3

3 Probability and Life Distributions for Reliability Analysis

In reliability engineering, data are often collected from analysis of incoming parts and materials, tests during and after manufacturing, fielded products, and warranty returns. If the collected data can be modeled, then properties of the model can be used to make decisions for product design, manufacture, reliability assessment, and logistics support (e.g., maintainability and operational availability).

In this chapter, discrete and continuous probability models (distributions) are introduced, along with their key properties. Two discrete distributions (binomial and Poisson) and five continuous distributions (exponential, normal, lognormal, Weibull, and gamma) that are commonly used in reliability modeling and hazard rate assessments are presented.

3.1 Discrete Distributions

A discrete random variable is a random variable with a finite (or countably infinite) set of values. If a discrete random variable (X) has a set of discrete possible values ($x_1, x_2, \ldots x_n$), a probability mass function (pmf), $f(x_i)$, is a function such that

$$\begin{aligned} &f(x_i) \geq 0, \quad \text{for all } i \\ &\sum_{i=1}^{n} f(x_i) = 1 \\ &f(x_i) = P\{X = x_i\}. \end{aligned} \tag{3.1}$$

The cumulative distribution function (cdf) is written as:

$$F(x_i) = P\{X \leq x_i\}. \tag{3.2}$$

Reliability Engineering, First Edition. Kailash C. Kapur and Michael Pecht.

The mean, μ, and variance, σ^2, of a discrete random variable are defined using the pmf as (see also Chapter 2):

$$\mu = E[X] = \sum_i x_i f(x_i) \tag{3.3}$$

$$\sigma^2 = V[X] = \sum_i (x_i - \mu)^2 f(x_i) = \sum_i x_i^2 f(x_i) - \mu^2. \tag{3.4}$$

3.1.1 Binomial Distribution

The binomial distribution is a discrete probability distribution applicable in situations where there are only two mutually exclusive outcomes for each trial or test. For example, for a roll of a die, the probability is one to six that a specified number will occur (success) and five to six that it will not occur (failure). This example, known as a "Bernoulli trial," is a random experiment with only two possible outcomes, denoted as "success" or "failure." Of course, success or failure is defined by the experiment. In some experiments, the probability of the result not being a certain number may be defined as a success.

The pmf, $f(x)$, for the binomial distribution gives the probability of exactly k successes in m attempts:

$$f(k) = \binom{m}{k} p^k q^{m-k}, \quad 0 \le p \le 1, \quad q = 1 - p, \quad k = 0, 1, 2, \ldots, m, \tag{3.5}$$

where p is the probability of the defined success, q (or $1 - p$) is the probability of failure, m is the number of independent trials, k is the number of successes in m trials, and the combinational formula is defined by

$$\binom{m}{k} \equiv C_k^m = \frac{m!}{k!(m-k)!}, \tag{3.6}$$

where ! is the symbol for factorial. Since $(p + q)$ equals 1, raising both sides to a power j gives

$$(p+q)^j = 1. \tag{3.7}$$

The general equation is

$$\sum_{k=0}^{m} f(k) = F(m) = (p+q)^m = 1. \tag{3.8}$$

The binomial expansion of the term on the left in Equation 3.7 gives the probabilities of j or less number of successes in j trials, as represented by the binomial distribution. For example, for three components or trials each with equal probability of success (p) or failure (q), Equation 3.7 becomes:

$$(p+q)^3 = p^3 + 3p^2q + 3pq^2 + q^3 = 1. \tag{3.9}$$

The four terms in the expansion of $(p + q)3$ give the values of the probabilities for getting 3, 2, 1, and no successes, respectively. That is, for $m = 3$ and the *probability of success* $= p$, $f(3) = p^3$, $f(2) = 3p^2q$, $f(1) = 3pq^2$, and $f(0) = q^3$.

The binomial expansion is also useful when there are products with different success and failure probabilities. The formula for the binomial expansion in this case is

$$\prod_{i=1}^{m}(p_i + q_i) = 1, \tag{3.10}$$

where i pertains to the ith component in a system consisting of m components. For example, for a system of three different components, the expansion takes the form

$$\begin{aligned}(p_1 + q_1)(p_2 + q_2)(p_3 + q_3) = {} & p_1p_2p_3 + (p_1p_2q_3 + p_1q_2p_3 + q_1p_2q_3) \\ & + (p_1q_2q_3 + q_1p_2q_3 + q_1q_2p_3) + q_1q_2q_3 = 1,\end{aligned} \tag{3.11}$$

where the first term on the right side of the equation gives the probability of success of all three components, the second term (in parentheses) gives the probability of success of any two components, the third term (in parentheses) gives the probability of success of any one component, and the last term gives the probability of failure for all components.

The cdf for the binomial distribution, $F(k)$, gives the probability of k or fewer successes in m trials. It is defined by using the pmf for the binomial distribution,

$$F(k) = \sum_{i=0}^{k}\binom{m}{i}p^i q^{(m-i)}. \tag{3.12}$$

For a binomial distribution, the mean, μ is given by

$$\mu = mp \tag{3.13}$$

and the variance is given by

$$\sigma^2 = mp(1 - p). \tag{3.14}$$

Example 3.1

An engineer wants to select four capacitors from a large lot of capacitors in which 10 percent are defective. What is the probability of selecting four capacitors with:

(a) Zero defective capacitors?
(b) Exactly one defective capacitor?
(c) Exactly two defective capacitors?
(d) Two or fewer defective capacitors?

Solution:
Let success be defined as "getting a good capacitor." Therefore, $p = 0.9$, $q = 0.1$, and $m = 4$. Using Equation 3.5 and Equation 3.6, f (4) is the probability of all four being

good (no defectives)—that is, based on four components (trials), the values of p and q are equal for all the capacitors.

$$f(4) = \binom{4}{4}(0.9)^4(0.1)^0 = 0.6561.$$

Another way to solve this problem is by defining success as "getting a certain number of defective capacitors" with $p = 0.1$ and thus $q = 0.9$. In this case, $f(0)$ gives the probability that there will be no defectives in the four selected samples. That is,

(a) $f(0) = \binom{4}{0}(0.1)^0(0.9)^4 = 0.6561$

Continuing with the latter approach, the solution to problems (b), (c), and (d), respectively, are:

(b) $f(1) = \binom{4}{1}(0.1)^1(0.9)^3 = 0.2916$

(c) $f(2) = \binom{4}{2}(0.1)^2(0.9)^2 = 0.0486$

(d) $F(2) = f(0) + f(1) + f(2) = 0.9963.$

Example 3.2

Consider a product with a probability of failure in a given test of 0.1. Assume 10 of these products are tested.

(a) What is the expected number of failures that will occur in the test?
(b) What is the variance in the number of failures?
(c) What is the probability that no product will fail?
(d) What is the probability that two or more products will fail?

Solution:
Here $m = 10$, and $p = 0.1$.

(a) The expected number of failures is the mean,

$$\mu = mp = (10 \times 0.1) = 1.$$

(b) The variance is:

$$\sigma^2 = mp(1-p) = [10 \times 0.1 \times (1-0.1)] = 0.9.$$

(c) The probability of having no failures is the pmf with $k = 0$. That is,

$$f(0) = \binom{10}{0} \times 0.1^0 \times (1-0.1)^{10} = 0.349.$$

(d) The probability of having two or more failures is the same as 1 minus the probability of having zero or one failures. It is given by:

$$\begin{aligned}\Pr(\text{two or more failures}) &= [1-\{f(0)+f(1)\}]\\ &= \left[1-0.349-\left\{10\times 0.1\times(1-0.1)^9\right\}\right]\\ &= 0.264.\end{aligned}$$

Example 3.3

An electronic automotive control module consists of three identical microprocessors in parallel. The microprocessors are independent of each other and fail independently. For successful operation of the module, at least two microprocessors must operate normally. The probability of success of each microprocessor for the duration of the warranty is 0.95. Determine the failure probability of the control module during warranty.

Solution:
The module fails when two or more microprocessors fail. In other words, the module fails when only one or none of the microprocessors is working. So the probability of failure of the module during warranty will be given by:

$$\Pr(\text{module fails during warranty}) = [f(0)+f(1)],$$

where $m = 3$ components, $k = 0$ or 1 is the total number of working components, $p = 0.95$, and $q = 0.05$. Therefore:

$$\begin{aligned}\Pr(\text{module fails during warranty}) &= \left[(0.05)^3+\left\{3\times 0.95\times(0.05)^2\right\}\right]\\ &= 0.00725.\end{aligned}$$

Example 3.4

The probability of a Black Hawk helicopter surviving a mission is 0.91. If seven helicopters are sent on a mission and five must succeed for mission success, what is the probability of mission success?

Solution:
This is also called a 5-out-of-7 system in reliability (see Chapter 17). If the number of successes is five or more, the mission will be a success. Hence, the probability of mission success or mission reliability is

$$\begin{aligned}R_S &= \sum_{i=k}^{m}\binom{m}{i}R^i(1-R)^{m-i} = \sum_{i=5}^{7}\binom{7}{i}R^i(1-R)^{7-i}\\ &= \binom{7}{5}0.91^5(0.09)^2+\binom{7}{6}0.91^6(0.09)^1+\binom{7}{7}0.91^7(0.09)^0\\ &= 0.1061+0.3577+0.5168 = 0.9806.\end{aligned}$$

3.1.2 Poisson Distribution

In situations where the probability of success (p) is very low and the number (m) of samples tested (i.e., the number of Bernoulli trials conducted) is large, it is cumbersome to evaluate the binomial coefficients. A Poisson distribution is useful in such cases.

The pmf of the Poisson distribution is given as:

$$f(k) = \frac{\mu^k}{k!} e^{-\mu}; k = 0, 1, 2, \ldots, \tag{3.15}$$

where μ is the mean and also the variance of the Poisson random variable.

For a Poisson distribution for m Bernoulli trials, with the probability of success in each trial equal to p, the mean and the variance are given by:

$$\mu = mp, \sigma^2 = mp. \tag{3.16}$$

The Poisson distribution is widely used in industrial and quality engineering applications. It is also the foundation of some of the attribute control charts. For example, it is used in applications such as determination of particles of contamination in a manufacturing environment, number of power outages, and flaws in rolls of polymers.

Example 3.5

Solve Example 3.2 using the Poisson distribution approximation.

Solution:
The expected number of failures is the same as the mean,

$$\mu = (10)(0.1) = 1.$$

The variance is also equal to 1.

The probability of obtaining no failures is the same as the pmf with $k = 0$,

$$f(0) = e^{-\mu} = e^{-1} = 0.3678.$$

The probability of getting two or more failures is the same as 1 minus the probability of obtaining zero or one failures. It is given by:

$$\begin{aligned} \Pr(\text{two or more failures}) &= [1 - \{f(0) + f(1)\}] \\ &= [1 - \{0.3678 + e^{-1}\}] = 0.2642. \end{aligned}$$

Note the differences from Example 3.2, because m is not very large.

3.1.3 Other Discrete Distributions

Other discrete distributions that are used in reliability analysis include the geometric distribution, the negative binomial distribution, and the hypergeometric distribution.

With the geometric distribution, the Bernoulli trials are conducted until the first success is obtained. The geometric distribution has the "lack of memory" property, implying that the count of the number of trials can be started at any trial without affecting the underlying distribution. In this regard, this distribution is similar to the continuous exponential distribution, which will be described later.

With the negative binomial distribution (a generalization of the geometric distribution), the Bernoulli trials are conducted until a certain number of successes are obtained. Negative binomial distribution is, however, conceptually different from the binomial distribution, since the number of successes is predetermined, and the number of trials is a random variable.

With the hypergeometric distribution, testing or sampling is conducted without replacement from a population that has a certain number of defective products. The hypergeometric distribution differs from the binomial distribution in that the population is finite and the sampling from the population is made without replacement.

3.2 Continuous Distributions

If the range of a random variable, X, extends over an interval (either finite or infinite) of real numbers, then X is a continuous random variable. The cdf is given by:

$$F(x) = P\{X \leq x\}. \tag{3.17}$$

The probability density function (pdf) is analogous to pmf for discrete variables, and is denoted by $f(x)$, where $f(x)$ is given by (if $F(x)$ is differentiable):

$$f(x) = \frac{d}{dx} F(x), \tag{3.18}$$

which yields

$$F(x) = \int_{-\infty}^{x} f(u)\,du. \tag{3.19}$$

The mean, μ, and variance, σ^2, of a continuous random variable are defined over the interval from $-\infty$ to $+\infty$ in terms of the probability density function as (see Chapter 2):

$$\mu = \int_{-\infty}^{+\infty} x f(x)\,dx \tag{3.20}$$

$$\sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{+\infty} x^2 f(x)\,dx - \mu^2. \tag{3.21}$$

Reliability is concerned with the time to failure random variable T and thus X is replaced by T. Thus, Equation 3.19 corresponds to Equation 2.5 and Equation 3.20 corresponds to Equation 2.46.

Example 3.6

The pdf for the time to failure of an appliance is given by:

$$f(t)=\frac{1}{16}t\cdot e^{-t/4},$$

where t is in years, and $t > 0$.

(a) What is the probability of failure in the first year?
(b) What is the probability of the appliance lasting at least 5 years?
(c) If no more than 5% of the appliances will require warranty service, what is the maximum number of months for which the appliance should be warranted?

Solution:

(a) For the given pdf, the cdf is

$$F(t)=\frac{1}{16}\int_0^t \tau\cdot e^{-\tau/4}\cdot d\tau=1-\left(\frac{t}{4}+1\right)e^{-t/4}.$$

The probability of failure during the first year = $F(1)$ = 0.0265.

(b) The probability of lasting more than 5 years is = [1 – $F(5)$] = [1 – 0.3554] = 0.6446.
(c) For this case, $F(t_0)$ has to be less than or equal to 0.05, where t_0 is the warranty period. From the above results, we find that the time has to be more than 1 year. Also, $F(2)$ is equal to 0.09, hence the warranty period should be between 1 and 2 years. We can find that for no more than 5% warranty service, t_0 = 1.42 years. Therefore, the warranty should be set at no greater than 17 months.

Example 3.7

The time-to-failure random variable, T, of a product follows the following probability density function:

$$\begin{aligned} F(t) &= \frac{t}{80{,}000}, && 0\le t\le 400 \\ &= 0, && \text{otherwise.} \end{aligned}$$

We give solutions to the following four parts.

(a) Find the standard deviation for the time-to-failure random variable.

Solution:

$$E[T] = \int_0^{400} t \cdot \frac{t}{80{,}000} dt = \left. \frac{t^3}{240{,}000} \right|_0^{400} = 266.67$$

$$E[T^2] = \int_0^{400} t^2 \cdot \frac{t}{80{,}000} dt = \left. \frac{t^4}{320{,}000} \right|_0^{400} = 80{,}000,$$

then variance V[T] and the standard deviation are given by (see Eq. 2.35 in Chapter 2)

$$V[T] = E[T^2] - (E[T])^2 = 8888.89$$
$$\text{Standard deviation} = \sqrt{V[T]} = 94.28.$$

(b) Find the coefficient of skewness of the distribution for the time-to-failure random variable.

Solution:

$$\mu_1 = \mu = \mu_1' = E[T]$$
$$\mu_2' = E[T^2] \quad \mu_2 = V[T]$$
$$\mu_3' = \int_0^{400} t^3 \frac{t}{80{,}000} dt = \left. \frac{t^5}{400{,}000} \right|_0^{400} = 25.6 \times 10^6.$$

Using Equation 2.52 (Chapter 2), we have

$$\begin{aligned} \mu_3 &= \mu_3' - 3\mu\mu_2' + 3\mu^2\mu_1' - \mu^3 \\ &= \mu_3' - 3\mu\mu_2' + 2\mu^3 = -474{,}074. \end{aligned}$$

Hence,

$$\alpha_3 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{-474{,}074}{(8888.89)^{1.5}} = -0.5657.$$

The above triangular distribution is negatively skewed, which is good in terms of reliability because the time to failure is a "larger the better" characteristic for the product.

(c) Find the B_5 and B_{50} life of the product based on the above probability density function.

Solution:
Using Equation 2.5 and Equation 2.40, give

$$F(t) = \int_0^t \frac{u}{80{,}000} du = \frac{t^2}{160{,}000}$$

$$F(B_5) = 0.05 = \frac{B_5^2}{160{,}000}$$

$$B_5 = 89.44$$

$$F(B_{50}) = 0.5 = \frac{B_{50}^2}{160{,}000}$$

$$B_{50} = 282.843.$$

(d) Draw the failure rate (or hazard rate) curve for the above product by evaluating it at $t = 0$, 50, 100, 300, 400.

Solution:
Using Equation 2.5, the following table can be developed and the hazard rate function $h(t)$ is drawn as shown in Figure 3.1.

Value of *h*(*t*) vs. *t*

t	*f(t)*	*R(t)*	*h(t)*
0	0	1	0
50	0.000625	0.984375	0.000635
100	0.00125	0.9375	0.001333
300	0.00375	0.4375	0.008571
400	0.005	0	Infinity

The values of the hazard function are given in Figure 3.1. It is clear that for the above triangular distribution, the failure rate is increasing; such distributions have the property of an increasing failure rate (IFR). Many products that wear or deteriorate with time will exhibit IFR behavior.

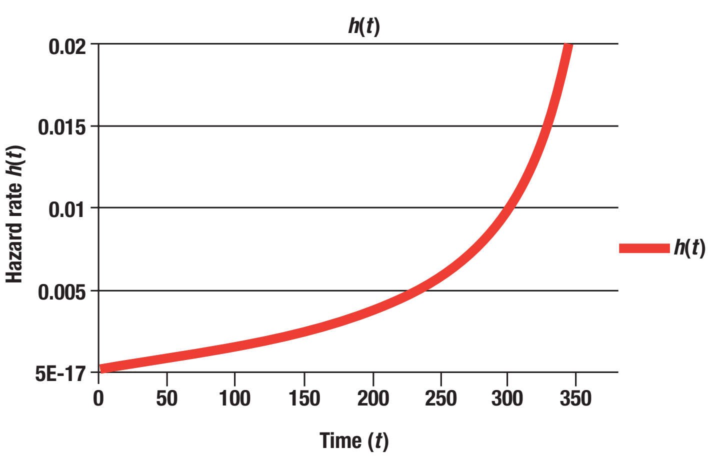

Figure 3.1 Hazard rate function, $h(t)$.

3.2.1 Weibull Distribution

The Weibull distribution is a continuous distribution developed in 1939 by Waloddi Weibull (1939), and who presented it in detail in 1951 (Weibull 1951). The Weibull distribution is widely used for reliability analyses because a wide diversity of hazard rate curves can be modeled with it. The distribution can also be approximated to other distributions under special or limiting conditions. The Weibull distribution has been applied to life distributions for many engineered products, and has also been used for reliability testing, material strength, and warranty analysis.

The probability density function for a three-parameter Weibull probability distribution function is

$$f(t) = \beta\eta^{-\beta}(t-\gamma)^{\beta-1} e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}, \tag{3.22}$$

where $\beta > 0$ is the shape parameter, $\eta > 0$ is the scale parameter, which is also denoted by θ in many references and books, and γ is the location or time delay parameter. The reliability function is given by

$$R(t) = \int_t^{\infty} f(\tau)d\tau = e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}. \tag{3.23}$$

It can be shown that Equation 3.23 gives, for a duration $t = \gamma + \eta$, starting at time $t = 0$, a reliability value of $R(t) = 36.8\%$, regardless of the value of β. Thus, for any Weibull failure probability density function, 36.8% of the products survive for $t = \gamma + \eta$.

The time to failure of a product with a specified reliability, R, is given by

$$t = \gamma + \eta[-\ln R(t)]^{1/\beta}. \tag{3.24}$$

The hazard rate function for the Weibull distribution is given by

$$h(t) = \frac{f(t)}{R(t)} = \frac{\beta}{\eta}\left[\frac{t-\gamma}{\eta}\right]^{\beta-1}. \tag{3.25}$$

The conditional reliability function is (see Eq. 2.39):

$$\begin{aligned} R(t, t_1) &= \frac{R(t+t_1)}{R(t_1)} \\ &= \exp\left\{-\left[\frac{t+t_1-\gamma}{\eta}\right]^{\beta} + \left[\frac{t_1-\gamma}{\eta}\right]^{\beta}\right\}. \end{aligned} \tag{3.26}$$

Equation 3.26 gives the reliability for a new mission of duration t for which t_1 hours of operation were previously accumulated up to the beginning of this new mission. It is seen that the Weibull distribution is generally dependent on both the age at the beginning of the mission and the mission duration (unless $\beta = 1$). In fact, this is true for most distributions, except for the exponential distribution (discussed later).

Table 3.1 Weibull distribution parameters

Location	γ	
Shape parameter	β	
Scale parameter	η	
Mean (arithmetic average)	$\gamma + \eta + \Gamma(1/\beta + 1)$	
Median (B_{50}, or time at 50% failure)	$\gamma + \eta(\ln 2)^{1/\beta}$	
Mode (highest value of $f(t)$)	for $\beta > 1$	$\gamma + \eta(l - 1/\beta)^{1/\beta}$
	for $\beta = 1$	γ
Standard deviation	$\eta\sqrt{\Gamma\left(\frac{2}{\beta}+1\right)-\Gamma^2\left(\frac{1}{\beta}+1\right)}$	

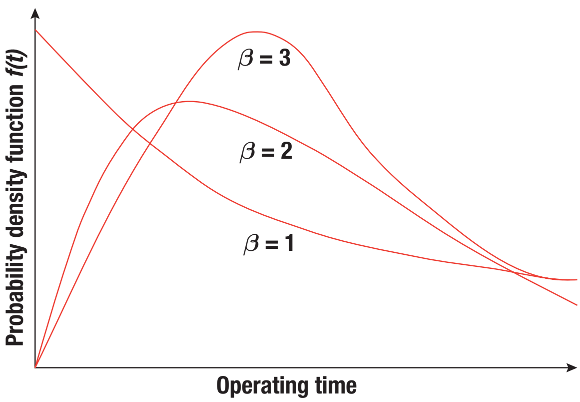

Figure 3.2 Effects of shape parameter β on probability density function, where $\eta = 1$ and $\gamma = 0$.

Table 3.1 lists the key parameters for a Weibull distribution and values for mean, median, mode, and standard deviation. The function Γ is the gamma function, for which the values are available from statistical tables and also are provided in Appendix B.

The shape parameter of a Weibull distribution determines the shape of the hazard rate function. With $0 < \beta < 1$, the hazard rate decreases as a function of time, and can represent early life failures (i.e., infant mortality). A $\beta = 1$ indicates that the hazard rate is constant and is representative of the "useful life" period in the "idealized" bathtub curve (see Figure 2.6). A $\beta > 1$ indicates that the hazard rate is increasing and can represent wearout. Figure 3.2 shows the effects of β on the probability density function curve with $\eta = 1$ and $\gamma = 0$. Figure 3.3 shows the effect of β on the hazard rate curve with $\eta = 1$ and $\gamma = 0$.

The scale parameter η has the effect of scaling the time axis. Thus, for a fixed γ and β, an increase in η will stretch the distribution to the right while maintaining its starting location and shape (although there will be a decrease in the amplitude, since the total area under the probability density function curve must be equal to unity). Figure 3.4 shows the effect of η on the probability density function for $\beta = 2$ and $\gamma = 0$.

The location parameter locates the distribution along the time axis and thus estimates the earliest time to failure. For $\gamma = 0$, the distribution starts at $t = 0$. With $\gamma > 0$, this implies that the product has a failure-free operating period equal to γ. Figure 3.5 shows the effects of γ on the probability density function curve for $\beta = 2$ and $\eta = 1$. Note that if γ is positive, the distribution starts to the right of the $t = 0$

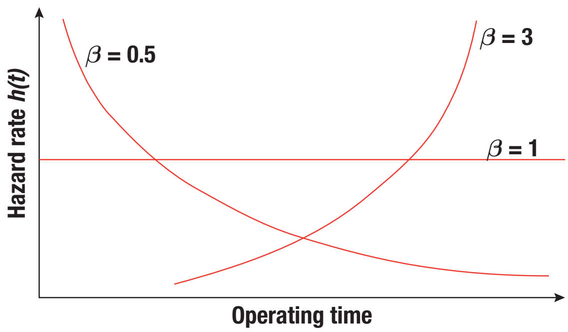

Figure 3.3 Dependence of hazard rate on shape parameter, where $\eta = 1$ and $\gamma = 0$.

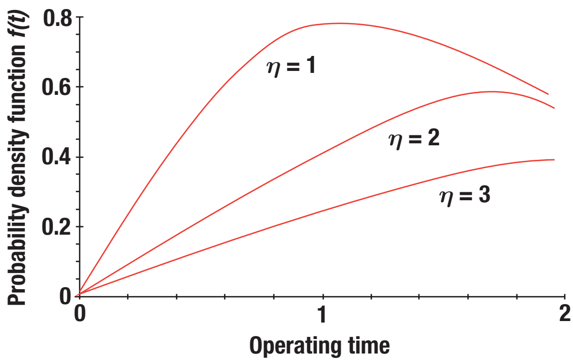

Figure 3.4 Effects of scale parameter η on the pdf of a Weibull distribution, where $\beta = 2$ and $\gamma = 0$.

line, or the origin. If γ is negative, the distribution starts to the left of the origin, and could imply that failures had occurred prior to the time $t = 0$, such as during transportation or storage. Thus, there is a probability mass $F(0)$ at $t = 0$, and the rest of the distribution for $t > 0$ is given in Figure 3.5. The Weibull distribution can also be formulated as a two-parameter distribution with $\gamma = 0$.

The reliability function for the two-parameter Weibull distribution is

$$R(t) = \int_t^{\infty} f(\tau)d\tau = e^{-\left(\frac{t}{\eta}\right)^{\beta}} \tag{3.27}$$

and the hazard rate function is

$$h(t) = \frac{f(t)}{R(t)} = \frac{\beta}{\eta}\left[\frac{t}{\eta}\right]^{\beta-1}. \tag{3.28}$$

The *two-parameter Weibull distribution* can be used to model skewed data. When $\beta < 1$, the failure rate for the Weibull distribution is decreasing and hence can be used to model infant mortality or a debugging period, situations when the reliability in terms of failure rate is improving, or reliability growth. When $\beta = 1$, the Weibull distribution is the same as the exponential distribution. When $\beta > 1$, the failure rate is increasing, and hence can model wearout and the end of useful life. Some examples of this are corrosion life, fatigue life, or the life of antifriction bearings, transmission gears, and electronic tubes.

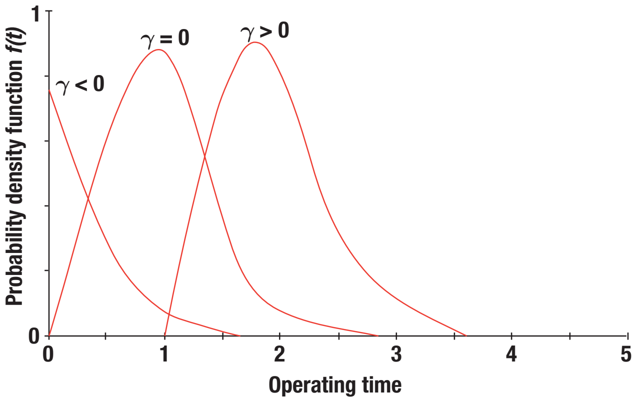

Figure 3.5 Effects of location parameter γ, where $\beta = 2$ and $\eta = 1$.

The *three-parameter Weibull distribution* is a model when there is a minimum life or when the odds of the component failing before the minimum life are close to zero. Many strength characteristics of systems do have a minimum value significantly greater than zero. Some examples are electrical resistance, capacitance, and fatigue strength.

Example 3.8

Assume that the time to failure of a product can be described by the Weibull distribution, with estimated parameter values of $\eta = 1000$ hours, $\gamma = 0$, and $\beta = 2$. Estimate the reliability of the product after 100 hours of operation. Also determine the MTTF.

Solution:
From Equation 3.27, we have:

$$R(100) = e^{-(100/1000)^2} = 0.990.$$

And from Table 3.1 we have

$$\text{MTTF} = 1000\Gamma(1/2+1) = 1000\Gamma(1.50) = 886 \text{ hours}.$$

where the value of $\Gamma(1.50)$ can be found from the table in Appendix B.

Example 3.9

Suppose that the life distribution for miles to failure for a give failure mode for the transmission of a GM Cadillac model follows the two-parameter Weibull distribution with $\eta = 150{,}000$ mi, $\beta = 4.5$.

(a) Find the mean miles between failures or the expected life in miles of these transmissions.

Solution:
Using the table in Appendix B, we have

$$\begin{aligned} E[T] &= \eta\Gamma\left(1+\frac{1}{\beta}\right) = 150{,}000\Gamma\left(1+\frac{1}{4.5}\right) \\ &= 150{,}000 \times \Gamma(1.2222) = 150{,}000 \times 0.912573 \\ &= 136{,}886 \text{ mi.} \end{aligned}$$

(b) Find the standard deviation for the miles to failure random variable.

Solution:
Again, using the table in Appendix B,

$$\begin{aligned} V[T] &= \eta^2\left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)\right] \\ &= 150{,}000^2\left[\Gamma(1.4444) - \Gamma(1.2222)^2\right] \\ &= 150{,}000^2 \times \left(0.8858 - 0.912573^2\right) = 1.19 \times 10^9 \end{aligned}$$

$$\text{Standard deviation} = \sqrt{V(T)} = \sqrt{1.19 \times 10^9} = 34{,}513 \text{ mi.}$$

(c) If GM gives a warranty for 70,000 mi on these transmissions, what percent of these transmissions will fail during the warranty period?

Solution:

$$1 - R(70{,}000) = 1 - e^{-\left(\frac{70{,}000}{150{,}000}\right)^{4.5}} = 0.03188.$$

Thus, 3.188% of the transmissions will fail during the warranty period for the given failure mode.

Example 3.10

Suppose that the life distribution (life in years of continuous use) of hard disk drives for a computer system follows a two-parameter Weibull distribution with the following parameters: $\beta = 3.10$ and $\eta = 5$ years.

(a) The manufacturer gives a warranty for 1 year. What is the probability that a disk drive will fail during the warranty period?

Solution:

$$\begin{aligned} F(1) &= 1 - R(1) = 1 - \exp\left[-\left(\frac{1}{5}\right)^{3.10}\right] \\ &= 1 - 0.993212 = 0.006788. \end{aligned}$$

(b) Find the mean life and the median life (B_{50}) for the disk drive.

Solution:

$$\begin{aligned}\text{mean} &= 5\Gamma\left(1+\frac{1}{3.10}\right) = 5\Gamma(1.32258)\\ &= 5\times 0.89431 = 4.47155 \text{ years.}\end{aligned}$$

And to find the median life, we have

$$\begin{aligned}R(B_{50}) &= \exp\left[-\left(\frac{B_{50}}{5}\right)^{3.10}\right] = 0.5\\ \left(\frac{B_{50}}{5}\right)^{3.10} &= -\ln(0.5) = 0.693147\\ B_{50} &= 0.888492\times 5 = 4.44246 \text{ years.}\end{aligned}$$

(c) By what time will 95% of the disk drives fail? (Find the B_{95} life).

Solution:

$$\begin{aligned}R(B_{95}) &= \exp\left[-\left(\frac{B_{95}}{5}\right)^{3.10}\right] = 0.05\\ \left(\frac{B_{95}}{5}\right)^{3.10} &= -\ln(0.05) = 2.99573\\ B_{95} &= 1.42466\times 5 = 7.12329 \text{ years.}\end{aligned}$$

Example 3.11

The failure rate of a component, in failures per year, is given by:

$$h(t) = 0.003t^2, \quad t \geq 0.$$

(a) Find an expression for the reliability function and the probability density function for the time to failure of the component.

Solution:
Using Equation. 2.35 and Equation 2.36 in Chapter 2, we have

$$\begin{aligned}h(t) &= 0.003t^2\\ H(t) &= \int_0^t 0.003x^2 dx = 0.001t^3\\ R(t) &= \exp\left[-0.001t^3\right]\\ f(t) &= 0.003t^2 \exp\left[-0.001t^3\right].\end{aligned}$$

This is easily recognizable as a Weibull distribution with the following values of the parameters:

$$\text{Weibull: } \beta = 3, \eta = 10.$$

(b) Find the expected life (MTTF) for the component.

Solution:

$$\begin{aligned} E[T] &= \theta\Gamma\left(1+\frac{1}{\beta}\right) \\ &= 10\Gamma\left(1+\frac{1}{3}\right) = 10\times 0.89298 = 8.9298 \text{ years.} \end{aligned}$$

(c) Find the B_{10} (the 10th percentile) for the life of the component.

Solution:
We need to find the value of t, such that the item has a 10% chance of failing. This is equivalent to finding the point at which $R(t) = 0.9$. Solving for t:

$$\begin{aligned} 0.9 &= e^{-0.001(B_{10})^3} \\ \ln 0.9 &= -0.001(B_{10})^3 \\ B_{10} &= \sqrt[3]{\frac{-\ln(0.9)}{0.001}} = 4.723 \text{ years.} \end{aligned}$$

3.2.2 Exponential Distribution

The exponential distribution is a single-parameter distribution that can be viewed as a special case of a Weibull distribution, where $\beta = 1$. The probability density function has the form

$$f(t) = \lambda_0 e^{-\lambda_0 t}, \quad t \geq 0, \tag{3.29}$$

where λ_0 is a positive real number, often called the constant failure rate. The parameter λ_0 is typically an unknown that must be calculated or estimated based on statistical methods discussed later in this section. Figure 3.6 gives a graph for an exponential distribution, with $\lambda_0 = 0.10$. Table 3.2 summarizes the key parameters for the exponential distribution.

Once λ_0 is known, the reliability can be determined from the probability density function as

$$R(t) = \int_t^\infty f(\tau)d\tau = \int_t^\infty \lambda_0 e^{-\lambda_0 t} d\tau = e^{-\lambda_0 t}. \tag{3.30}$$

The cdf or unreliability is given by

$$F(t) = Q(t) = 1 - \exp[-\lambda_0 t]. \tag{3.31}$$

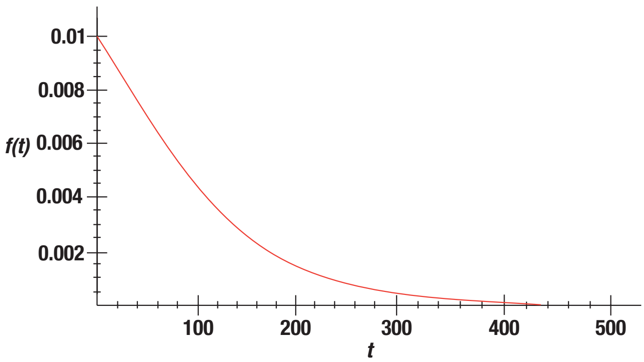

Figure 3.6 An example of an exponential distribution.

Table 3.2 Exponential distribution parameter

Scale parameter	$1/\lambda_0$
Median (B_{50})	$0.693/\lambda_0$
Mode (highest value of $f(t)$)	0
Standard deviation	$1/\lambda_0$
Mean	$1/\lambda_0$

As mentioned, the hazard rate for the exponential distribution is constant:

$$h(t) = \frac{f(t)}{R(t)} = \frac{1}{e^{-\lambda_0 t}}\left(\lambda_0 e^{-\lambda_0 t}\right) = \lambda_0. \tag{3.32}$$

The conditional reliability is

$$R(t, t_1) = \frac{R(t+t_1)}{R(t_1)} = e^{-\lambda_0(t+t_1)}/e^{-\lambda_0 t_1} = e^{-\lambda_0 t}. \tag{3.33}$$

Equation 3.33 shows that previous usage (e.g., tests or missions) do not affect future reliability. This “as good as new” result stems from the fact that the hazard rate is a constant and the probability of a product failing is independent of the past history or use of the product.

The mean time to failure (MTTF) for an exponential distribution, also denoted by θ, is determined from the general equation for the mean of a continuous distribution:

$$\mathrm{MTTF} = \int_0^\infty R(t)\,dt = \int_0^\infty e^{-\lambda_0 t}\,dt = \frac{1}{\lambda_0}. \tag{3.34}$$

Thus, the MTTF or the MTBF is inversely proportional to the constant failure rate, and thus the reliability can be expressed as

$$R(t) = e^{-t/MTBF}. \tag{3.35}$$

The MTBF is sometimes misunderstood to be the life of the product or the time by which 50% of products will fail. For a mission time of t = MTBF, the reliability calculated from Equation 3.30 gives $R(\text{MTBF}) = 0.368$. Thus, only 36.8% of the products survive a mission time equal to the MTBF.

Example 3.12

Show that the exponential distribution is a special case of the Weibull distribution.

Solution:
From Equation 3.22, set $\beta = 1$ and $\gamma = 0$

$$f(t) = \frac{1}{\eta} e^{-\frac{t}{\eta}}.$$

Thus, in this case, the Weibull distribution reduces to the single-parameter exponential distribution with $\lambda_0 = 1/\eta$. The reliability and the hazard rate functions simplify to:

$$R(t) = e^{-\frac{t}{\eta}}$$
$$h(t) = \frac{1}{\eta},$$

where

$$\eta = \frac{1}{\lambda_0}.$$

If $\beta = 1$ and $\gamma > 0$, then the Weibull distribution is the same as the exponential distribution with minimum life, γ, or is also a two-parameter exponential distribution.

Example 3.13

Consider an electronic product that exhibits a constant hazard rate. If the MTBF is 5 years, at what time will 10% of the products fail?

Solution:
Using Equation 3.35 with $R = 0.90$ and MTBF $\approx$ 43,800 hours (5 years), we solve for t, where t is in hours. Thus, $t = -[(\text{MTBF}) \times \ln(R)] \approx 4600$ hours, or nearly half a year.

Example 3.14

Here we consider a mixture of exponential distributions. The pdf for the life of a device is given by the following probability density function, which is a mixture of two exponential distributions.

$$f(t) = \frac{1}{4} e^{-t} + \frac{3}{2} e^{-2t}, \quad t \geq 0.$$

(a) Prove that the above function is a valid pdf.

Solution:

$$\int_0^\infty \left(\frac{1}{4}e^{-t}+\frac{3}{2}e^{-2t}\right)dt = -\frac{1}{4}e^{-t}-\frac{3}{2}\times\frac{1}{2}e^{-2t}\Big|_0^\infty$$

$$= -\frac{1}{4}(0-1)-\frac{3}{4}(0-1)=1.$$

Therefore, the above function is a valid pdf.

(b) Find the probability that a device will last at least 3 hours.

Solution:

$$f(t)=\frac{1}{4}e^{-t}+\frac{3}{2}e^{-2t},\ t\geq 0.$$

$$R(t)=\int_t^\infty f(\tau)d\tau=\int_t^\infty\left(\frac{1}{4}e^{-\tau}+\frac{3}{2}e^{-2\tau}\right)d\tau$$

$$=\left(-\frac{1}{4}e^{-\tau}-\frac{3}{4}e^{-2\tau}\right)\Big|_t^\infty=\frac{1}{4}e^{-t}+\frac{3}{4}e^{-2t}$$

$$R(3)=\frac{1}{4}e^{-3}+\frac{3}{4}e^{-6}=0.01431.$$

Alternatively: $R(t) = 1 - F(t)$.

$$F(t)=\int_0^t\left(\frac{1}{4}e^{-\tau}+\frac{3}{2}e^{-2\tau}\right)d\tau=\left(-\frac{1}{4}e^{-\tau}-\frac{3}{4}e^{-2\tau}\right)\Big|_0^t$$

$$=-\frac{1}{4}e^{-t}-\frac{3}{4}e^{-2t}+1$$

$$F(3)=-\frac{1}{4}e^{-3}-\frac{3}{4}e^{-6}+1=0.98569$$

$$R(3)=1-F(3)=1=1-0.98569=0.01431.$$

(c) Find the expected life or the MTBF of the device.

Solution:

$$\text{MTBF}=\int_0^\infty R(t)dt=\int_0^\infty\left(\frac{1}{4}e^{-t}+\frac{3}{4}e^{-2t}\right)dt$$

$$=\left(-\frac{1}{4}e^{-t}-\frac{3}{8}e^{-2t}\right)\Big|_0^\infty=\frac{1}{4}+\frac{3}{8}=\frac{5}{8}\text{ hours.}$$

3.2.3 Estimation of Reliability for Exponential Distribution

For reliability tests in which the hazard rate is assumed to be constant, and the time to failure can be assumed to follow an exponential distribution, the constant failure

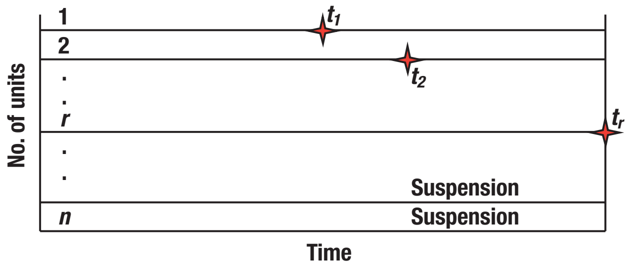

Figure 3.7 Failure-truncated test. Failures are denoted by ✦.

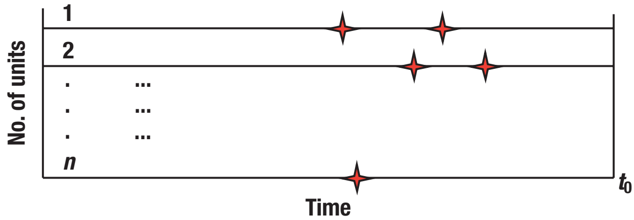

Figure 3.8 Time-truncated test. ✦, Failures.

rate can be estimated by life testing. There are various ways to test the items. Figure 3.7 gives an example of a failure-truncated test, in which n items on individual test stands are monitored to failure. The test ends as soon as there are r failures (without replacement $r \leq n$), as shown in Figure 3.7.

The total time on test, T_T, considering both failed and unfailed (or suspended) units, is calculated by the following equation:

$$T_T = \sum_{i=1}^{r} t_i + (n-r)t_r. \tag{3.36}$$

Another test situation is called time-truncated testing. In Figure 3.8, there are n test stands (or n items on test in a test chamber). The units are monitored and replaced as soon as they fail. Testing for these units continues until some predetermined time, t_0. In this case, the total time on test is

$$T_T = nt_0. \tag{3.37}$$

Then the point estimator (minimum variance unbiased estimator) for θ, the MTBF, is

$$\hat{\theta} = \frac{T_T}{r}. \tag{3.38}$$

Further details are given in Chapter 13. Also, the point estimator for λ is

$$\hat{\lambda} = \frac{1}{\hat{\theta}}. \tag{3.39}$$

Chapter 13 will present the methodology for the point estimation and confidence interval for several test situations and underlying life distributions.

Example 3.15

Seven prototypes are monitored for some failure during development testing or as fielded products. The failures on the products are fixed (it is assumed that we can renew the product) and testing continues. Then testing is stopped at the times given below for each product:

Product no.	Hours when failures are recorded	Hours when testing is stopped
01	2467; 3128; 3283; 7988	8012
02	None	6147
03	1870; 6121; 6175	9002
04	3721; 4393; 5848; 6425; 6353	11,000
05	498	4651
06	184; 216; 561; 2804	5012
07	2342; 4213	12,718

Estimate the MTBF for this product.

Solution:
In this case, T_T, the total time on test, is obtained by adding all the hours when testing was stopped:

$$T_T = 8012 + \cdots + 12{,}718 = 56{,}542 \text{ hours.}$$

During this total test period, there were 19 failures. Thus, the point estimator for the MTBF, under the assumption that the time between failures follows the exponential distribution, is

$$\hat{\theta} = 56{,}542/19 = 2975 \text{ hours.}$$

Example 3.16

Estimate the MTBF (point estimator) or ($\hat{\theta}$) for the following reliability test situations:

(a) Failure terminated, with no replacement. Twelve items were tested until the fourth failure occurred, with failures at 200, 500, 625, and 800 hours.

(b) Time terminated, with no replacement. Twelve items were tested up to 1000 hours, with failures at 200, 500, 625, and 800 hours.

(c) Failure terminated, with replacement. Eight items were tested until the third failure occurred, with failures at 150, 400, and 650 hours.

(d) Time terminated, with replacement. Eight items were tested up to 1000 hours, with failures at 150, 400, and 650 hours.

(e) Mixed replacement/nonreplacement. Six items were tested through 1000 hours on six different test stands. The first failure on the test stand occurred at 300 hours, and its replacement failed after an additional 400 hours. On the second test stand, failure occurred at 350 hours, and its replacement failed after an additional 500 hours. On the third test stand, failure occurred at 600 hours, and its replacement did not fail up to the completion of the test. The items on the other three test stands did not fail for the duration of the test.

Solution:

(a) $\text{MTBF}(e) = \hat{\theta} = (200 + 500 + 625 + 800 + 8(800))/4 = 2{,}131$ hours
(b) $\text{MTBF}(e) = \hat{\theta} = (200 + 500 + 625 + 800 + 8(1000))/4 = 2{,}531$ hours
(c) $\text{MTBF}(e) = \hat{\theta} = (8)(650)/3 = 1{,}733$ hours
(d) $\text{MTBF}(e) = \hat{\theta} = (8)(1000)/3 = 2{,}667$ hours
(e) $\text{MTBF}(e) = \hat{\theta} = (700 + 850 + 1000 + (3)(1000))/5 = 1{,}110$ hours.

Example 3.17

Forty modules were placed on life test for 20 days (24 hours per day). Failed boards were replaced on the test stands with new ones. The test produced two failures. Estimate the MTBF or the failure rate for the modules.

Solution:
In this case, the total time on test is

$$T_T = 20 \times 24 \times 40 = 19{,}200 \text{ hours}$$

$$\hat{\theta} = T_T/\mathrm{r} = 19{,}200/2 = 9600 \text{ hours}$$

$$\hat{\lambda} = \mathrm{r}/T_T = 2/19{,}200 = 1.04 \times 10 - 4 \text{ failures per hour.}$$

3.2.4 The Normal (Gaussian) Distribution

The normal distribution occurs whenever a random variable is affected by a sum of random effects, such that no single factor dominates. This motivation is based on central limit theorem, which states that under mild conditions, the sum of a large number of random variables is approximately normally distributed. It has been used to represent dimensional variability in manufactured goods, material properties, and measurement errors. It has also been used to assess product reliability.

The normal distribution has been used to model various physical, mechanical, electrical, or chemical properties of systems. Some examples are gas molecule velocity, wear, noise, the chamber pressure from firing ammunition, the tensile strength of aluminum alloy steel, the capacity variation of electrical condensers, electrical power consumption in a given area, generator output voltage, and electrical resistance.

The probability density function for the normal distribution is based on the following Gaussian function:

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\left(-\frac{1}{2}\right)\left(\frac{t-\mu}{\sigma}\right)^2\right], \quad -\infty \le t \le +\infty, \tag{3.40}$$

where the parameter μ is the mean or the MTTF, and σ is the standard deviation of the distribution. The parameters for a normal distribution are listed in Table 3.3.

Table 3.3 Normal distribution parameters

Mean (arithmetic average)	μ
Median (B_{50} or 50th percentile)	μ
Mode (highest value of $f(t)$)	μ
Location parameter	μ
Shape parameter/standard deviation	σ
s (an estimate of σ)	$B_{50} - B_{16}$

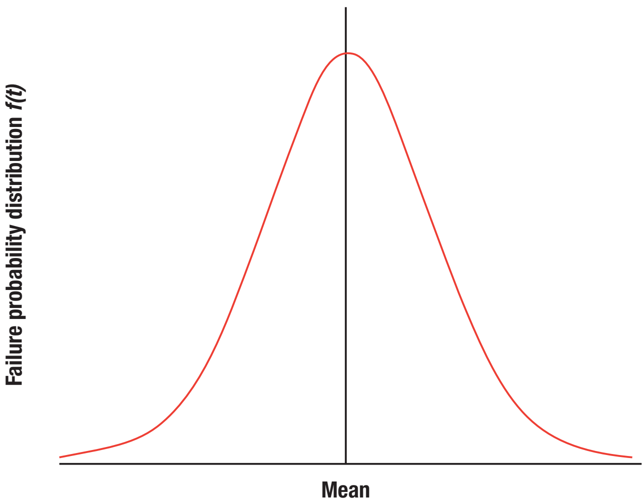

Figure 3.9 Probability density function for normal distribution.

Figure 3.9 shows the shape of the probability density function for the normal distribution.

The cdf, or unreliability, for the normal distribution is:

$$F(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left[\left(-\frac{1}{2}\right)\left(\frac{x-\mu}{\sigma}\right)^2\right] dx. \tag{3.41}$$

A normal random variable with mean equal to zero and variance of 1 is called a standard normal variable (Z), and its pdf is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \tag{3.42}$$

where $z \equiv (t - \mu)/\sigma$.

The properties of the standard normal variable, in particular the cumulative probability distribution function, are tabulated in statistical tables (provided in Appendix C). Table 3.4 provides the percentage values of the areas under the normal curve at different distances from the mean in terms of multiples of σ. For example,

$$P[X \leq \mu - 3\sigma] = 0.00135 \tag{3.43}$$

and

Table 3.4 Areas under the normal curve

$\mu - 1\sigma = 15.87\%$	$\mu + 1\sigma = 84.130\%$
$\mu - 2\sigma = 2.28\%$	$\mu + 2\sigma = 97.720\%$
$\mu - 3\sigma = 0.135\%$	$\mu + 3\sigma = 99.865\%$
$\mu - 4\sigma = 0.003\%$	$\mu + 4\sigma = 99.997\%$

$$P[X \leq \mu + 3\sigma] = 0.99865. \tag{3.44}$$

There is no closed-form solution to the integral of Equation 3.41, and, therefore, the values for the area under the normal distribution curve are obtained from the standard normal tables by converting the random variable, t, to a random variable, z, using the transformation:

$$z = \frac{t-\mu}{\sigma}, \tag{3.45}$$

given by Equation 3.42. We have

$$F(t) = \Phi(z) = \Phi\left(\frac{t-\mu}{\sigma}\right) \tag{3.46}$$

or

$$R(t) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right) \tag{3.47}$$

and

$$h(t) = \frac{\phi[(t-\mu)/\sigma]}{\sigma R(t)}, \tag{3.48}$$

where $\phi(.)$ is the pdf for the standard normal distribution and $\Phi(z)$ is the cdf for the standard normal random variable Z.

From Equation 3.48, we can prove that the normal distribution has an increasing hazard rate (IHR). The normal distribution has been used to describe the failure distribution for products that show wearout and that degrade with time. The life of tire tread and the cutting edges of machine tools fit this description. In these situations, life is given by a mean value of μ, and the variability about the mean value is defined through standard deviation. When the normal distribution is used, the probabilities of a failure occurring before or after this mean time are equal because the mean is the same as the median.

Example 3.18

A machinist estimates that there is a 90% probability that the washer in an air compressor will fail between 25,000 and 35,000 cycles of use. Assuming a normal distribution for washer degradation, find the mean life and standard deviation of the life of the washers.

Solution:
Assuming that 5% of the failures are at fewer than 25,000 cycles and 5% are at more than 35,000 cycles, the mean of the distribution will be centered at 30,000 cycles of use, that is, $\mu = 30{,}000$.

In this condition:

$$\Phi(z_1) = 0.05,\ z_1 = \frac{25{,}000 - \mu}{\sigma}$$
$$\Phi(z_2) = 0.95,\ z_2 = \frac{35{,}000 - \mu}{\sigma}.$$

From the normal distribution table, $z_1 = -1.65$, and $z_2 = 1.65$. Hence, $-1.65\sigma = 25{,}000 - \mu$ and $1.65\sigma = 35{,}000 - \mu$.

Solving the above two equations with the mean value of 30,000 cycles results in a σ of 3030 cycles.

Example 3.19

The time for failure due to fungi growth is normally distributed with mean $\mu = 2.8$ hours and standard deviation $\sigma = 0.6$ hours.

(a) What is the probability that the failure due to fungi growth will occur in 1.5 hrs?
(b) If we accept a probability of failure due to fungi growth of only 10%, after what time from the start should the fungi be analyzed?

Solution:

(a) The probability that the fungi will grow in less than 1.5 hours is given by:

$$\mathrm{P}\{\mathrm{T} < 1.5\} = \mathrm{Q}(1.5) = \Phi(z),$$

$$z = (t - \mu)/\sigma = (1.5 - 2.8)/0.6 = -2.1667.$$

From the standard normal table, $\Phi(-2.1667) = 0.0151$.

(b) For this condition, $F(t) = 0.1 = \Phi(z)$, then from the standard normal table, z is approximately -1.28. Therefore, $-t + \mu = 1.28\sigma$, hence $t = 2.03$ hours.

Example 3.20

A component has the normal distribution for time to failure, with $\mu = 20{,}000$ hours and $\sigma = 3000$ hours.

(a) Find the probability that the component will fail between 14,000 hours and 15,000 hours.

Solution:

$$
\begin{aligned}
&P[14{,}000 \le T \le 15{,}000] \\
&= P\left[\frac{14{,}000 - 20{,}000}{3000} \le \frac{T-\mu}{\sigma} \le \frac{15{,}000 - 20{,}000}{3000}\right] \\
&= P[-2 \le Z \le -1.667] \\
&= \Phi(-1.667) - \Phi(-2) \\
&= (1 - 0.95225) - (1 - 0.97725) \\
&= 0.04775 - 0.02275 = 0.025.
\end{aligned}
$$

(b) Find the failure rate of a component that has been working for 14,000 hours.

Solution:

$$
\begin{aligned}
f(14{,}000) &= \phi\left(\frac{14{,}000 - 20{,}000}{3000}\right) \Big/ 3000 \\
&= \frac{1.7997 \times 10^{-5}}{3000} = 5.999 \times 10^{-9} \\
\text{pdf } \phi(14{,}000) &= \phi\left(\frac{14{,}000 - 20{,}000}{3000}\right) = 1.7997 \times 10^{-5} \\
&\quad \text{based on MS Excel evaluation} \\
R(t) &= P\left[Z > \frac{14{,}000 - 20{,}000}{3000}\right] = P[Z > -2] = \Phi(2) = 0.977249 \\
h(t) &= \frac{f(t)}{R(t)} = 6.13865 \times 10^{-9} \text{ failures/cycle}.
\end{aligned}
$$

Example 3.21

The time to failure random variable for a light bulb made by Company X follows a normal distribution, with $\mu = 1600$ hours and $\sigma = 250$ hours.

(a) Find the B_{10} life of these light bulbs.

Solution:
Setting the value of reliability at B_{10} equal to 0.90, we have

$$
\begin{aligned}
R(B_{10}) &= 0.9 \\
z &= -1.28 = \frac{B_{10} - 1600}{250} \\
B_{10} &= (-1.28)(250) + 1600 = 1280 \text{ hours}.
\end{aligned}
$$

(b) Find the reliability of the light bulbs for 1100 hours.

Solution:

$$
R(1100) = \Pr\left(Z \ge \frac{1100 - 1600}{250}\right) = \Pr(Z \ge -2.0) = 0.9773.
$$

(c) What is the failure rate or hazard rate of a light bulb that has not failed for 1100 hours?

Solution:

$$h(1100) = \frac{f(1100)}{R(1100)} = \frac{\phi(-2.0)/250}{\Phi(-2.0)} = \frac{0.05399/250}{0.9773}$$
$$= 0.0002210 \text{ failures/hour}.$$

where:

$$\phi\left(\frac{1100-1600}{250}\right) = \phi(-2) = 0.05399.$$

based on MS Excel evaluation, where:

$$\phi(-2.0) = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{(-2)^2}{2}\right)} = (0.39894)(0.13533) = 0.05399.$$

Alternatively,

$$f(1100) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{(-2)^2}{2}\right)}$$
$$= \frac{0.05399}{250} = 0.0002160$$
$$h(1100) = \frac{f(1100)}{R(1100)}$$
$$= \frac{0.0002160}{0.9773} = 0.0002210.$$

(d) Company X has 5 million light bulbs in the field that have been in use for 1100 hours and have not failed so far. How many light bulbs will fail in the next day (or 24 hours)? Assume that light bulbs are used on average for 10 hours per day.

Solution:
Using the concepts covered in Section 2.2.1 of Chapter 2, we have $N_s(1100) = 5 \times 10^6$ and $\Delta t = 10$ hours.

$$h(1100) = 0.0002210 =$$
$$\frac{N_S(1100) - N_S(1100+10)}{N_S(1100) \times \Delta t} = \frac{N_S(1100) - N_S(1100+10)}{5 \times 10^6 \times 10}.$$

Then the number of failures between 1100 and 1110 hours is given by:

$$(0.0002210)(5 \times 10^6)(10) = 11{,}050 \text{ light bulbs}.$$

3.2.5 The Lognormal Distribution

For a continuous random variable, there may be a situation in which the random variable is a product of a series of random variables. The lognormal distribution is a positively skewed distribution and has been used to model situations where large occurrences are concentrated at the tail (left) end of the range. Some examples are the amount of electricity used by different customers, the downtime of systems, the time to repair, the light intensities of light bulbs, the concentration of chemical process residues, and automotive mileage accumulation by different customers. For example, the wear on a system may be proportional to the product of the magnitudes of the loads acting on it. Thus, a random variable may be modeled as a lognormal random variable if it can be thought of as the multiplicative product of many independent random variables each of which is positive. If a random variable has lognormal distribution, then the logarithm of the random variable is normally distributed. If X is a random variable with a normal distribution, then $Y = e^X$ has a lognormal distribution; or if Y has lognormal distribution, then $X = \log Y$ has normal distribution.

Suppose Y is the product of n independent random variables given by

$$Y = Y_1 Y_2 Y_3 \ldots\ldots Y_n. \tag{3.49}$$

Taking the natural logarithm of Equation 3.49 gives

$$\ln Y = \ln Y_1 + \ln Y_2 + \ln Y_3 + \cdots + \ln Y_n. \tag{3.50}$$

Then $\ln Y$ may have approximately normal distribution based on the central limit theorem.

The lognormal distribution has been shown to apply to many engineering situations, such as the strengths of metals and the dimensions of structural elements, and to biological parameters, such as loads on bone joints. Lognormal distributions have been applied in reliability engineering to describe failures caused by fatigue and to model time to repair for maintainability analysis. The probability density function for the lognormal distribution is:

$$f(t) = \frac{1}{\sigma t \sqrt{2\pi}} \exp\left[\left(-\frac{1}{2}\right)\left(\frac{\ln t - \mu}{\sigma}\right)^2\right], \tag{3.51}$$

where σ is the standard deviation of the logarithms of all times to failure, and μ is the mean of the logarithms of all times to failure. If random variable T follows a lognormal distribution with parameters μ and σ, then $\ln T$ follows a normal distribution so that

$$E[\ln T] = \mu \text{ and } V[\ln T] = \sigma^2. \tag{3.52}$$

The cdf (unreliability) for the lognormal distribution is:

$$\begin{aligned} Q(t) &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^t \frac{1}{x} \exp\left[\left(-\frac{1}{2}\right)\left(\frac{\ln x - \mu}{\sigma}\right)^2\right] dx \\ &= \Phi\left(\frac{\ln t - \mu}{\sigma}\right). \end{aligned} \tag{3.53}$$

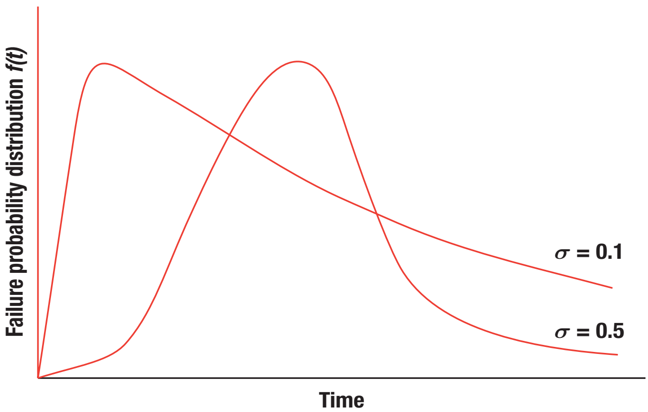

Figure 3.10 Lognormal probability density function where $\sigma = 0.1$ and $\sigma = 0.5$.

Table 3.5 Lognormal distribution parameters

Mean	$\exp[\mu + 0.5\sigma^2]$
Variance	$\left(e^{\sigma^2} - 1\right)e^{2\mu+\sigma^2}$
Median (B_{50} or time at 50% failures)	$B_{50} = e^{\mu}$
Mode (highest value of $f(t)$)	$t = \exp[\mu - \sigma^2]$
Location parameter	e^{μ}
Shape parameter	σ
s (estimate of σ)	$\ln(B_{50}/B_{16})$

The probability density function for two values of σ are as shown in Figure 3.10. The key parameters for the lognormal distribution are provided in Table 3.5.

The MTTF for a population for which the time to failure follows a lognormal distribution is given by

$$\text{MTTF} = \exp\left[\mu + \frac{\sigma^2}{2}\right] \tag{3.54}$$

and the failure rate is given by

$$h(t) = \phi\left(\frac{\ln t - \mu}{\sigma}\right) \Big/ t\sigma R(t). \tag{3.55}$$

The hazard rate for the lognormal distribution is neither always increasing nor always decreasing. It takes different shapes, depending on the parameters μ and σ. We can prove that the hazard rate of a lognormal distribution is increasing on average (called IHRA).

From the basic properties of the logarithm operator, it can be shown that if variables X and Y are distributed lognormally, then the product random variable $Z = XY$ is also lognormally distributed.

Example 3.22

A population of industrial circuit breakers was found to have a lognormal failure distribution with parameters $\mu = 3$ and $\sigma = 1.8$. What is the MTTF of the population? What is the estimate of reliability of these circuit breakers for continuous operation over 30 years?

Solution:
From Equation 3.54 for the MTTF,

$$\text{MTTF} = \exp(3 + 0.5 \times (1.8)2) = 101.5 \text{ years.}$$

For a 30-year operation (from Eq. 3.53),

$$z = \frac{\ln(30) - 3}{1.8} = \frac{3.41 - 3}{1.8} = 0.223.$$

Hence, from the table of standard normal distribution, the estimate of reliability for a 30-year operation is given by:

$$R(30) = [1 - \Phi(z)] = [1 - \Phi(0.223)] = [1 - 0.588] = 0.412.$$

Example 3.23

The time to repair a copy machine follows the lognormal distribution with $\mu = 2.50$ and $\sigma = 0.40$. Time is in minutes.

(a) Find the probability that the copy machine will be repaired in 20 minutes.

Solution:

$$P[T \le 20] = P[\ln T \le \ln 20] = P\left[Z \le \frac{\ln 20 - 2.5}{0.40}\right]$$
$$= P[Z \le 1.23933] = 0.89239.$$

(b) Find the median value, or B50 life, for the time to repair a random variable.

$$P[T \le B_{50}] = 0.5 = P[Z \le 0]$$
$$0 = \frac{\ln T - 2.5}{0.40}$$
$$T = 12.185.$$

3.2.6 Gamma Distribution

The probability density function for the gamma distribution is given by

$$f(t) = \frac{\lambda^\eta}{\Gamma(\eta)} t^{\eta-1} e^{-\lambda t}, \quad t \ge 0, \tag{3.56}$$

where $\Gamma(\eta)$ is the gamma function (values for this function are given in Appendix B). The gamma distribution has two parameters, η and λ, where η is called the shape parameter and λ is called the scale parameter. The gamma distribution reduces to the exponential distribution if $\eta = 1$. Adding η exponential distributions, $\eta \geq 1$, with the same parameter λ, provides the gamma distribution. Thus, the gamma distribution can be used to model time to the ηth failure of a system if the underlying system/component failure distribution is exponential with parameter λ. We can also state that if T_i is exponentially distributed with parameter λ, $i = 1, 2, \ldots, \eta$, then $T = T_1 + T_2 + \cdots + T_\eta$ has a gamma distribution with parameters λ and η. This distribution could be used if we wanted to determine the system reliability for redundancy with identical components all having a constant failure rate.

From Equation 3.56, the cumulative distribution or the unreliability function is

$$F(t) = \int_0^t \frac{\lambda^\eta}{\Gamma(\eta)} \tau^{\eta-1} e^{-\lambda t} d\tau, \quad t \geq 0. \tag{3.57}$$

If η is an integer, then the gamma distribution is also called the Erlang distribution, and it can be shown by successive integration by parts that

$$F(t) = \sum_{k=\eta}^{\infty} \frac{(\lambda t)^k \exp[-\lambda t]}{k!}. \tag{3.58}$$

Then,

$$R(t) = \sum_{k=0}^{\eta-1} \frac{(\lambda t)^k \exp[-\lambda t]}{k!} \tag{3.59}$$

and

$$h(t) = \frac{f(t)}{R(t)}. \tag{3.60}$$

Also,

$$E(T) = \frac{\eta}{\lambda} \tag{3.61}$$

and

$$V(T) = \frac{\eta}{\lambda^2}. \tag{3.62}$$

The failure rate for the gamma distribution is decreasing when $\eta < 1$, is constant when $\eta = 1$(because it is an exponential distribution), and is increasing when $\eta > 1$.

Example 3.24

The time to a major failure in hours for a copy machine follows a gamma distribution with parameters $\eta = 3$ and $\lambda = 0.002$.

(a) What is the expected life, or mean time between failures (MTBF), for the copy machine?

Solution:
Using Equation 3.61, we have

$$\text{MTBF} = \frac{\eta}{\lambda} = \frac{3}{0.002} = 1500 \text{ hours.}$$

(b) What is the reliability of the copy machine for 500 hours of continuous operation?

Solution:
Using Equation 3.59,

$$\begin{aligned} R(t) &= \sum_{k=0}^{\eta-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \\ R(500) &= \sum_{k=0}^{3-1} \frac{(0.002 \times 500)^k e^{-0.002\times 500}}{k!} \\ &= 0.919698. \end{aligned}$$

(c) What is the failure rate of a copy machine that has been working for 500 hours?

Solution:
Using Equation 3.56 and Equation 3.60, we have

$$\begin{aligned} f(t) &= \frac{\lambda^\eta}{\Gamma(\eta)} t^{\eta-1} e^{-\lambda t} \\ f(500) &= \frac{0.002^3}{\Gamma(3)} 500^{3-1} e^{-(0.002*500)} = 0.000368 \\ h(500) &= f(500)/R(500) \\ &= 0.0004001 \text{ failures per hour.} \end{aligned}$$

Thus, the failure rate is 0.0004 failures per hour, or 4 failures per 10,000 hours of total use.

3.3 Probability Plots

Probability plotting is a method for determining whether data (observations) conform to a hypothesized distribution. Typically, computer software is used to assess the

Table 3.6 Examples of cdf estimates for $N = 20$

Rank order (i)	Estimate of cumulative distribution function or unreliability: Midpoint plotting position	Expected plotting position	Median plotting position	Median rank
1	2.5	4.8	3.4	3.406
2	7.5	9.5	8.3	8.251
3	12.5	14.3	13.2	13.147
4	17.5	19.0	18.1	18.055
5	22.5	23.8	23.0	22.967
6	27.5	28.6	27.9	27.880
7	32.5	33.3	32.8	32.795
8	37.5	38.1	37.7	37.710
9	42.5	42.8	42.6	42.626
10	47.5	47.6	47.5	47.542
11	52.5	52.4	52.5	52.458
12	57.5	57.1	57.4	57.374
13	62.5	61.9	62.3	62.289
14	67.5	66.7	67.2	67.205
15	72.5	71.4	72.1	72.119
16	77.5	76.4	77.0	77.033
17	82.5	80.1	81.9	81.945
18	87.5	85.7	86.8	86.853
19	92.5	90.5	91.7	91.749
20	97.5	95.2	96.6	96.594

hypothesized distribution and determine the parameters of the underlying distribution. The method used by the software tools is analogous to using constructed probability plotting paper to plot data. The time-to-failure data is ordered from the smallest to the largest in value in an appropriate metric (e.g., time to failure and cycles to failure). An estimate of the percent of unreliability is selected. The data are plotted against a theoretical distribution in such a way that the points should form a straight line if the data come from the hypothesized distribution. The data are plotted on probability plotting papers (these are distribution specific), with ordered times to failure in the x-axis and the estimate of percent unreliability as the y-axis. A best-fit straight line is drawn through the plotted data points.

The time to failure data used for the x-axis is obtained from the field or testing. The estimate of unreliability against which to plot this time-to-failure data is not that obvious. Several different techniques, such as "midpoint plotting position," "expected plotting position," "median plotting position," "median rank," and Kaplan–Meier ranks (in software) are used for this estimate. Table 3.6 provides estimates for unreliability based on different estimation schemes for a sample size of 20.

The median rank value for the ith failure, Q_i, is given by the solution to the following equation:

$$\sum_{k=i}^{N} \frac{N!}{k!(N-k)!}(1-Q_i)^{N-k} Q_i^k = 0.5, \tag{3.63}$$

where N is the sample size, i is the failure number, and Q_i is the median rank (or estimate of unreliability at the failure time of the ith failure). Equation 3.64, which estimates the median plotting positions, can be used in place of the median rank as an approximation:

$$Q_i = \frac{100 \times (i - 0.3)}{N + 0.4}. \tag{3.64}$$

The axes used for the plots are not linear. The axes are different for each probability distribution and are created by linearizing the cmf or unreliability function, typically by taking the logarithm of both sides repeatedly. For example, mathematical manipulation based on Equation 3.27 for a two-parameter Weibull distribution will result in an ordinate (y-axis) as log log reciprocal of $R(t) = 1 - Q(t)$ scale and the abscissa as a log scale of time to failure, and is derived below:

$$\begin{aligned} Q(t) &= 1 - e^{-\left(\frac{t}{\eta}\right)^{\beta}} \\ \ln(1 - Q(t)) &= \ln\left(e^{-\left(\frac{t}{\eta}\right)^{\beta}}\right) \\ \ln(-\ln(1 - Q(t))) &= \beta \ln\left(\frac{t}{\eta}\right) \\ y &= \beta x - \beta \ln(\eta), \end{aligned} \tag{3.65}$$

where $x = \ln(t)$ and $y = \ln(-\ln(1 - Q(t)))$.

Once the probability plots are prepared for different distributions, the goodness of fit of the plots is one factor in determining which distribution is the right fit for the data. Probability distributions for data analysis should be selected based on their ability to fit the data and for physics-based reasons. There should be a physics-based argument for selection of a distribution that draws from the failure model for the mechanism(s) that caused the failures. These decisions are not always clear-cut. For example, the lognormal and the Weibull distribution both model fatigue failure data well, and hence it is often possible for both to fit the failure data; thus, experience-based engineering judgments need to be made.

There is no reason to assume that all the time-to-failure data taken together need to fit only one failure distribution. Since the failures in a product can be caused by more than one mechanism, it is possible that some of the failures are caused by one mechanism and the others by a different mechanism. In that case, no single probability distribution will fit the data well. Even if it appears that one distribution fits all the data, that distribution may not have good predictive ability. That is why it may be necessary to separate the failures by mechanisms into sets and then fit separate distributions for each set.

Table 3.7 shows times to failure separated into two groups by failure mechanism. Figure 3.11 shows the Weibull probability plots for the competing failure mechanism data. Note that the shape and scale factors for the two sets are distinct, with one set having a decreasing hazard rate ($\beta = 0.67$) and the other set having an increasing hazard rate ($\beta = 4.33$). If the data are plotted together, the result shows an almost constant hazard rate. However, spare part and support decisions made based on results from a combined data analysis can be misleading and counterproductive.

Table 3.7 Time to failure data separated by failure mechanism

Ordered Data	State F or S	Time to F or S	Failure Mechanism Group
1	F	2	V
2	F	10	V
3	F	13	V
4	F	23	V
5	F	23	V
6	F	28	V
7	F	30	V
8	F	65	V
9	F	80	V
10	F	88	V
11	F	106	V
12	F	143	V
13	F	147	W
14	F	173	V
15	F	181	W
16	F	212	W
17	F	245	W
18	F	247	V
19	F	261	V
20	F	266	W
21	F	275	W
22	F	293	W
23	S	300	
24	S	300	
25	S	300	
26	S	300	
27	S	300	
28	S	300	
29	S	300	
30	S	300	

F, failure; S, suspension; V, failure mechanism 1; W, failure mechanism 2.

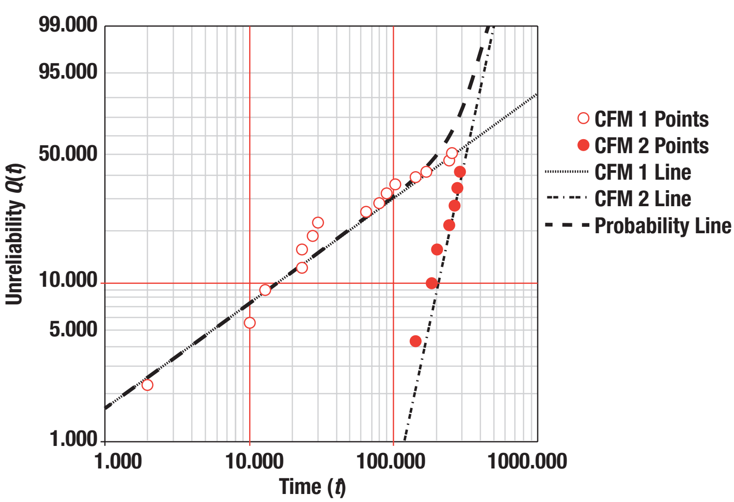

Figure 3.11 Weibull probability plot for competing failure mechanism data shown in Table 3.7. $\beta_1 = 0.67$, $\eta_1 = 450$; $\beta_2 = 4.33$, $\eta_2 = 340$.

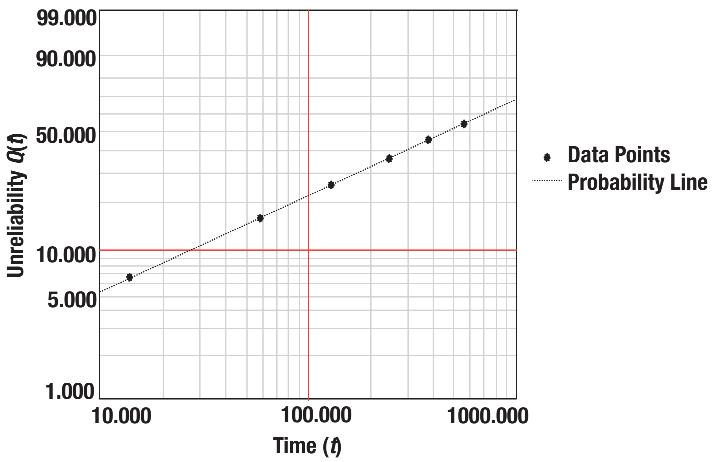

Figure 3.12 Two-parameter Weibull probability plot for time-to-failure data shown in Table 3.8. $\beta_1 = 0.65$, $\eta_1 = 825$.

Table 3.8 Test Data for Example 3.25

Sample number	Time to failure (hours)	Sample number	Time to failure (hours)
1	14	6	563
2	58	7	–
3	130	8	–
4	245	9	–
5	382	10	–

Example 3.25

Figure 3.12 shows reliability test data for 10 identical products out of which six products failed within the test duration of 600 hors. The time to failure is plotted on two-parameter Weibull probability plotting paper. Using the plot, estimate the following:

(a) The unreliability and reliability at the end of 50 hours.
(b) The reliability for a new period of 50 hours, starting after the end of the previous 50-hour period.
(c) The longest duration that will provide a reliability of 95% assuming the operation starts at 50 hours.

Solution:

(a) For this example, we find that $\beta = 0.65$, and η is estimated to be 825 hours. It is now possible to write the equation for the reliability and use it for analysis. The plotted straight line can also be used to determine the reliability values directly.

From Figure 3.12, the unreliability estimate for a mission time of 50 hours can be read directly from the straight line. The value is $Q(50) = 15\%$. Thus, the reliability for this duration is $R(50) = 1 - Q(50) = 85\%$.

(b) The reliability for a new 50-hour period starting with an age of 50 hours is given by the conditional reliability equation as

$$R(50,50)=\frac{R(50+50)}{R(50)}=\frac{R(100)}{R(50)}=\frac{0.78}{0.85}=91.7\%,$$

where $R(100) = 1 - Q(100)$ can be taken directly from the curve.

(c) For a mission time, t, that starts after a 50-hour period and must have a reliability of 95%,

$$R(t,50)=\frac{R(t+50)}{R(50)}=\frac{R(t+50)}{0.85}=0.95$$

or

$$R(t+50)=0.95\times 0.85=0.808.$$

To obtain this reliability, the unreliability is 0.192 or 19.2%. From the curve, the time to obtain this unreliability is about 75 hours. Thus, $50 + t = 75$ gives a maximum new mission time of 25 hours in order to have a reliability of 95%.

When the life data contains two or more life segments—such as infant mortality, useful life, and wearout—a mixed Weibull distribution can be used to fit parts of the data with different distribution parameters. A curved or S-shaped Weibull probability plot (in either two or three parameters) is an indication that a mixed Weibull distribution may be present.

Statistical analysis provides no magical way of projecting into the future. The results from an analysis are only as good as the assumed model and assumptions, including how failure is defined, the validity of the data, how the model is used, and taking into consideration the tail of the distribution and the limits of extrapolations and interpolations. The following example demonstrates the absurdity of extrapolating times to failure beyond their reasonable limits.

Example 3.26

A Weibull probability plot was made for a population collected over the first 10 years of its life containing failures (see Figure 3.13).

(a) Estimate the percentage of this population expected to fail by 300 years.
(b) Does the answer make sense if the time-to-failure data is for human mortality? Explain.

Solution:

(a) The results show that the probability of failure at 300 years is approximately 2%.
(b) The mortality data for over a billion people for a 10-year period from the time of birth fits a Weibull distribution very well. This looks impressive, but is nevertheless all wrong. It is clear that this data should not be used for making any

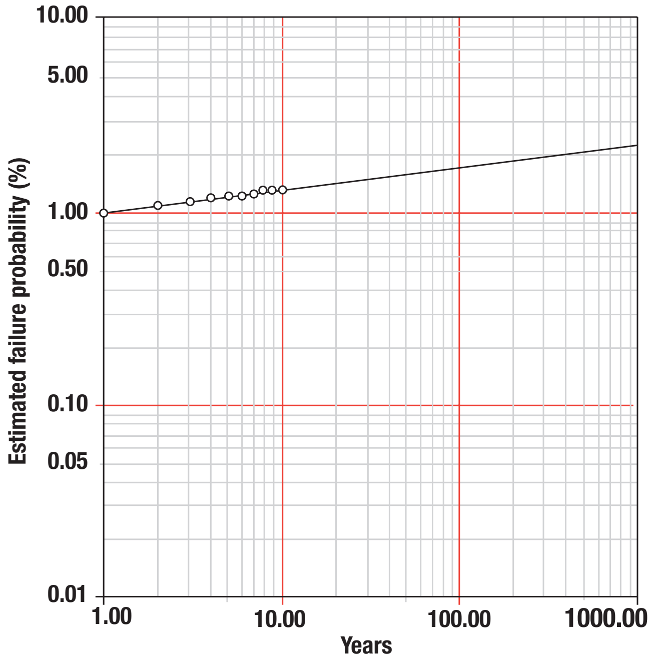

Figure 3.13 Weibull probability plot of time-to-failure data for Example 3.26.

judgment on human longevity, even though all the calculations are correct. The mortality pattern of humans in the first 10 years of life cannot be extrapolated, because the mortality pattern changes with age. This is also often true for engineered goods. Failures that occur in postmanufacturing tests are often caused by defects introduced in manufacturing. The first 10 years of time-to-failure data will result in a shape factor (β) of less than one. However, during early childhood through a large part of adulthood, the shape factor will be close to one, where most deaths can be considered random (e.g., caused by many causes such as accidents). Then the population will enter a wearout stage during which people die from old age. Complete human mortality data should be modeled using a mixed Weibull distribution.

3.4 Summary

The reliability function is used to describe the probability of successful system operation during a system's life. A natural question is then, "What is the shape of a reliability function for a particular system?" There are basically three ways in which this can be determined:

1. Test many systems to failure using a mission profile identical to use conditions. This would provide an empirical curve based on the histogram that can give some idea about the nature of the underlying life distribution.

2. Test many subsystems and components to failure under use conditions recreated in the test environment. This empirically provides the component reliability functions. Then derive analytically or numerically or through simulation the system reliability function. (Chapter 17 covers topics related to system reliability.)
3. Based on past experience with similar systems, hypothesize the underlying failure distribution. Fewer systems can be tested to determine the parameters needed to adapt the failure distribution to a particular situation. However, this will not account for new failure mechanisms or new use conditions.

In some cases, the failure physics involved in a particular situation may lead to the hypothesis of a particular distribution. For example, fatigue of certain metals tends to follow either a lognormal or Weibull distribution. Once a distribution is selected, the parameters for a particular application can be ascertained using statistical or graphical procedures.

In this chapter, various distributions were presented. However, the most appropriate distribution(s) for a particular failure mechanism or product that exhibits certain failure mechanisms must be determined by the actual data, and not guessed. The distribution(s) that best fit the data and that also make sense in terms of the failure processes should be used.

Problems

3.1 Prove that for a binomial distribution in which the number of trials is m and the probability of success in each trial is p, the mean and the variance are equal to mp and $mp(1 - p)$, respectively.

3.2 Prove that for a Poisson distribution, the mean and the variance are equal to the Poisson parameter μ.

3.3 Compare the results of Examples 3.2 and 3.5. What is the reason for the differences?

3.4 Consider a system that has seven components; the system will work if any five of the seven components work. Each component has a reliability of 0.930 for a given period. Find the reliability of the system.

3.5 For an exponential distribution, show that the time to 50% failure is given by $0.693/\lambda_0$.

3.6 For an exponential distribution, show that the standard deviation is equal to $1/\lambda_0$.

3.7 Show that for a two-parameter Weibull distribution, for $t = \eta$, the reliability $R(t) = 0.368$, irrespective of β.

3.8 The front wheel roller bearing life for a car is modeled by a two-parameter Weibull distribution with the following two parameters: $\beta = 3.7$, $\theta = 145{,}000$ mi. What is the 100,000-mi reliability for a bearing?

3.9 The life distribution (life in years of continuous use) of hard disk drives for a computer system follows the Weibull distribution with the following parameters:

$$\beta = 2.7 \quad \text{and} \quad \theta = 5.5\ \text{years.}$$

(a) The manufacturer gives a warranty for 1 year. What is the probability that a disk drive will fail during the warranty period?
(b) Find the mean life and the median life (B_{50}) for the disk drive.
(c) By what time will 99% of the disk drives fail? (That is, find the B_{99} life.)

3.10 The life distribution for miles to failure for the engine of a Lexus car follows the Weibull distribution with

$$\beta = 3.8 \quad \text{and} \quad \theta = 185{,}000\ \text{mi.}$$

(a) Find the mean miles between failures, or the expected life for the engine.
(b) Find the standard deviation for miles to failure.
(c) What percent of these engines will fail by 100,000 mi?
(d) What is the failure rate of an engine that has a life of 100,000 mi?

If a certain model has 200,000 engines in the field with a life of 100,000 mi, how many engines on average will fail in the next 100 mi of use out of the 200,000 engines?

3.11 A component has the normal distribution for time to failure, with $\mu = 26{,}000$ hours and $\sigma = 3500$ hours.

(a) Find the probability that the component will fail between 22,000 hours and 23,000 hours.
(b) Find the failure rate of a component that has been working for 22,000 hours.

3.12 The time to failure random variable for a battery follows a normal distribution, with $\mu = 800$ hours and $\sigma = 65$ hours.

(a) Find the B_{10} life of these batteries.
(b) Find the probability that a battery will fail between 700 and 710 hours, given that it has not failed by 700 hours.
(c) What is the failure rate or hazard rate of a battery that has a life of
    (i) 700 hours
    (ii) 710 hours.

3.13 The time to failure for the hard disk drives for a computer system follows a normal distribution with

$$\mu = \text{mean life} = 14{,}000\ \text{hours}$$

$$\sigma = \text{standard deviation} = 1500\ \text{hours.}$$

(a) A manufacturer gives a warranty for 1 year of continuous use, or 365 $\times$ 24 hours of use. What percentage of hard disk drives will fail during this warranty period?

(b) What is the failure or hazard rate of a drive that has been working successfully for 1 year of continuous use?

(c) An IT manager of a large company, based on field surveys and inventory management, finds that the company has 250,000 of these drives on which the warranty has just expired—that is, they are working today after one year of continuous use. What is the expected number of these drives that will fail in the next 24 hours?

3.14 The time to repair a communication network system follows a lognormal distribution with $\mu = 3.50$ and $\sigma = 0.75$. The time is in minutes.

(a) What is the probability that the communication network will be repaired by 60 minutes?

(b) Find the B_{20} value (the 20th percentile) for the time-to-repair random variable.

(c) Find the mean time to repair (MTTR) for the communication network.

3.15 The time to repair a copy machine follows the lognormal distribution with $\mu = 2.70$ and $\sigma = 0.65$. Time is in minutes.

(a) Find the probability that the copy machine will be repaired in 30 minutes.

(b) Find the median value or B_{50} life for the time-to-repair random variable.

3.16 The time to failure for a copy machine follows a gamma distribution with parameters $\eta = 2$ and $\lambda = 0.004$.

(a) What is the expected or mean time between failures (MTBF) for the copy machine?

(b) What is the reliability of the copy machine for 200 hours of continuous operation?

(c) What is the failure rate of a copy machine that has been working for 200 hours?

3.17 Describe two examples of systems that require a failure-free operating period, without any maintenance. What are the timeframes involved?

3.18 Describe two examples of systems that require a failure-free operating period, but may allow a maintenance period. Discuss the timeframes.

3.19 Show that the mode of the three parameter Weibull distribution is for

$$t = \gamma + \eta(1 - 1/\beta)^{1/\beta}$$

for $\beta > 1$.

3.20 A company knows that approximately 3 out of every 1000 processors that it manufactures are defective. What is the probability that out of the next 20 processors selected (at random):

(a) All 20 are working processors?
(b) Exactly 2 defective processors?
(c) At most 2 defective processors?
(d) At least 18 are defective?