4	Total
A_1	−12.8	−0.3	13.1	20.6	20.6
A_2	−18.7	−6.0	8.0	23.1	6.4
Total	−31.5	−6.3	21.1	43.7	27.0

(*b*)

S_{AB_l}: interaction between the linear effect of B and the levels of A (A_1 and A_2)

S_{e_1}: primary error (error between experiments)

$$S_A = \frac{1}{16}(20.6 - 6.4)^2$$

$$= 12.60 \qquad (f = 1) \tag{34.4}$$

$$S_{B_l} = \frac{(-3B_1 - B_2 + B_3 + 3B_4)^2}{[(-3)^2 + (-1)^2 + 1^2 + 3^2](4)}$$

$$= \frac{[(-3)(31.5) - (1)(-6.3) + (1)(21.1) + (3)(43.7)]^2}{80}$$

$$= 800.11 \qquad (f = 1) \tag{34.5}$$

$$S_{B_q} = \frac{[(1)(-31.5) + (-1)(-6.3) + (-1)(21.1) + (1)(43.7)]^2}{(4)(4)}$$

$$= 0.42 \qquad (f = 1) \tag{34.6}$$

$$S_{B_c} = \frac{[(-1)(-31.5) + (3)(-6.3) + (-3)(21.1) + (1)(43.7)]^2}{(20)(4)}$$

$$= 0.61 \qquad (f = 1) \tag{34.7}$$

Next, the comparisons of the linear effects of B for A_1 and A_2 are calculated:

$$\begin{aligned} L(A_1) &= (-3)(-12.8) - (1)(-0.3) + (1)(13.1) + (3)(20.6) \\ &= 113.6 \\ L(A_2) &= (-3)(-18.7) - (1)(-6.0) + (1)(8.0) + (3)(23.1) \\ &= 139.4 \end{aligned} \tag{34.8}$$

The number of units for $L(A_1)$ and $L(A_2)$ are both equal to

$$[(-3)^2 + (-1)^2 + 1^2 + 3^2](2) = 40$$

$$S_{AB_l} = \frac{L(A_1)^2 + L(A_2)^2}{40} - \frac{[L(A_1) + L(A)_2)]^2}{80}$$

$$= \frac{113.6^2 + 13.94^2}{40} - \frac{253.0^2}{80}$$

$$= 8.32 \qquad (f = 1) \tag{34.9}$$

$$S_{e_1} = S_{T_1} - S_A - S_{B_l} - S_{B_q} - S_{B_c} - S_{AB_l}$$

$$= 3.98 \qquad (f = 2) \tag{34.10}$$

Next, the *variation within repetitions* (also called the *variation of the secondary error*), namely S_{e_2}, is calculated:

$$S_{e_2} = \left[(-6.0)^2 + (-6.8)^2 - \frac{(-12.8)^2}{2} \right] + \cdots$$
$$+ \left(12.1^2 + 11.0^2 - \frac{23.1^2}{2} \right)$$
$$= (-6.0) + (-6.8)^2 + \cdots + 12.1^2 + 11.0^2 - \text{CF}$$
$$- \left[\frac{(-12.8)^2 + \cdots + 23.1^2}{2} - \text{CF} \right]$$
$$= S_T - S_{T_1}$$
$$= 833.50 - 826.04$$
$$= 7.46 \qquad (f - 8) \tag{34.11}$$

The analysis of variance is shown in Table 34.3. (*Note:* When e_1 is significant as tested against e_2, then B_q, B_c, and e_1 are pooled to get a new primary error variance, which is used to test A, B_1, and AB_1.)

Comparing the primary error against the secondary error, we find it to be insignificant. Hence, we deem that the error between experiments does not exist, so e_1 and e_2 are pooled.

The results from the analysis of variance enable the following conclusions to be made:

1. There is a slight difference between the two brands.
2. The bouncing height increases linearly according to a rise in temperature.
3. The bouncing height increases linearly according to a rise in temperature, but the trend differs slightly between A_1 and A_2.

Estimation is made as follows: The linear equations of B are formed for A_1 and A_2, respectively. For case A_1,

Table 34.3
ANOVA table

Source	f	S	V	S'	ρ (%)
A	1	12.60	12.60	11.56	1.4
B					
l	1	800.11	800.11	799.07	95.9
q	1	0.42	0.42		
c	1	0.61	0.61		
AB_1	1	8.32	8.32	7.28	0.9
E_1	2	3.98	1.99		
E_2	8	7.46	0.932		
(e)	(12)	(12.47)	(1.04)	15.59	1.8
Total	15	833.50		833.50	100.0

$$\hat{\mu} = \overline{A}_1 + b_1(B - \overline{B}) \tag{34.12}$$

$$\begin{aligned}\overline{A}_1 &= \text{working mean} + \frac{1}{8} A_1 \\ &= 105.0 + \frac{20.6}{8} \\ &= 107.6\end{aligned} \tag{34.13}$$

$$\begin{aligned}\hat{b}_1 &= \frac{-3B_1 - B_2 + B_3 + 3B_4}{r(\lambda S)h} \\ &= \frac{(-3)(-12.8) - (1)(-0.3) + (1)(13.1) + (3)(20.6)}{(2)(10)(10)} \\ &= \frac{113.6}{200} \\ &= 0.568\end{aligned} \tag{34.14}$$

$$\begin{aligned}\overline{B} &= \frac{1}{4}(0 + 10 + 20 + 30) \\ &= 15\end{aligned} \tag{34.15}$$

From the above, the relationship between the bouncing height of golf ball A_1 and temperature, B, is

$$\hat{\mu} = 107.6 + 0.568(B - 15) \tag{34.16}$$

Case A_2 is similar to case A_1:

$$\overline{A}_2 = 105.0 + \frac{6.4}{8} = 105.8$$

$$\hat{b}_1 = \frac{139.4}{200} = 0.697$$

$$\hat{\mu} = 105.8 + 0.697(B - 15) \tag{34.17}$$

The results above show that the effect of bouncing by temperature for A_2 (Eagle) is about 23% higher than A_1 (Dunlop).

It is said that professional golfers tend to have lower scores during cold weather if their golf balls are kept warm. The following is the calculation to show how much the distance will be increased if a golf ball is warmed to 20°C.

To illustrate, B = 5°C and B = 20°C are placed into equations (34.16) and (34.17). When B = 5°C,

$$\begin{aligned}\text{Dunlop:}\quad \hat{\mu} &= 107.6 + (0.568)(5 - 15) \\ &= 101.92 \\ \text{Eagle:}\quad \hat{\mu} &= 105.8 + 0.697(5 - 15) \\ &= 98.83\end{aligned} \tag{34.18}$$

When $B = 20°C$,

$$\text{Dunlop:} \quad \hat{\mu} = 107.6 + (0.568)(20 - 15) = 110.44$$
$$\text{Eagle:} \quad \hat{\mu} = 105.8 + (0.697)(20 - 15) = 109.28 \qquad (34.19)$$

Hence, the increase in distance is

$$\text{Dunlop:} \quad (110.44 \div 101.92 - 1)(100) = 8.4\%$$
$$\text{Eagle:} \quad (109.28 \div 98.83 - 1)(100) = 10.6\% \qquad (34.20)$$

34.3. Different Numbers of Repetitions

In this section, analysis with an unequal number of data for each combination of A and B is described. In the experiments shown in Section 34.2, it seldom happens that an unequal number of repetitions occurs; but it can happen, especially when the data are obtained from questionnaires.

Table 34.4 shows the tensile strength of a certain product from three different manufacturers:

A_1: foreign product

A_2: our company's products

A_3: another domestic company's products

at four different temperatures: $B_1 = -30°C$, $B_2 = 0°C$, $B_3 = 30°C$, and $B_4 = 60°C$. The numbers of repetitions are 1 for A_1, 5 for A_2 (in A_2, 4 for B_3 and 3 for B_4), and 3 for A_3. A working mean of 80 has been subtracted. Now, how would we analyze these data?

First, calculate the mean for each combination. From the repeated data of each combination in Table 34.4, the mean is calculated to get Table 34.5. It is

Table 34.4
Tensile strength (kg/mm²)

	B_1	B_2	B_3	B_4
A_1	20	8	0	−9
A_2	22	12	−2	−12
	25	8	0	−14
	28	10	3	−13
	25	9	0	
	26	12		
A_3	17	6	−8	−20
	23	8	−6	−18
	20	4	−3	−22

Table 34.5
Table of means

	B_1	B_2	B_3	B_4	Total
A_1	20.0	8.0	0.0	−9.0	19.0
A_2	25.2	10.2	0.2	−13.0	22.6
A_3	20.0	6.0	−5.7	−20.0	0.3
Total	65.2	24.2	−5.5	−42.0	41.9

recommended that the means be calculated to at least one decimal place. Then do an analysis of variance using the means.

$$\text{CF} = \frac{41.9^2}{12} = 146.30 \tag{34.21}$$

$$S_A = \frac{19.0^2 + 22.6^2 + 0.3^2}{4} - \text{CF} = 71.66 \tag{34.22}$$

$$S_B = \frac{65.2^2 + 24.2^2 + \cdots + (-42.0)^2}{3} - \text{CF} \tag{34.23}$$

$$S_{T_1} = 20.2^2 + 8.0^2 + \cdots + (-20.0)^2 - \text{CF} = 2175.31 \tag{34.24}$$

S_{T_1} is the total variation between the combinations of A and B. From S_{T_1}, subtract S_A and S_B to get the variation, called the *interaction* between A and B and denoted by S_{AB}. S_A indicates the difference between the companies as a whole.

To make a more detailed comparison, the following contrasts are considered:

$$L_1(A) = \frac{A_1}{4} - \frac{A_2 + A_3}{8} \tag{34.25}$$

$$L_2(A) = \frac{A_2}{4} - \frac{A_3}{4} \tag{34.26}$$

L_1 shows the comparison between foreign and domestic products, and L_2 compares our company and another domestic company. Since L_1 and L_2 are orthogonal to each other, their variations, $S_{L1(A)}$ and $L_{S2(A)}$, are

$$S_A = S_{L_1}(A) + S_{L_2}(A) \tag{34.27}$$

where

$$S_{L_1}(A) = \frac{[L_1(A)]^2}{\text{sum of coefficients squared}} = \frac{2A_1 - A_2 - A_3}{(2^2)(4) + (-1)^2(8)}$$

$$= \frac{[(2)(19.0) - 22.6 - 0.3]^2}{24} = \frac{15.1^2}{24}$$

$$= 9.50 \tag{34.28}$$

$$S_{L_2}(A) = \frac{[L_2(A)]^2}{\text{sum of coefficients squared}} = \frac{(A_2 - A_3)^2}{(1^2)(4) + (-1)^2(4)}$$

$$= \frac{(22.6 - 0.3)^2}{8}$$

$$= 62.16 \qquad (34.29)$$

Next, we want to know whether or not the relationship between temperature and tensile strength is expressed by a linear equation. The linear component of temperature B is given by

$$S_{B_l} = \frac{(-3B_1 - B_2 + B_3 + 3B_4)^2}{r(\lambda^2 S)}$$

$$= \frac{[(-3)(65.2) - 24.2 + (-5.5) + (3)(-42.0)]^2}{(3)(20)}$$

$$= 2056.86 \qquad (34.30)$$

Let the variation of B, except its linear component, be B_{res}:

$$S_{B_{\text{res}}} = S_B - S_{B_l} = 2064.01 - 2056.86 = 7.15 \qquad (34.31)$$

For the linear coefficients of temperature, their differences between foreign and domestic, or between our company and another domestic company, are found by testing $S_{L_1(A)B_l}$ and $S_{L_2(A)B_l}$, respectively.

$$S_{L_1}(A)B_l = \frac{[2L(A_1) - L(A_2) - L(A_3)]^2}{[2^2 + (-1)^2(2)][(-3)^2 + (-1)^2 + 1^2 + 3^2]}$$

$$= \frac{[(2)(-95.0) + 124.6 + 131.7]^2}{120}$$

$$= 36.63 \qquad (34.32)$$

$$S_{L_2}(A)B_l = \frac{[L(A_2) - L(A_3)]^2}{[1^2 + (-1)^2][(-3)^2 + (-1)^2 + 1^2 + 3^2]}$$

$$= \frac{(-124.6 + 131.7)^2}{40}$$

$$= 1.26 \qquad (34.33)$$

where

$$L(A_i) = -3A_iB_1 - A_iB_2 + A_iB_3 + 3A_iB_4 \qquad (34.34)$$

The primary error variation, S_{e_1}, is obtained from S_{T_1} by subtracting all of the variations of the factorial effects that we have considered.

$$S_{e_1} = S_{T_1} - S_A - S_B - S_{L_1(A)B_1} - S_{L_2(A)B_1}$$

$$= 2175.31 - 71.66 - 2064.01 - 36.63 - 1.26$$

$$= 1.75 \qquad (f = 4) \qquad (34.35)$$

Next, the variation within repetitions, S_{e_2}, is calculated. The error variation of each combination of A and B in Table 34.4 is calculated such that the number of repetitions is different for each combination. These variations are summed and then divided by the harmonic mean of the numbers or repetitions. This is denoted by $\bar{r}$.

$$S_{e_2} = \frac{1}{r}\,[S_{11}\text{ (error variation within the repetitions of }A_1B_1) + \cdots$$
$$+\, S_{34}\text{ (error variation within the repetions of }A_3B_4)]$$
$$= \frac{1}{r}\left\{\left(20 - \frac{20^2}{1}\right) + \cdots + \left[(-20)^2 + (-18)^2 + (-22)^2 - \frac{(-20-18-22)^2}{3}\right]\right\}$$
$$= \frac{1}{r}\,[\text{(sum of the individual data squared)} - \text{(variation between the combinations of }A\text{ and }B)]$$
$$= \frac{1}{r}\,(93.02) \tag{34.36}$$

$$\frac{1}{r} = \frac{1}{\text{number of combinations of }A\text{ and }B}$$
$$\text{(sum of recipricals of the number of repetitions of }A_iB_j)$$
$$= \frac{1}{12}\left(\frac{1}{1} + \frac{1}{1} + \cdots + \frac{1}{3}\right) = \frac{1}{1.8997} \tag{34.37}$$

Putting equation (34.37) into equation (34.36) yields

$$S_{e_2} = 48.97 \tag{34.38}$$

Since S_A and S_B were obtained from the mean values, S_{e_2} is multiplied by $1/\bar{r}$. From the results above, the analysis of variance can be obtained, as shown in Table 34.6. In the table, S_T was obtained by

$$S_T = S_A + S_B + \cdots + S_{e_2} = 2224.28$$

The degrees of freedom for S_{e_2} are obtained by summing the degrees of freedom for the repetitions A_1B_1, A_1B_2, ... , A_3B_4 to be $0 + \cdots + 2 = 21$.

The analysis of variance shows that $L_{2(\mathrm{A})}$, B_1, and $L_{1(\mathrm{A})\mathrm{B}_1}$ are significant. The significance of $L_{2(\mathrm{A})}$ means there is a significant difference between the two domestic companies. It has also been determined that the influence of temperature to tensile strength is linear; the extent to which temperature affects tensile strength differs between the foreign and domestic products.

Table 34.6
ANOVA table

Source	f	S	V	S'	ρ (%)
A					
$L_1(A)$	1	9.50	9.50[a]		
$L_2(A)$	1	62.16	62.16	59.793	2.69
B					
l	1	2056.86	2056.86	2054.493	92.37
Res	2	7.15	3.58[a]		
AB					
$L_1(A)B_1$	1	36.63	36.63	34.263	1.54
$L_2(A)B_1$	1	1.26	1.26[a]		
e_1	4	1.75	0.438[a]		
e_2	21	48.97	2.332[a]		(3.46)
e'	(29)	(68.63)	(2.367)	(75.731)	(3.40)
Total	32	2224.28		2224.28	100.00

[a] Pooled data.

In the case of foreign products, the influence per 1°C, or b_1, is given by

$$\hat{b} = \frac{L(A_1)}{r(\lambda S)h} = \frac{(-3)(20) - 8.0 + 0.0 + (3)(-9.0)}{(1)(10)(30)} = \frac{-95.0}{300} = -0.317 \qquad (34.39)$$

Similarly, b_2 for domestic products is

$$\hat{b}_2 = \frac{L(A)_2 + L(A_3)}{r(\lambda S)h} = \frac{-124.6 - 131.7}{(2)(10)(30)} = \frac{-256.3}{600} = -0.427 \qquad (34.40)$$

Therefore, the tensile strength is expressed by the linear equation of temperature.

For foreign products,

$$\begin{aligned}\hat{\mu}_1 &= \overline{A}_1 + b_1(B - \overline{B}) \\ &= 80 + 4.75 - (0.317)(B - 15) \\ &= 84.75 - (0.317)(B - 15) \qquad (34.41)\end{aligned}$$

Our company's products:

$$\begin{aligned}\hat{\mu}_2 &= \overline{A}_2 + b_2(B - \overline{B}) \\ &= 80 + 5.65 - (0.427)(B - 15) \\ &= 85.65 - (0.427)(B - 15) \qquad (34.42)\end{aligned}$$

Another domestic company's products:

$$\begin{aligned}\hat{\mu}_3 &= \overline{A}_3 + b_3(B - \overline{B}) \\ &= 80 + 0.075 - (0.427)(B - 15) \\ &= 80.075 - (0.427)(B - 15) \qquad (34.43)\end{aligned}$$