CHAPTER 11

Problem-Solving Methods

11.1 GENERAL

There are three main types of problems in geotechnical engineering: failure load problems, deformation problems, and flow problems. Each problem can be solved by performing experimental modeling, by doing theoretical modeling, or by using experience. The best solutions are those that have a theoretical framework, are calibrated against and correlated with experimental measurements, and are verified by experience at full scale. Experience is obtained by years of practice. As the saying goes, good judgment comes from experience, but experience comes from bad judgment. An attempt can be made at teaching experience by letting engineers, who have been practicing successfully for a long time, discuss case histories—including failures—in a classroom environment. Theoretical modeling includes continuum mechanics closed-form solutions, numerical simulations, dimensional analysis, probabilistic analysis, and risk analysis. Experimental modeling includes the use of scaled models, centrifuge models, and/or full-scale models. In all cases, fundamental laws and constitutive laws help in solving the problem.

11.2 DRAWING TO SCALE AS A FIRST STEP

One very important first step in solving a geotechnical engineering problem (or any engineering problem in general) is to always start by making a drawing to scale of the problem. If this step is not taken, the engineer may not get a proper sense of the issues at hand. For example, if one is designing a pile foundation under a building with the pile tips bearing into a sand layer, making a drawing to scale helps the engineer evaluate whether the sand layer is thick enough to prevent serious compression of the layers below. Failing to make that drawing properly, and instead drawing only a sample single pile bearing into the sand layer, may give the false impression of a thick sand layer (Figure 11.1). Also, if you draw a driven pile as a thick, short vertical member instead of the actual slender member, the issue of buckling will not come to your attention. Embankments typically have side slopes of 2 to 1 or 3 to 1, yet when they are sketched on a piece of paper, these slopes are often drawn too steep. By making a drawing to scale, you give yourself a better chance of recognizing some of the problems associated with the physical dimensions of the project. Always make a drawing to scale as a first step in solving an engineering problem.

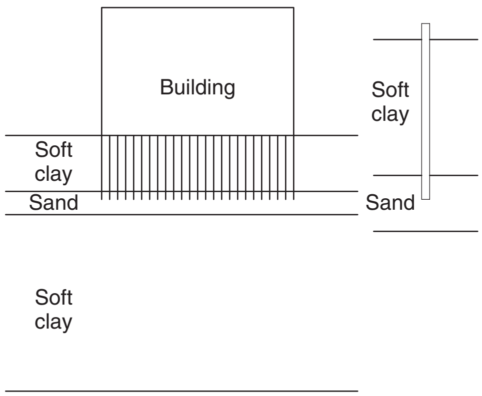


Figure 11.1 Make a drawing to scale.

11.3 PRIMARY LAWS

Two main types of laws are used to solve problems: fundamental laws and constitutive laws. *Fundamental laws* are valid no matter what material is being considered. They apply equally to soil, concrete, steel, or marshmallow. Fundamental laws include, for example, force and moment equilibrium, conservation of energy, and conservation of mass. The *constitutive laws* describe the behavior of the material. They are different for each material, whether it is soil, concrete, steel, or peanut butter. Constitutive laws include, for example, elasticity, plasticity, and viscoelasticity. Shear strength laws such as the Mohr-Coulomb criterion belong to the class of constitutive laws. Most theoretical problems are solved by making use of a combination of fundamental laws and constitutive laws. Other laws exist, such as the similitude laws used in dimensional analysis and the probabilistic laws used in risk analysis.

Geotechnical Engineering: Unsaturated and Saturated Soils Jean-Louis Briaud

11.4 CONTINUUM MECHANICS METHODS

The basic and general steps in developing a theoretical solution to a soil problem are to describe the problem precisely, identify the variables, write the applicable equations, and solve for the unknowns. If there are more equations than unknowns, then one must choose which equations are most important to satisfy. If there are more unknowns than equations, it is time to make reasonable assumptions to generate new equations. The reasonableness of the assumptions should then be verified by experimentation at model scale or (even better) at full scale. In soil mechanics, there are three main types of problems: failure problems, deformation problems, and flow problems.

11.4.1 Solving a Failure Problem: Limit Equilibrium, Method of Characteristics, Lower and Upper Bound Theorems

A typical solution to the problem of finding a failure load (e.g., ultimate bearing capacity of a footing) or a failure moment (e.g., slope stability) is to use the limit equilibrium analysis. In such a failure analysis, the step-by-step process advances as follows:

1. Assume a reasonable failure mechanism. If such a failure mechanism is not obvious, an experiment can be performed to observe the failure mechanism.
2. Draw a free body diagram of the failing body (soil mass) and identify the external forces and external moments applied to the failing body.
3. Write the applicable fundamental equations. These are equations that are valid for all problems and independent of the type of material involved. They include equilibrium equations (three forces, three moments), conservation of mass, and conservation of energy, among others.
4. Write the applicable constitutive equations. These are the equations describing the behavior of the material under load. Constitutive laws include, for example, elasticity, plasticity, and viscoelasticity. The shear strength equation for soils, which states that the shear strength is a function of the effective stress on the failure plane, is another example of a constitutive law.
5. Count the number of equations and the number of unknowns. If there are as many equations as there are unknowns, proceed to the next step. If there are more equations than unknowns (rare), choose which equations are most important to satisfy. If there are more unknowns than equation, formulate assumptions that lead to additional equations. These assumptions should be based on engineering judgment, or experience, or experimental observations. The reasonableness of these assumptions should be verified by comparing the solution to observed full-scale behavior.
6. Combine all equations and solve for the unknown (failure load or failure moment).

There can be as many solutions as there are assumed failure mechanisms, so obviously it becomes important to choose the failure mechanism that most closely duplicates the real one. This is where observation of full-scale failures becomes very useful.

Another solution is to use the method of characteristics. *Characteristics* are lines in the physical soil mass where the partial differential equation collapses into an ordinary differential equation. The equilibrium equations at the element level typically lead to partial differential equations. The method of characteristics simplifies these equations to the point where the problem is easier to solve. The method of characteristics can help to calculate failure loads for simple geometries.

Yet another solution is to use the bound theorems and apply them to soil masses. There are two such theorems: the lower bound theorem and the upper bound theorem. The lower bound theorem states that if any stress distribution throughout the soil mass can be found which is everywhere in equilibrium internally, does not violate the yield condition, and balances the external loads, the soil mass will safely carry the external loads. The upper bound theorem states that if an estimate of the failure load of a soil mass is made by equating internal rate of energy to the rate at which external forces do work in any postulated but kinematically admissible mechanism of deformation of the soil mass, the estimate will be either high or correct. In short, the lower bound theorem involves guessing a stress field that leads to a lower bound of the failure load; the upper bound theorem involves guessing a velocity or displacement field that leads to an upper bound estimate of the failure load.

11.4.2 Examples of Solving a Failure Problem

The first exemple problem is to find the ultimate pressure p_u that a strip footing of width B (Figure 11.2) can exert on the surface of a saturated clay that has a shear strength s equal to the undrained shear strength s_u because the loading is rapid. The steps described in section 11.4.1 for the limit equilibrium method are followed.

1. A cylindrical failure surface, as shown in Figure 11.3*a*, seems reasonable. This failure mechanism has been observed in many old silo failures.
2. The failing soil mass is the half cylinder shown in Figure 11.3*b* together with its free body diagram. All external forces and stresses are shown on the diagram, including the weight of the mass. Note that the weight is always an external force.

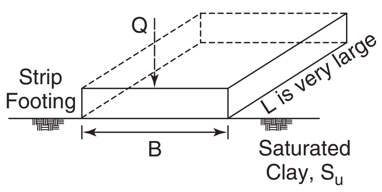


Figure 11.2 Strip footing example.

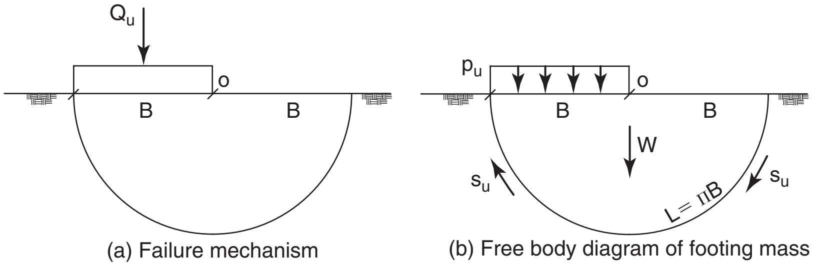


Figure 11.3 Failure load for a strip footing.

3. The most useful fundamental equation in this case is moment equilibrium around point O on Figure 11.3*b*.

$$M@O = 0 = p_u B\frac{B}{2} - s\ \pi B\ B \tag{11.1}$$

4. The constitutive equation in this case is the shear strength equation, which states that the shear strength s is equal to the undrained shear strength of the clay.

$$s = s_u \tag{11.2}$$

5. There are two unknowns (p_u and s) and two equations, so the problem can be solved.
6. Now we combine the equations and obtain:

$$p_u = 2\pi s_u \tag{11.3}$$

Other failure mechanisms are plausible and would lead to slightly different estimates of p_u.

The second problem is the one of a vertical wall with a height H supporting a clean, dry sand backfill with a friction angle φ (Figure 11.4). It is assumed that there is no friction between the wall and the backfill. The wall exerts a horizontal load P against the sand. As the wall moves very slightly away from the sand, the load P decreases and there is a point where the sand behind the wall starts to fail. At that point, the load is P_a and the question is to find the load P_a corresponding to impending failure of the sand. Note that the problem is a plane strain problem; therefore, all the loads will be line loads expressed in kN/m.

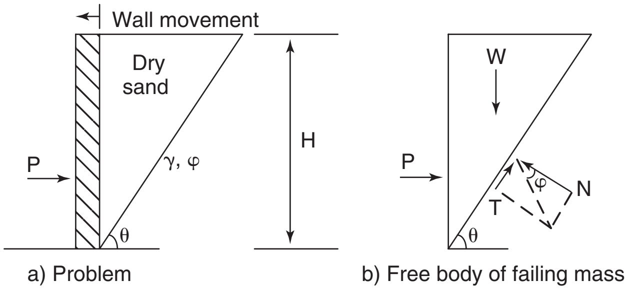


Figure 11.4 Example of a wall moving away from the backfill.

The steps described in section 11.4.1 for the limit equilibrium method are followed.

1. The soil is assumed to fail as a wedge making an angle θ with the horizontal, as shown in Figure 11.4. This failure mechanism has been observed in model scale and centrifuge experiments.
2. The failing soil mass is the wedge; its free-body diagram is shown in Figure 11.4. All external forces are shown on the diagram, including the action of the wall P, the weight of the soil mass W, the normal force N, and the shear force T on the failure plane. Note that the shear force T is acting uphill because the wedge is falling down along that plane and the soil outside of the wedge is resisting that tendency.
3. The most useful fundamental equations in this case are vertical and horizontal equilibrium of forces:

$$\sum F_v = 0 = W - N\cos\theta - T\sin\theta \tag{11.4}$$

$$\sum F_h = 0 = P - N\sin\theta + T\cos\theta \tag{11.5}$$

4. The constitutive equations in this case are the shear strength equation of the sand and the expression of the weight of the wedge. The shear strength equation states that the ultimate shear force T comes from the friction generated by the normal force N on the failure plane. The weight of the wedge is equal to the area of the wedge times the unit weight of the sand γ:

$$T = N\tan\varphi \tag{11.6}$$

$$W = \frac{\gamma H^2}{2\tan\theta} \tag{11.7}$$

5. There are four unknowns (W, N, T, P) and four equations, so the problem can be solved.
6. Now we combine the equations and obtain:

$$P = \frac{\gamma H^2}{2}\left(\frac{\sin\theta\cos\theta - \tan\varphi\cos^2\theta}{\sin\theta\cos\theta + \tan\varphi\sin^2\theta}\right) \tag{11.8}$$

There is one more issue to resolve. The load P depends on θ, yet there is a unique value of θ associated with the failure load P_a. This is the load at which the wedge fails behind the

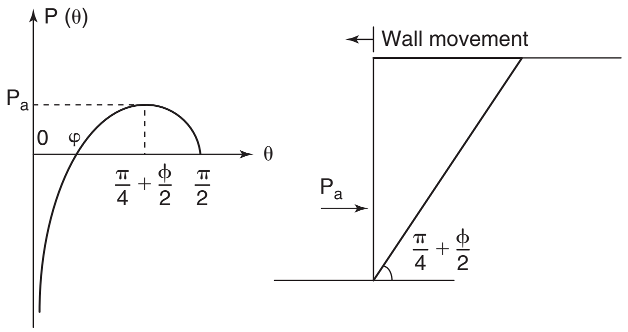


Figure 11.5 Load P as a function of the wedge angle θ; wall moves away from backfill.

wall, and this load corresponds to the θ value that maximizes P (Figure 11.5). In other words, the wedge that needs the maximum support will fail first. The maximum value of P is obtained by setting $\frac{dP}{d\theta} = 0$. This derivative is:

$$\frac{dP}{d\theta} = \frac{\gamma H^2}{2}\left(\frac{\sin\theta\cos\theta - \sin(\theta-\varphi)\cos(\theta-\varphi)}{\sin^2\theta\cos^2(\theta-\varphi)}\right) = 0 \tag{11.9}$$

There are two solutions to Eq. 11.9: one is $\varphi = 0$, which is not realistic, and the other one is:

$$\theta = \frac{\pi}{4} + \frac{\varphi}{2} \tag{11.10}$$

The load P_a can then be obtained from Eq. 11.8.

$$P_a = \frac{\gamma H^2}{2}\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right) \tag{11.11}$$

This problem is repeated but now with the wall being pushed into the sand (Figure 11.6) instead of pulled away from the sand as in the previous case. The question is to find the load P_p that corresponds to the failure of the soil mass. Only one thing changes in the derivation: the direction of the shear force T on the failure plane. Because the wedge will now move up along the failure plane, the soil outside the wedge will exert a shear force acting toward the bottom of the wedge. Therefore, in the equations T is replaced by −T and the problem leads to the situation shown in Figure 11.7. The failure load P_p is the load corresponding to the value of θ that minimizes P; that is, the wedge offering the minimum resistance is the wedge that will fail first.

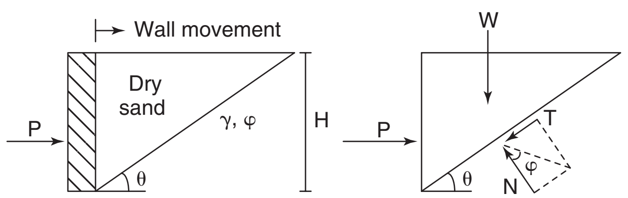


Figure 11.6 Example of a wall moving toward the backfill.

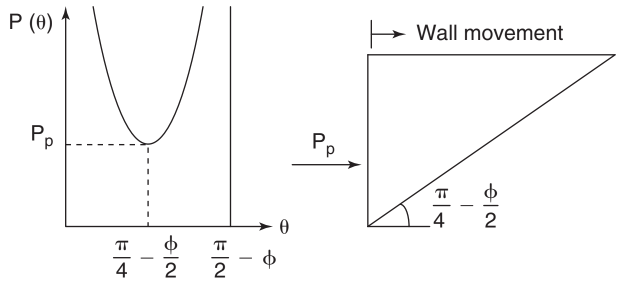


Figure 11.7 Load P as a function of the wedge angle θ; wall moves into backfill.

$$\theta = \frac{\pi}{4} - \frac{\varphi}{2} \tag{11.12}$$

$$P_p = \frac{\gamma H^2}{2}\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) \tag{11.13}$$

11.4.3 Solving a Deformation Problem

A typical solution to a deformation problem proceeds through the following steps:

1. Zoom in at the infinitesimal element level. This element has dimensions expressed in differential lengths.
2. The knowns and unknowns (e.g., loads, displacements, stresses, and strains) are identified on the element, including their variation from one side of the element to the other. This variation involves derivatives expressing the rate of change of the variable in one direction over a small distance.
3. The fundamental equations are written using the knowns and unknowns identified in step 2. These are equations that are true for all materials. They include equilibrium equations (three forces and three moments), conservation of mass, and conservation of energy, among others.
4. The constitutive equations are written using the knowns and unknowns identified in step 2. These equations describe the behavior of the material involved in the deformation. They include elasticity equations, plasticity equations, and viscosity equations, among others.
5. All equations are regrouped into the governing differential equations (GDEs).
6. The boundary conditions are expressed mathematically. If the problem is a dynamic problem, the boundary conditions involve both space and time.
7. The GDEs are solved in closed-form solutions if they are simple enough and through numerical solutions such as the finite difference method if they are too complicated. The boundary conditions are used to define the constants involved in the solution.

11.4.4 Example of Solving a Deformation Problem

The exemple problem is to find the horizontal displacement y(z) of a pile as a function of z if the pile is loaded in

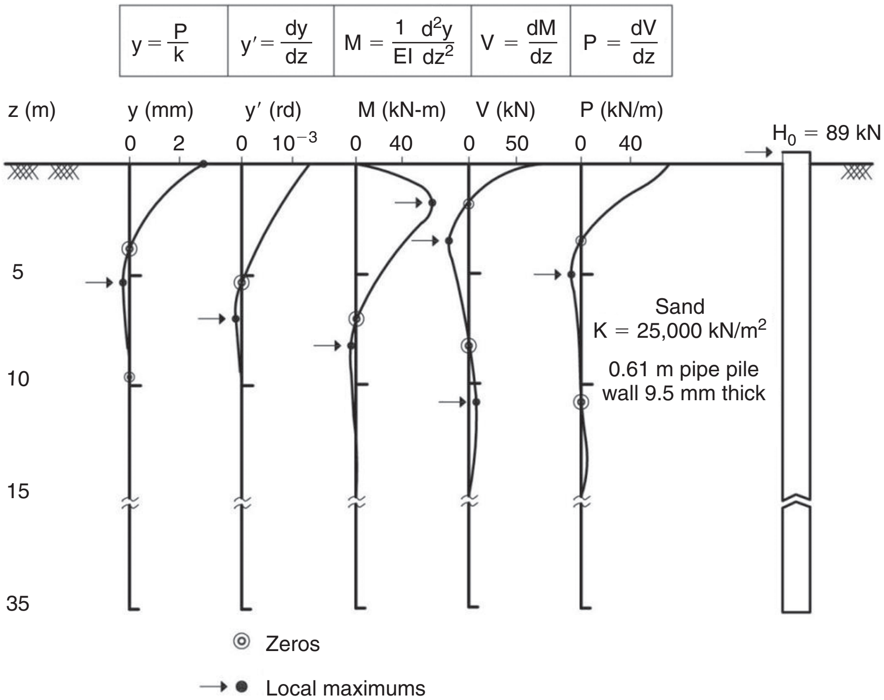


Figure 11.8 Horizontally loaded pile example.

overturning by a horizontal load H_o and an overturning moment M_o applied at the ground surface (Figure 11.8). For this simple example, the influence of the axial load will be ignored.

The solution proceeds by following the steps described in section 11.4.3.

1. Zoom in at the element level. In this case, we will select an element of the pile that is dz long (Figure 11.9).
2. The forces and moments acting on the element are shown on the element (Figure 11.9). These actions are the shear V (kN) and moment M (kN.m) at both ends of the element, and the soil resistance P (kN/m) as a line load. Some of these quantities change by a little bit from one end of the element to the other. This little bit is expressed mathematically as $\frac{\partial V}{\partial z}dz$ for the shear force V, for example, expressing that the change is equal to the rate of change of V times the distance dz. Because V is dependent only on z, $\frac{\partial V}{\partial z}dz$ can be simply written as dV.
3. The fundamental equations that are most useful in this case are horizontal equilibrium and moment equilibrium. Let's write horizontal equilibrium first (Figure 11.9):

$$\sum F_H = 0 = Pdz + V - (V + dV) \tag{11.14}$$

or

$$P = \frac{dV}{dz} \tag{11.15}$$

So, horizontal equilibrium states that the line load P on a pile is equal to the first derivative of the shear V. Now

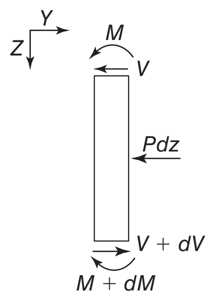


Figure 11.9 Element of horizontally loaded pile.

let's write moment equilibrium around point O (Figure 11.9). Again, Because M is only a function of z, $\frac{\partial M}{\partial z}dz$ can be simply written as dM.

$$\sum M@o = 0 = M + dM - M - V\frac{dz}{2} - (V + dV)\frac{dz}{2} \tag{11.16}$$

Neglecting the higher-order term, we are left with:

$$V = \frac{dM}{dz} \tag{11.17}$$

So, moment equilibrium states that the shear in a pile is equal to the first derivative of the bending moment.

4. The constitutive equations describe the behavior of the pile and of the soil. The pile behavior is described by relating the bending moment M applied to the pile to the curvature generated in the pile. This curvature is expressed by the

second derivative of y. The proportionality constant between the moment and the curvature is the bending stiffness E_pI, where E_p is the modulus of the pile material and I is the moment of inertia of the pile cross section around the axis of the moment. Again, because y is only a function of z, $\frac{\partial^2 y}{\partial z^2}$ can be more simply written as $\frac{d^2 y}{dz^2}$.

$$M = E_p I \frac{d^2 y}{dz^2} \tag{11.18}$$

Note that the unit of $\frac{d^2 y}{dz^2}$ is 1/m because $d^2 y$ is a little piece of y and dz^2 is the square of a little piece of z. This extends to the n^{th} derivative; the unit of $\frac{d^n y}{dz^n}$ is $1/m^{(n-1)}$ because $d^n y$ is still a little piece of y while dz^n is the n^{th} power of a little piece of z. For the constitutive equation describing the soil behavior, a simple linear relationship is used between the line load P (kN/m) characterizing the soil resistance and the deflection y (m) of the soil-pile interface. The proportionality constant is a spring constant K (kN/m^2) that characterizes the stiffness of the soil:

$$P = -Ky \tag{11.19}$$

The minus sign is there because P and y are in opposite directions (Figure 11.9).

5. The governing differential equations can now be assembled by regrouping the fundamental and constitutive equations:

$$P = -Ky = \frac{dV}{dz} = \frac{d^2 M}{dz^2} = E_p I \frac{d^4 y}{dz^4} \tag{11.20}$$

or

$$y + \frac{E_p I}{K}\frac{d^4 y}{dz^4} = 0 \tag{11.21}$$

6. The boundary conditions are stated for both ends of the pile. To simplify the solution of the differential equation, it is assumed that the pile is infinitely long and that the deflection is zero at the infinite end. At the top of the pile, the horizontal force and the overturning moment are known. The boundary conditions are:

a. z = infinity, y = 0
b. z = 0, $M = M_o$,
c. z = 0, $V = H_o$

7. Now we need to solve the differential equation. The solution y(z) has to be a function that becomes the same function when differentiated four times. This means a combination of exponential and trigonometric functions. The solution is therefore of the general form:

$$y(z) = e^{-\frac{z}{l_o}}\left(a \sin\frac{z}{l_o} + b\cos\frac{z}{l_o}\right) + e^{\frac{z}{l_o}}\left(c \sin\frac{z}{l_o} + d\cos\frac{z}{l_o}\right) \tag{11.22}$$

The l_o parameter is required because of the need to match the factor E_pI/K in the differential equation 11.21. Applying boundary condition 6a gives c = d = 0. Applying boundary conditions 6b and 6c requires that the expressions of V and M be derived using Eqs. 11.17 and 11.18:

$$y(z) = e^{-\frac{z}{l_o}}\left(a \sin\frac{z}{l_o} + b\cos\frac{z}{l_o}\right) \tag{11.23}$$

$$\frac{dy}{dz} = \frac{1}{l_o}e^{-\frac{z}{l_o}}\left(-(a+b)\sin\frac{z}{l_o} + (a-b)\cos\frac{z}{l_o}\right) \tag{11.24}$$

$$\frac{d^2 y}{dz^2} = -\frac{2}{{l_o}^2}e^{-\frac{z}{l_o}}\left(-b\sin\frac{z}{l_o} + a\cos\frac{z}{l_o}\right) \tag{11.25}$$

$$\frac{d^3 y}{dz^3} = \frac{2}{{l_o}^3}e^{-\frac{z}{l_o}}\left((a-b)\sin\frac{z}{l_o} + (a+b)\cos\frac{z}{l_o}\right) \tag{11.26}$$

$$\frac{d^4 y}{dz^4} = -\frac{4}{{l_o}^4}e^{-\frac{z}{l_o}}\left(a\sin\frac{z}{l_o} + b\cos\frac{z}{l_o}\right) \tag{11.27}$$

It can be seen from Eq. 11.23 and Eq. 11.27 that:

$$\frac{d^4 y}{dz^4} = -\frac{4}{l_o^4}\,y \tag{11.28}$$

which compared to Eq. 11.21 leads to:

$$l_o = \left(\frac{4E_pI}{K}\right)^{\frac{1}{4}} \tag{11.29}$$

Now boundary condition 6b can be written as:

$$\frac{d^2 y}{dz^2}_{@z=0} = \frac{M_o}{E_pI} = -\frac{2}{l_o^2}e^{-\frac{0}{l_o}}\left(-b\sin\frac{0}{l_o} + a\cos\frac{0}{l_o}\right) \tag{11.30}$$

and

$$a = -\frac{M_o\, l_o^2}{2E_pI} \tag{11.31}$$

Then:

$$\frac{d^3 y}{dz^3}_{@z=0} = \frac{H_o}{E_pI} = \frac{2}{l_o^3}e^{-\frac{0}{l_o}}\left((a-b)\sin\frac{0}{l_o} + (a+b)\cos\frac{0}{l_o}\right) \tag{11.32}$$

and

$$a + b = \frac{H_o l_o^3}{2E_pI} \tag{11.33}$$

so

$$b = \frac{H_o l_o^3}{2E_pI} + \frac{M_o\, l_o^2}{2E_pI} \tag{11.34}$$

Now we can build the equations for the deflection y, the slope y′, the bending moment M, the shear force V, and the line load P:

$$y(z) = e^{-\frac{z}{l_o}}\left(-\frac{M_o\, l_o^2}{2E_pI}\sin\frac{z}{l_o} + \left(\frac{H_o l_o^3}{2E_pI} + \frac{M_o\, l_o^2}{2E_pI}\right)\cos\frac{z}{l_o}\right) \tag{11.35}$$

But

$$E_pI = \frac{Kl_o^4}{4} \tag{11.36}$$

Therefore, finally:

$$y(z) = \frac{2H_o}{l_oK}e^{-\frac{z}{l_o}}\cos\frac{z}{l_o} + \frac{2M_o}{l_o^2K}e^{-\frac{z}{l_o}}\left(\cos\frac{z}{l_o} - \sin\frac{z}{l_o}\right) \tag{11.37}$$

$$y'(z) = -\frac{2H_o}{l_o^2K}e^{-\frac{z}{l_o}}\left(\cos\frac{z}{l_o} + \sin\frac{z}{l_o}\right) - \frac{4M_o}{l_o^3K}e^{-\frac{z}{l_o}}\cos\frac{z}{l_o} \tag{11.38}$$

$$M(z) = H_o l_o e^{-\frac{z}{l_o}}\sin\frac{z}{l_o} + M_o e^{-\frac{z}{l_o}}\left(\cos\frac{z}{l_o} + \sin\frac{z}{l_o}\right) \tag{11.39}$$

$$V(z) = H_o e^{-\frac{z}{l_o}}\left(\cos\frac{z}{l_o} - \sin\frac{z}{l_o}\right) - \frac{2M_o}{l_o}e^{-\frac{z}{l_o}}\sin\frac{z}{l_o} \tag{11.40}$$

where y(z) is the pile displacement at a depth z, y′(z) is the pile slope at z, M(z) is the bending moment at z, and V(z) is the shear force at z; H_o and M_o are the shear and moment at the ground surface, K is the soil spring constant, and l_o is the transfer length given by Eq. 11.29. The profiles of y(z), y′(z), M(z), and V(z) corresponding to Eq. 11.37 to 11.40 are shown in Figure 11.8 as a function of depth for a real pile.

11.4.5 Solving a Flow Problem

A typical solution to a flow problem proceeds through the following steps:

1. Zoom in at the infinitesimal element level. This element has dimensions expressed in differential lengths.
2. The knowns and unknowns (flow velocities, volumes, total head, and water stress, for example) are identified on the element, including their variation from one side of the element to the other. This variation involves derivatives expressing the rate of change of the variable in one direction over a small distance.
3. The fundamental equations are written using the knowns and unknowns identified in step 2. These equations are true for all materials. The most useful in this case is the conservation of mass equation.
4. The constitutive equations are written using the knowns and unknowns identified in step 2. These equations describe the behavior of the material involved in the flow. The main equation in this case is Darcy's law in the three dimensions.
5. All equations are regrouped into the governing differential equations.
6. The boundary conditions are expressed mathematically. These boundary conditions are usually in the form of total head or flow conditions. If the problem is a transient flow problem, the initial conditions are also stipulated.
7. The governing differential equations are solved in closed-form solutions if they are simple enough, and through numerical solutions such as the finite difference method if they are too complicated. The boundary conditions are used to define the constants involved in the solution.

11.4.6 Example of Solving a Flow Problem

One example of a flow problem is the flow of water out of a saturated soil layer when it is loaded by a long embankment (Figure 11.10*a*). Before loading, the layer is under an at-rest state of stress with a vertical effective stress σ'_{ov} and an initial water stress u_{wo}. Both σ'_{ov} and u_{wo} vary with the depth z. When the vertical stress is increased by $\Delta\sigma$ due to the embankment loading, the water stress increases by an amount called the *excess pore pressure* u_{we}. The excess pore pressure u_{we} is high at first and decreases as a function of time while the water drains out. The settlement takes place as a result of this water drainage (Figure 11.10*b*). The problem is to predict the variation of the excess pore pressure u_{we} as a function of time t and the settlement ΔH of the embankment as a function of time t.

The following simplifying assumptions are made:

a. The soil is saturated with water
b. The water is incompressible
c. The soil skeleton is linear elastic (linear stress-strain relation)
d. The soil particles are incompressible
e. Darcy's law governs the flow of water through the soil
f. The water drains at the top and at the bottom of the layer
g. The flow is in the vertical direction only

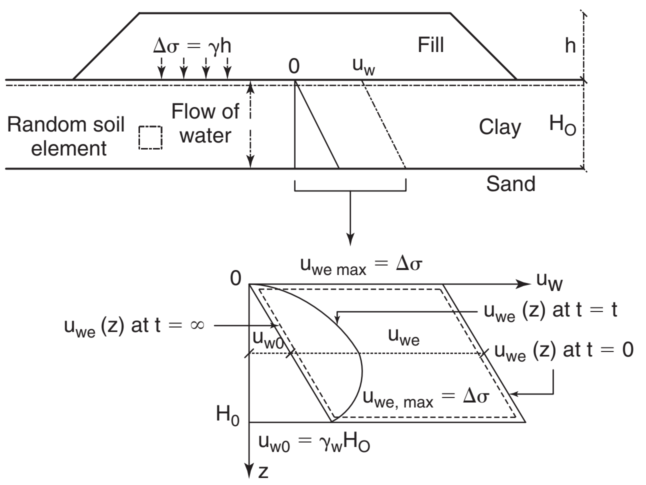


Figure 11.10 Embankment example.

h. The increase in stress $\Delta\sigma$ in the layer due to the embankment is constant within the layer
i. The excess water stress u_{we} increases by $\Delta\sigma$ when the embankment is placed
j. No lateral soil movement takes place

With these assumptions, the solution proceeds as follows:

1. Zoom in at the element level. In this case, we will select an element of soil with an elementary volume V equal to dx dy dz (Figure 11.11).

2. Considering the element of Figure 11.11, the water velocity in the z direction is v_z when it enters the element and $v_z + dv_z$ when it exits the element. It is assumed that the water does not flow in the y direction because of the plain strain condition induced by the infinitely long embankment. It is also assumed that there is no flow in the x direction because the total head gradient is much higher in the z direction than in the x direction. Because the water velocity is proportional to the total head gradient (Darcy's law), most of the water goes in the vertical direction. Also shown on the element is the change of volume dV of the element during a time dt. This change of volume corresponds to the water loss and also to the compression of the element, given that the soil is saturated.

3. The fundamental equation in this case is the conservation of mass equation expressing that, during a time dt, the volume of water entering the element plus the water squeezed out of the element due to the stress applied is equal to the volume of water exiting the element. Use is made of the flow equation (Qdt = vAdt):

$$v_z\,dx\,dy\,dt + dV = (v_z + dv_z)\,dx\,dy\,dt \tag{11.41}$$

$$\frac{dV}{Vdt} = \frac{dv_z}{dz} \tag{11.42}$$

Another fundamental equation is conservation of energy, which leads to the relationship between the total head h_t, the elevation head h_e, and the pressure head h_p. Note that the velocity head is neglected because water flows very slowly through soils:

$$h_t = h_e + h_p \tag{11.43}$$

and by differentiation

$$dh_t = dh_e + dh_p \tag{11.44}$$

Note that for the element, the elevation head h_e is constant and therefore $dh_e = 0$. Note also that, by definition:

$$h_p = \frac{u_{wo} + u_{we}}{\gamma_w} \tag{11.45}$$

Because u_{wo} is constant:

$$dh_p = \frac{du_{we}}{\gamma_w} \tag{11.46}$$

Combining the previous observation, we get:

$$dh_t = dh_p = \frac{du_{we}}{\gamma_w} \tag{11.47}$$

The effective stress in the element is:

$$\sigma' = \sigma - (u_{wo} + u_{we}) \tag{11.48}$$

By differentiation and noting that both σ and u_{wo} are constant during the loading and subsequent drainage:

$$d\sigma' = -du_{we} \tag{11.49}$$

4. The first constitutive equation describes how fast the water flows through the soil (Darcy's law):

$$v_z = ki = -k\frac{dh_t}{dz} \tag{11.50}$$

and by taking the first derivative of v_z with respect to z:

$$\frac{dv_z}{dz} = -k\frac{d^2h_t}{dz^2} \tag{11.51}$$

The second constitutive equation describes how much the soil compresses under stress (stress-strain relationship):

$$d\sigma' = M\frac{dV}{V} \tag{11.52}$$

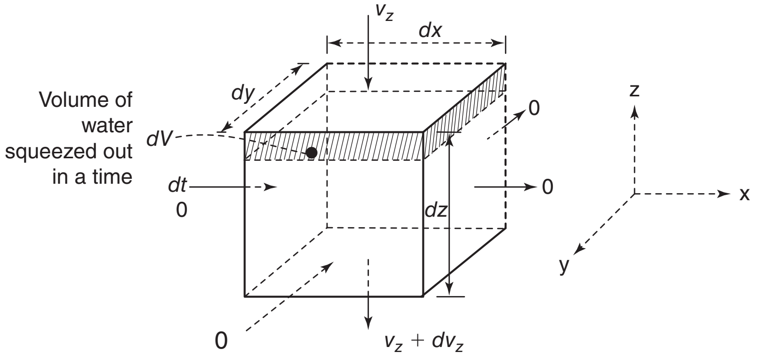


Figure 11.11 Element of soil under the embankment.

The strain in this case is the volumetric strain ($\varepsilon_v = dV/V$) and M is the constrained modulus because the soil is not allowed to expand laterally.

5. By regrouping Eqs. 11.42, 11.47, 11.49, 11.51, and 11.52, the governing differential equation is obtained:

$$\frac{1}{V}\frac{dV}{dt} = \frac{dv_z}{dz} = \frac{1}{M}\frac{d\sigma'}{dt} = -\frac{1}{M}\frac{du_{we}}{dt} = -k\frac{d^2h_t}{dz^2} = -\frac{k}{\gamma_w}\frac{d^2u_{we}}{dz^2} \tag{11.53}$$

$$\frac{du_{we}}{dt} = \frac{k\ M}{\gamma_w}\frac{d^2u_{we}}{dz^2} \tag{11.54}$$

The coefficient of consolidation c_v is expressed in m^2/s and is defined as:

$$c_v = \frac{k\,M}{\gamma_w} \tag{11.55}$$

and the governing differential equation for this problem is:

$$\frac{du_{we}}{dt} = c_v\frac{d^2u_{we}}{dz^2} \tag{11.56}$$

6. Now we need to organize the space and time boundary conditions. The space boundary conditions state that the excess water stress u_{we} at the ground surface is zero because the water can drain freely at that location. Also, the excess water stress u_{we} is zero at the bottom of the layer because the water can drain freely at that depth:

$$u_{we@z=0} = 0 \quad \text{at any time } t \tag{11.57}$$

$$u_{we@z=H_o} = 0 \quad \text{at any time } t \tag{11.58}$$

The time boundary conditions state that the excess water stress u_{we} is equal to the increase in total stress $\Delta\sigma$ at time $t = 0$ and then equal to 0 at time t = infinity:

$$u_{we@t=0} = \Delta\sigma \quad \text{at any depth } z \tag{11.59}$$

$$u_{we@t=\infty} = 0 \quad \text{at any depth } z \tag{11.60}$$

7. This is the step where we solve the governing differential equation (11.56) and apply the boundary conditions. To simplify the mathematical process, it is convenient to use the following transformation into dimensionless variables:

$$Z = \frac{z}{H_d} \tag{11.61}$$

$$U = 1 - \frac{u_{we}}{u_{we(\max)}} \tag{11.62}$$

$$T = \frac{c_v t}{{H_d}^2} \tag{11.63}$$

where z is the depth below ground surface, H_d is the longest drainage path, U is the degree of consolidation at depth z and time t, u_{we} is the excess water stress at depth z and time t, $u_{we(\max)}$ is the maximum excess water stress at time $t = 0$ at any depth taken as equal to $\Delta\sigma$, T is the time factor, and t is the time. Note that the maximum drainage length is equal to the layer thickness H_o if the water can only drain on one side (top or bottom of the layer), but is equal to $0.5H_o$ if the water can drain at both ends. With these transformed variables, the GDE (Eq. 11.56) becomes:

$$\frac{dU}{dT} = \frac{d^2U}{dZ^2} \tag{11.64}$$

The solution to this partial differential equation, together with the space and time boundary conditions, is a Fourier series expansion of the form (Terzaghi 1943):

$$U = 1 - \sum_{m=0}^{m=\infty} \frac{2}{M}\sin(MZ)\ \exp(-M^2T) \quad \text{with} \quad M = \frac{\pi}{2}(2m+1) \tag{11.65}$$

The graphical representation of U as a function of Z and T is shown in Figure 11.12.

It is also useful to define the average degree of consolidation U_{av}:

$$U_{av} = 1 - \frac{\int_0^H u_{we}dz}{\int_0^H u_{we\max}dz} = 1 - \sum_{m=0}^{m=\infty}\frac{2}{M^2}\exp(-M^2T) \quad \text{with} \quad M = \frac{\pi}{2}(2m+1) \tag{11.66}$$

The average degree of consolidation represents the ratio of the area under the excess water stress profile at time t over the same area at time $t = 0$ (Figure 11.13).

The graphical representation of U_{av} as a function of T is shown in Figures 11.14 and 11.15.

Equation 11.52 indicates that the volumetric strain dV/V in the layer is linearly proportional to the increase in effective stress $d\sigma'$. Because the soil is assumed not to move laterally, the volumetric strain is also the vertical strain dH/H. Also, because the total stress is constant, the increase in effective stress is equal to the decrease in excess water stress (Eq. 11.49). Therefore, the average degree of consolidation U_{av} can be rewritten as:

$$U_{av} = 1 - \frac{u_{we}(average)}{u_{we(\max)}(average)} = \frac{u_{we(\max)}(av) - u_{we}(av)}{u_{we(\max)}(av)} = \frac{\Delta\sigma'(av)}{\Delta\sigma'_{\max}(av)} = \frac{M\frac{\Delta H}{H}}{M\frac{\Delta H_{\max}}{H}} = \frac{\Delta H}{\Delta H_{\max}} \tag{11.67}$$

This means that U_{av} represents the settlement of the structure divided by the maximum settlement. In contrast, because T is a function of the time t, the complete settlement vs.

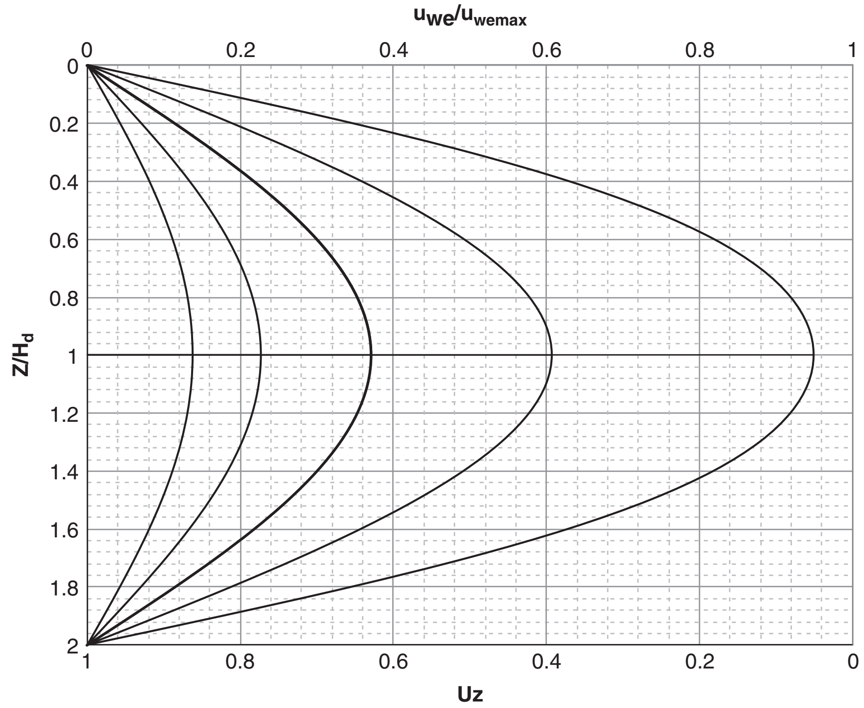


Figure 11.12 Degree of consolidation and excess water stress as a function of depth and time.

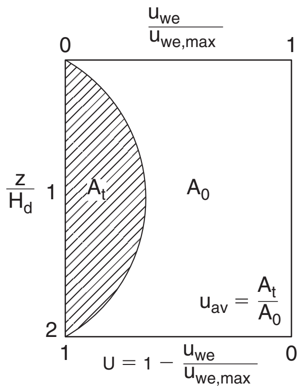


Figure 11.13 Definition of the average degree of consolidation.

time curve ($\Delta H/\Delta H_{max}$ vs. t) can be constructed by using the U vs. T curves. An example is shown in Figure 11.16. The following equations have been proposed to approximate Eq. 11.66:

For $U_{av} < 0.6$:

$$T = \frac{\pi}{4}{U_{av}}^2 \quad or \quad \Delta H = \Delta H_{\max}\frac{2}{H_d}\sqrt{\frac{c_v t}{\pi}} \tag{11.68}$$

For $U_{av} > 0.6$:

$$T = -0.933\log(1 - U_{av}) - 0.085 \quad or$$

$$\Delta H = \Delta H_{\max}\left(1 - 10^{-\left(\frac{\frac{c_v t}{H_d{}^2}+0.085}{0.933}\right)}\right) \tag{11.69}$$

11.5 NUMERICAL SIMULATION METHODS

Numerical solutions typically require the use of a computer because of the complexity and amount of the mathematics involved. They tend to work as follows. The soil space or the foundation is discretized into many small elements (linear, surface, or volume). The points forming the geometry of these elements are the *nodes*. The unknowns (e.g., stresses, strains, displacements, forces, moments, flow velocity, head) have to be calculated at all the nodes. The governing differential equations are transformed into algebraic equations that must be written as many times as there are nodes in the discretized soil space. This usually yields a large number of equations organized in matrix form. From this matrix equation, the unknowns must be extracted and solved for; this often requires an inversion process of the main matrix and can only be done by computers. The output of these numerical solutions is in the form of large tables that give the calculated values of the unknowns at each node within the soil mass. Numerical methods (Jing and Hudson 2002; Bobet 2010) include the finite difference method (FDM), the finite element method (FEM), the boundary element method (BEM), and the discrete element method (DEM).

11.5.1 Finite Difference Method

The finite difference method is very powerful in solving differential equations. The main idea is to replace the differential equation by incremental algebraic equations. This is done by using algebraic expressions of the derivatives of the functions involved in the governing differential equation. In Figure 11.17, the function y(z) has values $y_{i-2}, \ldots, y_{i+2}$

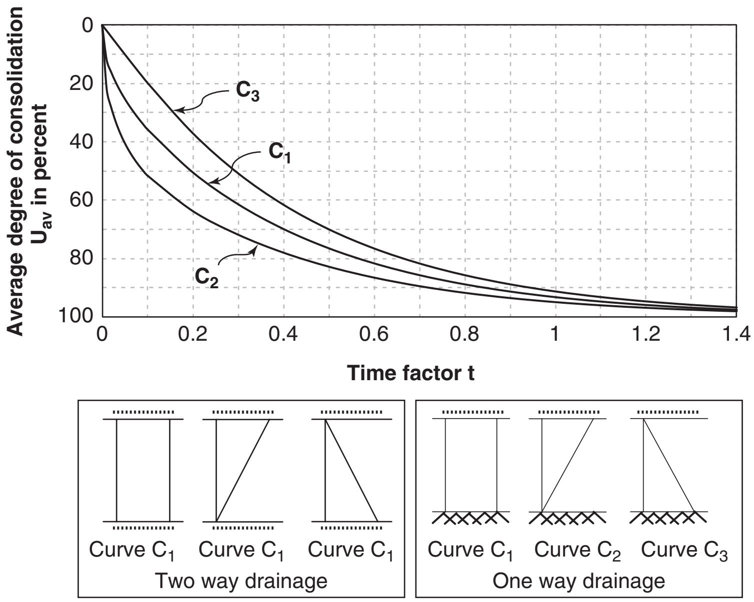


Figure 11.14 Average degree of consolidation U_{av} vs. time factor T on natural scale for different stress increase profiles.

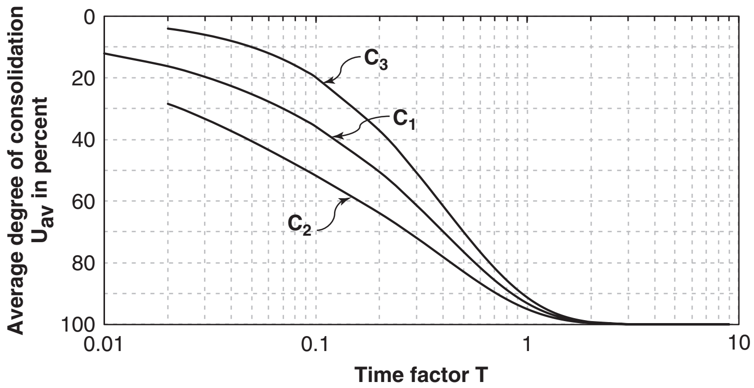


Figure 11.15 Average degree of consolidation U_{av} vs. time factor T on semilog scale.

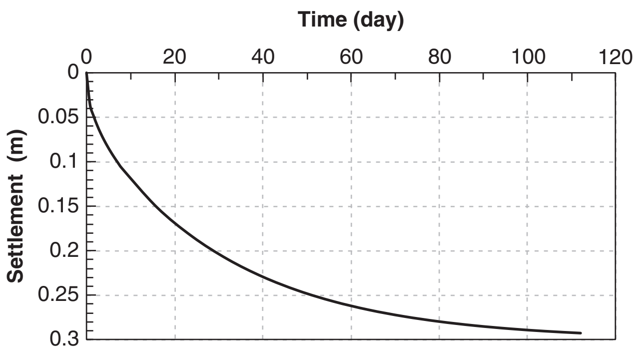


Figure 11.16 Example of settlement vs. time curve obtained from the U_{av} vs. T curve.

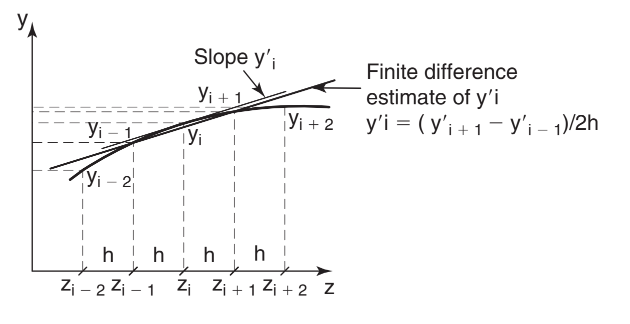


Figure 11.17 Derivative expressed by the central finite difference formulation.

corresponding to values of z equal to $z_{i-2}, \ldots, z_{i+2}$ respectively. The values of z are separated by a constant distance h. The first derivative of y evaluated at $z = z_i$ can be expressed as the slope of the tangent at z_i:

$$\frac{dy}{dz}_{@z_i} = y_i' = \frac{y_{i+1} - y_{i-1}}{2h} \tag{11.70}$$

This expression is called the *central difference expression* of the derivative, as it balances the influence of both sides of the function with respect to point i (Figure 11.17).

The forward difference would be:

$$\frac{dy}{dz}_{@z_i} = y_i' = \frac{y_{i+1} - y_i}{h} \tag{11.71}$$

and the backward difference would be:

$$\frac{dy}{dz}_{@z_i} = y_i' = \frac{y_i - y_{i-1}}{h} \tag{11.72}$$

The second derivative can be expressed using the same approach. Indeed, the second derivative is the first derivative of the first derivative. This gives the following expression, using a forward and a backward formulation for y′ to end up with a centered formulation of y″.

$$\frac{d^2y}{dz^2}_{@z_i} = \frac{dy'}{dz}_{@z_i} = y_i'' = \frac{y_{i+1}' - y_i'}{h} = \frac{\dfrac{y_{i+1} - y_i}{h} - \dfrac{y_i - y_{i-1}}{h}}{h} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \tag{11.73}$$

Using the same process, the third derivative can be expressed as:

$$\frac{d^3y}{dz^3}_{@z_i} = \frac{dy''}{dz}_{@z_i} = y_i''' = \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2h^3} \tag{11.74}$$

and the fourth derivative:

$$\frac{d^4y}{dz^4}_{@z_i} = \frac{dy'''}{dz}_{@z_i} = y_i'''' = \frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4} \tag{11.75}$$

A typical finite difference solution proceeds through the following steps:

1. The structure or soil mass involved is broken down into small elements of chosen finite dimensions. Each element has a number and each node at the boundaries of these elements has a number.
2. The knowns and unknowns (loads, displacements, stresses, strains, velocities, and heads, for example) are identified for each node and given a subscript corresponding to the node number.
3. The governing differential equation is written in algebraic finite difference form as many times as there are nodes in the structure or soil mass.
4. The space and time boundary conditions are also expressed in terms of the algebraic expressions of the variables.
5. All equations are regrouped into a matrix equation.
6. The matrix equation is solved to extract the unknown quantities. This usually requires that the matrix be inverted. Considering the size of these matrices, a computer is required for this step.
7. The solution is presented in the form of a table that gives the sought quantities at all the nodes.

11.5.2 Examples of Finite Difference Solutions

The example is to solve the governing differential equation by using the FDM for the problem of section 11.4.4: a pile subjected to a horizontal force H_o and an overturning moment M_o applied at the ground surface. The GDE is (Eq. 11.21):

$$y + \frac{E_p I}{K} \frac{d^4y}{dz^4} = 0 \tag{11.76}$$

The solution to this problem is the function y(z) describing the horizontal deflection of the pile as a function of the depth z. The process consists of the following steps:

1. The pile is discretized into elements as shown in Figure 11.18. The displacement at node i is y_i. There are a total of $n + 1$ unknown values of the horizontal displacement y (y_0 to y_n).

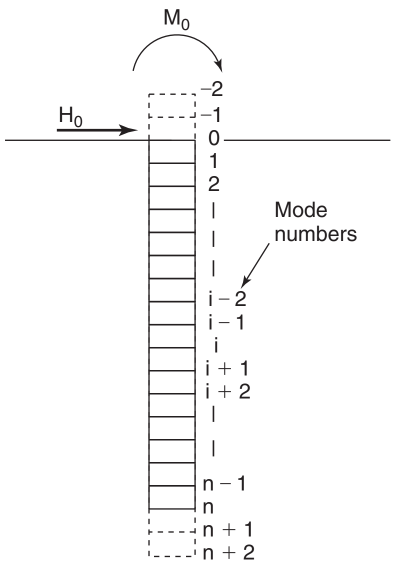


Figure 11.18 Pile discretized into numbered elements and nodes.

2. The GDE is written for each node using the expressions of the derivatives presented in section 11.5.1:

$$y_i + \frac{E_p I}{K}\frac{d^4 y}{dz^4}_{@z_i} = y_i + \frac{E_p I}{K}\left(\frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4}\right) = 0 \quad (11.77)$$

or

$$y_{i+2} - 4y_{i+1} + \left(6 + \frac{Kh^4}{E_p I}\right) y_i - 4y_{i-1} + y_{i-2} = 0 \quad (11.78)$$

Because there are n + 1 nodes along the pile (0 to n), Eq. 11.78 theoretically could be written n + 1 times. That is not the case here, because Eq. 11.78 involves 5 nodal values of the displacement y, so in fact Eq. 11.78 can only be written n − 3 times. Because there are n + 1 values of the horizontal displacement y, we are missing four equations. Can the boundary conditions help us?

3. The boundary conditions are that the horizontal load is H_o at the ground surface and zero at the bottom of the pile and that the moment is M_o at the ground surface and zero at the bottom of the pile. To express these four boundary conditions, additional and fictitious nodes are created. These are nodes −1 and −2 at the top of the pile and nodes n + 1 and n + 2 at the bottom of the pile (Figure 11.18). The fact that the moment is M_o at the ground surface and zero at the bottom of the pile is written as:

$$M_{@z=0} = E_p I \frac{d^2 y}{dz^2}_{@z=0} = E_p I\left(\frac{y_1 - 2y_0 + y_{-1}}{h^2}\right) = M_o \quad (11.79)$$

$$M_{@z=L} = E_p I \frac{d^2 y}{dz^2}_{@z=L} = E_p I\left(\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}\right) = 0 \quad (11.80)$$

The fact that the shear force is H_o at the ground surface and zero at the bottom of the pile is written as:

$$V_{@z=0} = E_p I \frac{d^3 y}{dz^3}_{@z=0} = E_p I\left(\frac{y_2 - 2y_1 + 2y_{-1} - y_{-2}}{h^3}\right) = H_o \quad (11.81)$$

$$V_{@z=L} = E_p I \frac{d^3 y}{dz^3}_{@z=L} = E_p I\left(\frac{y_{n+2} - 2y_{n+1} + 2y_{n-1} - y_{n-2}}{h^3}\right) = 0 \quad (11.82)$$

The boundary conditions lead to four new equations, but we have also created four new unknowns (y_{-2}, y_{-1}, y_{n+1}, and y_{n+2}). Thus, the new count is n + 5 unknowns and n + 1 equations. The extra four equations are created because the additional nodes allow the GDE to be written four more times. Now we have n + 5 unknowns and n + 5 equations. These n + 5 equations are written in matrix form as:

$$[K][Y] = [C] \quad (11.83)$$

where $[K]$ is an n + 5 by n + 5 matrix of the coefficients of the y_i values in the algebraic equations corresponding to the GDE and the boundary conditions, $[Y]$ is a n + 5 long column matrix of the y values (y_{-2} to y_{n+2}), and $[C]$ is a n + 5 column matrix of the constants in the n + 5 GDE equations. Because the y values are the unknowns to be solved for, the $[K]$ matrix must be inverted and the solution is:

$$[Y] = [K]^{-1}[C] \quad (11.84)$$

This solution is illustrated by solving for the deflection and pressure distribution for a retaining wall as shown in Figure 11.19.

The units for this problem are not stated, because as long as the units are consistent the solution is independent of the units. The bending stiffness of the wall is 10,000 and the element height is 1. The soil reaction curves at each node must be prepared (Figure 11.20). The reaction curves represent the relationship between the line load P on the wall and the horizontal displacement y of the wall. A number of simplifying assumptions will be made to facilitate the solution.

At node 0, the reaction curve shows that the line load P_o is equal to zero for all y values:

$$P_0 = 0 \quad (11.85)$$

At node 1, the reaction curve is taken as a constant equal to 60. In fact, the reaction curve at node 1 should reflect the mobilization of the active pressure if the wall moves away from the soil and of the passive pressure if the wall moves into the soil. However, because the active pressure is the pressure that will be mobilized considering the problem, and because the active pressure requires very little movement to be mobilized, it is reasonable to assume that the movement will be large enough that the line load on the wall will

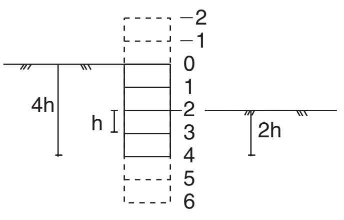


Figure 11.19 Wall discretized into numbered elements and nodes.

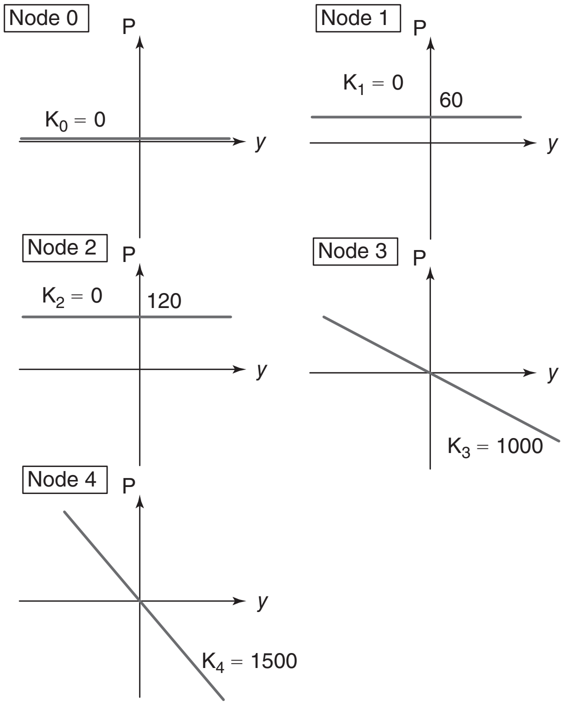


Figure 11.20 Reaction curves for the wall at each node.

correspond to the active pressure for a large range of lateral displacements:

$$P_1 = 60 \tag{11.86}$$

The same reasoning applies to the reaction curve at node 2, where the pressure is twice as high and the line load is equal to 120:

$$P_2 = 120 \tag{11.87}$$

At node 3, the reaction curve is as shown in Figure 11.20. It indicates that the line load is linearly proportional to the lateral displacement of the wall. Again, this reaction curve should reflect the influence of the active and passive pressures on both sides of the wall. The simplifying assumption in this case is that the passive resistance dominates the behavior of the wall below the excavation level. Knowing that it takes much larger displacements to mobilize the passive resistance than the active pressure, it is likely that below the excavation depth the wall will be in the range of displacement where a linear assumption is reasonable. Therefore, the reaction curve at node 3 is characterized by:

$$P_3 = -K_3 y = -1000y \tag{11.88}$$

The reason for the minus sign is that when the deflection increases to the right ($\Delta y > 0$), the line load decreases ($\Delta P < 0$). The same reasoning applies for the reaction curve at node 4, but with a higher stiffness K_4, as node 4 is deeper in the soil and therefore likely stiffer:

$$P_4 = -K_4 y = -1500y \tag{11.89}$$

As you can see, this problem has been greatly simplified compared to the real problem. The reason is that without such simplifications, the mathematics would become quite complicated.

Now the problem is clearly defined and we can proceed with the step-by-step procedure:

1. The wall has been discretized as shown in Figure 11.19.
2. The line loads and the horizontal displacements are numbered from 0 at the top of the wall to 4 at the bottom of the wall.
3. The GDE is the same as the one for the horizontally loaded pile (Eq. 11.21):

$$P - E_p I \frac{d^4 y}{dz^4} = 0 \tag{11.90}$$

Expressed in finite difference formulations, it becomes:

$$P_i - E_p I \left(\frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4} \right) = 0 \tag{11.91}$$

4. The boundary conditions are that the moment and the shear force are zero at both ends of the wall. This requires adding two fictitious nodes at both ends of the wall, as shown in Figure 11.19. The equations for the shear and moment are:

$$M = E_p I \left(\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \right) = 0 \tag{11.92}$$

$$V = E_p I \left(\frac{y_{n+2} - 2y_{n+1} + 2y_{n-1} - y_{n-2}}{h^3} \right) = 0 \tag{11.93}$$

5. Now all the equations can be written and assembled in a matrix:

$$\begin{bmatrix} -1 & 2 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 6.1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 6.15 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} y_{-2} \\ y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.006 \\ 0.012 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{11.94}$$

Table 11.1 Results of the Finite Difference Solution for the Simulated Wall

Node number (depth)	Horizontal deflection y of wall	Line load P on wall
1	2.72	0
2	1.95	60
3	1.18	120
4	0.42	−420
5	−0.32	480

The first two equations in the matrix equation are the two equations for the moment and shear boundary conditions at the top of the wall; then there are five GDEs written at five nodes; and the last two equations are the two equations for the moment and shear boundary conditions at the bottom of the wall. Now it is time to invert the matrix to obtain the $[Y]$ matrix as the solution to the problem.

6. The computer does that for us, and the deflections y at each node are calculated. The line loads on the wall are obtained by using the relationship between the load and the deflection given by the reaction curves of Eqs. 11.85 to 11.89. The results of this finite difference solution are presented in Table 11.1.

The deflection profile y(z) and the line load profile P(z) are shown in Figure 11.21. The profile P(z) shows that the wall is in horizontal equilibrium because the area under the left side of the profile is equal to the areas under the right side of the profile. This is the way it should be, as horizontal equilibrium was one of the fundamental equations we used in setting up the solution.

11.5.3 Finite Element Method

The finite element method (FEM) is another powerful numerical method to solve geotechnical problems (Clough 1960; Desai and Abel 1972; Zienkiewicz et al. 2005). The output, like most numerical methods, will be in the form of tables giving the unknown quantities at discrete locations in the soil mass. The general steps in developing a solution to a finite element simulation are as follows:

1. Discretize the soil mass into finite elements connected by nodes.
2. Choose the functions describing the variation of the unknowns across each element and between its nodes.
3. Write the strain-displacement equations.
4. Write the stress-strain equations for the soil.
5. Derive the equations governing the behavior of the soil element.
6. Assemble the element equations into the global matrix equation.
7. Introduce the boundary conditions into the global matrix equation.
8. Solve the global matrix equation for the unknowns.

Each step is discussed in more detail here.

1. *Discretize the soil mass into finite elements connected by nodes.* In this step the soil mass is subdivided into a number of small elements (Figure 11.22). The sides of the elements intersect at the nodes. Each element and each node is numbered in sequence. The size of the elements is influenced by a number of factors, including how fast the stress changes from one point to another of the soil mass (stress gradient). Various shapes of elements exist: lines, triangles, quadrilaterals, parallelepipeds, or brick elements. One of the big advantages of the FEM is that irregular boundaries do not present a big problem.

2. *Choose the functions describing the variation of the unknowns across each element and between its nodes.* These are called *interpolation functions* or *shape functions*. The solution of the FEM will give the answers at the nodes (Fig. 11.23), but we need to be able to calculate the unknowns everywhere in the mass to establish the general equations. The interpolation functions relate for example the displacement anywhere in the element to the displacements at the nodes. These interpolation functions are typically in the form of polynomials. It is more convenient, however, to write them in the following form:

$$u_x(x, y) = H_1 u_{x1} + H_2 u_{x2} + H_3 u_{x3} + H_4 u_{x4} = \sum_{i=1}^{\#nodes} H_i u_{xi} \tag{11.95}$$

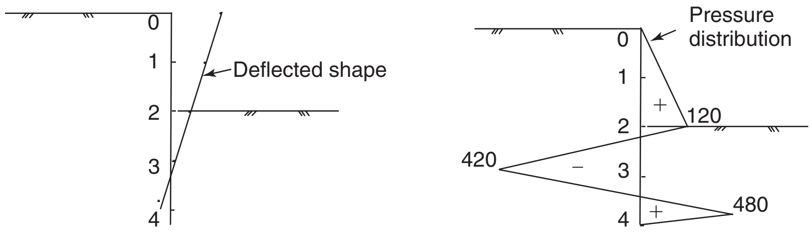


Figure 11.21 Wall deflection and line load.

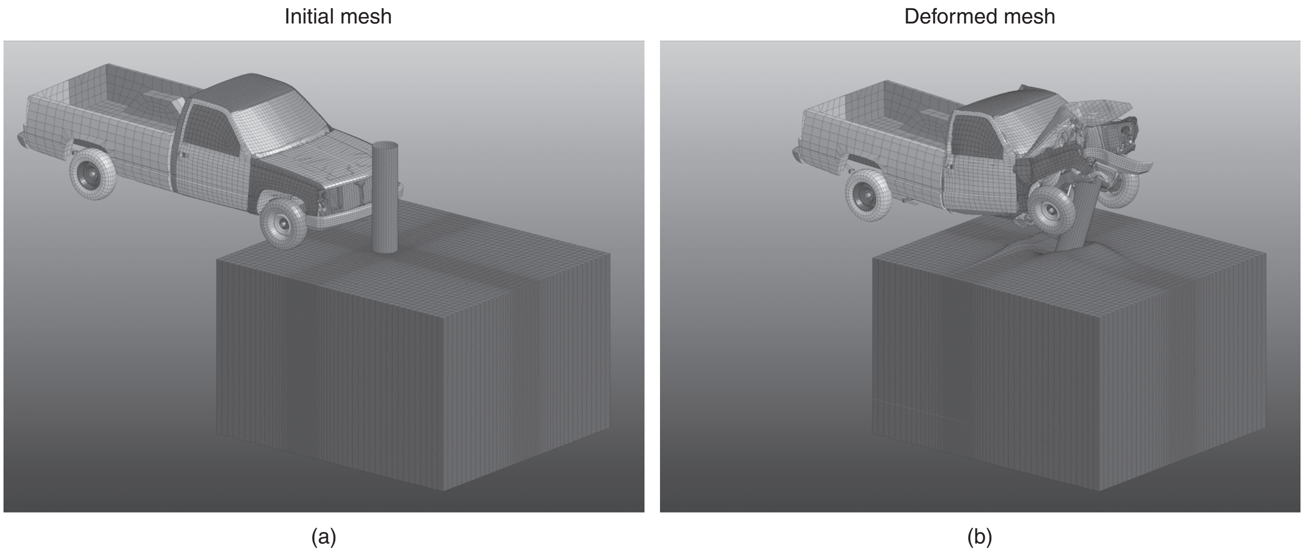


Figure 11.22 Example of finite element mesh: (*a*) Initial mesh. (*b*) Deformed mesh.

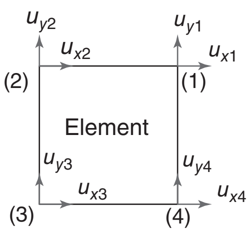


Figure 11.23 Element in plane strain.

$$u_y(x, y) = H_1 u_{y1} + H_2 u_{y2} + H_3 u_{y3} + H_4 u_{y4} = \sum_{i=1}^{\#nodes} H_i u_{yi} \tag{11.96}$$

where $u_x(x, y)$ is the displacement in the x direction of any point within the element with coordinates x and y, u_{xi} is the displacement in the x direction of node i, $u_y(x, y)$ is the displacement in the y direction of any point within the element with coordinates x and y, u_{yi} is the displacement in the y direction of node i, and the H_is are the interpolation functions. Equations 11.95 and 11.96 would be for an element with four nodes and plain strain condition in the z direction. They describe the shape of the displacement surface across the element.

In matrix form:

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} H_1 & 0 & H_2 & 0 & H_3 & 0 & H_4 & 0 \\ 0 & H_1 & 0 & H_2 & 0 & H_3 & 0 & H_4 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

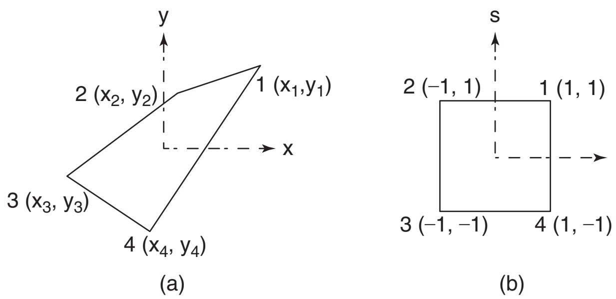


Figure 11.24 Finite element representation in real and natural coordinates: (*a*) Real coordinates. (*b*) Natural coordinates.

or

$$[u] = [H][u_i] \tag{11.97}$$

Note that the $[u]$ matrix is the matrix of displacements as variables, whereas the $[u_i]$ matrix is the matrix of displacements at the nodes. The $[H]$ matrix is called the *shape function matrix*. Note also that these matrices are written for the element and not for the entire soil mass.

Regarding the coordinates x and y, it is more convenient to use natural coordinates (Figure 11.24). As can be seen, regardless of the element's original shape, the transformation leads to a set of coordinates varying from −1 to +1 along each face. Also, the element is square. The interpolation functions for a four-node element in natural coordinates are:

$$H_1 = \frac{1}{4}(1 + r)(1 + s) \tag{11.98}$$

$$H_2 = \frac{1}{4}(1 - r)(1 + s) \tag{11.99}$$

$$H_3 = \frac{1}{4}(1-r)(1-s) \tag{11.100}$$

$$H_4 = \frac{1}{4}(1+r)(1-s) \tag{11.101}$$

where r and s are the natural coordinates (Figure 11.24).

In the general case, coordinates can be expressed in terms of interpolation functions as follows:

$$x = \sum_{i=1}^{\#nodes} H_i x_i \tag{11.102}$$

$$y = \sum_{i=1}^{\#nodes} H_i y_i \tag{11.103}$$

3. *Write the strain-displacement equations.* There are typically 9 equations: 3 force equilibrium equations and 6 constitutive equations linking the stresses to the strains. The other equations are the 3 moment equilibrium equations, but they simply lead to the fact that shear stresses on perpendicular planes are equal and in opposite directions so they have already been used up. However, there are 15 unknowns: 6 stresses, 6 strains, and 3 displacements. So we are short 6 equations. What saves the day is that the 6 strains are defined on the basis of the 3 displacements, so this adds 6 strain-displacement equations. In the end we have 15 unknowns and 15 equations.

Recalling Eq. 11.95, the normal strain in the x direction is ε_{xx}:

$$\varepsilon_{xx} = \frac{\partial u_x(x,y)}{\partial x} = \left[\frac{\partial H}{\partial x}\right][u_{xi}] \tag{11.104}$$

The same equation holds true for ε_{yy}:

$$\varepsilon_{yy} = \frac{\partial u_y(x,y)}{\partial y} = \left[\frac{\partial H}{\partial y}\right][u_{yi}] \tag{11.105}$$

For the shear strain γ_{xy}, the equation becomes:

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \left[\frac{\partial H}{\partial y}\right][u_{xi}] + \left[\frac{\partial H}{\partial x}\right][u_{yi}] \tag{11.106}$$

or, in matrix form:

$$[\varepsilon] = [B][u_i] \tag{11.107}$$

where $[\varepsilon]$ is the strain matrix (3×1 vector for a two-dimensional problem), $[B]$ is the matrix containing the derivatives of the interpolation functions H_i (3×8 for a two-dimensional problem), and $[u_i]$ is the matrix of nodal displacements (8×1 for a two-dimensional problem).

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial H_1}{\partial x} & 0 & \frac{\partial H_2}{\partial x} & 0 & \frac{\partial H_3}{\partial x} & 0 & \frac{\partial H_4}{\partial x} & 0 \\ 0 & \frac{\partial H_1}{\partial y} & 0 & \frac{\partial H_2}{\partial y} & 0 & \frac{\partial H_3}{\partial y} & 0 & \frac{\partial H_4}{\partial y} \\ \frac{\partial H_1}{\partial y} & \frac{\partial H_1}{\partial x} & \frac{\partial H_2}{\partial y} & \frac{\partial H_2}{\partial x} & \frac{\partial H_3}{\partial y} & \frac{\partial H_3}{\partial x} & \frac{\partial H_4}{\partial y} & \frac{\partial H_4}{\partial x} \end{bmatrix} \times \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} \tag{11.108}$$

Because the interpolation functions H_i are defined in natural coordinates, the derivatives $\partial H_i/\partial x$ and $\partial H_i/\partial y$ can be related to $\partial H_i/\partial r$ and $\partial H_i/\partial s$ through the Jacobian matrix [J] as follows:

$$\begin{bmatrix} \frac{\partial H_i}{\partial x} \\ \frac{\partial H_i}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial H_i}{\partial r} \\ \frac{\partial H_i}{\partial s} \end{bmatrix} \tag{11.109}$$

where the Jacobian matrix $[J]$ is described as follows:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \tag{11.110}$$

Recalling Eq. 11.102 and Eq. 11.103, the components of this Jacobian matrix are written as follows:

$$\frac{\partial x}{\partial r} = \sum \frac{\partial H_i}{\partial r}.x_i \tag{11.111}$$

$$\frac{\partial x}{\partial s} = \sum \frac{\partial H_i}{\partial s}.x_i \tag{11.112}$$

$$\frac{\partial y}{\partial r} = \sum \frac{\partial H_i}{\partial r}.y_i \tag{11.113}$$

$$\frac{\partial y}{\partial s} = \sum \frac{\partial H_i}{\partial s}.y_i \tag{11.114}$$

4. *Write the stress-strain equations for the soil.* These are the constitutive equations, the ones that are specific to the soil involved. One of the simplest constitutive laws is the case where the stresses are linearly related to the strains (elasticity):

$$[\sigma] = [C][\varepsilon] \tag{11.115}$$

where $[\sigma]$ is the stress matrix, which is a 3×1 matrix for a two-dimensional problem and a 6×1 matrix for a three-dimensional problem; $[\varepsilon]$ is the strain matrix, which is a 3×1 matrix for a two-dimensional problem and a 6×1 matrix for a three-dimensional problem; and $[C]$ is the soil stiffness matrix, which is a 3×3 matrix for a two-dimensional problem and a 6×6 matrix for a three-dimensional problem. In elasticity and for three dimensions, Eq. 11.115 is written as:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{bmatrix} = \frac{E}{(1-2\nu)(1+\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}-\nu & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2}-\nu & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}-\nu\end{bmatrix} \times \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\end{bmatrix} \quad (11.116)$$

In the case of two dimensions, the C matrix becomes:

$$C = \frac{E}{(1-2\nu)(1+\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1}{2}-\nu\end{bmatrix} \quad (11.117)$$

5. *Derive the equations governing the behavior of the soil element.* You may have noticed that we have not yet written any fundamental equations such as the equilibrium equations. We do that in this step, but it is done by using another technique called a *variational principle* that includes the equilibrium equations. More specifically, we use the minimum total potential energy principle (MTPE) along with the virtual work technique. The MTPE principle states that the actual displacement solution of a deformable body is the solution that renders the TPE functional Π minimum, meaning that the derivative of Π is equal to zero. The two types of energies involved in the TPE are the work done by the external forces W and the internal strain energy U of the deformable soil mass. The TPE is minimum (system in equilibrium) when the change in work done by the external forces δW is equal to the change in internal strain energy δU:

$$\delta\Pi = 0 \quad \rightarrow \quad \delta U = \delta W \quad (11.118)$$

The increment of virtual internal strain energy δU for a bar is:

$$\delta U = \sigma A\,\delta\varepsilon\,dx = \delta\varepsilon\,\sigma\,dV \quad (11.119)$$

where σ is the axial stress, A is the cross-sectional area, $\delta\varepsilon$ is a virtual axial strain, and dx and dV are an infinitesimal length and volume of the bar. This is generalized for the three-dimensional soil element as:

$$\delta U = \int_V \delta[\varepsilon]^T[\sigma]dV \quad (11.120)$$

The change in work is calculated by assuming that the soil mass is subjected to virtual small displacements (virtual work). The increment in virtual external work δW for a bar is:

$$\delta W = F_{body}\delta u + F_{boundary}\delta u = bdV\delta u + tdA\delta u \quad (11.121)$$

where F_{body} is the body force, δu is a virtual displacement, F is the boundary force, b is the body force density (e.g., unit weight), t is the boundary tractions (e.g., pressure), and dV and dA are an infinitesimal volume and area of the bar. This is generalized for the three-dimensional soil element as:

$$\delta W = \int_V \delta[u]^T[b]dV + \int_V \delta[u]^T[t]dA \quad (11.122)$$

Then the principle of virtual work states that the expressions in Eq. 11.120 and 11.122 are equal:

$$\int_V \delta[\varepsilon]^T[\sigma]dV = \int_V \delta[u]^T[b]dV + \int_V \delta[u]^T[t]dA \quad (11.123)$$

Using Eqs. 11.107 and 11.115, we get:

$$\int_V \delta[u]^T[B]^T[C][B]\delta[u]dV = \delta[u]^T\left(\left[F_{body}\right] + \left[F_{boundary}\right]\right) = \delta[u]^T[F] \quad (11.124)$$

The element stiffness matrix K^e is defined as:

$$[K^e] = \int_V [B]^T[C][B]dV \quad (11.125)$$

To calculate the integral on the right side of Eq. 11.125, we select integration points where all the components of the B and C matrices are evaluated. In the special case of a plane strain problem, the components of the stiffness matrix can be reduced to the following expression:

$$t\iint_{Area} f_{mn}(x,y)dxdy = \sum_{i=1}^{2}\sum_{j=1}^{2} t \times f_{mn}(r_i, s_j).\det J.w_i.w_j \quad (11.126)$$

where t is the thickness of the element (1 in plane strain cases), $f_{mn}(x, y)$ is the function found at the intersection of the m row and n column of the B^tCB matrix of Eq. 11.125 expressed in real coordinates, $f_{mn}(r, s)$ is the same function but expressed in natural coordinates, i and j are the running indices identifying the location of the integration point, r_i and s_j are the natural coordinates of the chosen integration points on the element, w_i and w_j are the weighting factors that depend on the number and location of the integration points, and det J is the determinant of the Jacobian matrix. In the general case, the thickness is not a constant and must be calculated at each integration point by using the interpolation functions (see problem 11.7). Figure 11.25 shows an example of four integration points.

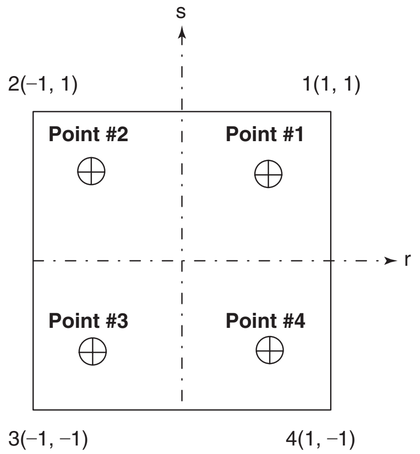


Figure 11.25 Four integration points.

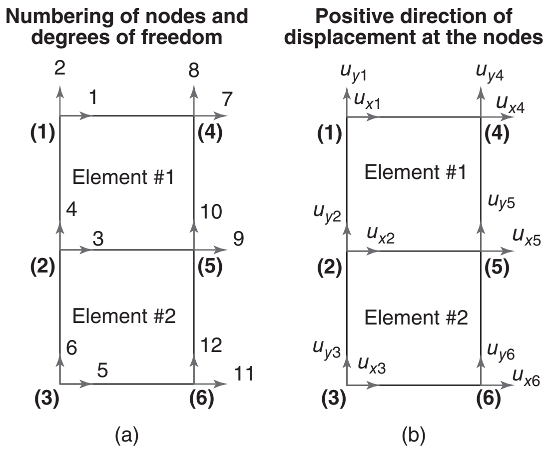


Figure 11.26 Two 2D FEM elements and numbering the degrees of freedom: (*a*) Number of nodes and degrees of freedom. (*b*) Positive direction of displacements at nodes.

Because Eq. 11.124 must be satisfied for any kinematically admissible virtual displacement field $[u]$, we must have:

$$[K^e][u] = [F] \tag{11.127}$$

In Eq. 11.127, most of the displacements u are unknown and most of the forces are either zero or known. This is the equation governing the behavior of the element. If the element were a spring, K would be the spring constant, but in the case of the three-dimensional element, K is a square matrix.

6. *Assemble the element equations into the global matrix equation.* Equation 11.126 is the equation for one element. There are as many such matrix equations as there are elements in the mesh. They must be assembled to form the stiffness matrix for the entire soil mass. To do so, we specify that the body must remain continuous during the deformation. This means that each node can have only one displacement vector common to all elements containing this node. At each node, we also have only one body force and one external force value. The following example illustrates how the global matrix is assembled.

Consider the two elements of Figure 11.26. The stiffness matrices for the 2 elements and their assembly into the global matrix of the soil mass of the 2 elements are shown in Figure 11.27. As can be seen, the coefficients of the individual element matrices are labeled K^i_{jk}. The index i designates the element number, j refers to the node number corresponding to the force F_j, and k refers to the number of the node where a displacement u_k contributes an additional displacement at node j. With these definitions for the indices, the stiffness coefficients for adjacent elements are simply added when they refer to the same j and k values while coming from different elements i. This simple example is extended to all nodes in the mesh to form the global stiffness matrix $[K]$. Then the global governing equation for the entire soil mass is:

$$[K][u] = [F] \tag{11.128}$$

Figs. 11.26 and 11.27 show how to assemble the global matrix for two four-node elements.

7. *Introduce the boundary conditions into the global matrix equation.* Equation 11.128 describes how the soil mass will behave in general terms. The boundary conditions make the problem specific. These boundary conditions (also called *constraints*) are given in the way of specified values of displacements, forces, temperatures, or any other parameters that affect the problem. In dynamics, these conditions involve the same types of parameters, but all of them are associated with a specific time. Examples of boundary conditions include requiring no movement at a node ($u^i_x = u^i_y = u^i_z = 0$), no external force at a node ($F^i_x = F^i_y = F^i_z = 0$), or movement at a node allowed only in one direction, or a single force applied at a node. The specified values of displacement and forces go directly into the $[u]$ and $[F]$ matrices. Of course, for problems other than deformation problems, the boundary conditions are different and can be in terms of specified flow velocities, heat flux, and so on.

8. *Solve the global matrix equation for the unknowns.* The matrix equation to be solved is:

$$[K][u] = [F] \tag{11.129}$$

In a three-dimensional problem, the $[K]$ matrix is a 3n × 3n matrix where n is the number of nodes; the $[u]$ matrix is a 3n × 1 matrix; and the $[F]$ matrix is also a 3n × 1 matrix. The reason it is 3n is that there are 3 directions at each node with 3 associated displacements and 3 associated forces. The displacement vectors and the force vectors will be:

$$\begin{bmatrix} u^1_x \\ u^1_y \\ u^1_z \\ . \\ . \\ . \\ u^n_x \\ u^n_y \\ u^n_z \end{bmatrix} \quad \begin{bmatrix} F^1_x \\ F^1_y \\ F^1_z \\ . \\ . \\ . \\ F^n_x \\ F^n_y \\ F^n_z \end{bmatrix}$$

Stiffness Matrix of Element #1

$$
\begin{bmatrix}
K^1_{77} & K^1_{78} & K^1_{71} & K^1_{72} & K^1_{73} & K^1_{74} & K^1_{79} & K^1_{7,10} \\
 & K^1_{88} & K^1_{81} & K^1_{82} & K^1_{83} & K^1_{84} & K^1_{89} & K^1_{8,10} \\
 & & K^1_{11} & K^1_{12} & K^1_{13} & K^1_{14} & K^1_{19} & K^1_{1,10} \\
 & & & K^1_{22} & K^1_{23} & K^1_{24} & K^1_{29} & K^1_{2,10} \\
 & & & & K^1_{33} & K^1_{34} & K^1_{39} & K^1_{3,10} \\
 & SYM & & & & K^1_{44} & K^1_{49} & K^1_{4,10} \\
 & & & & & & K^1_{99} & K^1_{9,10} \\
 & & & & & & & K^1_{10,10}
\end{bmatrix}
$$

(a)

Stiffness Matrix of Element #2

$$
\begin{bmatrix}
K^2_{99} & K^2_{9,10} & K^2_{93} & K^2_{94} & K^2_{95} & K^2_{96} & K^2_{9,11} & K^2_{9,12} \\
 & K^2_{10,10} & K^2_{10,3} & K^2_{10,4} & K^2_{10,5} & K^2_{10,6} & K^2_{10,11} & K^2_{10,12} \\
 & & K^2_{33} & K^2_{34} & K^2_{35} & K^2_{36} & K^2_{3,11} & K^2_{3,12} \\
 & & & K^2_{44} & K^2_{45} & K^2_{46} & K^2_{4,11} & K^2_{4,12} \\
 & & & & K^2_{55} & K^2_{56} & K^2_{5,11} & K^2_{5,12} \\
 & SYM & & & & K^2_{66} & K^2_{6,11} & K^2_{6,12} \\
 & & & & & & K^2_{11,11} & K^2_{11,12} \\
 & & & & & & & K^2_{12,12}
\end{bmatrix}
$$

(b)

Assembled Global Stiffness Matrix

$$
\begin{array}{c}
1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12
\end{array}
\begin{bmatrix}
K^1_{11} & K^1_{12} & K^1_{13} & K^1_{14} & 0 & 0 & K^1_{17} & K^1_{18} & K^1_{19} & K^1_{1,10} & 0 & 0 \\
 & K^1_{22} & K^1_{23} & K^1_{24} & 0 & 0 & K^1_{27} & K^1_{28} & K^1_{29} & K^1_{2,10} & 0 & 0 \\
 & & K^1_{33}+K^2_{33} & K^1_{34}+K^2_{34} & K^2_{35} & K^2_{36} & K^1_{37} & K^{11}_{38} & K^1_{39}+K^2_{39} & K^1_{3,10}+K^2_{3,10} & K^2_{3,11} & K^2_{3,12} \\
 & & & K^1_{44}+K^2_{44} & K^2_{45} & K^2_{46} & K^1_{47} & K^1_{48} & K^1_{49}+K^2_{49} & K^1_{4,10}+K^2_{4,10} & K^2_{4,11} & K^2_{4,12} \\
 & & & & K^2_{55} & K^2_{56} & 0 & 0 & K^2_{59} & K^2_{5,10} & K^2_{5,11} & K^2_{5,12} \\
 & & & & & K^2_{66} & 0 & 0 & K^2_{69} & K^2_{6,10} & K^2_{6,11} & K^2_{6,12} \\
 & & & & & & K^1_{77} & K^1_{78} & K^1_{79} & K^1_{7,10} & 0 & 0 \\
 & & & & & & & K^1_{88} & K^1_{89} & K^1_{8,10} & 0 & 0 \\
 & & SYM & & & & & & K^1_{99}+K^2_{99} & K^1_{9,10}+K^2_{9,10} & K^2_{9,11} & K^2_{9,12} \\
 & & & & & & & & & K^1_{10,10}+K^2_{10,10} & K^2_{10,11} & K^2_{10,12} \\
 & & & & & & & & & & K^2_{11,11} & K^2_{11,12} \\
 & & & & & & & & & & & K^2_{12,12}
\end{bmatrix}
$$

(c)

Figure 11.27 Assembling the global stiffness matrix: (*a*) Stiffness of matrix of element #1. (*b*) Stiffness matrix of element #2. (*c*) Assembled global stiffness matrix.

In these vectors most of the unknowns are the displacements at the nodes, except for the displacement boundary conditions. However, most of the forces at the nodes are known and are zero. Remember that we are talking about the external forces, not the internal forces. The soil experiences stresses all over its mass, but the external forces at the nodes are zero except at supports or at boundary conditions. This distinction between internal forces and external forces is critically important and can be illustrated as follows.

Consider a simply supported beam resting on rigid supports at both ends. Place a heavy load in the center of the beam. If the beam is in equilibrium, the external moment is zero everywhere along the beam, but the internal moment (bending moment) is significant along most of the beam. You know the displacement at both ends (zero), but you do not know the force (support reaction). Along the rest of the beam, you do not know the displacement, but you know the force, which is zero except in the center where the force is equal to the applied external load.

The same principle applies to the finite element method and Eq. 11.128. The displacement matrix $[u]$ is largely unknown and the external force matrix $[F]$ is largely known. Therefore, because we want to know $[u]$, it will be necessary to invert the stiffness matrix $[K]$ to get the displacements:

$$[u] = [K]^{-1}[F] \tag{11.130}$$

Because the global stiffness matrix is very large, this operation can require a lot of time when the mesh has many elements. Techniques for optimizing this operation have been developed in mathematics, including matrix banding. This banding is affected by the numbering of the nodes and it is always desirable to ensure that neighboring nodes do not have very different numbers.

One issue arises with a boundary condition that specifies a displacement: say, $u_i = \delta$. An example may be a support where no movement is allowed. In this case, the displacement is zero but the force is unknown. To solve the matrix problem (Eq. 11.130), all unknowns must be in the displacement matrix and all values in the force matrix must be known. To satisfy this mathematical need, the following trick is applied. The known displacement is entered in the displacement matrix as an unknown u_i. The corresponding force is entered as the value of the known displacement δ to form the modified force matrix F' and the corresponding row (row i) in the K matrix is set to be all zeroes except for the diagonal value, which is 1. The same applies to column i, because the matrix is symmetrical. That way the i^{th} equation simply says that $u_i = \delta$. This is repeated for all such cases and gives rise to a new matrix K′. The matrix K' is inverted and all displacements at all nodes are found by:

$$[u] = [K']^{-1}[F'] \tag{11.131}$$

Then the complete force matrix F is found as the matrix product Ku:

$$[F] = [K][u] \tag{11.132}$$

Once the displacement matrix is obtained, the strains and stresses can be obtained by using the strain-displacements relationships (Eq. 11.107) and the stress-strain relationships (Eq. 11.115).

11.5.4 Example of Finite Element Solution

Use the FEM to solve the deformation field for a test performed on an elastic soil. The height of the sample is 0.1 m, the width is 0.05 m, and the length is infinite. The major principal stress is 300 kPa and the minor principal stress is 100 kPa. The modulus is 40 MPa and the Poisson's ratio is 0.35. Consider a plane strain geometry and use two four-noded elements. Use numerical integration with four points to construct the stiffness matrix.

Step 1: Discretize the soil mass into finite elements connected by nodes

The elements are shown in Figure 11.28. The element dimensions are $a = 0.05$ m and $b = 0.05$ m; the soil properties are $E = 40{,}000$ kPa and $\mu = 0.35$.

Step 2: Choose the interpolation functions in natural coordinates

Recalling Eqs. 11.98 to 11.101, these functions are considered:

$$H_1 = \frac{1}{4}(1+r)(1+s) \tag{11.133}$$

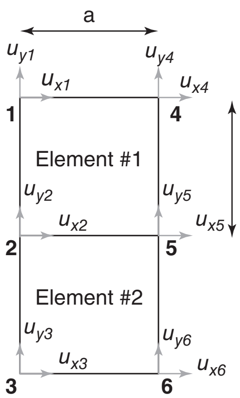


Figure 11.28 Triaxial test in plane strain.

$$H_2 = \frac{1}{4}(1-r)(1+s) \tag{11.134}$$

$$H_3 = \frac{1}{4}(1-r)(1-s) \tag{11.135}$$

$$H_4 = \frac{1}{4}(1+r)(1-s) \tag{11.136}$$

Step 3: Write the strain-displacement equations

$$[\varepsilon] = [B][u_i] \tag{11.137}$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial H_1}{\partial x} & 0 & \frac{\partial H_2}{\partial x} & 0 & \frac{\partial H_3}{\partial x} & 0 & \frac{\partial H_4}{\partial x} & 0 \\ 0 & \frac{\partial H_1}{\partial y} & 0 & \frac{\partial H_2}{\partial y} & 0 & \frac{\partial H_3}{\partial y} & 0 & \frac{\partial H_4}{\partial y} \\ \frac{\partial H_1}{\partial y} & \frac{\partial H_1}{\partial x} & \frac{\partial H_2}{\partial y} & \frac{\partial H_2}{\partial x} & \frac{\partial H_3}{\partial y} & \frac{\partial H_3}{\partial x} & \frac{\partial H_4}{\partial y} & \frac{\partial H_4}{\partial x} \end{bmatrix} \times \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} \tag{11.138}$$

Constructing the [B] Matrix.

a. Calculate the inverse of the Jacobian matrix used in the transformation from natural coordinates to real coordinates.

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{b}{2} \end{bmatrix} = \begin{bmatrix} 0.025 & 0 \\ 0 & 0.025 \end{bmatrix} \tag{11.139}$$

Therefore:

$$\det J = 6.25 * 10^{-4}$$

and

$$J^{-1} = \left(\frac{1}{\det J}\right) \cdot \begin{bmatrix} \frac{b}{2} & 0 \\ 0 & \frac{a}{2} \end{bmatrix} = \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix} \tag{11.140}$$

b. Obtain the relation between the derivatives of the interpolation functions in real coordinates and in natural coordinates:

$$\begin{bmatrix} \frac{\partial H_i}{\partial x} \\ \frac{\partial H_i}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial H_i}{\partial r} \\ \frac{\partial H_i}{\partial s} \end{bmatrix} \tag{11.141}$$

$$\begin{bmatrix} \dfrac{\partial H}{\partial x} \\ \dfrac{\partial H}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{b}{2} & 0 \\ 0 & \dfrac{a}{2} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial H}{\partial r} \\ \dfrac{\partial H}{\partial s} \end{bmatrix} = \begin{bmatrix} \dfrac{b}{2} \cdot \dfrac{\partial H}{\partial r} \\ \dfrac{a}{2} \cdot \dfrac{\partial H}{\partial s} \end{bmatrix} \tag{11.142}$$

c. Select the natural coordinates of integration points r and s for a four-node element. This information is found in most FEM books (e.g., Zienkiewicz 2005).

$$r = \begin{bmatrix} \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{3}} \\ -\dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \end{bmatrix} \tag{11.143}$$

$$s = \begin{bmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \\ -\dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{3}} \end{bmatrix} \tag{11.144}$$

d. Compute the components of the matrix [B] at the four integration points (Figure 11.29):

Point #1. Recalling Eqs. 11.133 to 11.136, the derivatives of the interpolation function are:

$$\frac{\partial H}{\partial r} = \left[\frac{1}{4}(1+s) \quad -\frac{1}{4}(1+s) \quad -\frac{1}{4}(1+s) \quad \frac{1}{4}(1+s)\right] \tag{11.145}$$

$$\frac{\partial H}{\partial s} = \left[\frac{1}{4}(1+r) \quad \frac{1}{4}(1+r) \quad -\frac{1}{4}(1+r) \quad -\frac{1}{4}(1+r)\right] \tag{11.146}$$

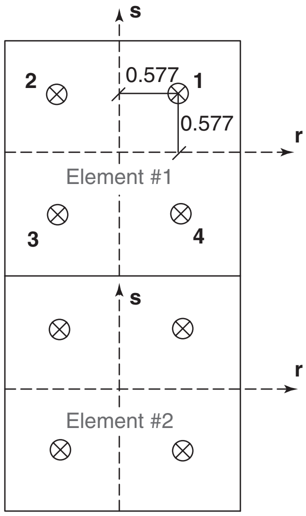


Figure 11.29 The integration points.

For integration point #1. the natural coordinates are:

$$r = \frac{1}{\sqrt{3}}$$
$$s = \frac{1}{\sqrt{3}}$$
$$\frac{\partial H}{\partial r} = \left[\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \times \frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right)\right]$$
$$= \left[0.394 \quad -0.394 \quad -0.105 \quad 0.105\right] \tag{11.147}$$
$$\frac{\partial H}{\partial s} = \left[\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad \frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right)\right]$$
$$= \left[0.394 \quad 0.105 \quad -0.105 \quad -0.394\right] \tag{11.148}$$

Point #2.

$$r = -\frac{1}{\sqrt{3}}$$
$$s = \frac{1}{\sqrt{3}}$$
$$\frac{\partial H}{\partial r} = \left[\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right) \times \frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right)\right]$$
$$= \left[0.394 \quad -0.394 \quad -0.105 \quad 0.105\right] \tag{11.149}$$
$$\frac{\partial H}{\partial s} = \left[\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right) \quad \frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right)\right]$$
$$= \left[0.105 \quad 0.394 \quad -0.394 \quad -0.105\right] \tag{11.150}$$

Point #3.

$$r = -\frac{1}{\sqrt{3}}$$
$$s = -\frac{1}{\sqrt{3}}$$
$$\frac{\partial H}{\partial r} = \left[\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \times \frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right)\right]$$
$$= \left[0.105 \quad -0.105 \quad -0.394 \quad 0.394\right] \tag{11.151}$$

$$\frac{\partial H}{\partial s} = \left[\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right) \quad \frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \right.$$
$$\left. \times -\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right)\right]$$
$$= \begin{bmatrix}0.105 & 0.394 & -0.394 & -0.105\end{bmatrix} \tag{11.152}$$

Point #4.

$$r = \frac{1}{\sqrt{3}}$$
$$s = -\frac{1}{\sqrt{3}}$$

$$\frac{\partial H}{\partial r} = \left[\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \right.$$
$$\left. \times \frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right)\right]$$
$$= \begin{bmatrix}0.105 & -0.105 & -0.394 & 0.394\end{bmatrix} \tag{11.153}$$

$$\frac{\partial H}{\partial s} = \left[\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad \frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right) \right.$$
$$\left. -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right)\right]$$
$$= \begin{bmatrix}0.394 & 0.105 & -0.105 & -0.394\end{bmatrix} \tag{11.154}$$

Now Eqs. 11.138, 11.140, and Eqs. 11.145 to 154 are combined to create the B matrix. For example, the top right element of matrix B is $40 \times 0.394 = 15.773$.

Step 4. Write the stress-strain equations for the soil and obtain the constitutive matrix

Recalling Eq. 11.116:

$$C = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}\begin{bmatrix}1 & \frac{\mu}{(1-\mu)} & 0\\ \frac{\mu}{(1-\mu)} & 1 & 0\\ 0 & 0 & \frac{(1-2\mu)}{2(1-\mu)}\end{bmatrix}$$
$$= 10^4 * \begin{bmatrix}6.419 & 3.457 & 0\\ 3.457 & 6.419 & 0\\ 0 & 0 & 1.481\end{bmatrix} \tag{11.156}$$

Step 5. Derive the equations governing the behavior of the soil element

Recalling Eqs. 11.124 and 11.126:

$$[K^e] = \int_V [B]^T[C][B]dV \tag{11.157}$$

$$[K^e][u] = [F] \tag{11.158}$$

and recalling the numerical integration from Eq. 11.125:

$$K_e = \int_v B^T C B dv = \sum_{i=1}^{2}\sum_{j=1}^{2} B_{ij}^T C_{ij} B_{ij} \det J.w_i.w_j.t \tag{11.159}$$

$$B_{\substack{i=1\\j=1}} = \begin{bmatrix}15.773 & 0 & -15.773 & 0 & -4.226 & 0 & 4.226 & 0\\ 0 & 15.773 & 0 & 4.226 & 0 & -4.226 & 0 & -15.773\\ 15.773 & 15.773 & 4.226 & -15.773 & -4.226 & -4.226 & -15.773 & 4.226\end{bmatrix}$$
$$B_{\substack{i=1\\j=2}} = \begin{bmatrix}15.773 & 0 & -15.773 & 0 & -4.226 & 0 & 4.226 & 0\\ 0 & 4.226 & 0 & 15.773 & 0 & -15.773 & 0 & -4.226\\ 4.226 & 15.773 & 15.773 & -15.773 & -15.773 & -4.226 & -4.226 & 4.226\end{bmatrix}$$
$$B_{\substack{i=2\\j=1}} = \begin{bmatrix}4.226 & 0 & -4.226 & 0 & -15.773 & 0 & 15.773 & 0\\ 0 & 4.226 & 0 & 15.773 & 0 & -15.773 & 0 & -4.226\\ 4.226 & 4.226 & 15.773 & -4.226 & -15.773 & -15.773 & -4.226 & 15.773\end{bmatrix}$$
$$B_{\substack{i=2\\j=2}} = \begin{bmatrix}4.226 & 0 & -4.226 & 0 & -15.773 & 0 & 15.773 & 0\\ 0 & 15.773 & 0 & 4.226 & 0 & -4.226 & 0 & -15.773\\ 15.773 & 4.226 & 4.226 & -4.226 & -4.226 & 15.773 & -15.773 & 15.773\end{bmatrix} \tag{11.155}$$

For two-point Gauss integration, w_i, and w_j are equal to 1. In the case of plane strain, the thickness t of the elements is taken as 1. Therefore, the stiffness matrix for each element is as follows:

$$K^e = 10^4 \times \begin{bmatrix} 2.63 & 1.23 & -1.89 & 0.49 & -1.32 & -1.23 & 0.58 & -0.49 \\ & 2.63 & -0.49 & 0.57 & -1.23 & -1.31 & 0.49 & -1.89 \\ & & 2.63 & -1.23 & 0.58 & 0.49 & -1.31 & 1.23 \\ & & & 2.63 & 0.49 & -1.89 & 1.23 & -1.31 \\ & & & & 2.63 & 1.23 & -1.89 & 0.49 \\ & SYM & & & & 2.63 & -0.49 & 0.58 \\ & & & & & & 2.63 & -1.23 \\ & & & & & & & 2.63 \end{bmatrix} \tag{11.160}$$

Step 6. Assemble the element equations into the global matrix equation

The global stiffness matrix equation K_g is based on the connected degrees of freedom shown in Figure 11.28, and is assembled as:

$$K_g = 10^4 \times \begin{bmatrix} 2.63 & -1.23 & 0.58 & 0.49 & 0 & 0 & -1.89 & 0.49 & -1.32 & 1.23 & 0 & 0 \\ & 2.63 & -0.49 & -1.89 & 0 & 0 & 0.49 & 0.57 & 1.23 & -1.32 & 0 & 0 \\ & & 5.27 & 0 & 0.58 & 0.49 & -1.32 & -1.23 & 0.39 & 0 & -1.32 & 1.23 \\ & & & 5.27 & -0.49 & -1.89 & -1.23 & -1.32 & 0 & 1.15 & 1.23 & -1.32 \\ & & & & 2.63 & 1.23 & 0 & 0 & -1.32 & -1.23 & -1.89 & 0.49 \\ & & & & & 2.63 & 0 & 0 & -1.23 & -1.32 & -0.49 & 0.57 \\ & & & & & & 2.63 & 1.23 & 0.57 & -0.49 & 0 & 0 \\ & & & & & & & 2.63 & 0.49 & -1.89 & 0 & 0 \\ & & SYM & & & & & & 5.27 & 0 & 0.57 & -0.49 \\ & & & & & & & & & 5.27 & 0.49 & -1.89 \\ & & & & & & & & & & 2.63 & -1.23 \\ & & & & & & & & & & & 2.63 \end{bmatrix} \tag{11.161}$$

Step 7. Introduce the boundary conditions into the global matrix equation

Referring to Figure 11.28, the degrees of freedom of the triaxial sample at nodes (3) and (6) should be constrained in both directions. Moreover, nodes (1) and (4) can only deform vertically. Thus, the rows and columns associated with those degrees of freedom should be zero.

Step 8. Solve the global matrix equation for the unknowns

The triaxial sample is subjected to a confining pressure σ_3 and a vertical pressure σ_1. For this problem, $\sigma_3 = 100$ kPa, and $\sigma_1 = 300$ kPa. The force components applied at the nodes due to the confining pressure and the vertical stress are:

At nodes 2 and 5

$$P_{horizontal} = \sigma_3 \times \frac{b}{2} \times 2 = 100 \times \frac{0.05}{2} \times 2 = 5\ \text{kN/m} \tag{11.162}$$

At nodes 1 and 4

$$P_{vertical} = \sigma_1 \times \frac{a}{2} = 300 \times \frac{0.05}{2} = 7.5\ \text{kN/m} \tag{11.163}$$

Now the force matrix is assembled as:

$$F' = \begin{bmatrix} F'_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F'_{x3} \\ F'_{y3} \\ F'_{x4} \\ F_{y4} \\ F_{x5} \\ F_{y5} \\ F'_{x6} \\ F'_{y6} \end{bmatrix} = \begin{bmatrix} 0 \\ -7.5 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ -7.5 \\ -5 \\ 0 \\ 0 \\ 0 \end{bmatrix} (\text{kN}) \tag{11.164}$$

Note that in fact the forces F_{x1}, F_{x3}, F_{y3}, F_{x4}, F_{x6}, and F_{y6} are actually unknown, but they are set equal to zero because of the mathematical trick mentioned at the end of section 11.5.3 and because the corresponding displacements are zero. Note also that the matrix K' will have zeroes on rows corresponding to the displacement boundary conditions, except the diagonal, which will have a 1. The same applies to the corresponding columns. The 12×12 matrix K' is inverted by the computer and the displacement vector u is found as $K'^{1} \times F'$:

$$u = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \\ u_{x5} \\ u_{y5} \\ u_{x6} \\ u_{y6} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5134 \\ -0.0428 \\ -0.2567 \\ 0 \\ 0 \\ 0 \\ -0.5134 \\ 0.0428 \\ -0.2567 \\ 0 \\ 0 \end{bmatrix} \text{(mm)} \tag{11.165}$$

Then we can obtain the force vector by $K \times u$:

$$F = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \\ F_{x5} \\ F_{y5} \\ F_{x6} \\ F_{y6} \end{bmatrix} = \begin{bmatrix} 3.62 \\ -7.5 \\ 5 \\ 0 \\ 3.62 \\ 7.5 \\ -3.62 \\ -7.5 \\ -5 \\ 0 \\ -3.62 \\ 7.5 \end{bmatrix} \text{(kN)} \tag{11.166}$$

11.5.5 Boundary Element Method

The FDM and the FEM model the continuum by discretizing the entire body of the soil mass. The boundary element method (BEM) (Crouch and Starfield 1983; Brebbia et al. 1984) is different in that it models the continuum by discretizing only the boundaries of the continuum (Figure 11.30). The mathematical technique for the BEM consists of replacing the governing differential equations valid over the entire soil mass by integral equations that consider only the boundary values. If the soil mass extends to infinity, the FEM requires a boundary at some distance from the imposed loading or deformation. No artificial boundaries are needed in the BEM; this is an advantage of the BEM over the FEM and the FDM. Another advantage is that for a 3D problem, only the boundary surface need be discretized; this reduces the problem from a 3D problem (volume) to a 2D (surface) problem. This is attractive if the boundary surface is small compared to the volume of soil to be simulated. The BEM is particularly well suited to addressing static continuum problems with small boundary-to-volume ratios, with elastic behavior, and with stresses or displacements applied to the boundaries (Bobet 2010).

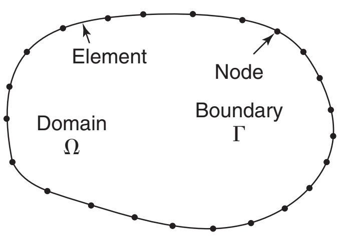


Figure 11.30 Discretization with the boundary element method.

11.5.6 Discrete Element Method

The discrete element method (DEM), also called the distinct element method (Cundall and Strack 1979; Ghaboussi and Barbosa 1990) differs from the finite element method in that it does not assume that the soil mass is a continuum; rather, it treats the soil mass as an assembly of particles of various sizes (Figure 11.31). Obviously, this is an improvement that gets us closer to reality for soils. The DEM addresses three issues during the calculations: the representation of the contacts, the representation of the solid material, and the detection and revisions of the contacts during deformation. Each soil particle is subjected to the forces transmitted at the contacts by adjacent particles and to its own body forces (gravity). The representation of the contact is usually handled through the use of spring and dashpot models (Figure 11.32). The springs have a stiffness k_n for the normal force and k_s for the shear force. The dashpots have damping factor c_n for the normal force and c_s for the shear force.

The solution proceeds in small time steps and the finite difference method (FDM) is used in the solution (see sections 11.5.1 and 11.5.2). The steps are:

1. The state of all the particles in the soil mass is known at time t. This includes contact forces, displacements, velocities, and accelerations.

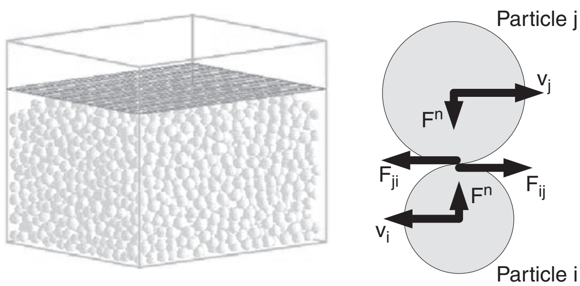


Figure 11.31 Distinct element method: (a) DEM domain. (b) Particle interaction. (*a*: Courtesy of C. Couroyer PhD Thesis, 2000, University of Surrey, Guildford, Surrey, UK.)

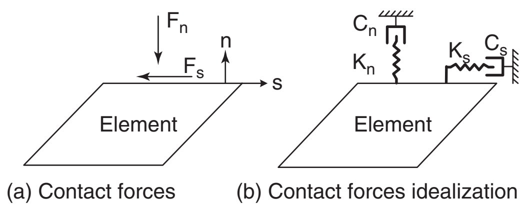


Figure 11.32 DEM element and idealized contact models.

2. A time increment Δt is considered. This time increment has to be small enough for the solution to be numerically stable. The following condition can be used (Hart et al. 1998):

$$\Delta t < 0.1\sqrt{\frac{m_{\min}}{2k_{\max}}} \tag{11.167}$$

where m_{min} is the smallest particle mass and k_{max} is the largest stiffness of all contacts. In the DEM, time comes into play for both dynamic and static problems. Even in a static problem, it takes time for the deformations to take place.

3. The differential equations of motion are then used to obtain the displacement and rotation of the particles at time $t + \Delta t$. The accelerations of the particles are calculated assuming that the forces and moments are constant over Δt:

$$\ddot{u}_i^t = \frac{\sum F_i^t}{m_i} \tag{11.168}$$

$$\ddot{\theta}_i^t = \frac{\sum M_i^t}{I_i} \tag{11.169}$$

where $\ddot{u}_i^t$ and $\ddot{\theta}_i^t$ are the linear and angular acceleration of particle i at time t respectively, F_i^t and M_i^t are the resultant force and resultant moment on particle i at time t respectively, and m_i and I_i are the mass and the moment of inertia of particle i respectively. Then the velocities of the particles are calculated assuming that the accelerations are constants over $-\Delta t/2$ and $+\Delta t/2$:

$$\dot{u}_i^{t+\frac{\Delta t}{2}} = \dot{u}_i^{t-\frac{\Delta t}{2}} + \ddot{u}_i^t \Delta t \tag{11.170}$$

$$\dot{\theta}_i^{t+\frac{\Delta t}{2}} = \dot{\theta}_i^{t-\frac{\Delta t}{2}} + \ddot{\theta}_i^t \Delta t \tag{11.171}$$

where $\dot{u}_i$ and $\dot{\theta}_i$ are the linear and angular velocities respectively. Then the displacements and rotations of the particles are calculated assuming that the velocities are constant over Δt:

$$u_i^{t+\Delta t} = u_i^t + \dot{u}_i^{t+\frac{\Delta t}{2}} \Delta t \tag{11.172}$$

$$\theta_i^{t+\Delta t} = \theta_i^t + \dot{\theta}_i^{t+\frac{\Delta t}{2}} \Delta t \tag{11.173}$$

where u_i and θ_i are the displacement and the rotation respectively.

4. The equations representing the behavior of the contacts are then used to update the forces and moments. Figure 11.32 gives a common model for the contact normal forces F_n and the contact shear forces F_s:

$$F_n^{t+\Delta t} = k_n \Delta u_n^{\Delta t} + c_n \Delta \dot{u}_n^{\Delta t} \tag{11.174}$$

$$F_s^{t+\Delta t} = k_s \Delta u_s^{\Delta t} + c_s \Delta \dot{u}_s^{\Delta t} \tag{11.175}$$

where k_s and k_n are the stiffnesses in the normal and shear directions respectively, c_n and c_s are the damping factors in the normal and shear directions respectively, Δu_n and Δu_s are the incremental displacements in the normal and shear directions respectively, and $\Delta \dot{u}_n$ and $\Delta \dot{u}_s$ are the incremental velocities in the normal and shear directions. The shear force F_s cannot exceed the shear strength of the soil, so the following condition is checked at each increment:

$$F_s^{t+\Delta t} \leq c' A_c + F_n^{t+\Delta t} \tan \varphi' \tag{11.176}$$

where c' is the effective stress cohesion intercept, A_c is the contact area, and φ' is the effective stress friction angle.

5. The cycle of calculations in 1 through 4 is repeated many times. The final solution is obtained when a chosen tolerance in the difference between two consecutive sets of calculations is achieved.

The DEM is quite efficient with these calculations. The calculations are done through a straightforward process solving one equation at a time, and no large matrix has to be inverted. Where the computing power and storage capacity are required is in recognizing and keeping track of all the contacts between elements from one step to the next. The DEM is very useful for soils and fissured rock masses.

11.6 PROBABILITY AND RISK ANALYSIS

All the methods discussed so far are deterministic in nature, which means that they give one precise answer for one problem. Considering the fact that uncertainty exists in every step taken in arriving at a solution, it makes sense to calculate the uncertainty associated with the solution or predicted value. This is called the *probabilistic approach*.

11.6.1 Background

This subsection reviews some basic concepts of statistics because they are useful in the steps described for the general procedure. When many values of a certain variable are collected—such as the undrained shear strength s_u of a clay at a site and at a given depth, for example—they will vary and can be organized in a table from the lowest to the highest value (Table 11.2). These values s_{ui} can then be regrouped into sets of increments or ranges, as shown in Table 11.2. A *histogram* is a plot of the number of times the variable is found in each increment as a function of the value of the variable (Figure 11.33a and b). Note that a different histogram is generated if a different increment magnitude is selected.

A distinction is made between the variable X and the values of that variable x_i. The mean μ of a set of values

Table 11.2 Values of Undrained Shear Strength and Histogram Input

Undrained strength value (kPa)	Number of values (10 kPa increments)	Number of values (20 kPa increments)
49	1 value between 40 and 50	1 value between 40 and 60
62	2 values between 60 and 70	6 values between 60 and 80
67		
73	4 values between 70 and 80	
75		
76		
79		
81	3 values between 80 and 90	4 values between 80 and 100
85		
86		
93	1 value between 90 and 100	
105	1 value between 100 and 110	1 value between 100 and 120

$(x_1, x_2, x_3, \ldots, x_n)$ is defined as follows and is called the *expected value* E(X) of X:

$$\mu = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n} = E(X) \qquad (11.177)$$

The standard deviation σ is a measure of the deviation of the values with respect to the mean. It is given by:

$$\sigma = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_n - \mu)^2}{n-1}} = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n-1}} \qquad (11.178)$$

The reason for using the squares is that the difference $(x_i - \mu)$ can be positive or negative and might cancel out during summation, thereby not giving a true rendition of the scatter around the mean. We could have used the absolute values of the difference, but that is not what is chosen in practice. Also, the reason for using $(n - 1)$ rather than n is the fact that only $(n - 1)$ values of $(x_i - \mu)$ are independent, as the sum of the n values of $(x_i - \mu)$ is equal to zero. This is called the *Bessel correction*. The square of the standard deviation σ^2 is called the *variance* v and the ratio of the standard deviation to the mean is the *coefficient of variation* CoV. The CoV is a measure of the scatter in the data. The CoV of structural dead loads may be around 0.05, whereas the CoV of soil data may be around 0.3:

$$CoV = \frac{\sigma}{\mu} \qquad (11.179)$$

For normal distributions, the inverse of the CoV is the *reliability index* β. The reliability index tells us how many standard deviations the mean is from the zero origin. It is very useful in reliability analysis and engineering code calibration. In this case, the variable is the difference between the resistance R and the load L and the reliability index β tells us how many standard deviations $\sigma_{(R-L)}$ the mean $\mu_{(R-L)}$ is from failure $(R - L = 0)$. It serves as an indication of the safety level (reliability index).

$$\beta = \frac{\mu}{\sigma} \qquad (11.180)$$

For distributions different from normal distributions, the generalized reliability index is still used, but is defined differently.

If the number of values of x_i increases, the histogram becomes smoother; if the number becomes infinity, a smooth function is obtained. This function is $f(x)$ and is called the *probability density function* (PDF) (Figure 11.33c). It is defined as the function $f(x)$ that satisfies:

$$P(a < X < b) = \int_a^b f(x)dx \qquad (11.181)$$

where $P(a < X < b)$ is the probability that X will be between a and b. The curves on Figure 11.34 are examples of the function $f(x)$. The area under the curve between two values a and b is the probability that X will fall between those two values. The function also satisfies:

$$P(-\infty < X < +\infty) = \int_{-\infty}^{+\infty} f(x)dx = 1 \qquad (11.182)$$

Recall that for the histogram, the distribution depended on the increment selected for the variable. The same happens for f(x): Different functions will be obtained depending on the units used for the variable axis. However, the integral in Eq. 11.181 will be the same because it is a relative measure. The *cumulative distribution function* (CDF) gives the value:

$$P(X < x) = \int_{-\infty}^{x} f(x)dx \qquad (11.183)$$

One of the most commonly used PDFs is the normal distribution. The normal distribution function is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad (11.184)$$

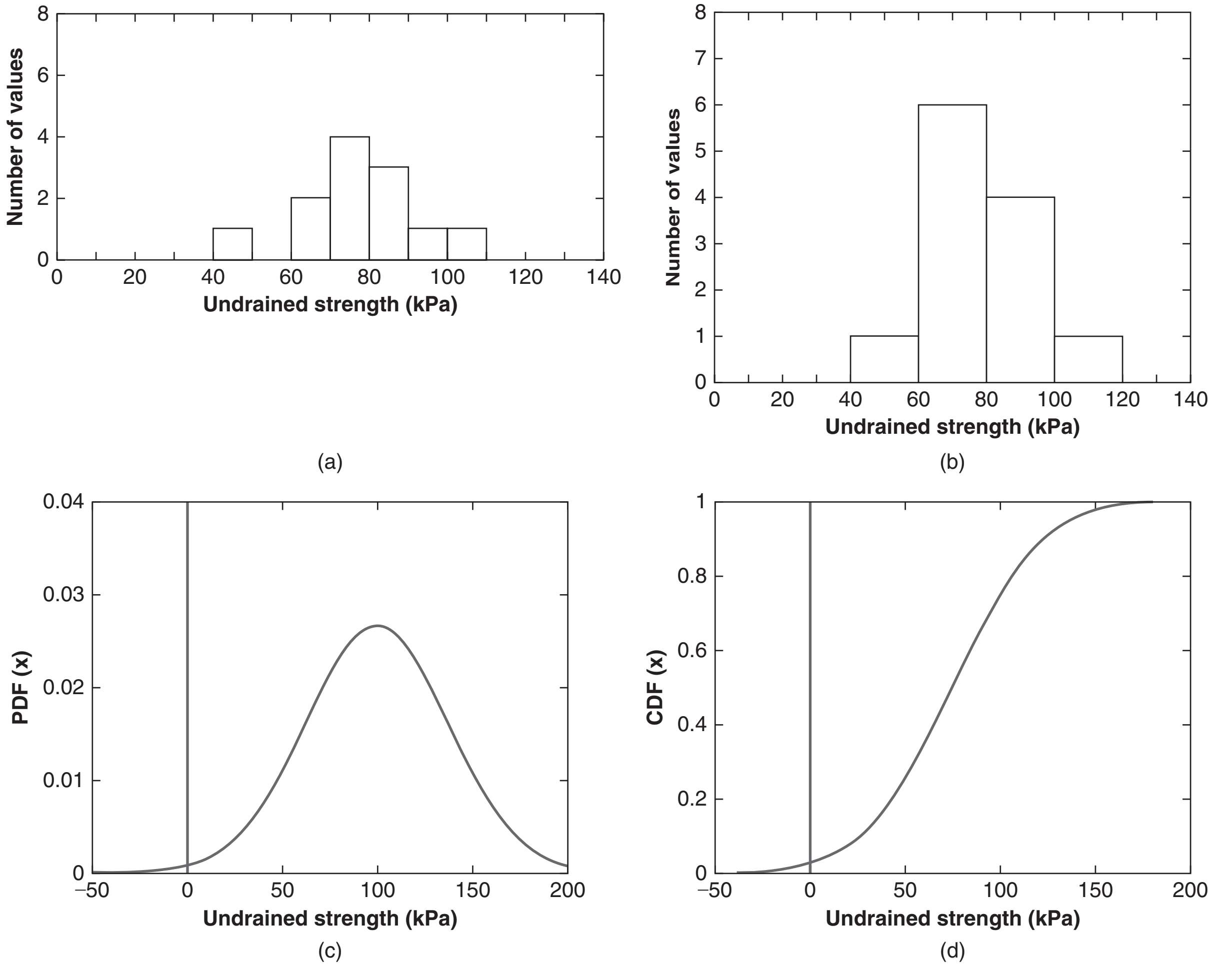


Figure 11.33 Histogram for two values of the variable increment.

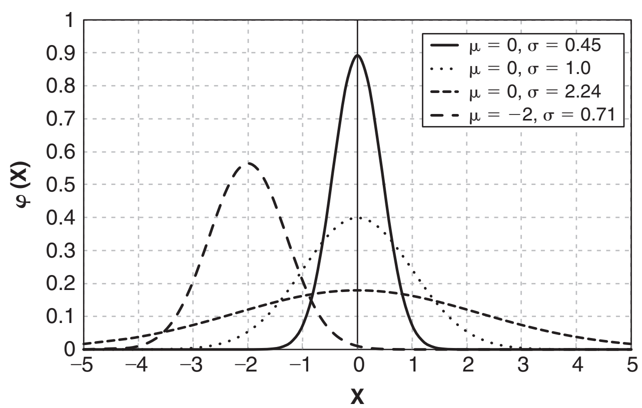


Figure 11.34 Examples of probability density function for normal distributions.

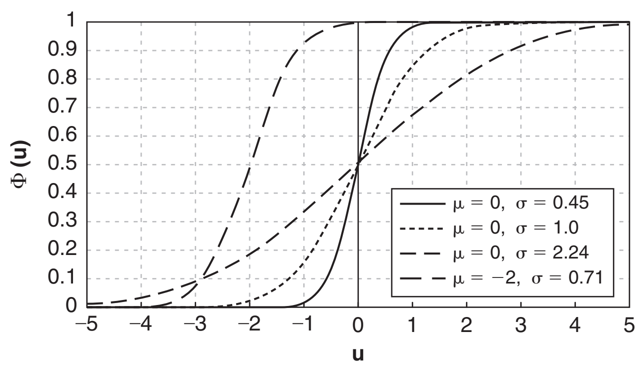


Figure 11.35 Examples of cumulative distribution function for normal distributions.

The corresponding CDF is:

$$F(x) = \frac{1}{2}\left(1 + erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right) \tag{11.185}$$

The "erf" function is called the *error function*; it does not have a closed-form expression, but can be tabulated. Figure 11.34 shows normal distributions and Figure 11.35 shows cumulative distributions.

It is often advantageous to normalize the variable. The standard normal variable (SNV) is denoted u:

$$u = \frac{x - \mu}{\sigma} \tag{11.186}$$

Therefore, the mean and the standard deviation of the SNV are 0 and 1 respectively. The PDF and CDF for the SNV are:

$$\text{PDF} \quad \varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \tag{11.187}$$

$$\text{CDF} \quad \Phi(u) = \frac{1}{2}\left(1 + erf\left(\frac{u}{\sqrt{2}}\right)\right) \tag{11.188}$$

Values of the CDF function $\Phi(u)$ for the Standard Normal Variable u are presented in Table 11.3. So, if you wish to find out the probability $P(X < x)$ that a normally distributed variable X is less than a chosen value x, the steps are:

1. Obtain the mean μ and standard deviation σ of X
2. Calculate the value of the standard normal variable $u = (x - \mu)/\sigma$
3. Look in Table 11.3 to find $\Phi(u)$
4. Then

$$\begin{aligned}\Phi(u) = P(U < u) = P\left(\frac{X - \mu}{\sigma} < \frac{x - \mu}{\sigma}\right) \\ = P(X < x)\end{aligned} \tag{11.189}$$

5. Remember that $\Phi(u)$ has the following properties:

$$P(U < u) = 1 - P(U < -u) \quad so$$

$$\Phi(u) = 1 - \Phi(-u) \tag{11.190}$$

$$P(U < u) = P(U > -u) \tag{11.191}$$

Figure 11.36 shows some useful areas under the normal distribution.

Another distribution that is very commonly used is the *lognormal distribution* (Figures 11.37 and 11.38). This distribution of a variable X is defined as a distribution such that the LnX (natural logarithm) is normally distributed. The probability density function of the lognormal distribution is therefore:

$$f(x) = \frac{1}{x\sigma_{Lnx}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{Lnx - \mu_{Lnx}}{\sigma_{Lnx}}\right)^2} \tag{11.192}$$

Note that the distribution differs slightly from the normal distribution equation. This is because the function is $f(x)$ rather than $f(Lnx)$. The function $f(Lnx)$ would have the same expression as Eq. 11.184, but $f(x)$ is equal to $f(Lnx)$ times the derivative of Lnx with respect to x, which brings about the additional $1/x$. In this case, the mean and standard deviation of the lognormal distribution are:

$$\mu_{Lnx} = Ln\left(\frac{{\mu_x}^2}{\sqrt{{\mu_x}^2 + {\sigma_x}^2}}\right) \tag{11.193}$$

$$\sigma_{Lnx} = \sqrt{Ln\left(1 + \frac{{\sigma_x}^2}{{\mu_x}^2}\right)} \tag{11.194}$$

Table 11.3 can be used to obtain the probability $P(X < x)$ that a lognormal distributed variable X is smaller than a chosen value x. The process takes place as follows:

1. Obtain the mean μ_x and the standard deviation σ_x of X
2. Obtain the mean μ_{Lnx} and standard deviation σ_{Lnx} of LnX. This can be done by using Eqs. 11.193 and 11.194 once μ_x and σ_x are known.
3. Calculate the value of the standard normal variable $u = \left(\frac{Lnx - \mu_{Lnx}}{\sigma_{Lnx}}\right)$
4. Look in Table 11.3 to find $\Phi(u)$Then

$$\begin{aligned}\Phi(u) = P(U < u) = P\left(\frac{LnX - \mu_{Lnx}}{\sigma_{Lnx}} < \frac{Lnx - \mu_{Lnx}}{\sigma_{Lnx}}\right) \\ = P(LnX < Lnx) = P(X < x)\end{aligned} \tag{11.195}$$

5. Remember that $\Phi(u)$ has the following properties:

$$P(U < u) = 1 - P(U < -u) \quad so$$

$$\Phi(u) = 1 - \Phi(-u) \tag{11.196}$$

and

$$P(U < u) = P(U > -u) \tag{11.197}$$

11.6.2 Procedure for Probability Approach

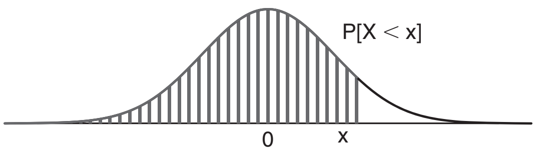
A method of calculating the uncertainty associated with a predicted value usually proceeds as follows:

1. First, the uncertainty associated with each variable involved in the solution is quantified. This quantification process often requires that the mean μ and standard deviation σ of each variable be determined, or that the mean μ and the coefficient of variation $CoV = \sigma/\mu$ be determined. Soil properties tend to have coefficients of variation on the order of 0.3 to 0.4.
2. Deterministic approaches may use mean values of the variables to obtain the mean value of the predicted value. In probabilistic approaches, a second set of equations is organized dealing with the relationship between standard deviations. There are special mathematical rules of operation to combine the standard deviations of the contributing variables and obtain the standard deviation of the variable to be predicted. If the expression of the variable to be predicted as a function of the contributing variables is too complicated, one may have to use numerical probabilistic simulations such as the Monte Carlo simulation.

Table 11.3 Values of the Areas under the Distribution of the Standard Normal Variable

The table gives the cumulative probability up to the standarized normal value of x

$$P[X < x] = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2}X^2\right) dX$$



x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
x	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

3. The Monte Carlo simulation consists of drawing values of the contributing variables at random from the range of possible values (using a random number generator), respecting the distribution of these variables, and calculating the value of the function to be predicted. This drawing process is repeated thousands of times and the values obtained are organized into a distribution for the predicted function from which a mean and a standard deviation are calculated.

4. Once the standard deviation of the predicted function is known, one can find out the probability that the predicted function value will be higher or lower than a chosen target.

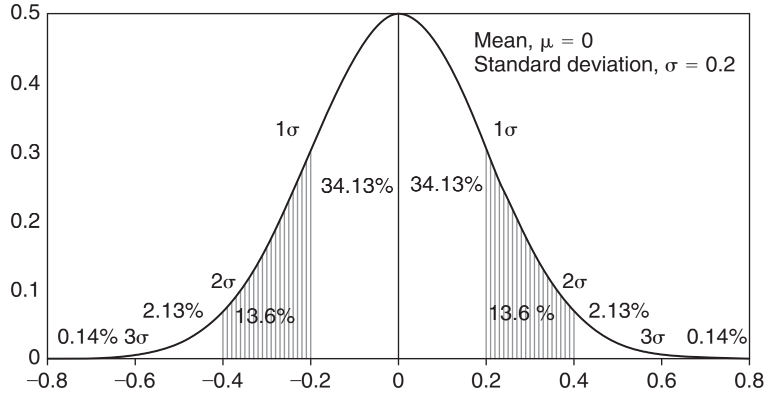


Figure 11.36 Useful areas under the normal distribution.

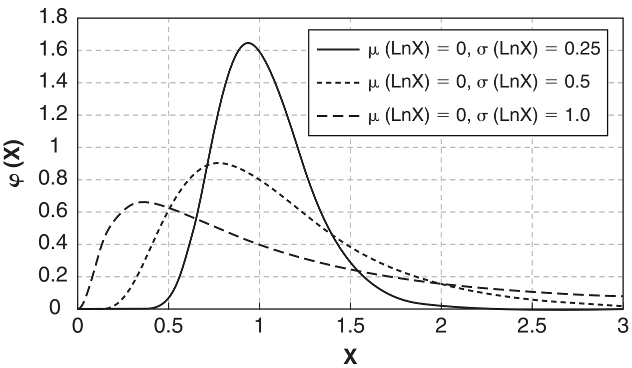


Figure 11.37 Examples of probability density function for lognormal distributions.

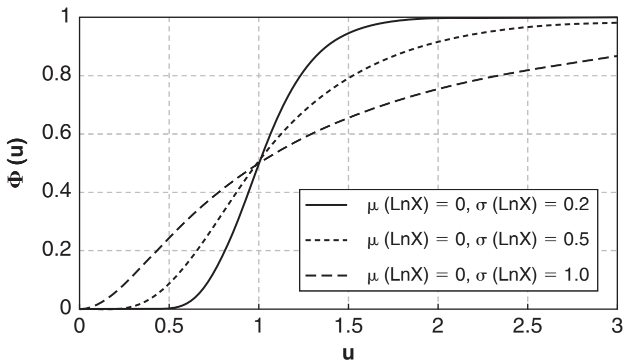


Figure 11.38 Examples of cumulative distribution function for lognormal distributions.

11.6.3 Risk and Acceptable Risk

There is a very important distinction to be made between probability of failure and risk. The *probability of failure* is simply the probability that something is going to collapse (e.g., a bridge, a slope, a building, a dam). *Risk* is defined as the probability of failure multiplied by the value of the consequence. It uses units of the value of the consequence, typically fatalities or dollars lost:

$$R = P_{(F<1)}C \tag{11.198}$$

where R is the risk, P is the probability of failure, and C the value of the consequence. For example, if a slope exists at a very steep angle, it likely would have a high probability of failure. If it were located in the middle of a deserted area and it failed, no one would die and the economic loss would be minimal; therefore the risk would be small. If the same slope were in the middle of a busy city with many buildings and people around it, the number of people dying and the economic loss from destroyed building, utilities, and transportation facilities would be significant; therefore the risk would be very high even though the probability of failure was the same. That illustrates the distinction between probability of failure and risk.

This brings up the point of what is an acceptable risk. First of all, it is not possible to design a structure (for example, a tunnel or an earth dam) that has zero risk associated with its engineering life. This is due to the facts that any calculation is associated with some uncertainty; that the engineering profession's knowledge, though having made great strides, is still incomplete in many respects; that human beings are not error free; and that the engineer designs the structure for conditions that do not include extremely unlikely events such as a falling satellite hitting the structure at the same time as an earthquake, a hurricane, and a 500-year flood during rush hour.

Most modern codes have been written with an accepted probability of failure of about 1 chance in 1000 (structural engineering); it may be estimated that geotechnical engineering operates at a somewhat higher risk than that. In any case, the choice of an acceptable risk is difficult because so many factors enter into the decision. One of those factors is the evaluation of how many fatalities are acceptable. Though few people are prepared to say that any fatality is acceptable,

Table 11.4 Approximate Probability of Human Death

Activity	Probability of death*
Heart disease	0.25
Cancer	0.23
Stroke	0.036
Car	0.012
Suicide	0.009
Fire	0.0009
Airplane	0.0002
Bicycle	0.0002
Lightning	0.00001
Earthquake	0.000009
Flood	0.000007

*These numbers represent the number of deaths due to that activity in one year in the USA divided by the total number of death in the USA during that same year. *Sources*: New York Times, Center for Disease Control, National Safety Council.

it is a matter of public record that some fatalities do occur because of civil engineering decisions. These fatalities can be due to malpractice or to unforeseen events. The choice of an acceptable risk involves other disciplines beyond geotechnical engineering, including philosophy, politics, and social sciences. One of the very difficult steps required in estimating an acceptable risk is what price to put on human life. It is not uncommon to use a number like \$1 million, because that is an average life insurance value for many people. The probabilities of death in the USA for various human activities (Table 11.4) help frame the acceptable numbers in this domain.

Note that the statistics do not always match the human perception. For example, the probability of dying in a car accident is much higher than the probability of dying in an airplane accident, yet people tend to be much more afraid of flying than of driving their cars. Figure 11.39 shows the annual risk associated with various activities in geotechnical engineering and in everyday life. The annual probability of failure (PoF) is on the vertical axis, and there are two scales on the horizontal axis: lives lost or fatalities per year (F) and dollars lost per year (D). Because the two do not necessarily correspond, the activities are shown as bubbles rather than precise points on the graphs. Since the risk is the product of the probability times the value of the consequence, two risk values can be defined:

$$R(fatalities) = PoF \times F \tag{11.199}$$

$$R(dollars\ lost) = PoF \times D \tag{11.200}$$

Therefore, the annual risk is constant on diagonals in Figure 11.39. The red, blue, and green lines correspond to a high, medium, and low annual risk. The numbers are shown in Table 11.5. These data indicate that 0.001 fatalities per year and \$1000 US per year may be acceptable target risk values.

A more advanced way to formulate the risk is:

$$R = T \times V \times C \tag{11.201}$$

where T is the threat, V the vulnerability, and C the value of the consequence.

As can be seen in this case, the probability of failure is split into two components. The *threat* is the probability that a certain event will occur (big flood or big earthquake), whereas the *vulnerability* is the probability that failure will occur if the event occurs. Vulnerability is the part of the system where

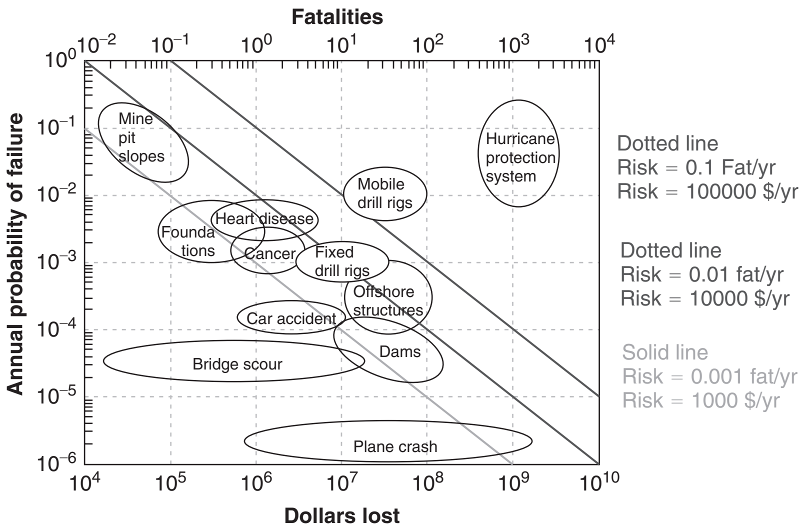


Figure 11.39 Risk associated with various engineering and human activities (Yao 2013).

Table 11.5 Annual Risks for the USA (risk = PoF × value of the consequence)

Annual risk level	Fatalities/year in USA	Dollars lost/year in USA
Low	0.001	1000
Medium	0.01	10000
High	0.1	100000

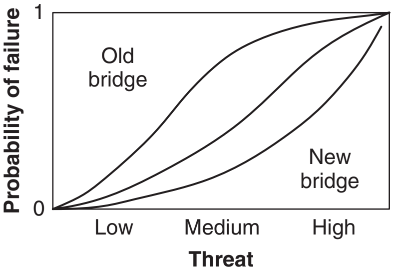


Figure 11.40 Fragility curves.

one has the most control. Fragility curves (Figure 11.40) link the probability of failure to the severity of the threat; they quantify the vulnerability function V.

11.6.4 Example of Probability Approach

A slope stability analysis is used as an example of the probability approach. In a *deterministic analysis*, a single factor of safety is calculated. In a *probabilistic analysis*, a mean factor of safety is calculated from the mean values of the soil parameters, the slope geometry, and the water stress conditions. Then a standard deviation of the factor of safety is obtained from the standard deviations of the parameters involved in the calculations. This is done either by mathematical calculations from the individual standard deviations of the parameters involved in the factor of safety of the slope, if the problem is simple enough; or by numerical simulations, such as the Monte Carlo simulation, if the problem involves several layers or complicated geometry and water stress conditions. Knowing the standard deviation of the factor of safety, one can calculate the probability that the calculated factor of safety will be below 1; this is the probability of failure of the slope.

The deterministic approach gives only one factor of safety, whereas the probabilistic approach gives a mean factor of safety and a probability of failure. This added information can be very valuable for the engineer who must accept or reject the calculated value of the factor of safety. Indeed, one could have the same mean factor of safety but drastically different probabilities of failure depending on whether the soil parameters are known with good precision (low coefficient of variation) or with poor precision (high coefficient of variation). For example, a mean factor of safety of 1.5 with a coefficient of variation of 0.5 would likely be unacceptable, whereas a mean factor of safety of 1.5 with a coefficient of variation of 0.05 would likely be acceptable. Yet no distinction could be made on the basis of the mean value alone.

Let us say that a slope has a mean factor of safety equal to 1.5 ($\mu_F = 1.5$) and a standard deviation equal to 0.45 ($\sigma_F = 0.45$). The coefficient of variation is 0.3 ($CoV_F = 0.3$). Let's further assume that F follows a lognormal distribution. The question is what is the probability of failure P(F < 1)? We follow the steps of section 11.6.2:

1. The mean and standard deviation of F are 1.5 and 0.45 respectively.
2. The mean and standard deviation of LnF are calculated as follows:

$$\mu_{Lnx} = Ln\left(\frac{1.5^2}{\sqrt{1.5^2 + 0.45^2}}\right) = 0.362 \qquad (11.202)$$

$$\sigma_{Lnx} = \sqrt{Ln\left(1 + \frac{0.45^2}{1.5^2}\right)} = 0.294 \qquad (11.203)$$

3. Calculate the value of the standard normal variable U for F = 1:

$$u = \frac{LnF - \mu_{LnF}}{\sigma_{LnF}} = \frac{Ln1 - 0.362}{0.294} = -1.231 \qquad (11.204)$$

4. Table 11.3 does not gives the value of $\Phi(-1.231)$, but it gives the value of $\Phi(1.231) = 0.8907$. Because $\Phi(u) = 1 - \Phi(-u)$, then $\Phi(-1.231) = 1 - 0.891 = 0.109$. Therefore, the probability of failure is 0.109.

This process can be repeated a number of times for different values of the factor of safety, and a plot of the factor of safety versus the probability of failure can be generated (Figure 11.41). Using that plot, if we wish to operate at a

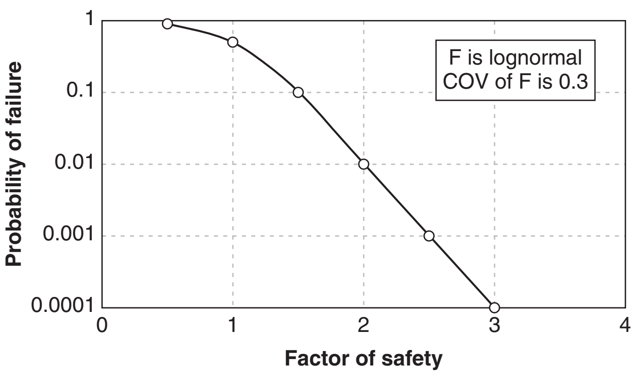


Figure 11.41 Probability of failure vs. mean factor of safety for a slope.

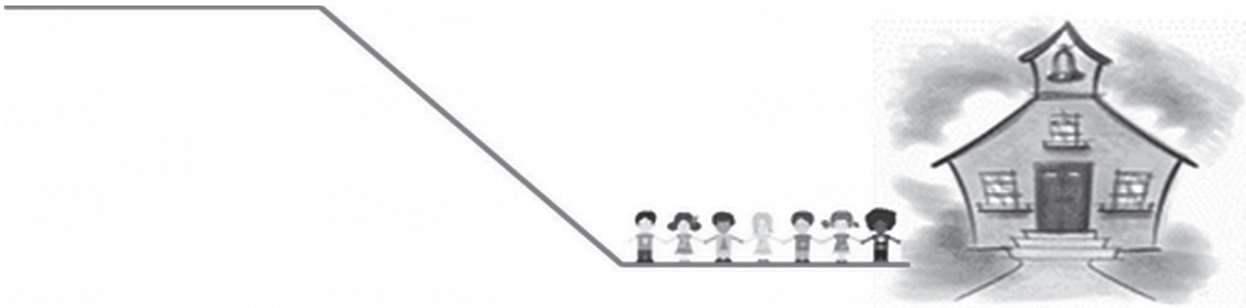


Figure 11.42 Slope and consequence of failure.

probability of failure of 0.001, then we would have to use a factor of safety of 2.5. Now let's say that the slope failure would have some serious consequences (Figure 11.42), such as 10 fatalities and \$5 million. For a factor of safety of 1.5, we would calculate a risk of $0.109 \times 10 = 1.09$ fatalities and $0.109 \times 5\text{M\$} = \$545{,}000$. If we wish to operate at a risk level of 0.01 fatalities/year (Figure 11.39), then we need to have: Risk = 0.01 fatalities = $P(F < 1) \times 10$ fatalities or $P(F < 1) = 0.001$; therefore $F = 2.5$ (Figure 11.41). If we wish to operate at a risk level of \$10,000/year, then we need to have: Risk = \$10000 = $P(F < 1) \times 5\text{M\$}$ or $P(F < 1) = 0.002$ and $F = 2.35$ (Figure 11.41). In this case, the fatality-risk criterion controls.

11.7 REGRESSION ANALYSIS

Let's say that we have data presented on an x—y scatter plot (Figure 11.43) representing n x_i values and n corresponding y_i values. It is often desirable to find the best fit line for the data presented so that y can best be predicted for any value of x. This is *regression analysis*. The basic concepts are presented here for a linear regression where the best fit line to be found is a straight line $y = ax + b$. The first step is to define what is meant by *best fit*. The most common definition is that the sum of the squares of the differences d_i (Figure 11.43) between the predicted values of y and the measured value of y is minimum. The sum of the squares is:

$$f(a,b) = \sum_{i=1}^{n} d_i^{\,2} = \sum_{i=1}^{n} (y_i - ax_i - b)^2 \tag{11.205}$$

This function f(a,b) is minimum when the partial derivatives with respect to a and to b are zero:

$$\frac{\partial f(a,b)}{\partial a} = 0 = a\sum x_i^{\,2} + b\sum x_i - \sum x_i y_i \tag{11.206}$$

$$\frac{\partial f(a,b)}{\partial b} = 0 = a\sum x_i + bn - \sum y_i \tag{11.207}$$

These two equations give a and b as:

$$a = \frac{\sum x_i \sum y_i - n\sum x_i y_i}{\left(\sum x_i\right)^2 - n\sum x_i^{\,2}} \tag{11.208}$$

$$b = \frac{\sum x_i^{\,2}\sum y_i - \sum x_i \sum x_i y_i}{n\sum x_i^{\,2} - \left(\sum x_i\right)^2} \tag{11.209}$$

If you look at a scatter plot and "eyeball" the regression line, you tend to minimize the normal distance (NM on Figure 11.44) between the data points and the best fit line rather than the vertical distance. This is called an *orthogonal regression*.

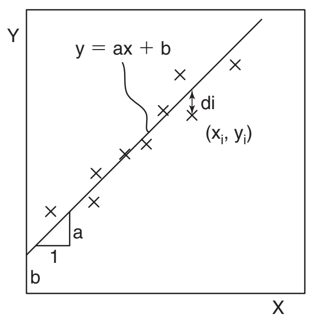


Figure 11.43 Regression minimizing the vertical distance.

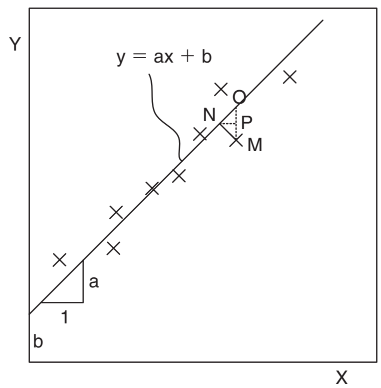


Figure 11.44 Regression minimizing the orthogonal distance.

In this case the expression for the sum of the squares of the distances becomes:

$$f(a,b) = \sum_{i=1}^{n} d_i{}^2 = \sum_{i=1}^{n}((x_{iM} - x_{iN})^2 + y_{iM} - y_{iN})^2)$$
$$= \sum_{i=1}^{n} \frac{1}{1+a^2}(y_{iM} - ax_{iM} - b)^2 \qquad (11.210)$$

Then the derivative with respect to b gives:

$$\frac{\partial f(a,b)}{\partial b} = 0 \quad or \quad \sum (y_i - ax_i - b) = 0 \qquad (11.211)$$

or

$$b = \frac{\sum y_i}{N} - a\frac{\sum x_i}{N} = \overline{y} - a\overline{x} \qquad (11.212)$$

Eliminating b from Eq. 11.210 gives:

$$f(a) = \frac{\sum (y_i - \overline{y})^2 - 2a\sum (x_i - \overline{x})(y_i - \overline{y}) + a^2 \sum (x_i - \overline{x})^2}{1 + a^2} \qquad (11.213)$$

Then we set:

$$\frac{\partial f(a)}{\partial a} = 0 \qquad (11.214)$$

which gives the following equation:

$$a^2 \sum (x_i - \overline{x})(y_i - \overline{y}) - a\left(\sum (y_i - \overline{y})^2 - \sum (x_i - \overline{x})^2\right) - \sum (x_i - \overline{x})(y_i - \overline{y}) = 0 \qquad (11.215)$$

which leads to the solution for a:

$$a = \frac{\sum (y_i - \overline{y})^2 - \sum (x_i - \overline{x})^2 + \sqrt{\left(\sum (y_i - \overline{y})^2 - \sum (x_i - \overline{x})^2\right)^2 + 4\left(\sum (x_i - \overline{x})(y_i - \overline{y})\right)^2}}{2\sum (x_i - \overline{x})(y_i - \overline{y})} \qquad (11.216)$$

Note that if the regression line is forced to go through the origin, then:

$$b = 0 \quad and \quad a = \frac{\sum y_i}{\sum x_i} \qquad (11.217)$$

The coefficient of regression r^2 is a measure of how well the regression equation predicts the data. It is given by:

$$r^2 = 1 - \frac{\sum (y_i - ax_i - b)^2}{\sum (y_i - \mu_y)^2} \qquad (11.218)$$

It tells us how well the regression line predicts the data compared to a simple average. Values close to 1 indicate that y = ax + b is a very good predictor of the data, whereas values close to zero indicate that y = ax + b is a very poor predictor of the data and that you might as well use the mean regardless of the value of x.

11.8 ARTIFICIAL NEURAL NETWORK METHOD

The artificial neural network (ANN) (De Wilde 1996; Schalkoff 1997) gets its name from the human brain, where neurons interact with each other to process information and make decisions. ANN can be thought of as a very sophisticated regression analysis where a data set is input and, after calculations through a number of neuron layers involving mathematical functions, is converted into a desired output (Figure 11.45). For example, there are many bridges in the USA for which the foundation type and depth are unknown. Let's say that you wish to predict the type and depth of the foundation on the basis of related information, such as the dead load of the bridge, the number of lanes, the length of the span, the foundation depth of neighboring bridges, the dates when the bridge was designed and built, the soil type and the soil strength if borings are available, and so on. This input becomes a set of numbers fitting in a layer of initial neurons and related in some fashion to the type and depth of the unknown foundation. A set of mathematical functions to be chosen by the user are placed in the next layer of neurons waiting for the arrival of the input data; these mathematical functions will transform the input values into a new set of values that is in turn sent to the next layer of neurons. Each time the data set goes through a new layer of neurons, it is mathematically transformed and then sent to the next set of neurons, where it undergoes new mathematical transformation. The output layer of neurons for this example would contain the type and depth of the foundation.

The neurons in a layer are connected only to the previous layer and to the next layer of neurons. Any given neuron is connected to some of the neurons in the previous layer and some of the neurons in the following layer but not necessarily

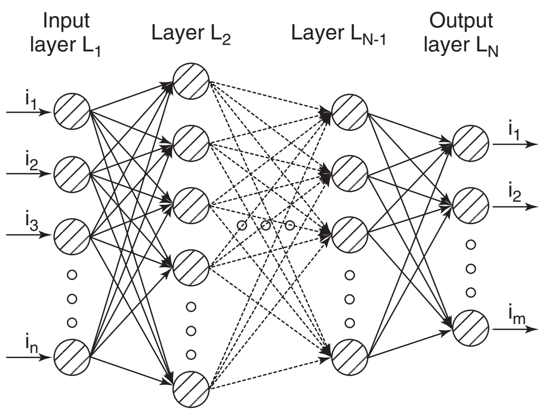


Figure 11.45 Artificial neural network. (After Bobet 2010.)

to all of them. The mathematical functions f that operate the transformation in a neuron are for example (Bobet 2010):

$$i_{jk} = f\left(\sum_{h\in L_{k-1}} \left(w_{hj} i_{h(k-1)} + \theta_{jk}\right)\right) = f(o_j) \quad with \quad j \in L_k \tag{11.219}$$

where i_{jk} is the information to be calculated and be stored in neuron j of layer k, often called the *state* of neuron jk; $i_{h(k-1)}$ is the known information stored in neuron h of layer $(k-1)$; w_{hj} is the weight factor associated with the connection between neuron h in layer $k-1$ and neuron j in layer k (note that w_{jh} does not exist, as there is no connection back from neuron h to neuron j); θ_{jk} is the bias or threshold value associated with neuron jk; and o_j is the argument associated with neuron jk in the function f. Although these functions are chosen by the user at will, certain functions are more popular than others. This is the case of the sigmoidal function:

$$f(o_j) = \frac{1}{1+e^{-o_j}} \tag{11.220}$$

Once the functions are in place, the ANN must be “trained,” which means that the constants in the functions must be determined. This is done by minimizing the error E between the input data and the output predictions through a process similar to the regression analysis discussed in section 11.7:

$$E = \sum_{m\in L_N} (d_m - f(o_m))^2 \tag{11.221}$$

where d_m is the data for neuron m of layer n and $f(o_m)$ is the predicted value for neuron m of layer n. Once the ANN is trained, it can be used to make predictions concerning the type of data that was used to train it. However, the accuracy is tied to the quality of the ANN and the experience of the developer. Using ANN outside of the range of values used to train it can lead to serious errors.

11.9 DIMENSIONAL ANALYSIS

Units are essential to quantify engineering parameters. Unfortunately, there are several unit systems, and this often makes it difficult to communicate across countries using different systems. The most common system in the world is the SI unit system (Système International), but the U.S. customary unit system is still used in the USA. These systems were developed in the late 1700s (SI system in France) and the early 1800s (Imperial system in the UK). Although there are seven units in a system of units (Chapter 1), four are used commonly in engineering: length, mass, time, and temperature. In the SI system, the practice is to use the meter, the kilogram, the second, and the degree Celsius. In the U.S. customary system, the practice is to use the foot, the pound, the second, and the degree Fahrenheit. These four units are called *primary units*, from which derived units can be obtained. The Newton is a unit of force; it is not a primary unit but rather a derived unit, as it is a combination of mass, length, and time ($F = ma = mass \times length/time^2$). Stress is also not a primary unit, but rather a derived unit, as it uses mass, length, and time. One way to avoid worrying about units is to nondimensionalize the parameters used in a problem. Strains are an example of such nondimensional quantities. Strains are the same regardless of the system of units used. In geotechnical engineering, we tend to use dimensional parameters, whereas in hydraulic engineering the trend is toward using nondimensional parameters. As a result, the difference in unit systems does not affect hydraulic engineering as much as geotechnical engineering.

11.9.1 Buckingham Π Theorem

Dimensional analysis is a very useful tool when dealing with mechanics problems. It goes back to the work of Newton and Fourier, but culminated with Buckingham (an American physicist born in 1867) and his famous Π theorem in 1915. This theorem states that a function describing a relationship among n quantities, x_i, such as $f_1(x_1, x_2, x_3, \ldots, x_n) = 0$. where m primary units are required to express the x_i quantities, can be reduced to the form $f_2(\Pi_1, \Pi_2, \Pi_3, \ldots, \Pi_{n-m}) = 0$, where Π_i are nondimensional products of powers of the x_i of the form $\pi_i = x_1{}^a x_2{}^b \ldots x_n{}^c$. This means that the number n of variables necessary to describe a function can be reduced by the number m of primary units necessary to describe these variables. For example, if we have 5 variables ($n = 5$) with units of mass, length, and time ($m = 3$), then only 2 variables enter the function and are necessary to describe the solution.

The advantages of dimensional analysis include:

- Forcing us to think through a problem at the front end
- Providing insight about a solution
- Reducing the number of required experiments or simulations
- Providing a basis for direct scaling from model tests to prototype predictions
- Helping in memorizing formulas
- Helping in transforming empirical formulas from one system of units to another
- Detecting errors in equations revealed by lack of dimensional homogeneity
- Providing a mechanism for converting a formula from one unit system to another
- Interpreting the behavior of scale models
- Guiding the selection of experiments
- Obtaining partial solutions to complex problems

The procedure for applying the Buckingham Π theorem is as follows:

1. Identify all the n independent variables influencing the solution to the problem.
2. Identify the m primary units involved in these n independent variables and form m primary unit groups. In

each primary unit group, list all the variables containing that primary unit.

3. Select one variable from each group as a repeating variable. Do not select the variable that is to be predicted and do not select the same variable from each group.
4. Form the (n − m) Π terms as products of the repeating variables and each one of the nonrepeating variable in turn. Each variable is raised to a power exponent.
5. Determine the exponents of the power such that the products are dimensionless.
6. Write the function that links the Π terms and formulate the expression of the solution.

A certain degree of art and experience is associated with judicious use of the Π theorem, and it does require some trial and error, but as the following example will show it is worth the effort.

11.9.2 Examples of Dimensional Analysis

The problem is to use dimensional analysis to find the general expression of the function giving the lateral displacement at the top of an infinitely long pile subjected to a horizontal load applied at the top of the pile and placed in an elastic soil. Figure 11.46 shows the problem and the variables.

1. The independent variables are shown in Table 11.6 with their dimensions. There are 5 independent variables.

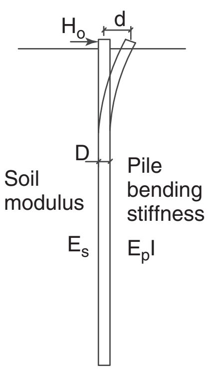


Figure 11.46 Laterally loaded pile problem.

Table 11.6 Variables and Their Dimensions

Quantity	Symbol	Dimension
Displacement	d	L
Force	H_o	F
Soil modulus	E_s	F/L^2
Pile bending stiffness	E_pI	$F\,L^2$
Pile diameter	D	L

2. There are 2 primary units, as listed in Table 11.6. We therefore form 2 primary unit groups of variable. For example,
   a. L group: d, E_s, E_pI, D
   b. F group: H_o, E_s, E_pI
3. We select one variable in each group, for example D in the L group and E_s in the F group. These are the repeating variables.
4. Because there are 5 variables and 2 primary units, we have $5 - 2 = 3$ Π terms. To obtain the 3 Π terms, we form the power product of the 2 repeating variables plus 1 of the remaining variables. The Π terms are:
   a. $\Pi_1 = D^a E_s{}^b d^c$
   b. $\Pi_2 = D^d E_s{}^e H_o{}^f$
   c. $\Pi_3 = D^g E_s{}^h E_p I^i$
5. Now we need to find the exponent of the powers in the Π terms such that they are dimensionless.
   a. For $\Pi_1 = D^a E_s{}^b d^c$ in terms of dimensions $L^a (F/L^2)^b L^c$
      For this term to be dimensionless, we must have $b = 0$ and $a - 2b + c = 0$
      This gives $b = 0$ and $a = -c$. We then set one exponent to a convenient value: say, $a = 1$ and Π_1 becomes $\Pi_1 = d/D$.
   b. For $\Pi_2 = D^d E_s{}^e H_o{}^f$ in terms of dimensions $L^d (F/L^2)^e F^f$
      For this term to be dimensionless, we must have $e + f = 0$ and $d - 2e = 0$
      This gives $e = -f$ and $d = 2e = -2f$. We chose $f = 1$ for this example, so Π_2 becomes $\Pi_2 = H_o/(E_s D^2)$.
   c. For $\Pi_3 = D^g E_s{}^h E_p I^i$ in terms of dimensions $L^g (F/L^2)^h (FL^2)^i$
      For this term to be dimensionless, we must have $h + i = 0$ and $g - 2h + 2i = 0$
      This gives $h = -i$ and $g = -4i$. We chose $i = 1$ for this example, so Π_3 becomes $\Pi_3 = E_p I_p/(E_s D^4)$
6. Then we can say that $g(\Pi_1, \Pi_2, \Pi_3) = 0$ or $f_1(d/D, H_o/(E_s D^2), E_p I_p/(E_s D^4)) = 0$. This can be rewritten as $d/D = f_2(H_o/(E_s D^2), E_p I_p/(E_s D^4))$. Because the problem is linear (linear soil and linear pile), we can write:

$$\frac{d}{D} = \frac{H_o}{E_s D^2} f_3 \left(\frac{E_p I}{E_s D^4} \right) \tag{11.222}$$

Although the pile displacement cannot be calculated with this function, the result is still very helpful. For example, if we wish to find the function f_3, all we need to do is vary $E_pI/(E_sD^4)$. Without the dimensional analysis, we would have to vary many combinations of the four variables. Discovering the general expression of the solution saves a lot of research time in this case.

11.10 SIMILITUDE LAWS FOR EXPERIMENTAL SIMULATIONS

11.10.1 Similitude Laws

Experiments play a very important role in geotechnical engineering. From laboratory testing to in situ testing, from scaled models to centrifuge testing, all contribute to a better understanding of the problem. This section deals with similitude laws or scaling laws as they are used in scaled models and centrifuge tests. When facing a geotechnical problem where few established design procedures exist, the engineer may elect to perform scaled model tests to predict the behavior of the full-scale prototype. These scaled tests may be done by using models tested under one gravity (1g tests) or centrifuge tests.

A *geotechnical centrifuge* is a large rotating arm at the end of which is a swinging bucket (Figure 11.47). In that bucket is a model of the real problem (slope, retaining wall, foundation). The rotating arm spins at high speeds (e.g., 60 rpm); the swinging bucket first swings upward and then flies nearly horizontally. The centrifugal acceleration artificially increases the stresses in the sample. These high stresses make the sample behave as if it were much larger than it truly is, so the full-scale structure can be simulated and studied. Similitude laws must be evaluated and satisfied to ensure that a true similitude exists between the model scale (simulation) and the full scale, also called *prototype scale* (reality). If such a similitude is satisfied, the results from the scaled model can easily be extrapolated to the full-scale behavior.

To achieve similitude, each dimensionless term (Π terms from section 11.9) must be equal in the prototype and in the model:

$$\Pi_i \text{ model} = \Pi_i \text{ prototype} \tag{11.223}$$

11.10.2 Example of Similitude Laws Application for a Scaled Model

We wish to run a model test to predict the ultimate load P_u of a square footing of size B embedded at a depth d in clay with an undrained shear strength s_u and a unit weight γ. There are 5 parameters and 2 primary units. Therefore, there are 3 Π terms and the dimensional analysis gives:

$$f(\Pi_1, \Pi_2, \Pi_3) = f\left(\frac{P_u}{B^2 s_u}, \frac{d}{B}, \frac{s_u}{B\gamma}\right) = 0 \tag{11.224}$$

The subscript m will be used for the model parameters, and the subscript p will be used for the prototype. To simplify the experiment, we would like to use the same clay as the one found at the site for the prototype:

$$s_{um} = s_{up} \tag{11.225}$$

The scaled model will be n times smaller than the prototype:

$$n\, B_m = B_p \tag{11.226}$$

To satisfy the similitude, we now have to ensure that all Π terms are equal for the model and for the prototype. First we check the Π_1 term:

$$\Pi_{1m} = \Pi_{1p} \quad or \quad \frac{P_{um}}{{B_m}^2 s_{um}} = \frac{P_{up}}{{B_p}^2 s_{up}} \tag{11.227}$$

$$P_{um} = \frac{P_{up}}{n^2} \tag{11.228}$$

Therefore, we can expect the prototype ultimate load to be n^2 times larger than the load measured in the scaled model. Now let's look at the Π_2 term:

$$\Pi_{2m} = \Pi_{2p} \quad or \quad \frac{d_m}{B_m} = \frac{d_p}{B_p} \tag{11.229}$$

This is satisfied by geometric scaling. Finally let's look at the Π_3 term:

$$\Pi_{3m} = \Pi_{3p} \quad or \quad \frac{s_{um}}{B_m \gamma_m} = \frac{s_{up}}{B_p \gamma_p} \tag{11.230}$$

$$\gamma_m = n\gamma_p \tag{11.231}$$

Therefore, to satisfy similitude we will have to find a clay with the same undrained shear strength but with a unit weight n times larger than the unit weight of the prototype soil. This is very difficult to achieve, but there is an artificial way to do this using the centrifuge.

11.10.3 Example of Similitude Laws Application for a Centrifuge Model

Let's continue the example of section 11.10.2 and recognize that:

$$\gamma = \rho g \tag{11.232}$$

where ρ is the mass density of the clay and g is the acceleration due to gravity (9.81 m/s^2). Satisfying the Π_3 term leads to:

$$g_m = n g_p \tag{11.233}$$

Therefore, we can satisfy all similitude requirements by using a gravitational field n times larger for the model. This can be achieved in a geotechnical centrifuge (section 11.10.1) by spinning the bucket of the centrifuge fast enough to generate a centrifugal acceleration equal to ng. This is very useful, because in geotechnical engineering body forces play an important role, unlike in structures. These body forces affect the stability of slopes, tunnels, and mines, for example. Also, the strength of soils is greatly affected by the stress level; indeed, the shear strength depends on the effective stress on the plane of failure. Thus, the centrifuge plays an important role in solving geotechnical engineering problems. It is not without difficulties, however, as shown in the next example.

Consider the problem of flow through soil. The constitutive law for the soil is Darcy's law.

$$v = ki = k\frac{dh_t}{dx} \tag{11.234}$$

where v is the discharge velocity, k is the soil hydraulic conductivity, i is the hydraulic gradient, and dh_t is the change in total head over the distance dx. The hydraulic conductivity is dependent not only on the soil but also on the fluid. Indeed, as you can imagine for the same hydraulic gradient, water would flow faster through sand than thick oil. In fact, the hydraulic conductivity k is expressed as:

$$k = \frac{\gamma_w K}{\mu} \tag{11.235}$$

where γ_w is the unit weight of the flowing fluid, K is a property of the soil, and μ is the dynamic viscosity of the fluid. We can write:

$$\frac{k_m}{k_p} = \frac{\dfrac{\gamma_{wm} K_m}{\mu_m}}{\dfrac{\gamma_{wp} K_p}{\mu_p}} \tag{11.236}$$

Because we wish to use the same soil and the same fluid in the model and the prototype, $K_m = K_p$ and $\mu_m = \mu_p$. However, $\gamma_{wm} = n\,\gamma_{wp}$ because of the gravitational field, so we get:

$$k_m = n\,k_p \tag{11.237}$$

This means that the soil should be n times more pervious in the model than in the prototype—a conflict, as we wish to use the same soil.

Now consider the coefficient of consolidation c_v. The expression comes from Eq. 11.55:

$$c_v = \frac{k\,M}{\gamma_w} \tag{11.238}$$

where M is the constrained modulus. Because we are using the same soil in the model and in the prototype, $M_m = M_p$ and we can write:

$$\frac{c_{vm}}{c_{vp}} = \frac{\dfrac{k_m M_m}{\gamma_{wm}}}{\dfrac{k_p\,M_p}{\gamma_{wp}}} = \frac{\dfrac{n k_p M_m}{n\gamma_{wp}}}{\dfrac{k_p\,M_p}{\gamma_{wp}}} = 1 \tag{11.239}$$

So the coefficient of consolidation is the same. Now let's look at the time factor T (Eq. 11.63), which is one of the dimensionless terms to be satisfied:

$$T_m = T_p = \frac{c_{vm} t_m}{{H_{dm}}^2} = \frac{c_{vp} t_p}{{H_{dp}}^2} \tag{11.240}$$

We know that c_v is unchanged and that the drainage length H_d will scale geometrically, so:

$$t_m = \frac{t_p}{n^2} \tag{11.241}$$

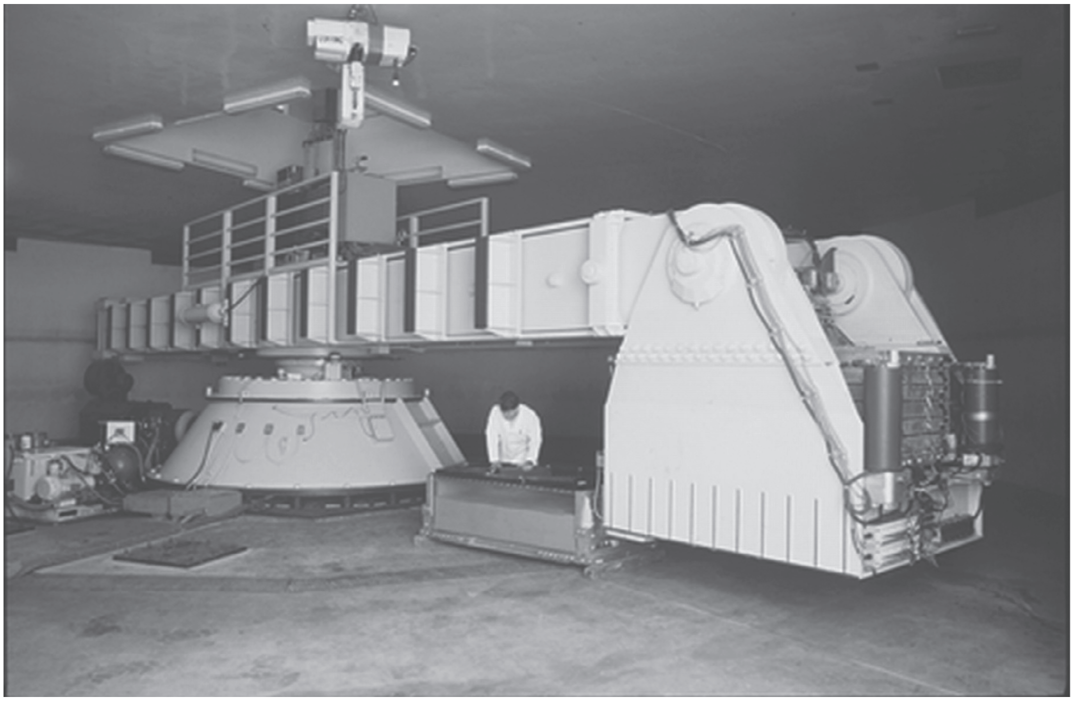


Figure 11.47 Geotechnical centrifuge. (Courtesy of the Center for Geotechnical Modeling, University of California-Davis, USA.)

Therefore, the time scale in the model is n^2 times faster than in the prototype. If the model is 100 times smaller, the consolidation time will be 10,000 times faster. Note that the scaling of time is not always n^2 and depends on the problem. For example, the scaling of time for a dynamic event is n, not n^2.

11.11 TYPES OF ANALYSES (DRAINED–UNDRAINED, EFFECTIVE STRESS–TOTAL STRESS, SHORT-TERM–LONG-TERM)

With respect to water and air drainage, geotechnical engineering analyses can be

- Effective stress or total stress analyses
- Drained or undrained analyses
- Long-term or short-term analyses

An effective stress analysis is the best approach in all geotechnical engineering problems, but it is not always the simplest, and sometimes the added complexity may not be necessary. In the effective stress analysis, the soil is considered to be made of three distinct phases (water, air, solids) and the stresses in the three phases are handled separately. The effective stress analysis is always appropriate and applicable, but is often difficult because it requires knowledge of the water stress and even the air stress. For example, it is perfectly appropriate to do an effective stress analysis to solve a problem involving the undrained behavior of a saturated clay, but the water stress must be known in the soil mass.

A total stress analysis consists of considering that the soil is monophase. This is the approach taken when dealing with concrete or steel. For soils, such an analysis is appropriate when dealing with the undrained behavior of saturated clays, for example, because in this case the two phases involved (solids and water) remain bound together, as there is no water movement. An undrained analysis is simply the analysis of a soil that does not drain; it is a common analysis for clays that are loaded rapidly. In extreme cases, the liquefaction of sands during earthquakes can also be considered an undrained behavior.

In a drained analysis, the water stress remains equal to hydrostatic; it is a common assumption for the slow loading of sands or the very slow loading of clays. A long-term analysis is similar to a drained analysis because in the long term all soils become drained. A short-term analysis may be an undrained analysis for some soils (clays) and a drained analysis for others (static loading of clean sands).

PROBLEMS

11.1 A vertical wall is supporting a clean, dry sand backfill with a unit weight γ and effective angle of internal friction φ' (Figure 11.1s). It is assumed that there is no friction between the wall and the backfill. The wall exerts a horizontal load P against the sand. As the wall is pushed into the sand, the load P increases and there is a point where the sand behind the wall fails. At that point, the load is P_p corresponding to the passive earth pressure and the question is to find the load P_p corresponding to impending failure of the sand. Note that the problem is a plane strain problem.

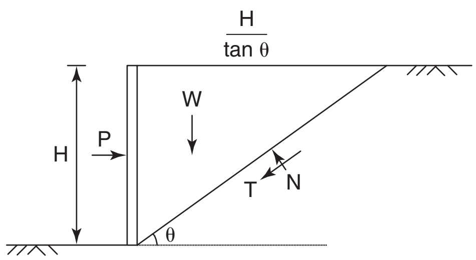


Figure 11.1s Free-body diagram of the failing soil mass.

11.2 A slope is made of a saturated clay with a total unit weight γ, and an undrained shear strength s_u. The slope makes an angle β with the horizontal. Choose 2 circles along which the slope could fail and calculate the factor of safety of that slope against rotation failure along the 2 circles. Why are the 2 factors of safety not the same? Describe how you would find the minimum factor of safety for this slope.

11.3 A pile has a diameter D, a length L, and a modulus E_p. It is subjected to a vertical load Q. The soil generates a constant pile soil friction f. At the pile point the soil generates a point pressure $p = k_p$ w, where w is the vertical displacement of the point and k_p is a constant.

a. Develop the governing differential equation.
b. Find the expression for the top displacement by the finite difference method.

11.4 Develop the closed-form solution for the expansion of an infinitely long cylindrical cavity in an elastic soil space. The soil is weightless and has a Poisson's ratio ν and a modulus E. The cavity has an initial radius r_o. The goal is to generate the curve that gives the radial stress σ_r as a function of the relative increase in cavity radius $\Delta r/r_o$.

11.5 Develop the closed-form solution for the expansion of a spherical cavity in an elastic soil space. The soil is weightless and has a Poisson's ratio ν and a modulus E. The cavity has an initial radius r_o. The goal is to generate the curve that gives the radial stress σ_r as a function of the relative increase in cavity radius $\Delta r/r_o$.

11.6 Develop the solution for the flow of water through a saturated soil sample in a constant head permeameter. The goal is to find the excess water stress anywhere and at any time in the sample.

11.7 Use the finite element method to construct the global stiffness matrix for triaxial test performed on an elastic soil. The major principal stress is 300 kPa and the minor principal stress is 100 kPa. The modulus is 40 MPa and the Poisson's ratio is 0.35. The height and diameter of the sample are 0.1 m and 0.05 m respectively. Consider an axisymmetric geometry and use two four-noded elements.

11.8 Two weightless particles of fine sand have a diameter of 1 mm and are placed in the corner of a container as shown in Figure 11.2s. The vertical load applied on the top particle is 0.4 kN. Find all forces between the particles, the wall, and the ground surface. Calculate the contact stress between the two particles if the contact area is 0.005 mm^2. The angles θ_1 and θ_2 are equal to 45°.

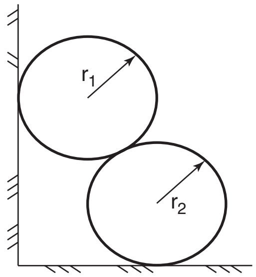


Figure 11.2s Discrete element problem.

11.9 A slope is to be designed for a target probability of failure of 0.001. Plot the mean factor of safety μ versus the coefficient of variation CoV_F in the following cases:
  a. F follows a normal distribution.
  b. F follows a lognormal distribution.

11.10 A levee system is to be designed to meet a risk of 0.001 fatalities/yr and \$1000/yr. It protects a city where 500,000 people could die and where the potential economic loss is \$200 billion if the system fails. What would you recommend for the design annual probability of failure of the levee system?

11.11 A levee system is to be designed to meet a risk of 0.001 fatalities/yr and \$1000/yr. It protects farmland where 100 people and a few cows could die and where the total potential economic loss is \$200 million. What would you recommend for the design probability of failure of the levee system?

11.12 The set of data (y, x) shown in Table 11.1s is plotted and a linear regression (y = ax + b) is performed. Calculate the values of a and b by:
  a. Minimizing the vertical distance between the measured and predicted y values.
  b. Minimizing the normal distance between the measured data and the regression line.
  c. Compare the results.

Table 11.1s Data Set

Data point number	x value	y value
1	2.1	7.4
2	4.5	10.1
3	4.8	11.7
4	5.3	12.4
5	5.7	13.1
6	6.2	16.7
7	7.8	23.4

11.13 Use consistent units to find the relationship between the shear wave velocity v_s, the mass density ρ, and the shear modulus of elasticity G.

11.14 The following empirical equations are used in sands to obtain the ultimate pressure p_u under a driven pile point and the ultimate friction f_u on a driven pile side. Use normalization to give these formulas with p_u and f_u in the U.S. customary system.

$$p_u(\text{kPa}) = 1000\ (N(\text{bl/ft}))^{0.5}$$
$$f_u(\text{kPa}) = 5\ (N(\text{bl/ft})^{0.7}$$

11.15 Perform a dimensional analysis for a square footing embedded at a depth d in a clay with an undrained shear strength s_u. The footing size is B and the failure load is Q_u.

Problems and Solutions

Problem 11.1

A vertical wall is supporting a clean, dry sand backfill with a unit weight γ and effective angle of internal friction φ' (Figure 11.1s). It is assumed that there is no friction between the wall and the backfill. The wall exerts a horizontal load P against the sand. As the wall is pushed into the sand, the load P increases and there is a point where the sand behind the wall fails. At that point, the load is P_p corresponding to the passive earth pressure and the question is to find the load P_p corresponding to impending failure of the sand. Note that the problem is a plane strain problem.

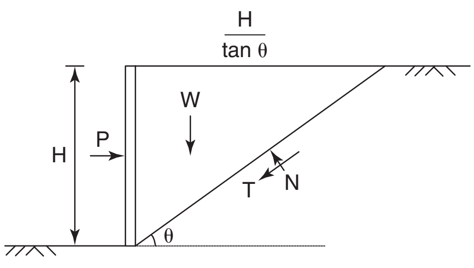


Figure 11.1s Free-body diagram of the failing soil mass.

Solution 11.1

The free-body diagram of the failing soil mass is shown in Figure 11.1s. All the external forces are shown on the diagram, including the weight of the soil mass W, the normal force N, and the shear force T on the failure plane. Also, the force generated by the wall is shown in the diagram as P. Here the failure plane is assumed to be at an angle θ from the horizontal plane. Note that the direction of the shear force T is acting toward the bottom of the wedge, because the soil has the tendency to move upward along the failure surface. The equilibrium equations are set up as follows:

$$W - N\cos\theta + T\sin\theta = 0$$

$$P - N\sin\theta - T\cos\theta = 0$$

The constitutive equations in this case are the shear strength equation of the sand and the expression of the weight of the wedge:

$$W = \frac{\gamma H^2}{2\tan\theta}$$

$$T = N\tan\varphi$$

We can then obtain N and P as:

$$N = \frac{W}{\cos\theta - \tan\varphi\sin\theta} = \frac{\gamma H^2}{2\tan\theta\ (\cos\theta - \tan\varphi\sin\theta)}$$

and

$$P = \frac{\gamma H^2(\sin\theta\cos\theta + \tan\varphi\cos^2\theta)}{2(\sin\theta\cos\theta - \tan\varphi\sin^2\theta)}$$

The maximum value of P, which is P_p, is obtained by setting $\frac{dP}{d\theta} = 0$:

$$\frac{dP}{d\theta} = \frac{\gamma H^2}{2} \times \frac{(-\cos 2\theta + \tan\varphi\sin 2\theta)\tan\varphi}{(\sin\theta\cos\theta - \tan\varphi\sin^2\theta)^2} = \frac{\gamma H^2(\sin 2\theta\sin^2\varphi - \cos 2\theta\sin\varphi\cos\varphi)}{2\sin^2\theta\ \cos^2(\theta+\varphi)} = 0$$

There are two solutions to this equation: one is $\varphi = 0$, which is not realistic, and the other one is:

$$\theta = \frac{\pi}{4} - \frac{\varphi}{2}.$$

The load P_p can then be expressed as:

$$P_p = \frac{\gamma H^2}{2}\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right).$$

Problem 11.2

A slope is made of a saturated clay with a total unit weight γ, and an undrained shear strength s_u. The slope makes an angle β with the horizontal. Choose 2 circles along which the slope could fail and calculate the factor of safety of that slope against rotation failure along the 2 circles. Why are the 2 factors of safety not the same? Describe how you would find the minimum factor of safety for this slope.

Solution 11.2

Case 1: The circle is chosen as shown in Figure 11.3s. The center of the circle is 20 m horizontally away from the edge of the slope, and 10 m vertically above the top of the slope. The radius of the circle is 20 m.

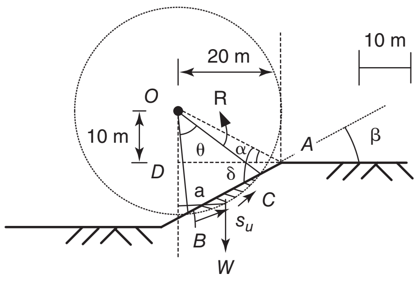


Figure 11.3s Illustration of the slope potential failure surface (case 1).

$$F.S. = \frac{M_r}{M_d} = \frac{s_u \cdot l \cdot R}{W \cdot a} = \frac{s_u \cdot \theta \cdot R^2}{W \cdot a} \quad \text{(a unit width of the soil slice is analyzed)}$$

Here, θ is in radians, a is the arm of the weight of the failure area (hatched), W is the weight of the failing soil, s_u is the undrained shear strength of the soil, and R is the radius of the circle.

In triangle ODA,

$$|OA| = \sqrt{|OD|^2 + |DA|^2} = \sqrt{10^2 + 20^2} = 22.4 \text{ m}$$

$$\alpha = \arctan\left(\frac{10}{20}\right) = 26.6^\circ$$

Therefore,

$$\delta = \alpha + \beta = 26.6 + 30 = 56.6^\circ$$

In triangle OCA (OBA),

$$|OC|^2 = |OA|^2 + |AC|^2 - 2 \times |OA| \times |AC| \times \cos\delta$$

and

$$20^2 = 22.4^2 + |AC|^2 - 2 \times 22.4 \times |AC| \times \cos 56.6^\circ$$

Therefore,

$$|AC| = 5.2 \text{ m}$$

and with the same method:

$$|AB| = 19.4 \text{ m}$$

$$|BC| = 14.2 \text{ m}$$

In triangle OBC,

$$\cos\theta = \frac{|OC|^2 + |OB|^2 - |BC|^2}{2 \times |OC| \times |OB|} = \frac{20^2 + 20^2 - 14.2^2}{2 \times 20 \times 20} = 0.748$$

$$\theta = 42^\circ = 0.733 \text{ rad}$$

The weight of the circular segment can be calculated as:

$$W = \gamma \cdot A \cdot 1 = \gamma \cdot \frac{R^2}{2}(\theta - \sin\theta) \cdot 1 = 19 \times \frac{20^2}{2}(0.733 - \sin 0.733) \times 1 = 242.8 \text{ kN}$$

$$a = \frac{4R \sin^3\left(\frac{\theta}{2}\right)}{3(\theta - \sin\theta)} \times \sin\beta = \frac{4 \times 20 \times \sin^3\left(\frac{0.733}{2}\right)}{3(0.733 - \sin 0.733)} \times \sin 30^\circ = 9.6 \text{ m}$$

The safety of factor can be obtained as follows:

$$F.S. = \frac{M_r}{M_d} = \frac{s_u \cdot l \cdot R \cdot 1}{W \cdot a} = \frac{s_u \cdot \theta \cdot R^2 \cdot 1}{W \cdot a} = \frac{50 \times 0.733 \times 20^2 \times 1}{242.8 \times 9.6} = 6.3$$

Case 2: The circle is chosen as shown in Figure 11.4s. The center of the circle is at 24.2 m distance horizontally away from the edge of the slope, and 16.2 m distance vertically above the top surface of the slope. The radius of the circle is defined to be 36.5 m.

$$F.S. = \frac{M_r}{M_d} = \frac{s_u \cdot l \cdot R \cdot 1}{W_1 \cdot a + W_2 \cdot b} = \frac{s_u \cdot \theta \cdot R^2 \cdot 1}{W_1 \cdot a + W_2 \cdot b} \quad \text{(unit width of the soil slice is analyzed)}$$

Here, θ is in radians, a is the moment arm of the weight of the failure area (circular segment), W_1 is the weight of the circular segment, b is the moment arm of the weight of triangle ABC, W_2 is the weight of triangle ABC, s_u is the undrained shear strength of the soil, and R is the radius of the circle.

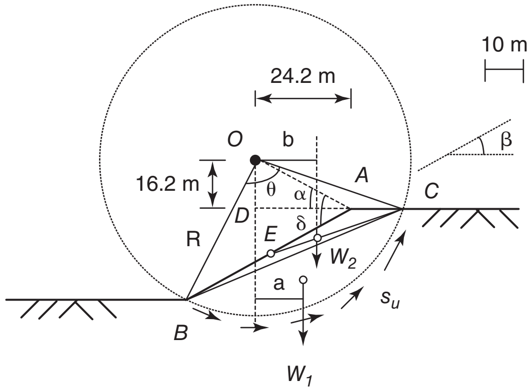


Figure 11.4s Illustration of the slope potential failure surface (case 2).

In triangle ODA,

$$|OA| = \sqrt{|OD|^2 + |DA|^2} = \sqrt{16.2^2 + 24.2^2} = 29.1 \text{ m}$$
$$\alpha = \arctan \frac{16.2}{24.2} = 33.8^\circ$$

Therefore,

$$\delta = \alpha + \beta = 33.8 + 30 = 63.8^\circ$$

In triangle OAC,

$$|OC|^2 = |OA|^2 + |AC|^2 - 2 \times |OA| \times |AC| \times \cos(180^\circ - \alpha)$$
$$36.5^2 = 29.1^2 + |AC|^2 - 2 \times 29.1 \times |AC| \times \cos(180^\circ - 33.8^\circ)$$

Therefore,

$$|AC| = 8.5 \text{ m}$$

In triangle OBA,

$$|OA|^2 + |AB|^2 - 2 \times |OA| \times |AB| \times \cos\delta = |OB|^2$$
$$29.1^2 + |AB|^2 - 2 \times 29.1 \times |AB| \times \cos 63.8^\circ = 36.5^2$$
$$|AB| = 38.4 \text{ m}$$

In triangle ABC,

$$|BC| = \sqrt{|AB|^2 + |AC|^2 - 2 \times |AB| \times |AC| \times \cos 150^\circ} = \sqrt{38.4^2 + 8.5^2 - 2 \times 38.4 \times 8.5 \times \cos 150^\circ} = 46 \text{ m}$$
$$\cos \angle ACB = \frac{8.5^2 + 46^2 - 38.4^2}{2 \times 8.5 \times 46} = 0.913$$
$$\angle ACB = 24.1^\circ$$

In triangle OBC,

$$\cos\theta = \frac{|OB|^2 + |OC|^2 - |BC|^2}{2 \times |OC| \times |OB|} = \frac{36.5^2 + 36.5^2 - 46^2}{2 \times 36.5 \times 36.5} = 0.206$$

$$\theta = 78^\circ = 1.36 \text{ rad}$$

The weight of the circular segment can be calculated as:

$$W_1 = \gamma \cdot A \cdot 1 = \gamma \cdot \frac{R^2}{2}(\theta - \sin\theta) = 19 \times \frac{36.5^2}{2}(1.36 - \sin 1.36) = 4836 \text{ kN/m}$$

$$a = \frac{4R\sin^3\left(\frac{\theta}{2}\right)}{3(\theta - \sin\theta)} \times \sin\angle ACB = \frac{4 \times 36.5 \times \sin^3\left(\frac{1.36}{2}\right)}{3(1.36 - \sin 1.36)} \times \sin 24.1^\circ = 12.9 \text{ m}$$

E is the center point of segment AB:

$$|AE| = \frac{1}{2}|AB| = \frac{1}{2} \times 38.4 = 19.2 \text{ m}$$

$$|CE| = \sqrt{|AE|^2 + |AC|^2 - 2 \times |AE| \times |AC| \times \cos 150^\circ} = \sqrt{19.2^2 + 8.5^2 - 2 \times 19.2 \times 8.5 \times \cos 150^\circ} = 26.9 \text{ m}$$

$$\cos\angle ACE = \frac{8.5^2 + 26.9^2 - 19.2^2}{2 \times 8.5 \times 26.9} = 0.934,$$

$$\angle ACE = 20.9^\circ$$

$$b = |DA| + |AC| - \frac{2}{3}|CE|\cos\angle ACE = 24.2 + 8.5 - \frac{2}{3} \times 26.9 \times \cos 20.9^\circ = 15.9 \text{ m}$$

$$W_2 = \gamma \cdot A \cdot 1 = \gamma \cdot \frac{1}{2}|AC| \cdot |AB| \cdot \sin 150^\circ \cdot 1 = 19 \times \frac{1}{2} \times 8.5 \times 38.4 \times \sin 150^\circ \times 1 = 1550.4 \text{ kN}$$

The safety of factor can be obtained as follows:

$$F.S. = \frac{M_r}{M_d} = \frac{s_u \cdot l \cdot R \cdot 1}{W_1 \cdot a + W_2 \cdot b} = \frac{s_u \cdot \theta \cdot R^2 \cdot 1}{W_1 \cdot a + W_2 \cdot b} = \frac{50 \times 1.36 \times 36.5^2 \times 1}{4836 \times 12.9 + 1550.4 \times 15.9} = 1.04$$

The two results are different because different potential failure circles are chosen to perform the calculation. The minimum factor of safety should be obtained by repeating this trial-and-error process until the minimum factor of safety is found. Choose different locations for the center of the circle and different radii, and perform the calculations following the procedure used previously until the minimum factor of safety is found. It is recommended that you use a software program to minimize the time spent on the calculations!

Problem 11.3

A pile has a diameter D, a length L, and a modulus E_p. It is subjected to a vertical load Q. The soil generates a constant pile soil friction f. At the pile point the soil generates a point pressure $p = k_p w$, where w is the vertical displacement of the point and k_p is a constant.

a. Develop the governing differential equation
b. Find the expression for the top displacement by the finite difference method.

Solution 11.3 (Figure 11.5s)

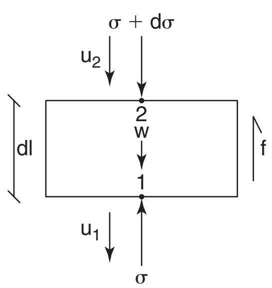


Figure 11.5s Pile element.

a.

$$\left.\begin{array}{l}\text{Equilibrium} : A_s.d\sigma + \gamma.dl.A_s = f.dl.p_p \Rightarrow \dfrac{d\sigma}{dl} = \dfrac{f}{A_s}.p_p - \gamma \\ \text{Constitutive} : \varepsilon = \dfrac{\overbrace{u_2 - u_1}^{du}}{dl} = \dfrac{\sigma}{E} \Rightarrow \dfrac{d\sigma}{dl} = \dfrac{d^2u}{dl^2}E\end{array}\right\} \dfrac{d^2u}{dl^2}E = \dfrac{f}{A_s}p_p - \gamma$$

A_s: Area of pile section σ: Compressive stress
f: friction stress p_p: Perimeter of pile
E: Young's modulus of pile Y: Unit weight of pile

$$\text{Governing differential equation} : \frac{d^2u}{dl^2} = \frac{f}{EA_s}p_p - \frac{\gamma}{E}$$

b. Figure 11.6s Finite difference application:

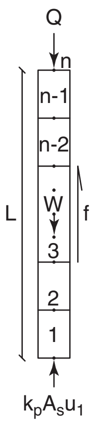


Figure 11.6s Pile discretization.

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta l^2} = \frac{f}{EA_s}p_p - \frac{\gamma}{E} \rightarrow u_{i+1} = \left(\frac{f}{EA_s}p_p - \frac{\gamma}{E}\right).\Delta l^2 + 2u_i - u_{i-1}$$

Boundary condition:

$$\sigma_1 = k_p u_1 \rightarrow \frac{u_2 - u_1}{\Delta l} = \frac{k_p u_1}{E} \rightarrow u_2 = \left(1 + \frac{k_p \Delta l}{E}\right)u_1$$

Equilibrium:

$$k_p.A_s.u_1 = Q + W - fL.p_p \rightarrow u_1 = \frac{Q + W - f.L.p_p}{k_p.A_s}$$

$$u_3 = \left(\frac{f}{EA_s}p_p - \frac{\gamma}{E}\right).\Delta l^2 + 2u_2 - u_1 \rightarrow u_3 = \left(\frac{f}{EA_s}p_p - \frac{\gamma}{E}\right).\Delta l^2 + 2\left(1 + \frac{k_p\Delta l}{E}\right)u_1 - u_1$$

$$= \left(\frac{f}{EA_s}p_p - \frac{\gamma}{E}\right).\Delta l^2 + \left(1 + 2\frac{k_p\Delta l}{E}\right)u_1$$

$$u_4 = \left(\frac{f}{EA_s}p_p - \gamma\right).\Delta l^2 + 2u_3 - u_2 \rightarrow u_4$$

$$= \left(\frac{f}{EA_s}p_p - \frac{\gamma}{E}\right).\Delta l^2 + 2\left[\left(\frac{f}{EA_s}p_p - \frac{\gamma}{E}\right).\Delta l^2 + \left(1 + 2\frac{k_p\Delta l}{E}\right)u_1\right] - \left(1 + \frac{k_p\Delta l}{E}\right)u_1$$

$$= 3\left(\frac{f}{EA_s}p_p - \frac{\gamma}{E}\right).\Delta l^2 + \left(1 + 3\frac{k_p\Delta l}{E}\right)u_1$$

$$u_5 = \left(\frac{f}{EA_s}p_p - \frac{\gamma}{E}\right).\Delta l^2 + 2u_4 - u_3 \to u_5$$

$$= \left(\frac{f}{EA_s}p_p - \frac{\gamma}{E}\right).\Delta l^2 + 2\left[3\left(\frac{f}{EA_s}p_p - \frac{\gamma}{E}\right).\Delta l^2 + \left(1 + 3\frac{k_p\Delta l}{E}\right)u_1\right]$$

$$-\left[\left(\frac{f}{EA_s}p_p - \frac{\gamma}{E}\right).\Delta l^2 + \left(1 + 2\frac{k_p\Delta l}{E}\right)u_1\right]$$

$$= 6\left(\frac{f}{EA_s}p_p - \frac{\gamma}{E}\right).\Delta l^2 + \left(1 + 4\frac{k_p\Delta l}{E}\right)u_1$$

$$u_n = \frac{(n-1)(n-2)}{2}\left(\frac{f}{EA_s}p_p - \frac{\gamma}{E}\right).\Delta l^2 + \left[1 + (n-1)\frac{k_p\Delta l}{E}\right]u_1$$

$$\Delta l = \frac{L}{n-1}$$

$$u_n = \frac{(n-2)}{2(n-1)}\left(\frac{f}{EA_s}p_p - \frac{\gamma}{E}\right).L^2 + \left[1 + \frac{k_pL}{E}\right]\left(\frac{Q + W - f.L.p_p}{k_p.A_s}\right)$$

$$n \to \infty \quad u_{top} = \frac{Q.L}{EA_s} - \frac{f.p_p.L^2}{2EA_s} + \frac{W.L}{2EA_s} + \left(\frac{Q + W - f.L.p_p}{k_pA_s}\right)$$

Problem 11.4

Develop the closed-form solution for the expansion of an infinitely long cylindrical cavity in an elastic soil space. The soil is weightless and has a Poisson's ratio ν and a modulus E. The cavity has an initial radius r_o. The goal is to generate the curve that gives the radial stress σ_r as a function of the relative increase in cavity radius $\Delta r/r_o$.

Solution 11.4

The geometry of the problem indicates that this is an axisymmetric problem and a plane strain problem in the vertical direction. The initial state of stress is σ_{ov} in the vertical direction and σ_{oh} in the radial direction at any point in the soil space. After applying the pressure p at the cavity surface, the stresses in the mass become:

$$\sigma_{rr} = \sigma_{oh} + \Delta\sigma_{rr}$$
$$\sigma_{\theta\theta} = \sigma_{oh} + \Delta\sigma_{\theta\theta}$$
$$\sigma_{zz} = \sigma_{ov} + \Delta\sigma_{zz}$$

where σ_{rr} and $\sigma_{\theta\theta}$ are the radial stress and the hoop stress respectively at a distance r from the axis of the cylinder, and $\Delta\sigma_{rr}$ and $\Delta\sigma_{\theta\theta}$ are the increments of the radial and hoop stress above the at-rest stress value.

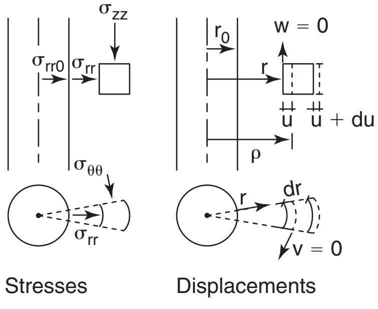


Figure 11.7s Element of soil around an expanding cylindrical cavity.

The radial displacement u is the only displacement type in this problem, as there are no displacements in the hoop direction or in the vertical direction. The radial strain is ε_{rr}, the hoop strain is $\varepsilon_{\theta\theta}$, and the vertical strain is ε_{zz}, which is zero because of the plane strain condition in the z direction. The relationships between the displacement and the strains for small strain theory are given in the following. Note that the minus sign is there to keep compression positive, because u decreases when r increases:

$$\varepsilon_{rr} = -\frac{du}{dr} \tag{11.1s}$$

$$\varepsilon_{\theta\theta} = -\frac{u}{r} \tag{11.2s}$$

$$\varepsilon_{zz} = 0 \tag{11.3s}$$

The equations of equilibrium reduce to:

$$\frac{d\sigma_{rr}}{dr} + \frac{\Delta\sigma_{rr} - \Delta\sigma_{\theta\theta}}{r} = 0 \tag{11.4s}$$

The constitutive equations are:

$$\varepsilon_{rr} = \frac{1}{E}(\Delta\sigma_{rr} - \nu(\Delta\sigma_{\theta\theta} + \Delta\sigma_{zz})) \tag{11.5s}$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}(\Delta\sigma_{\theta\theta} - \nu(\Delta\sigma_{zz} + \Delta\sigma_{rr})) \tag{11.6s}$$

$$\varepsilon_{zz} = \frac{1}{E}(\Delta\sigma_{zz} - \nu(\Delta\sigma_{rr} + \Delta\sigma_{\theta\theta})) \tag{11.7s}$$

By combining Eqs. 11.1s to 11.7s, the governing differential equation is obtained as:

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = 0 \tag{11.8s}$$

Assume that $u = r^n$. Then, by plugging it into Eq. 11.4s, we can get $n = 1$, $n = -1$.

$$\text{So, } u = \frac{A}{r} + Br$$

From boundary conditions

$$u = 0 \text{ when } r = \infty,\ \text{ we get } B = 0$$

$$u = u_0, \text{ when } r = r_0,\ \text{ we get } A = u_0 r_0$$

Therefore,

$$u = \frac{u_0 r_0}{r}$$

The strains are:

$$\varepsilon_r = -\frac{du}{dr} = \frac{u_0 r_0}{r^2}$$

$$\varepsilon_\theta = -\frac{u}{r} = -\frac{u_0 r_0}{r^2}$$

Note that from Eqs. 11.1s to 11.7s:

$$\Delta\sigma_r = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\frac{u_0 r_0}{r^2} - \frac{\nu}{1-\nu}\frac{u_0 r_0}{r^2}\right] = \frac{E}{1+\nu}\frac{u_0 r_0}{r^2}$$

Therefore,

$$\Delta\sigma_{r(r=r_0)} = \frac{E}{1+\nu}\frac{u_0 r_0}{{r_0}^2} = \frac{E}{1+\nu}\frac{u_0}{r_0}$$

and

$$\sigma_{rr} = \sigma_{oh} + 2G\frac{u_o r_o}{r^2} \qquad \sigma_{\theta\theta} = \sigma_{oh} - 2G\frac{u_o r_o}{r^2} \qquad \sigma_{zz} = \sigma_{ov}$$

In a pressuremeter test, the relative increase in radius ($u_o/r_o = \varepsilon_{\theta o}$) of the cavity is measured along with the pressure exerted on the cavity wall σ_{rro}. Therefore, the pressuremeter curve is a direct plot of a stress-strain curve of the soil.

Problem 11.5

Develop the closed-form solution for the expansion of a spherical cavity in an elastic soil space. The soil is weightless and has a Poisson's ratio ν and a modulus E. The cavity has an initial radius r_o. The goal is to generate the curve that gives the radial stress σ_r as a function of the relative increase in cavity radius $\Delta r/r_o$.

Solution 11.5

1. The equilibrium equation in spherical coordinates is:

$$\sigma_r(r\mathrm{d}\theta)(r\mathrm{d}\phi) - \left(\sigma_r + \frac{\partial\sigma_r}{\partial r}\mathrm{d}r\right)(r+\mathrm{d}r)^2\mathrm{d}\theta\mathrm{d}\phi + \left(\sigma_\theta + \sigma_\theta + \frac{\partial\sigma_\theta}{\partial\theta}\partial\theta\right)\frac{\mathrm{d}\theta}{2}r\mathrm{d}\phi\mathrm{d}r$$
$$+ \left(\sigma_\phi + \sigma_\phi + \frac{\partial\sigma_\phi}{\partial\phi}\partial\phi\right)\frac{\mathrm{d}\phi}{2}r\mathrm{d}\theta\mathrm{d}r = 0$$

Ignoring the higher terms:

$$-\frac{\partial\sigma_r}{\partial r} - \frac{2\sigma_r}{r} + \frac{\sigma_\theta}{r} + \frac{\sigma_\phi}{r} = 0$$

For $\sigma_\theta = \sigma_\varphi$, it becomes:

$$\frac{\partial\sigma_r}{\partial r} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0$$

2. Obtain stress-strain relationships (constitutive equations) in the elastic range.

Due to symmetry, $\begin{matrix}\sigma_\theta = \sigma_\phi \\ \varepsilon_\theta = \varepsilon_\phi\end{matrix}$ and σ_θ, σ_ϕ, and σ_r are principal stresses.

Constitutive Equations

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu(\sigma_\theta + \sigma_\phi)) = -\frac{\mathrm{d}u}{\mathrm{d}r}$$
$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu(\sigma_r + \sigma_\phi)) = -\frac{u}{r}$$
$$\varepsilon_\phi = \frac{1}{E}(\sigma_\phi - \nu(\sigma_r + \sigma_\theta)) = -\frac{u}{r}$$

Again, the minus signs are there to keep compression positive. So:

$$\varepsilon_r = \frac{1}{E}(\sigma_r - 2\nu\sigma_\theta) = -\frac{\mathrm{d}u}{\mathrm{d}r}$$
$$\varepsilon_\theta = \varepsilon_\phi = \frac{1}{E}((1-\nu)\sigma_\theta - \nu\sigma_r) = -\frac{u}{r}$$

Obtain

$$\sigma_r = f(u, r, \mathrm{d}u, \mathrm{d}r)$$
$$\sigma_\theta = g(u, r, \mathrm{d}u, \mathrm{d}r)$$

Solve for $u = F(r)$ with the appropriate boundary conditions.
Because

$$\varepsilon_r = -\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{1}{E}(\sigma_r - 2\nu\sigma_\theta)$$

we have:

$$-E\frac{\mathrm{d}u}{\mathrm{d}r} + 2\nu\sigma_\theta = \sigma_r$$

$$-E\frac{u}{r} = (1-\nu)\sigma_\theta - \nu\left[-E\frac{\mathrm{d}u}{\mathrm{d}r} + 2\nu\sigma_\theta\right]$$

$$-E\frac{u}{r} = (1-\nu)\sigma_\theta + \nu E\frac{\mathrm{d}u}{\mathrm{d}r} - 2\nu^2\sigma_\theta$$

$$-E\left(\frac{u}{r} + \frac{\nu\mathrm{d}u}{\mathrm{d}r}\right) = \sigma_\theta(1-\nu-2\nu^2)$$

$$\sigma_\theta = -\frac{E}{(1-\nu-2\nu^2)}\left[\frac{u}{r} + \nu\frac{\mathrm{d}u}{\mathrm{d}r}\right]$$

Solve for

$$\sigma_r = -E\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{2\nu E}{(1-\nu-2\nu^2)}\left[\frac{u}{r} + \nu\frac{\mathrm{d}u}{\mathrm{d}r}\right]$$

$$\sigma_r = -E\frac{\mathrm{d}u}{\mathrm{d}r}\left[\frac{1-\nu-2\nu^2+2\nu^2}{\left(1-\nu-2\nu^2\right)}\right] - \frac{2\nu E}{(1-\nu-2\nu^2)}\frac{u}{r}$$

$$\sigma_r = -\frac{E(1-\nu)}{(1-\nu-2\nu^2)}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{2\nu E}{(1-\nu-2\nu^2)}\frac{u}{r}$$

$$\frac{\partial\sigma_r}{\partial r} = -\frac{E(1-\nu)}{(1-\nu-2\nu^2)}\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{2\nu E}{(1-\nu-2\nu^2)}\frac{u}{r^2} - \frac{\mathrm{d}u}{\mathrm{d}r}\frac{2\nu E}{(1-\nu-2\nu^2)}\frac{1}{r}$$

$$(\sigma_r - \sigma_\theta)\frac{2}{r} = -\frac{2}{r}\left\{\frac{E}{\left(1-\nu-2\nu^2\right)} \times \left[(1-\nu)\frac{\mathrm{d}u}{\mathrm{d}r} + 2\nu\frac{u}{r} - \frac{u}{r} - \frac{\nu\mathrm{d}u}{\mathrm{d}r}\right]\right\}$$

$$= -\frac{2E}{r(1-\nu-2\nu^2)} \times \left[(1-2\nu)\frac{\mathrm{d}u}{\mathrm{d}r} + (2\nu-1)\frac{u}{r}\right]$$

So

$$\frac{\partial\sigma_r}{r} + \frac{2}{r}(\sigma_r - \sigma_\theta) = -\frac{E}{(1-\nu-2\nu^2)}\left\{(1-\nu)\frac{\mathrm{d}^2u}{\mathrm{d}r^2} - 2\nu\frac{u}{r^2} + \frac{2(1-2\nu)}{r}\frac{\mathrm{d}u}{\mathrm{d}r} + 2(2\nu-1)\frac{u}{r^2} + \frac{1}{r}2\nu\frac{\mathrm{d}u}{\mathrm{d}r}\right\}$$

$$= -\frac{E}{(1-\nu-2\nu^2)}\left\{(1-\nu)\frac{\mathrm{d}^2u}{\mathrm{d}r^2} - 2(1-\nu)\frac{u}{r^2} + \frac{2(1-\nu)}{r}\frac{\mathrm{d}u}{\mathrm{d}r}\right\} = 0$$

and

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} - 2\frac{u}{r^2} + \frac{2}{r}\frac{\mathrm{d}u}{\mathrm{d}r} = 0$$

or

$$r^2\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + 2r\frac{\mathrm{d}u}{\mathrm{d}r} - 2u = 0$$

Solve the differential equation:

$$u = r^n$$

$$r^2(n)(n-1)r^{n-2} + 2r(n)r^{n-1} - 2r^n = 0$$

$$\left.\begin{array}{l} n(n-1) + 2n - 2 = 0 \\ n(n-1) + 2(n-1) = 0 \end{array}\right\} n = -1, n = 1$$

$$u = Ar + \frac{B}{r^2}$$

Apply the boundary conditions. When

$$r = \infty \rightarrow u = 0$$
$$A = 0$$
$$u = \frac{B}{r^2}$$

When

$$r = r_0, u = u_0$$
$$B = u_0 r_o^2$$
$$u = \frac{u_0 r_0^2}{r^2}$$

The strains can be expressed as:

$$\left.\begin{array}{l} \varepsilon_\theta = -\frac{u}{r} = -\frac{u_0 r_0^2}{r^3} = \varepsilon_\phi \\ \varepsilon_r = -\frac{du}{dr} = 2\frac{u_0 r_0^2}{r^3} \end{array}\right\} \frac{\Delta V}{V} = \varepsilon_r + 2\varepsilon_\theta = 2\frac{u_0 r_0^2}{r^3} - 2\frac{u_0 r_0^2}{r^3} = 0$$

Recall the equilibrium equation:

$$\frac{\partial \sigma_r}{\partial r} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0$$

The preceding derivation assumes an unstressed initial state. If the soil is under a hydrostatic initial state of stress equal to $p_o(\sigma_\phi = \sigma_r = \sigma_\theta = p_0)$, then the preceding solutions are in terms of stress increments as follows:

$$\sigma_r = p_0 + \Delta\sigma_r$$
$$\sigma_\theta = p_0 + \Delta\sigma_\theta$$
$$\sigma_\phi = p_0 + \Delta\sigma_\phi$$

Thus, the equilibrium equation can be written as:

$$\frac{\partial \sigma_r}{\partial r} + \frac{2}{r}(\Delta\sigma_r - \Delta\sigma_\theta) = 0$$

As seen previously, we have:

$$\Delta\sigma_r = -\frac{E}{(1-\nu-2\nu^2)}\left[(1-\nu)\frac{\mathrm{d}u}{\mathrm{d}r} + 2\nu\frac{u}{r}\right]$$
$$\frac{\mathrm{d}u}{\mathrm{d}r} = -2\frac{u_0 r_0^2}{r^3}$$
$$\frac{u}{r} = \frac{u_0 r_0^2}{r^3}$$

Therefore,

$$\Delta\sigma_r = -\frac{E}{(1-\nu-2\nu^2)}\frac{u_0 r_0^2}{r^3}(-2(1-\nu)+2\nu) = -\frac{2(2\nu-1)E}{(1-\nu-2\nu^2)}\frac{u_0 r_0^2}{r^3}$$
$$= -\frac{2(2\nu-1)}{(1+\nu)(1-2\nu)}E\left(\frac{u_0 r_0^2}{r^3}\right)$$
$$\Delta\sigma_r = \frac{2E}{(1+\nu)}\frac{u_0 r_0^2}{r^3}$$

At

$$r = r_0 \rightarrow \Delta\sigma_r = \frac{2E}{1+\nu}\frac{u_0}{r_0}$$

Therefore,

$$\Delta\sigma_{r(r=r_0)} = \frac{2E}{1+\nu}\frac{u_0 r_0^2}{{r_0}^3} = \frac{2E}{1+\nu}\frac{u_0}{r_0} = -4G\varepsilon_{\theta 0}$$

and by the same process:

$$\Delta\sigma_{\theta(r=r_0)} = 2G\varepsilon_{\theta 0}$$

Problem 11.6

Develop the solution for the flow of water through a saturated soil sample in a constant head permeameter. The goal is to find the excess water stress anywhere and at any time in the sample.

Solution 11.6

Conservation of mass law: Qdt = vAdt

Darcy's law: vdl = kdh

$$dh = \frac{Q}{kA}dl$$

$$\int_{h_1}^{h_2} dh = \frac{Q}{kA}\int_{l_1}^{l_2} dl \rightarrow \Delta h = \frac{Q}{kA}\Delta l$$

$$\Delta u_w = \gamma_w \Delta h = \gamma_w \frac{Q}{kA}\Delta l$$

Note that the total head h decreases through the sample; therefore so will u_w. See Figure 9.69.

Problem 11.7

Use the finite element method to construct the global stiffness matrix for a triaxial test performed on an elastic soil. The major principal stress is 300 kPa and the minor principal stress is 100 kPa. The modulus is 40 MPa and the Poisson's ratio is 0.35. The height and diameter of the sample are 0.1 m and 0.05 m respectively. Consider an axisymmetric geometry and use two four-noded elements.

Solution 11.7

Refer to section 11.5.3 for the equations used in this problem

Step 1: The selected elements are shown in Figure 11.8s.
The element dimensions are $a = 0.025$ m and $b = 0.05$ m. The soil properties are $E = 40000$ kPa and $\mu = 35$.

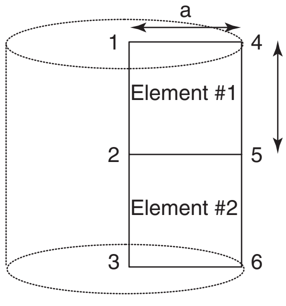


Figure 11.8s Triaxial test mesh.

Step 2: Choose the interpolation or shape functions. The equations for these functions are:

$$H_1 = \frac{1}{4}(1+r)(1+s)$$
$$H_2 = \frac{1}{4}(1-r)(1+s)$$
$$H_3 = \frac{1}{4}(1-r)(1-s)$$
$$H_4 = \frac{1}{4}(1+r)(1-s)$$

Step 3: Write the strain-displacement equations:

$$[\varepsilon] = [B][u_i]$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial H_1}{\partial x} & 0 & \frac{\partial H_2}{\partial x} & 0 & \frac{\partial H_3}{\partial x} & 0 & \frac{\partial H_4}{\partial x} & 0 \\ 0 & \frac{\partial H_1}{\partial y} & 0 & \frac{\partial H_2}{\partial y} & 0 & \frac{\partial H_3}{\partial y} & 0 & \frac{\partial H_4}{\partial y} \\ \frac{\partial H_1}{\partial y} & \frac{\partial H_1}{\partial x} & \frac{\partial H_2}{\partial y} & \frac{\partial H_2}{\partial x} & \frac{\partial H_3}{\partial y} & \frac{\partial H_3}{\partial x} & \frac{\partial H_4}{\partial y} & \frac{\partial H_4}{\partial x} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Construct the [B] matrix:

a. Calculate the inverse of the Jacobian matrix used in the transformation from natural coordinates to real coordinates:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{b}{2} \end{bmatrix} = \begin{bmatrix} 0.0125 & 0 \\ 0 & 0.025 \end{bmatrix}$$

Therefore,

$$\det J = 3.125 * 10^{-4}$$

and

$$J^{-1} = \left(\frac{1}{\det J}\right) \cdot \begin{bmatrix} \frac{b}{2} & 0 \\ 0 & \frac{a}{2} \end{bmatrix} = \begin{bmatrix} 80 & 0 \\ 0 & 40 \end{bmatrix}$$

$$\frac{\partial H}{\partial r} = \left[\frac{1}{4}(1+s) \quad -\frac{1}{4}(1+s) \quad -\frac{1}{4}(1+s) \quad \frac{1}{4}(1+s)\right]$$

$$\frac{\partial H}{\partial s} = \left[\frac{1}{4}(1+r) \quad \frac{1}{4}(1+r) \quad -\frac{1}{4}(1+r) \quad -\frac{1}{4}(1+r)\right]$$

b. Obtain the relation between the derivatives of the interpolation functions in real coordinates and in natural coordinates:

$$\begin{bmatrix} \frac{\partial H_i}{\partial x} \\ \frac{\partial H_i}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial H_i}{\partial r} \\ \frac{\partial H_i}{\partial s} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{b}{2} & 0 \\ 0 & \frac{a}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial H}{\partial r} \\ \frac{\partial H}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{b}{2} \cdot \frac{\partial H}{\partial r} \\ \frac{a}{2} \cdot \frac{\partial H}{\partial s} \end{bmatrix}$$

c. Select the natural coordinates r and s of the integration points for a four-node element:

$$r = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$s = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

d. Compute the components of the matrix [B] at the four integration points:

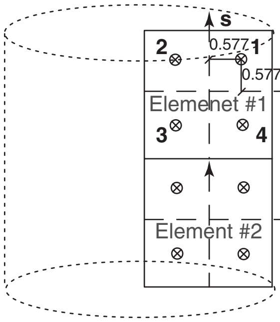


Figure 11.9s The integration points.

Intg. Point #1. The derivatives of the interpolation function are:

$$\frac{\partial H}{\partial r} = \left[\frac{1}{4}(1+s) \quad -\frac{1}{4}(1+s) \quad -\frac{1}{4}(1+s) \quad \frac{1}{4}(1+s)\right]$$
$$\frac{\partial H}{\partial s} = \left[\frac{1}{4}(1+r) \quad \frac{1}{4}(1+r) \quad -\frac{1}{4}(1+r) \quad -\frac{1}{4}(1+r)\right]$$

For integration point #1, the natural coordinates are $r = \frac{1}{\sqrt{3}}$ and $s = \frac{1}{\sqrt{3}}$

$$\frac{\partial H}{\partial r} = \left[\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad \frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right)\right]$$
$$= \left[0.394 \quad -0.394 \quad -0.105 \quad 0.105\right]$$
$$\frac{\partial H}{\partial s} = \left[\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad \frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right)\right]$$
$$= \left[0.394 \quad 0.105 \quad -0.105 \quad -0.394\right]$$

In the case of plane strain (Section 11.5.4), the thickness t of the elements was 1. However, in the case of axisymmetric geometry, the thickness varies across the element and must be evaluated at each integration point. If you look at the element in plan view, it looks like a piece of pizza with an angle θ. For convenience, we take a value of 1 radian for this angle. The thickness t of the element at a radius x_i is equal to x_i times θ. Because θ is 1 rd, the thickness is simply equal to x_i. Therefore, the equation for the thickness t is:

$$t = [H][x] = \left[H_1 \quad H_2 \quad H_3 \quad H_4\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = H_1.x_1 + H_2.x_2 + H_3.x_3 + H_4.x_4$$

where x_i represents the real coordinates of the nodes. For elements 1 and 2, the x matrices are:

$$element\ \#1 \rightarrow [x] = \begin{bmatrix} 0.025 \\ 0 \\ 0 \\ 0.025 \end{bmatrix} \quad \text{and } element\ \#2 \rightarrow [x] = \begin{bmatrix} 0.025 \\ 0 \\ 0 \\ 0.025 \end{bmatrix}$$

$$H_1 = \frac{1}{4}(1+r)(1+s) = \frac{1}{4}(1+\frac{1}{\sqrt{3}})(1+\frac{1}{\sqrt{3}}) = 0.622$$

$$H_2 = \frac{1}{4}(1-r)\,(1+s) = \frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right)(1+\frac{1}{\sqrt{3}} = 0.1667$$

$$H_3 = \frac{1}{4}(1-r)\,(1-s) = \frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right)\left(1-\frac{1}{\sqrt{3}}\right) = 0.0446$$

$$H_4 = \frac{1}{4}(1+r)\,(1-s) = \frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right)\left(1-\frac{1}{\sqrt{3}}\right) = 0.1667$$

$$t = \begin{bmatrix} 0.622 & 0.1667 & 0.0446 & 0.1667 \end{bmatrix} \begin{bmatrix} 0.025 \\ 0 \\ 0 \\ 0.025 \end{bmatrix} = 0.0197\ \text{(m)}$$

Intg Point #2: For integration point #2, the natural coordinates are $r = -\frac{1}{\sqrt{3}}$ and $s = \frac{1}{\sqrt{3}}$

$$\frac{\partial H}{\partial r} = \left[\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right) \quad \frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right)\right]$$
$$= \begin{bmatrix} 0.394 & -0.394 & -0.105 & 0.105 \end{bmatrix}$$

$$\frac{\partial H}{\partial s} = \left[\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right) \quad \frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right)\right]$$
$$= \begin{bmatrix} 0.105 & 0.394 & -0.394 & -0.105 \end{bmatrix}$$

$$H_1 = \frac{1}{4}(1+r)\,(1+s) = \frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{\sqrt{3}}\right) = 0.1667$$

$$H_2 = \frac{1}{4}(1-r)\,(1+s) = \frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{\sqrt{3}}\right) = 0.622$$

$$H_3 = \frac{1}{4}(1-r)(1-s) = \frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right)\left(1-\frac{1}{\sqrt{3}}\right) = 0.1667$$

$$H_4 = \frac{1}{4}(1+r)\,(1-s) = \frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right)\left(1-\frac{1}{\sqrt{3}}\right) = 0.0446$$

$$t = \begin{bmatrix} 0.1667 & 0.622 & 0.1667 & 0.0446 \end{bmatrix} \begin{bmatrix} 0.025 \\ 0 \\ 0 \\ 0.025 \end{bmatrix} = 0.0053\ \text{(m)}$$

Intg. Point #3: For integration point #3, the natural coordinates are $r = -\frac{1}{\sqrt{3}}$ and $s = -\frac{1}{\sqrt{3}}$

$$\frac{\partial H}{\partial r} = \left[\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right) \quad \frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right)\right]$$
$$= \begin{bmatrix} 0.105 & -0.105 & -0.394 & 0.394 \end{bmatrix}$$

$$\frac{\partial H}{\partial s} = \left[\frac{1}{4}\left(1 - \frac{1}{\sqrt{3}}\right) \quad \frac{1}{4}\left(1 + \frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1 + \frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1 - \frac{1}{\sqrt{3}}\right)\right]$$
$$= \begin{bmatrix} 0.105 & 0.394 & -0.394 & -0.105 \end{bmatrix}$$
$$H_1 = \frac{1}{4}(1+r)(1+s) = \frac{1}{4}\left(1 - \frac{1}{\sqrt{3}}\right)\left(1 - \frac{1}{\sqrt{3}}\right) = 0.0446$$
$$H_2 = \frac{1}{4}(1-r)(1+s) = \frac{1}{4}\left(1 + \frac{1}{\sqrt{3}}\right)\left(1 - \frac{1}{\sqrt{3}}\right) = 0.1667$$
$$H_3 = \frac{1}{4}(1-r)(1-s) = \frac{1}{4}\left(1 + \frac{1}{\sqrt{3}}\right)\left(1 + \frac{1}{\sqrt{3}}\right) = 0.622$$
$$H_4 = \frac{1}{4}(1+r)(1-s) = \frac{1}{4}\left(1 - \frac{1}{\sqrt{3}}\right)\left(1 + \frac{1}{\sqrt{3}}\right) = 0.1667$$
$$t = \begin{bmatrix} 0.0446 & 0.1667 & 0.622 & 0.1667 \end{bmatrix} \begin{bmatrix} 0.025 \\ 0 \\ 0 \\ 0.025 \end{bmatrix} = 0.0053 \text{ (m)}$$

Intg. Point #4: For integration point #4, the natural coordinates are $r = \frac{1}{\sqrt{3}}$ and $s = -\frac{1}{\sqrt{3}}$

$$\frac{\partial H}{\partial r} = \left[\frac{1}{4}\left(1 - \frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1 - \frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1 + \frac{1}{\sqrt{3}}\right) \quad \frac{1}{4}\left(1 + \frac{1}{\sqrt{3}}\right)\right]$$
$$= \begin{bmatrix} 0.105 & -0.105 & -0.394 & 0.394 \end{bmatrix}$$
$$\frac{\partial H}{\partial s} = \left[\frac{1}{4}\left(1 + \frac{1}{\sqrt{3}}\right) \quad \frac{1}{4}\left(1 - \frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1 - \frac{1}{\sqrt{3}}\right) \quad -\frac{1}{4}\left(1 + \frac{1}{\sqrt{3}}\right)\right]$$
$$= \begin{bmatrix} 0.394 & 0.105 & -0.105 & -0.394 \end{bmatrix}$$
$$H_1 = \frac{1}{4}(1+r)(1+s) = \frac{1}{4}\left(1 + \frac{1}{\sqrt{3}}\right)\left(1 - \frac{1}{\sqrt{3}}\right) = 0.1667$$
$$H_2 = \frac{1}{4}(1-r)(1+s) = \frac{1}{4}\left(1 - \frac{1}{\sqrt{3}}\right)\left(1 - \frac{1}{\sqrt{3}}\right) = 0.0446$$
$$H_3 = \frac{1}{4}(1-r)(1-s) = \frac{1}{4}\left(1 - \frac{1}{\sqrt{3}}\right)\left(1 + \frac{1}{\sqrt{3}}\right) = 0.1667$$
$$H_4 = \frac{1}{4}(1+r)(1-s) = \frac{1}{4}\left(1 + \frac{1}{\sqrt{3}}\right)\left(1 + \frac{1}{\sqrt{3}}\right) = 0.622$$
$$t = \begin{bmatrix} 0.1667 & 0.0446 & 0.1667 & 0.622 \end{bmatrix} \begin{bmatrix} 0.025 \\ 0 \\ 0 \\ 0.025 \end{bmatrix} = 0.0197 \text{ (m)}$$

Then we assemble the B matrix:

$$B_{\substack{i=1 \\ j=1}} = \begin{bmatrix} 31.547 & 0 & -31.547 & 0 & -8.453 & 0 & 8.453 & 0 \\ 0 & 15.773 & 0 & 4.266 & 0 & -4.266 & 0 & -15.773 \\ 15.773 & 31.547 & 4.266 & -31.547 & -4.266 & -8.453 & -15.773 & 15.773 \\ 31.547 & 0 & 31.547 & 0 & 8.453 & 0 & 8.453 & 0 \end{bmatrix}$$

$$B_{\substack{i=1\\j=2}} = \begin{bmatrix} 31.547 & 0 & -31.547 & 0 & -8.453 & 0 & 8.453 & 0 \\ 0 & 4.226 & 0 & 15.733 & 0 & -15.733 & 0 & -4.226 \\ 4.226 & 31.547 & 15.733 & -31.547 & -15.733 & -8.453 & -4.226 & 8.453 \\ 8.453 & 0 & 117.735 & 0 & 31.547 & 0 & 2.2650 & 0 \end{bmatrix}$$

$$B_{\substack{i=2\\j=1}} = \begin{bmatrix} 8.453 & 0 & -8.453 & 0 & -31.547 & 0 & 31.547 & 0 \\ 0 & 4.226 & 0 & 15.773 & 0 & -15.773 & 0 & -4.226 \\ 4.226 & 8.453 & 15.773 & -8.453 & -15.773 & -31.547 & -4.226 & 31.547 \\ 2.2650 & 0 & 31.547 & 0 & 117.735 & 0 & 8.453 & 0 \end{bmatrix}$$

$$B_{\substack{i=2\\j=2}} = \begin{bmatrix} 8.453 & 0 & -8.453 & 0 & -31.547 & 0 & 31.547 & 0 \\ 0 & 15.773 & 0 & 4.266 & 0 & -4.266 & 0 & -15.773 \\ 15.773 & 8.453 & 4.266 & -8.453 & -4.266 & -31.547 & -15.773 & 31.547 \\ 8.453 & 0 & 8.453 & 0 & 31.547 & 0 & 31.547 & 0 \end{bmatrix}$$

Step 4: Write the stress-strain equations for the soil using the constitutive matrix.

$$C_{4X4} = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1 & \frac{\mu}{(1-\mu)} & \frac{\mu}{(1-\mu)} & 0 \\ \frac{\mu}{(1-\mu)} & 1 & \frac{\mu}{(1-\mu)} & 0 \\ \frac{\mu}{(1-\mu)} & \frac{\mu}{(1-\mu)} & 1 & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2(1-\mu)} \end{bmatrix}$$

$$= 10^4 * \begin{bmatrix} 6.419 & 3.457 & 3.457 & 0 \\ 3.457 & 6.419 & 3.457 & 0 \\ 3.457 & 3.457 & 6.419 & 0 \\ 0 & 0 & 0 & 1.418 \end{bmatrix}$$

Step 5: Derive the equations governing the behavior of the soil element.

$$[K^e] = \int_V [B]^T [C][B] dV$$

$$[K^e][u] = [F]$$

The numerical integration equation is:

$$K_e = \int_v B^T C B dv = \sum_{i=1}^{2} \sum_{j=1}^{2} B_{ij}^T C_{ij} B_{ij} \det J . w_i . w_j . t$$

For a 2-point Gauss quadrature integration, the weighing factors w_i, w_j are equal to 1 and the element stiffness matrix is:

$$K^e = 10^3 \times \begin{bmatrix} 1.12 & 0.8 & -0.42 & -.44 & -0.48 & -0.52 & 0.11 & 0.16 \\ & 1.02 & -0.26 & -0.54 & -0.45 & -0.55 & -0.08 & 0.07 \\ & & 0.92 & 0.19 & 0.43 & 0.05 & -0.05 & 0.09 \\ & & & 0.46 & 0.16 & 0.2 & 0.16 & -0.11 \\ & & & & 1.2 & 0.53 & -0.27 & -0.24 \\ & SYM & & & & 0.75 & -0.05 & -0.39 \\ & & & & & & 0.57 & 0.135 \\ & & & & & & & 0.45 \end{bmatrix}$$

Now the global stiffness matrix K_g can be assembled:

$$K_g = 10^3 \times \begin{bmatrix} 0.92 & 0.12 & 0.43 & 0.046 & 0 & 0 & -0.42 & -0.26 & -0.05 & 0.09 & 0 & 0 \\ & 0.45 & 0.17 & 0.2 & 0 & 0 & -0.45 & -0.54 & 0.16 & -0.11 & 0 & 0 \\ & & 2.12 & 0.65 & 0.43 & 0.04 & -0.48 & -0.45 & -0.68 & -0.5 & -0.05 & 0.09 \\ & & & 1.2 & 0.16 & 0.2 & -0.5 & -0.5 & -0.5 & -0.93 & 0.15 & -0.11 \\ & & & & 1.2 & 0.52 & 0 & 0 & -0.48 & -0.45 & -0.27 & -0.27 \\ & & & & & 0.75 & 0 & 0 & -0.5 & -0.55 & -0.05 & -0.39 \\ & & & & & & 1.12 & 0.8 & 0.11 & 0.16 & 0 & 0 \\ & & & & & & & 1.02 & -0.08 & 0.06 & 0 & 0 \\ & & SYM & & & & & & 1.67 & 0.79 & 0.11 & 0.16 \\ & & & & & & & & & 1.47 & -0.08 & 0.06 \\ & & & & & & & & & & 0.54 & -0.015 \\ & & & & & & & & & & & 0.44 \end{bmatrix}$$

Problem 11.8

Two weightless particles of fine sand have a diameter of 1 mm and are placed in the corner of a container as shown in Figure 11.2s. The vertical load applied on the top particle is 0.4 kN. Find all forces between the particles, the wall, and the ground surface. Calculate the contact stress between the two particles if the contact area is 0.005 mm^2. The angles θ_1 and θ_2 are equal to 45°.

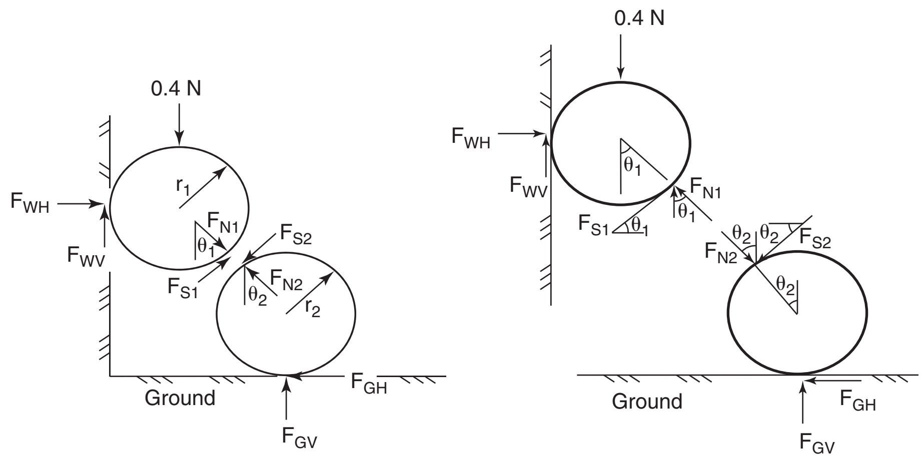


Figure 11.2s Discrete element problem.

Solution 11.8

Ball 1:

$$\sum F_V = F_{WV} + F_{S1}\sin\theta_1 + F_{N1}\cos\theta_1 - Q = 0$$

$$\sum F_H = F_{WH} + F_{S1}\cos\theta_1 - F_{N1}\sin\theta_1 = 0$$

$$\sum M_{center} = F_{S1}r_1 - F_{WV}r_1 = 0$$

Ball 2:

$$\sum F_v = F_{GV} - F_{N2}\cos\theta_2 - F_{S2}\sin\theta_2 = 0$$

$$\sum F_H = F_{N2}\sin\theta_2 - F_{S2}\cos\theta_2 - F_{GH} = 0$$

$$\sum M_{center} = F_{S2}r_2 - F_{GH}r_2 = 0$$

Also, at the contact between the two balls:

$$F_{S1} = F_{S2}$$
$$F_{N1} = F_{N2}$$

There are 8 unknown forces and 8 equations:

$$F_{WV} = F_{S1}$$
$$F_{WV} + F_{WV}\sin\theta_1 + F_{N1}\cos\theta_1 - Q = 0$$
$$F_{N1} = \frac{Q - F_{WV}(1+\sin\theta_1)}{\cos\theta_1}$$
$$F_{S2} = F_{GH}$$
$$F_{N2}\sin\theta_2 - F_{S2}\cos\theta_2 - F_{S2} = 0$$
$$F_{N2}\sin\theta_2 = F_{S2}(\cos\theta_2 + 1)$$
$$F_{N2} = \frac{F_{S2}(\cos\theta_2 + 1)}{\sin\theta_2}$$

For Q = 0.4 kN, $r_1 = r_2 = 0.5$ mm, $\theta_1 = \theta_2 = 45^\circ$

$$F_{WV} = 0.117 \text{ kN}$$
$$F_{S1} = F_{S2} = 0.117 \text{ kN}$$
$$F_{N1} = F_{N2} = 0.283 \text{ kN}$$
$$F_{WH} = 0.117 \text{ kN}$$
$$F_{GH} = 0.117 \text{ kN}$$
$$F_{GV} = 0.283 \text{ kN}$$

If the two balls touch over an area with a unit width of 0.05 mm, the stress distribution is:

$$P = \frac{F_N}{A} = \frac{0.283}{0.05 \times 10^{-6}} = 5660 \text{ kPa}$$

Problem 11.9

A slope is to be designed for a target probability of failure of 0.001. Plot the mean factor of safety μ versus the coefficient of variation CoV_F in the following cases:

a. F follows a normal distribution.
b. F follows a lognormal distribution.

Solution 11.9

a. Normal distribution (Figure 11.10s)
   Probability of failure = 0.001. The mean of F is μ and the standard deviation is σ

$$P(F < 1) = 0.001 \Rightarrow P\left(\frac{F-\mu}{\sigma} < \frac{1-\mu}{\sigma}\right) = 0.001 \Rightarrow \frac{1-\mu}{\sigma} = -3.1$$
$$\sigma = \mu.COV$$
$$1 - \mu = -3.1\mu.COV \rightarrow \mu = \frac{1}{(-3.1COV + 1)}$$

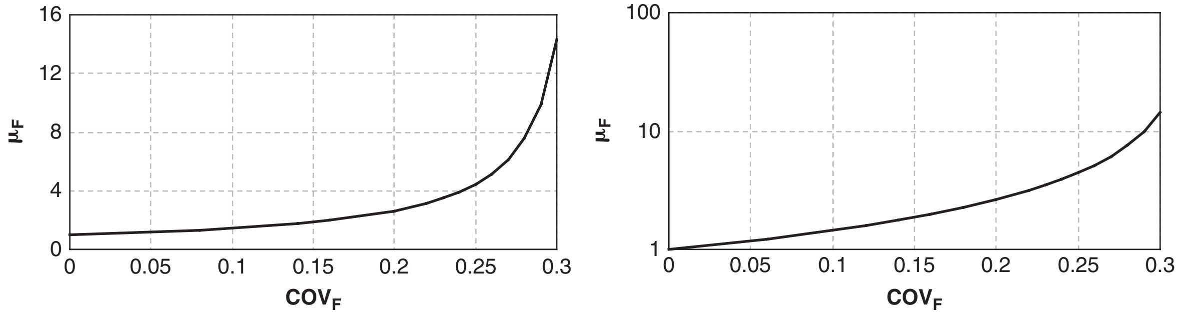


Figure 11.10s μ_F vs. CoV_F for a probability of failure of 0.001 when F follows a normal distribution.

b. Lognormal distribution (Figure 11.11s)

$$P(F<1)=0.001 \Rightarrow P\left(\frac{\ln F-\mu_{\ln F}}{\sigma_{\ln F}}<\frac{\ln 1-\mu_{\ln F}}{\sigma_{\ln F}}\right)=0.001 \Rightarrow \frac{\ln 1-\ln\left(\dfrac{\mu^2}{\sqrt{\mu^2+\mu^2.COV^2}}\right)}{\sqrt{\ln(1+COV^2)}}=-3.1$$

$$\frac{\ln\left(\dfrac{\mu}{\sqrt{1+COV^2}}\right)}{\sqrt{\ln(1+COV^2)}}=3.1 \rightarrow \ln(\mu)=3.1\sqrt{\ln(1+COV^2)}+\ln\sqrt{1+COV^2}$$

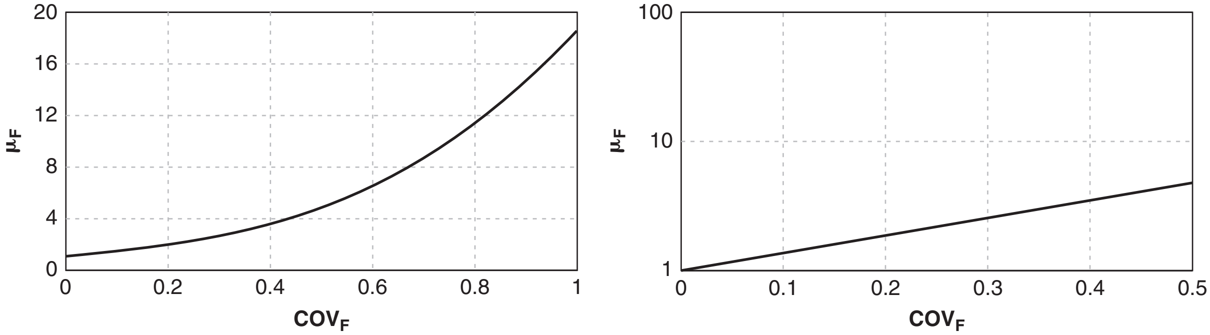


Figure 11.11s μ_F versus CoV_F for a probability of failure of 0.001 when F follows a lognormal distribution.

Problem 11.10

A levee system is to be designed to meet a risk of 0.001 fatalities/yr and \$1000/yr. It protects a city where 500,000 people could die and where the potential economic loss is \$200 billion if the system fails. What would you recommend for the design annual probability of failure of the levee system?

Solution 11.10

R(fatalities) = PoF × F

R(dollars lost) = PoF × D

R: risk

PoF: annual probability of failure

F: lives lost or fatalities if failure occurs = 500,000

D: dollars lost if failure occurs = 200×10^9 \$.

R(fatalities) = 0.001 fatalities/yr

R(dollars lost) = 1000 \$/yr

$$\frac{0.001 \text{ fatality/yr}}{500{,}000 \text{ people}} = 2 \times 10^{-9}$$

$$\frac{1000 \text{ dollars/yr}}{200 \times 10^{9}} = 5 \times 10^{-9}$$

The recommended annual probability of failure is 2×10^{-9}, as it is the most demanding of the two: fatalities control.

Problem 11.11

A levee system is to be designed to meet a risk of 0.001 fatalities/yr and \$1000/yr. It protects farmland where 100 people and a few cows could die and where the total potential economic loss is \$200 million. What would you recommend for the design probability of failure of the levee system?

Solution 11.11

R(fatalities) = PoF × F

R(dollars lost) = PoF × D

R: risk

PoF: annual probability of failure

F: lives lost or fatalities if failure occurs = 100

D: dollars lost if failure occurs = 100×10^{6} \$.

R(fatalities) = 0.001 fatalities/yr

R(dollars lost) = 1000 \$/yr

$$\frac{0.001 \text{ fatality/yr}}{100 \text{ people}} = 1 \times 10^{-5}$$

$$\frac{1000 \text{ dollars/yr}}{200 \times 10^{6}} = 5 \times 10^{-6}$$

The recommended annual probability of failure is 5×10^{-6}, as it is the most demanding of the two: economic loss controls.

Problem 11.12

The set of data (y, x) shown in Table 11.1s is plotted and a linear regression (y = ax + b) is performed. Calculate the values of a and b by:

a. Minimizing the vertical distance between the measured and predicted y values.
b. Minimizing the normal distance between the measured data and the regression line.
c. Compare the results.

Table 11.1s Data Set

Data point number	x value	y value
1	2.1	7.4
2	4.5	10.1
3	4.8	11.7
4	5.3	12.4
5	5.7	13.1
6	6.2	16.7
7	7.8	23.4

Solution 11.12

a. Vertical distance:

$$a = \frac{\sum xi \sum yi - n \sum xiyi}{\left(\sum xi\right)^2 - n \sum xi^2} = 2.77$$

$$b = \frac{\sum xi^2 \sum yi - \sum xi \sum xiyi}{n \sum xi^2 - \left(\sum xi\right)^2} = -0.86$$

b. Normal distance:

$$a = \frac{\sum (y_i - \overline{y})^2 - \sum (x_i - \overline{x})^2 + \sqrt{\left(\sum (y_i - \overline{y})^2 - \sum (x_i - \overline{x})^2\right)^2 + 4\left(\sum (x_i - \overline{x})(y_i - \overline{y})\right)^2}}{2\sum (x_i - \overline{x})(y_i - \overline{y})} = 2.06$$

$$b = \frac{\sum y_i}{N} - a\frac{\sum x_i}{N} = \overline{y} - a\overline{x} = 2.82$$

c. The plot of the linear regression and the orthogonal regression are shown in Figure 11.12s.

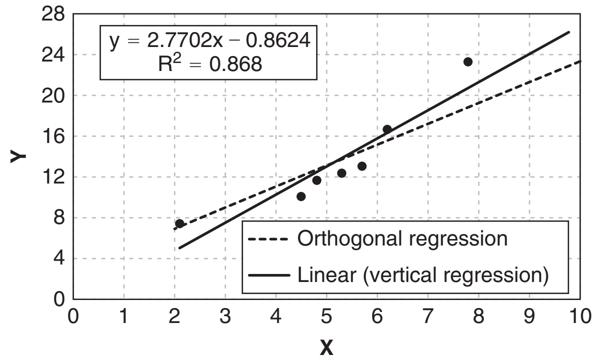


Figure 11.12s Regression plots.

Problem 11.13

Use consistent units to find the relationship between the shear wave velocity v_s, the mass density ρ, and the shear modulus of elasticity G.

Solution 11.13

If we use length, time, and force as primary units, we have

Variable	Dimension
V_s	L/T
G	F/L^2
ρ	FT^2/L^4

We hypothesize that $v_s = f(G, \rho)$. Thus, G and ρ must appear to cancel the force dimension, because F does not appear in v_s. Let's try the ratio $G/\rho = (F/L^2)/(FT^2/L^4) = L^2/T^2$. This ratio has the units of velocity squared, so a reasonable guess is $(v_s)^2 = G/\rho$ or $v_s = (G/\rho)^{0.5}$. This is, of course, correct.

Problem 11.14

The following empirical equations are used in sands to obtain the ultimate pressure p_u under a driven pile point and the ultimate friction f_u on a driven pile side. Use normalization to give these formulas with p_u and f_u in the U.S. customary system.

$$p_u(\text{kPa}) = 1000\ (N(\text{bl/ft}))^{0.5}$$
$$f_u(\text{kPa}) = 5\ (N(\text{bl/ft})^{0.7}$$

Solution 11.14

We first normalize the right-hand term.

$$p_u(\text{kPa}) = 1000\left(\frac{N\ (\text{bl/ft})}{1(\text{bl/ft})}\right)^{0.5}$$

Now the blow count N is normalized and the factor 1000 is in kPa. It can be changed to tsf, for example, by recalling that 100 kPa = 1.0443 tsf. In that case:

$$p_u(\text{tsf}) = 1000\ \text{kPa}\left(\frac{1.0443\ \text{tsf}}{100\ \text{kPa}}\right)\left(\frac{N\ (\text{bl/ft})}{1(\text{bl/ft})}\right)^{0.5} = 10.443\ (N(\text{bl/ft}))^{0.5}$$

We repeat the process for the friction:

$$f_u(\text{kPa}) = 5\left(\frac{N\ (\text{bl/ft})}{1(\text{bl/ft})}\right)^{0.7}$$
$$f_u(\text{tsf}) = 5\ \text{kPa}\left(\frac{1.0443\ \text{tsf}}{100\ \text{kPa}}\right)\left(\frac{N\ (\text{bl/ft})}{1(\text{bl/ft})}\right)^{0.7} = 0.0522\ (N(\text{bl/ft}))^{0.7}$$

Problem 11.15

Perform a dimensional analysis for a square footing embedded at a depth d in a clay with an undrained shear strength s_u. The footing size is B and the failure load is Q_u.

Solution 11.15 (Figure 11.13s)

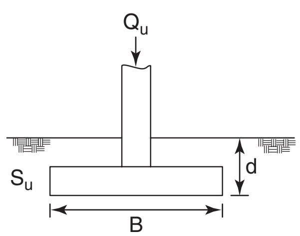


Figure 11.13s Square footing on clay.

The independent variables are shown in Table 11.2s with their dimensions. There are four independent variables.

Table 11.2s Independent Variables and Dimensions

Quantity	Variable	Dimension
Load	Q_u	F
Embedment	d	L
Footing size	B	L
Undrained shear strength	S_u	F/L^2

There are two primary units, as listed in Table 11.2s. We therefore form two primary unit groups of variables:

a. L group: d, B, S_u
b. F group: $Q_{u,}$ S_u

We select one variable from each group; for example, B in the L group and S_u in the F group. These are the repeating variables.

Because there are 4 variables and 2 primary units, we have $4 - 2 = 2\,\pi$ terms. To obtain the $2\,\pi$ terms, we form the power:

a. $\pi_1 = B^a S_u^b d^c$
b. $\pi_2 = B^a S_u^b Q_u^c$

Now, we find the exponent and we determine that the $2\,\pi$ terms are:

$$\pi_1 = \frac{d}{B}$$

$$\pi_2 = \frac{Q_u}{B^2 S_u}$$

Then we can say that $g(\pi_1, \pi_2) = 0$, or:

$$f\left(\frac{d}{B}, \frac{Q_u}{B^2 S_u}\right) = 0$$

We can also write this expression as:

$$Q_u = \frac{d}{B} f_1(B^2 S_u)$$

Notice that if the embedment $d = 0$, then the expression becomes:

$$Q_u = C_1 B^2 S_u$$

where C_1 is a constant that is approximately 6.0.