

10 Tolerance Design

- 10.1. Introduction 208
- 10.2. Tolerance Design for Nominal-the-Best and Smaller-the-Better Characteristics 208
- 10.3. Tolerance Design for Larger-the-Better Characteristics 210
- 10.4. Tolerance Design for Multiple Factors 212

10.1. Introduction

In most cases of parameter design, quality can be improved without cost increases. In tolerance design, quality is improved by upgrading raw materials or component parts that increase cost. In other words, tolerance design is used to compare the total loss caused by quality and cost.

10.2. Tolerance Design for Nominal-the-Best and Smaller-the-Better Characteristics

After the system and parameter designs have been completed, tolerance design is conducted to complete the process. At the product design stage, tolerance design must include noise factors associated with deterioration and environmental conditions.

Example

Values of the linear thermal coefficient, b (percentage of expansion per 1°C), and wear, β , per year (percentage of wear per year) of three materials, A_1 , A_2 , and A_3 , are as shown in Table 10.1. If the dimension changes by 6%, there will be a problem in the market, and the loss, A_0 , in this case is \$180. Among A_1 , A_2 , and

Table 10.1

Material characteristics

Material	b (%)	β (%)	Price
A_1	0.08	0.15	\$1.80
A_2	0.03	0.06	3.50
A_3	0.01	0.05	6.30

A_3 , which is the best material? The standard deviation, σ_x , of the temperature condition x at which the material is used is 15°C , and the design life is 20 years.

Because the dimension of the product at the time of shipment is equal to the target value, m , at the standard temperature, the variance, σ^2 , of the sum of variability due to temperature and variability due to deterioration is given by

$$\sigma^2 = b^2\sigma_x^2 + \frac{T^2}{3}\beta^2 \quad (10.1)$$

The second term of equation (10.1) is determined because the variance is given by the following equation, where the deterioration per year is β and the design life is T years:

$$\sigma^2 = -\frac{1}{T} \int_0^T (\beta t)^2 dt = \frac{\beta^2}{3} T^2 \quad (10.2)$$

By substituting $\sigma_x = 15^\circ\text{C}$ and $T = 20$ years in equation (10.1), the variances, σ^2 , of materials A_1 , A_2 , and A_3 are

$$\begin{aligned} A_1: \sigma^2 &= (0.08^2)(15^2) + \frac{20^2}{3}(0.15^2) \\ &= 4.440 \end{aligned} \quad (10.3)$$

$$\begin{aligned} A_2: \sigma^2 &= (0.03^2)(15^2) + \frac{20^2}{3}(0.06^2) \\ &= 0.6825 \end{aligned} \quad (10.4)$$

$$\begin{aligned} A_3: \sigma^2 &= (0.01^2)(15^2) + \frac{20^2}{3}(0.05^2) \\ &= 0.3558 \end{aligned} \quad (10.5)$$

Table 10.2 was obtained in this way, and the total loss is the sum of price and quality. The optimum solution is material A_2 .

The quality level L is obtained by the equation

$$L = \frac{A_0}{\Delta_0^2} \sigma^2 = \frac{180}{6^2} \sigma^2 \quad (10.6)$$

Table 10.2
Tolerance design

Material	b (%)	β (%)	Price	σ^2	Quality Level	Total Loss
A_1	0.08	0.15	\$1.80	4.44	\$22.20	\$24.00
A_2	0.03	0.06	3.50	0.6825	\$3.41	\$6.91
A_3	0.01	0.05	6.30	0.3558	\$1.78	\$8.08

Because materials and parts cannot be adjusted later, the optimum solution is approximately at the point where the quality level and the price balance. The optimum solution, A_2 , is used to determine equation (10.7), or more generally, the safety factor. It is important to balance quality and cost in advance for rational determination of the tolerance.

Minimizing the sum of production cost and quality is tolerance design for selection of production methods and production tools. It is important to minimize the sum of the quality evaluation level, Q , and the product cost, P . The values of loss, A , due to failure are the cost of materials, parts, and products after balancing quality and cost in the tolerance design. In this case, the safety factor, ϕ , is given by

$$\phi = \sqrt{\frac{A_0}{A}} \quad (10.7)$$

10.3. Tolerance Design for Larger-the-Better Characteristics

The loss function of larger-the-better characteristics is given by

$$L = \frac{A_0 \Delta_0^2}{y^2} = A_0 \Delta_0^2 \sigma^2 \quad (10.8)$$

where σ^2 is the square average of the inverse of the larger-the-better characteristic values.

□ Example

Let's assume that both the strength and price of a pipe are proportional to the cross section of the pipe. A resin pipe is broken at a stress of 5000 kg, and the loss in this case is \$300,000. The strength of a unit area b is set at 80 kg/mm² and the

price per unit area a is set at \$40 (per/mm²). For a cross-sectional area of x mm², the sum L_T of the price and the quality is given as

$$L_T = ax + A_0\Delta_0^2 \frac{1}{(bx)^2} \quad (10.9)$$

Setting the differential of L_T with x equal to zero and solving for x yields the x that minimizes this value:

$$x = \left(\frac{2A_0\Delta_0^2}{ab^2} \right)^{1/3} \quad (10.10)$$

By substituting $a = \$40$, $b = 80$, $A_0 = \$300,000$, and $\Delta_0 = 5000$, we obtain

$$\begin{aligned} x &= \left[\frac{(2)(300,000)(5000^2)}{(40)(80^2)} \right]^{1/3} \\ &= 388 \text{ mm}^2 \end{aligned} \quad (10.11)$$

Therefore, the price is

$$(388)(40) \approx \$15,520 \quad (10.12)$$

and the quality level is

$$\begin{aligned} &(300,000) \cdot (5000^2) \left(\frac{1}{(80)(388)^2} \right) \\ &\approx \$7800 \end{aligned} \quad (10.13)$$

The total loss is

$$L = 15,520 + 7800 = \$23,320 \quad (10.14)$$

Because the strength deteriorates in an actual situation, the loss is determined using the following variability σ^2 :

$$\begin{aligned} \sigma^2 &= \frac{1}{(bx)^2} \frac{1}{T} \int_0^T e^{2\beta t} dt \\ &= \frac{1}{(bx)^2} \frac{1}{2\beta T} (e^{2\beta T} - 1) \end{aligned} \quad (10.15)$$

where T is the design life and β is the deterioration in one year. That is not calculated here. If the characteristic values fluctuate when tested under various noise conditions, the average of the square of the inverse of these values is taken as the target value of larger-the-better characteristics.

The calculation above was made from one type of resin. The same calculation can be made for other types, and the one that gives the minimum total loss will be selected.

10.4. Tolerance Design for Multiple Factors

In the preceding sections we described the tolerance design for one factor, such as material type (A_1 , A_2 , and A_3) or a certain resin pipe. In general, many factors can be used simultaneously to conduct tolerance design in any one case.

In parameter design, optimum control factor levels are determined to assure that the system output becomes least sensitive to noise factors. This means that even if wider tolerances are used around these factor levels, the system output will still yield minimal variability. Once this testing is done, the proper tolerances around these factor levels need to be fine-tuned further, since some tolerances may further reduce output variability. Tolerances that do not greatly affect variability should remain the same.

Determining which tolerances to tighten in tolerance design should be a conscious trade-off between quality improvement (variability reduction) and the cost of upgrading. Use analysis of variance to estimate the quality improvement as the tolerance of a factor is tightened, and use the quality loss function to translate the quality improvement into monetary units. Then compare the improvement to the cost of upgrading to decide whether the tolerance should be tightened to “add value.” If the improvement is greater than the cost increase, use the tighter tolerance; otherwise, allow the tolerance to remain as is.

The primary steps in tolerance design are:

1. Conduct an experiment using existing tolerances.
2. Perform an analysis of variance (ANOVA) on the experimental data, and obtain the current total variance and the percent contribution from each factor.
3. Establish the loss function for the system output response, and calculate the current total loss.
4. For each factor, calculate the existing monetary loss using the loss function.
5. For each factor, calculate the new monetary loss and the quality improvement (in loss) using the upgraded tolerance; then compare to the cost increase to decide whether or not the upgraded tolerance should be used.
6. For factors to be upgraded, calculate the total quality improvement (in loss) and the total upgrade cost to determine the total net gain.

□ Example

In the design of an engine control circuit, the output response is y = number of on signals per minute. The target is 600, and the specification is ± 60 . If the output is out of specification, the average repair cost is \$2.50. The optimum nominal values of the factors obtained through parameter design are shown in Table 10.3 with the existing tolerances. The tolerances of resistors, transistors, and condensers can be upgraded through the respective cost increases shown in Table 10.4.

The steps for tolerance design are implemented as follows.

Table 10.3

Factors, nominal values, and tolerances

Tolerance Factor		Nominal Value	σ
P	Resistor R ₁	2200	5%
Q	Resistor R ₂	470	5%
R	Resistor R ₃	100k	5%
S	Resistor R ₄	10k	5%
T	Resistor R ₅	1500	5%
U	Resistor R ₆	10k	5%
V	Transistor T ₁	180 (h_{FE})	50
W	Transistor T ₂	180 (h_{FE})	50
X	Condenser C ₁	0.68	20%
Y	Condenser C ₂	10.00	20%
Z	Voltage	6.6 V	0.3

Step 1. Conduct an experiment using the existing tolerances. Based on the nominal values and existing tolerances, the factor levels for the experiment are set as shown in Table 10.5. In two-level tolerance design experiments, the factor levels are established as

$$\text{level 1} = \text{nominal} - \sigma$$

$$\text{level 2} = \text{nominal} + \sigma \quad (10.16)$$

Table 10.4

Upgrade cost and tolerance

Component	Grade	Cost	σ
Resistor	Low	Base	5%
	High	\$2.75	1%
Transistor	Low	Base	50
	High	\$2.75	25
Condenser	Low	Base	20%
	High	\$5.50	5%

Table 10.5
Factor levels

Tolerance Factor		Level 1	Level 2
<i>P</i>	Resistor R_1	2090	2310
<i>Q</i>	Resistor R_2	446.5	493.5
<i>R</i>	Resistor R_3	95k	10.5k
<i>S</i>	Resistor R_4	9.5k	10.5k
<i>T</i>	Resistor R_5	1425	1575
<i>U</i>	Resistor R_6	9.5k	10.5k
<i>V</i>	Transistor T_1	130	230
<i>W</i>	Transistor T_2	130	230
<i>X</i>	Condenser C_1	0.544	0.816
<i>Y</i>	Condenser C_2	8.00	12.00
<i>Z</i>	Voltage	6.3	6.9

Table 10.6
Experimental layout

L_{12}	<i>P</i> 1	<i>Q</i> 2	<i>R</i> 3	<i>S</i> 4	<i>T</i> 5	<i>U</i> 6	<i>V</i> 7	<i>W</i> 8	<i>X</i> 9	<i>Y</i> 10	<i>Z</i> 11	No. of on Signals min
1	1	1	1	1	1	1	1	1	1	1	1	588
2	1	1	1	1	1	2	2	2	2	2	2	530
3	1	1	2	2	2	1	1	1	2	2	2	597
4	1	2	1	2	2	1	2	2	1	1	2	637
5	1	2	2	1	2	2	1	2	1	2	1	613
6	1	2	2	2	1	2	2	1	2	1	1	630
7	2	1	2	2	1	1	2	2	1	2	1	584
8	2	1	2	1	2	2	2	1	1	1	2	617
9	2	1	1	2	2	2	1	2	2	1	1	601
10	2	2	2	1	1	1	1	2	2	1	2	621
11	2	2	1	2	1	2	1	1	1	2	2	579
12	2	2	1	1	2	1	2	1	2	2	1	624

where as in three-level experiments, the factor levels are established as

$$\begin{aligned}\text{level 1} &= \text{nominal} - \sqrt{3/2} \sigma \\ \text{level 2} &= \text{nominal} \\ \text{level 3} &= \text{nominal} + \sqrt{3/2} \sigma\end{aligned}\quad (10.17)$$

Use an L_{12} orthogonal array for the experiment to obtain the data, as shown in Table 10.6.

Step 2. Perform the analysis of variance on the experimental data, and obtain the current total variance and the percent contribution from each factor. Use the 12 data points from the experiment for the ANOVA calculations. First calculate the total sum of squares and the total variance. Then, for each factor, calculate the sum of squares, variance, pure sum of squares, and percent contribution. Start by calculating, the correction factor, CF:

$$\begin{aligned}\text{CF} &= \frac{(\text{sum of all data points})^2}{\text{total number of data points}} \\ &= \frac{(588 + 530 + \dots + 624)^2}{12} \\ &= 4,345,236.75\end{aligned}\quad (10.18)$$

The total sum of squares, S_T , is

$$\begin{aligned}S_T &= y_1^2 + y_2^2 + \dots + y_{12}^2 - \text{CF} \\ &= 588^2 + 530^2 \dots + 624^2 - 4,345,236.75 \\ &= 4,354,695 - 4,345,236.75 \\ &= 9458.25\end{aligned}\quad (10.19)$$

Calculate the current total variance, V_T , as follows:

$$V_T = \frac{S_T}{f_T} \quad (10.20)$$

where f_T shows total degrees of freedom; in this case,

$$\begin{aligned}f_T &= (\text{number of experiments}) - 1 \\ &= 12 - 1 \\ &= 11\end{aligned}\quad (10.21)$$

Therefore,

$$\begin{aligned}V_T &= \frac{9458.25}{11} \\ &= 859.84\end{aligned}\quad (10.22)$$

Calculate the level sums of all the factors as shown in Table 10.7. Calculate the sum of squares for factor P :

$$\begin{aligned} S_p &= P_1^2 + \frac{P_2^2}{n} - CF \\ &= 3595.00^2 + \frac{3626.00^2}{6} - 4,345,236.75 \\ &= 80.08 \end{aligned} \quad (10.23)$$

where n is the number of data points per each factor level, in this case, 6. The degrees of freedom of factor P are:

$$\begin{aligned} f_p &= (\text{number of levels}) - 1 \\ &= 2 - 1 \\ &= 1 \end{aligned} \quad (10.24)$$

The variance of factor P , V_p , is

$$\begin{aligned} V_p &= \frac{S_p}{f_p} \\ &= \frac{80.08}{1} \\ &= 80.08 \end{aligned} \quad (10.25)$$

Perform similar calculations for the other factors. The results are shown in the ANOVA table, Table 10.8.

Table 10.7

Response table: Level sums

Factor	Level 1	Level 2	Factor	Level 1	Level 2
P	3595.00	3626.00	V	3599.00	3622.00
Q	3517.00	3704.00	W	3635.00	3586.00
R	3559.00	3662.00	X	3618.00	3603.00
S	3593.00	3628.00	Y	3694.00	3527.00
T	3532.00	3689.00	Z	3640.00	3581.00
U	3651.00	3570.00			

Table 10.8
ANOVA table

Source	<i>f</i>	<i>S</i>	<i>V</i>
<i>P</i>	1	80.08	80.08
<i>Q</i>	1	2914.08	2914.08
<i>R</i>	1	884.08	884.08
<i>S</i>	1	102.08	102.08
<i>T</i>	1	2054.08	2054.08
<i>U</i>	1	546.75	546.75
<i>V</i>	1	44.08	44.08
<i>W</i>	1	200.08	200.08
<i>X</i>	1	18.75	18.75
<i>Y</i>	1	2324.08	2324.08
<i>Z</i>	1	290.08	290.08
Total	11	9458.25	859.84

Since the effects (sum of squares) of factors *P*, *V*, and *X* are relatively small, “pool” them together to represent $S_{(e)}$, the pooled error sum of squares:

$$\begin{aligned}
 S_{(e)} &= S_P + S_V + S_X \\
 &= 80.08 + 44.08 + 18.75 \\
 &= 142.91
 \end{aligned}
 \tag{10.26}$$

The pooled degrees of freedom, $f_{(e)}$, are

$$\begin{aligned}
 f_{(e)} &= f_P + f_V + f_X \\
 &= 1 + 1 + 1 \\
 &= 3
 \end{aligned}
 \tag{10.27}$$

Thus, the pooled error variance, $V_{(e)}$, is

$$\begin{aligned}
 V_{(e)} &= \frac{S_{(e)}}{f_{(e)}} \\
 &= \frac{142.91}{3} \\
 &= 47.64
 \end{aligned}
 \tag{10.28}$$

Next, calculate the pure sum of squares, S'_Q , and the percent contribution, $\rho(\%)_Q$, for each factor. For example:

$$\begin{aligned} S'_Q &= S_Q - V_{(e)} f_Q \\ &= 2914.08 - (47.64)(1) \\ &= 2866.44 \end{aligned} \quad (10.29)$$

$$\begin{aligned} \rho(\%)_Q &= \frac{S'_Q}{S_T} \\ &= \frac{2866.44}{9458.25} \\ &= 30.31 \end{aligned} \quad (10.30)$$

The rearranged ANOVA table is shown in Table 10.9.

Step 3. Establish the loss function for the system output response, and calculate the current total loss. For the output response, y = (number of on signals)/minute, the specification Δ_0 is 60, and the average repair cost A_0 is \$250. Calculate the proportional constant k in the loss function as

$$\begin{aligned} k &= \frac{A_0}{\Delta_0^2} \\ &= \frac{250}{60^2} \\ &= 0.0694 \end{aligned} \quad (10.31)$$

The loss function can now be established as

$$L = 0.0694V_T \quad (10.32)$$

The current total loss (Table 10.10) is

$$\begin{aligned} L_{T(\text{current})} &= (0.0694)(859.84) \\ &= \$59.67 \text{ per circuit} \end{aligned} \quad (10.33)$$

Step 4. For each factor, calculate the existing monetary loss using the loss function. Calculate the loss due to each factor, for example, Q :

$$\begin{aligned} L_{Q(\text{current})} &= L_{T(\text{current})}\rho(\%)_Q \\ &= (59.67)(30.31\%) \\ &= \$18.09 \text{ per circuit} \end{aligned} \quad (10.34)$$

Make similar calculations for the other factors.

Step 5. For each factor, calculate the new monetary loss, the quality improvement (in loss) using the upgraded tolerance, then compare to the cost increase to decide if the upgraded tolerance should be used. Recall the tolerances and cost increases shown in Table 10.4. If we consider upgrading the tolerance of factor Q ,

Table 10.9

Rearranged ANOVA table

Source	<i>f</i>	<i>S</i>	<i>V</i>	<i>S'</i>	<i>ρ</i> (%)
<i>P</i>	1	80.08	80.08		
<i>Q</i>	1	2914.08	2914.08	2866.4	30.31
<i>R</i>	1	884.08	884.08	836.44	8.84
<i>S</i>	1	102.08	102.08	54.44	0.58
<i>T</i>	1	2054.08	2054.08	2006.44	21.21
<i>U</i>	1	546.75	546.75	499.11	5.28
<i>V</i>	1	44.08	44.08		
<i>W</i>	1	200.08	200.08	152.44	1.61
<i>X</i>	1	18.75	18.75		
<i>Y</i>	1	2324.08	2324.08	2276.44	24.07
<i>Z</i>	1	290.08	290.08	242.44	2.56
(e)	3	142.91	47.64	524.03	
Total	11	9458.25	859.84		100.00

the improvement ratio is $(1\%/5\%)^2$. Then the new loss due to factor *Q* after upgrading is

$$\begin{aligned}
 L_{Q(\text{new})} &= L_{Q(\text{current})} \left(\frac{1\%}{5\%} \right)^2 \\
 &= (18.09)(0.04) \\
 &= \$0.72 \text{ per circuit} \qquad (10.35)
 \end{aligned}$$

Table 10.10

Current loss

Factor	<i>ρ</i> (%)	<i>L</i> _{current}
<i>Q</i>	30.31	18.09
<i>R</i>	8.84	5.27
<i>S</i>	0.58	0.35
<i>T</i>	21.21	12.66
<i>U</i>	5.28	3.15
<i>W</i>	1.61	0.96
<i>Y</i>	24.07	14.36

Therefore,

$$\begin{aligned}\text{quality improvement} &= L_{Q(\text{current})} - L_{Q(\text{new})} \\ &= 0.1809 - 0.072 \\ &= \$17.37 \text{ per circuit}\end{aligned}\quad (10.36)$$

The upgrade cost for factor Q is \$2.75. Thus, the

$$\begin{aligned}\text{net gain} &= (\text{quality improvement}) - (\text{cost increase}) \\ &= 17.37 - 2.75 \\ &= \$14.62 \text{ per circuit}\end{aligned}\quad (10.37)$$

In this case, it does pay off to use the upgraded tolerance for factor Q. Similar calculations and justification are done for the other factors as shown in Table 10.11.

Step 6. For factors to be upgraded, calculate the total quality improvement (in loss), the total upgrade cost, and therefore the total new gain. Since factors Q, R, T, U, and Y are to be upgraded,

$$\begin{aligned}\text{total quality improvement} &= 17.37 + 5.06 + 12.15 + 3.02 + 13.46 \\ &= \$51.06 \text{ per circuit}\end{aligned}\quad (10.38)$$

The total upgrade cost for these factors is

$$\begin{aligned}\text{total upgrade cost} &= 2.75 + 2.75 + 2.75 + 2.75 + 5.50 \\ &= \$16.50 \text{ per circuit}\end{aligned}\quad (10.39)$$

Therefore,

$$\begin{aligned}\text{total net gain} &= (\text{total quality improvement}) - (\text{total upgrade cost}) \\ &= 51.06 - 16.50 \\ &= \$34.56 \text{ per circuit}\end{aligned}\quad (10.40)$$

Table 10.11
Upgrading decision making

Factor	L_{current}	L_{new}	Quality Improvement	Upgrade Cost	Net Gain	Upgrade Decision
Q	18.09	0.72	\$17.37	\$2.75	\$14.62	Yes
R	5.27	0.21	5.06	2.75	2.31	Yes
S	0.35	0.07	0.34	2.75	-2.41	No
T	12.66	0.51	12.15	2.75	9.40	Yes
U	3.15	0.13	3.02	2.75	0.27	Yes
W	0.96	0.24	0.72	2.75	-2.03	No
Y	14.36	0.90	13.46	5.50	7.96	Yes