23 Feedback Control Based on Product Characteristics

23.1. Introduction

Under feedback control, the characteristics of a product produced by a production process are checked, and if a difference between a characteristic and a target is greater than a certain level, the process is adjusted. If the difference is not so great, production is continued without adjustment. This type of control system can be applied without modification to cases where the process is controlled automatically. Even when a process is controlled automatically, routine control by human beings is necessary.

23.2. System Design of Feedback Control Based on Quality Characteristics

Many factories today use a system in which the characteristics of a product are checked and the process condition is controlled if the quality level is off-target. However, when it is costly to adjust process conditions, feedback about an entire process based on quality engineering is rarely used. The optimal design of a feedback controller to maintain quality is described below.

The following parameters are defined:

- *Specification of objective characteristic:* $m \pm \Delta$
- *Loss due to a defective:* A (dollars)
- *Checking (measuring) cost:* B (dollars)
- *Adjustment cost:* C (dollars)
- *Current checking interval:* n_0 (units)
- *Current mean adjustment interval (observed):* u_0 (units)

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- ❑ *Current adjustment (control) limit:* D_0
- ❑ *Time lag of the checking method:* l (units)
- ❑ *Optimal checking interval:* n (units)
- ❑ *Estimate of mean adjustment interval:* u (units)

If checking is done by n-unit intervals at a cost of B dollars, the checking cost per product is given as

$$\text{checking cost} = \frac{B}{n} \quad \text{dollars} \tag{23.1}$$

The average checking interval depends on the individual process: in particular, the stability of the process given any adjustment limit, D. A production process changes in a complex manner due to various factors (e.g., external disturbances, tool wear, measurement error). Here the average adjustment interval, u, is assumed to be proportional to the squared adjustment limit, D. This assumption is an approximation since we can assume an optimal adjustment limit, D, and estimate the average adjustment interval, u, based on that assumption. We can validate this assumption later by comparing it with the data observed. Furthermore, the proportion coefficient, λ, is determined based on past performance. We first determine the current (initial) adjustment limit, D_0, then investigate the mean adjustment interval, u_0, under D_0, and then determine the proportion coefficient, λ, using the following equation:

$$\lambda = \frac{u_0}{D_0^2} \tag{23.2}$$

(Many Japanese firms use $D_0 = \Delta/3$, while many U.S. firms use $D_0 = D$. Which is more rational can be determined only through comparison with the actual optimal value, D.) After deciding on an optimal adjustment limit, D, the mean adjustment interval, u, is estimated by

$$u = u_0 \frac{D^2}{D_0^2} \tag{23.3}$$

Under this assumption, the checking adjustment costs are calculated based on the use of equations (23.1) and (23.3):

$$\frac{B}{n} + \frac{C}{u} = \frac{B}{n} + \frac{D_0^2 C}{u_0 D^2} \tag{23.4}$$

On the other hand, what would happen to the quality level if checking and adjustment were performed? Assuming the absence of checking error, the loss is given by

$$\frac{A}{\Delta^2}\left[\frac{D^2}{3} + \left(\frac{n+1}{2} + 1\right)\frac{D^2}{u}\right] \tag{23.5}$$

The characteristic is considered to be distributed uniformly within the adjustment limit, $m \pm D$; therefore, $D^2/3$ is given by the variance of the characteristic value:

$$\frac{1}{2D}\int_{m-D}^{m+D} (y - m)^2 \, dy \tag{23.6}$$

When checking is done with interval n, and the last reading is within the adjustment limit but the current measurement is outside the adjustment limit, the average number of products outside the limit is roughly equal to $(n + 1)/2$. Although the mean of those falling outside the adjustment limit is greater than D, D is used as an approximation, since the difference is not great. Therefore, the variance when the measurement is outside the adjustment limit is obtained by dividing:

$$\left(\frac{n+1}{2} + 1\right) D^2 \tag{23.7}$$

with mean adjustment interval u,

$$\left(\frac{n+1}{2} + 1\right) \frac{D^2}{u} \tag{23.8}$$

Therefore, the quality level in terms of money is given by

$$\frac{A}{\Delta^2}\left[\frac{D^2}{3} + \left(\frac{n+1}{2} + 1\right) \frac{D^2}{u}\right] \tag{23.9}$$

Total loss L, which is the sum of checking and adjustment costs [i.e., equation (23.4)] and the quality level [i.e., equation (23.9)], is

$$L = \frac{B}{n} + \frac{C}{u} + \frac{A}{\Delta^2}\left[\frac{D^2}{3} + \left(\frac{n+1}{2} + 1\right) \frac{D^2}{u}\right] \tag{23.10}$$

After substituting equation (23.3), the loss, L, becomes

$$L = \frac{B}{n} + \frac{D_0^2 C}{u_0 D^2} + \frac{A}{\Delta^2}\left[\frac{D_0^2}{3} + \left(\frac{n_0+1}{2} + 1\right) \frac{D_0^2}{u_0}\right] \tag{23.11}$$

In particular, if n, u, and D are current n_0, u_0, and D_0, equation (23.11) becomes

$$L_0 = \frac{B}{n_0} + \frac{C}{u_0} + \frac{A}{\Delta^2}\left[\frac{D_0^2}{3} + \left(\frac{n_0+1}{2} + 1\right) \frac{D_0^2}{u_0}\right] \tag{23.12}$$

The optimal checking interval, n, and the optimal adjustment limit, D, are derived by differentiating equation (23.11) with respect to n and D and by solving them after equating them to zero:

$$\frac{dL}{dn} = -\frac{B}{n^2} + \frac{A}{\Delta^2}\left(\frac{1}{2}\right)\frac{D_0^2}{u_0} = 0 \tag{23.13}$$

Thus, the optimal checking interval is

$$n = \sqrt{\frac{2u_0 B}{A}}\,\frac{\Delta}{D_0} \tag{23.14}$$

The optimal adjustment limit D is found from

$$\frac{dL}{dD} = -2\frac{D_0^2 C}{u_0 D^3} + \frac{A}{\Delta^2}\left(\frac{2}{3}\right) D = 0 \tag{23.15}$$

$$D = \left(\frac{3C}{A}\frac{D_0^2}{u_0}\Delta^2\right)^{1/4} \tag{23.16}$$

In practice, the values of n and D can be rounded to convenient numbers. By substituting these numbers into equation (23.10), the loss, L, of the optimal system is derived. Here, u is estimated according to equation (23.3).

❒ Example

Consider the control of a certain component part dimension. The parameters are as follows.

- ❑ *Specification:* $m \pm 15$ μm
- ❑ *Loss due to a defective:* $A = 80$ cents
- ❑ *Checking cost:* $B = \$1.50$
- ❑ *Time lag:* $l = 1$ unit
- ❑ *Adjustment cost:* $C = \$12$
- ❑ *Current checking interval:* $n_0 = 600$ units, once every two hours
- ❑ *Current adjustment limit:* $\pm D_0 = \pm 5$ μm
- ❑ *Current mean adjustment interval:* $u_0 = 1200$ units, twice a day

(a) Derive the optimal checking interval, n, and the optimal adjustment limit, D, and estimate the extent of improvement from the current situation.

(b) Estimate the process capability index C_p.

(c) Assuming that it takes 3 minutes for checking and 15 minutes for adjustment, calculate the required worker-hours for this process to two digits after the decimal.

(a) We begin the derivation as follows:

$$n = \sqrt{\frac{2u_0B}{A}\frac{\Delta}{D_0}} = \sqrt{\frac{(2)(1200)(1.50)}{0.80}\left(\frac{15}{5}\right)}$$

$$= 201 \longrightarrow 200 \text{ units} \tag{23.17}$$

$$D = \left(\frac{3C}{A}\frac{D_0^2}{u_0}\Delta^2\right)^{1/4} = \left[\frac{(3)(12)}{0.80}\left(\frac{5^2}{1200}\right)(15^2)\right]^{1/4}$$

$$= 3.8 \longrightarrow 4.0\ \mu\text{m} \tag{23.18}$$

Thus, loss function L of feedback control is

$$L = \frac{B}{n} + \frac{C}{u} + \frac{A}{\Delta^2}\left[\frac{D^2}{3} + \left(\frac{n+1}{2} + l\right)\frac{D^2}{u}\right] \tag{23.19}$$

Substituting

$$u = u_0\frac{D^2}{D_0^2} = (1200)\left(\frac{4^2}{5^2}\right)$$

$$= 768 \text{ units} \tag{23.20}$$

into equation (23.19) yields

$$L = \frac{1.50}{200} + \frac{12}{768} + \frac{0.80}{15^2}\left[\frac{4^2}{3} + \left(\frac{201}{2} + l\right)\left(\frac{4^2}{768}\right)\right]$$
$$= 0.0075 + 1.0056 + 1.0090 + 0.75$$
$$= \$0.0496 \tag{23.21}$$

Under the current conditions,

$$L_0 = \frac{B}{n_0} + \frac{C}{u_0} + \frac{A}{\Delta^2}\left[\frac{D_0^2}{3} + \left(\frac{n_0 + 1}{2} + l\right)\frac{D_0^2}{u_0}\right]$$
$$= \frac{1.50}{600} + \frac{12}{1200} + \frac{0.80}{15^2}\left[\frac{5^2}{3} + \left(\frac{601}{2} + l\right)\left(\frac{5^2}{1200}\right)\right]$$
$$= 0.0025 + 0.01 + 0.0296 + 2.23$$
$$= \$0.0644 \tag{23.22}$$

Assuming that the total hours of operation per year are 2000, a production rate of 300 units per hour will result in a yearly improvement of

$$(0.0644 - 0.0496)\,(300)\,(2000) = \$8880 \tag{23.23}$$

(b) Although this demonstration of the improvement effect should be adequate to demonstrate the effectiveness of feedback control, the process capability index is derived below for reference. The square root of the term in brackets of in equation (23.19) is an estimate of the standard deviation, σ:

$$\sigma = \sqrt{\frac{D^2}{3} + \left(\frac{n + 1}{2} + 1\right)\frac{D^2}{u}} \tag{23.24}$$

Since $\Delta = 15$ and the definition of the process capability index is [tolerance ÷ (6 × standard deviation)]

$$C_p = \frac{(2)(15)}{\sqrt{(4^2/3) + [(201/2) + 1]\,(4^2/768)}\;(6)} \approx 1.83 \tag{23.25}$$

The process capability index is about 1.8. This number deteriorates if there is measurement error. The C_p value of the current condition is

$$C_p = \frac{(2)(15)}{\sqrt{(5^2/3) + [(601/2) + 1]\,(5^2/1200)}\;(6)} = 1.31 \tag{23.26}$$

(c) Under the optimal checking interval, checking is done every 40 minutes; it takes 3 minutes for checking. Therefore, in one 8-hour day, there will be 12 checks, amounting to 36 minutes. The estimated adjustment interval is once every 768 pieces and

$$\frac{(300)(8)}{768} \approx 3.1 \tag{23.27}$$

times a day. It takes (15 minutes)(3.1) = 46.5 minutes per day. Assuming that work minutes per day per worker is 480 minutes, the number of personnel required is

$$\frac{36 + 46.5}{480} = 0.17 \text{ person} \tag{23.28}$$

Similar calculations are done for each process, and an allocation of processes per worker is determined. For example, one person can be in charge of seven processes (i.e., M_1, M_2, ... , M_7), and the checking interval is 200, 150, 180, 300, 100, 190, and 250, respectively. These can be rearranged with +30% flexibility to obtain the following schedule.

- *Once every 0.5 hour* (once every 150 pieces): M_1, M_2, M_3, M_5, M_6
- *Once every 1.0 hour* (once every 300 pieces): M_4, M_7

One should create an easily understandable checking system, and based on the system, the number of checking persons is determined.

23.3. Batch Production Process

Batch production is processing of a large number of products or a large volume of products at one time. If liquid, gas, or powder is flowing continuously in a pipe, it can be analyzed, as in Section 23.2, by setting the unit of production as 1 liter, and so on. On the other hand, if a film is deposited on 100 wafers at a time, the mean film thickness of the batch can be controlled, although one cannot control the film thickness among different wafers in the same batch. Control of the mean film thickness is done by estimating the mean of the entire batch and controlling process.

If many units are processed in one step, the unit of production is the number of units in a batch. Checking is done for one or several units from a batch to estimate the average of the batch. When this method is used to estimate the mean of all the products, we have to be concerned about the problem of estimation error. Let σ_m^2 be the measurement (estimated) error variance, and the loss function, L, be as follows:

$$L = \frac{B}{n} + \frac{C}{u} + \frac{A}{\Delta^2}\left[\frac{D^2}{3} + \left(\frac{n+1}{2} + 1\right)\frac{D^2}{u} + \sigma_m^2\right] \tag{23.29}$$

Measurement error in feedback control is counted as an independent error item.

❒ Example

In an injection molding process that produces 12 units in one shot, an average product dimension is estimated once every 100 shots and adjusted relative to its target value. It takes 1 hour of production to produce 100 shots, and the loss per defective is 30 cents. The tolerance is ±120 μm; current adjustment limit is ±50 μm; adjustment cost is $18; mean adjustment interval is 800 shots; cost B for the estimation of average size is $4; and time lag l is four shots.

(a) Derive the optimal checking interval, n, and optimal adjustment limit, D, and obtain the possible gain per shots and per year (200,000 shots/year).

(b) Suppose that the standard deviation of the estimation error of the current measuring methods is 15 μm. Compare this with a new measuring method whose estimation error is supposed to be one-third of the current error and whose cost is $7 ($3 more than the current cost). Which method is better, and by how much? Assume that the time lag is the same for both alternatives.

(c) The standard deviation between cavities is 6 μm. Obtain the process capability index when the process is controlled with the new measuring method.

(a) Since 12 units are produced at a time, 12 is treated as the unit of production. The same treatment will be used for a mass-produced chemical product (e.g., a 10-ton batch). A is the loss when all units in a batch are defective. Thus, the parameters can be summarized as follows:

- *Tolerance of the dimension:* $\Delta = 120$ μm
- *Loss due to defective:* $A = (30)(12) = \$3.60$
- *Cost of estimating the average dimension:* $B = \$4$
- *Adjusting cost:* $C = \$18$
- *Time lag:* $l = 4$ shots
- *Mean dimension checking interval:* $n_0 = 100$ shots
- *Current adjustment limit:* $\pm D_0 = \pm 50$ (μm)
- *Current mean adjustment interval:* $u_0 = 800$ shots

Derive the optimal checking interval, n, and optimal adjustment limit, D:

$$n = \sqrt{\frac{2u_0B}{A}\frac{\Delta}{D}} = \sqrt{\frac{(2)(800)(4)}{3.60}}\left(\frac{120}{50}\right) = 101 \rightarrow 100 \text{ shots} \qquad (23.30)$$

Thus, the current level of n is optimal:

$$D = \left(\frac{3C}{A}\frac{D_0^2}{u_0}\Delta^2\right)^{1/4} = \left[\frac{(3)(18)}{3.60}\left(\frac{50^2}{800}\right)(120^2)\right]^{1/4} = 29 \rightarrow 30 \ \mu\text{m} \qquad (23.31)$$

Thus, the adjustment limit, 30, which is smaller than the current 50, is better.

The loss under the current method, L_0, is

$$L_0 = \frac{B}{n_0} + \frac{C}{u_0} + \frac{A}{\Delta^2}\left[\frac{D_0^2}{3} + \left(\frac{n+1}{2} + 1\right)\frac{D_0^2}{u_0}\right]$$

$$= \frac{4}{100} + \frac{18}{800} + \frac{3.60}{120^2}\left[\frac{50^2}{3} + \left(\frac{101}{2} + 4\right)\left(\frac{50^2}{800}\right)\right]$$

$$= 0.04 + 0.0225 + 0.2083 + 4.26$$

$$= 31 \text{ cents} \qquad (23.32)$$

Under the optimal conditions, we have that

$$u = u_0 \frac{D^2}{D_0^2} = (800)\left(\frac{30^2}{50^2}\right) = 288 \text{ shots} \qquad (23.33)$$

and the loss will be

$$L = \frac{4}{100} + \frac{18}{288} + \frac{3.60}{120^2}\left[\frac{30^2}{3} + \left(\frac{101}{2} + 4\right)\left(\frac{30^2}{288}\right)\right]$$

$$= 0.04 + 0.0625 + 0.075 + 4.26$$

$$= 22 \text{ cents} \qquad (23.34)$$

Therefore, use of the optimal system results in an improvement over the current system by

$$(31 - 0.22)(100)(2000) = \$18{,}000 \qquad (23.35)$$

(b) For the current checking method, the loss due to measurement error (its standard deviation),

$$\frac{A}{\Delta^2}\sigma_m^2 = \frac{3.60}{120^2}(15^2) = 6 \text{ cents} \qquad (23.36)$$

is added to equation (23.34):

$$0.22 + 0.06 = 28 \text{ cents} \qquad (23.37)$$

On the other hand, the new measuring method costs \$7 rather than \$4, and thus it is necessary to derive the optimal checking interval, n, again:

$$n = \sqrt{\frac{(2)(800)(7)}{3.60}}\left(\frac{120}{50}\right)$$

$$= 134 \rightarrow 150 \text{ shots once every 1.5 hours} \qquad (23.38)$$

Therefore, the loss, L, is, by setting $\sigma_m = 15/3 = 5\ \mu m$,

$$L = \frac{B}{n} + \frac{C}{u} + \frac{A}{\Delta^2}\left[\frac{D^2}{3} + \left(\frac{n+1}{2} + 1\right)\frac{D^2}{u} + \sigma_m^2\right]$$
$$= \frac{7}{150} + \frac{(18)}{288} + \frac{3.60}{120^2}\left[\frac{30^2}{3} + \left(\frac{151}{2} + 4\right)\left(\frac{30^2}{288}\right) + 5^2\right]$$
$$= 0.0467 + 0.0625 + 0.075 + 6.21 + 0.62$$
$$= 25 \text{ cents} \tag{23.39}$$

Because measuring accuracy is inferior using the current method, the new measuring method results in an improvement of

$$(0.28 - 0.25)\,(100)\,(2000) \approx \$6000 \tag{23.40}$$

(c) The on-line control of a production process does not allow control of variability among cavities created by one shot. Therefore, if σ_c is the standard deviation among cavities, the overall standard deviation, σ, is

$$\sigma = \sqrt{\frac{D^2}{3} + \left(\frac{n+1}{2} + 1\right)\frac{D^2}{u} + \sigma_m^2 + \sigma_c^2}$$
$$= \sqrt{\frac{30^2}{3} + \left(\frac{151}{2} + 4\right)\left(\frac{30^2}{288}\right) + 5^2 + 6^2}$$
$$= 24.7\ \mu\text{m} \tag{23.41}$$

Therefore, the process capability index, C_p, is

$$C_p = \frac{(2)\,(120)}{(6)\,(24.7)} = 1.6 \tag{23.42}$$

23.4. Design of a Process Control Gauge (Using a Boundary Sample)

When we are confronted with quality problems where continuous data are not available, control of process conditions and preventive maintenance become important. However, we cannot be 100% certain about such a process, even if we control process conditions as much as possible. Additionally, we cannot even be sure if it is rational to control all process conditions. This is analogous to a health checkup, where it is irrational to check for symptoms of all diseases, and thus we do not attempt to control problems that occur rarely. Even if continuous data are not given for certain quality items, it is possible in many cases to judge the quality level by a factor such as appearance. Thus, instead of attempting to control all process conditions, we should control major conditions only, together with the use of preventive maintenance based on quality data. The method of preventive maintenance can be applied to the preventive method for delivered products.

Suppose that tolerance, Δ, is specified for the appearance of a product. The appearance is currently controlled at D_0, using a boundary sample. Similar to the example in Section 23.2, parameters are defined as follows.

A: loss per defective

Δ: tolerance

B: checking cost

C: adjustment cost

n_0: current checking interval

u_0: current mean adjustment interval

D_0: current adjustment limit

l: time lag

The optimal checking interval, n, and optimal adjustment limit, D, are given by the following:

$$n = \sqrt{\frac{2u_0B}{A}\frac{\Delta}{D_0}} \tag{23.43}$$

$$D = \left(\frac{3C}{A}\frac{D_0^2}{u_0}\Delta^2\right)^{1/4} \tag{23.44}$$

If pass/fail is determined using a boundary or a limit sample without an intermediate boundary sample, we have the following:

$$D_0 = \Delta \text{ (tolerance)} \tag{23.45}$$

We often want to conduct preventive maintenance by providing an appropriate limit sample before actually producing defectives. This is called the *design method for preventive maintenance* with inspection, described fully below.

When continuous data are not available, inspection is done with a boundary sample or a gauge. However, it is still possible to devise a method in which a boundary sample or a gauge midway between pass and fail is prepared so that process adjustment is required if a product exceeds the boundary sample. Suppose that the intermediate condition can be expressed quantitatively as a continuous value, and its ratio with the tolerance limit is ϕ, that is,

$$\phi = \frac{\text{value of intermediate condition}}{\Delta} = \frac{D}{\Delta} \tag{23.46}$$

Accordingly, the adjustment limit D can be expressed as

$$D = \phi\Delta \tag{23.47}$$

Since the current method is $D_0 = \Delta$, the optimal adjustment limit, D, and optimal checking interval, n, are determined by substituting $D_0 = \Delta$, and $u_0 = \bar{u}$ in the formula for the optimal adjustment limit, D:

$$D = \left(\frac{3C}{A} + \frac{D_0^2}{u_0}\,\Delta^2\right)^{1/4} = \left(\frac{3C}{A}\,\frac{\Delta^2}{\bar{u}}\,\Delta^2\right)^{1/4} = \left(\frac{3C}{A\bar{u}}\right)^{1/4}\Delta \tag{23.48}$$

From equation (23.47),

$$D = \phi\Delta \tag{23.49}$$

Thus, ϕ is given by

$$\phi = \left(\frac{3C}{A\bar{u}}\right)^{1/4} \tag{23.50}$$

Furthermore, the optimal checking interval, n, is

$$n = \sqrt{\frac{2u_0 B}{AD_0^2}}\,\Delta \tag{23.51}$$

By substituting

$$D_0 = \Delta$$

$$u_0 = \bar{u}\,\frac{D_0^2}{\Delta^2} = \bar{u}\,\frac{\Delta^2}{\Delta^2} = \bar{u} \tag{23.52}$$

the following formula is obtained. The optimal checking interval, n, is determined independently from the adjustment limit, as shown by:

$$n = \sqrt{\frac{2\bar{u}B}{A\Delta^2}}\,\Delta = \sqrt{\frac{2\bar{u}B}{A}} \tag{23.53}$$

Furthermore, the loss function, L is obtained based on the formula for continuous values. That is,

$$L = \frac{B}{n} + \frac{C}{u} + \frac{A}{\Delta^2}\left[\frac{D^2}{3} + \left(\frac{n+1}{2} + 1\right)\frac{D^2}{u}\right] \tag{23.54}$$

and by substituting equation (23.47) into (23.54).

$$L = \frac{B}{n} + \frac{C}{u} + \frac{A}{\Delta^2}\left[\frac{1}{3}(\phi\Delta)^2 + \left(\frac{n+1}{2} + 1\right)\frac{(\phi\Delta)^2}{u}\right] = \frac{B}{n} + \frac{C}{u} + A\left[\frac{\phi^2}{3} + \left(\frac{n+1}{2} + 1\right)\frac{\phi^2}{u}\right] \tag{23.55}$$

❒ Example

Consider a characteristic value that cannot be measured easily, and product pass/fail is evaluated using a boundary sample. The loss, A, when the product is not accepted is \$1.80; the mean failure interval, $\bar{u}$, is 2300 units; the adjustment cost, C is \$120; the measuring cost, B, is \$4; and the time lag, l, is 2 units.

In the following, we design the optimal preventive maintenance method. The parameters are:

$$A = \$1.80$$

$$B = \$4$$

$$C = \$120$$

$$\bar{u} = 2300 \text{ units}$$

$$l = 2 \text{ units}$$

From equation (23.50),

$$\phi = \left(\frac{3C}{A\bar{u}}\right)^{1/4} = \left(\frac{(3)(120)}{(1.80)(2300)}\right)^{1/4} = 0.54 \rightarrow 0.5 \qquad (23.56)$$

This is about one-half of the unacceptable boundary samples. In other words, optimal control should be done with a boundary sample that is half as bad as the unacceptable boundary sample. That level is D.

$$D = 0.5\Delta \qquad (23.57)$$

Next, derive the optimal checking interval, n, and the loss function, L. From equation (23.53),

$$n = \sqrt{\frac{2\bar{u}B}{A}} = \sqrt{\frac{(2)(2300)\,(4)}{1.80}} \approx 100 \text{ units} \qquad (23.58)$$

This means that the optimal checking interval of inspection is 100. Loss does not vary much, even if we change the checking interval by 20 to 30% around the optimal value. In this case, if there are not sufficient staff, one can stretch the checking interval to 150.

Next is the loss function. Under current conditions,

$$L_0 = \frac{B}{n_0} + \frac{C}{u_0} + \frac{A}{\Delta^2}\left[\frac{D_0^2}{3} + \left(\frac{n_0 + 1}{2} + 1\right)\frac{D_0^2}{u_0}\right] \qquad (23.59)$$

and since

$$u_0 = \overline{u} = 2300 \text{ units} \tag{23.60}$$

$$D_0 = \Delta \tag{23.61}$$

the current optimal checking interval, n_0, is

$$\begin{aligned} n_0 &= \sqrt{\frac{2u_0B}{AD_0^2}}\,\Delta = \sqrt{\frac{2\overline{u}B}{A\Delta^2}}\,\Delta \\ &= \sqrt{\frac{2\overline{u}B}{A}} = \sqrt{\frac{(2)(2300)(4)}{1.80}} \\ &\approx 100 \text{ units} \end{aligned} \tag{23.62}$$

Therefore, the loss L of the current control method is

$$\begin{aligned} L_0 &= \frac{B}{n_0} + \frac{C}{u_0} + \frac{A}{\Delta^2}\left[\frac{D_0^2}{3} + \left(\frac{n_0+1}{2} + 1\right)\frac{D_0^2}{u_0}\right] \\ &= \frac{B}{n_0} + \frac{C}{\overline{u}_0} + \frac{A}{\Delta^2}\left[\frac{\Delta^2}{3} + \left(\frac{n_0+1}{2} + 1\right)\frac{\Delta^2}{\overline{u}}\right] \\ &= \frac{B}{n_0} + \frac{A}{3} + \frac{n_0+1}{2}\,\frac{A}{\overline{u}} + \frac{C}{\overline{u}} + \frac{lA}{\overline{u}} \\ &= \frac{4}{100} + \frac{1.80}{3} + \frac{101}{2}\left(\frac{1.80}{2300}\right) + \frac{120}{2300} + \frac{(2)(1.80)}{2300} \\ &= 0.04 + 0.6 + 0.04 + 0.052 + 0.002 \\ &= 73 \text{ cents} \end{aligned} \tag{23.63}$$

Furthermore, under optimal conditions,

$$L = \frac{B}{n} + \frac{C}{u} + \frac{A}{\Delta^2}\left[\frac{D^2}{3} + \left(\frac{n+1}{2} + 1\right)\frac{D^2}{u}\right] \tag{23.64}$$

and A, B, C, n, and l are the same as in equation (23.63):

$$D = \phi\Delta \tag{23.65}$$

$$u = \overline{u}\,\frac{D^2}{\Delta^2} = \overline{u}\,\frac{(\phi\Delta)^2}{\Delta^2} = \overline{u}\phi^2 \tag{23.66}$$

Thus, by substituting into the loss function,

$$L = \frac{B}{n} + \frac{C}{u} + \frac{\phi^2 A}{3} + \frac{n+1}{2}\,\frac{\phi^2 A}{u} + \frac{l\phi^2 A}{u} \tag{23.67}$$

the following,

$$u = \bar{u}\phi^2 = (2300)\,(0.5^2) = 575 \text{ units} \tag{23.68}$$

$$\phi^2 A = (0.5^2)\,(1.80) = 45 \text{ cents} \tag{23.69}$$

we obtain

$$\begin{aligned} L &= \frac{4}{100} + \frac{120}{575} + \frac{0.45}{3} + \frac{101}{2}\left(\frac{0.45}{575}\right) + \frac{(2)(0.45)}{575} \\ &= 0.04 + 0.21 + 0.15 + 0.04 + 0.002 \\ &= 44 \text{ cents} \end{aligned} \tag{23.70}$$

Compared with the current method, the optimal method results in a gain of

$$73 - 44 = 29 \text{ cents} \tag{23.71}$$

If the yearly production were 500,000 units, there would be a savings of \$146,500.