2

2 Reliability Concepts

In Chapter 1, the reliability of a product was defined as "the ability of a product to perform as intended (i.e., without failure and within specified performance limits) for a specified time, in its life cycle conditions." This chapter presents the fundamental definitions and measures needed for quantifying and communicating the reliability of a product. The focus in this chapter is on reliability and unreliability functions, the probability density function, hazard rate, conditional reliability function, percentiles of life, and time-to-failure metrics.

The purpose of, and the need for, a particular product determines the kind of reliability measures that are most meaningful and most useful. In general, a product may be required to perform various functions, each having a different reliability. In addition, at any given time (or number of cycles, or any other measure of the use of a product), the product may have a different probability of successfully performing the required function under the stated conditions.

2.1   Basic Reliability Concepts

For a constant sample size, n_0, of identical products that are tested or being monitored, if n_f products have failed and the remaining number of products, n_S, are still operating satisfactorily at any time, t, then

$$n_S(t) + n_f(t) = n_0. \tag{2.1}$$

The factor t in Equation 2.1 can pertain to age, total time elapsed, operating time, number of cycles, distance traveled, or be replaced by a measured quantity that could range from $-\infty$ to ∞ for any general random variable. This quantity is called a variate in statistics. Variates may be discrete (for the life of a product, the range is from 0 to ∞; e.g., number of cycles) or continuous when they can take on any real value within a certain range of real numbers.

The ratio of failed products per sample size is an estimate of the unreliability, $\hat{Q}(t)$, of the product at any time t:

$$\hat{Q}(t) = \frac{n_f(t)}{n_0},\tag{2.2}$$

where the caret above the variable indicates that it is an estimate. Similarly, the estimate of reliability, $\hat{R}(t)$, of a product at time t is given by the ratio of operating (not failed) products per sample size or the underlying frame of reference:

$$\hat{R}(t) = \frac{n_S(t)}{n_o} = 1 - \hat{Q}(t).\tag{2.3}$$

As fractional numbers, $\hat{R}(t)$ and $\hat{Q}(t)$ range in value from zero to unity; multiplied by 100, they give the estimate of the probability as a percentage.

Example 2.1

A semiconductor fabrication plant has an average output of 10 million devices per week. It has been found that over the past year 100,000 devices were rejected in the final test.

(a) What is the unreliability of the semiconductor devices according to the conducted test?
(b) If the tests reject 99% of all defective devices, what is the chance that any device a customer receives will be defective?

Solution:
The total number of devices produced in a year is:

(a) $n_0 = 52 \times 10 \times 10^6 = 520 \times 10^6$
The number of rejects (failures), n_f, over the same period is:

$$n_f = 1 \times 10^5.$$

Therefore, from Equation 2.2, an estimate for device unreliability is:

$$\hat{Q}(t) = \frac{n_f(t)}{n_0} = \frac{1 \times 10^5}{520 \times 10^6} \approx 1.92 \times 10^{-4},$$

or 1 chance in 5200.
(b) If the rejected devices represent 99% of all the defective devices produced, then the number of defectives that passed testing is:

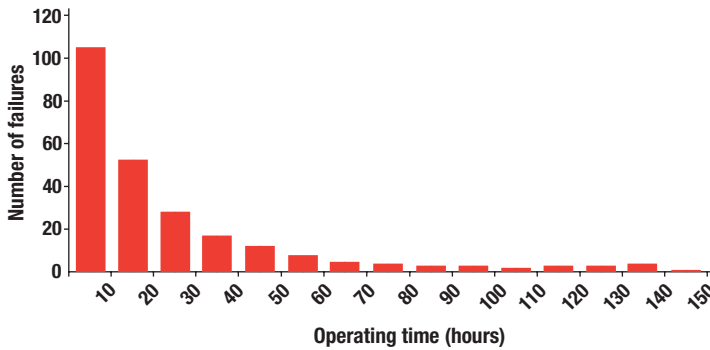$$x_d = \left[\frac{1 \times 10^5}{0.99} - (1 \times 10^5) \right] \approx 1010.$$

Figure 2.1 Frequency histogram or life characteristic curve for data from Table 2.2.

Therefore, the probability of a customer getting a defective device, or the unreliability of the supplied devices on first use, is:

$$\hat{Q}(t) = \frac{1010}{\left(520 \times 10^6\right) - \left(1 \times 10^5\right)} \approx 1.94 \times 10^{-6},$$

or 1 chance in 515,000.

Reliability estimates obtained by testing or monitoring samples in the field generally exhibit variability. For example, light bulbs designed to last for 10,000 hours of operation that are all installed at the same time in the same room are unlikely to fail at exactly the same time, let alone at exactly 10,000 hours. Variability in both the measured product response as well as the time of operation is expected. In fact, product reliability assessment is often associated with the measurement and estimation of this variability.

The accuracy of a reliability estimate at a given time is improved by increasing the sample size, n_0. The requirement of a large sample is analogous to the conditions required in experimental measurements of probability associated with coin tossing and dice rolling. This implies that the estimates given by Equation 2.2 and Equation 2.3 approach actual values for $R(t)$ and $Q(t)$ as the sample size becomes infinitely large. Thus, the practical meanings of reliability and unreliability are that in a large number of repetitions, the proportional frequency of occurrence of success or failure will be approximately equal to the $\hat{R}(t)$ and $\hat{Q}(t)$ estimates, respectively.

The response values for a series of measurements on a certain product parameter of interest can be plotted as a histogram in order to assess the variability. For example, Table 2.1 lists a series of time to failure results for 251 samples that were tested in 11 different groups. These data are summarized as a frequency table in the first two columns of Table 2.2, and a histogram was created from those two columns (Figure 2.1). In the histogram, each rectangular bar represents the number of failures in the interval. This histogram represents the life distribution curve for the product.

The ratios of the number of surviving products to the total number of products (i.e., the reliability at the end of each interval) are calculated in the fourth column of Table 2.2 and are plotted as a histogram in Figure 2.2. As the sample size increases, the intervals of the histogram can be reduced, and often the plot will approach a smooth curve.

Table 2.1 Measured time to failure data (hours) for 251 samples

Group number										
1	2	3	4	5	6	7	8	9	10	11
Data										
1	1	1	1	1	1	1	1	1	1	1
1	1	2	2	2	2	2	2	2	2	2
2	3	2	2	3	3	3	3	3	3	3
3	3	3	3	3	3	3	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6	6
6	6	7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	9	9	9
9	9	9	9	10	10	11	11	11	11	11
11	12	12	12	12	12	12	13	13	13	13
13	14	14	14	14	15	15	15	15	15	15
16	16	16	16	17	17	17	17	17	18	18
18	18	18	18	18	18	19	19	19	19	20
20	20	20	21	21	22	22	23	23	24	24
25	25	26	26	27	27	27	28	28	28	28
28	28	29	29	29	29	29	29	30	31	31
32	32	33	33	34	34	35	35	36	36	36
36	37	38	39	41	41	42	42	43	44	45
46	47	48	49	49	51	52	53	54	55	56
58	59	62	64	65	66	67	69	72	76	78
79	83	85	89	93	97	99	105	107	111	115
117	120	125	126	131	131	137	140	142	–	–

Table 2.2 Grouped and analyzed data from Table 2.1

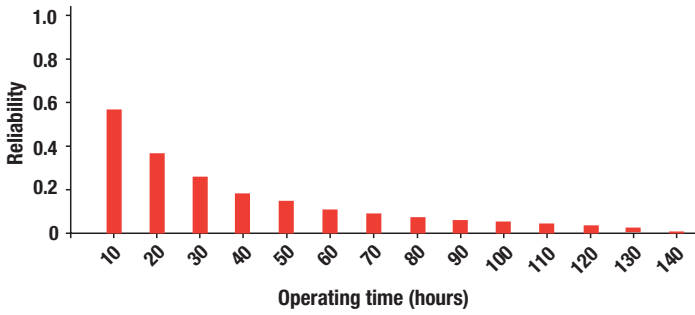| Operating time interval (hours)	Number of failures in the interval	Number of surviving products at the end of the interval	Relative frequency	Estimate of reliability at the end of the interval	Estimate of hazard rate in each interval (failures/hour)
0–10	105	146	0.418	0.582	0.042
11–20	52	94	0.207	0.375	0.036
21–30	28	66	0.112	0.263	0.030
31–40	17	49	0.068	0.195	0.026
41–50	12	37	0.048	0.147	0.024
51–60	8	29	0.032	0.116	0.022
61–70	6	23	0.024	0.092	0.021
71–80	4	19	0.016	0.076	0.017
81–90	3	16	0.012	0.064	0.016
91–100	3	13	0.012	0.052	0.019
101–110	2	11	0.008	0.044	0.015
111–120	3	8	0.012	0.032	0.027
121–130	3	5	0.012	0.020	0.038
131–140	4	1	0.016	0.004	0.080
Over 140	1	0	0.004	0.000	–

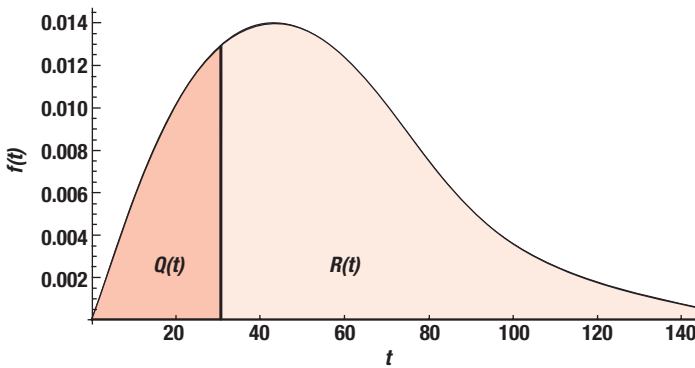**Figure 2.2** Reliability histogram of data from Table 2.1.
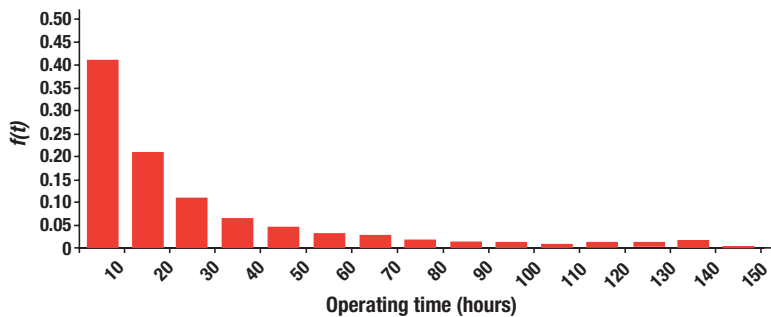

Figure 2.3 Probability density function.

Figure 2.4 Probability density function for the data in Table 2.1.

2.1.1 Concept of Probability Density Function

One reliability concern is the life of a product from a success and failure point of view. The random variable used to measure reliability is the time to failure (T) random variable. If we assume time t as continuous, the time to failure random variable has a probability density function $f(t)$. Figure 2.3 shows an example of a probability density function (pdf).

The ratio of the number of product failures in an interval to the total number of products gives an estimate of the probability density function corresponding to the interval. For the data in Table 2.1, the estimate of the probability density function for each interval is evaluated in the fourth column of Table 2.2. Figure 2.4 shows the

estimate of the probability density function for the data in Table 2.1. The sum of all values in the pdf is equal to unity (e.g., the sum of all values in column four of Table 2.2 is equal to 1).

The probability density function is given by:

$$f(t) = \frac{1}{n_0}\frac{d[n_f(t)]}{dt} = \frac{d[Q(t)]}{dt}.$$

(2.4)

Integrating both sides of this equation gives the relation for unreliability in terms of $f(t)$,

$$Q(t) = \frac{n_f(t)}{n_0} = \int_0^t f(\tau)d\tau,$$

(2.5)

where the integral is the probability that a product will fail in the time interval $0 \leq \tau \leq t$. The integral in Equation 2.5 is the area under the probability density function curve to the left of the time line at some time t (see Figure 2.3). The reliability at any point in time, called the reliability function, is

$$R(t) = \text{Probability}\left[\text{Product life} > t\right] = P[T > t]$$
$$= 1 - P[T \leq t].$$

(2.6)

$P[T \leq t]$ is the cumulative probability of failure, denoted by $F(t)$, and is called the cumulative distribution function (cdf), as explained above.

Similarly, the percentage of products that have not failed up to time t is represented by the area under the curve to the right of t by

$$R(t) = \int_t^\infty f(\tau)d\tau.$$

(2.7)

Since the total probability of failures must equal 1 at the end of life for a population, we have

$$\int_0^\infty f(t)dt = 1.$$

(2.8)

Figure 2.5 gives an example of the cdf and the reliability function and their relationships. The cdf is a monotonically nondecreasing function, and thus $R(t)$ is a monotonically nonincreasing function.

Example 2.2

From the histogram in Figure 2.4:

(a) Calculate the unreliability of the product at a time of 30 hours.
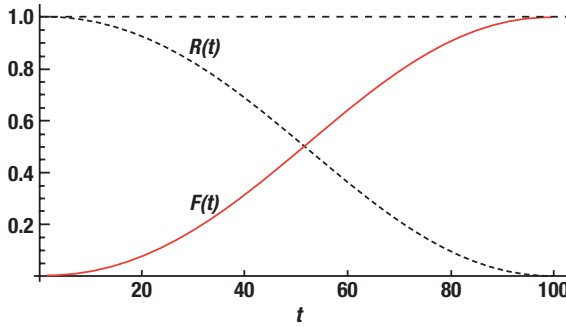(b) Also calculate the reliability.

Figure 2.5 Example of $F(t)$ and $R(t)$.

Solution:

(a) For the discrete data represented in this histogram, the unreliability is the sum of the failure probability density function values from $t = 0$ to $t = 30$. This sum, as a percentage, is 73.7%.

(b) The reliability is equal to 26.3% and can be read from column 5 of Table 2.2. The sum of reliability and unreliability must always be equal to 100%.

Example 2.3

A product has a maximum life of 100 hours, and its pdf is given by a triangular distribution, as shown in the figure below. Develop the pdf, cdf, and the reliability function for this product.
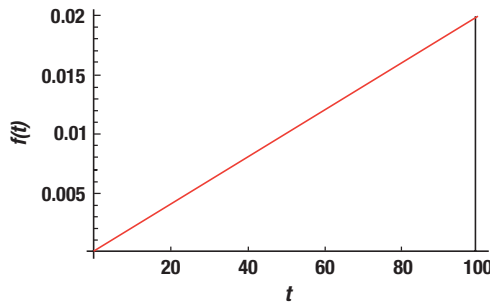
Solution:
Its pdf, cdf, and reliability function, respectively, are given below:

$$f(t) = \begin{cases} \dfrac{t}{5,000}, & \text{for } 0 \leq t \leq 100 \\ 0, & \text{otherwise} \end{cases}$$

$$F(t) = \int_0^t f(\tau)d\tau = \int_0^t \dfrac{\tau}{5,000}d\tau = \begin{cases} 0, & \text{for } t < 0 \\ \dfrac{t^2}{10,000}, & \text{for } 0 \leq t \leq 100 \\ 1, & \text{for } t > 100 \end{cases}$$

25

$$R(t) = 1 - F(t) = \begin{cases} 1, & \text{for } t < 0 \\ 1 - \dfrac{t^2}{10{,}000}, & \text{for } 0 \leq t \leq 100 \\ 0, & \text{for } t > 100. \end{cases}$$

2.2   Hazard Rate

The failure of a population of fielded products can arise from inherent design weaknesses, manufacturing- and quality control-related problems, variability due to customer usage, the maintenance policies of the customer, and improper use or abuse of the product. The hazard rate, $h(t)$, is the number of failures per unit time per number of nonfailed products remaining at time t. An idealized (though rarely occurring) shape of the hazard rate of a product is the bathtub curve (Figure 2.6). A brief description of each of the three regions is given in the following:

1. *Infant Mortality Period.* The product population exhibits a hazard rate that decreases during this first period (sometimes called "burn-in," "infant mortality," or the "debugging period"). This hazard rate stabilizes at some value at time t_1 when the weak products in the population have failed. Some manufacturers provide a burn-in period for their products, as a means to eliminate a high proportion of initial or early failures.

2. *Useful Life Period.* The product population reaches its lowest hazard rate level and is characterized by an approximately constant hazard rate, which is often referred to as the "constant failure rate." This period is usually considered in the design phase.

3. *Wear-Out Period.* Time t_2 indicates the end of useful life and the start of the wear-out phase. After this point, the hazard rate increases. When the hazard rate becomes too high, replacement or repair of the population of products should be conducted. Replacement schedules are based on the recognition of this hazard rate.

Optimizing reliability must involve the consideration of the actual life-cycle periods. The actual hazard rate curve will be more complex in shape and may not even exhibit all of the three periods.
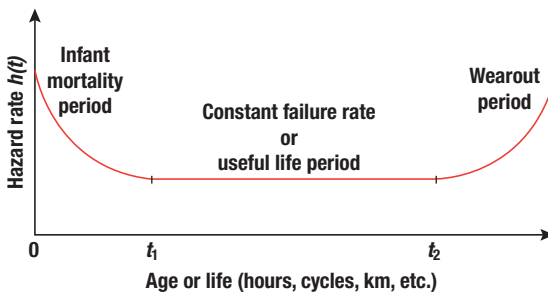

Figure 2.6   Idealized bathtub hazard rate curve.

2.2.1 Motivation and Development of Hazard Rate

Suppose N items are put on test at time $t = 0$. Let $N_S(t)$ be the random variable denoting the number of products functioning at time t. $N_S(t)$ follows the binomial distribution (see Chapter 3) with parameters N and $R(t)$, where $R(t)$ is the reliability of a product at time t. Denoting the expected value of $N_S(t)$ by $\overline{N_S}(t)$, we have

$$E[N_S(t)] = \overline{N_S}(t) = NR(t) \tag{2.9}$$

or

$$R(t) = \frac{\overline{N_S}(t)}{N}. \tag{2.10}$$

Also, we have

$$F(t) = 1 - R(t) = \frac{N - \overline{N_S}(t)}{N}. \tag{2.11}$$

And by differentiating, we have

$$
\begin{aligned}
f(t) &= \frac{dF(t)}{dt} = -\frac{1}{N}\frac{d\overline{N_S}(t)}{dt} \\
&= \lim_{\Delta t \to 0} \frac{\overline{N_S}(t) - \overline{N_S}(t + \Delta t)}{N \Delta t}.
\end{aligned}
\tag{2.12}
$$

Equation 2.12 illustrates that the failure pdf is normalized in terms of the size of the original population, N. However, it is often more meaningful to normalize the rate with respect to the average number of units successfully functioning at time t, since this indicates the hazard rate for those surviving units. If we replace N with $\overline{N_S}(t)$, we have the hazard rate or "instantaneous" failure rate, which is given by Equation 2.13:

$$
\begin{aligned}
h(t) &= \lim_{\Delta t \to 0} \frac{\overline{N}_s(t) - \overline{N}_s(t + \Delta t)}{\overline{N}_s(t) \Delta t} \\
&= \frac{N}{\overline{N}_s(t)} f(t) = \frac{f(t)}{R(t)}.
\end{aligned}
\tag{2.13}
$$

Thus, the hazard rate is the rate at which failures occur in a certain time interval for those items that are working at the start of the interval. If N_1 units are working at the beginning of time t, and after the time increment Δt, N_2 units are working, that is, if $(N_1 - N_2)$ units fail during Δt, then the failure rate $\hat{h}(t)$ at time t is given by:

$$\hat{h}(t) \approx \frac{N_1 - N_2}{N_1 \Delta t}. \tag{2.14}$$

Or, in words,

$$\text{Hazard rate} = \frac{\text{of failures in the given time interval}}{\text{of survivors at the start of interval} \times \text{interval length}}.$$

Hazard rate is thus a relative rate of failure, in that it does not depend on the original sample size. From Equation 2.13, a relation for the hazard rate in terms of the reliability is:

$$h(t) = \frac{-1}{R(t)} \frac{dR(t)}{dt} \tag{2.15}$$

because

$$f(t) = -\frac{dR(t)}{dt}. \tag{2.16}$$

Integrating Equation 2.15 over an operating time from 0 to t and noting that $R(t = 0) = 1$ gives:

$$\int_0^t h(\tau)d\tau = -\int_0^t \frac{1}{R(\tau)}dR(\tau) = -\ln R(t) \tag{2.17}$$

$$R(t) = e^{-\int_0^t h(\tau)d\tau}. \tag{2.18}$$

2.2.2 Some Properties of the Hazard Function

Some properties of the hazard rate are valuable for understanding reliability. We can prove that

$$\int_0^t h(\tau)d\tau \xrightarrow[t\to\infty]{} \infty. \tag{2.19}$$

In order to prove it, first note that

$$h(t) = \frac{f(t)}{R(t)} = \frac{1}{R(t)}\left[-\frac{d}{dt}R(t)\right]. \tag{2.20}$$

Hence,

$$\begin{aligned}
\int_0^t h(\tau)d\tau &= -\int_0^t \frac{1}{R(\tau)}\left[\frac{d}{d\tau}R(\tau)\right]d\tau \\
&= -\ln[R(\tau)]\big|_0^t \\
&= -\ln[R(\tau)] + \ln[R(0)].
\end{aligned} \tag{2.21}$$

Now, $R(t) \to 0$ as $t \to \infty$, hence $-\ln[R(t)] \to \infty$ as $t \to \infty$, and $\ln[R(0)] = \ln[1] = 0$. Thus,

$$\int_0^\infty h(t)dt \to \infty. \tag{2.22}$$

We also note:

$$\int_0^{t\to\infty} h(\tau)d\tau = \int_0^{t\to\infty} \frac{f(\tau)}{R(\tau)}d\tau = \int_0^{t\to\infty} \frac{f(\tau)}{1-F(\tau)}d\tau. \tag{2.23}$$

We can let $u = 1 - F(\tau)$, and then we have

$$du = -f(\tau)d\tau. \tag{2.24}$$

So,

$$-\int_1^0 \frac{du}{u} = -\ln u \Big|_1^0 \to \infty. \tag{2.25}$$

The rate at which failures occur in a certain time interval $[t_1, t_2]$ is called the hazard (or failure) rate during that interval. This time-dependent function is a conditional probability defined as the probability that a failure per unit time occurs in the interval $[t_1, t_2]$ given that a failure has not occurred prior to t_1. Thus, the hazard rate is

$$\frac{R(t_1) - R(t_2)}{(t_2 - t_1)R(t_1)}. \tag{2.26}$$

If we redefine the interval as $[t, t + \Delta t]$, the above expression becomes:

$$\frac{R(t) - R(t + \Delta t)}{\Delta t \cdot R(t)}. \tag{2.27}$$

The "rate" in the above definitions is expressed as failures per unit "time," where "time" is generic in the sense that it denotes units of product usage, which might be expressed in hours, cycles, or kilometers of usage.

The hazard function, $h(t)$, is defined as the limit of the failure rate as Δt approaches zero:

$$h(t) = \lim_{\Delta t \to 0} \frac{R(t) - R(t + \Delta t)}{\Delta t \cdot R(t)} = \frac{1}{R(t)}\left(-\frac{d}{dt}R(t)\right) = \frac{f(t)}{R(t)}. \tag{2.28}$$

Thus, $h(t)$ can be interpreted as the rate of change of the conditional probability of failure given that the system has survived up to time t.

The importance of the hazard function is that it indicates the change in failure rate over the life of a population of devices. For example, two designs may provide the same reliability at a specific point in time; however, the hazard rates can differ over time. Accordingly, it is often useful to evaluate the cumulative hazard function, $H(t)$. $H(t)$ is given by:

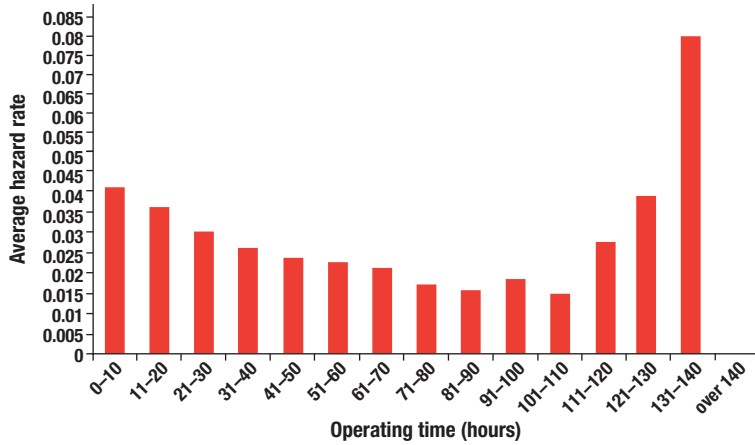$$H(t) = \int_{\tau=0}^{t} h(\tau)d\tau. \tag{2.29}$$

Figure 2.7 Hazard rate histogram of data from Table 2.1.

Both $R(t)$ and $F(t)$ are related to $h(t)$ and $H(t)$, and we can develop the following relationships:

$$h(t) = \frac{f(t)}{R(t)} = \frac{1}{R(t)}\left(-\frac{d}{dt}R(t)\right) = -\frac{d\ln[R(t)]}{dt} \tag{2.30}$$

or

$$-d\ln(R(t)) = h(t)dt. \tag{2.31}$$

Integrating both sides leads to the following relationship:

$$-\ln[R(t)] = \int_{\tau=0}^{t} h(\tau)d\tau = H(t) \tag{2.32}$$

or

$$R(t) = \exp\left(-\int_{\tau=0}^{t} h(\tau)d\tau\right) = \exp(-H(t)). \tag{2.33}$$

Using the data from Table 2.1 and Equation 2.14, an estimate (over Δt) of the hazard rate is calculated in the last column of Table 2.2. Figure 2.7 is the histogram of hazard rate versus time.

Example 2.4

The failure or hazard rate of a component is given by (life is in hours):

$$h(t) = \begin{cases} 0.015, & t \leq 200 \\ 0.025, & t > 200. \end{cases}$$

Thus, the hazard rate is piecewise constant.

Find an expression for the reliability function of the component.

Solution:

Using Equation 2.18 or Equation 2.33, we have

$$R(t) = \exp\left[-\int_0^t h(\tau)d\tau\right].$$

For

$$0 \leq t \leq 200 : R(t) = \exp\left[-\int_0^t 0.015 d\tau\right] = \exp[-0.015t].$$

For

$$t > 200 : R(t) = \exp\left[-\left(\int_0^{200} 0.015 d\tau + \int_{200}^t 0.025 d\tau\right)\right]$$

$$= [-(0.015(200) + 0.025t - 0.025(200))]$$

$$= \exp[-(0.025t - 2)] = \exp[2 - 0.025t].$$

The four functions $f(t)$, $F(t)$, $R(t)$, and $h(t)$ are all related. If we know any one of these four functions, we can develop the other three using the following equations:

$$h(t) = \frac{f(t)}{R(t)} \tag{2.34}$$

$$R(t) = \exp\left[-\int_0^t h(u)du\right] \tag{2.35}$$

$$f(t) = h(t)\exp\left[-\int_0^t h(u)du\right] \tag{2.36}$$

$$Q(t) = F(t) = 1 - R(t). \tag{2.37}$$

2.2.3 Conditional Reliability

The conditional reliability function $R(t, t_1)$ is defined as the probability of operating for a time interval of duration, t, given that the nonrepairable system has operated for a time t_1 prior to the beginning of the interval. The conditional reliability can be expressed as the ratio of the reliability at time $(t + t_1)$ to the reliability at t_1, where t_1 is the "age" of the system at the beginning of a new test or mission. That is,

$$R(t, t_1) = P[(t + t_1) > T \mid T > t_1] = \frac{P[(t + t_1) > T]}{P[T > t_1]} \qquad (2.38)$$

or

$$R(t, t_1) = \frac{R(t + t_1)}{R(t_1)}. \qquad (2.39)$$

For a product with a decreasing hazard rate, the conditional reliability will increase as the age, t_1, increases. The conditional reliability will decrease for a product with an increasing hazard rate. The conditional reliability of a product with a constant rate of failure is independent of age. This suggests that a product with a constant failure rate can be treated "as good as new" at any time.

Example 2.5

The reliability function for a system is assumed to be an exponential distribution (see Chapter 3) and is given by

$$R(t) = e^{-\lambda_0 t},$$

where λ_0 is a constant (i.e., a constant hazard rate).

Calculate the reliability of the system for mission time, t, given that the system has already been used for 10 years.

Solution:
Using Equation 2.39

$$R(t, 10) = \frac{R(t + 10)}{R(10)} = \frac{e^{-\lambda_0(t + 10)}}{e^{-\lambda_0 10}} = e^{-\lambda_0 t} = R(t).$$

That is, the system reliability is "as good as new," regardless of the age of the system.

Example 2.6

If T is a random variable representing the hours to failure for a device with the following pdf:

$$f(t) = t \exp\left(\frac{-t^2}{2}\right), \quad t \geq 0.$$

(a) Find the reliability function.

Solution:
To develop the reliability function, $R(t)$, we have

$$R(t) = \int_t^\infty f(\tau)d\tau = \int_t^\infty \tau \exp(-\tau^2/2)d\tau.$$

Let $u = \tau^2/2$, du $= \tau d\tau$; then we have

$$R(t) = \int_{\frac{t^2}{2}}^{\infty} \exp(-u)\,du = \exp\left(\frac{-t^2}{2}\right), \quad t \geq 0.$$

(b) Find the hazard function.

Solution:
To develop the hazard function $h(t)$, we have

$$h(t) = f(t)/R(t) = t, \quad t \geq 0.$$

Thus the hazard rate is linearly increasing with a slope of 1.

(c) If 50 devices are placed in operation and 27 are still in operation 1 hour later, find approximately the expected number of failures in the time interval from 1 to 1.1 hours using the hazard function.

Solution:
To answer this question, we can use the information in Section 2.2.1, and we have

$$N_S(0) = 50, N_S(1\,\text{hour}) = 27, \Delta N = N_S(1) - N_S(1.1) = ?$$

For small Δt, the expected number failing can be calculated using Equation 2.14:

$$\Delta N = N_S(t) - N_S(t + \Delta t) \approx h(t)\Delta t\, N_S(t)$$
$$= 1.0 \times 0.1 \times 27 = 2.7.$$

Note that by the using the concept of conditional reliability, we also get,

$$P[T > 1.1 | T > 1.0] = \frac{R(1.1)}{R(1.0)} = \frac{0.54607}{0.60653} = 0.8904$$

or

$$\Delta N = 27 \times (1 - 0.9) = 2.7.$$

2.3 Percentiles Product Life

The reliability of a product can be experienced in terms of percentiles of life. Because this approach was originally used to specify the life of bearings, the literature often uses the symbol B_α, where the B_α life is the time by which α percent of the products fail, or:

$$F(B_\alpha) = \frac{\alpha}{100} \tag{2.40}$$

or

$$R(B_\alpha) = 1 - \frac{\alpha}{100}. \tag{2.41}$$

For example, B_{10} life is the 10th percentile of life of the product. Thus,

$$F(B_{10}) = \frac{10}{100} = 0.10. \tag{2.42}$$

Similarly, B_{95} is the 95th percentile of life of the product and is given by

$$F(B_{95}) = \frac{95}{100} = 0.95 \tag{2.43}$$

or

$$R(B_{95}) = 1 - \frac{95}{100} = 0.05. \tag{2.44}$$

Median life is the 50th percentile of life and is denoted by B_{50}. Thus, the median life, M, of a probability distribution is the time at which the area under the distribution is divided in half (i.e., the time to reach 50% reliability). That is,

$$\int_0^M f(t)dt = 0.50. \tag{2.45}$$

Example 2.7

The failure rate or hazard rate of a component is:

$$h(t) = 0.02t^{1.7}, \quad t \geq 0.$$

The failure rate is in failures per year.

(a) What is the reliability function of this component and what is the value of the reliability for a period of 2 years?

Solution:

$$R(t) = \exp\left[-\int_0^t h(\tau)d\tau\right] = \exp\left[-\int_0^t 0.02\tau^{1.7}d\tau\right] = e^{-0.007407t^{2.7}}$$

$$R(2) = e^{-0.007407 \times 2^{2.7}} = e^{-0.048131} = 0.953009.$$

(b) What is the median life or B_{50} life of this component?

Solution:

$$R(B_{50}) = 0.50 = e^{-0.007407(B_{50})^{2.7}}$$

$$\ln 0.50 = -0.007407(B_{50})^{2.7}$$

$$B_{50} = \left(\frac{-\ln 0.50}{0.007407}\right)^{1/2.7} = 5.37 \text{ years.}$$

2.4 Moments of Time to Failure

The mean or expected value of T, a measure of the central tendency of the random variable, also known as the first moment, is denoted as $E[T]$ or μ, and given by

$$E[T] = \mu = \int_{-\infty}^{\infty} tf(t)dt. \tag{2.46}$$

Higher order moments are discussed in the following section.

2.4.1 Moments about Origin and about the Mean

The kth moment about the origin of the random variable T is

$$\mu_k' = E[T^k] = \int_{-\infty}^{\infty} t^k f(t)dt, \quad k = 1, 2, 3, \dots. \tag{2.47}$$

Notice that the first moment about the origin is just the mean. That is,

$$E[T] = \mu_1' = \mu. \tag{2.48}$$

The kth moment about the mean of the random variable T is

$$\mu_k = E\left[(T-\mu)^k\right] = \int_{-\infty}^{\infty} (t-\mu)^k f(t)dt \tag{2.49}$$

$$k = 2, 3, 4, \dots.$$

For large k, the above integration can be tedious. The equation to derive the kth moment about the mean is:

$$\mu_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mu^j \mu_{k-j}' \tag{2.50}$$

where

$$C_j^k = \binom{k}{j} = \frac{k!}{j!(k-j)!}. \tag{2.51}$$

2.4.2 Expected Life or Mean Time to Failure

For a given underlying probability density function, the mean time to failure (MTTF) is the expected value for the time to failure. It is defined as

$$E[T] = \text{MTTF} = \int_0^\infty t f(t)\,dt. \tag{2.52}$$

It can also be shown that MTTF is equivalent to

$$\text{MTTF} = \int_0^\infty R(t)\,dt. \tag{2.53}$$

Thus, $E[T]$ is the first moment or the center of gravity of the probability density function (like the fulcrum of a seesaw). $E[T]$ is also called the mean time between failures (MTBF), when the product exhibits a constant hazard rate; that is, the failure probability density function is an exponential.

The MTTF should be used only when the failure distribution function is specified, because the value of the reliability function at a given MTTF depends on the probability distribution function used to model the failure data. Furthermore, different failure distributions can have the same MTTF while having very different reliability functions.

The first few failures that occur in a product or system often have the biggest impact on safety, warranty, and supportability, and consequently on the profitability of the product. Thus, the beginning of the failure distribution is a much more important concern for reliability than the mean.

2.4.3 Variance or the Second Moment about the Mean

Information on the dispersion of the values with respect to the mean is expressed in terms of variance, standard deviation, or coefficient of variation. The variance of the random variable T, a measure of variability or spread in the data about the mean, is also known as the second central moment and is denoted as $V[T]$. It can be calculated as

$$\mu_2 = V[T] = E\big[(T - E[T])^2\big] = \int_{-\infty}^\infty \big(t - E[T]^2\big) f(t)\,dt. \tag{2.54}$$

Using Equation 2.50, we have

$$\mu_2 = \mu_2' - 2\mu\mu_1' + \mu^2\mu_0' = \mu_2' - \mu^2$$

$$\text{because} \quad \mu_0' = \int_0^\infty t^0 f(t)\,dt = 1 \quad \text{and} \quad \mu = \mu_1'. \tag{2.55}$$

Since the second moment about the origin is $E[T^2] = \mu_2'$, we can write the variance of a random variable in terms of moments about the origin as follows:

$$V[T] = E[T^2] - \{E[T]\}^2 = \mu'_2 - \mu^2. \tag{2.56}$$

The positive square root of the variance is called the standard deviation, denoted by σ, and is written as

$$\sigma = \sqrt[+]{V[T]}. \tag{2.57}$$

Although the standard deviation value is expressed in the same units as the mean value, its value does not directly indicate the degree of dispersion or variability in the random variable, except in reference to the mean value. Since the mean and the standard deviation values are expressed in the same units, a nondimensional term can be introduced by taking the ratio of the standard deviation and the mean. This is called the coefficient of variation and is denoted as $CV[T]$:

$$\alpha_2 = CV[T] = \frac{\mu_2^{1/2}}{\mu} = \frac{\sigma}{\mu}. \tag{2.58}$$

2.4.4 Coefficient of Skewness

The degree of symmetry in the probability density function can be measured using the concept of skewness, which is related to the third moment, μ_3. Since it can be positive or negative, a nondimensional measure of skewness, known as the coefficient of skewness, can be developed to avoid dimensional problems as given below:

$$\alpha_3 = \frac{\mu_3}{\mu_2^{3/2}}. \tag{2.59}$$

If α_3 is zero, the distribution is symmetrical about the mean; if α_3 is positive, the dispersion is more above the mean than below the mean; and if it is negative, the dispersion is more below the mean. If a distribution is symmetrical, then the mean and the median are the same. If the distribution is negatively skewed, then the median is greater than the mean. And if the distribution is positively skewed, then the mean is greater than the median.

For reliability, we want products to last longer and hence we should design products so that the life distribution is negatively skewed. For maintainability, we want to restore the function of the system in a small amount of time, and hence the time to repair or restoration should follow a positively skewed distribution.

2.4.5 Coefficient of Kurtosis

Skewness describes the amount of asymmetry, while kurtosis measures the concentration (or peakedness) of data around the mean and is measured by the fourth central moment. To find the coefficient of kurtosis, divide the fourth central moment by the square of the variance to get a nondimensional measure. The coefficient of kurtosis represents the peakedness or flatness of a distribution and is defined as:

$$\alpha_4 = \frac{\mu_4}{\mu_2^2}. \tag{2.60}$$

The normal distribution (see Chapter 3) has $\alpha_4 = 3$, and hence sometimes we define a coefficient of kurtosis as

$$\alpha_4 - 3 = \frac{\mu_4}{\mu_2^2} - 3, \tag{2.61}$$

to compare the peakness or flatness of the distribution with a normal distribution.

Example 2.8

For the triangular life distribution given in Example 2.3, calculate the $E[T]$, $V[T]$, and standard deviation.

Solution:
We have

$$f(t) = \begin{cases} \dfrac{t}{5,000}, & \text{for } 0 \leq t \leq 100 \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$E[T] = \int_0^\infty t f(t) dt$$

$$= \int_0^{100} t \frac{t}{5,000} dt = \frac{1}{5,000} \frac{t^3}{3}\Big|_0^{100} = \frac{1}{5,000} \frac{100^3}{3}$$

$$= \frac{2}{3} \cdot 100 = 66.67 \text{ hours}$$

and

$$E[T^2] = \int_0^\infty t^2 f(t) dt$$

$$= \int_0^{100} t^2 \frac{t}{5,000} dt = \frac{1}{5,000} \frac{t^4}{4}\Big|_0^{100} = \frac{100^4}{20,000} = 5,000$$

so,

$$V[T] = E[T^2] - (E[T])^2$$

$$= 5,000 - \left(\frac{200}{3}\right)^2 = \frac{5,000}{9} = 555.55.$$

The standard deviation, σ, is 23.57, and the coefficient of variation is $23.57/66.67 = 0.354$.

Example 2.9

The failure rate per year of a component is given by:

$$h(t) = 0.003t^2, \quad t \geq 0.$$

(a) Find an expression for the reliability function and the probability density function for the time to failure of the component.

Solution:

$$R(t) = \exp\left(-\int_0^t h(\tau)d\tau\right) = \exp\left(-\int_0^t 0.003\tau^2 d\tau\right)$$
$$= \exp(-0.001t^3)$$

and for the probability density function, we have

$$f(t) = h(t)R(t) = 0.003t^2 \exp(-0.001t^3).$$

(b) Find the B_{20} (the 20th percentile) for the life of the component.

Solution:
We have

$$0.80 = \exp(-0.001B_{20}^3)$$
$$B_{20} = \left(\frac{\ln 0.80}{-0.001}\right)^{1/3} = 6.065 \text{ years.}$$

(c) Find the expected life (MTTF) for the component.

Solution:

$$E[T] = \int_0^\infty R(t)dt = \int_0^\infty t \cdot f(t)dt = \int_0^\infty 0.003t^3 \exp(-0.001t^3)dt.$$

Let $u = 0.00t^3$, $du = 0.003t^2 dt$

$$E[T] = \frac{1}{0.001^{1/3}} \int_0^\infty u^{(1/3+1)-1} e^{-u} du = \frac{1}{0.001^{1/3}} \Gamma(1.333) = 10 \times 0.89302 = 8.9302 \text{ years}$$

where the value of the gamma function is found from the table in Appendix B.

2.5   Summary

The fundamental reliability concepts presented in this chapter include reliability and unreliability functions, the probability density function, hazard rate, conditional reliability function, percentiles of life, and time-to-failure metrics. The proper reliability

measure for a product is determined by the specific purpose, and need, for a product. A single product may perform separate functions that each have a different level of reliability. In addition, a single product can have different reliability values at different times during its lifetime, depending on various operational and environmental conditions. The concepts presented in this chapter represent the basis for successful implementation of a reliability program in an engineering system.

Problems

2.1 Following the format of Table 2.1, record and calculate the different reliability metrics after bending 30 paper clips 90° back and forth to failure. Thus, the number of bending cycles is the underlying random variable. Plot the life characteristics curve, the estimate of the probability density function, the reliability and unreliability, and the hazard rate. Do you think your results depend on the amount of bend in the paper clip? Explain.

2.2 A warranty reporting system reports field failures. For the rear brake drums on a particular pickup truck, the following (coded) data were obtained. For the data provided, plot the hazard rate, the failure probability density function, and the reliability function. Assume that the population size is 2680 and that the data represent all of the failures.

Kilometer interval	Number of failures
$M < 2000$	707
$2000 \leq M < 4000$	532
$4000 \leq M < 6000$	368
$6000 \leq M < 8000$	233
$8000 \leq M < 10{,}000$	231
$10{,}000 \leq M < 12{,}000$	136
$12{,}000 \leq M < 14{,}000$	141
$14{,}000 \leq M < 16{,}000$	78
$16{,}000 \leq M < 18{,}000$	101
$18{,}000 \leq M < 20{,}000$	46
$20{,}000 \leq M < 22{,}000$	51
$22{,}000 \leq M < 24{,}000$	56

2.3 Consider the piecewise linear bathtub hazard function defined over the three regions of interest given below. The constants in the expressions are determined so that they satisfy the normal requirements for $h(t)$ to be a hazard function.

$$h(t) = \begin{cases} h_1(t) = b_1 - c_1 t, & 0 \leq t \leq t_1 \\ h_2(t) = b_1 - c_1 t_1 - c_2(t - t_1), & t_1 \leq t \leq t_2 \\ h_3(t) = b_1 - c_1 t_1 - c_2(t_2 - t_1) + c_3(t - t_2), & t_2 \leq t \leq \infty. \end{cases}$$

Develop the equations for the reliability function and the probability density function for the time to failure random variable based on the above hazard function.

2.4 Consider the following functions:

(a) e^{-at}

(b) e^{at}

(c) ct^5

(d) dt^{-3}

where a, c, and d are positive constants.

 Which of the above functions can serve as hazard function models? Also, develop mathematical expressions for the probability density function and the reliability function for the valid hazard functions.

2.5 Prove that

$$\text{MTTF} = \int_0^\infty tf(t)dt = \int_0^\infty R(t)dt.$$

2.6 The time to failure random variable, t, for a product follows the following probability density function, where time is in years:

$$f(t) = \begin{cases} \dfrac{t}{200}, & 0 \le t \le 20 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find the standard deviation for the time to failure random variable.

(b) Find the B_{10} and B_{50} life of the product based on the above probability density function.

(c) Draw the failure rate (or hazard rate) curve for the above product by evaluating it at $t = 0, 1, 2, 5, 10, 15, 20$.

(d) Find the coefficient of skewness, α_3, for this life distribution of the product.

2.7 The hazard rate or failure rate of a product is given by

$$h(t) = 0.002t, \quad t \ge 0.$$

The failure rate is in failures per year.

(a) Find an expression for the reliability function and the probability density function for the time to failure of the product.

(b) Find the B_{10} (the 10th percentile) life of the product.

(c) Find the expected value for the life of the product.

2.8 The failure rate of a component is given by:

$$h(t) = 0.006t^2, \quad t \ge 0.$$

The failure rate is in failures per year.

(a) Find an expression for the reliability function and the probability density function for the time to failure for the component.
(b) Find the B_{20} (the 20th percentile) for the life of the component.

2.9 Calculate the MTTF for a failure probability density function given by:

$$f(t) = \begin{cases} 0, & \text{for } (t < t_1) \\ \dfrac{1}{t_2 - t_1}, & \text{for } (t_1 \le t \le t_2) \\ 0, & \text{for } (t > t_2) \end{cases}.$$

2.10 The failure or hazard rate of a component is given by (life is in hours):

$$h(t) = \begin{cases} 0.015, & t \le 200 \\ 0.025, & t > 200. \end{cases}$$

Find the expected life or MTTF for the component.

2.11 The failure density function for a group of components is:

$$f(t) = 0.25 - \left(\frac{0.25}{8} \right) t, \quad \text{for } 0 \le t \le 8$$

($f(t)$ is 0 otherwise). Time is in years.

(a) Show how this is a valid pdf.
(b) Find $F(t)$, $h(t)$, and $R(t)$.
(c) Find MTTF.
(d) Find B_{10} and B_{90} for the life of the components.
(e) Find the probability that this component fails within the first year of operation.

2.12 Assume that the system in Example 2.5 is a car. Do the results in the example 2.5 make sense? Why? Provide some examples of systems where the results may be more appropriate.

2.13 What does the conditional reliability reduce to if the hazard rate is a constant?

2.14 The time to failure random variable T of a product follows the following probability density function:

$$f(t) = \begin{cases} \dfrac{t}{80,000}, & 0 \le t \le 400 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the standard deviation for the time to failure random variable.

(b) Find the coefficient of skewness for the distribution for the time to failure random variable.

(c) Find the B_5 and B_{50} life of the product based on the above probability density function.

Draw the failure rate (or hazard rate) curve for the above product by evaluating it at:

$$t = 0, 50, 100, 300, 400.$$

2.15 The failure rate or hazard rate of a component is:

$$h(t) = 0.02t^{1.7}, \quad t \geq 0.$$

The failure rate is in failures per year.

(a) What is the reliability function of this component for a period of 2 years?

(b) What is the median life or B_{50} life of this component?

(c) What is the expected life for this component?

2.16 Calculate the coefficient of kurtosis for the probability density function given in Problem 2.9, where $t_1 = 3$ and $t_2 = 10$.

2.17 If the unreliability for a part is given as:

$$F(t) = \begin{cases} 0, & t < 0 \\ 0.5t^2 + 0.5t, & 0 \leq t \leq 1 \\ 1, & 1 < t \end{cases}.$$

What is the hazard rate as a function of time?