# CHAPTER 91

# Economic Risk Analysis*

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## 1. INTRODUCTION

Various methods of analysis for economic justification are shown in Chapter 90 based on the assumption that all of the component cash flows for the proposed investment are known and certain. However, in most cases the amount and timing of these cash flows are estimated, and uncertainties exist in the estimation process. Furthermore, there is usually more uncertainty with some component cash flows than others, and some of these component flows affect the economic criteria more than others. Thus, additional methodologies and concepts are needed for economic analysis when explicit information on the effects of uncertainties in the timing and amounts of the cash flows is important. These methodologies and concepts are the focus of this chapter.

Numerous factors contribute to the uncertainties in the estimates of the amount and timing of component cash flows. Delivery or construction delays, unexpected bottlenecks in new projects, inflationary or recessionary pressures, labor negotiations, and problems in R&D are but a few examples of changes that can and do occur to alter the amounts and timing of disbursements and receipts of monies. Although these possibilities are usually recognized during the early planning phases of a project, the actual cash flows are uncertain, and there is a risk associated with the resulting project's present worth, benefit–cost ratio, or other measure of economic merit being used. Since this economic risk is as important to the decision maker as the other aspects of economic analysis, explicit information regarding the risk should be developed as part of the analysis. Approaches to this form of analysis and some of the relevant techniques are described in this chapter.

A variety of measures have been proposed for dealing with a noncertain operating environment, that is, where the relevant parameters of the analytical model cannot be assumed with certainty. The relevant literature is very extensive, and an encyclopedic treatment is beyond the scope of this chapter. Our discussion will be limited, therefore, to a limited number of concepts related for their popularity among practitioners and because they are representative of the spectrum of possible approaches to this issue. We begin with *sensitivity analysis,* that technique which, surveys show, appears to be most commonly used in industry.

## 2. SENSITIVITY ANALYSIS

*Sensitivity analysis* is the process whereby one or more system input variables are changed and corresponding changes in the system output, or figure of merit, are observed. If a decision is changed as a certain input is varied over a reasonable range of possible values, the decision is said to be *sensitive* to that input; otherwise it is *insensitive.*

The term *break-even analysis* is often used to express the same concept for a single input variable. Here, the value of the input variable at which the decision is changed is determined. If the break-even point lies within the range of expected values, the decision is said to be sensitive to that point. Thus, sensitivity and the break-even point are directly related.

### 2.1. Numerical Example: Certainty Analysis

A manufacturing firm is considering the introduction of a new product to be produced and sold over a 15-year period. The initial cost of capital facilities is \$100,000; the anticipated net salvage value at the end of 15 years is \$20,000. It is expected that 7000 units will be produced each year at a cost of $10 per unit and sold at $12 per unit. The firm's minimum attractive rate of return (MARR) is 10% per year.

The anticipated "profitability" of this proposed investment can be measured by present worth (PW) as follows.

$$\text{PW} = Q(r - c)(P/A, i, N) - P + S(P/F, i, N) \tag{1}$$

where $Q$ = quantity sold per year
$r$ = revenue per unit
$c$ = cost per unit
$P$ = initial cost of capital facilities
$S$ = net salvage value of capital facilities
$N$ = project life, in years
$i$ = MARR, the discount rate per year

[Note the factor $(P/A, i, N)$ is the *functional* form of the uniform series present worth factor, the *algebraic* form of which is $((1 + i)^N - 1)/(i(1 + i)^N)$. Similarly, the functional form of the single payment present worth factor, $(P/F, i, N)$, represents the algebraic form $(1 + i)^{-N}$. See Chapter 90 for additional discussion.]

Assuming the "certainty estimates" for these seven parameters as described in the preceding paragraph, the solution is

$$\begin{aligned}\text{PW} &= 7000(\$12 - \$10)(P/A, 10\%, 15) - \$100{,}000 + \$20{,}000(P/F, 10\%, 15)\\ &= \$14{,}000(7.606) - \$100{,}000 + \$20{,}000(0.2394)\\ &= \$11{,}273\end{aligned}$$

Since the PW is positive, we conclude that the proposal appears to be economically attractive. This result, of course, is based on the presumption that all of the parameter values assumed for the analysis will in fact occur as anticipated.

## 2.2. Classical Sensitivity Analysis: Single Variable

### *2.2.1. Algebraic Solution*

Suppose there is some reason to question the validity of the assumption concerning the number of units produced and sold annually. Additional investigation, for example, may suggest that the "certainty estimate" of 7,000 units per year is questionable; it now appears that this parameter value could occur anywhere over the range of 6,000 to 7,500 units. With this new information the resulting range of values for the present worth is

$$\begin{aligned}\text{Min PW} &= 6000(\$2)(7.606) - \$95{,}212 = -\$3{,}940\\ \text{Max PW} &= 7500(\$2)(7.606) - \$95{,}212 = \$18{,}878\end{aligned}$$

The break-even point can be determined by determining that value of $Q = Q_0$ such that PW = 0:

$$\text{PW} \equiv 0 = Q_0(\$2)(7.606) - \$95{,}212$$

Solving,

$$Q_0 = \$95{,}212/\$15.212 = 6259 \text{ units}$$

Since the break-even point lies within the range ($6000 < 6259 < 7500$), the decision *is* sensitive to the estimate for $Q$.

### *2.2.2. Graphical Presentation*

Sensitivity analyses are usually presented in graphical format. Indeed, it is this "power of pictures" that probably accounts for its widespread popularity. The graphical portrayal of sensitivity of PW to the variable $Q$ in our example is illustrated in Figure 1. The linear function in the figure is the graph of

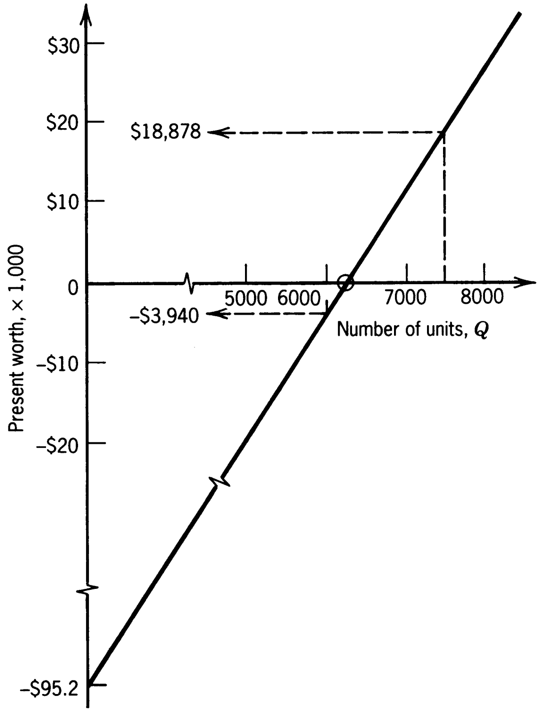


**Figure 1** Present Worth as a Function of Number of Units Produced and Sold Annually. (Break-even = 6259 units)

$$\text{PW} = \$15.212Q - \$95{,}212 \qquad 0 \leq Q \leq 9{,}000$$

Note the break-even point at $Q_0 = 6259$. Also note that the range for $Q$ is highlighted at $Q(\text{min}) = 6000$ and $Q(\text{max}) = 7500$.

### *2.2.3. Percent Deviation Graph*

An alternative approach is a plot of the figure of merit—here, the present worth (PW)—as a function of the *percent deviation* of the variable of interest In the example let $p_Q$ = the percent deviation of $Q$ such that

$$\begin{aligned} \text{PW} &= \$15.212(7000)(1 - p_Q) - \$95{,}212 \\ &= \$11{,}272 - \$106{,}485 p_Q \end{aligned} \tag{2}$$

The function is graphed in Figure 2. Also shown in the figure are similar graphs for percent deviation for revenue per unit ($p_r$), cost per unit ($p_c$), and the number of units produced and sold annually ($p_Q$).

Although percent deviation graphs for one or more variables may be shown in a single illustration, it should be emphasized that sensitivity to only one variable at a time is being examined. The graph of PW as a function of $p_Q$, for example, is based on the assumptions that *all* other variables ($r$, $c$, $P$, $S$, $N$, $i$) are held constant at their "certainty estimates." When sensitivity to $p_r$ is being examined, we set $Q = 7000$. And so on.

One notable advantage of the percent deviation graph is that it makes apparent the relative degree of sensitivity for the various parameters. The greater the slope (steepness of the function) the more likely is the decision to be sensitive to that parameter, that is, the break-even point for percent deviation will be relatively small. In Figure 2 it is apparent that the decision is somewhat more sensitive to per unit revenue ($r$) and cost ($c$) and is relatively insensitive to number of years of service ($N$). This conclusion may be misleading, however, because it is based on the presumption of equal

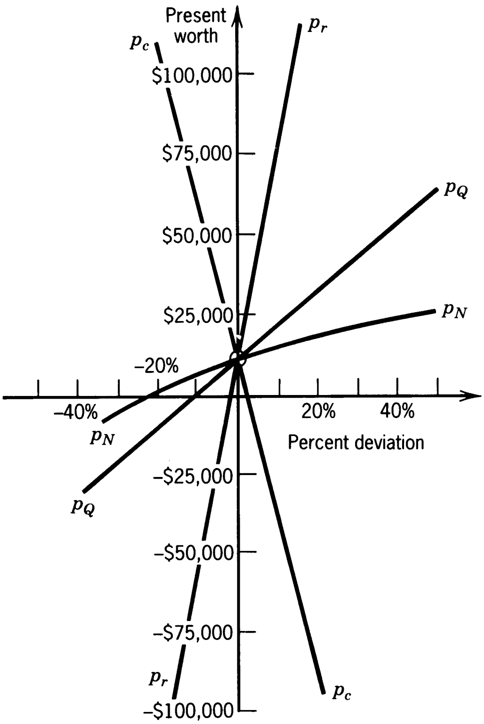


**Figure 2** Present Worth as Function of Percent Deviation in Estimates for $r$, $c$, $Q$, and $N$.

likelihoods of deviation for the various parameters. To illustrate, we found that the break-even percent deviations are about $-23\%$ for $p_N$ and $-11\%$ for $p_Q$. But suppose that there is evidence to suggest that:

| Parameter | Certainty Estimate | Range | Deviation |
|---|---|---|---|
| Quantity | 7000 units | 6000 to 7500 | $-14$ to $+7\%$ |
| Life | 15 years | 10 to 18 | $-33$ to $+20\%$ |

Thus, it would appear that the decision maker would be well advised to give careful attention to the assumption concerning service life ($N$) as well as quantity produced ($Q$). The point here is that the *range* of interest for percent deviation may be different for different parameters.

## 2.3. Sensitivity to Two Parameters Considered Simultaneously

Suppose that our decision maker in this example is concerned about the sensitivity to the revenue per unit ($r$) as well as the number of units produced and sold ($Q$). Considering these two parameters, now variables, simultaneously,

$$\text{PW} = Q(r - \$10)(7.606) - \$95{,}212 \tag{3}$$

As before, assume $6000 \leq Q \leq 7500$, and assume further that $\$11.25 \leq r \leq \$12.50$.

One approach to sensitivity analysis for two variables considered simultaneously is to construct a three-dimensional graph with the $x$ and $y$ axes representing the two variables and the $z$ axis serving as the figure of merit. The combined function is now a *surface* and we now have a break-even *line.* But three-dimensional graphs are difficult to construct and generally harder to interpret. A useful alternative is a variant of the two-dimensional graph as illustrated in Figure 3. One of the two variables is represented along the $x$ axis. The second variable is reflected by a family of curves, specifically, curves based on the maximum and minimum values of the variable.

The two functions plotted in Figure 3 are

$$\text{PW} = Q(\$12.50 - \$10)(7.606) - \$95{,}212$$

and

$$\text{PW} = Q(\$11.25 - \$10)(7.606) - \$95{,}212$$

These represent the upper and lower bounds of the $r$ variable, respectively. Two additional vertical lines are drawn at the lower and upper bounds of the $Q$ variable, at 6000 and 7500 units. The polygon thus formed contains all possible combinations of $r$ and $Q$, and the maximum and minimum values of the figure of merit (PW) can be readily determined. The decision is insensitive if the polygon lies either wholly above the $x$ axis (PW = \$0) or wholly below the $x$ axis.

One problem in the interpretation of sensitivity graphs can be illustrated by this numerical example. It would appear from Figure 3 that, since the area of the polygon lying above the $x$ axis is roughly the same as the area lying below the line, the likelihood of making money on this project (PW > \$0) is about the same as the likelihood of losing money. Implicit in this conclusion is the assumption that all points in the polygon are equally probable. But this is not necessarily the case. Indeed, it would be reasonable to assume that there is an inverse relationship between price per unit and quantity sold, so that $Q$ would decrease as $r$ increases. This dependency is not reflected in the graph.

The simultaneous consideration of sensitivity to two variables can also be displayed in a percent deviation format. In Figure 4, the percent deviations for each of the variables are shown on the $x$ and $y$ axes. The function plotted is

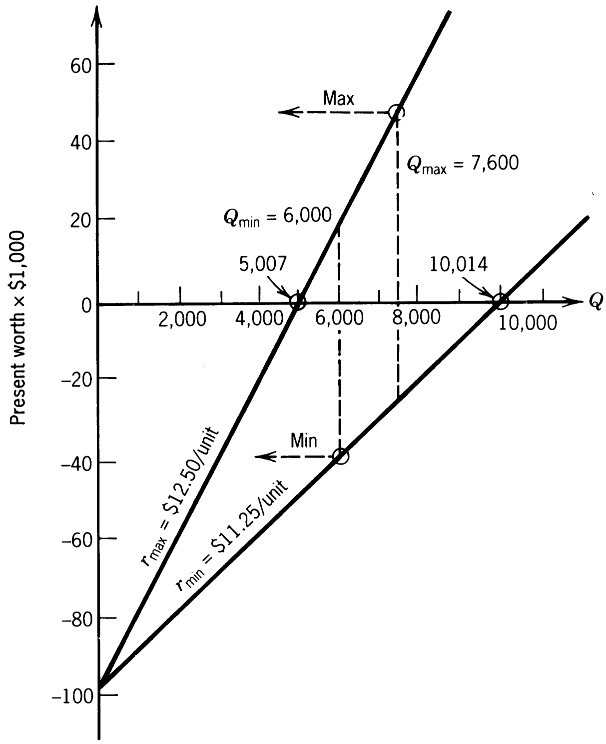


**Figure 3** Present Worth as a Function of Units Produced ($Q$) and Revenue per Unit ($r$).

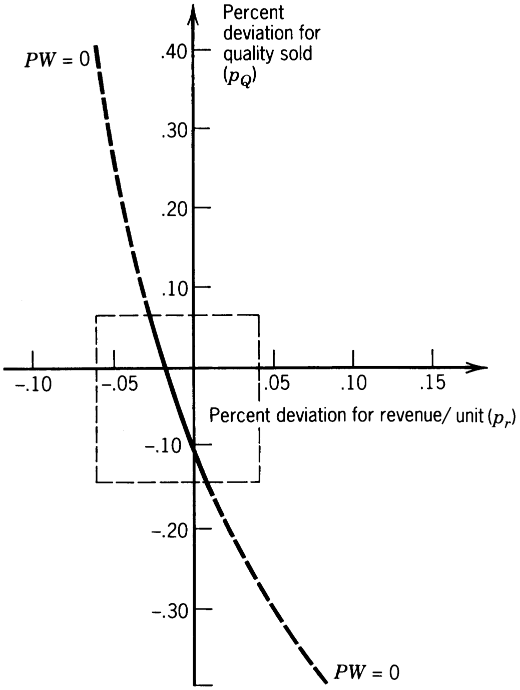


**Figure 4** Percent Deviation Graph for Two Variables: Quantity Sold and Revenue per Unit.

$$\text{PW} = 7000(1 + p_Q)[\$12(1 + p_r) - \$10](7.606) - \$95{,}212 = 0$$

This is an *indifference curve,* the locus of all points ($p_r$, $p_Q$) such that PW = \$0, which is in fact the *break-even line.* The solid portion of the line represents the set of possible outcomes: ($-6\% \le p_r \le +4\%$) and ($-14\% \le p_Q \le +7\%$).

## 2.4. Sensitivity to More Than Two Parameters

Using the previous example, suppose that, in addition to uncertainty about quantity sold ($Q$) and revenue per unit ($r$), there is also uncertainty as to the cost per unit ($c$). Suppose that the following range of values is possible:

| Parameter | Minimum | Most likely | Maximum |
|---|---|---|---|
| Quantity | 6000 | 7000 | 7500 |
| Revenue/unit | \$11.25 | \$12.00 | \$12.50 |
| Cost/unit | \$ 9.00 | \$10.00 | \$11.00 |

As mentioned previously, a percent deviation graph, as in Figure 2, permits the plotting of the figure of merit (PW) as a function of the percent deviation from the most likely value for any number of parameters. However, the *interactive* effects of the parameter are ignored.

It is possible, of course, to reduce the original problem to a series for two-dimensional graphs. Here, for example, consider: (1) PW as a function of quantity, assuming $C$ = \$9.00, and a family of curves for $r$ = \$11.25 and \$12.50; and (2) PW as a function of quantity assuming $C$ = \$11.00, and a family of curves for $r$ = \$11.25 and \$12.50. This approach suffers from two defects. First, although a series of smaller problems is solved, we are not testing for the sensitivity of *all* parameters *simultaneously.* Second, the number of graphs required grows exponentially as the number of uncertain parameters increases arithmetically.

A second approach is based on the *a fortiori* ("strength of the argument") principle. If it can be shown that a certain course of action is indicated regardless of the input assumptions, then it has been proven, *a fortiori,* that there can be no other possible outcome. To illustrate, the following is a computation of both the minimum and maximum possible values for PW, given the ranges for the input assumptions:

$$\text{Min PW} = 6000(\$11.25 - \$11.00)(7.606) - \$95{,}212 = -\$83{,}803$$
$$\text{Max PW} = 7500(\$12.50 - \$9.00)(7.606) - \$95{,}212 = \$104{,}446$$

If both present worths had been negative, we would have proven, *a fortiori,* that the proposal should be rejected on economic grounds. Conversely, if both PW values had been positive, an "accept" decision would have been indicated.

Unfortunately, this test of extreme values rarely yields a clear result, and the *a fortiori* argument cannot be used. Nevertheless, analysts would be well advised to try this approach before proceeding further. The calculations can be completed relatively easily, and the few cases for which a clear signal *is* indicated more than justify the time involved.

## 3. RISK ANALYSIS

### 3.1. Alternative Risk Measures

A number of different statistics have been proposed for the measure of "riskiness" of proposed plans, programs, and projects. Perhaps the most widely used measure is the *variance* (*or standard deviation*) *of the prospective return,* where return is generally the present worth, internal rate of return, and so forth. The variance ($\sigma^2$) of the distribution for a continuous random variable $x$ is given by

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \qquad (4)$$

Large variance signifies large risk; relatively small variance indicates relatively small risk. In general, everything else being equal, risk is to be minimized.

An alternative view is that the *semivariance* is a preferred statistic as it focuses on the variability in negative return, that is, on the reduction of losses. The semivariance ($S_h$) of a distribution for a random variable $X$ is given by

$$S_h = \int_{-\infty}^{h} (h - x)^2 f(x) dx \qquad (5)$$

Still another measure of risk is the *probability of loss,* a statistic that measures the probability that the return will lie below some predetermined critical level, $h$. The probability of loss ($L$) for a continuous random variable is given by

$$L = \int_{-\infty}^{h} f(x) dx \qquad (6)$$

Limited space precludes a full discussion of these (and other) risk measures. Therefore, in the remainder of this section we will limit our remarks to the variance, a statistic that has proven most popular in use as well as in the literature of engineering economy.

### 3.2. Determining the Probability Distribution for Present Worth

#### *3.2.1. Expected Present Worth*

Consider an uncertain stream of cash flows, $A_j$, occurring at the end of periods 1, 2, . . . $j$, . . . , $N$. If the project life, $N$, the discount rate, $i$, and the amounts and timing of the cash flows are known with certainty, then

$$\text{PW} = \sum_{j=0}^{N} A_j(1 + i)^{-j} \qquad (7)$$

Now suppose that the cash flows are random variables with associated probability or density functions $f(A_j)$. The PW is a function of random variables, so it is itself a random variable with mean, $\mu_p$,

$$\mu_p = \text{Exp[PW]} = \sum_{j=0}^{N} \mu_j(1 + i)^{-j} \tag{8}$$

where $\mu_j = \text{Exp}[A_j]$ for $j = 0, 1, \ldots, N$.

### 3.2.2. *Variance of Present Worth*

The variance of the PW distribution depends upon the degree of correlation between the individual cash flows. In general

$$\sigma_p^2 = \text{Var[PW]} = \sum_{j=0}^{N} \sigma_j^2(1 + i)^{-2j} + 2 \sum_{j=0}^{N-1} \sum_{k=j+1}^{N} \rho_{jk}\sigma_j\sigma_k(1 + i)^{j+k} \tag{9}$$

where $\rho_{jk}$ is the correlation coefficient between cash flows, $A_j$ and $A_k$ and $\sigma_j$ and $\sigma_k$ are the standard deviations of the distribution of $A_j$ and $A_k$, respectively. This formulation is intractable in practice because of the difficulty, if not impossibility, in estimating the correlation coefficients. However, formulations of the variance under the two extreme cases—independent cash flows ($\rho_{jk} = 0$) and perfectly correlated cash flows ($\rho_{jk} = 1$)—is helpful, as will be shown.

*3.2.2.1. Independent Cash Flows* If there is no causative or consequential relationship between the cash flows, they are said to be independent and

$$\sigma_p^2 = \sum_{j=0}^{N} \sigma_j^2(1 + i)^{-2j} \tag{10}$$

A numerical example, summarized in Table 1, illustrates Eqs. (9) and (10). This is a five-period project life with means ($\mu_j$) and variances ($\sigma_j^2$) of the cash flows as shown. The results: $\mu_p = \$69.44$ and $\sigma_p = \sqrt{\$802.9325} = \$28.34$.

*3.2.2.2. Perfectly Correlated Cash Flows* Cash flows in any two periods, $x$ and $y$, are perfectly correlated if, given that $A_x$ is the actual value of $\mu_x + d\sigma_x$, then

$$A_y = \mu_y + d\sigma_y$$

In words, if random factors cause $A_x$ to deviate from its mean value by $d$ standard deviations, the same factors will cause $A_y$ to deviate from its mean in the same direction by $d$ standard deviations. Under these conditions

$$\sigma_p = \sum_{j=0}^{N} \sigma_j(1 + i)^{-j} \tag{11}$$

To illustrate, consider the example summarized in Table 2. Assuming a 10% discount rate, the expected value of the PW of the five cash flows is \$625.92, and the standard deviation of the PW is \$75.82. Note that the expected value of PW is given by Eq. (8) and is independent of the degree of correlation.

**TABLE 1 Numerical Example: Determining the Mean and Variance of PW Given Probabilistic Cash Flows and 10% Discount Rate—Independent Cash Flows**

| End of Period | Cash Flow Estimates | | | Present Worth at 10% | |
|---|---|---|---|---|---|
| $j$ | Mean $\mu_j$ | Variance $\sigma_j^2$ | | $\mu^j(1.10)^{-j}$ | $\sigma_j^2(1.10)^{-2j}$ |
| 0 | −\$400 | \$\$ $0^2$ | | −\$400.00 | \$\$ 0.0000 |
| 1 | 100 | $10^2$ | | 90.91 | 82.6446 |
| 2 | 130 | $15^2$ | | 107.44 | 153.6780 |
| 3 | 160 | $20^2$ | | 120.21 | 225.7896 |
| 4 | 130 | $20^2$ | | 88.79 | 186.6030 |
| 5 | 100 | $20^2$ | | 62.09 | 154.2173 |
| | | | Totals | \$ 69.44 | \$\$802.9325 |

**TABLE 2 Numerical Example: Determining the Mean and Standard Deviation of PW Given Probabilistic Cash Flows and 10% Discount Rate—Perfectly Correlated Cash Flows**

| End of Period | Cash Flows | | | Present Worth at 10% | |
|---|---|---|---|---|---|
| | Mean | Standard Deviation | | | |
| $j$ | $\mu_j$ | $\sigma_j$ | | $\mu_j(1.0)^{-j}$ | $\sigma_j(1.10)^{-j}$ |
| 1 | $100 | 20 | | $ 90.91 | $18.18 |
| 2 | 150 | 20 | | 123.97 | 16.53 |
| 3 | 200 | 20 | | 150.26 | 15.03 |
| 4 | 200 | 20 | | 136.60 | 13.66 |
| 5 | 200 | 20 | | 124.18 | 12.42 |
| | | | Totals | $625.92 | $75.82 |

*3.2.2.3. Combining Independent and Perfectly Correlated Cash Flows* Suppose that it is feasible, in a given problem situation, to identify two types of cash flows: those that are statistically independent and those that are perfectly correlated. In this case the variance of the PW distribution is the sum of (a) the sum of the variances of the independent cash flows, discounted, and (b) the sum of the variances of each of the subsets of perfectly correlated cash flows, where the variance of each subset is the square of the sum of the standard deviations of the cash flows in that subset. That is,

$$\sigma_p^2 = \sum_{j=0}^{N} \sigma_j^2(1 + i)^{-2j} + \sum_{k-1}^{M} \left\{ \sum_{j=0}^{N} [\sigma_{jk}(1 + i)^{-j}] \right\}^2 \qquad (12)$$

where $\sigma_j^2$ = variance of the distribution of the independent $A_j$'s
$\sigma_{jk}$ = standard deviation of the distribution of the perfect correlated cash flows in subset $k$, $k = 1, 2, \ldots, M$

Returning to the previous example (Table 2), suppose that there is a cash flow $A_0$ such that $\mu_0 = -\$500$ and $\sigma_0 = \$10$, and $A_0$ is independent of the positive cash flows in periods 1 through 5. All cash flows for the proposal are now completely specified, and

$$\mu_p = \sum_{j=0}^{5} \mu_j(1.10)^{-j} = -\$500 + \$625.93 = \$125.93$$

$$\sigma_p = \$10 + \$75.82 = \$85.82$$

Note in this example that $M = 1$; there is only one subset of perfectly correlated cash flows.

## 3.3. Cash Flows with Uncertain Timing

### *3.3.1. Single Cash Flow*

Consider a single (impulse) cash flow, $F$, occurring at time $t$. If both $t$ and $F$ are deterministic, and assuming that interest is compounded/discounted continuously at nominal rate $r$ per period, then the present worth is given by

$$\text{PW} = Fe^{-n} \qquad (13)$$

If the timing, $t$, is a random variable with probability density function $f(t)$, then

$$\mu_p = \text{Exp[PW]} = F \int_0^\infty f(t)e^{-n}\, dt \qquad (14)$$

and

$$\sigma_p^2 = \text{Var[PW]} = F^2 \int_0^\infty f(t)e^{-n}\, dt - \{F[\text{Exp}(e^{-n})]\}^2 \qquad (15)$$

When the cash flow, $F$, is also a random variable with known $\mu_F$ and $\sigma_F^2$, then the PW is the product of two random variables. Determination of $\mu_p$ and $\sigma_p^2$ results from a straightforward application of probability theory with respect to products of random variables.

### *3.3.2. Uncertain Initiation and Duration*

Consider a uniform continuous cash flow, $A$, which begins at time $m$ and continues for an uncertain duration $t$. Assume that $m$ and $t$ are statistically independent random variables with known probability functions $f(m)$ and $f(t)$. It may be shown that

$$\mu_p = (\overline{A}/r)[\text{Exp}(e^{-rm})][1 - \text{Exp}(e^{-rt})]$$

and

$$\sigma_p^2 = (\overline{A}/r)^2[\text{Var}(e^{-rm}) + \text{Var}(e^{-r(m+t)})] \tag{16}$$

To illustrate, consider a uniform cash flow of \$1000 per year beginning at some uncertain time $m$ and continuing for a duration of $t$ years. The delay to initiation is *uniformly* distributed between 6 months and 1 year. The project duration is *gamma* distributed with mean of 3 years and standard deviation of 1 year; the parameters of the gamma distribution yielding these statistics are $a = 3$ and $b = 9$. The nominal interest rate is 10% compounded continuously. It is assumed that the initiation time and project duration are independent random variables. Our problem is to determine the equivalent present value of these cash flows. (This problem is taken from Park and Sharp-Bette (1990, p. 411)

This problem may be solved by use of integral calculus in connection with Eq. (16). However, it may be instructive to use *Laplace transform* methodology to evaluate $\mu_p$ and $\sigma_p^2$. If a function $f(x)$ is considered to be piecewise continuous, then the Laplace transform of the function, written $\mathcal{L}\{f(x)\}$, is defined as a function $F(s)$ of the variable $r$ by the integral

$$\mathcal{L}\{f(x)\} = F(r) = \int_0^\infty f(x)e^{-rx}\,dx = \text{Exp}(e^{-rx})$$

over the range of values of $r$ for which the integral exists. For the *uniform* distribution

$$F(x) = \frac{e^{-ra} - e^{-rb}}{r(b - a)}$$

and for the *gamma* distribution

$$F(r) = \left[1 + \left(\frac{r}{a}\right)\right]^{-b}$$

Returning to our example, for the uniformly distributed delay time, $m$,

$$\text{Exp}(e^{-rm}) = \mathcal{L}(r)_m = \frac{e^{-0.10(0.5)} - e^{-0.10(1.0)}}{0.10(1.0 - 0.5)} = 0.92784$$

and

$$\begin{aligned}\text{Var}(e^{-rm}) &= \mathcal{L}(2r) - \mathcal{L}(r)^2\\ &= \frac{e^{-0.2(0.5)} - e^{-0.2(1.0)}}{0.2(1.0 - 0.5)} - (0.92784)^2\\ &= 0.86107 - 0.86089 = 0.00018\end{aligned}$$

Similarly, considering the gamma-distributed random variable, $t$,

$$\text{Exp}(e^{-rt}) = \mathcal{L}(r)_t = [1 + (0.10/3)]^{-9} = 0.74445$$

and

$$\begin{aligned}\text{Var}(e^{-rt}) &= [1 + (0.20/3)]^{-9} - (0.74445)^2\\ &= 0.559425 - 0.554206 = 0.00522\end{aligned}$$

Next, we must determine

$$\begin{aligned}\text{Var}(e^{-r(m+t)}) &= \text{Var}(e^{-rm}e^{-rt}) \\ &= \mathcal{E}(r)_m^2[\text{Var}(e^{-rt}) + \mathcal{E}(r)_t^2[\text{Var}(e^{-rm})] + [\text{Var}(e^{-rm})][\text{Var}(e^{-rt})] \\ &= (0.92784)^2(0.00522) + (0.74445)^2(0.00018) + (0.0018)(0.00522) \\ &= 0.00459\end{aligned}$$

Finally, using Eq. (16),

$$\begin{aligned}\mu_p &= (\$1{,}000/0.10)(0.92784)(1 - 0.74455) \\ &= \$2{,}371\end{aligned}$$

and

$$\begin{aligned}\sigma_p^2 &= (\$1{,}000/0.10)^2(0.00018 + 0.00459) \\ &= \$477{,}000\end{aligned}$$

or

$$\sigma_p = \$691$$

### 3.4. Uncertain Project Life and Uncertain Cash Flow

Consider the case in which the amounts and timing of cash flows are random variables with known means and variances, and the project life, also, is a random variable, $N$. Here, $N$ must be integer valued and, of course, must be positive. If cash flows are statistically independent,

$$\mu_p = \text{Exp}[\text{PW}] = \sum_{N=1}^{\infty} \left[ \sum_{j=1}^{N} \mu_j(1 + i)^{-j} \right] p_N \tag{17}$$

where $p_N$ is the probability mass function for $N$. Moreover,

$$\sigma_p^2 = \text{Var}[\text{PW}] = \sum_{k=1}^{N} \left\{ \sum_{j=0}^{k} \text{Var}(X_j) + \left[ \sum_{j=0}^{k} \text{Exp}(x_j) \right]^2 \right\} P_k - \mu_p^2 \tag{18}$$

where

$$\text{Exp}(X_j) = \mu_j(1 + i)^{-j} \tag{19a}$$

$$\text{Var}(X_j) = \sigma_j^2(1 + i)^{-2j} \tag{19b}$$

To illustrate, consider the problem summarized in Table 3. There are three risky, independent, end-of-year cash flows; the means and variances of their respective probability functions are given in columns (2) and (3) of Table 3. Project life, $N$, is also a random variable, with probability mass function as shown in columns (6) and (7) of the table. A 10% discount rate is assumed. Determination of the expected present worth, $\mu_p$, based on Eq. 17, is summarized in the table. Here, $\mu_p = -\$82.64$. The variance of present worth, $\sigma_p^2$, based on Eqs. 18 and 19, may be shown to be

$$\sigma_p^2 = \$121{,}687 \qquad \text{or} \qquad \sigma_p = \$349$$

### 3.5. Other Models

There are a variety of other analytical models for assessing risky investments. The randomness ("riskiness") of cash flow amounts, timing, project life, and discount rate are considered singly and/or in combination. The complexity of the analytical procedure is roughly a function of the number of variables considered as well as the assumptions concerning mutual independence between random variables. In almost all cases the *mean* and *variance* of the distribution of the figure of merit are of primary concern. In some instances it is also possible to approximate the statistical *distribution* as well. Space limitations preclude an exhaustive review of the extant literature. For further readings consult the bibliography at the end of this chapter.

### 3.6. Analysis Based on the Probability Distribution for Present Worth

As before, the mean and variance of the probability distribution for the present worth statistic (PW) are denoted by $\mu_p$ and $\sigma_p^2$, respectively. These are measures of central tendency and variability, or

**TABLE 3 Numerical Example: Both Cash Flows ($A_j$) and Project Life ($N$) Are Random Variables**

| End of Year $j$ (1) | Mean $\mu_j$ (2) | Variance $\sigma_j^2$ (3) | Present Worth at 10% $\mu_j(1.10)^{-j}$ (4) | Present Worth at 10% $\sigma_j^2(1.10)^{-2j}$ (5) |
|---|---|---|---|---|
| 0 | −$1,000 | $ $50^2$ | −$1,000.00 | $ 2,500.00 |
| 1 | 500 | $100^2$ | 454.55 | 8,264.46 |
| 2 | 800 | $200^2$ | 661.16 | 27,320.54 |
| **Project Life $N$ (6)** | **Probability $p_N$ (7)** | | **$\sum_{j=0}^{N} \mu_j(1.10)^{-j}$ (8)** | **$p_N \sum_{j=0}^{N} \mu_j(1.10)^{-j}$ (9)** |
| 0 | 0.00 | | −$1,000.00 | $ 0 |
| 1 | 0.30 | | − 545.45 | − 163.64 |
| 2 | 0.70 | | 115.71 | 81.00 |
| Totals | 1.00 | | | −$ 82.64 |

From Park and Sharp-Bette 1990, p. 416.

dispersion, of the PW distribution. Under certain conditions the underlying probability distribution may be fully or partially characterized. When such is the case, it may be useful to describe the riskiness of the figure of merit in terms other than the variance of the distribution; for example, the probability that the PW will exceed some specified critical level.

### *3.6.1. Discrete Distribution for Present Worth*

Consider a two-period problem as summarized in Figure 5. A cash outlay of $100 occurs at the start of period 1 ($j = 0$). There are two possible discrete cash flows at end of period 1: $A_1$ = $50 with probability 0.5 or $A_1$ = $70 with probability 0.5. *If* $A_1$ = $50, there are two possibilities for the cash flow at end of period 2: either $A_2$ = $60 with probability 0.3 or $A_2$ = $80 with probability 0.7. *If* $A_1$ = $70, then either $A_2$ = $44 with probability 0.4 or $A_2$ = $90 with probability 0.6. The diagram of the possible outcomes shown in Figure 5 is sometimes known as a *probability tree.*

There are four possible present worths (outcomes), each with an associated joint probability. Assuming a 10% discount rate:

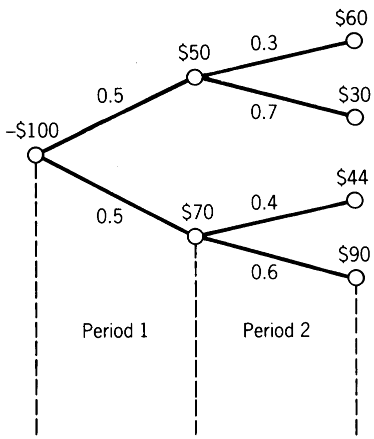


**Figure 5** Cash Rows and Their Probabilities. Example problem from Park and Sharpe-Bette 1990, p. 419.

| Outcome | $A_0$ | $A_1$ | $A_2$ | $\sum_{j=0}^{2} A_j(1.10)^{-j}$ | Joint Probability |
|---|---|---|---|---|---|
| 1 | −$100 | $50 | $60 | −$4.96 | 0.5 × 0.3 = 0.15 |
| 2 | −100 | 50 | 80 | 11.57 | 0.5 × 0.7 = 0.35 |
| 3 | −100 | 70 | 44 | 0 | 0.5 × 0.4 = 0.20 |
| 4 | −100 | 70 | 90 | 38.02 | 0.5 × 0.6 = 0.30 |

The remainder of the analysis is summarized in Table 4. Note that column (4) reflects the calculation of Exp[PW] and column (5) reflects the calculation of Exp[(PW)²]. Moreover

$$\sigma_p^2 = \text{Exp}[(\text{PW})^2] - [\text{Exp}(\text{PW})]^2 \tag{20}$$

as discussed previously.

Now, suppose that it is of interest to determine the probability that this investment will be profitable, that is, PW > $0. Only two of the possible outcomes, 2 and 4, meet this requirement, and, as they are independent events, the sum of their probabilities is

$$\begin{aligned}\text{Prob}[\text{PW} > \$0] &= \text{Prob}[\text{PW} = \$11.57] + \text{Prob}[\text{PW} = \$38.02] \\ &= 0.35 + 0.30 = 0.65\end{aligned}$$

### *3.6.2. Using Only the Mean and Variance of the PW Distribution*

*Tchebycheff's* (sometimes written Chebyshev's) *inequality* states that

$$\text{Prob}[\mu - k\sigma < X < \mu + k\sigma] \geq 1 - 1/k^2 \tag{21}$$

where $X$ is any random variable having mean $\mu$ and variance $\sigma^2$ and $k$ is a positive constant. This is a useful relationship when only the mean and variance of the distribution are known. In terms of an unknown PW distribution with known mean $\mu_p$ and variance $\sigma_p^2$,

$$\text{Prob}[\mu_p - k\sigma_p < \text{PW} < \mu_p + k\sigma_p] \geq 1 - 1/k^2 \tag{22}$$

To illustrate, suppose that the mean and variance of the PW distribution have been determined to be $800 and ($50)², respectively. The analyst has been asked to determine the probability that the PW lies between two values, say, between $600 and $1000. Note here that

$$\mu_p - k\sigma_p = \$800 - k(\$50) = \$600$$

and

**TABLE 4 Determining the Expected Present Worth**

| Outcome (1) | PW at 10% (2) | Joint Probability (3) | (4) = (2) × (3) | (5) = (2)² × (3) |
|---|---|---|---|---|
| 1 | −$ 4.96 | 0.15 | −$ 0.744 | $$ 3.690 |
| 2 | 11.57 | 0.35 | 4.050 | 46.853 |
| 3 | 0 | 0.20 | 0 | 0 |
| 4 | 38.02 | 0.30 | 11.406 | 433.656 |
| | Totals | 1.00 | $14.712 | $$484.199 |

$$\text{Exp}[\text{PW}] = \underline{\$14.712}$$

$$\text{Var}[\text{PW}] = \$\$484.199 - (\$14.712)^2 = \$\$267.756$$

$$\sigma_P = \sqrt{\text{Var}[\text{PW}]} = \underline{\$16.363}$$

$$\mu_p + k\sigma_p = \$800 + k(\$50) = \$1000$$

from which it is apparent that $k = 4$. Thus

$$\text{Prob}[\$600 < \text{PW} < \$1000] \geq 1 - 1/16 \quad \text{or} \quad 0.9375$$

Put somewhat differently, in the absence of any knowledge as to the *shape* of the distribution, the probability is *at least* 0.9375 that the random variable lies with $\pm 2\sigma$ of the mean.

### *3.6.3. When the Normal Distribution Can Be Assumed*

Consider a stream of risky cash flows $A_j$ occurring at the ends of periods 1, 2, . . . $j$, . . . $N$. The project life $N$ and the discount rate $i$ are known with certainty. The only stochastic variable here is the amount of the cash flow. The resulting PW is a random variable with mean given by Eq. (8) and, assuming independent cash flows, with variance given by Eq. (10). Under some general conditions application of the central limit theorem leads to the result that

$$Z_N = \frac{\text{PW} - \Sigma_{j=0}^{N} \mu_j(1 + i)^{-j}}{\sqrt{\Sigma_{j=0}^{N} \sigma_j^2}} \tag{23}$$

is approximately normally distributed, with $\mu = 0$ and $\sigma = 1$, as $N$ approaches infinity. The "general condition" may be summarized as follows: the terms $A_j$, taken individually, contribute a negligible amount to the variance of the sum, and it is unlikely that any single $A_j$ makes a relatively large contribution to the sum.

The terms $A_j$ may have essentially any distribution. As a general rule of thumb, if the $A_j$'s are approximately normally distributed, then the central limit theorem is a very good approximation when $N \geq 4$. If the distribution of the $A_j$'s has no prominent mode(s), that is, approximately uniformly distributed, then $N \geq 12$ is a reasonable rule of thumb for applicability of the central limit theorem.

To illustrate the application of Eq. (23), consider the numerical example given in Table 1. It was determined that $\mu_p = \$69.44$ and $\sigma_p = \sqrt{\$\$802.9325} = \$28.34$. The probability distribution for PW is shown in part (*a*) of Figure 6; the equivalent standardized normal distribution is shown in part (*b*).

Consider the question: What is the probability that this proposal will result in a present worth greater than $50?

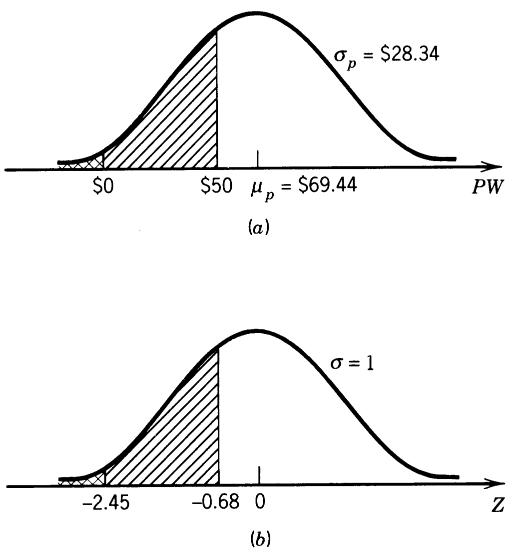


**Figure 6** Probability Distribution for Present Worth.

## TABLE 5 Cumulative Standard Normal Distribution

$$\phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 | z |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.500 00 | 0.503 99 | 0.507 98 | 0.511 97 | 0.515 95 | 0.519 94 | 0.523 92 | 0.527 90 | 0.531 88 | 0.535 86 | 0.0 |
| 0.1 | 0.539 83 | 0.543 79 | 0.547 76 | 0.551 72 | 0.555 67 | 0.555 62 | 0.563 56 | 0.567 49 | 0.571 42 | 0.575 34 | 0.1 |
| 0.2 | 0.579 26 | 0.583 17 | 0.587 06 | 0.590 95 | 0.594 83 | 0.598 71 | 0.602 57 | 0.606 42 | 0.610 26 | 0.614 09 | 0.2 |
| 0.3 | 0.617 91 | 0.621 72 | 0.625 51 | 0.629 30 | 0.633 07 | 0.636 83 | 0.640 58 | 0.644 31 | 0.648 03 | 0.651 73 | 0.3 |
| 0.4 | 0.655 42 | 0.659 10 | 0.662 76 | 0.666 40 | 0.670 03 | 0.673 64 | 0.677 24 | 0.680 82 | 0.684 38 | 0.687 93 | 0.4 |
| 0.5 | 0.691 46 | 0.694 97 | 0.698 47 | 0.701 94 | 0.705 40 | 0.708 84 | 0.712 26 | 0.715 66 | 0.719 04 | 0.722 40 | 0.5 |
| 0.6 | 0.725 75 | 0.729 07 | 0.732 37 | 0.735 65 | 0.738 91 | 0.742 15 | 0.745 37 | 0.748 57 | 0.751 75 | 0.754 90 | 0.6 |
| 0.7 | 0.758 03 | 0.761 15 | 0.764 24 | 0.767 30 | 0.770 35 | 0.773 37 | 0.776 37 | 0.779 35 | 0.782 30 | 0.785 23 | 0.7 |
| 0.8 | 0.778 14 | 0.791 03 | 0.793 89 | 0.796 73 | 0.799 54 | 0.802 34 | 0.805 10 | 0.807 85 | 0.810 57 | 0.813 27 | 0.8 |
| 0.9 | 0.815 94 | 0.818 94 | 0.821 21 | 0.823 81 | 0.826 39 | 0.828 94 | 0.831 47 | 0.833 97 | 0.836 46 | 0.838 91 | 0.9 |
| 1.0 | 0.841 34 | 0.843 75 | 0.846 13 | 0.848 49 | 0.850 83 | 0.853 14 | 0.855 43 | 0.857 69 | 0.859 93 | 0.862 14 | 1.0 |
| 1.1 | 0.864 33 | 0.866 50 | 0.868 64 | 0.870 76 | 0.872 85 | 0.874 93 | 0.876 97 | 0.879 00 | 0.881 00 | 0.882 97 | 1.1 |
| 1.2 | 0.884 93 | 0.886 86 | 0.888 77 | 0.890 65 | 0.892 51 | 0.894 35 | 0.896 16 | 0.897 96 | 0.899 73 | 0.901 47 | 1.2 |
| 1.3 | 0.903 20 | 0.904 90 | 0.906 58 | 0.908 24 | 0.909 88 | 0.911 49 | 0.913 08 | 0.914 65 | 0.916 21 | 0.917 73 | 1.3 |
| 1.4 | 0.919 24 | 0.920 73 | 0.922 19 | 0.923 64 | 0.925 06 | 0.926 47 | 0.927 85 | 0.929 22 | 0.930 56 | 0.931 89 | 1.4 |
| 1.5 | 0.933 19 | 0.934 48 | 0.935 74 | 0.936 99 | 0.938 22 | 0.939 43 | 0.940 62 | 0.941 79 | 0.942 95 | 0.944 08 | 1.5 |
| 1.6 | 0.945 20 | 0.946 30 | 0.947 38 | 0.948 45 | 0.949 50 | 0.950 53 | 0.951 54 | 0.952 54 | 0.953 52 | 0.954 48 | 1.6 |
| 1.7 | 0.955 43 | 0.956 37 | 0.957 28 | 0.958 18 | 0.959 07 | 0.959 94 | 0.960 80 | 0.961 64 | 0.962 46 | 0.963 27 | 1.7 |
| 1.8 | 0.964 07 | 0.964 85 | 0.965 62 | 0.966 37 | 0.967 11 | 0.967 84 | 0.968 56 | 0.969 26 | 0.969 95 | 0.970 62 | 1.8 |
| 1.9 | 0.971 28 | 0.971 93 | 0.972 57 | 0.973 20 | 0.973 81 | 0.974 41 | 0.975 00 | 0.975 58 | 0.976 15 | 0.976 70 | 1.9 |
| 2.0 | 0.977 25 | 0.977 78 | 0.978 31 | 0.978 82 | 0.979 32 | 0.979 82 | 0.980 30 | 0.980 77 | 0.981 24 | 0.981 69 | 2.0 |
| 2.1 | 0.982 14 | 0.982 57 | 0.983 00 | 0.983 41 | 0.983 82 | 0.984 22 | 0.984 61 | 0.985 00 | 0.985 37 | 0.985 74 | 2.1 |
| 2.2 | 0.986 10 | 0.986 45 | 0.986 79 | 0.987 13 | 0.987 45 | 0.987 78 | 0.988 09 | 0.988 40 | 0.988 70 | 0.988 99 | 2.2 |
| 2.3 | 0.989 28 | 0.989 56 | 0.989 83 | 0.990 10 | 0.990 36 | 0.990 61 | 0.990 86 | 0.991 11 | 0.991 34 | 0.991 58 | 2.3 |
| 2.4 | 0.991 80 | 0.992 02 | 0.992 24 | 0.992 45 | 0.992 66 | 0.992 86 | 0.993 05 | 0.993 24 | 0.993 43 | 0.993 61 | 2.4 |
| 2.5 | 0.993 79 | 0.993 96 | 0.994 13 | 0.994 30 | 0.994 46 | 0.994 61 | 0.994 77 | 0.994 92 | 0.995 06 | 0.995 20 | 2.5 |
| 2.6 | 0.995 34 | 0.995 47 | 0.995 60 | 0.995 73 | 0.995 85 | 0.995 98 | 0.996 09 | 0.996 21 | 0.996 32 | 0.996 43 | 2.6 |
| 2.7 | 0.996 53 | 0.996 64 | 0.996 74 | 0.996 83 | 0.996 93 | 0.997 02 | 0.997 11 | 0.997 20 | 0.997 28 | 0.997 36 | 2.7 |
| 2.8 | 0.997 44 | 0.997 52 | 0.997 60 | 0.997 67 | 0.997 74 | 0.997 81 | 0.997 88 | 0.997 95 | 0.998 01 | 0.998 07 | 2.8 |
| 2.9 | 0.998 13 | 0.998 19 | 0.998 25 | 0.998 31 | 0.998 36 | 0.998 41 | 0.998 46 | 0.998 51 | 0.998 56 | 0.998 61 | 2.9 |
| 3.0 | 0.998 65 | 0.998 69 | 0.998 74 | 0.998 78 | 0.998 82 | 0.998 86 | 0.998 89 | 0.998 93 | 0.998 97 | 0.999 00 | 3.0 |
| 3.1 | 0.999 03 | 0.999 06 | 0.999 10 | 0.999 13 | 0.999 16 | 0.999 18 | 0.999 21 | 0.999 24 | 0.999 26 | 0.999 29 | 3.1 |
| 3.2 | 0.999 31 | 0.999 34 | 0.999 36 | 0.999 38 | 0.999 40 | 0.999 42 | 0.999 44 | 0.999 46 | 0.999 48 | 0.999 50 | 3.2 |
| 3.3 | 0.999 52 | 0.999 53 | 0.999 55 | 0.999 57 | 0.999 58 | 0.999 60 | 0.999 61 | 0.999 62 | 0.999 64 | 0.999 65 | 3.3 |
| 3.4 | 0.999 66 | 0.999 68 | 0.999 69 | 0.999 70 | 0.999 71 | 0.999 72 | 0.999 73 | 0.999 74 | 0.999 75 | 0.999 76 | 3.4 |
| 3.5 | 0.999 77 | 0.999 78 | 0.999 78 | 0.999 79 | 0.999 80 | 0.999 81 | 0.999 81 | 0.999 82 | 0.999 83 | 0.999 83 | 3.5 |
| 3.6 | 0.999 84 | 0.999 85 | 0.999 85 | 0.999 86 | 0.999 86 | 0.999 87 | 0.999 87 | 0.999 88 | 0.999 88 | 0.999 89 | 3.6 |
| 3.7 | 0.999 89 | 0.999 90 | 0.999 90 | 0.999 90 | 0.999 91 | 0.999 91 | 0.999 92 | 0.999 92 | 0.999 92 | 0.999 92 | 3.7 |
| 3.8 | 0.999 93 | 0.999 93 | 0.999 93 | 0.999 94 | 0.999 94 | 0.999 94 | 0.999 94 | 0.999 95 | 0.999 95 | 0.999 95 | 3.8 |
| 3.9 | 0.999 95 | 0.999 95 | 0.999 96 | 0.999 96 | 0.999 96 | 0.999 96 | 0.999 96 | 0.999 96 | 0.999 97 | 0.999 97 | 3.9 |

*Source:* Adapted from W. W. Hines and D.C. Montgomery, *Probability and Statistics in Engineering and Management Science,* 2nd Ed., John Wiley & Sons, New York, 1980; from Park and Sharp-Bette 1990.

$$\text{Prob[PW} > \$50] = 1 - \text{Prob[PW} < \$50]$$
$$= 1 - \text{Prob}\left[Z < \frac{\$50.00 - \$69.44}{\$28.34}\right]$$
$$= 1 - \text{Prob}[Z < -0.686] \simeq 1 - 0.75 = 0.25$$

An abbreviated version is included here in Table 5.

Consider a second question: What is the probability that this proposal will result in a loss?

$$\text{Prob[PW} < \$0] = \text{Prob}\left[Z < \frac{\$0 - \$69.44}{\$28.34}\right]$$
$$= \text{Prob}[Z < -2.45] \simeq 0.01$$

Note that the probability of a loss is identical here to the probability that the proposal's internal rate of return will be less than the minimum attractive rate of return.

### 3.7. Comparing Risky Proposals

As indicated previously, decision makers are generally risk avoiders. Additional risk, as measured by the variance of the figure of merit, is to be avoided whenever possible. Thus there are two criteria to be considered simultaneously: the figure of merit, for example, present worth, as measured by the expected value (mean, $\mu$) of the distribution; and the riskiness of the outcome as measured by the variance ($\sigma^2$) of the distribution. The former is to be maximized, and the latter is to be minimized.

Consider two mutually exclusive alternations. Let ($\mu_1$, $\sigma_1^2$) and ($\mu_2$, $\sigma_2^2$) represent the mean and variance of alternatives I and II, respectively. The appropriate decision rules are as follows:

| | | |
|---|---|---|
| Case A: | If $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$<br>or $\mu_1 > \mu_2$ and $\sigma_1 \leq \sigma_2$ | Choose I over II |
| Case B: | If $\mu_1 = \mu_2$ and $\sigma_1 > \sigma_2$<br>or $\mu_1 < \mu_2$ and $\sigma_1 \geq \sigma_2$ | Choose II over I |
| Case C: | If $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$<br>or $\mu_1 > \mu_2$ and $\sigma_1 > \sigma_2$ | Conclusion ambiguous |

There is no clear conclusion in case C because the riskier alternative is also the one with the larger expected return. When this situation arises, trade-offs must be made between risk and return. There is little theoretical guidance short of converting to utility theory (discussed later). Additional discussion is beyond the scope of this chapter.

## 4. DECISION THEORY APPLICATIONS

The approach to risk analysis outlined in the previous section is based on the premise that the decision maker desires to (a) maximize expected return and (b) minimize risk. This section presents some additional principles of choice that may be appealing under certain conditions. A simple numerical example is used as a basis for the discussion.

### 4.1. Problem Statement

The International Manufacturing Company (IMC) is considering five mutually exclusive alternatives for constructing a new manufacturing plant in a certain Asian country. The cost of each alternative, stated in terms of present worth (or net present value) of cost, depend on the outcome of negotiations that are currently under way between IMC, lending agencies, and the government of the host country. IMC analysts have concluded that four specific mutually exclusive outcomes are possible, and they have computed the present worth (cost) for each alternative—outcome combination. These are shown as cell values in the *cost matrix* in Table 6.

If the future is known with certainty, then the least costly alternative may be selected by any of the methods presented in Chapter 91. For example, if it is known that outcome $s_3$ will definitely occur, then $a_3$ should be selected because it will result in the lowest present worth (cost). On the other hand, if $a_3$ is selected and $s_4$ perversely occurs, choosing $a_3$ will have resulted in the most costly event.

Assume that sufficient information exists to warrant statements about the relative probabilities of the possible future outcomes. Specifically, these probabilities (expected relative frequencies) are

**TABLE 6 Cost Matrix for Illustrative Problem (Cell entries are multiples of $1,000,000)**

| | | Possible Outcomes | | | |
|---|---|---|---|---|---|
| | | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
| Alternatives | $a_1$ | 18 | 11 | 10 | 10 |
| | $a_2$ | 16 | 16 | 16 | 16 |
| | $a_3$ | 14 | 14 | 8 | 20 |
| | $a_4$ | 9 | 12 | 17 | 16 |
| | $a_5$ | 10 | 13 | 17 | 18 |

$$P[s_1] = 0.3 \qquad P[s_3] = 0.2$$

$$P[s_2] = 0.4 \qquad P[s_4] = 0.1$$

Given this additional information, which alternative should be selected? A number of principles that may be applied in this situation are discussed later.

A problem statement of this type is known as a *decision under risk* because the underlying probability distribution for the future scenarios, or *states of nature,* is known or can be assumed. "Risk," in the previous section, was used in a more general sense to characterize the absence of certainty. The term was used analogously to "randomness" or uncertainty. Here, in a more limited sense, a problem statement in which the underlying distribution for the $s_j$'s is *not* known or assumed is a *decision under uncertainty*.

## 4.2. Dominance

Before applying *any* of the principles of choice, it is first desirable (although not absolutely necessary) to apply the *dominance principle* to determine which alternatives, if any, are dominated. If, of two alternatives, one would never be preferred no matter what future occurs, it is said to be dominated and may be removed from any further consideration. From the example, consider $a_4$ and $a_5$:

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ | |
|---|---|---|---|---|---|
| $a_4$ | 9 | 12 | 17 | 16 | (in $ million) |
| $a_5$ | 10 | 13 | 17 | 18 | |

Since $a_5$ is always at least as costly as $a_4$, irrespective of which future outcome occurs, $a_5$ may be ignored in the remaining discussion.

If one alternative dominates all others, it is said to be *globally dominant,* and the decision maker need look no further; the optimal solution has been found. Unfortunately, globally dominant alternatives are rare. But in any event, the dominance principle is frequently effective in reducing the number of alternatives to be considered.

## 4.3. Principles for Decisions under Risk

### *4.3.1. The Principle of Expectation*

The *principle of expectation* states that the alternative to be selected is the one that has the minimum expected cost (or maximum expected profit or revenue). In general,

$$\text{Min } E[C(a_i)] = \sum_j C(a_i|s_j)p_j \tag{24a}$$

or

$$\text{Max } E[R(a_i)] = \sum_j R(a_i|s_j)p_j \tag{24b}$$

where $C(a_i|s_j)$ = total cost of alternative $a_i$ given that states of nature $s_j$ occurs
$R(a_i|s_j)$ = total net return of alternative $a_i$ given that state of nature $s_j$ occurs
$p_j$ = probability that state $s_j$ will occur

From the example it may be shown that

$$E[C(a_1)] = \$12{,}800{,}000 \qquad E(C(a_3)] = \$13{,}400{,}000$$
$$E[C(a_2)] = \$16{,}000{,}000 \qquad E[C(a_4)] = \$12{,}500{,}000$$

Here, $a_4$ should be selected because it yields the minimum expected cost.

Principles that depend on determination of expected values by the mathematics of probability theory are frequently criticized on the grounds that the theory holds only when trials are repeated many times. It is argued that, for certain types of decisions—for example, whether to finance a major expansion—expectation is meaningless since this type of decision is not made very often. According to the counterargument, even if the firm is not faced with a large number of repetitive decisions, it should apply the principle to many different decisions and thus realize the long-run effects. Moreover, even if the decision is unique, the only way to approach decisions for which probabilities are known is to behave as if the decision were a repetitive one and thus minimize expected cost or maximize expected revenue or profit.

### *4.3.2. The Principle of Most Probable Future*

Assume that the future event to expect is the most likely event. Thus, observing that $s_2$ has the highest probability of occurring, assume that it will in fact occur. In this case $a_1$ (with present worth (cost) = \$11,000,000 is the least costly of the four available alternatives.

This principle is particularly appealing in cases in which one future is significantly more probable than all other possibilities.

### *4.3.3. The Aspiration Level Principle*

The *aspiration level principle* requires the establishment of a goal, or "level of aspiration." Thus the alternative that maximizes the probability that the goal will be met or exceeded should be selected. To illustrate, suppose that the management of IMC wishes to minimize the probability that present worth (cost) will exceed \$15,000,000. (This is identical to the requirement that it maximize the probability that costs will *not* exceed \$15,000,000.) The probabilities are

$$\text{Prob}[C(a_1) > \$15{,}000{,}000] = 0.3$$
$$\text{Prob}[C(a_2) > \$15{,}000{,}000] = 0.3 + 0.4 + 0.2 + 0.1 = 1.0$$
$$\text{Prob}[C(a_3) > \$15{,}000{,}000] = 0.1$$
$$\text{Prob}[C(a_4) > \$15{,}000{,}000] = +0.2 + 0.1 = 0.3$$

Thus, the aspiration level will be met if $a_1$ is selected.

Clearly, the selection from mutually exclusive alternatives is a matter of which principle is used to guide the decision.

## 4.4. Principles for Decisions under Uncertainty

This section examines a number of principles of choice that may be used when the relative likelihoods of future states of nature *cannot* be estimated. These principles will be demonstrated by using the example problem introduced earlier.

### *4.4.1. The Minimax (or Maximin) Principle*

The *minimax principle* is pessimistic in the extreme. It assumes that, if any alternative is selected, the worst possible outcome will occur. The maximum cost associated with each alternative is examined, and the alternative that *minimizes* the *maximum* cost is selected. In general, the mathematical formulation of the minimax principle is

$$\underset{i}{\text{Min}}[\underset{j}{\text{Max}}(C_{ij})] \tag{25a}$$

where $C_{ij}$ is the cost that results when alternative $i$ is selected and state of nature $j$ occurs. From the example

| Alternative ($a_j$) | $\underset{j}{\text{Max}}\ C_{ij}$ |
|---|---|
| $a_1$ | 18 (in \$ million) |
| $a_2$ | 16 (in \$ million) |
| $a_3$ | 20 (in \$ million) |
| $a_4$ | 17 (in \$ million) |

If the minimax principle is adopted, $a_2$ is indicated because it results in minimum costs, assuming the worst possible conditions.

The mirror image of the minimax principle, the *maximin principle,* may be applied when the matrix contains *profits* or *revenue* measures. In this case the most pessimistic view suggests that the alternative to select is the one that *maximizes* the *minimum* profit or revenue associated with each alternative. The mathematical formulation of the maximin principle is

$$\underset{i}{\text{Max}}[\underset{j}{\text{Min}}(R_{ij})] \tag{25b}$$

where $R_{ij}$ is the revenue or profit resulting from the combination of $a_i$ and $s_j$.

### *4.4.2. The Minimin (or Maximax) Principle*

The *minimin principle* is based on the view that the best possible outcome will occur when a given alternative is selected. It is optimistic in the extreme. The minimum cost associated with each alternative is examined, and the alternative that *minimizes* the *minimum* cost is selected. The mathematical formulation is

$$\underset{i}{\text{Min}}[\underset{j}{\text{Min}}(C_{ij})] \tag{26a}$$

From the example

| Alternative ($a_1$) | $\underset{j}{\text{Min}}\ C_{ij}$ |
|---|---|
| $a_1$ | 10 (in \$ million) |
| $a_2$ | 16 (in \$ million) |
| $a_3$ | 8 (in \$ million) |
| $a_4$ | 9 (in \$ million) |

Alternative $a_3$ minimizes the minimum cost.

As a corollary to the minimin principle, the *maximax principle* is appropriate when the decision maker is extremely optimistic and the matrix contains measures of profit or revenue. The maximum profit (or revenue) associated with each alternative is examined, and the alternative that *maximizes* the *maximum* profit (or revenue) is selected. The mathematical formulation is

$$\underset{i}{\text{Max}}[\underset{j}{\text{Max}}(R_{ij})] \tag{26b}$$

### *4.4.3. The Hurwicz Principle*

It may be argued that decision makers need not be either completely optimistic or pessimistic, in which case the *Hurwicz principle* permits selection of a position between the two extremes. When evaluating costs, $C_{ij}$ the *Hurwicz criterion* for alternative $a_i$ is given by

$$\text{Min}\ H(a_j) = \alpha[\underset{j}{\text{Min}}(C_{ij})] + (1 - \alpha)[\underset{j}{\text{Max}}(C_{ij})] \tag{27a}$$

where $\alpha$ is the "index of optimism" such that $0 \leq \alpha \leq 1$. Extreme pessimism is defined by $\alpha = 0$; extreme optimism is defined by $\alpha = 1$. The value of $\alpha$ used in any particular analysis is selected by the decision maker based on subjective judgment. The alternative that *minimizes* the quantity $H(a_i)$ is the alternative to select.

When evaluating profits or revenues, $R_{ij}$, the expression for the Hurwicz criterion is

$$\text{Max } H(a_i) = \alpha[\underset{j}{\text{Max}}(R_{ij})] + (1 - \alpha)[\underset{j}{\text{Min}}(R_{ij})] \tag{27b}$$

The values of $H(\alpha_j)$ are plotted in Figure 7 for the sample problem. We may determine, either graphically or algebraically, that $a_2$ will be chosen for $0 \leq \alpha \leq 0.125$, $a_4$ will be selected for $0.125 \leq \alpha \leq 0.75$, and $a_3$ is least costly for $0.75 \leq \alpha \leq 1.00$.

### *4.4.4. The Laplace Principle (Insufficient Reason)*

The *Laplace principle,* sometimes known as the *principle of insufficient reason,* assumes that the probabilities of future events occurring are equal. That is, in the absence of any information to the contrary, it is assumed that all future outcomes are equally likely to occur. The expected cost (or profit/revenue) of each alternative is then computed, and the alternative that yields the minimum expected cost (or maximum expected profit/revenue) is selected. The mathematical expression for this principle is

$$\underset{i}{\text{Min}} \left\{ \left(\frac{1}{k}\right) \sum_{j=1}^{k} C_{ij} \right\} \tag{28a}$$

when the figure of merit is expressed as a cost or as

$$\underset{i}{\text{Max}} \left\{ \left(\frac{1}{k}\right) \sum_{j=1}^{k} R_{ij} \right\} \tag{28b}$$

when the figure of merit is expressed as revenue or profit.

Returning to our example, the insufficient reason assumption yields $p_1 = p_2 = p_3 = p_4 = 0.25$. With these probabilities:

$$E[C(a_1)] = \$12{,}250{,}000 \qquad E[C(a_3)] = \$14{,}000{,}000$$

$$E[C(a_2)] = \$16{,}000{,}000 \qquad E[C(a_4)] = \$13{,}500{,}000$$

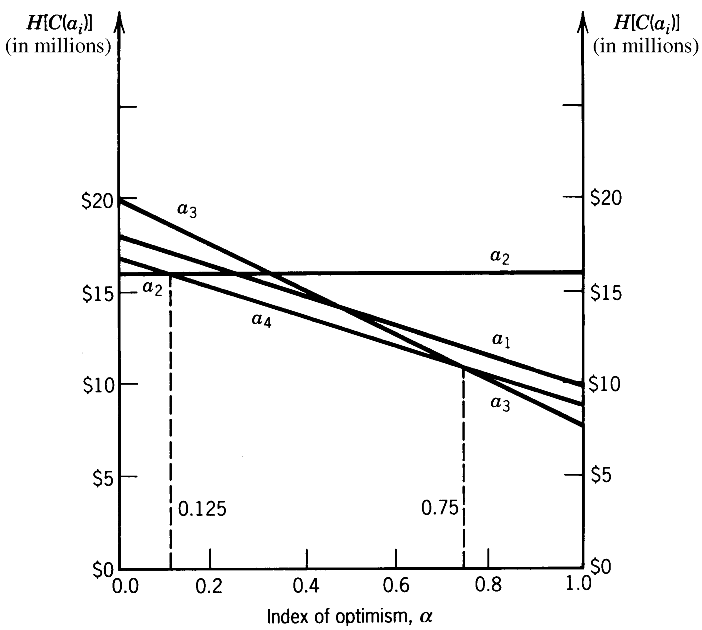


**Figure 7** Sample Problem—Hurwicz Criterion as Function of Index of Optimism.

Alternative $a_1$ should therefore be selected because it results in the minimum expected present worth (cost).

### *4.4.5. The Savage Principle (Minimax Regret)*

The *Savage principle,* or *principle of minimax regret,* is based on the assumption that the decision maker's primary interest is the *difference* between the actual outcome and the outcome that would have occurred had it been possible to accurately predict the future. Given these difference, or *regrets,* the decision maker then adopts a conservative position and selects the alternative that minimizes the maximum potential regret for each alternative.

A *regret matrix* is constructed, having for its cell values either

$$C_{ij} - [\underset{i}{\text{Min}}(C_{ij})] \tag{29a}$$

for cost data or

$$[\underset{i}{\text{Max}}(R_{ij})] - R_{ij} \tag{29b}$$

for revenue or profit data. In either case these cell values, or regrets, represent the differences between (a) the outcome if alternative $a_j$ is selected and state of nature $s_j$ subsequently occurs and (b) the outcome that would have been achieved had it been known in advance which state of nature would occur, so that the best alternative could have been selected. To illustrate, consider alternative $a_1$ and state of nature $s_1$: $C_{11}$ = \$18,000,000. However, if we had known *a priori* that state $s_1$ would in fact occur, we would have selected $a_4$, incurring a cost of only \$9,000,000. The difference (\$18,000,000 − \$9,000,000) is a measure of "regret" about selecting $a_1$ when we could have selected $a_4$ (had we known the state of nature in advance). The complete regret matrix for the example is given in Table 7.

The alternative that *minimizes* the *maximum* regret is preferred. That is, for cost data,

$$\underset{i}{\text{Min}}\ \underset{j}{\text{Max}}\{C_{ij} - [\text{Min}(C_{ij})]\} \tag{30a}$$

or, when the cell values are based on revenue or profit data,

$$\underset{i}{\text{Min}}\ \underset{j}{\text{Max}}\{[\underset{i}{\text{Max}}(R_{ij})] - R_{ij}\} \tag{30b}$$

Equation (30a) is applicable for the sample problem, yielding the following:

| Alternative | Maximum Regret |
|---|---|
| $a_1$ | 9 (in \$ millions) |
| $a_2$ | 8 (in \$ millions) |
| $a_3$ | 10 (in \$ millions) |
| $a_4$ | 9 (in \$ million) |

Thus, according to this principle of choice, $a_2$ should be preferred.

**TABLE 7 Regret Matrix for Sample Problem (Cell values are multiples of \$1,000,000)**

| | | Possible Outcomes | | | |
|---|---|---|---|---|---|
| | | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
| Alternatives | $a_1$ | 18 − 9 = 9 | 11 − 11 = 0 | 10 − 8 = 2 | 10 − 10 = 0 |
| | $a_2$ | 16 − 9 = 7 | 16 − 11 = 5 | 16 − 8 = 8 | 16 − 10 = 6 |
| | $a_3$ | 14 − 9 = 5 | 14 − 11 = 3 | 8 − 8 = 0 | 20 − 10 = 10 |
| | $a_4$ | 9 − 9 = 0 | 12 − 11 = 1 | 17 − 8 = 9 | 16 − 10 = 6 |

### 4.5. Summary of Results

There is no special reason why the principles of choice discussed in the preceding sections should yield the same solution. Indeed, each of the alternatives in this example problem were selected at least once.

Decision Under Risk

| Principle | Solution |
|---|---|
| Expectation | $a_4$ |
| Most probable future | $a_1$ |
| Aspiration level | $a_3$ |

Decisions Under Uncertainty

| Principle | Solution |
|---|---|
| Minimax | $a_2$ |
| Minimin | $a_3$ |
| Hurwicz ($0.1215 < \alpha < 0.75$) | $a_4$ |
| Laplace (insufficient reason) | $a_1$ |
| Savage (minimax regret) | $a_2$ |

Is one principle more "correct" than any other? There is no simple answer to this question—the choice of principle largely depends on the predisposition of the decision maker and the specific decision situation. Each principle has certain obvious advantages, and each is deficient in one or more desirable characteristics. Nevertheless, the principles in this section are useful because they shed some light on the subjective decision process and make the available information explicit to the decision process.

## 5. DECISION TREES

Decision tree methodology is useful for the evaluation of problems characterized by *sequential* decisions, each of which involves a variety of outcomes. The pictorial representation of this problem is suggestive of a tree lying on its side, with the branches in the tree representing successions of outcomes. The graphic portrayal of the problem structure is both its primary asset as well as its principal disadvantage. The ability to communicate complex dependencies is of great value, of course. However, the number of sequential decisions and outcomes (branches) is necessarily limited by the graphic medium (CRT screen, 8½ × 11 in. paper, etc.). These features will be apparent from the following discussion of deterministic and stochastic decision trees.

### 5.1. Deterministic Decision Trees

Consider a problem of retirement and replacement over a three-year planning horizon. The existing equipment, the defender, is now two years old. Replacement decisions are to be made now, one year hence, and two years hence. Whichever equipment is in service three years from now will be removed from service and sold in any event. Initial costs, salvage values, and operating costs for each of the 3 years are summarized in Table 8. Note that the first row of the table represents the defender: It has a current salvage value of \$50, \$40 after one year, \$30 after two years, and \$20 if sold at the end of the third year.

**TABLE 8 Input Data for Deterministic Example (All cash flows have been discounted—they are shown as their PW equivalents)**

| Year | Initial Cost if Purchased at | Salvage Value if Sold at End of Year | | | Operating Cost in Year | | |
|---|---|---|---|---|---|---|---|
| $j$ | Start of Year $j$ | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | \$50[a] | \$40 | \$ 30 | \$ 20 | \$90 | \$95 | \$100 |
| 1 | 100[b] | 80 | 65 | 55 | 50 | 60 | 70 |
| 2 | 120[b] | | 100 | 85 | | 45 | 55 |
| 3 | 130[b] | | | 110 | | | 40 |

[a] Current salvage value of defender.
[b] Challengers at start of years 1, 2, and 3.

The decision tree for this problem is shown in Figure 8. The three decision points are represented by squares, the branches are the decisions, and the economic consequence of each branch is shown at that line. The solution begins at the end of the tree, that is, at the latest decision point, which is decision 3 in this case.

| Decision Point | Alternative | Monetary Outcome, 3rd Year | Choice |
|---|---|---|---|
| 3*a* | Keep | −\$100 + \$20 = −\$80 | |
| | Replace | \$30 − \$130 − \$40 + \$110 = −\$30 | Replace |
| 3*b* | Keep | −\$55 + \$85 = \$30 | |
| | Replace | \$100 − \$130 − \$40 + \$110 = \$40 | Replace |
| 3*c* | Keep | −\$70 + \$55 = −\$15 | |
| | Replace | \$65 − \$130 − \$40 + \$110 = \$5 | Replace |
| 3*d* | Keep | −\$55 + \$85 = \$30 | |
| | Replace | \$100 − \$130 − \$40 + \$110 = \$40 | Replace |

Next, the procedure rolls back to the preceding decision point, the beginning of the second year. The monetary outcomes are cumulative, that is, the economic consequences in year 2 are added to those of the optimal decisions at the beginning of year 3.

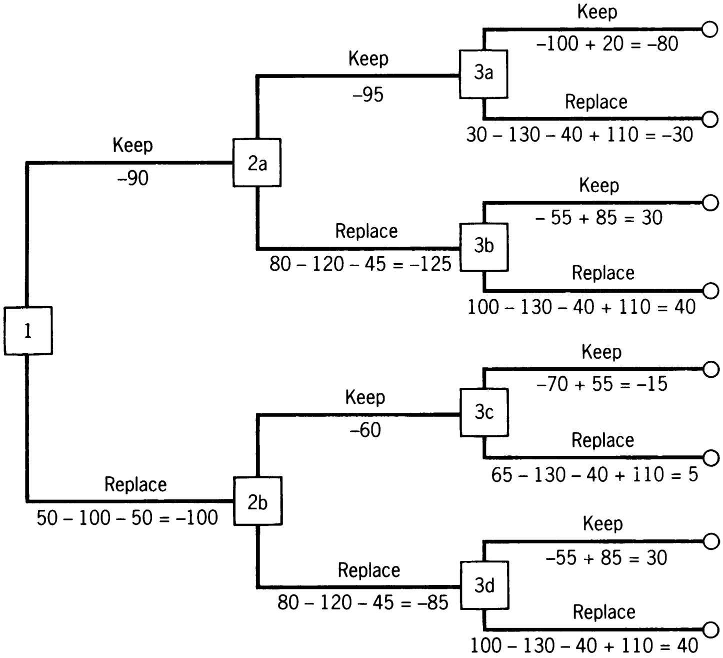


**Figure 8** Decision Tree for Deterministic Example.

| Decision Point | Alternative | Monetary Outcome, 2nd + 3rd Year | Choice |
|---|---|---|---|
| 2*a* | Keep | (−\$95) + (−\$30) = −\$125 | |
| | Replace | (\$40 − \$120 − \$45) + (\$40) = −\$85 | Replace |
| 2*b* | Keep | (−\$60) + (\$5) = −\$55 | |
| | Replace | (\$80 − \$120 − \$45) + (\$40) = −\$45 | Replace |

The process continues until the first decision point. Here:

| Decision Point | Alternative | Monetary Outcome, 1st, 2nd, 3rd Year | Choice |
|---|---|---|---|
| 1 | Keep Replace | (−\$90) + (−\$85) = −\$175 | |
| | | (\$50 − \$100 − \$50) + (−\$45) = −\$145 | Replace |

The optimal solution is now complete. The optimal path through the tree is determined by beginning at the first decision point and continuing through subsequent decisions in accordance with the indicated decisions. In this example the optimal solution is *replace, replace,* and *replace* at the start of years 1, 2, and 3, respectively.

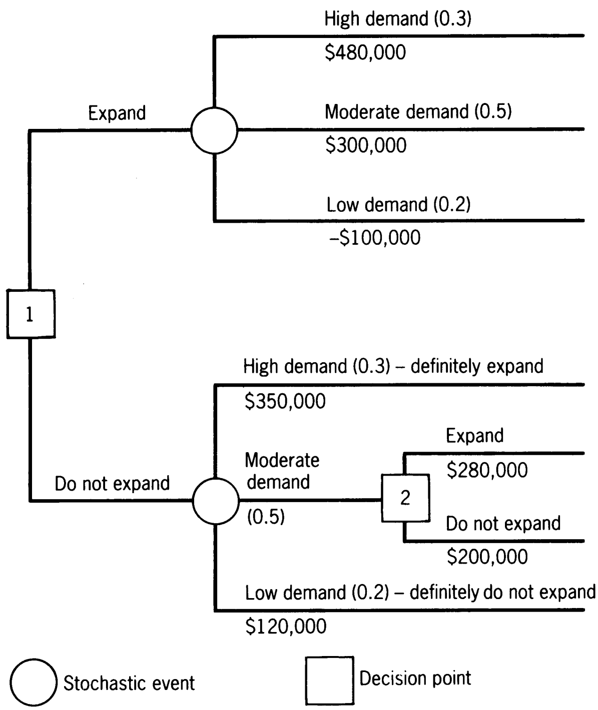


**Figure 9** Decision Tree for Stochastic Example. (All cash flows have been discounted. They are shown here in their PW equivalents.)

### 5.2. Stochastic Decision Trees

One of the most useful features of decision trees is their ability to illustrate variable outcomes when their probabilities of occurrence can be estimated. To be effective in a graphic format, the outcomes must be characterized as discrete random variables with a relatively small number of possibilities.

A very simple example of a stochastic decision tree is illustrated in Figure 9. The firm is considering expansion of a production line for the manufacture of a certain product. If the decision is made to forgo expansion at the present time, a subsequent opportunity will arise in about a year from now. The delay, perhaps, might be warranted based on the demand experienced over the next year. All dollar values shown in the tree are present value equivalents. There are three possible demand outcomes as shown—high, moderate, and low, with probabilities 0.3, 0.5, and 0.2, respectively.

At decision point 2 it is clear that plant expansion is preferred as PW(expand) = \$280,000 and PW(do not expand) = \$200,000. Rolling back to decision point 1, it is now necessary to evaluate the *expected* present worth through each of the branches emanating from that point.

$$\begin{aligned}\text{Exp[PW, expand now]} &= 0.3(\$480{,}000) + 0.5(\$300{,}000) + 0.2(-\$100{,}000)\\ &= \$274{,}000\\ \text{Exp[PW, do not expand]} &= 0.3(\$230{,}000) + 0.5(\$280{,}000) + 0.2(\$120{,}000)\\ &= \$233{,}000\end{aligned}$$

Thus, based on this analysis, it would appear that expansion at the present time is warranted.

The solution can be effected, of course, without reference to the decision tree. It is the graphic character of the tree that permits the analyst to articulate the sequential and stochastic nature of events and outcomes and to communicate these interrelationships to decision makers.

## 6. DIGITAL COMPUTER (MONTE CARLO) SIMULATION

The statistical procedures related to risk analysis suffer from at least one important drawback: The analytical techniques necessary to derive the mean, variance, and possibly the probability distribution of the figure of merit may be extremely difficult to implement. Indeed, the complexity of many real-world problems precludes the use of these computational techniques altogether; computations may be intractable, or the necessary underlying assumptions may not be met. Under these conditions analysts may find *digital computer* (*Monte Carlo*) *simulation* especially useful. (Strictly speaking, *Monte Carlo* simulation and *digital computer* simulation are not synonymous. Monte Carlo simulation is a sampling technique used in the digital computer simulation of systems behavior. However, in recent years, practitioners have tended to blur this semantic distinction, using the terms interchangeably.)

The objective of digital computer simulation is to generate a probability distribution for the figure of merit, generally present worth or rate of return, given the probability distributions for the various components of the analysis. The decision maker can thus compare expected returns as well as the variability of returns for two or more alternatives. Moreover, probability statements can be made, in this form: The probability is $x$ that project $y$ will result in a profit in excess of $z$.

### 6.1. Sampling from a Discrete Distribution

Suppose that ‘‘annual operating savings,’’ $A$, is a discrete random variable with probabilities as given in Table 9. The associated *cumulative distribution function* (CDF), also given in the table, represents the probability that the annual operating savings will be less than or equal to some given value.

Our problem now is one of sampling from this distribution, using either the probability function or its associated CDF in order to preserve precisely all the characteristics of the original distribution. We can do this by obtaining, say, 100 perfectly matched balls, numbered from 00 to 99. We want to label four of the balls ‘‘\$2400,’’ eight of the balls ‘‘\$2500,’’ and so on. The number of balls labeled with a particular amount is proportional to their relative probability in the original distribution:

| Ball Numbers | Number of Balls | Labels | Ball Numbers | Number of Balls | Label |
|---|---|---|---|---|---|
| 01–04 | 4 | \$2400 | 49–64 | 16 | \$2900 |
| 05–12 | 8 | 2500 | 65–80 | 16 | 3000 |
| 13–22 | 10 | 2600 | 81–90 | 10 | 3100 |
| 23–34 | 12 | 2700 | 91–98 | 8 | 3200 |
| 35–48 | 14 | 2800 | 99–00 | 2 | 3300 |

**TABLE 9 Probabilities and Cumulative Distribution Function for Sample Problem**

| Event $i$ | Annual Savings $\bar{A}_i$ | Probability of Occurrence[a] $P(\bar{A}_i)$ | Cumulative Distribution Function (CDF) $\underline{P}$(Ann. Savings $\leq \bar{A}_i$) |
|---|---|---|---|
| 1 | $2400 | 0.04 | 0.04 |
| 2 | 2500 | 0.08 | 0.12 |
| 3 | 2600 | 0.10 | 0.22 |
| 4 | 2700 | 0.12 | 0.34 |
| 5 | 2800 | 0.14 | 0.48 |
| 6 | 2900 | 0.16 | 0.64 |
| 7 | 3000 | 0.16 | 0.80 |
| 8 | 3100 | 0.10 | 0.90 |
| 9 | 3200 | 0.08 | 0.98 |
| 10 | 3300 | 0.02 | 1.00 |

[a] The probability function for a *discrete* random variable is sometimes known as a *probability mass function.* The equivalent function for a *continuous* random variable is a *probability density function* (pdf).

Now we can put all the balls into a large jar, shake it thoroughly so that the balls are completely mixed, and then draw out a single ball. Then, the result of this "random sample" is recorded and the ball is placed back in the jar to select our next sample. As we continue this process through a large number of samples, or trials, we can expect the resulting frequency distribution to approximate that of the original population.

Of course, in practical applications, the sampling process does not consist of drawing balls from a jar. There are a variety of more elegant procedures, generally based on successive iterations of a predetermined formula. An alternative approach that is useful when the number of samples to be drawn is relatively small is to reference a *table of random numbers*. Such tables have been developed and the results recorded in tabular format. (A table of three-digit random numbers appears in Table 10.) Inasmuch as the numbers in the table are randomly generated, users may enter the table at any point and proceed in any direction.

## 6.2. Sampling from a Normal Distribution

The *normal distribution* is frequently used to describe the probabilities of certain continuous random variables. The probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\left\{\exp\left[\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)\right]\right\} \tag{31}$$

where $\mu$ = mean and $\sigma$ = standard deviation of the distribution and $x$ is the particular value of the random variable. A particular normal distribution is fully described by the parameter $\mu$ and $\sigma$, where $\mu$ is a measure of central tendency and $\sigma$ is a measure of dispersion.

The *standard normal distribution* results from the special case wherein $\mu = 0$ and $\sigma = 1$. The area under the curve from $-\infty$ to $+\infty$ is exactly 1.0. If one can develop a table of random numbers for a uniform distribution over the interval 0–1, it is possible to map a set of equivalent values for the standard normal distribution, as in Figure 10. The value along the ordinate represents the probability that the random variable $X$ lies in the interval $-\infty$ to $x$. For any random number we can compute the equivalent value $x$. This latter value is called the *random normal deviate*.

Tables of random normal deviates exist that contain values from which one may generate a random sample for *any* normal distribution with known parameters $\mu$ and $\sigma$. (See Table 11.) The simulated event, then, is given by

$$\{\text{Simulated event (Sample)}\} = \mu + \{\text{Random normal deviate}\}\sigma \tag{32}$$

## 6.3. General Framework

The previous sections discussed the process whereby random samples are drawn from an underlying probability distribution. The general procedure can be described in four steps:

*Step 1:* Determine the probability distribution(s) for the significant factors, as illustrated in Figure 11.

**TABLE 10 Random Numbers**[a]

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 139 | 407 | 027 | 030 | 530 | 687 | 694 | 017 | 943 | 787 |
| 073 | 886 | 255 | 332 | 037 | 264 | 341 | 948 | 462 | 774 |
| 075 | 259 | 224 | 042 | 332 | 890 | 196 | 693 | 988 | 467 |
| 254 | 352 | 917 | 614 | 273 | 643 | 994 | 956 | 128 | 193 |
| 096 | 119 | 694 | 625 | 095 | 727 | 846 | 565 | 868 | 405 |
| 459 | 637 | 289 | 778 | 407 | 468 | 234 | 472 | 567 | 681 |
| 577 | 111 | 813 | 903 | 194 | 321 | 019 | 757 | 959 | 726 |
| 062 | 868 | 748 | 951 | 815 | 863 | 435 | 621 | 154 | 365 |
| 895 | 362 | 955 | 001 | 004 | 798 | 091 | 394 | 637 | 554 |
| 438 | 170 | 667 | 256 | 871 | 953 | 972 | 528 | 265 | 370 |
| 424 | 995 | 495 | 044 | 900 | 283 | 436 | 601 | 275 | 016 |
| 963 | 666 | 423 | 819 | 951 | 864 | 219 | 317 | 274 | 820 |
| 539 | 136 | 809 | 158 | 257 | 900 | 430 | 504 | 249 | 235 |
| 011 | 483 | 389 | 765 | 429 | 720 | 553 | 115 | 557 | 840 |
| 615 | 910 | 272 | 467 | 450 | 776 | 447 | 227 | 934 | 337 |
| 958 | 745 | 941 | 218 | 680 | 646 | 347 | 045 | 488 | 555 |
| 026 | 442 | 257 | 096 | 854 | 034 | 862 | 896 | 705 | 447 |
| 178 | 578 | 454 | 305 | 080 | 768 | 977 | 233 | 443 | 091 |
| 149 | 856 | 142 | 171 | 844 | 800 | 051 | 635 | 937 | 689 |
| 047 | 106 | 304 | 149 | 003 | 210 | 819 | 804 | 796 | 572 |
| 357 | 279 | 299 | 816 | 794 | 199 | 389 | 569 | 005 | 190 |
| 939 | 454 | 864 | 876 | 825 | 097 | 246 | 882 | 922 | 123 |
| 027 | 834 | 106 | 157 | 081 | 356 | 250 | 823 | 284 | 073 |
| 230 | 747 | 510 | 611 | 920 | 554 | 634 | 594 | 197 | 869 |
| 532 | 647 | 935 | 317 | 078 | 396 | 009 | 523 | 148 | 464 |
| 294 | 111 | 617 | 479 | 664 | 707 | 358 | 063 | 996 | 936 |
| 248 | 843 | 163 | 423 | 162 | 443 | 042 | 793 | 974 | 488 |
| 506 | 670 | 559 | 604 | 431 | 680 | 793 | 415 | 692 | 449 |
| 551 | 546 | 165 | 599 | 706 | 623 | 723 | 758 | 136 | 270 |
| 242 | 550 | 713 | 112 | 597 | 599 | 314 | 775 | 663 | 531 |
| 814 | 883 | 315 | 971 | 087 | 061 | 427 | 544 | 008 | 935 |
| 876 | 874 | 453 | 128 | 536 | 588 | 296 | 268 | 281 | 309 |
| 413 | 977 | 988 | 663 | 678 | 882 | 530 | 275 | 967 | 607 |
| 784 | 769 | 154 | 777 | 623 | 772 | 114 | 018 | 923 | 907 |
| 723 | 954 | 560 | 800 | 855 | 210 | 407 | 076 | 386 | 412 |
| 340 | 360 | 190 | 184 | 234 | 276 | 143 | 151 | 964 | 450 |
| 119 | 939 | 405 | 508 | 993 | 172 | 432 | 073 | 641 | 475 |
| 920 | 770 | 938 | 474 | 743 | 226 | 758 | 792 | 778 | 064 |
| 976 | 057 | 899 | 910 | 468 | 891 | 980 | 389 | 108 | 921 |
| 898 | 126 | 771 | 771 | 526 | 746 | 333 | 066 | 740 | 873 |
| 669 | 432 | 416 | 134 | 653 | 493 | 427 | 152 | 160 | 875 |
| 649 | 553 | 066 | 201 | 957 | 961 | 245 | 098 | 226 | 003 |
| 573 | 190 | 331 | 302 | 924 | 103 | 147 | 484 | 173 | 461 |
| 549 | 174 | 196 | 889 | 412 | 997 | 868 | 013 | 610 | 577 |
| 062 | 457 | 020 | 541 | 656 | 846 | 516 | 512 | 522 | 805 |

[a] This table was prepared by Mr. Ken Molay using a PDP 22 computer with a TOPS 10 operating system. To produce these random numbers, a congruential multiplicative generator was used, based on a seed of system time in milliseconds.

*Step 2:* Using Monte Carlo simulation, select random samples from these factors according to their relative probabilities of occurring in the future. (See Figure 12.) Note that the selection of one factor (price, for example) may determine the probability distribution of another factor (total amount demanded, for example).

*Step 3:* Determine the figure of merit (rate of return or present worth, for example) for each combination of factors. One trial consists of one calculation of the figure of merit.

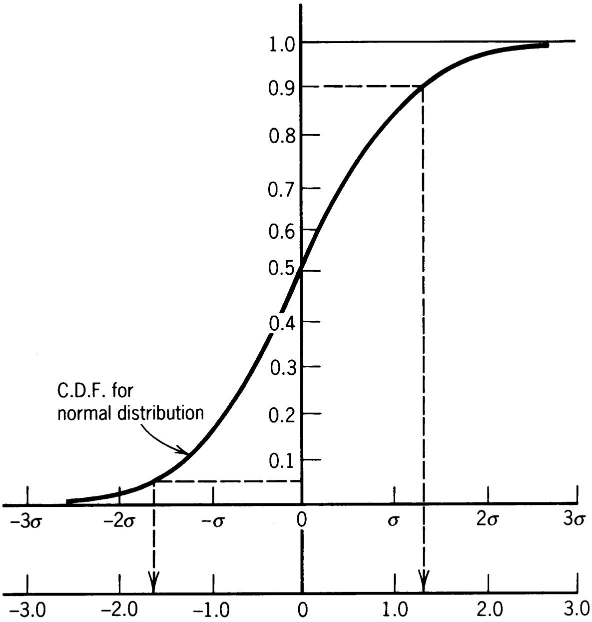


**Figure 10** Sampling from a Normal Distribution.

*Step 4:* Repeat the process, that is, conduct a series of trials, building a frequency histogram with the results, as in Figure 13. Continue until you are reasonably satisfied that the histogram yields a clear portrayal of the investment risk. (There is no universally accepted rule for determining the optimum number of trials. It is clearly less expensive to produce a smaller number of trials, yet a larger number of trials yields more information. Substantial literature is addressed to this interesting problem, but additional discussion is not warranted here.)

## 6.4. Numerical Example

A certain investment, if purchased, will result in annual operating savings described by the probability distribution shown in column (b) of Table 12. The project life is described by the probability distribution shown in column (e), and the initial cost is a normally distributed random variable with $\mu$ = \$25,000 and $\sigma$ = \$1000. The discount rate to be used in the analysis, a certainty estimate, is 0.10. It is assumed that the annual savings, project life, and initial cost are independent random variables. If there are no other relevant consequences of the proposed investment, the expected net present value is given by the equation

$$\mathrm{PW} = \bar{A}(\mathrm{P}/\bar{A}, 10\%, N) - P \qquad (33)$$

where $\bar{A}$ = annual operating savings, $N$ = project life, and $P$ = initial cost.

Columns (c) and (f) of Table 12 contain the random numbers corresponding to the relative probabilities in columns (b) and (e), respectively. Note that two-digit random numbers are used. Inasmuch as the specified accuracy of the probability distributions is two significant digits, the corresponding random numbers must be specified by *at least* two digits. (A three-digit random number, say 843, could be rounded to 84 or simply truncated after the first two digits.) For the first variable, annual operating savings ($\bar{A}$), there are 100 random numbers, the first four of which correspond to the event $\bar{A}$ = \$2400. The next eight numbers, 05 through 12, correspond to the event $\bar{A}$ = \$2500, and so on.

The results of 10 simulated trials are shown in Table 13. Consider the first trial, for example: a random number, 09, is drawn from the table of random numbers in Table 10. As shown in Table 12, this corresponds to the event $\bar{A}$ = \$2500. Next, a new random number, 52, is drawn, which corresponds to the event $N$ = 30. Note that *the same random number cannot be used for both random variables because they are independent.* The third random variable, $P$, is normally distributed, so a random normal deviate (RND) is drawn from Table 11. This number, 0.464, indicates a simulated value for

**TABLE 11 Random Normal Deviates**[a]

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0.199 | −0.066 | −0.205 | 0.455 | −2.023 | −0.131 | −0.032 | 1.050 |
| 1.344 | 0.421 | −0.599 | −0.575 | 0.231 | −0.455 | 1.977 | 2.029 |
| −0.362 | 1.112 | −0.200 | 0.072 | 1.044 | 1.399 | 0.910 | −1.630 |
| −0.451 | −0.413 | −0.159 | 1.421 | 0.286 | 0.499 | 1.402 | 0.750 |
| −1.477 | −0.149 | −1.234 | −0.644 | −1.753 | −0.895 | 1.393 | 0.853 |
| −0.392 | 0.977 | 0.603 | 0.851 | −1.161 | 0.206 | 0.294 | −0.270 |
| 1.341 | 0.009 | −1.489 | 0.499 | 0.695 | −1.284 | −0.542 | 0.682 |
| −0.993 | 1.078 | 0.194 | 0.231 | 0.615 | −1.436 | −0.019 | 0.928 |
| −0.708 | −0.134 | −0.308 | 1.797 | −0.354 | −0.445 | 0.019 | 1.355 |
| −0.336 | 2.044 | 0.199 | −0.401 | −0.929 | −1.964 | −0.746 | −0.229 |
| 0.307 | −0.998 | −1.083 | 0.104 | −1.385 | −1.224 | 0.428 | 0.607 |
| −1.361 | −0.203 | 0.675 | −0.761 | −0.092 | −1.309 | −0.966 | −0.335 |
| 0.467 | −0.256 | 0.788 | 0.72 | −0.349 | −1.401 | 0.205 | 1.043 |
| 0.373 | −1.472 | 0.334 | −0.361 | −2.519 | −0.658 | −0.249 | −1.017 |
| 1.517 | 0.615 | −1.414 | −0.665 | −0.701 | −0.105 | −0.78 | −0.266 |
| −1.659 | −0.902 | −0.883 | −1.679 | −0.197 | −1.329 | 0.596 | −0.419 |
| 1.078 | 0.274 | 0 | 0.926 | −1.557 | −0.610 | 1.554 | −0.139 |
| −0.388 | −1.048 | −1.135 | −0.878 | −1.705 | 0.275 | 0.535 | −0.488 |
| 0.008 | 0.184 | −0.208 | 0.236 | −0.134 | −0.705 | 0.202 | 0.354 |
| −2.998 | −0.165 | −0.295 | −0.282 | −0.709 | 1.024 | 0.029 | 1.179 |
| 0.051 | −1.229 | −1.265 | 0.440 | 0.593 | 0.276 | 1.053 | −0.125 |
| 0.536 | −0.367 | 2.430 | 0.312 | 0.431 | 0.987 | 0.335 | 0.505 |
| 1.761 | 0.349 | −1.039 | −0.814 | 0.299 | −0.057 | 0.970 | 1.705 |
| 0.365 | 0.250 | 1.426 | −1.042 | −0.822 | −1.065 | 0.708 | −0.144 |
| 0.921 | 0.190 | 0.385 | 1.674 | 0.483 | −0.863 | −0.743 | 2.513 |
| −1.308 | −0.892 | 1.333 | −0.127 | −0.590 | −1.590 | −0.470 | 0.159 |
| 0.647 | 0.879 | 0.094 | −0.464 | 0.093 | 0 | 0.614 | 0.393 |
| −0.603 | −0.333 | −0.373 | −0.523 | −0.058 | −1.294 | 0.321 | −1.855 |
| −0.214 | −0.699 | −0.292 | 0.928 | 0.363 | 0.035 | 0.645 | −1.243 |
| 1.223 | −0.868 | −0.397 | −0.047 | 0.870 | −0.613 | 0.174 | 1.602 |
| −0.649 | −0.244 | 0.008 | −0.611 | 0.958 | −0.940 | 2.080 | 0.964 |
| −2.215 | 1.712 | 0.941 | 0.537 | −1.221 | 0.263 | 0.893 | 1.171 |
| 0.630 | −0.602 | −0.401 | 0.922 | −0.734 | 1.992 | −0.310 | −1.030 |
| −0.516 | 0.539 | 1.148 | −0.373 | −0.805 | 1.855 | −0.115 | −0.773 |
| 0.764 | −1.190 | −0.150 | 0.396 | 1.620 | 0.575 | −0.049 | −0.279 |
| −0.519 | 0.772 | 0.817 | 1.003 | 0.306 | −1.761 | −0.841 | −1.099 |
| −0.144 | 1.254 | −0.661 | 0.890 | 0.645 | 1.618 | −1.800 | −0.297 |
| 0.469 | 0.514 | −0.304 | −0.166 | 1.145 | 1.018 | −0.080 | 0.030 |
| 1.871 | 0.048 | −0.075 | 0.105 | −0.617 | −1.945 | 1.378 | 0.782 |
| −2.306 | −1.901 | 1.636 | −0.725 | 0.264 | 0.169 | −0.337 | −0.208 |

[a] This table was prepared by Mr. Shay Bao Lai using an Apple II computer. The algorithm for producing these random normal deviates is from A. M. Law and W. D. Kelton, *Simulation Modeling and Analysis,* McGraw-Hill, New York, 1982, pp. 258–259.

$$P = \mu + (\text{RND})\sigma$$
$$= \$25{,}000 + (0.464)(\$1000) = \$25{,}464$$

The present worth for the first trial can now be computed:

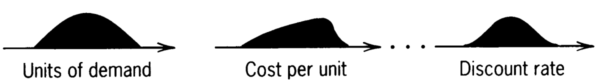


**Figure 11** Probability Functions for Inputs.

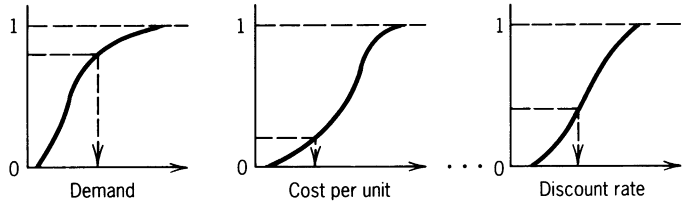


**Figure 12** Cumulative Distribution Functions.

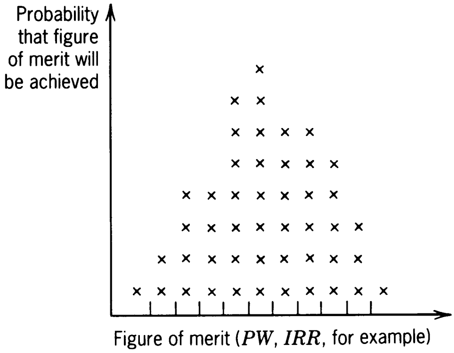


**Figure 13** Frequency Histogram for Figure of Merit.

$$\text{PW} = \$2500(P/\bar{A}, 10\%, 30) - \$25{,}464 = -\$737$$

A frequency distribution can be developed from the resulting PW values, and relevant statistics can be computed. In this example the cumulative average PW after 10 trials is \$2023; 40% of the trials result in a negative PW. The minimum value simulated was −\$737; the maximum value simulated was \$6761. Of course, if this were an actual application, the number of trials would be much larger, perhaps several thousand or more, and we would have considerably greater confidence in the resulting statistics.

In this particular example there are relatively few random variables, the relationships are not complex, and the variables are independent. Hence it is possible to compute the *theoretical expected value* for the net present value. That is

$$E[\text{PW}] = \text{E}[\bar{A}](P/\bar{A}, 10\%, E[N]) - E[P] \tag{34}$$

where

$$\begin{aligned} E[\bar{A} &= 0.04(\$2400) + 0.08(\$2500) + \ldots + 0.02(\$3300) \\ &= \$2848 \\ E[N] &= 0.05(25) + 0.10(26) + \ldots + 0.05(35) \\ &= 30 \\ E[P] &= \$25{,}000 \end{aligned}$$

Thus,

$$E[\text{PW}] = \$2848(P/A, 10\%, 30) - \$25{,}000 = \$3170$$

We can expect, therefore, that the cumulative average PW will approach \$3170 as the number of trials increases. Although we can compute the theoretical *mean,* it is not possible in this case to compute the theoretical *distribution* for the PW. However, an approximate distribution can be developed through simulation.

**TABLE 12 Example Simulation Problem: Input Data**

| Annual operating savings (a) | Probability (b) | Corresponding random numbers (c) | Project life (d) | Probability (e) | Corresponding random numbers (f) |
|---|---|---|---|---|---|
| $2400 | 0.04 | 01–04 | 25 | 0.05 | 01–05 |
| 2500 | 0.08 | 05–12 | 26 | 0.10 | 06–15 |
| 2600 | 0.10 | 13–22 | 27 | 0.10 | 16–25 |
| 2700 | 0.12 | 23–34 | 28 | 0.10 | 26–35 |
| 2800 | 0.14 | 35–48 | 29 | 0.10 | 36–45 |
| 2900 | 0.16 | 49–64 | 30 | 0.10 | 46–55 |
| 3000 | 0.16 | 65–80 | 31 | 0.10 | 56–65 |
| 3100 | 0.10 | 81–90 | 32 | 0.10 | 66–75 |
| 3200 | 0.08 | 91–98 | 33 | 0.10 | 76–85 |
| 3300 | 0.02 | 99–00 | 34 | 0.10 | 86–95 |
| | | | 35 | 0.05 | 96–00 |
| | 1.00 | | | 1.00 | |

**TABLE 13 Example Simulation Problem: Simulated Trials**

| Trial | Random Number | Operating Savings | Random Number | Project Life (years) | Random Number Deviate | Initial Cost | Present Worth | Cumulative Average PW |
|---|---|---|---|---|---|---|---|---|
| 1 | 09 | $2500 | 52 | 30 | 0.464 | $25,464 | −$737 | −$737 |
| 2 | 54 | 2900 | 80 | 33 | 0.137 | 25,137 | 3979 | 1621 |
| 3 | 42 | 2800 | 45 | 29 | 2.455 | 27,455 | 72 | 1105 |
| 4 | 01 | 2400 | 68 | 32 | −0.323 | 24,677 | −689 | 656 |
| 5 | 80 | 3000 | 59 | 31 | −0.068 | 24,932 | 4904 | 1506 |
| 6 | 06 | 2500 | 48 | 30 | 0.296 | 25,269 | −569 | 1160 |
| 7 | 06 | 2500 | 12 | 26 | −0.288 | 24,712 | −682 | 897 |
| 8 | 26 | 2700 | 35 | 28 | 0.060 | 24,940 | 1,426 | 963 |
| 9 | 57 | 2900 | 91 | 34 | −2.526 | 22,474 | 6761 | 1607 |
| 10 | 79 | 3000 | 89 | 34 | −0.531 | 24,469 | 5774 | 2023 |

## 7. OTHER APPROACHES FOR DEALING WITH THE UNCERTAIN/RISKY FUTURE

As indicated at the beginning of this chapter, risk and uncertainty are inherent in the general problem of resource allocation because all decisions depend on estimates about the noncertain future. Thus, risk and uncertainty have occupied the attention of a great many theoreticians and practitioners. A substantial number of approaches have been proposed, several of which were summarized earlier. Now four additional approaches are briefly identified. The first three are widely used in industry, despite certain important shortcomings; the fourth requires detailed discussion beyond the scope of this text.

### 7.1. Increasing the Minimum Attractive Rate of Return

Some analysts advocate adjusting the minimum attractive rate of return to compensate for risky investments, suggesting that, since the future is uncertain, stipulation of a minimum attractive rate of return of, say, $i + \Delta i$ will ensure that $i$ will be earned in the long run. Since some investments will not turn out as well as expected, they will be compensated for by the incremental ‘‘safety margin,’’ $\Delta i$. This approach, however, fails to come to grips with the risk or uncertainty associated with estimates for specific alternatives, and thus an element $\Delta i$ in the minimum attractive rate of return penalizes all alternatives equally.

### 7.2. Differentiating Rates of Return by Risk Class

Rather than building a safety margin into a single minimum attractive rate of return, some firms establish several risk classes with separate standards for each class. For example, a firm may require low-risk investments to yield at least 15% and medium-risk investments to yield at least 20%, and it may define a minimum attractive rate of return of 25% for high-risk proposals. The analyst then judges which class a specific proposal belongs in, and the relevant minimum attractive rate of return is used in the analysis. Although this approach is a step away from treating all alternatives equally, it is less than satisfactory in that it fails to focus attention on the uncertainty associated with the individual proposals. No two proposals have precisely the same degree of risk, and grouping alternatives by class obscures this point. Moreover, the attention of the decision maker should be directed to the causes of uncertainty, that is, to the individual estimates.

### 7.3. Decreasing the Expected Project Life

Still another procedure frequently employed to compensate for uncertainty is to decrease the expected project life. It is argued that estimates become less and less reliable as they occur further and further into the future; thus shortening project life is equivalent to ignoring those distant, unreliable estimates. Furthermore, distant consequences are more likely to be favorable than unfavorable; that is, distant estimated cash flows are generally positive (resulting from net revenues) and estimated cash flows near date zero are more likely to be negative (resulting from startup costs). Reducing expected project life, however, has the effect of penalizing the proposal by precluding possible future benefits, thereby allowing for risk in much the same way that increasing the minimum attractive rate of return penalizes marginally attractive proposals. Again, this procedure is to be criticized on the basis that it obscures uncertain estimates.

### 7.4. Utility Models

In essence, utility is a single metric on the unit interval denoting the degree of desirability of an item or a quantity of items with respect to a completely defined collection of such items. Thus an item or group of items with the greatest desirability would have a utility of, say, 100, and at a least desirable item, a zero utility. All items and groups within the collection range between these extremes in an ordered fashion. Amounts of monetary receipts and disbursements would provide a utility function from 0 to 100. A monetary gamble would be reviewed in this theory as a linear combination of the amount won and lost in the gamble, with the expected utility associated with winning. Once a person's utility function is derived, the theory of utility denotes how one should act in order to remain consistent with his or her denoted goals. Accordingly, utility theory is a description of normative economic behavior based on several stated axioms.

A number of advocates of this theory have therefore recommended that utility functions be established and economic risk analysis conducted with respect to this theory. That is, projects with the greatest expected utility should be selected by rational economic decision makers. There are many compelling features to this approach. However, it also involves the required development of the utility function, which is not a simple task; the question of whose utility function should represent the firm; and other perplexing problems. Also, it has been shown that current methods of risk cash flow analysis do represent a reasonable and rational approximation of the utility theory approach. There are also challenges to the axioms of existing theories of utility. Because of these and other detractions, the utility theory approach has not enjoyed popularity among many practitioners.

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## ADDITIONAL READING

There exists a very substantial literature relevant to the topics discussed in this chapter. An exhaustive compilation is neither feasible nor, in this context, particularly useful. The references included here have been selected because of their historical importance and/or their value as additional material to augment the rather limited discussion in this *Handbook.* See especially Buck (1989) for an exceptionally comprehensive bibliography.

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