CHAPTER 12

Soil Constitutive Models

A *soil model* is a mathematical representation of the behavior of the soil under load. The model typically relates the stresses applied to the strains experienced by the soil as a result. The simplest of these relationships is the theory of elasticity.

12.1 ELASTICITY

12.1.1 Elastic Model

The theory of *elasticity* states that stresses and strains are linearly related (Figure 12.1).

Because there are 6 stresses and 6 strains, the matrix relating the stresses to the strains is made of 36 constants. Satisfying isotropy and symmetry reduces those 36 constants to only 2: the *modulus of elasticity* E (also called *Young's modulus*), and the *Poisson's ratio* ν. The equations are:

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})) \tag{12.1}$$

$$\varepsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})) \tag{12.2}$$

$$\varepsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})) \tag{12.3}$$

$$\varepsilon_{xy} = \frac{1+\nu}{E}\tau_{xy} = \frac{\gamma_{xy}}{2} \tag{12.4}$$

$$\varepsilon_{yz} = \frac{1+\nu}{E}\tau_{yz} = \frac{\gamma_{yz}}{2} \tag{12.5}$$

$$\varepsilon_{zx} = \frac{1+\nu}{E}\tau_{zx} = \frac{\gamma_{zx}}{2} \tag{12.6}$$

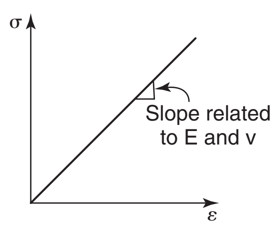

Figure 12.1 Linear elasticity stress-strain curve.

where σ_{ii} is the normal stress on the plane perpendicular to i in the direction of i, τ_{ij} is the shear stress on the plane perpendicular to i in the direction of j, ε_{ii} is the normal strain associated with the normal stress σ_{ii}, ε_{ii} is the shear strain associated with the shear stress τ_{ij}, γ_{ij} is the engineering shear strain associated with the shear stress τ_{ij}, E is Young's modulus or modulus of elasticity, and ν is Poisson's ratio. Young's modulus is named after Thomas Young, a British physician and physicist who made his contribution around the turn of the 1800s. Poisson's ratio is named after Simeon Poisson, a French mathematician and physicist who lived around the turn of the 1800s and had Lagrange and Laplace as his doctoral advisors at the École Polytechnique in Paris. In matrix form, the elasticity equations are:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} \tag{12.7}$$

or

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \times \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix} \tag{12.8}$$

Note that the normal strain in one direction is affected by the normal stress in that direction and also by the normal

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stresses in the other two directions–yet this is not true for the shear strains. Indeed, the shear strain is affected by the shear stress in that direction, but not by the shear stresses in the other two directions. Note also that although ε_{xy} is the shear strain, the engineering shear strain γ_{xy} (=2 ε_{xy}) is often used in practice.

Other elasticity moduli have been defined from E and ν. They are the shear modulus G, the bulk modulus K, and the constrained modulus M. The *shear modulus G* can be obtained by performing a simple shear test; it is defined as the ratio of the shear stress τ over the corresponding engineering shear strain γ. The *bulk modulus K* is obtained when a soil sample is subjected to an all-around (hydrostatic) pressure σ; it is defined as the ratio of the pressure σ over the volumetric strain generated $\varepsilon_v = \Delta V/V$. The *constrained modulus M* is obtained when a soil sample is subjected to a vertical normal stress in a cylinder that prevents any lateral movement; it is defined as the ratio of the normal stress applied over the vertical strain obtained. The relationships are as follows:

$$\text{Shear modulus} \quad G = \frac{\tau_{xy}}{\gamma_{xy}} = \frac{\tau_{xy}}{2\varepsilon_{xy}} = \frac{E}{2(1+\nu)} \tag{12.9}$$

$$\text{Bulk modulus} \quad K = \frac{\sigma}{\Delta V/V} = \frac{\frac{1}{3}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})}{\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}} = \frac{E}{3(1-2\nu)} \tag{12.10}$$

$$\text{Constrained modulus} \quad M = \frac{\sigma_{xx}}{\varepsilon_{xx}} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \tag{12.11}$$

The term *plane strain* means that the normal strain in one direction is zero throughout the soil mass. The term *plane stress* means that the normal stress in one direction is zero throughout the soil mass. Such conditions lead to an additional equation, as setting the normal strain in one direction equal to zero (for example) gives a relationship between the normal stresses in the three directions.

One of the advantages of the elastic model is the associated superposition principle, which is possible because the equations are linear. Table 12.1 indicates some of the possible superposition operations. The superposition principle is not applicable to nonlinear theories, such as the theory of plasticity.

Table 12.1 Superposition Principle Operations

Force	Stress	Strain	Displacement
F_1	σ_1	ε	u_1
λF_1	$\lambda\sigma_1$	$\lambda\varepsilon_1$	λu_1
F_2	σ_2	ε_2	u_2
$F_1 + F_2$	$\sigma_1 + \sigma_2$	$\varepsilon_1 + \varepsilon_2$	$u_1 + u_2$

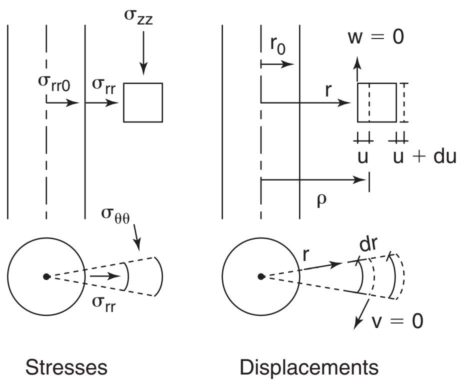

Figure 12.2 Element of soil around an expanding cylindrical cavity.

12.1.2 Example of Use of Elastic Model

The problem is to solve the expansion of an infinitely long cylinder subjected to a pressure p in an elastic soil space (Figure 12.2). The geometry of the problem indicates that this is an axisymmetric problem and a plane strain problem in the vertical direction. The initial state of stress is σ_{ov} in the vertical direction and σ_{oh} in the radial direction at any point in the soil space. After applying the pressure p at the cavity surface, the stresses in the mass become:

$$\sigma_{rr} = \sigma_{oh} + \Delta\sigma_{rr} \tag{12.12}$$

$$\sigma_{\theta\theta} = \sigma_{oh} + \Delta\sigma_{\theta\theta} \tag{12.13}$$

$$\sigma_{zz} = \sigma_{ov} + \Delta\sigma_{zz} \tag{12.14}$$

where σ_{rr} and $\sigma_{\theta\theta}$ are the radial stress and the hoop stress respectively at a distance r from the axis of the cylinder, and $\Delta\sigma_{rr}$ and $\Delta\sigma_{\theta\theta}$ are the increments of the radial and hoop stress above the at-rest stress value.

The radial displacement u is the only displacement type in this problem, as there are no displacements in the hoop direction or in the vertical direction. The radial strain is ε_{rr}, the hoop strain is $\varepsilon_{\theta\theta}$, and the vertical strain is ε_{zz}, which is zero because of the plane strain condition in the z direction. The relationships between the displacement and the strains for small-strain theory are:

$$\varepsilon_{rr} = -\frac{du}{dr} \tag{12.15}$$

$$\varepsilon_{\theta\theta} = -\frac{u}{r} \tag{12.16}$$

$$\varepsilon_{zz} = 0 \tag{12.17}$$

The minus sign is used in Eqs. 12.15 and 12.16 because compression has been chosen to be positive. In fact, u is positive but decreases with radial distance; hence, if the minus sign were not there, du/dr would be negative and associated with compression considering the loading for this problem. The equations of equilibrium reduce to:

$$\frac{d\sigma_{rr}}{dr} + \frac{\Delta\sigma_{rr} - \Delta\sigma_{\theta\theta}}{r} = 0 \tag{12.18}$$

The constitutive equations are:

$$\varepsilon_{rr} = \frac{1}{E}(\Delta\sigma_{rr} - \nu(\Delta\sigma_{\theta\theta} + \Delta\sigma_{zz})) \tag{12.19}$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}(\Delta\sigma_{\theta\theta} - \nu(\Delta\sigma_{zz} + \Delta\sigma_{rr})) \tag{12.20}$$

$$\varepsilon_{zz} = \frac{1}{E}(\Delta\sigma_{zz} - \nu(\Delta\sigma_{rr} + \Delta\sigma_{\theta\theta})) \tag{12.21}$$

By combining equations 12.12 to 12.21, the governing differential equation is obtained as:

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = 0 \tag{12.22}$$

The boundary conditions are:

$$u = 0 \text{ at } r = \text{infinity}$$

$$u = u_o \text{ at } r = r_o$$

The solution that satisfies Eq. 12.22 and the boundary conditions is:

$$u = \frac{u_o r_o}{r} \tag{12.23}$$

$$\varepsilon_{rr} = \frac{u_o r_o}{r^2} \qquad \varepsilon_{\theta\theta} = -\frac{u_o r_o}{r^2} \qquad \varepsilon_{zz} = 0 \tag{12.24}$$

$$\sigma_{rr} = \sigma_{oh} + 2G\frac{u_o r_o}{r^2} \qquad \sigma_{\theta\theta} = \sigma_{oh} - 2G\frac{u_o r_o}{r^2} \qquad \sigma_{zz} = \sigma_{ov} \tag{12.25}$$

At the wall of the cylindrical cavity, the equations become:

$$u_o = u_o \tag{12.26}$$

$$\varepsilon_{rro} = \frac{u_o}{r_o} \qquad \varepsilon_{\theta\theta o} = -\frac{u_o}{r_o} \qquad \varepsilon_{zzo} = 0 \tag{12.27}$$

$$\sigma_{rro} = \sigma_{oh} + 2G\frac{u_o}{r_o} \qquad \sigma_{\theta\theta o} = \sigma_{oh} - 2G\frac{u_o}{r_o} \qquad \sigma_{zzo} = \sigma_{ov} \tag{12.28}$$

In a pressuremeter test, the relative increase in radius ($u_o/r_o = \varepsilon_{\theta\theta o}$) of the cavity is measured along with the pressure exerted on the cavity wall σ_{rro}. Therefore, the pressuremeter curve is a direct plot of a stress-strain curve of the soil (Figure 12.3).

In the case of large-strain theory, the large-strain definitions require use of the current radius ρ, the initial radius r, and the displacement u:

$$\rho = r + u \tag{12.29}$$

Then the strains can be defined as:

$$\text{Radial strain} \quad \alpha_r = \frac{1}{2}\left(\frac{d\rho^2 - dr^2}{d\rho^2}\right) \tag{12.30}$$

$$\text{Hoop strain} \quad \alpha_\theta = \frac{1}{2}\left(\frac{\rho^2 - r^2}{\rho^2}\right) \tag{12.31}$$

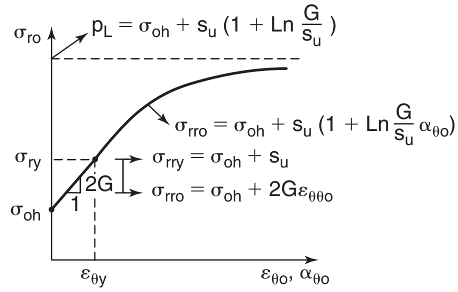

Figure 12.3 Expansion of a cylindrical cavity.

and the solution becomes:

$$\sigma_{rr} = \sigma_{oh} + 2G\alpha_\theta = \sigma_{oh} + G\left(\frac{\rho^2 - r^2}{\rho^2}\right) = \sigma_{oh} + G\left(\frac{\Delta V}{V}\right) \tag{12.32}$$

where ΔV is the increase in volume of the cylinder having an initial radius r and V is the current volume of the cylinder having a current radius ρ and an initial radius r.

12.2 LINEAR VISCOELASTICITY

When load is applied to a linear elastic material, the stresses, strains, and displacements occur instantaneously and remain constant with time. *Viscoelasticity* introduces the influence of time on the deformation process (Figure 12.4). *Linear viscoelasticity* further simplifies the phenomenon by allowing superposition of the elastic deformation and the time-dependent deformation. A good way to understand viscoelasticity is to start by studying simple models.

12.2.1 Simple Models: Maxwell and Kelvin-Voigt Models

Simple, one-dimensional models help to understand the potential use of linear viscoelasticity (Figure 12.5). These models make use of mechanical elements such as a spring and a *dashpot* (also called *damper*). The spring behavior is governed by $\sigma = k\ \varepsilon$ where σ is the axial stress applied, k is the spring stiffness, and ε is the axial strain. The dashpot behavior is governed by $\sigma = \eta\ (d\varepsilon/dt)$ where η is the viscosity and $d\varepsilon/dt$ the strain rate. The dashpot behavior is very similar to the behavior of a shock absorber in a car suspension. If you load it fast, it generates a stiff response; if you load it

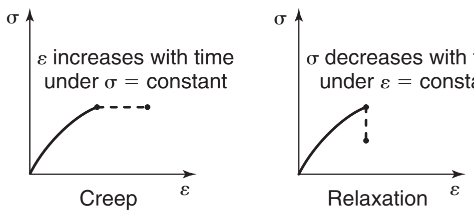

Figure 12.4 Creep and relaxation of viscous models.

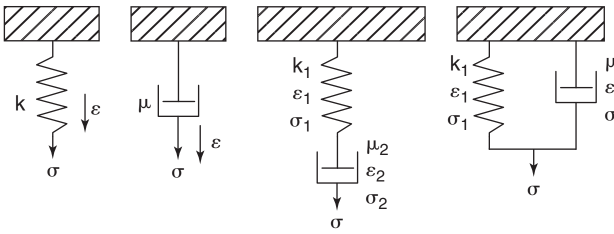

Figure 12.5 One-dimensional viscoelastic models: *(a)* Spring, $\sigma =$ k ε. *(b)* Dashpot, $\sigma = \eta$ (dε/dt). *(c)* Maxwell, $\sigma = \sigma_1 = \sigma_2$, $\varepsilon = \varepsilon_1 + \varepsilon_2$. *(d)* Kelvin-Voigt, $\sigma = \sigma_1 + \sigma_2$, $\varepsilon = \varepsilon_1 = \varepsilon_2$.

slowly, it offers little resistance. These mechanical elements can be combined to represent a more complex behavior. The Maxwell model is made of a spring and a dashpot in series, whereas the Kelvin-Voigt model is made of a spring and a dashpot in parallel. The Maxwell model is named after James Maxwell, a British physicist and mathematician of the mid-1800s. The Kelvin-Voigt model is named after William Thompson, First Baron Kelvin, a British physicist and engineer of the late 1800s; and Woldemar Voigt, a German physicist of the late 1800s.

Two basic phenomena can be investigated with these models: creep and relaxation (Figure 12.4). *Creep* refers to the increase in strain as a function of time when a constant stress is applied. For example, creep could occur in the soil under a high embankment. *Relaxation* refers to the decrease in stress as a function of time when a constant strain is applied. For example, relaxation of the horizontal total stress could occur against the side of a pile after driving. To find out how the Maxwell model creeps and relaxes, we write (Figure 12.5):

$$\sigma = \sigma_1 = \sigma_2 \tag{12.33}$$

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \tag{12.34}$$

Therefore, the governing equation for the Maxwell model is:

$$\frac{d\varepsilon}{dt} = \frac{1}{k}\frac{d\sigma}{dt} + \frac{\sigma}{\eta} \tag{12.35}$$

Creep occurs under a constant stress σ_o. If that stress is applied instantaneously, only the spring deflects and the initial value of the strain is $\varepsilon_o = \sigma_o$/k. Therefore:

$$\int_{\sigma_o/k}^{\varepsilon} d\varepsilon = \frac{\sigma_o}{\eta}\int_0^t dt \quad and \quad \varepsilon = \frac{\sigma_o}{k} + \frac{\sigma_o}{\eta}t \tag{12.36}$$

which shows that the Maxwell model creeps linearly. This does not fit well with observed soil behavior. Relaxation occurs under constant strain ε_o. If that strain is applied instantaneously, only the spring deflects and the initial value of the stress is $\sigma_o =$ k ε_o, Therefore:

$$\int_{k\varepsilon_o}^{\sigma} \frac{d\sigma}{\sigma} = -\frac{k}{\eta}\int_0^t dt \quad and \quad \sigma = k\varepsilon_o e^{-\frac{t}{\eta/k}} \tag{12.37}$$

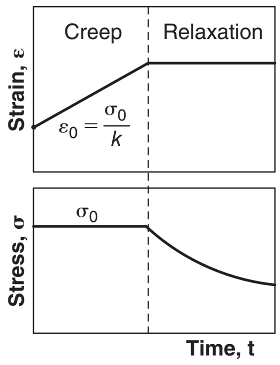

Figure 12.6 Creep and relaxation of the Maxwell model.

Equation 12.37 shows a relaxation process that is closer to what one would expect in actual soils. Figure 12.6 summarizes the behavior of the Maxwell model.

To find out how the Kelvin-Voigt model creeps and relaxes, we write (Figure 12.5):

$$\sigma = \sigma_1 + \sigma_2 \tag{12.38}$$

$$\varepsilon = \varepsilon_1 = \varepsilon_2 \tag{12.39}$$

Therefore, the governing equation for the Maxwell model is:

$$\sigma = k\varepsilon + \eta\frac{d\varepsilon}{dt} \tag{12.40}$$

Creep occurs under a constant stress σ_o. If that stress is applied instantaneously, the dashpot is infinitely stiff, all the stress is carried by the dashpot, no strain occurs initially under σ_o, and $\varepsilon_o = 0$. After an infinite time, however, the dashpot carries no load, the stress is entirely carried by the spring, and $\varepsilon_{t = \text{infinity}} = \sigma_o/k$. Therefore:

$$\int_0^{\varepsilon} \frac{d\varepsilon}{\varepsilon - \dfrac{\sigma_o}{k}} = -\frac{k}{\eta}\int_0^t dt \quad and \quad \varepsilon = \frac{\sigma_o}{k}\left(1 - e^{-\frac{t}{\eta/k}}\right) \tag{12.41}$$

which shows that the Kelvin-Voigt model creeps in a way consistent with what can be expected for actual soils. Relaxation occurs under constant strain ε_o, therefore there is no contribution from the dashpot and the stress is simply:

$$\sigma = k\varepsilon_o \tag{12.42}$$

The Kelvin-Voigt model does not relax. Figure 12.7 summarizes the behavior of the Kelvin-Voigt model.

12.2.2 General Linear Viscoelasticity

The simple models from section 12.2.1 indicate that stress behavior over time is related to the strain through a function called the *relaxation modulus function G*(t). Similarly, the strain behavior over time of a viscoelastic material is related to the stress through a function called the *creep*

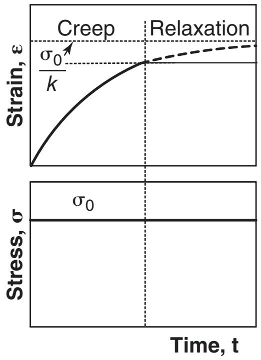

Figure 12.7 Creep and relaxation of the Kelvin-Voigt model.

compliance function $J(t)$. For example, Eq. 12.37 indicates that the relaxation modulus function $G(t)$ for the Maxwell model is:

$$G(t) = \frac{\sigma(t)}{\varepsilon_o} = ke^{-\frac{t}{\eta/k}} \tag{12.43}$$

and that the creep compliance function $J(t)$ for the Kelvin-Voigt model (Eq. 12.41) is:

$$J(t) = \frac{\varepsilon(t)}{\sigma_o} = \frac{1}{k}\left(1 - e^{-\frac{t}{\eta/k}}\right) \tag{12.44}$$

Ludwig Boltzmann, an Austrian physicist of the late 1800s, generalized these observations by proposing a superposition principle that can be explained as follows. At time $t_1' = 0$, a constant stress σ_1 is applied and the strain induced is (Figure 12.8):

$$\varepsilon_1(t) = J(t)\sigma_1 \tag{12.45}$$

Then at a time t_2' an increment of stress $(\sigma_2 - \sigma_1)$ is imposed and the strain increase is:

$$\varepsilon_2(t) = J(t - t_2')(\sigma_2 - \sigma_1) \tag{12.46}$$

Note here that the function J is the same as in equation 12.41 and independent of the stress level. This is the property of linear viscoelasticity. Again, at a time t_3', an increment of stress $(\sigma_3 - \sigma_2)$ is imposed and the strain increase is:

$$\varepsilon_3(t) = J(t - t_3')(\sigma_3 - \sigma_2) \tag{12.47}$$

And so on, so that in the end the total strain is:

$$\varepsilon(t) = \sum_{i=1}^{n} \varepsilon_i(t) = \sum_{i=1}^{n} J(t - t_i')(\sigma_i - \sigma_{i-1}) \tag{12.48}$$

For a continuous stress function $\sigma(t)$, Eq. 12.48 becomes:

$$\varepsilon(t) = \int_0^t J(t - t')\frac{d\sigma(t')}{dt'}dt' \tag{12.49}$$

This represents the viscous part of the strain, to which should be added the elastic part. So, in the end, the general form of the model is:

$$\varepsilon_{ij}(t) = \varepsilon_{ij(elastic)} + \int_0^t J(t - t')\frac{d\sigma_{ij}(t')}{dt'}dt' \tag{12.50}$$

Similarly, for the relaxation modulus the equation is:

$$\sigma_{ij}(t) = \sigma_{ij(elastic)} + \int_0^t G(t - t')\frac{d\varepsilon_{ij}(t')}{dt'}dt' \tag{12.51}$$

12.3 PLASTICITY

One way to model a soil is to consider that it behaves elastically at first, then reaches a yield point, and then continues to deform plastically until it reaches failure. Beyond the yield point, the soil can strain harden, strain soften, or be perfectly plastic (Figure 12.9).

If the material is perfectly plastic beyond the yield point, the yield criterion and the failure criterion are the same. If the material strain hardens, they are not the same; and if the material strain softens, the yield criterion and the failure criterion are the same but postyield behavior requires further calculations. It is accepted that strain can be decomposed into an elastic component and a plastic component. Furthermore, because plasticity is primarily a nonlinear theory, the calculations involve strain increments $d\varepsilon_{ij}$:

$$d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p \tag{12.52}$$

where $d\varepsilon_{ij}^e$ is the elastic part of the strain increment, and $d\varepsilon_{ij}^p$ is the plastic part of the strain increment. There are four elements to any plasticity method (Potts and Zdravkovic 1999; Davies

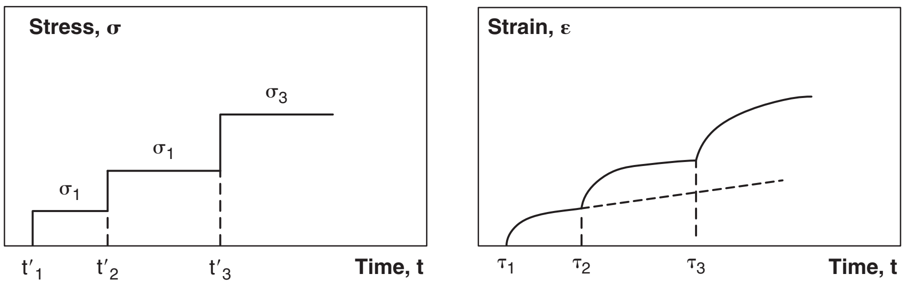

Figure 12.8 Boltzmann superposition principle.

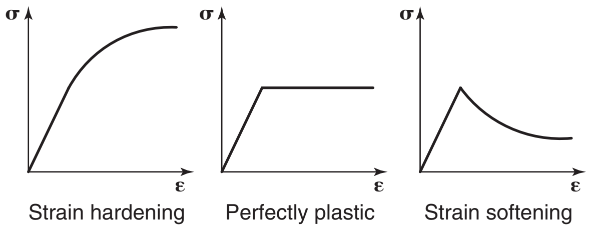

Figure 12.9 Plastic models.

and Selvadurai, 2002): coincidence of axes, yield function, plastic potential function, and hardening or softening rule. The coincidence of axes is a common assumption stating that the axes of the accumulated principal stress vectors (σ_1, σ_2, σ_3) and the axes of the plastic principal strain increment vectors ($d\varepsilon_1^p, d\varepsilon_2^p, d\varepsilon_3^p$) coincide. This is an extension of what is used in elasticity, but in plasticity it applies to the stress and the corresponding strain increment and not to the stress increment and the corresponding strain increment. The yield function and associated yield criterion give the combination of stresses that lead to yielding of the soil. The plastic potential function gives the direction of the plastic strain increments through a flow rule, and the hardening or softening rule gives the magnitude of the plastic strain increments.

12.3.1 Some Yield Functions and Yield Criteria

The combination of stresses that create yielding of the soil are given by the yield function, which is set equal to zero to give the yield criterion. The yield function involves a state parameter k:

$$Y(\sigma_{ij}, k) = 0 \tag{12.53}$$

The two most common yield criteria in soil mechanics are the Tresca yield criterion and the Mohr-Coulomb yield criterion. The *Tresca yield criterion* is named after Henri Tresca, a French mechanical engineer, who proposed it in 1864. When applied to soil mechanics and the undrained behavior of fine-grained soils, it states that yield will occur when the difference between the major principal stress and the minor principal stress reaches a value equal to two times the undrained shear strength s_u (Figure 12.10):

$$\sigma_1 - \sigma_3 - 2s_u = 0 \tag{12.54}$$

As can be seen in the Tresca criterion, s_u is the state parameter. It corresponds to the Mohr circle plotted in the shear stress vs. total stress set of axes reaching the undrained shear strength failure envelope.

The *Mohr-Coulomb yield criterion* is named after Otto Mohr, a German civil engineer of the late 1800s, and Charles de Coulomb, a French civil engineer of the late 1700s. It states that yield will occur when the Mohr circle reaches the line corresponding to the shear strength equation (Figure 12.11):

$$\tau_f - c' - \sigma' \tan\varphi' = 0 \tag{12.55}$$

As can be seen in the Mohr-Coulomb criterion, c' and φ' are the state parameters. This can be rewritten in terms of major and minor principal stresses by using the rectangular

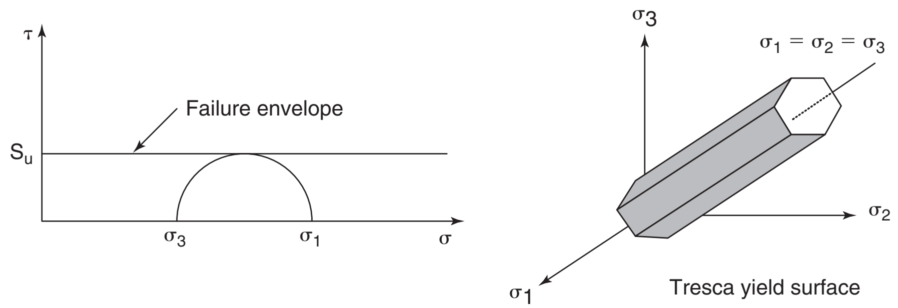

Figure 12.10 Tresca yield criterion.

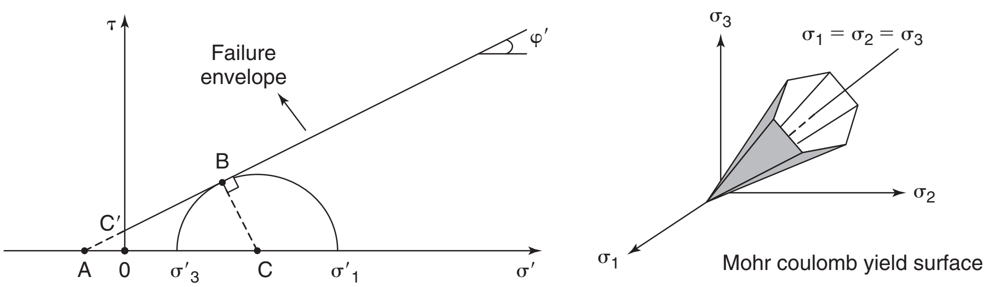

Figure 12.11 Mohr-Coulomb yield criterion.

triangle ABC in Figure 12.11.

$$\sin\varphi' = \frac{\frac{\sigma_1' - \sigma_3'}{2}}{\frac{c'}{\tan\varphi'} + \frac{\sigma_1' + \sigma_3'}{2}} \tag{12.56}$$

or

$$\sigma_1' - \sigma_3' - 2c'\cos\varphi' - (\sigma_1' + \sigma_3')\sin\varphi' = 0 \tag{12.57}$$

The Mohr circle starts at a stress state that corresponds to the soil equilibrium in situ. As the soil is loaded, it deforms elastically at first until the circle reaches the yield criterion envelope (shear strength equation). At that point, the circle cannot grow past the envelope, but it can grow along the envelope (strain hardening) or decrease in size along the envelope (strain softening). Note that in sand ($c' = 0$), the Mohr-Coulomb yield criterion simplifies to:

$$\frac{\sigma_1'}{\sigma_3'} - \frac{1+\sin\varphi}{1-\sin\varphi} = 0 \tag{12.58}$$

Yet another yield criterion is the *Von Mises criterion*, named after Richard Von Mises, an Austrian engineer in the early 1900s:

$$\sqrt{J_2} - k = 0 \tag{12.59}$$

where J_2 is the second stress invariant of the deviatoric tensor (Section 10.7) and k is a constant to be determined experimentally. The *Drucker-Prager criterion* is a generalization of the Von Mises criterion named after two American engineers of the mid-1900s. It introduces the influence of the mean stress on the strength of soils:

$$\sqrt{J_2} - A - BI_1 = 0 \tag{12.60}$$

where I_1 is the first invariant of the stress tensor.

These four yield criteria are compared (Figure 12.12) on the π plane, the plane perpendicular to the bisectrice of the three-dimensional space $\sigma_1 - \sigma_2 - \sigma_3$. This bisectrice has the equation $\sigma_1 = \sigma_2 = \sigma_3$.

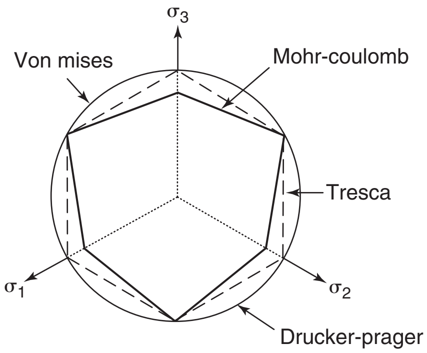

Figure 12.12 Yield criteria compared on the π plane.

12.3.2 Example of Use of Yield Criteria

Let's go back to the expansion of an infinite cylinder subjected to an internal pressure p and find out at what pressure the soil first yields. We will first use the Tresca criterion (undrained fine-grained soil behavior). The radial stress at the cavity wall σ_{rro} increases with p, because it is equal to p, and represents the major principal stress σ_1. The hoop stress at the cavity wall $\sigma_{\theta\theta o}$ decreases as much as the radial stress increases (Eq. 12.28), and represents the minor principal stress σ_3. The difference $\sigma_1 - \sigma_3$ increases as p increases, and when p reaches a value where the Tresca criterion is first satisfied, the soil yields.

$$\sigma_1 - \sigma_3 = 2s_u \tag{12.61}$$

where s_u is the undrained shear strength of the soil. A plastic zone is initiated around the cylindrical cavity and grows as the pressure continues to increase (Figure 12.13).

Using Eqs. 12.12 and 12.13 plus the observation that the increase in stress $\Delta\sigma$ in the radial direction is equal to the decrease in stress $\Delta\sigma$ in the hoop direction, we write:

$$\sigma_{oh} + \Delta\sigma - (\sigma_{oh} - \Delta\sigma) = 2s_u \quad or \quad \Delta\sigma = s_u \tag{12.62}$$

Therefore, the yield pressure p_y will be:

$$p_y = \sigma_{oh} + s_u \tag{12.63}$$

If we use the Mohr-Coulomb criterion for a soil with $c' = 0$ (drained behavior of a coarse-grained soil, for example), then we write that:

$$\frac{\sigma_1'}{\sigma_3'} = \frac{1+\sin\varphi}{1-\sin\varphi} = K_p \tag{12.64}$$

Using Eqs. 12.12 and 12.13 plus the observation that the increase in stress in the radial direction is equal to the decrease

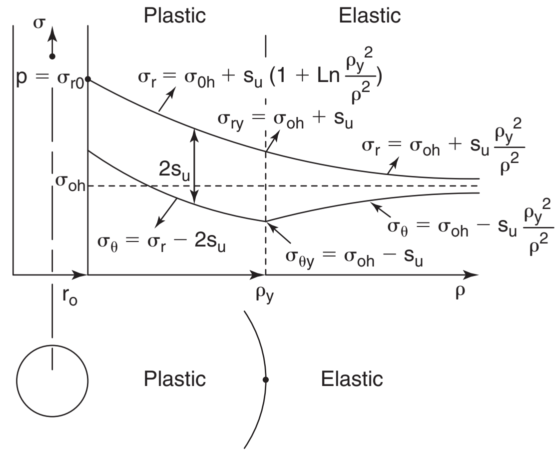

Figure 12.13 Elastic zone and plastic zone around an expanding cylindrical cavity.

in stress in the hoop direction, we write:

$$\frac{\sigma'_{oh} + \Delta\sigma'}{\sigma'_{oh} - \Delta\sigma'} = \frac{1+\sin\varphi}{1-\sin\varphi} \quad or \quad \Delta\sigma' = \sigma'_{oh}\sin\varphi \tag{12.65}$$

Therefore, the yield pressure p_y will be:

$$p_y = \sigma'_{oh}(1+\sin\varphi) \tag{12.66}$$

Note that this solution is presented in terms of effective stresses; thus, the water stress would have to be taken into account to obtain the total stress.

12.3.3 Plastic Potential Function and Flow Rule

Because the behavior in the plastic domain is nonlinear, the relationship is written in terms of strain increments $d\varepsilon_{ii}$. It is accepted that the strain increment can be separated into an elastic portion and a plastic portion (Figure 12.14):

$$d\varepsilon_{ij} = d\varepsilon^e_{ij} + d\varepsilon^p_{ij} \tag{12.67}$$

Now we need a way to predict the direction and magnitude of the plastic strain increments in the plastic region as we stress the soil beyond the yield point (if that is possible). As will be seen, the plastic potential gives the direction of the plastic strain increment, while the flow rule gives its magnitude. Von Mises proposed the existence of a plastic potential $P(\sigma_{ij}, m)$ function of the stress state at one point and material parameters m. This plastic potential is used to define a flow rule such that:

$$d\varepsilon^p_{ij} = \lambda\frac{dP(\sigma_{ij}, m)}{d\sigma_{ij}} \tag{12.68}$$

where λ is a proportionality constant. If $P(\sigma_{ij}, m)$ is set equal to zero, the equation defines a surface in the stress space and $\frac{dP}{d\sigma_{ij}}$ is a vector perpendicular to that surface. This is called the *normality rule*, indicating that the increment of plastic strain $d\varepsilon^p_{ij}$ is perpendicular to the plastic potential surface. Figure 12.15 shows plastic potential contours in the q − p′ plane where q is the deviator stress ($q = \sigma_1 - \sigma_3$ for a triaxial test) and p′ is the mean normal effective stress ($p' = 0.33(\sigma'_1 + 2\sigma'_3)$ for a triaxial test).

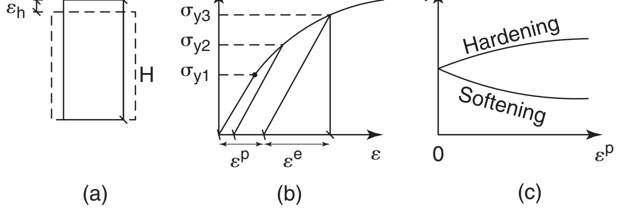

Figure 12.14 Elastic and plastic strains.

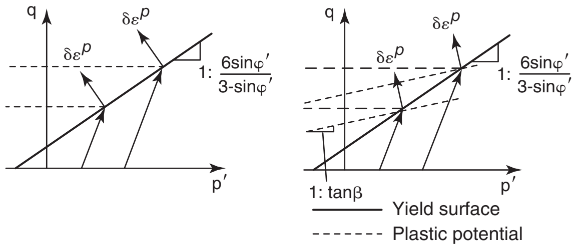

Figure 12.15 Plastic potential, yield surface, and normality rule.

A flow rule is said to be an *associated flow rule* if the plastic potential is equal to the yield function (Figure 12.15*a*):

$$P(\sigma_{ij}, m) = Y(\sigma_{ij}, k) \tag{12.69}$$

So, in the case of an associated flow rule, the plastic strain increment is perpendicular to the yield surface, because:

$$d\varepsilon^p_{ij} = \lambda\frac{dY}{d\sigma_{ij}} \tag{12.70}$$

For example, if we use the Tresca yield criterion, the plastic potential would be:

$$P(\sigma_{ij}, m) = Y(\sigma_{ij}, k) = \frac{\sigma_1 - \sigma_3}{2} - s_u \tag{12.71}$$

the derivatives $\frac{dY}{d\sigma_{ij}}$ would be:

$$\frac{dY}{d\sigma_1} = \frac{1}{2}, \quad \frac{dY}{d\sigma_2} = 0, \quad \frac{dY}{d\sigma_3} = -\frac{1}{2} \tag{12.72}$$

and the flow rule would be:

$$\begin{bmatrix} d\varepsilon^p_1 \\ d\varepsilon^p_2 \\ d\varepsilon^p_3 \end{bmatrix} = d\lambda \begin{bmatrix} +0.5 \\ 0 \\ -0.5 \end{bmatrix} \tag{12.73}$$

If the plastic potential $P(\sigma_{ij}, m)$ is different from the yield function $F(\sigma_{ij}, k)$, then the flow rule is said to be nonassociated (Figure 12.15*b*). Associated flow rules work well for pressure-nonsensitive soils (undrained behavior of fine-grained soils) but nonassociated flow rules work better for pressure-sensitive soils (effective stress approach for soils). Nonassociated flow rules require more complicated calculations and increase the computing time.

12.3.4 Hardening or Softening Rule

Now we know the direction of the plastic strain increment vector, because it has to be normal to the plastic potential surface. We need to determine its magnitude, which is done by using a hardening or softening rule. As mentioned before, beyond the yield point the soil can strain harden, strain soften, or be rigid plastic (Figure 12.9). The hardening or softening rule defines what happens to the yield function beyond yield. If the hardening/softening is due to the plastic strains, it is called *strain hardening/softening*; if it is due to the plastic work done, it is called *work hardening/softening*. The hardening/softening rule describes how the state parameters k vary with plastic strain. This relationship can then be used in Eq. 12.68 or 12.70 as appropriate. Figure 12.14 illustrates how the hardening rule can be obtained for a simple axial compression test.

12.3.5 Example of Application of Plasticity Method

Let's go back to the expansion of the infinite cylinder subjected to an internal pressure p and find out the relationship between stress and strain in the plastic domain beyond the yield pressure p_y. We will first take the case of the undrained behavior of fine-grained soils and use the Tresca criterion. Because the cavity expands beyond small-strain theories, we need to use large-strain definitions (section 12.1.2) to still be able to form a valid strain tensor. In the plastic zone (Figure 12.13), the constitutive law has changed from elasticity to plasticity, but the equilibrium equation is still valid:

$$\frac{d\sigma_{rr}}{d\rho} + \frac{\Delta\sigma_{rr} - \Delta\sigma_{\theta\theta}}{\rho} = 0 \tag{12.74}$$

The Tresca criterion gives:

$$\begin{aligned}\sigma_1 - \sigma_3 = \sigma_{rr} - \sigma_{\theta\theta} &= \sigma_{oh} + \Delta\sigma_{rr} - (\sigma_{oh} + \Delta\sigma_{\theta\theta}) \\ &= \Delta\sigma_{rr} - \Delta\sigma_{\theta\theta} = 2s_u\end{aligned} \tag{12.75}$$

This leads to the solution:

$$d\sigma_{rr} = -2s_u\frac{d\rho}{\rho} \quad \textit{and} \quad \sigma_{rr} = -2s_u Ln\,\rho + A \tag{12.76}$$

The constant A is defined by the boundary condition, which states that at the boundary between the elastic region and the plastic region the radial stress is equal to p_y, as given by Eq. 12.63. Therefore, A is found as:

$$A = \sigma_{oh} + s_u(1 + Ln\,\rho_y^2) \tag{12.77}$$

and the radial stress σ_{rr} in the plastic zone at a radial distance ρ from the axis of the cylinder (Figure 12.13) is given by:

$$\sigma_{rr} = \sigma_{oh} + s_u\left(1 + Ln\frac{\rho_y^2}{\rho^2}\right) \tag{12.78}$$

This equation gives the value of the radial stress anywhere in the plastic zone. At the cavity wall, the pressure is therefore:

$$\sigma_{rro} = \sigma_{oh} + s_u\left(1 + Ln\frac{\rho_y^2}{\rho_o^2}\right) \tag{12.79}$$

Now we want to evaluate the maximum pressure that can be resisted by the soil at the cavity wall, called the *limit pressure* p_L. This limit pressure p_L is reached when the entire soil mass has reached the yield criterion—in other words, when ρ_y becomes infinite. Therefore, we are looking for the limit:

$$\lim_{\rho_y \to \infty} \frac{\rho_y^2}{\rho_o^2} \tag{12.80}$$

For this we need a flow rule. Because we are dealing with the undrained behavior of a fine-grained soil, it makes sense to assume that there will be no volume change in the soil mass (a simple flow rule). Thus, the volume increase at radius r_o has to be the same as the volume increase at radius r_y, so that the soil mass in between the two radii does not change volume:

$$\rho_o^2 - {r_o}^2 = \rho_y^2 - r_y^2 \tag{12.81}$$

or

$$\alpha_{\theta o}\rho_o^2 = \alpha_{\theta y}\rho_y^2 \quad \text{and} \quad \frac{\alpha_{\theta o}}{\alpha_{\theta y}} = \frac{\rho_y^2}{\rho_o^2} \tag{12.82}$$

At the boundary between the elastic and plastic regions, both the yield criterion and the elastic solution must be satisfied. Using Eqs. 12.32 and 12.63:

$$\text{Elastic side of the boundary} \quad \sigma_{rry} = \sigma_{oh} + G\left(\frac{\rho_y^2 - r_y^2}{\rho_y^2}\right) \tag{12.83}$$

$$\text{Plastic side of the boundary} \quad \sigma_{rry} = p_y = \sigma_{oh} + s_u \tag{12.84}$$

Then

$$\frac{\rho_y^2 - r_y^2}{\rho_y^2} = \frac{s_u}{G} = \alpha_{\theta y} \tag{12.85}$$

and Eq. 12.79 becomes:

$$\sigma_{rro} = \sigma_{oh} + s_u\left(1 + Ln\frac{G}{s_u}\alpha_{\theta o}\right) \tag{12.86}$$

This equation gives the curve linking the radial stress at the cavity wall vs. the hoop strain at the cavity wall (the pressuremeter curve) (Figure 12.3). The limit of $\alpha_{\theta o}$ when ρ_o goes to infinity (limit pressure) is 1 because r_o^2 becomes negligible compared to ρ_o^2. Then the limit pressure p_L can be given from Eq. 12.86 as:

$$p_L = \sigma_{oh} + s_u\left(1 + Ln\frac{G}{s_u}\right) \tag{12.87}$$

12.4 COMMON MODELS

12.4.1 Duncan-Chang Hyperbolic Model

The *Duncan-Chang model* or DC model (Duncan and Chang 1970) is a nonlinear stress-dependent model where the

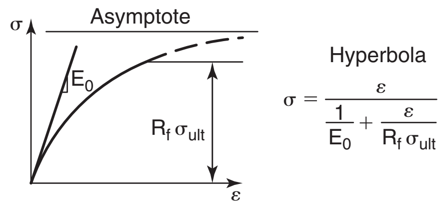

Figure 12.16 Duncan-Chang model.

stress-strain curve is described by a hyperbola (Figure 12.16):

$$\sigma = \frac{\varepsilon}{\dfrac{1}{E_o} + \dfrac{\varepsilon}{R_f \sigma_{ult}}} \tag{12.88}$$

where σ is typically taken as the deviator stress, ε is the axial strain, E_o is the initial tangent modulus, which depends on the stress level, σ_{ult} is the asymptotic value of the deviator stress, and R_f is a reduction factor such that R_F times σ_{ult} is the soil strength. The initial tangent modulus E_o increases when the mean confinement stress increases:

$$E_o = E_{o@p_a}\left(\frac{\sigma_m}{p_a}\right)^n \tag{12.89}$$

where $E_{o@p_a}$ is the initial tangent modulus for the reference pressure p_a (often taken as the atmospheric pressure), σ_m is the mean principal stress ($\sigma_m = 0.33(\sigma_1 + \sigma_2 + \sigma_3)$), and n is a soil-specific stress influence exponent. The nonlinearity of the model also recognizes the decrease in modulus with increase in strain. The volume change is characterized by a Poisson's ratio model dependent on the log of the confining stress. An unload-reload modulus E_{ur} is used to characterize the unload-reload path. The DC model uses the Mohr-Coulomb criterion as the failure criterion with a friction angle dependent on the confining stress level but does not directly include dilatancy. The soil parameters needed for the Duncan-Chang model are easily obtained from triaxial tests. Although this model does not have a plasticity framework, it is a very practical model.

12.4.2 Modified Cam Clay Model

Roscoe, Schofield, and Wroth (1958) at Cambridge University (UK) used the theory of plasticity to develop a complete stress-strain model for normally consolidated and lightly overconsolidated saturated clays, which they called the *Cam Clay model* (named after the River Cam, which passes through the campus of Cambridge University). This model was modified in 1965 (Roscoe and Burland 1968) and became known as the *Modified Cam Clay (MCC) model*. The MCC model is an elastic plastic strain hardening model based on critical-state soil mechanics (CSSM) theory, which makes the assumption that all stress paths end up at failure on the critical state line (CSL on Figure 12.17). On the critical state line (CSL), there is no more change in volume or stress. This line exists in the e–$Ln\,p'$ set of axes and in the $q - p'$ set of axes (Figure 12.17).

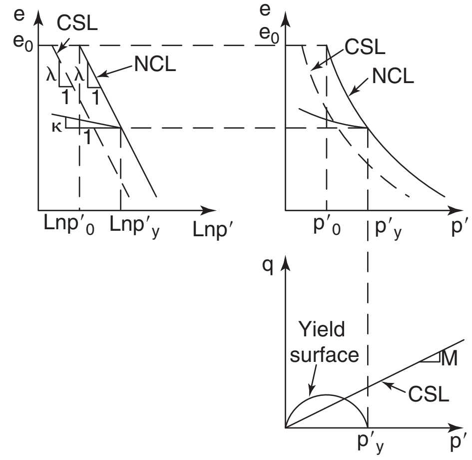

Figure 12.17 Modified Cam Clay model.

Recall that q is the deviator stress ($q = \sigma_1 - \sigma_3$ for the triaxial test) and p′ is the mean normal stress ($p' = 0.33(\sigma_1' + 2\sigma_3')$ for the triaxial test). The parameters defining these two lines are shown in Figure 12.17. In addition, a shear modulus G is necessary, as well as the initial state of the soil described by its initial void ratio e_o, its initial effective stress p_o', and its initial overconsolidation ratio (OCR).

Note that for the consolidation test, the axial strain ε and the change in void ratio Δe from e_o to e are linked by:

$$\varepsilon = \frac{\Delta e}{1 + e_o} \tag{12.90}$$

The normal compression line (NCL) describes the stress-strain curve as a straight line in the $e - \text{Ln}\ p'$ set of axes where e is the void ratio and p′ is the mean effective stress ($p' = 0.33(\sigma_1' + \sigma_2' + \sigma_3')$):

$$e = e_o - \lambda Ln\frac{p'}{p_o'} \tag{12.91}$$

where e_o is the initial void ratio corresponding to the initial mean effective stress p_o', e is the void ratio corresponding to the current mean effective stress p', and λ is the isotropic logarithmic compression index (slope of the line). The line in the e–Lnp′ set of axes corresponding to the critical state (the critical state line or CSL) links the critical void ratio e_c to Lnp′ and is assumed to have the same slope as the NCL. Recall that the critical void ratio is obtained at the end of loading when the soil reaches a state where no more volume change and no more stress increase or decrease occurs:

$$e_c = e_{co} - \lambda Ln\frac{p'}{p_o'} \tag{12.92}$$

The part of the strain recovered upon unload is the elastic component of the strain, e_e. This unload-reload line is considered to be a straight line in the e–Lnp' set of axes and is expressed as:

$$e^e = e_y - \kappa Ln \frac{p'}{p'_y} \tag{12.93}$$

where e_y is the void ratio corresponding to the yield stress p'_y, and κ is the swelling index (slope of the line). The critical state line in the $q - p'$ plot is:

$$q = M\, p' \tag{12.94}$$

where q is the deviator stress ($q = \sigma_1 - \sigma_3$ for a triaxial test), M is the critical state parameter, and p' is the mean confining stress ($p' = 0.33(\sigma'_1 + 2\sigma'_3)$ for a triaxial test). The plastic potential is the same as the yield function because the MCC model uses an associated flow rule. The yield function f is an ellipse (Figure 12.18) with the following equation:

$$f = q^2 - M^2(p'(p'_y - p')) \tag{12.95}$$

where f is the plastic potential and the yield function; it becomes the yield surface for $f = 0$. The direction of the plastic strain is perpendicular to the yield surface and the magnitude is given by the hardening rule (Figure 12.18). For the Cam Clay model, the hardening rule is an isotropic hardening rule. It is obtained from the increase in yield stress and from the recognition that the strain is made of the elastic part related to the swelling line and the plastic part associated with the difference between the normal compression line and the swelling line (Figure 12.19):

$$de^p = (\lambda - \kappa)\frac{dp'}{p'} \tag{12.96}$$

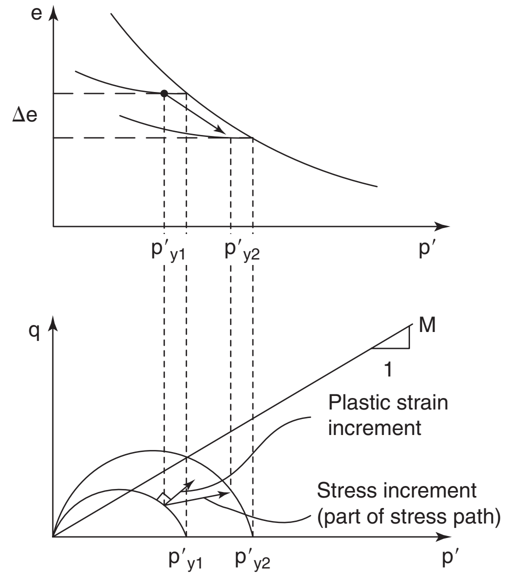

Figure 12.18 Strain hardening for Cam Clay model.

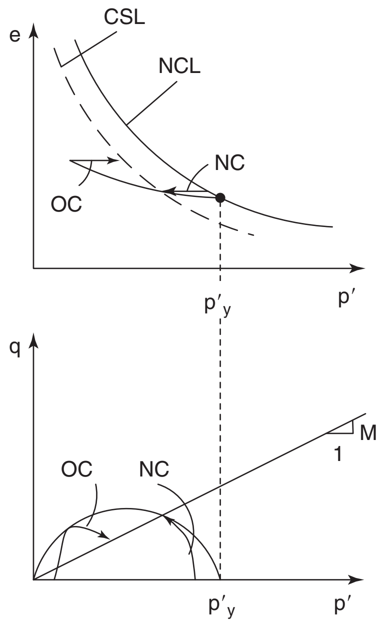

Figure 12.19 Evolution of stress path for NC and OC clays.

12.4.3 Barcelona Basic Model

Alonso, Gens, and Josa (1990) proposed a model to describe the behavior of unsaturated soils. It has become known as the Barcelona Basic Model (BBM), named after the city where the researchers' university is located. The BBM is an elastic plastic strain hardening model that makes use of two stress variables: the net normal stress ($p^* = \sigma - u_a$) and the net water tension or suction ($s = u_w - u_a$). The model is based on several observations of the behavior of unsaturated soils, including reversible swelling and shrinking at low confining pressures, collapse at high pressures, and increase in yield stress (preconsolidation pressure) with increase in net water tension. BBM becomes equal to the Modified Cam Clay model when the suction is equal to zero. Like the MCC model, the BBM uses a linear relationship between the void ratio e and the natural logarithm of the net normal stress p^*, called the normal compression loading (NCL) curve (Figure 12.20*a*). The BBM adds another NCL curve with a linear relationship between the void ratio e and the natural logarithm of the net water tension or suction s. The reference NCL curve in the $e - Lnp^*$ set of axes corresponds to a suction equal to zero (Figure 12.20) and has the same equation as in the MCC model except that the stress is now the net mean normal stress p* instead of the mean effective stress p′:

$$e = e_o - \lambda_o Ln \frac{p^*}{p^*_o} \tag{12.97}$$

where e is the void ratio corresponding to p^*, e_o is the initial void ratio corresponding to p^*_o, and λ_o is the compression index for zero suction. Then the NCL curve for a suction s different from zero is:

$$e = e_{so} - \lambda Ln \frac{p^*}{p_o^*} \tag{12.98}$$

where e_{so} is the initial void ratio corresponding to p_o^*, and λ is the compression index for a suction s.

The unload-reload line is also considered to be a straight line and is expressed as:

$$e^e = e_y - \kappa Ln \frac{p^*}{p_y^*} \tag{12.99}$$

where e^e is the void ratio after elastic rebound swelling, e_y is the void ratio corresponding to the yield stress p_y^*, and κ is the swelling index (slope of the line). The NCL curve is also presented in the $e - Lns$ set of axes (Figure 12.20*b*) and the equations are similar to those of the NCL curve in the $e - Lnp^*$ set of axes. However. p^* is replaced by s and the slopes λ_s and κ_s are defined as the compression index with respect to suction s and the swelling index with respect to suction s, respectively.

The yield stress p_y^* depends on the suction s (Figure 12.20*a*), and the curve linking the two is called the *loading-collapse curve* or *LC curve* (Figure 12.20*c*). The LC curve is a yield curve and the equation of this curve is given by:

$$Ln \frac{p_y^*}{p_r} = \left(\frac{\lambda_o - \kappa}{\lambda_\infty - (\lambda_\infty - \lambda_o)\, e^{-\beta s} - \kappa} \right) Ln \frac{p_{yo}^*}{p_r} \tag{12.100}$$

where p_y^* is the yield pressure at a suction s, p_{yo}* is the yield pressure at a suction $s = 0$, p_r is a reference pressure (atmospheric pressure, for example), λ_o is the compression index for a suction $s = 0$, λ_∞ is the compression index at very high suction, β is a coefficient controlling the rate of compression index λ with suction, and κ is the swelling index.

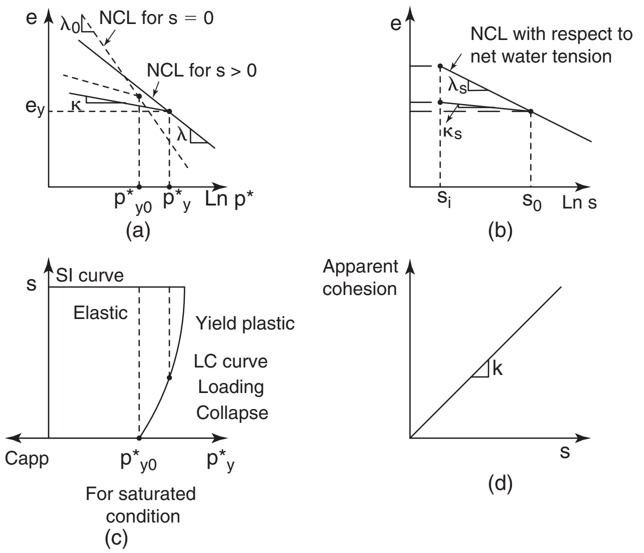

Figure 12.20 Elements of the Barcelona Basic Model.

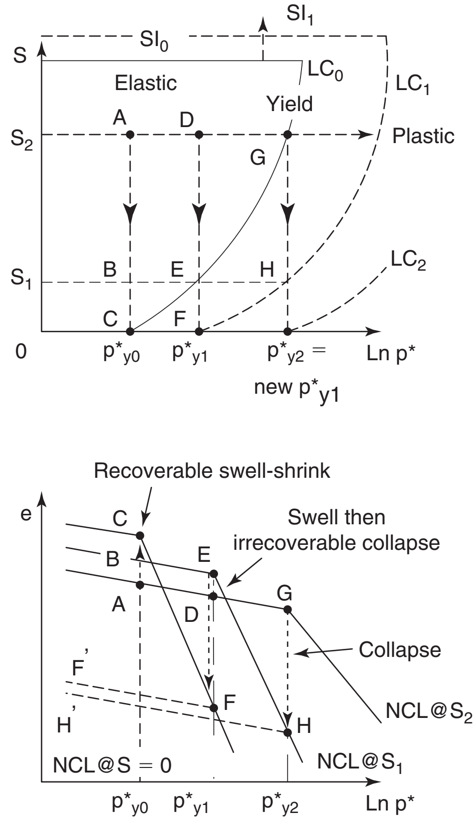

Figure 12.21 Loading collapse curve.

The LC curve dictates whether the sample will swell, shrink, or collapse. Figure 12.21 illustrates this point. To the left of the LC curve in Figure 12.21a, the soil behaves elastically; it yields on the LC curve and with strain hardening it deforms plastically to the right of the LC curve. If the soil is far inside the elastic domain, the soil will swell upon wetting (suction decreases) and the volume change will be reversible (path ABC on Figure 12.21b). If the soil is inside the elastic zone but not far enough from the LC curve, the soil can swell upon wetting (suction decreases) at first and then collapse (path DEF on Figure 12.21b). If the soil is outside the LC curve, the soil will collapse upon wetting (suction decreases) (path GH on Figure 12.21b). In all three cases, the suction decreases under constant total stress; this means that the water stress increases (e.g., from −1000 kPa to −100 kPa) and therefore the effective stress decreases. These three cases show that for unsaturated soils, there is not a single relationship between the volume change of the soil and the effective stress, unlike for saturated soils as postulated by Terzaghi. The inability of effective stress to explain this dual behavior has been the biggest obstacle in the development of a single effective stress model for unsaturated soils.

In the case of the swelling-collapse path DEF in Figure 12.21b, the soil moves from the NCL@s_2 curve on Figure 12.21a to the NCL@s_1 curve and finally comes to rest on the NCL@s = 0 curve. During that process, the value of the net normal stress does not change and remains equal to the yield stress p^*_{y1}; it starts associated with s_2 and ends up associated with s = 0. Therefore, the starting point of the LC curve for the soil after collapse is p^*_{y1}, which represents the new value of p^*_{y0}. This shows how the LC curve can evolve as the material experiences wetting or drying.

The LC curve is capped by the suction increase curve or SI curve (Figure 12.21), indicating that during drying the soil will reach a maximum suction value. This maximum suction is nearly independent of the net stress, and a horizontal line is chosen to represent this yield limit. Much like the LC curve, the SI curve can evolve with straining, wetting, or drying of the soil (Figure 12.21).

The critical state line (CSL) failure envelope in the $q - p'$ plot is the same as in the MCC model, but the suction increases the apparent cohesion c_{app} by a value linearly related to the suction s:

$$c_{app} = ks \tag{12.101}$$

where k is a constant of proportionality. The CSL equation (Eq. 12.90) is modified as follows:

$$q = Mp^* + ks \tag{12.102}$$

The shape of the yield surface is kept as an ellipse but is modified to include the contribution of the suction on the apparent cohesion (Figure 12.22):

$$q^2 - M^2(p^* + k\,s)(p^*_y - p^*) = 0 \tag{12.103}$$

The postyield behavior for the BBM is described through the strain hardening of the yield function. Much like in the MCC model, the incremental plastic strain is given by:

$$de^p = (\lambda - \kappa)\frac{dp^*}{p^*} \tag{12.104}$$

What is different with the BBM is that the flow rule that gives the direction of the incremental plastic strain is non-associated and given by a plastic potential:

$$G = \alpha q^2 - M^2(p^* + k\,s)(p^*_y - p^*) \tag{12.105}$$

where α is a parameter determined from the condition that the flow rule predicts zero lateral strains in a K_0 stress path (Alonso et al. 1990).

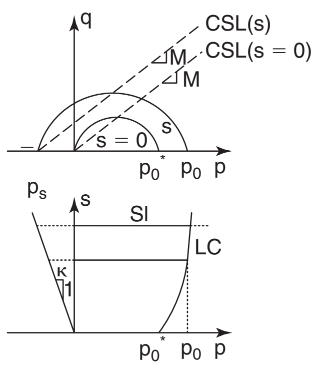

Figure 12.22 Increase in yield surface with increase in suction.

12.4.4 Water Stress Predictions

The prediction of water stresses for saturated and unsaturated cases in numerical methods can be classified in four categories:

1. **No water**. This is the case where the soil has no water. In this case the numerical simulations proceed on the basis of total stress or effective stress without distinction, as there is no difference between the two.
2. **Saturated and drained**. This is the case where the soil is saturated but the loading is slow enough that no water stress in excess of hydrostatic is generated. In this case the solution proceeds in terms of effective normal stress and the water stress is added at the end to obtain the total normal stress.
3. **Total stress approach**. In this case the numerical simulation proceeds in terms of total stresses regardless of the water regime. This is not a theoretically satisfying approach, as it does not recognize the basic and separate behavior of the soil skeleton and the water in the soil. This type of analysis can be accepted in the case of undrained behavior of saturated soils or in the case of high-water-tension soils. In both cases, it is approximately acceptable to consider the soil as a one-phase material, as it is likely that there is very little movement of the water in the soil mass. An exception is the liquefaction of loose sands.
4. **Saturated or unsaturated with water stress formulation**. This is the best and most appropriate way to simulate soil behavior, but it is also the most complicated. This approach requires one to formulate the flow of water through soil in the case of either a saturated soil or an unsaturated soil. This is the topic of Chapter 13.

PROBLEMS

12.1 Develop the expression for the bulk modulus K (hydrostatic compression) and the constrained modulus M (no lateral strain) by using the equations of elasticity linking the stresses and the strains.

12.2 A triaxial test is performed on an elastic soil and the result is plotted as major principal stress σ_1 versus axial strain ε_1. Is the slope of the line equal to the modulus E? If not, what is it? Give the expression of Poisson's ratio in terms of the stresses σ_1 and σ_3, and strains ε_1 and ε_3, for this test. What measurements would you have to make to back-calculate the modulus and Poisson's ratio from such a test?

12.3 Find the ultimate pressure that can be resisted by a soil subjected to a cylindrical expansion in the following case. The cylinder is infinitely long and the initial radius is r_o. The soil is a clay that behaves as a rigid plastic material with a yield criterion $\sigma_1 - \sigma_3 = 2s_u$. Beyond the yield criterion, the soil deforms without changing volume (undrained behavior of the clay).

12.4 Find the ultimate pressure that can be resisted by a soil subjected to a spherical expansion in the following case. The sphere has an initial radius equal to r_o. The soil is a clay that behaves as a rigid plastic material with a yield criterion $\sigma_1 - \sigma_3 = 2s_u$. Beyond the yield criterion, the soil deforms without changing volume (undrained behavior of the clay).

12.5 Find the ultimate pressure that can be resisted by a soil subjected to a cylindrical expansion in the following case. The cylinder is infinitely long and the initial radius is r_o. The soil is a sand that behaves as a rigid plastic material with a yield criterion $\sigma_1/\sigma_3 = K_p$. Beyond the yield criterion, the soil deforms without changing volume. (Although "no volume change" is not a common case in sand, it drastically simplifies the mathematics of this problem.)

12.6 Find the ultimate pressure that can be resisted by a soil subjected to a spherical expansion in the following case. The sphere has an initial radius r_o. The soil is a sand that behaves as a rigid plastic material with a yield criterion $\sigma_1/\sigma_3 = K_p$. Beyond the yield criterion, the soil deforms without changing volume. (Although "no volume change" is not a common case in sand, it drastically simplifies the mathematics of this problem.)

12.7 A Duncan-Chang (DC) model soil has an initial tangent modulus E_o equal to 100 MPa, a strength ratio R_f equal to 0.9, and a stress exponent n equal to 0.5. This DC soil is tested in a triaxial test with a confinement stress $\sigma_3 = 60$ kPa. The cohesion intercept if 5 kPa and the friction angle 34°.

a. Generate the complete σ_1 vs. ε_1 curve.
b. Derive the equation for the modulus as a function of stress level and strain level.

Problems and Solutions

Problem 12.1

Develop the expression for the bulk modulus K (hydrostatic compression) and the constrained modulus M (no lateral strain) by using the equations of elasticity linking the stresses and the strains.

Solution 12.1

The bulk modulus K:

$$K = \frac{\sigma}{\frac{\Delta V}{V}} = \frac{\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)}{\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}} = \frac{E}{3(1 - 2\upsilon)}$$

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \upsilon\left(\sigma_{yy} + \sigma_{zz}\right)\right)$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \upsilon\left(\sigma_{xx} + \sigma_{zz}\right)\right)$$

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \upsilon\left(\sigma_{xx} + \sigma_{yy}\right)\right)$$

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{1}{E}\left[\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) - 2\upsilon\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{1}{E}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)(1 - 2\upsilon)$$

If substituted in K formula:

$$K = \frac{\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)}{\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}} = \frac{\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)}{\frac{1}{E}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)(1 - 2\upsilon)} = \frac{E}{3(1 - 2\upsilon)}$$

The constrained modulus M:

$$M = \frac{\sigma_{xx}}{\varepsilon_{xx}} = \frac{E(1-\upsilon)}{(1+\upsilon)(1-2\upsilon)}$$

Because there is no lateral strain:

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \upsilon\left(\sigma_{xx}+\sigma_{zz}\right)\right) = 0 \rightarrow \sigma_{yy} = \upsilon\left(\sigma_{xx}+\sigma_{zz}\right) \quad \text{(I)}$$

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \upsilon\left(\sigma_{xx}+\sigma_{yy}\right)\right) = 0 \rightarrow \sigma_{zz} = \upsilon\left(\sigma_{xx}+\sigma_{yy}\right) \quad \text{(II)}$$

$$\textbf{(I)} + \textbf{(II)} \rightarrow \left(\sigma_{yy}+\sigma_{zz}\right) = 2\upsilon\sigma_{xx} + \upsilon\left(\sigma_{yy}+\sigma_{zz}\right)$$

$$\left(\sigma_{yy}+\sigma_{zz}\right) = \frac{2\upsilon\sigma_{xx}}{(1-\upsilon)}$$

By substituting in M formula:

$$M = \frac{\sigma_{xx}}{\varepsilon_{xx}} = \frac{\sigma_{xx}}{\frac{1}{E}\left(\sigma_{xx} - \upsilon\left(\sigma_{yy}+\sigma_{zz}\right)\right)} = \frac{\sigma_{xx}}{\frac{1}{E}\left(\sigma_{xx} - \frac{2\upsilon^2\sigma_{xx}}{(1-\upsilon)}\right)} = \frac{E\sigma_{xx}}{\sigma_{xx}\left(1-\frac{2\upsilon^2}{(1-\upsilon)}\right)}$$

$$M = \frac{E\sigma_{xx}}{\sigma_{xx}\left(\frac{1-\upsilon-2\upsilon^2}{(1-\upsilon)}\right)} = \frac{E(1-\upsilon)}{(1+\upsilon)(1-2\upsilon)}$$

Problem 12.2

A triaxial test is performed on an elastic soil and the result is plotted as major principal stress σ_1 versus axial strain ε_1. Is the slope of the line equal to the modulus E? If not, what is it? Give the expression of Poisson's ratio in terms of the stresses σ_1 and σ_3, and strains ε_1 and ε_3, for this test. What measurements would you have to make to back-calculate the modulus and Poisson's ratio from such a test?

Solution 12.2

No. The slope of the line is not the modulus E and is given by the following expression.

In a triaxial test, $\sigma_2 = \sigma_3$

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - 2\nu\sigma_3)$$

$$\frac{\sigma_1}{\varepsilon_1} = \frac{E\sigma_1}{\sigma_1 - 2\nu\sigma_3}$$

If σ_3 is equal to zero (unconfined compression test) then the slope is E. The Poisson's ratio is calculated as follows.

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - 2\nu\sigma_3)$$

$$\varepsilon_3 = \frac{1}{E}[\sigma_3 - 2\nu(\sigma_1+\sigma_3)]$$

$$\nu = \frac{\varepsilon_3\sigma_1 - \varepsilon_1\sigma_3}{2\varepsilon_3\sigma_3 - \varepsilon_1(\sigma_1+\sigma_3)}$$

The following measurements should be made to back-calculate the modulus and Poisson's ratio: confining pressure (σ_3), deviatoric stress ($\sigma_1 - \sigma_3$), axial strain (ε_1) or (ε_a) and radial strain (ε_3) or (ε_r).

Problem 12.3

Find the ultimate pressure that can be resisted by a soil subjected to a cylindrical expansion in the following case. The cylinder is infinitely long and the initial radius is r_o. The soil is a clay that behaves as a rigid plastic material with a yield criterion $\sigma_1 - \sigma_3 = 2s_u$. Beyond the yield criterion, the soil deforms without changing volume (undrained behavior of the clay).

Solution 12.3

Step 1

The elastic solution is summarized as follows:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

where $\sigma_r = \sigma_{oh} + \Delta\sigma_r, \sigma_\theta = \sigma_{oh} + \Delta\sigma_\theta$

$$\varepsilon_r = \frac{1}{E}(\Delta\sigma_r - \nu(\Delta\sigma_\theta + \Delta\sigma_z)) = -\frac{du}{dr}$$
$$\varepsilon_\theta = \frac{1}{E}(\Delta\sigma_\theta - \nu(\Delta\sigma_r + \Delta\sigma_z)) = -\frac{u}{r}$$
$$\varepsilon_z = \frac{1}{E}(\Delta\sigma_z - \nu(\Delta\sigma_r + \Delta\sigma_\theta)) = 0$$

The governing differential equation is

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = 0$$

By applying the boundary conditions, we have:

$$u = \frac{u_0 r_0}{r}$$

Then the strains are:

$$\varepsilon_r = \frac{du}{dr} = \frac{u_0 r_0}{r^2}$$
$$\varepsilon_\theta = \frac{u}{r} = -\frac{u_0 r_0}{r^2}$$
$$\Delta\sigma_{r(r=r_0)} = \frac{E}{1+\nu}\frac{u_0 r_0}{r_0^2} = \frac{E}{1+\nu}\frac{u_0}{r_0} = 2G\varepsilon_{\theta 0}$$

We know that:

$$\varepsilon_{\theta 0} = -\frac{u_0}{r_0} = -\frac{2\pi r_0 u_0}{2\pi r_0 r_0} = -\frac{1}{2}\frac{\Delta V}{V}$$

Therefore,

$$\sigma_{r(r=r_0)} = \sigma_{oh} - 2G\varepsilon_{\theta 0} = \sigma_{oh} + G\frac{\Delta V}{V} \text{ and } \sigma_\theta = \sigma_{oh} - G\frac{\Delta V}{V}$$

Step 2

In plasticity and for this problem, the yield criterion is Tresca:

$$\sigma_1 - \sigma_3 = 2s_u$$
$$\sigma_r - \sigma_\theta = 2s_u$$

We know that (equilibrium using the current radius ρ):

$$\frac{d\sigma_r}{d\rho} + \frac{\sigma_r - \sigma_\theta}{\rho} = 0$$
$$\frac{d\sigma_r}{d\rho} + \frac{2s_u}{\rho} = 0$$
$$d\sigma_r = -2s_u\frac{d\rho}{\rho}$$
$$\int d\sigma_r = \int -2s_u\frac{d\rho}{\rho}$$
$$\sigma_r = -2s_u \ln\rho + A$$

Boundary conditions:

$$\sigma_r = p_F @ \rho = \rho_F$$

$$A = p_F + 2s_u \ln \rho_F$$

$$\sigma_r = p_F - s_u \ln \frac{\rho^2}{\rho_F^2}$$

Compatibility equations at the plastic-elastic boundary:

$$\sigma_r = p_F - s_u \ln \frac{\rho_F^2}{\rho_F^2}$$

$$\sigma_r = p_F$$

When assuming no volume change, $\Delta V = const$, $\sigma_\theta = \sigma_{oh} - G\dfrac{\Delta V}{V}$, and $\sigma_r = \sigma_{oh} + G\dfrac{\Delta V}{V}$, so:

$$\sigma_r - \sigma_\theta = 2G\frac{\Delta V}{V} = 2s_u$$

at the interface., so:

$$p_F = \sigma_{oh} + s_u \text{ (for Tresca criterion)}$$

Step 3

Find p_L in plasticity condition (Figure 12.1s).

To get the limit pressure:

$$p_L = p_F - s_u \ln \frac{\rho_0^2}{\rho_F^2}$$

Let us look at no volume change, $\dfrac{\partial \Delta V}{\partial r} = 0$

$$\Delta V_0 = \Delta V_F = const$$

$$\pi\rho_0^2 - \pi r_0^2 = \pi\rho_F^2 - \pi r_F^2$$

$$\rho_F^2 - r_F^2 = \rho_0^2 - r_0^2 = const$$

$$\frac{\rho_F^2 - r_F^2}{\rho_F^2} = \frac{\Delta V_F}{V_F}$$

$$\frac{\rho_0^2 - r_0^2}{\rho_0^2} = \frac{\Delta V_0}{V_0} = \frac{\rho_F^2 - r_F^2}{\rho_0^2}$$

When $\rho_F \to \infty$, we have $\dfrac{\Delta V_0}{V_0} \to 1$

$$\frac{\rho_0^2}{\rho_F^2} = \frac{\Delta V_F}{V_F}$$

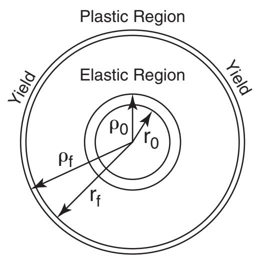

Figure 12.1s r and ρ definition.

We already know that:

$$G\frac{\Delta V}{V} = s_u$$
$$p_L = p_F - s_u \ln \frac{s_u}{G} = p_F + s_u \ln \frac{G}{s_u}$$
$$p_L = \sigma_{oh} + s_u \left(1 + s_u \ln \frac{G}{s_u}\right)$$

Problem 12.4

Find the ultimate pressure that can be resisted by a soil subjected to a spherical expansion in the following case. The sphere has an initial radius equal to r_o. The soil is a clay that behaves as a rigid plastic material with a yield criterion $\sigma_1 - \sigma_3 = 2s_u$. Beyond the yield criterion, the soil deforms without changing volume (undrained behavior of the clay).

Solution 12.4

Step 1

The elastic solution is summarized as follows (problem 11.5) Equilibrium in spherical space ($\sigma_\theta = \sigma_\phi$) gives:

$$\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_\theta}{r} = 0$$

where $\sigma_r = p_o + \Delta\sigma_r$, $\sigma_\theta = p_o + \Delta\sigma_\theta$, and p_o is the initial hydrostatic stress at rest in the soil.

$$\varepsilon_r = \frac{1}{E}(\Delta\sigma_r - 2\nu\Delta\sigma_\theta) = -\frac{du}{dr}$$
$$\varepsilon_\theta = \varepsilon_\phi = \frac{1}{E}[(1-\nu)\Delta\sigma_\theta - \nu\Delta\sigma_r] = -\frac{u}{r}$$

or

$$\Delta\sigma_\theta = -\frac{E}{(1+\nu)(1-2\nu)}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$
$$\Delta\sigma_r = -\frac{E}{(1+\nu)(1-2\nu)}\left[2\nu\frac{u}{r} + (1-\nu)\frac{du}{dr}\right]$$

The governing differential equation is:

$$r^2\frac{d^2u}{dr^2} + 2r\frac{du}{dr} - 2u = 0$$

By applying the boundary conditions we get:

$$u = \frac{u_0 r_0^2}{r^2}$$

The strains are:

$$\varepsilon_r = -\frac{du}{dr} = 2\frac{u_0 r_0^2}{r^3}$$
$$\varepsilon_\theta = -\frac{u}{r} = -\frac{u_0 r_0^2}{r^3}$$
$$\Delta\sigma_r = \frac{2E}{1+\nu}\frac{u_0 r_0^2}{r^3}$$

Therefore,

$$\Delta\sigma_{r(r=r_0)} = \frac{2E}{1+\nu}\frac{u_0}{r_0} = -4G\varepsilon_{\theta 0}$$

$$\frac{\Delta V}{V} = \frac{4\pi r^2 u}{\frac{4}{3}\pi r^3} = 3\frac{u}{r} = -3\varepsilon_\theta \rightarrow \Delta\sigma_r = -4G\varepsilon_\theta = \frac{4}{3}G\frac{\Delta V}{V}$$

By the same process:

$$\Delta\sigma_\theta = 2G\varepsilon_\theta = -\frac{2}{3}G\frac{\Delta V}{V}$$

Step 2

In plasticity and for this problem, the yield criterion is Tresca:

$$\sigma_r - \sigma_\theta = 2s_u$$

We know that (using the current radius ρ):

$$\frac{d\sigma_r}{d\rho} + 2\frac{\sigma_r - \sigma_\theta}{\rho} = 0$$

$$\frac{d\sigma_r}{d\rho} + \frac{4s_u}{\rho} = 0$$

$$d\sigma_r = -4s_u\frac{d\rho}{\rho}$$

$$\int d\sigma_r = \int -4s_u\frac{d\rho}{\rho}$$

$$\sigma_r = -4s_u \ln\rho + A$$

The boundary conditions are:

$$\sigma_r = p_F @ \rho = \rho_F$$

$$A = p_F + 4s_u \ln\rho_F$$

$$\sigma_r = p_F + 4s_u \ln\frac{\rho_F}{\rho}$$

From elasticity theory, we have already proved that:

$$\sigma_r = p_0 + 4G\frac{u_0 r_0^2}{r^3}$$

$$\sigma_\theta = p_0 - 2G\frac{u_0 r_0^2}{r^3}$$

At yield:

$$\sigma_r = p_F = p_0 + 4G\frac{u_0 r_0^2}{r_f^3} = p_0 + \frac{4}{3}G\frac{\Delta V}{V}$$

$$\sigma_\theta = p_0 - 2G\frac{u_0 r_0^2}{r_f^3}$$

At the boundary between the elastic zone and the plastic zone and using the Tresca criterion gives

$$\sigma_r = p_F$$
$$\sigma_r - \sigma_\theta = 6G\frac{u_0 r_0^2}{r_f^3} = 2G\frac{\Delta V}{V}$$
$$G\frac{\Delta V}{V} = s_u$$
$$p_F = p_0 + \frac{4}{3}s_u$$

Step 3

Find p_L in a plasticity condition:

$$p_L = p_F + 4s_u \ln\frac{\rho_F}{\rho_0}$$
$$\Delta V_0 = \Delta V_F = const$$

Furthermore

$$\frac{\rho_F^3 - r_F^3}{\rho_F^3} = \frac{\Delta V_F}{V_F}$$
$$\frac{\rho_0^3 - r_0^3}{\rho_0^3} = \frac{\Delta V_0}{V_0} = \frac{\rho_F^3 - r_F^3}{\rho_0^3}$$
$$\frac{\rho_F^3}{\rho_0^3} = \frac{\Delta V_0}{V_0}\frac{V_F}{\Delta V_F}$$

when $\rho_F \to \infty$ we have $\dfrac{\Delta V_0}{V_0} \to 1$ so at infitnite expansion

$$\frac{\rho_F^3}{\rho_0^3} = \frac{V_F}{\Delta V_F}$$
$$\frac{\Delta V}{V} = \frac{s_u}{G}$$
$$\frac{\rho_F}{\rho_0} = \left[\frac{G}{s_u}\right]^{1/3}$$
$$p_L = p_F + \frac{4}{3}s_u \ln\frac{G}{s_u}$$
$$p_L = p_0 + \frac{4}{3}s_u\left(1 + \ln\frac{G}{s_u}\right)$$

Problem 12.5

Find the ultimate pressure that can be resisted by a soil subjected to a cylindrical expansion in the following case. The cylinder is infinitely long and the initial radius is r_o. The soil is a sand that behaves as a rigid plastic material with a yield criterion $\sigma_1/\sigma_3 = K_p$. Beyond the yield criterion, the soil deforms without changing volume. (Although "no volume change" is not a common case in sand, it drastically simplifies the mathematics of this problem.)

Solution 12.5

Step 1

The elasticity constitutive model (problem 12.3, step 1) gives:

$$\sigma_r = p_0 + G\frac{\Delta V}{V} \text{ and } \sigma_\theta = p_0 - G\frac{\Delta V}{V}$$

Step 2

In plasticity and for this problem, the yield criterion is Mohr-Coulomb:

$$\frac{\sigma_\theta + \dfrac{c}{\tan\phi}}{\sigma_r + \dfrac{c}{\tan\phi}} = \frac{1-\sin\phi}{1+\sin\phi} = k_a \tag{12.1s}$$

From elasticity theory, we have already proved that:

$$\sigma_r = p_0 + 2G\frac{u_0 r_0}{r^2}$$
$$\sigma_\theta = p_0 - 2G\frac{u_0 r_0}{r^2}$$

At the boundary between the elastic and the plastic region, we have:

$$\sigma_r = p_F = p_0 + 2G\frac{u_0 r_0}{{r_F}^2} \tag{12.2s}$$
$$\sigma_\theta = p_0 - 2G\frac{u_0 r_0}{{r_F}^2} \tag{12.3s}$$

From Eq. 12.2s, we can get

$$r_F^2 = \frac{2Gu_0 r_0}{p_F - p_0}$$

Therefore,

$$\sigma_\theta = p_0 - \frac{2Gu_0 r_0}{2Gu_0 r_0}(p_F - p_0)$$
$$\sigma_\theta = 2p_0 - p_F \tag{12.4s}$$
$$\sigma_r = p_F \tag{12.5s}$$

Plugging Eqs. 12.4s and 12.5s into Eq. 12.1s, we get:

$$\frac{2p_0 - p_F + \dfrac{c}{\tan\phi}}{p_F + \dfrac{c}{\tan\phi}} = \frac{1-\sin\phi}{1+\sin\phi}$$

so

$$p_F = p_0 + p_0\sin\phi + c\ \cos\phi$$

Step 3

Find p_L in a plasticity condition.

From Eq. 12.1s, we get:

$$\sigma_\theta = \left(\sigma_r + \frac{c}{\tan\phi}\right)k_a - \frac{c}{\tan\phi} \tag{12.6s}$$

Plugging Eq. 12.6s into the equilibrium equation, we get:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \left(\sigma_r + \dfrac{c}{\tan\phi}\right)k_a + \dfrac{c}{\tan\phi}}{r} = 0$$
$$\frac{d\sigma_r}{dr} + \frac{\sigma_r(1-k_a)}{r} + \frac{c}{\tan\phi}(1-k_a)\frac{1}{r} = 0$$
$$r\frac{d\sigma_r}{dr} + \sigma_r(1-k_a) = \frac{c}{\tan\phi}(k_a - 1) \tag{12.7s}$$

The solution for σ_r includes a general solution and a particular solution. The general solution of σ_r is:

$$r\frac{d\sigma_r}{dr} + \sigma_r(1-k_a) = 0$$
$$\sigma_r = A\rho^{(k_a-1)}$$

The particular solution of σ_r is:

$$\sigma_r^* = -\frac{c}{\tan\phi}$$

The solution of σ_r is:

$$\sigma_r = A\rho^{(k_a-1)} - \frac{c}{\tan\phi}$$

Based on the boundary conditions:

$$\rho = \rho_F, \quad \sigma_r = p_F$$

therefore,

$$p_F = A\rho_F^{k_a-1} - \frac{c}{\tan\phi}$$

So,

$$A = \left(p_F + \frac{c}{\tan\phi}\right)\left(\frac{1}{\rho_F}\right)^{k_a-1}$$

Therefore,

$$\sigma_r = \left(p_F + \frac{c}{\tan\phi}\right)\left(\frac{\rho}{\rho_F}\right)^{k_a-1} - \frac{c}{\tan\phi}$$

The no volume change condition gives:

$$\Delta V_0 = \Delta V_F = const$$
$$\pi\rho_0^2 - \pi r_0^2 = \pi\rho_F^2 - \pi r_F^2$$
$$\rho_F^2 - r_F^2 = \rho_0^2 - r_0^2 = const$$
$$\frac{\rho_F^2 - r_F^2}{\rho_F^2} = \frac{\Delta V_F}{V_F}$$
$$\frac{\rho_0^2 - r_0^2}{\rho_0^2} = \frac{\Delta V_0}{V_0} = \frac{\rho_F^2 - r_F^2}{\rho_0^2}$$

when $\rho_F \to \infty$, we have $\dfrac{\Delta V_0}{V_0} \to 1$, Therefore at the limit pressure we have

$$\frac{\rho_0^2}{\rho_F^2} = \frac{\Delta V_F}{V_F}$$

In elasticity, $\sigma_r = p_0 + G\dfrac{\Delta V}{V}$; therefore at the elastic-plastic boundary,

$$\frac{\Delta V_F}{V_F} = \frac{p_F - p_0}{G} = \frac{p_0 + p_0\sin\phi + c\ \cos\phi - p_0}{G} = \frac{p_0\sin\phi + c\ \cos\phi}{G}$$

The limit pressure p_L corresponds to $\frac{\Delta V_o}{V_o} = 1$; therefore,

$$p_L = \left(p_F + \frac{c}{\tan\phi}\right)\left(\frac{\rho}{\rho_f}\right)^{k_a-1} - \frac{c}{\tan\phi}$$

$$p_L = \left(p_0 + p_0\sin\phi + c\ \cos\phi + \frac{c}{\tan\phi}\right)\left(\frac{G}{p_0\sin\phi + c\ \cos\phi}\right)^{\frac{1-k_a}{2}} - \frac{c}{\tan\phi}$$

Problem 12.6

Find the ultimate pressure that can be resisted by a soil subjected to a spherical expansion in the following case. The sphere has an initial radius r_o. The soil is a sand that behaves as a rigid plastic material with a yield criterion $\sigma_1/\sigma_3 = K_p$. Beyond the yield criterion, the soil deforms without changing volume. (Although "no volume change" is not a common case in sand, it drastically simplifies the mathematics of this problem.)

Solution 12.6

Step 1

The elasticity solution is presented in problem 12.4, step 1.

Step 2

In plasticity and for this problem, the yield criterion is Mohr-Coulomb:

$$\frac{\sigma_r + \frac{c}{\tan\phi}}{\sigma_\theta + \frac{c}{\tan\phi}} = \frac{1+\sin\phi}{1-\sin\phi} = k_p = \frac{1}{k_a} \tag{12.8s}$$

From elasticity theory, we have already proved that:

$$\sigma_r = p_0 + 4G\frac{u_0 r_0^2}{r^3}$$

$$\sigma_\theta = p_0 - 2G\frac{u_0 r_0^2}{r^3}$$

At yield:

$$\sigma_r = p_F = p_0 + 4G\frac{u_0 r_0^2}{r_f^3} = p_0 + \frac{4}{3}G\frac{\Delta V}{V} \tag{12.9s}$$

$$\sigma_\theta = p_0 - 2G\frac{u_0 r_0^2}{r_f^3} = p_0 - \frac{2}{3}G\frac{\Delta V}{V}$$

Combining Eq. 12.8s and 12.9s, we get.

$$p_F = \frac{3p_0(1+\sin\phi) + 4c\cos\phi}{3-\sin\phi} \tag{12.10s}$$

Step 3

Find p_L in plasticity. From Eq. 12.8s, we get:

$$\sigma_\theta = \left(\sigma_r + \frac{c}{\tan\phi}\right)k_a - \frac{c}{\tan\phi} \tag{12.11s}$$

Plugging Eq. 12.11s into the equilibrium equation, we get:

$$\begin{aligned}
&\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \left(\sigma_r + \dfrac{c}{\tan\phi}\right)k_a + \dfrac{c}{\tan\phi}}{r} = 0 \\
&\frac{d\sigma_r}{dr} + 2\frac{\sigma_r(1-k_a)}{r} + \frac{c}{\tan\phi}(1-k_a)\frac{2}{r} = 0 \\
&\frac{r}{(1-k_a)}\frac{d\sigma_r}{dr} + 2\sigma_r = -2\frac{c}{\tan\phi}
\end{aligned} \tag{12.12s}$$

The solution for σ_r includes a general solution and a particular solution. The general solution of σ_r is:

$$\begin{aligned}
&r\frac{d\sigma_r}{dr} + 2\sigma_r(1-k_a) = 0 \\
&\sigma_r = A\rho^{2(k_a-1)}
\end{aligned}$$

The particular solution of σ_r is:

$$\sigma_r^* = -\frac{c}{\tan\phi}$$

The solution of σ_r is:

$$\sigma_r = A\rho^{2(k_a-1)} - \frac{c}{\tan\phi}$$

At the elastic-plastic boundary, we have:

$$\rho = \rho_f,\ \sigma_r = p_F$$

therefore,

$$p_F = A\rho_f^{2(k_a-1)} - \frac{c}{\tan\phi}$$

So,

$$A = \left(p_F + \frac{c}{\tan\phi}\right)\rho_f{}^{-2(k_a-1)}$$

Therefore,

$$\sigma_r = \left(p_F + \frac{c}{\tan\phi}\right)\left(\frac{\rho}{\rho_f}\right)^{2(k_a-1)} - \frac{c}{\tan\phi}$$

by using p_F from Eq. 12.10s:

$$P_L = \sigma_r = (c + p_0\tan\phi)\left(\frac{3\,(1+\sin\phi)\cos\phi}{(3-\sin\phi)\sin\phi}\right)\left(\frac{r}{r_f}\right)^{2(k_a-1)} - \frac{c}{\tan\phi} \tag{12.13s}$$

Using the no volume change condition leads to

$$\begin{aligned}
&\Delta V_0 = \Delta V_F = const \\
&\rho_F^3 - r_F^3 = \rho_0^3 - r_0^3 \\
&\frac{\rho_F^3 - r_F^3}{\rho_F^3} = \frac{\Delta V_F}{V_F} \\
&\frac{\rho_0^3 - r_0^3}{\rho_0^3} = \frac{\Delta V_0}{V_0} = \frac{\rho_F^3 - r_F^3}{\rho_0^3} \\
&\frac{\rho_F^3}{\rho_0^3} = \frac{\Delta V_0}{V_0}\frac{V_F}{\Delta V_F}
\end{aligned}$$

when $\rho_F \to \infty$, we have $\dfrac{\Delta V_0}{V_0} \to 1$

$$\frac{\rho_F^3}{\rho_0^3} = \frac{V_F}{\Delta V_F} \tag{12.14s}$$

From Eqs. 12.9s and 12.10s:

$$G\frac{\Delta V}{V} = \frac{3p_0 \sin\phi + 3C\cos\phi}{3 - \sin\phi} \tag{12.15s}$$

By using Eqs. 12.13s, 12.14s, and 12.15s, we get:

$$p_L = (c + p_0 \tan\phi)\left(\frac{3\,(1+\sin\phi)\cos\phi}{(3-\sin\phi)\sin\phi}\right)\left[\frac{G}{c + p_0\tan\phi}\left(\frac{3-\sin\phi}{3\cos\phi}\right)\right]^{\frac{4}{3}\left(\frac{\sin\phi}{1+\sin\phi}\right)} - \frac{c}{\tan\phi}$$

Problem 12.7

A Duncan-Chang (DC) model soil has an initial tangent modulus E_o equal to 100 MPa, a strength ratio R_f equal to 0.9, and a stress exponent n equal to 0.5. This DC soil is tested in a triaxial test with a confinement stress $\sigma_3 = 60$ kPa. The cohesion intercept if 5 kPa and the friction angle 34°. Generate the complete $\sigma_1 - \sigma_3$ vs. ε_1 curve.

Solution 12.7

Given: $E_o = 100$ MPa, $R_f = 0.9$, $n = 0.5$, $c = 5$ kPa, $\varphi = 34^\circ$, and $\sigma_3 = 60$ kPa, we can get the σ_1 vs. ε_1 curve using the DC formulation:

$$\sigma_1 - \sigma_3 = \frac{\varepsilon_1}{\dfrac{1}{E_o} + \dfrac{\varepsilon_1}{R_f \sigma_{ult}}}$$

Based on the information from the triaxial test and the Mohr Coulomb failure criterion, the soil strength in term of deviator stress is computed as follows:

$$(\sigma_1 - \sigma_3)_f = \frac{2c\cos\phi + 2\sigma_3 \sin\phi}{1 - \sin\phi}$$

$$(\sigma_1 - \sigma_3)_f = \frac{2 \times 5 \times \cos 34 + 2 \times 60 \times \sin 34}{1 - \sin 34}$$

$$(\sigma_1 - \sigma_3)_f = 171 \text{ kPa}$$

The asymptotic value is given by

$$(\sigma_1 - \sigma_3)_{ult} = \frac{(\sigma_1 - \sigma_3)_f}{R_f} = \frac{171 \text{ kPa}}{0.9} = 190 \text{ kPa}$$

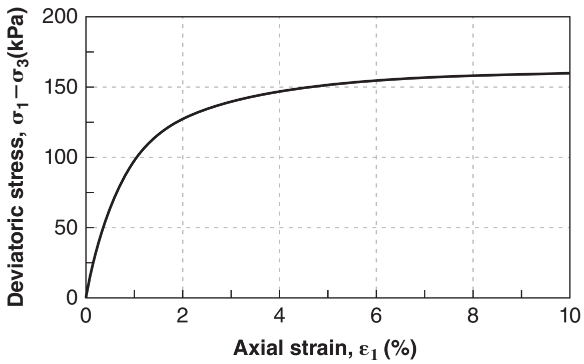

Figure 12.2s Stress-strain curve.