Computer Aided Process and Product Engineering
Luis Puigianer and Georges Heyen

5
Real Time Optimization

Vivek Dua, John D. Perkins, and Efstratios N. Pistikopoulos

Abstract

This chapter considers two real time optimization (RTO) problems. The first problem is concerned with the model based control of linear discrete time systems and the second problem considers the case when logical conditions are also involved in the first problem. These RTO problems are reformulated as multiparametric programs to obtain control variables as an explicit function of the state of the system. This reduces the real time Optimization problems to simple function evaluations.

5.1
Introduction

Real Time Optimization (RTO) of a system is typically concerned with the solution of the following problem (Marlin and Hrymak, 1997; Perkins, 1998):

$$\begin{aligned} &J(x) = \min_u f(x, u) \\ &\text{s.t. } h(u, x) = 0 \\ &g(u, x) \le 0 \\ &x \in X \end{aligned} \tag{1}$$

where x is the vector of the state of the system, u is the vector of control variables, f is a scalar objective function, such as cost, to be minimized, h is a vector representing the model of the system, g is a vector representing constraints, such as lower and upper bounds on x and u and X is a compact and convex set. Note that this problem is solved repetitively at regular time intervals.

Model Based Predictive Control (MPC) (Morari and Lee, 1999) is widely used by industry to address real time optimization problems with constraints on u and x. It is based on a receding horizon approach where a sequence of future control actions is computed based on a prediction of the future evolution of the system and applied to the system until new measurements become available. Then, a new sequence is determined which replaces the previous one – see Figure 5.1 where x^* is the desired

Computer Aided Process and Product Engineering. Edited by Luis Puigjaner and Georges Heyen

ISBN: 3-527-30804-0

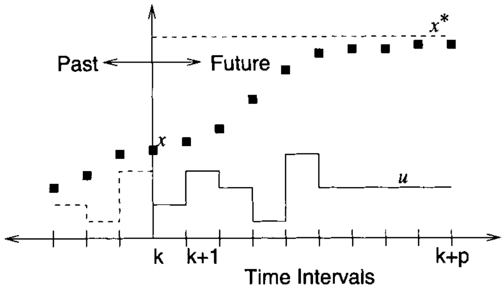


Figure 5.1 Model Based Predictive Control

state of the plant, k is the current time interval and $k + 1, \ldots, k + p$ are the future time intervals. Each sequence is evaluated by solving the optimization problem (1).

Real time optimization offers tremendous benefits but has large real time computational requirements which involve a repetitive solution of problem (1) at regular time intervals (see Figure 5.2). The rest of the chapter is organised as follows. In the next section a parametric programming approach is introduced which can be used to compute u as an explicit function of x. Section 5.3 considers the case when h is given by linear discrete state space equations and the case when u also involves 0–1 binary variables is addressed in section 5.4. The solution approaches presented in sections 5.3 and 5.4 reduce RTO to simple function evaluations.

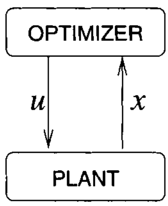
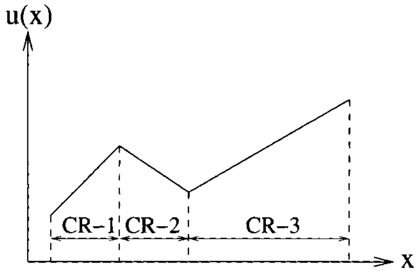


Figure 5.2 Real Time optimization

5.2 Parametric Programming

In an optimization framework, where the objective is to minimize or maximize a performance criterion subject to a given set of constraints and where some of the parameters in the optimization problem vary between specified lower and upper bounds, parametric programming is a technique for obtaining (i) the objective function and the optimization variables as a function of these parameters and (ii) the regions in the space of the parameters where these functions are valid (Fiacco, 1983; Gal, 1995; Acevedo and Pistikopoulos, 1996, 1997; Pertsinidis et al., 1998; Papalexandri and Dimkou, 1998; Acevedo and. Pistikopoulos, 1999; Dua and Pistikopoulos, 1999). Considering u as optimization variables and x as parameters in (1), parametric programming provides.



$$u(x) = \begin{cases} u^1(x) & \text{if } x \in CR^1 \\ u^2(x) & \text{if } x \in CR^2 \\ \vdots & \\ u^i(x) & \text{if } x \in CR^i \\ \vdots & \\ u^N(x) & \text{if } x \in CR^N \end{cases}$$

Figure 5.3 Parametric Optimization

such that $CR^i \cap CR^j = \phi$, $i \neq j$, $\forall i, j = 1, \ldots, N$ and $CRi \subseteq X$, $\forall i = 1, \ldots, N$. A CR^i is known as a Critical Region. For the case when f, g and h are linear and separable in u and x, the CRs are polyhedra and each CR corresponds to a unique set of active constraints (Dua et al., 2002). See Figure 5.3, where u is plotted as a function of x.

The procedure for obtaining $u^i(x)$ and CR^i depends upon whether f, g and h are linear, quadratic, nonlinear, convex, differentiable, or not, and also whether u is vector of continuous or mixed – continuous and integer – variables (Dua and Pistikopoulos, 2000; Dua et al., 2002; Dua and Pistikopoulos, 1999; Dua et al., 2003; Sakizlis et al., 2002b). Recently algorithms for the case when (1) involves (i) differential and algebraic equations (Sakizlis et al., 2002a) and (ii) uncertain parameters (Sakizlis et al., 2004) have also been proposed. The engineering significance of solving parametric programming problems is highlighted in the next motivating example.

5.2.1
Example 1

Consider the refinery blending and production problem depicted in Figure 5.4 (Edgar and Himmelblau, 1989). The objective is to maximize the profit for the operating conditions given in Table 5.1, where x_1 and x_2 are the parameters representing the additional maximum allowable production of gasoline and kerosene production respectively. This results in a multi-parametric linear programming problem given in Table 5.2, where u_i *and* u_2 are the flowrates of the crude oils-1 and 2 respectively, in bbl/day and the units of profit are \$/day. The solution of this problem by using the algorithm of Gal and Nedoma (1972) is given in Table 5.3. The engineering significance of obtaining this solution is as follows:

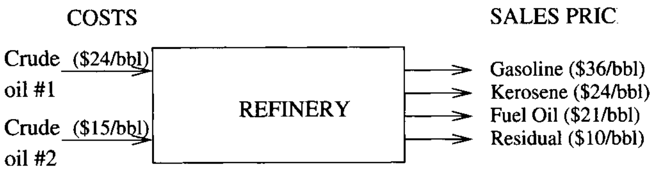


Figure 5.4 Crude Oil Refinery

(i) A complete map of all the optimal solutions, profit and crude oil flowrates as a function of x_1 and x_2, is available.
(ii) The space of x_1 and x_2 has been divided into two regions, CR^1 and CR^2, where the profiles of profit and flowrates of crude oils remain optimal and hence (a) one does not have to exhaustively enumerate the complete space of x_1 and x_2 and (b) the optimal solution can be obtained by simply substituting the value of x_1 and x_2 into the parametric profiles without any further optimization calculations.
(iii) The sensitivity of the profit to the parameters can be identified. In CR^1 the profit is more sensitive to x_2, whereas in CR^2 it is not sensitive to x_2 at all. Thus, for any value of x that lies in CR^2, any expansion in kerosene production will not affect the profit.

This type of Information is quite useful for solving real time optimization problems. In the next section it is shown that real time model based control and optimization problems can be reformulated as multi-parametric quadratic programming problems, the solution of which is given by optimal control variables as a function of the state variables. The real time optimization problem thus reduces to simple function evaluations.

Table 5.1 Refinery Data

	Volume % Yield		Maximum allowable production (bbl/day)
	Crude # 1	Crude # 2	
Gasoline	80	44	$24\,000 + x_1$
Kerosene	5	10	$2\,000 + x_2$
Fuel Oil	10	36	6 000
Residual	5	10	–
Processing Cost ($/bbl)	0.50	1.00	–

Table 5.2 Refinery Model

Profit = max $8.1\, u_1 + 10.8\, u_2$

s.t.
$$0.80\, u_1 + 0.44\, u_2 \leq 24\,000 + x_1$$
$$0.05\, u_1 + 0.10\, u_2 \leq 2\,000 + x_2$$
$$0.10\, u_1 + 0.36\, u_2 \leq 6\,000$$
$$u_1 \geq 0,\ u_2 \geq 0$$
$$0 \leq x_1 \leq 6000$$
$$0 \leq x_2 \leq 500$$

Table 5.3 Solution of the Refinery Example

i	CR^i	Optimal Solution
1	$-0.14\ x_1 + 4.21\ x_2 \leq 896.55$ $0 \leq x_1 \leq 6000$ $0 \leq x_2$	Profit (x) $= 4.66\ x_1 + 87.52\ x_2 + 286758.6$ $u_1 = 1.72\ x_1 - 7.59\ x_2 + 26206.90$ $u_2 = -0.86\ x_1 + 13.79\ x_2 + 6896.55$
2	$-0.14\ x_1 + 4.21\ x_2 \leq 896.55$ $0 \leq x_1 \leq 6000$ $x_2 \leq 500$	Profit (x) $= 7.53\ x_1 + 305409.84$ $u_1 = 1.48\ x_1 + 24590.16$ $u_2 = -0.41\ x_1 + 9836.07$

5.3 Parametric Control

Consider the following state-space representation of a given process model (Pistikopoulos et al., 2002):

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t), \end{cases} \tag{2}$$

subject to the following constraints:

$$\begin{aligned} & y_{\min} \leq y(t) \leq y_{\max} \\ & u_{\min} \leq u(t) \leq u_{\max}, \end{aligned} \tag{3}$$

where $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R^p$ are the state, input, and output vectors respectively, subscripts *min* and *max* denote lower and upper bounds respectively and (A, B) is stabilizable. Model based control problems for regulating to the origin can then be posed as the following optimization problems:

$$\begin{aligned} & \min_U J(U, x(t)) = x^T_{t+N_y|t} P x_{t+N_y|t} + \sum_{k=0}^{N_y-1} \left[x^T_{t+k|t} Q x_{t+k|t} + u^T_{t+k} R u_{t+k} \right] \\ & \text{s.t. } y_{\min} \leq y_{t+k|t} \leq y_{\max}, \ k = 1, \dots, N_c \\ & \quad u_{\min} \leq u_{t+k} \leq u_{\max}, \ k = 0, 1, \dots, N_c \\ & \quad x_{t|t} = x(t) \\ & \quad x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k}, \ k \geq 0 \\ & \quad y_{t+k|t} = Cx_{t+k|t}, \ k \geq 0 \end{aligned} \tag{4}$$

where $U \triangleq \{u_t, \dots, u_{t+N_u-1}\}$, $Q = Q^T \geq 0$, $R = R^T > 0$, $P \geq 0$, $N_y \geq N_u$ and the superscript T denotes the transpose of the corresponding vector or matrix. The problem (4) is solved repetitively at each time t for the current measurement $x(t)$ and the vector of predicted state variables, $x_{t+1|t}, \dots, x_{t+k|t}$ at time $t + 1, \dots, t + k$ respectively and corresponding control actions $u_t, \dots, u_{t+k-1}$ is obtained.

In the following paragraphs, a parametric programming approach which avoids a repetitive solution of (4) is presented. First, we do some algebraic manipulations to recast (4) in a form suitable for using and developing some new parametric programming concepts. By making the following substitution in (4):

$$x_{t+k|t} = A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{t+k-1-j} \tag{5}$$

the objective $J(U, x(t))$ can be formulated as the following Quadratic Programming (QP) problem:

$$\begin{aligned} &\min_U \frac{1}{2} U^T H U + x^T(t) F U + \frac{1}{2} x^T(t) Y x(t) \\ &\text{s.t. } GU \leq W + Ex(t) \end{aligned} \tag{6}$$

where $U \triangleq [u_t^T, \ldots, u_{t+N_u-1}^T]^T \in R^s$, $s \triangleq mN_u$, is the vector of optimization variables, $H = H^T > 0$, and H, F, Y, G, W, E are obtained from Q, R and (4)–(5). The QP problem (6) can now be formulated as the following Multi-parametric Quadratic Program (mp-QP):

$$\begin{aligned} &\mu(x) = \min_z \frac{1}{2} z^T H z \\ &\text{s.t. } Gz \leq W + Sx(t) \end{aligned} \tag{7}$$

where $z \triangleq U + H^{-1} F^T x(t)$, $z \in R^s$, represents the vector of optimization variables, $S \triangleq E + GH^{-1}$ and x represents the vector of parameters. The main advantage of writing (4) in the form given in (7) is that z (and therefore U) can be obtained as an affine function of x for the complete feasible space of x. To derive these results, we first state the following theorem.

Theorem 1 For the problem in (7) let x_0 be a vector of parameter values and (z_0, λ_0) a KKT pair, where $\lambda_0 = \lambda(x_0)$ is a vector of nonnegative Lagrange multipliers, λ, and $z_0 = z(x_0)$ is feasible in (7). Also assume that (i) linear independence constraint qualification and (ii) strict complementary slackness conditions hold. Then,

$$\begin{bmatrix} z(x) \\ \lambda(x) \end{bmatrix} = -(M_0)^{-1} N_0 (x - x_0) + \begin{bmatrix} z_0 \\ \lambda_0 \end{bmatrix} \tag{8}$$

where,

$$M_0 = \begin{pmatrix} H & G_1^T & \cdots & G_q^T \\ -\lambda_1 G_1 & -V_1 & & \\ \vdots & & \cdots & \\ -\lambda_p G_q & & & -V_q \end{pmatrix}$$

$$N_0 = (Y, \lambda_1 S_1, \ldots, \lambda_p S_p)^T$$

where G_i denotes the i^{th} row of G, S_i denotes the ith row of S, $V_i = G_i z_0 - W_i - S_i x_0$, W_i denotes the i^{th} row of W and Y is a null matrix of dimension $(s \times n)$.

See Pistikopoulos et al. (2002) for the proof. The space of x where this solution, (8), remains optimal is defined as the Critical Region (CR^0) and can be obtained as follows. Let CR^R represent the set of inequalities obtained (i) by substituting $z(x)$ into the inequalities in (7) and (ii) from the positivity of the Lagrange multipliers, as follows:

$$CR^R = \{Gz(x) \leq W + Sx(t),\ \lambda(x) \geq 0\}, \tag{9}$$

then CR^0 is obtained by removing the redundant constraints from CR^R as follows:

$$CR^0 = \Delta\{CR^R\} \tag{10}$$

where Δ is an operator which removes the redundant constraints – for a procedure to identify the redundant constraints, see Gal (1995). Since for a given space of state-variables, X, so far we have characterized only a subset of X i.e. $CR^0 \subseteq X$, in the next step the rest of the region CR^{rest}, is obtained as follows (Pistikopoulos et al., 2002):

$$CR^{\text{rest}} = X - CR^0. \tag{11}$$

The above steps, (8–11) are repeated and a set of $z(x)$, $\lambda(x)$ and corresponding CR^0s is obtained. The solution procedure terminates when no more regions can be obtained, i.e. when $CR^{rest} = \phi$. For the regions which have the same solution and can be unified to give a convex region, such a unification is performed and a compact representation is obtained. The continuity and convexity properties of the optimal solution are summarized in the next theorem.

Theorem 2 For the mp-QP problem, (7), the set offeasible parameters $X_f \subseteq X$ is convex, the optimal solution, $z(x) : X_f \rightarrow R^s$ is continuous and piecewise affine, and the optimal objective function $\mu(x) : X_f \rightarrow R$ is continuous, convex and piecewise quadratic.

See Pistikopoulos et al. (2002) for the proof. Based upon the above theoretical developments, an algorithm for the solution of an mp-QP of the form given in (7) to calculate U as an affine function of x and characterize X by a set of polyhedral regions, CRs, has been developed which is summarized in Table 5.4.

This approach provides a significant advancement in the solution and real time implementation of model based control problems. Since its application results in a complete set of control variables as a function of state-variables (from (8)) and the corresponding regions of validity (from (10)), which are computed off-line. Therefore during on-line optimization, no optimizer needs to be called and instead for the current state of the plant, the region, CR^0, where the value of the state variables is valid, can be identified by substituting the value of these state variables into the inequalities which define the regions. Then, the corresponding control variables can be computed by using a function evaluation of the corresponding affine function (see Figure 5.5). Figure 5.6 demonstrates how advanced controllers can be implemented on a simple hardware.

5.4 Hybrid Systems

Hybrid systems can be defined as systems comprising a number of interconnected continuous subsystems where the interconnections are determined by logical or discrete switchings. Each subsystem is governed by a unique set of differential and/or algebraic equations. In this section we focus on piecewise affine (PWA) systems (Bemporad and Morari, 1999). PWA systems are defined by partitioning the state and input space into polyhedral regions and associating with each region a different linear state update equation

$$x(t+1) = A^i x(t) + B^i u(t) + f^i \tag{12}$$

$$\text{if} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in p^i$$

where $i = 1, \ldots, s$, $x \in R^{n_c} \times \{0, 1\}^{n_l}$, $u \in R^{m_c} \times \{0, 1\}^{m_l}$, $\{P_i\}_{i=1}^{s}$ is a polyhedral partition of the set of the state and input space $P \subset R^{n+m}$, $n \triangleq n_c + n_l$, $m \triangleq m_c\ m_j$. P is assumed to be closed and bounded and $x_c \in R^{n_c}$ and $u_c \in R^{m_c}$ denote the continuous components of the state and input vector, respectively; $x_l \in \{0, 1\}^{n_l}$ and $u_l \in \{0, 1\}^{m_l}$ similarly denote the binary components.

Note that PWA models are not suitable for recasting analysis/synthesis problems into more compact optimization problems. For this purpose the Mixed Logical Dynamical (MLD) framework (Bemporad and Morari, 1999) is used. The general MLD form of a hybrid system is:

$$x(t+1) = Ax(t) + B_1 u(t) + B_2 \delta(t) + B_3 z(t) \tag{13}$$

$$y(t) = Cx(t) + D_1 u(t) + D_2 \delta(t) + D_3 z(t) \tag{14}$$

$$E_2 \delta(t) + E_3 z(t) \leq E_1 u(t) + E_4 x(t) + E_5 \tag{15}$$

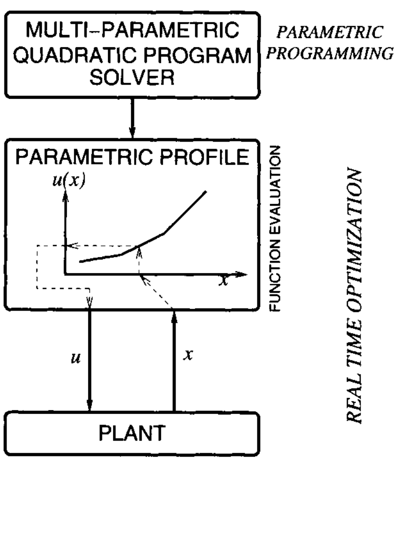


Figure 5.5 Real time optimization via parametric programming

Table 5.4 Solution Steps of the mp-QP Algorithm

Step 1	For a given space of x solve (7) by treating x as a free variable and obtain $[x_0]$.
Step 2	In (7) fix $x = x_0$ and solve (7) to obtain $[z_0, \lambda_0]$.
Step 3	Obtain $[z(x), \lambda(x)]$ from (8).
Step 4	Define CR^R as given in (9).
Step 5	From CR^R remove redundant inequalities and define the region of optimality CR^0 as given in (10).
Step 6	Define the rest of the region, CR^{rest}, as given in (11).
Step 7	If no more regions to explore, go to the next step, otherwise go to Step 1.
Step 8	Collect all the solutions and unify a convex combination of the regions having the same solution to obtain a compact representation.

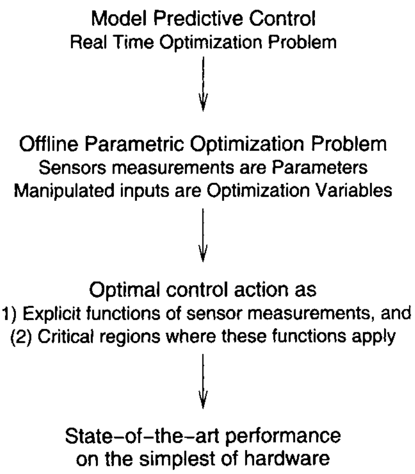


Figure 5.6 Achieving state-of-the-art control performance on simple hardware

where $x = [x_c^T \, x_l^T]^T \in R^{n_c} \times \{0, 1\}^{n_l}$ are the continuous and binary states, $u = [u_c^T \, u_l^T]^T \in R^{m_c} \times \{0, 1\}^{m_l}$ are the inputs, $y = [y_c^T \, y_l^T]^T \in R^{p_c} \times \{0, 1\}^{p_l}$ the outputs, and $\delta \in \{0, 1\}^{r_l}$, $z \in R^{r_c}$ represent auxiliary binary and continuous variables respectively. All constraints on the states, the inputs, the z and δ variables are summarized in the inequalities (15). Note that, although the description (13)–(14)–(15) seems to be linear, nonlinearity is hidden in the integrality constraints over the binary variables. MLD systems are a versatile framework to model various classes of systems. For a detailed description of such capabilities we defer the reader to Morari et al. (2003).

5.4.1
Predictive Control of MLD Systems

Let t be the current time, and $x(t)$ the current state. Consider the following optimal control problem

$$\min_{\{v_0^{T-1}\}} J(v_0^{T-1}, x(t)) \triangleq \sum_{k=0}^{T-1} ||v(k)||^2_{Q_1} + ||\delta(k|t)||^2_{Q_2} + ||z(k|t)||^2_{Q_3} + ||x(k|t)||^2_{Q_4} + ||y(k|t)||^2_{Q_5} \quad (16)$$

$$\begin{aligned} \text{s.t. } & x(k+1|t) = Ax(k|t) + B_1 v(k) + B_2\delta(k|t) + B_3 z(k|t) \\ & y(k|t) = Cx(k|t) + D_1 v(k) + D_2\delta(k|t) + D_3 z(k|t) \\ & E_2\delta(k|t) + E_3 z(k|t) \le E_1 v(k) + E_4 x(k|t) + E_5 \end{aligned} \quad (17)$$

where $v_0^{T-1} \triangleq [v^T(0), \ldots, v^T(T-1)]^T$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T \ge 0$, $Q_3 = Q_3^T \ge 0$, $Q_4 = Q_4^T > 0$ and $Q_5 = Q_5^T \ge 0$. $x(k|t) \triangleq x(t+k, x(t), v_0^{k-1})$ is the state predicted at time $t + k$ resulting from the input $u(t + k) = v(k)$ to (13–15) starting from $x(0|t) = x(t)$. $\delta(k|t)$, $z(k|t)$ and $y(k|t)$ are similarly defined. Assume for the moment that the optimal solution $\{v_t^*(k)\}_{k=0,\ldots,T-1}$ exists. According to the *receding horizon* philosophy mentioned above, set

$$u(t) = v_t^*(0), \quad (18)$$

disregard the subsequent optimal inputs $v_t^*(1), \ldots, v_t^*(T-1)$, and repeat the whole optimization procedure at time $t + 1$. Note that (16–17) is a Mixed Integer Quadratic Program (MIQP). This problem can be formulated as a Mixed Integer Linear Program (MILP) if 1 norm instead of the 2 norm is considered in the objective function. The repetitive somtion of the MIQP or MILP can be avoided by formulating (16–17) as a multiparametric program and solving it to obtain the control variables as a set of explicit functions of the current state of the system and the regions in the space of the state variables where the explicit functions remain valid (Bemporad et al., 2000; Sakizlis et al., 2002a). This is achieved by recasting (16–17) in a compact form as follows:

$$\begin{aligned} J(x(t)) &= \min_{\pi_c, \pi_d} \pi_c^T Q_c \pi_c + \phi^T \pi_d \\ \text{s.t. } & G_c\pi_c + G_d\pi_d \le S + Fx(t) \end{aligned} \quad (19)$$

where π_c and π_d are continuous and discrete variables of (16–17), Q_c, ϕ^T, G_c, G_d, S, F are constant matrices and vectors of appropriate dimensions and Q_c is symmetrie and positive definite. $x(t)$ is the state at the current time t. The objective is to obtain π_c and π_d as a function of $x(t)$ without exhaustively enumerating the entire space of $x(t)$. This can be achieved by using parametric programming. In the next section an algorithm for Multiparametric Mixed Integer Linear Programs (mp-MILP) is described. This reduces the real time hybrid system control problem to a function evaluation problem (Figure 5.7).

5.4.2 Multiparametric Mixed-Integer Linear Programming

Consider a multiparametric Mixed Integer Linear Programming (mp-MILP) problem of the following form:

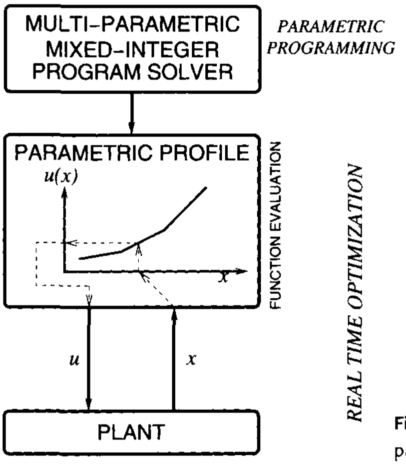


Figure 5.8 Real time optimization of hybrid systems via parametric programming

$$
\begin{aligned}
J(x(t)) &= \min_{\pi_c,\pi_d} \phi_1^T \pi_c + \phi_2^T \pi_d \\
\text{s.t. } & G_c\pi_c + G_d\pi_d \leq S + Fx(t)
\end{aligned}
\tag{20}
$$

where ϕ_1 and ϕ_2 are constant vectors.

5.4.2.1 Initialization

An initial feasible π_d is obtained by solving the following MILP:

$$
\begin{aligned}
&\min_{\pi_c,\pi_d,x(t)} \phi_1^T \pi_c + \phi_2^T \pi_d \\
&\text{s.t. } G_c\pi_c + G_d\pi_d \leq S + Fx(t)
\end{aligned}
\tag{21}
$$

where $x(t)$ is treated as a vector of free variable to find a starting feasible integer solution. Let the solution of (21) be given by $\pi_d = \bar{\pi}_d$.

5.4.2.2
Multiparametric LP Subproblem

Fix $\pi_d = \overline{\pi}_d$ (20) to obtain a multiparametric LP problem of the following form:

$$\begin{aligned} &\hat{J}(x(t)) = \min_{\pi_c} \phi_1^T \pi_c + \phi_2^T \overline{\pi}_d \\ &\text{s.t. } G_c\pi_c + G_d\overline{\pi}_d \leq S + Fx(t) \end{aligned} \tag{22}$$

The solution of (22) is given by a set of linear parametric profiles, $\hat{J}(x(t))^i$, where $\hat{J}(x(t))$ is convex, and corresponding critical regions, CR^i (Gal, 1995).

The final solution of the multiparametric LP subproblem in (22) which represents a parametric upper bound on the final solution is given by (i) a set of parametric profiles, $\hat{J}(x(t))^i$, and the corresponding critical regions, CR^i, and (ii) a set of infeasible regions where $\hat{J}(x(t))^i = \infty$.

5.4.2.3
MILP Subproblem

For each critical region, CR^i, obtained from the solution of the multiparametric LP subproblem in (22), an MILP subproblem is formulated as follows:

$$\begin{aligned} &\min_{\pi_c, \pi_d, x(t)} \phi_1^T \pi_c + \phi_2^T \pi_d \\ &\text{s.t. } G_c\pi_c + G_d\pi_d \leq S + Fx(t) \\ &\phi_1^T \pi_c + \phi_2^T \pi_d \leq \hat{J}(x(t))^i \\ &\pi_d \neq \overline{\pi}_d \\ &x \in CR^i \end{aligned} \tag{23}$$

The integer solution, $\pi_d = \overline{\pi}_d^1$, and the corresponding CRs, obtained from the solution of (23), are then recycled back to the multiparametric LP subproblem – to obtain another set of parametric profiles. Note that the integer cut, $\pi_d \neq \overline{\pi}_d$, and the parametric cut, $\phi_1^T \pi_c + \phi_2^T \pi_d \leq \hat{J}(x(t))^i$ are accumulated at every iteration.

If there is no feasible solution to the MILP subproblem (23) in a CR^i, that region is excluded from further consideration and the current upper bound in that region represents the final solution. Note also that the integer solution obtained from the solution of (23) is guaranteed to appear in the final solution, since it represents the minimum of the objective function at the point, in $x(t)$, obtained from the solution of (23). The final solution of the MILP subproblem is given by a set of integer solutions and their corresponding CR^is.

5.4.2.4
Comparison of Parametric Solutions

The set of parametric solutions corresponding to an integer solution, $\pi_d = \overline{\pi}_d$, which represents the current upper bound are then compared to the parametric solutions corresponding to another integer solution, $\pi_d = \overline{\pi}_d^1$, in the corresponding $C\,Rs$ in order to obtain the lower of the two parametric solutions and update the upper bound. This is achieved by employing the procedure proposed by Acevedo and Pistikopoulos (1997b).

5.4.2.5 Multiparametric MILP Algorithm

Based upon the above theoretical developments, the steps of the algorithm can be stated as follows:

Step 0 (Initialization) Define an initial region of $x(t)$, CR, with best upper bound $\hat{J}^*(x(t)) = \infty$, and an initial integer solution $\bar{\pi}_d$.

Step 1 (Multiparametric LP Problem) For each region with a new integer solution, $\bar{\pi}_d$:

- Solve multiparametric LP subproblem (22) to obtain a set of parametric upper bounds $\hat{J}(x(t))$ and corresponding critical regions, CR.
- If $\hat{J}(x(t)) \leq \hat{J}^*(x(t))$ for some region of $x(t)$, update the best upper bound function, $\hat{J}^*(x(t))$, and the corresponding integer solutions, π_d^*,
- If an infeasibility is found in some region CR, go to Step 2.

Step 2 (Master Subproblem) For each region CR, formulate and solve the MILP master problem in (23) by (i) treating $x(t)$ as a variable bounded in the region CR, (ii) introducing an integer cut, $\pi_d \neq \bar{\pi}_d$ and (iii) introducing a parametric cut, $\phi_1^T \pi_c + \phi_2^T \pi_d \leq \hat{J}(x(t))^i$. Return to Step 1 with new integer solutions and corresponding CRs.

Step 3 (Convergence) The algorithm terminates in a region where the solution of the MILP subproblem is infeasible. The final solution is given by the current upper bounds $\hat{J}^*(x(t))$ in the corresponding CRs. The $\pi_c(x(t))$ and $\pi_d(x(t))$ corresponding to $\hat{J}^*(x(t))$ are then used to obtain $u(x(t))$.

Note that the algorithms presented in this chapter have been implemented and tested on a number of real time optimization problems (PAROS, 2004).

5.5 Concluding Remarks

In this chapter it was shown how real time optimization problems can be recast as multiparametric programs. Linear discrete time optimization problems are recast as multiparametric quadratic programs and problem involving logical decisions as multiparametric mixed integer programs. Algorithms for solving the multiparametric programs were then presented to compute the optimal control actions as an explicit function of the state of the system. This reduces real time optimization problems to simple function evaluations.

References

1 *Acevedo J. Pistikopoulos E. N.* A parametric MINLP algorithm for process synthesis problems under uncertainty. Industrial and Engineering Chemistry Research 35 (1996) p. 147–158

2 *Acevedo J. Pistikopoulos E. N.* A multiparametric programming approach for linear process engineering problems under uncertainty. Industrial and Engineering Chemistry Research 36 (1997) p. 717–728

3 *Acevedo J. Pistikopoulos E. N.* An algorithm for multiparametric mixed integer linear programming problems. Operations Research Letters 24 (1999) p. 139–148

4 *Bemporad A. Borrelli F. Morari M.* Piecewise linear optimal controllers for hybrid systems, proceedings of the American Control Conference (2000) p. 1190–1194

5 *Bemporad A. Morari M.* Control of systems integrating logic, dynamics, and constraints. Automatica 35 (1999) p. 407–427

6 *Dua V. Bozinis N. A. Pistikopoulos E. N.* A multiparametric programming approach for mixed-integer quadratic engineering problems. Computers & Chemical Engineering 26 (2002) p. 715–733

7 *Dua V. Papalexandri K. P. Pistikopoulos E. N.* Global optimization issues in multiparametric continuous and mixed-integer optimization problems, accepted for publication in the Journal of Global Optimization, 30 (2004) p. 59

8 *Dua V. Pistikopoulos E. N.* Algorithms for the solution of multiparametric mixed-integer nonlinear optimization problems. Industrial and Engineering Chemistry Research 38 (1999) p. 3976–3987

9 *Dua V. Pistikopoulos E. N.* An algorithm for the solution of multiparametric mixed integer linear programming problems. Annals of Operations Research 99 (2000) p. 123–139

10 *Edgar T. F. Himmelblau D. M.* Optimization of chemical processes. McGraw Hill Book Co, Singapore 2000

11 *Fiacco A. V.* Introduction to Sensitivity and Stability Analysis in Nonlinear Programming. Academic Press, New York 1983

12 *Gal T.* Postoptimal Analyses, Parametric Programming, and Related Topics. de Gruyter, New York 1995

13 *Gal T. Nedoma J.* Multiparametric linear programming. Management Science 18 (1972) p. 406–422

14 *Marlin T. E. Hrymak A. N.* Real-time operations optimization of continuous processes. In: Kantor J. C. Garcia C. E. Carnahan B. (Eds.), 5th Int. Conf. Chem. Proc. Control. Vol. 93 of AIChE Symposium Series 1997

15 *Morari M. Baotic M. Borrelli F.* Hybrid systems modeling and control. European Journal of Control 9 (2003) p. 177–189

16 *Morari M. Lee J.* Model predictive control: past, present and future. Computers & Chemical Engineering 23 (1999) p. 667–682

17 *Papalexandri K. P. Dimkou T. I.* A parametric mixed integer optimization algorithm for multi-objective engineering problems involving discrete decisions. Industrial and Engineering Chemistry Research 37 (5) (1998) p. 1866–1882

18 *PAROS* Parametric Optimization Solutions Ltd. http://www.parostech.com 2004

19 *Perkins J. D.* Plant-wide optimization: opportunities and challenges. In: Pekny J. Blau G. (Eds.), 3rd Int. Conf. Foundations of Computer Aided Process Operations. Vol. 94 of AIChE Symposium Series 1998

20 *Pertsinidis A. Grossmann I. E. McRae G. J.* Parametric optimization of MILP programs and a framework for the parametric optimization of MINLPs. Computers & Chemical Engineering 22 (1998) p. S205

21 *Pistikopoulos E. N. Dua V. Bozinis N. A. Bemporad A. Morari M.* On-line optimization via off-line parametric optimization tools. Computers & Chemical Engineering 26 (2002) p. 175–185

22 *Sakizlis V. Dua V. Perkins J. D. Pistikopoulos E. N.* The explicit control law for hybrid systems via parametric programming, proceedings of the 2002 American Control Conference, Anchorage 2002a

23 *Sakizlis V. Dua V. Perkins J. D. Pistikopoulos E. N.* The explicit model-based control law for continuous time systems via parametric programming, proceedings of the 2002 American Control Conference, Anchorage 2002b

24 *Sakizlis V. Kakalis N. Dua V. Perkins J. D. Pistikopoulos E. N.* Design of robust model based controllers via parametric programming. Automatica 40 (2004) p. 189–201