CHAPTER 69
Statistical Process Control

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1. INTRODUCTION

Product-to-product and service-to-service variation is potentially observable in all organizations. For example, a group of size 13 running shoes vary from shoe to shoe in form, fit, and finish. Even though the variation may not be observable by the human eye, it is observable if a more accurate measurement system is employed. Consumers of the product declare the product or service to be of inferior quality. Product variation may become increasingly observable to the consumer in complex systems. For example, consumers readily agree that there exists variation of cars of the exact same model and installed accessories. Otherwise, consumers would not insist upon driving the vehicle before purchase and the producer would not have to provide a warranty period to satisfy the concern of purchasing a failure-prone vehicle (commonly called a ''lemon'').

Examples such as these provide motivation for professionals to consider the implications of product-to-product and service-to-service variation. Clearly, product variation is key to customer satisfaction and loyalty; therefore, successful producers realize this and take appropriate actions to control the variation or quality of the products or services provided.

To monitor the variation of a process or service, data are collected and analyzed for product critical performance metrics. Two types of data are common. Data may be measured on a continuous scale, for example, length, weight, and so on. Such measurements are called variable data. Alternatively, if process observations are of the classification type, they are called attribute data. Examples

of such data include the number of defects in an inspection unit, number of defective units in a sample, and so on.

Well-known and documented techniques exist to monitor product variation while it is within the producer's environment. Most of the techniques require the observations or data to be statistically independent. That is, the data for a specific performance measurement are assumed to have no relationship to prior or successive observations. It is assumed that no correlation exists between data collected prior to or following a specific observation. The techniques used to monitor such data are collectively called statistical process control (SPC). These techniques are utilized in consumer-oriented industries. Some of the more prominent or useful techniques are presented in this chapter. Specifically, seven tools for SPC are reviewed, and their applicability is examined. Furthermore, common and improved approaches for process capability analysis are presented.

2. TOOLSET FOR STATISTICAL PROCESS CONTROL

Ishikawa (1985) presents seven common tools for SPC. He claims that 95% of quality-related problems can be resolved with these tools. The tools have been called the ''Magnificent Seven'' and are as follows:

1. Histogram
2. Check sheet
3. Pareto chart
4. Cause-and-effect diagram
5. Defect concentration diagram
6. Scatter diagram
7. Control chart

In the following subsections, brief overviews and practical examples are presented with the exception of the control charts, which are the topics of detailed discussions in Sections 3, 4, and 5.

2.1. Histogram

The histogram is a graphical tool that presents the relative frequency (in number or percentage) of observations within predefined cells. In other words, the histogram provides a graphical representation of a population or process under examination. The natural spread or distribution, the central tendency, and the variation of the process are readily observable in the histogram. Since the data supporting the histogram are defined by individual observations, the comparison of the process variation to the allowable spread, as defined by specifications, is easily demonstrated by plotting the specifications on the histogram. The histogram is a simple tool that provides a wealth of information regarding the natural tendencies of a product or process under study. Furthermore, from a probability perspective, the histogram provides an empirical estimate of the probability density function defining the product or process.

Example 1 presents an example of the use of the histogram.

Example 1. The following data have been collected from your process. In particular, you produce a child's toy. You have collected 90 values of a critical performance measurement that reflects surface quality for the manufactured product. The upper specification limit (USL) and the lower specification limit (LSL) are known to be 54 and 47, respectively. It is desired to demonstrate the natural tendency, variability, and relationship to the specification limits of the process by constructing a histogram of the data. With the collected data, the histogram is calculated by accumulating the number of observations within prescribed bands. The number of observations can be used to determine the relative frequency of each cell as demonstrated below.

Data for Example 1
The resulting histogram may be plotted with most spreadsheet software systems.
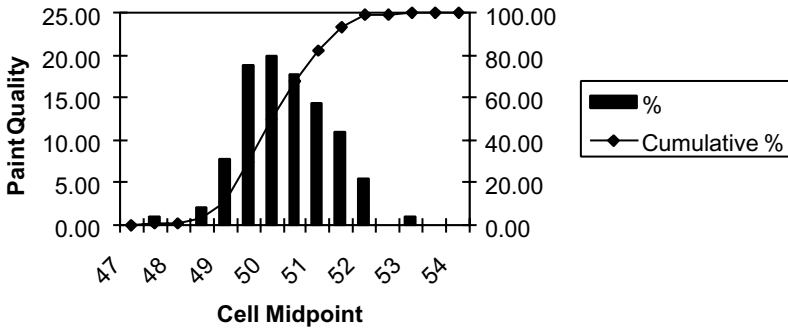For this example, the resulting histogram is as follows:

Figure 1   Histogram.

Cell Mid Point	# Observations	%	Cumulative %
47	0	0.00	0.00
47.5	1	1.11	1.11
48	0	0.00	1.11
48.5	2	2.22	3.33
49	7	7.78	11.11
49.5	17	18.89	30.00
50	18	20.00	50.00
50.5	16	17.78	67.78
51	13	14.44	82.22
51.5	10	11.11	93.33
52	5	5.56	98.89
52.5	0	0.00	98.89
53	1	1.11	100.00
53.5	0	0.00	100.00
54	0	0.00	100.00
Total =	90		

As observed, the paint quality measurements are contained within the specifications and are relatively symmetric.

2.2.   Check Sheet

The check sheet is a simple and useful tool that is often used in the early stages of a quality control program. It provides a uniform and consistent means for data collection and analysis. Like many of the traditional tools, the check sheet is defect oriented and is used to classify the types of defects frequently found for a product or service. It is used to spot problems areas by identifying defect types that frequently occur. As a result, a check sheet is often used as the input to the next tool, called the Pareto chart. The check sheet is most often tabular in format as presented in Example 2.

Example 2. For the same toy of Example 1, defect data have been collected on a weekly basis to attempt to classify the underlying sources of product defects. In particular, several defect types have been established, and the operators are given a matrix simply to track the number of defects of each category for a multiday study period. The resulting check sheet is presented as follows:

Toy 1 Check Sheet: Defect Count—Finished Surface									
Type	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7	Total	%	Cumulative %
Paint drips	11	9	10	11	10	12	17	80	42.3	42.3
Scratches	8	7	7	6	6	9	8	51	27.0	69.3
Handling marks	3	5	4	7	4	5	5	33	17.5	86.8
Chips	2	1	0	0	3	2	2	10	5.3	92.1
Poor color match	1	2	3	0	2	0	0	8	4.2	96.3
Poor assembly	2	1	0	0	1	0	0	4	2.1	98.4
Failure of functional test	0	1	0	0	1	0	0	2	1.1	99.5
Other	0	0	0	0	1	0	0	1	0.5	100.0

Total 189

As demonstrated above, the percent breakdown and accumulation of percentages are readily determined for each defect type. If the percent values are ranked from largest to smallest, a Pareto chart (discussed in the next section) is effectively computed, which makes the Pareto chart an obvious companion to the check sheet.

2.3. Pareto Chart

The Pareto chart was named after the Italian economist Vilfredo Pareto (1848–1923). Pareto observed that in certain cultures or economies a majority of the wealth was held by a small segment of the population.

In the context of quality or the observance of defects, time has proved over and over that there are usually a disproportionately small number of defect types that cause a majority of the problems. As a general rule of thumb, it is commonly observed that around 20% of the defect types can cause around 80% of the problems. Pareto analysis and the resulting Pareto chart are used to identify what is commonly known as the "vital few and the trivial many." Most quality professionals agree that the Pareto chart is one of the best tools, if not the best tool, within the Magnificent Seven. Montgomery (1996) states that "In general, the Pareto chart is one of the most useful of the 'magnificent seven'. Its applications to quality improvement are limited only by the ingenuity of the analyst."

In the context of the check sheet, the resulting Pareto chart is simply the relative frequency of the defect types as observed during a study period. Pareto charts are effectively the plot of the relative frequency or cumulative frequency, of defect types.

Example 3 provides an enlightening example in the use of the Pareto chart.

Example 3. From the toy data of Example 2, plot the associated Pareto chart. From Example 2, you see that the relative frequencies and cumulative frequencies have been calculated. This is, in fact, the Pareto analysis. The resulting Pareto chart is as follows:

Over 70% of the defect types are caused by two of the eight defect types. The resulting Pareto chart can be used to identify priorities for defect-reduction programs. Assuming that paint drops and scratches are of critical concern to the customer and are yielding excessive costs, paint drops and scratches should be the focus of the defect-reduction program.

2.4. Cause-and-Effect Diagram

Once a defect-reduction program has been defined and the focus of the study has been identified, it is necessary to determine the cause of each of the defect types. The cause-and-effect diagram has proven to be an invaluable team-oriented approach to brainstorm the cause of a specific product defect, or more generally a consumer concern. The cause-and-effect diagram is a graphical tool that is most often constructed in a team setting to identify the contributing factors and their relationship to a defect type. The cause-and-effect diagram is used to identify areas of future data collection and analysis to identify process-improvement strategies. The chart has also been called the fishbone diagram or the Ishikawa diagram (Ishikawa 1985).
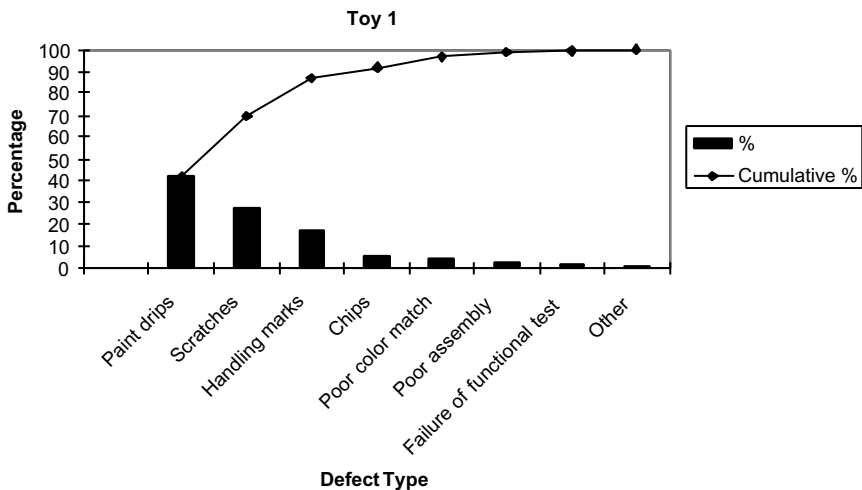

Figure 2   Pareto Chart.

The cause-and-effect diagram basically dissects a defect type into the associated causes. The causes can be further subdivided into additional groupings if the team believes such structure is necessary. Japanese-based tools such as the cause-and-effect diagram are useful for creative problem solving, and teams using such tools should be allowed flexibility in customizing tools for their specific needs. Example 4 presents the use of a cause-and-effect diagram.

Example 4. Within your facility, part accuracy is a consistent problem. You have established a cross-functional team to investigate the causes of this problem in an effort to identify opportunities of process improvement. The resulting cause-and-effect diagram is as follows:

2.5. Defect Concentration Diagram

Of the seven tools of the Magnificent Seven, the defect concentration diagram is most likely the simplest, and one of the most effective, for process-improvement activities. The defect concentration diagram is a schematic diagram of the part being produced. It provides all relevant views, and all defect types are recorded in accordance with a preset key. The operator simply records each observed defect on the defect concentration diagram. This diagram is often used in conjunction with the cause and effect diagram in an effort to identify process opportunities for improvement. The diagram is used to determine the specific location of defect types, and as a result, provides insightful information on process problems and the source of product defects.

Example 5 presents an example of a defect diagram.

Example 5. You produce a metal painted part that serves as the shell of the toy of Example 1. It is effectively a triangular column. The resulting defect concentration diagram for the operators is presented in Figure 4:

The defect concentration diagram is placed at a strategic point in the process, and the operator is instructed to document all observed defects according to location and type. For example, if five paint drips are observed on the top right corner of side 1, five A's would be placed on the top right corner of side 1 of the defect concentration diagram. The collective view of the completed defect concentration diagram provides conclusive evidence supporting the investigation of process-improvement activities.

2.6. Scatter Diagram

The sixth tool of the Magnificent Seven is the scatter diagram. For readers familiar with simple linear regression, the scatter diagram is considered the first step in determining the functional relationship between two variables. The scatter diagram is effectively the plotting of a process or product performance metric against some controllable variable. Most spreadsheet software systems provide the graphical tools to minimize the technical details of constructing such diagrams. Example 6 provides an illustration of the use of the scatter diagram.

Example 6. At the Department of Industrial Engineering at the University of Arkansas, miniature catapults are used to demonstrate the usefulness of statistical modeling. The teaching tool is effec-

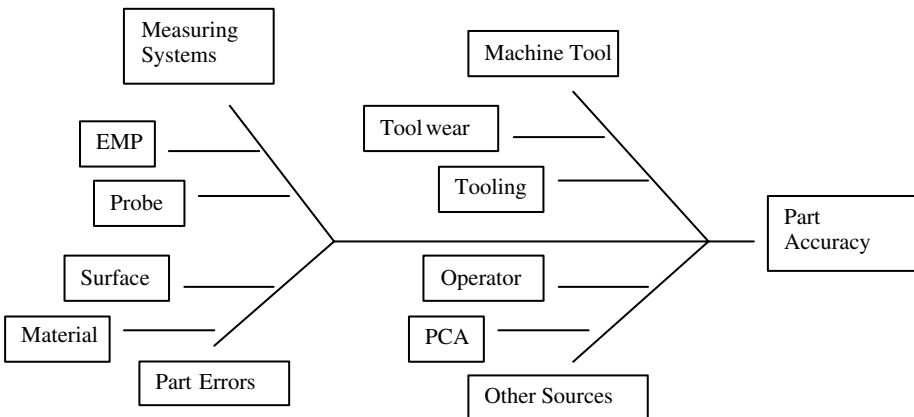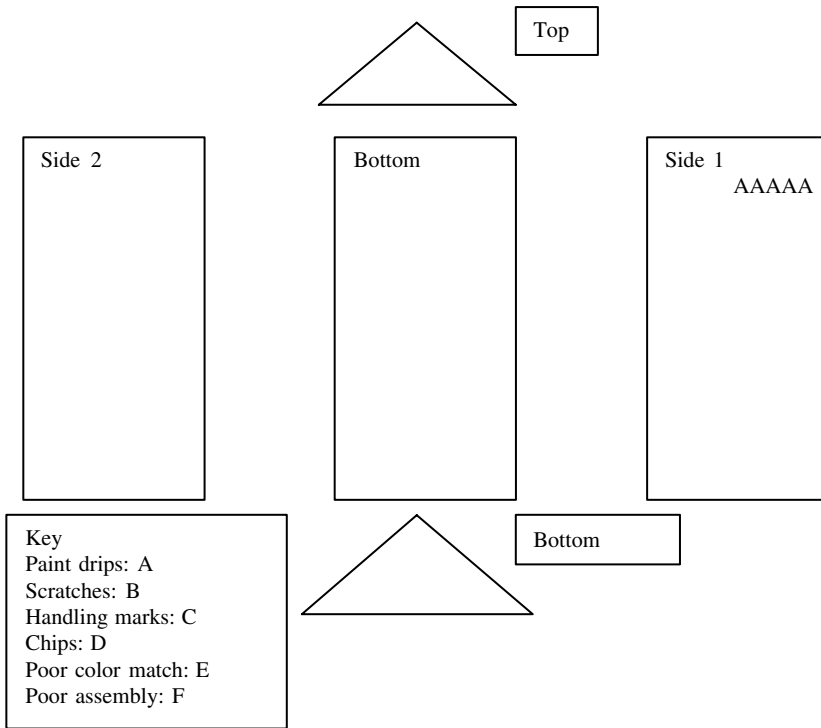**Figure 3**   Cause-and-Effect Diagram.

Figure 4 Defect Concentration Diagram.

tively a miniaturization of the classic weapon of ages past. As the arm of the catapult is pulled back further—the angle is called the throwing angle—a given projectile is launched to realize increasing distances. The first step in modeling this process is to demonstrate graphically the relationship between the throwing angle and the resulting throwing distances. In particular, the projectile is launched five times from six different throwing angles. The resulting scatter diagram is given in Figure 5.

There appears to be a fairly linear relationship between the throwing angle and the distance thrown.

3. OVERVIEW OF CONTROL CHARTS

As previously described, almost every product, process, or service varies. The control chart is used to monitor natural process variation in an effort to expose times that the process has shifted such that nonnatural variation is realized. Within the SPC domain, terms have been coined to define various types of variation. The natural variation in the process that cannot be avoided is commonly called common cause variation. It is assumed that such variation is not controllable. Alternatively, undesirable changes occur, causing a process to change in its natural behavior. The variation of the process may become excessively large in view of its common causes, or the central tendency of the data may shift in a positive or negative direction. Such unnatural variations in the data are known as special causes.

In view of these types of variation, it is desired to design SPC approaches such that the user maximizes opportunities for detecting special causes in the midst of the uncontrollable chance causes. In the application of SPC, samples or subgroups are collected and analyzed and are used for this judgment. The logical collection of the subgroups or samples such that opportunities for detecting special causes are maximized is commonly known as rational subgrouping. Most introductory texts in quality control have detailed discussion of this topic (i.e., Grant and Leavenworth 1980).

The tool that is used to monitor process variation over time is known as the control chart. Control charts originate from the work of Walter Shewhart (1927) and are often referred to as Shewhart control charts. Effectively, process observations based upon collected samples or subgroups, at fixed points in time, are plotted in accordance to time. As long as the current observation is within fixed

Class Data - SLR
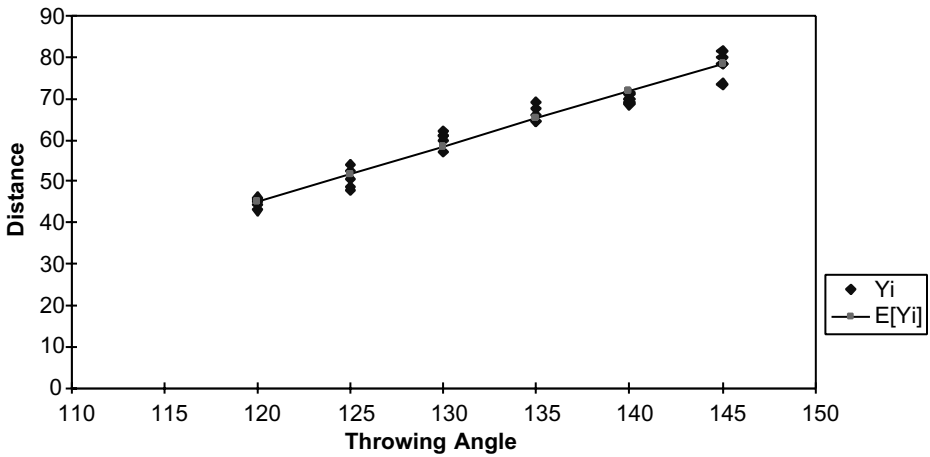
Figure 5 Scatter Diagram.

intervals, called control limits, or no unnatural trends in the data are observed (defined in Section 3.2), it is concluded that the process is operating in the presence of common causes only. Such a process is declared to be operating in a state of statistical control. Alternatively, if the process yields observations that exceed the control limits, or unnatural variation exists, it is concluded that special causes exist in addition to the common causes. Commonly, this process state is known as being out of control.

Figure 6 presents this graphical tool—the control chart. As discussed, there exists a line defining the central tendency as well as the control limits. This process is obviously operating under the presence of both special causes and common causes and should be judged out of control.

The basis of the centerline and the control limits are essentially the same for all charts with the exception of the type of data. Assume that the quality characteristic being monitored is Y_t, for which the mean is known to be $E[Y_t]$, and the variance is known to be $V[Y_t]$. The general centerline (CL) and resulting control limits are:

Figure 6 Example Control Chart.

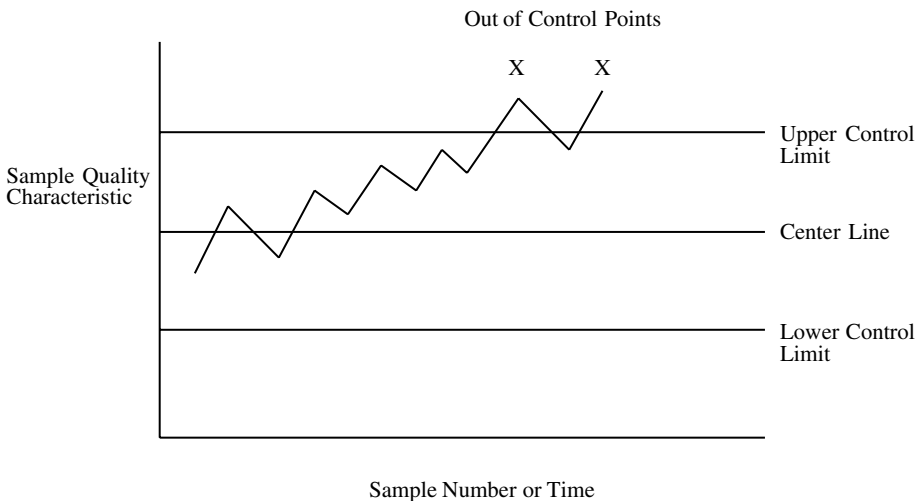$$\text{Upper control limit (LCL)} = E[Y_t] + k\sqrt{V[Y_t]}$$

$$\text{Centerline (CL)} = E[Y_t]$$

$$\text{Lower control limit (UCL)} = E[Y_t] - k\sqrt{V[Y_t]}$$

where k is defined by the user. In most applications within the United States, k is assumed to be 3 regardless of the type of data.

In Section 4, two control charts for variable data are presented. Both the $\overline{X}$ control chart for monitoring the process mean and the R control chart for monitoring the process variation are presented. In Section 5, two control charts for attribute data are discussed: the p control chart for monitoring percent nonconforming and the c chart for monitoring the number of defects in a sample. Furthermore, brief discussions of data patterns on control charts and recommended supplemental rules for judging nonrandom trends on a control chart are presented.

3.1. Data Patterns on Control Charts

Control charts are powerful tools for monitoring the variation of a process. Furthermore, the non-natural trends on a control chart can provide significant diagnostic information regarding the cause of a process disturbance. In this section, prevalent trends from out-of-control processes are presented and the suspect causes are briefly discussed. References such as AT&T (1985) and Montgomery (1996) provide more in-depth presentations of this discussion.

First, it is the assumption of most control chart applications that the process from which observations are collected is stable. That is, the statistical behavior of the process is time invariant in that the underlying distribution is fixed, yielding a fixed mean and variance. In some applications, process observations are collected such that multiple processes may feed the data-collection station. As a result, the observations may have a tendency to cluster around the control limits and be sparsely observed around the CL or the mean. This observation is commonly referred to as a mixture.

Secondly, processes may be overly sensitive to factors within the day; for example, temperature cycles. Such trends in the data are known as cycles and are not well suited for traditional applications of control charts, even though the cycling behavior is a natural part of the process variation. When faced with cycling or autocorrelation in the data, it is suggested the advanced techniques, such as the moving centerline exponentially weighted moving average chart, be considered, as presented in Montgomery and Mastrangelo (1991).

Third, a trend (see Figure 6) in the process is typically realized on the control chart as a generally increasing or decreasing trend in the data. Such trends in the data can be caused by process degradation, such as tool or machine wear. In some cases, such trends cannot be tolerated and can be adequately detected with standard control charts. In situations where such trends are a natural part of the process (i.e., tool wear), alternative approaches should be considered. The simplest approach is to fit a reasonable regression model to the process and monitor the process accordingly. Alternatively, Quesenberry (1988) develops an alternative SPC approach for a tool-wear process.

If a shift has occurred in the central tendency of the process, the process data will cluster around the new mean. Such shifts are often caused by changes in raw materials or process settings. In most all cases, such process behavior is not desirable and can be detected quickly with traditional control charts.

3.2. AT&T Runs Rules

Clearly, the term *nonrandom* trend must be consistently defined for the shop floor operators. Such definitions are made by the consistent adoption of rules for judging the current status of a process. Probably the most common set of rules for examining the state of the process is known as the AT&T runs rules (AT&T 1985). In AT&T (1985), these rules are called ''Tests for Instability.'' In applying these tests, each half of the control chart (the area between the CL and UCL and the CL and LCL) is divided into three equally spaced zones, as shown in Figure 7.

In accordance with the AT&T handbook, a process is judged out of control or unstable if any one of the following rules apply:

Test 1: A single point falls outside of the control limits (beyond zone A).
Test 2: Two out of three successive points fall in zone A or beyond for a given side.
Test 3: Four out of five successive points fall in zone B or beyond for a given side.
Test 4: Eight successive points fall in zone C or beyond for a given side.

If these rules are consistently employed throughout a plant, type I errors should be adequately controlled while ensuring reasonable detection of undesirable process shifts. A type I error, in the context of control charts, is the event that the operator falsely concludes that the process is out of control.
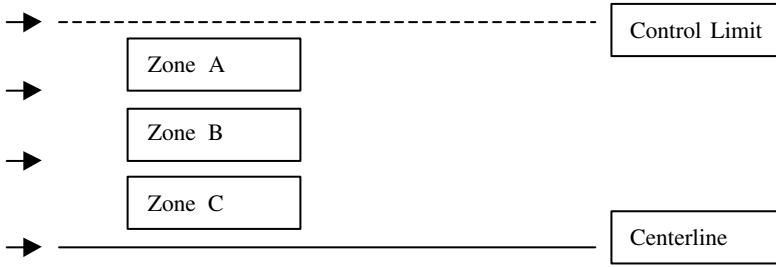
Figure 7 AT&T Runs Rules Zones.

4. CONTROL CHARTS FOR VARIABLES

4.1. $\overline{X}$ and R Control Charts

The most common means of monitoring variable data is with $\overline{X}$ the R and control chart combination. It is assumed that the observations collected from the process are independent and normally distributed. The $\overline{X}$ chart is utilized to monitor the process mean, and the R chart is used to monitor the process variation. Without exception, the $\overline{X}$ and R charts should always be used together. The normality assumption embeds the fact that two parameters, the mean and variance, completely characterize the process; therefore, both control charts are necessary to monitor the process completely.

From an applications perspective, n observations, $X_{i1}, X_{i2}, \ldots X_{in}$ are collected from the process at fixed points in time, i. These n observations from sampling interval i are commonly referred to as subgroup i. As a rule of thumb, the number of observations in each subgroup, n, should be between 4 and 6. Furthermore, for slight departures from normality, such sample averages, $\overline{X}$, can be assumed to be approximately normal, due to the central limit theorem (in as much as the underlying distribution is not excessively skewed). Estimates for the process mean and variance at a point in time i are based upon the sample average and range. That is,

$$\text{Sample average}_i = \overline{X}_i = \sum_{i=1}^{n} x_{ij}/n$$

$$\text{Range}_i = R_i = \text{maximum}(X_{i1}, X_{i2}, \ldots, X_{in}) - \text{minimum}(X_{i1}, X_{i2}, \ldots X_{in})$$

4.1.1. Control Limits: Standards Known

If the mean, μ, and the variance, σ^2, are known for the underlying process, the upper and lower control limits for these charts ($\overline{X}$ and R) can be determined as follows:

For the $\overline{X}$ chart:

$$\text{UCL}_{\overline{X}} = \frac{\mu + 3\sigma}{\sqrt{n}}$$

$$\text{CL}_{\overline{X}} = \mu$$

$$\text{LCL}_{\overline{X}} = \frac{\mu - 3\sigma}{\sqrt{n}}$$

For the R chart:

$$\text{UCL}_R = D_2\sigma$$

$$\text{CL}_R = d_2\sigma$$

$$\text{LCL}_R = D_1\sigma$$

where μ = the process mean
σ = the process standard deviation
$d_2, D_2,$ and D_1 = tabled values for a given n (subgroup size) and are included in the Appendix.

The AT&T runs rules should always be employed for the $\overline{X}$ chart and should be used for the R chart when the sample size is five or more (AT&T 1985). The first rule of the AT&T runs rules should always be used for both charts.

To illustrate the use of the $\overline{X}$ and R charts, the Example 7 is presented:

Example 7. Five observations have been collected from a process where the mean, μ, and the standard deviation, σ, are known to be 50 and 1, respectively. Construct and plot the necessary control limits and calculated observations for both the $\overline{X}$ and R charts. The data are collected and the resulting $\overline{X}_i$'s and R_i's are computed as follows:

i	X_{i1}	X_{i2}	X_{i3}	X_{i4}	X_{i5}	$\overline{X}_i$	R_i
1	51.69	49.59	52.51	51.07	50.01	50.97	2.92
2	49.37	49.13	49.77	49.61	49.89	49.55	0.77
3	50.59	50.08	50.4	47.82	48.67	49.51	2.77
4	51.24	49.79	50	49.89	51.98	50.58	2.19
5	50.78	48.82	50.59	51.48	49.12	50.16	2.66
6	49.86	48.43	49.1	48.76	49.45	49.12	1.43
7	50.86	50.78	50.43	52.12	49.7	50.78	2.42
8	50.94	51.61	50.2	49.08	51.63	50.70	2.55
9	49.25	49.46	51.71	49.08	51.55	50.21	2.63
10	50.52	50.92	49.06	50.43	51.46	50.48	2.40
11	49.45	49.02	51.23	49.96	49.9	49.91	2.20
12	50.12	51.06	50.99	50.98	50.09	50.65	0.97
13	49.32	49.93	51	50.68	48.14	49.81	2.86
14	50.04	50.1	49.08	49.14	50.3	49.73	1.22
15	50.25	49.78	50.51	49.37	48.21	49.62	2.30
16	50.07	49.69	48.81	50.72	50.25	49.91	1.91
17	49	50.25	49.93	48.87	49.32	49.47	1.38
18	49.64	51.22	49.84	50.51	50.68	50.38	1.58
19	51.5	49.49	48.24	48.02	50.86	49.62	3.48
20	51	49.58	49.57	50.08	51.75	50.39	2.19
21	49.66	49.41	47.43	49.68	49.39	49.11	2.25
22	48.64	48.53	50.03	49.85	48.77	49.16	1.50
23	50.35	49.34	51.2	50.82	49.72	50.29	1.87
24	51.49	51.44	50.78	48.91	49.86	50.50	2.57
25	49.81	49.45	49.43	51.46	49.76	49.98	2.03
26	49.82	49.99	51.3	49.4	50.28	50.15	1.90
27	48.59	49.31	49.95	50.63	48.78	49.45	2.04
28	51.67	51.13	49.33	50.93	48.61	50.33	3.05
29	50	50.85	49.53	50.7	49.3	50.08	1.55
30	50.2	48.74	50.42	51.27	48.57	49.84	2.71

The resulting control limits are as follows:
For the $\overline{X}$ chart:

$$\text{UCL} = 50 + \frac{3(1)}{\sqrt{5}} = 51.342$$

$$\text{CL} = 50$$

$$\text{LCL} = 50 - \frac{3(1)}{\sqrt{5}} = 48.658$$

For the R chart:

$$\text{UCL}_R = 4.918(1) = 4.918$$

$$\text{CL}_R = 2.326(1) = 2.326$$

$$\text{LCL}_R = 0(1) = 0$$

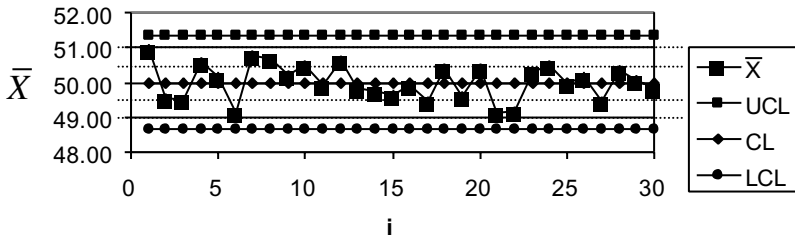The resulting control charts are given in Figures 8 and 9.

Figure 8 $\overline{X}$ Chart: Standards Known.

From the control charts above, the process exhibits stability using the AT&T runs rules on the $\overline{X}$ chart (see zones marked on Figure 8) and for rule number 1 for the R chart; therefore, the process is judged to be in a state of statistical control.

In most all cases, the process mean, μ, and the process standard deviation, σ, must be estimated from process data and the associated calculation of control limits must be adjusted. In the next section, the calculation of the control limits when the standards are not known is presented.

4.1.2. Control Limits: Standards Not Known

As stated previously, the true values of μ and σ are typically estimated with sampled data from a process. For the quality of estimation, it is typically assumed that at a minimum, 25 to 30 subgroups should be available before estimates of μ and σ are made ($\hat{\mu}$ and $\hat{\sigma}$). The following relationships are used to determine the necessary estimates:

$$\hat{\mu} = \overline{\overline{X}} = \sum_{i=1}^{m} \overline{X}_i / m$$

$$\hat{\sigma} = \overline{R} = \left[\sum_{i=1}^{m} R_i / m \right] \Big/ d_2$$

where m is the number of subgroups
d_2 is a table look-up value from the Appendix

The control limits when standards are unknown are calculated as follows:
For the $\overline{X}$ chart:

$$\text{UCL}_{\overline{x}} = \frac{\hat{\mu} + 3\hat{\sigma}}{\sqrt{n}}$$

$$\text{CL}_{\overline{x}} = \hat{\mu}$$

$$\text{LCL}_{\overline{x}} = \frac{\hat{\mu} - 3\hat{\sigma}}{\sqrt{n}}$$
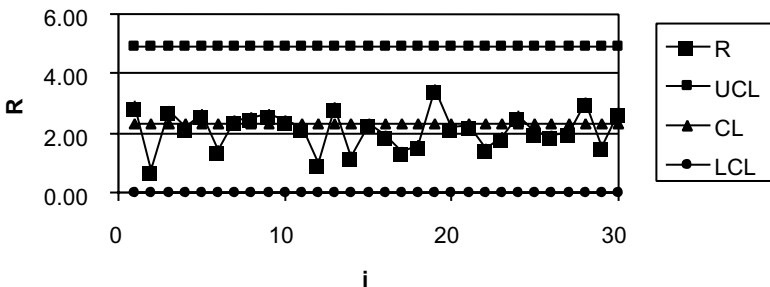

Figure 9 R Chart: Standards Known.

If A_2 is set equal to $3/d_2\sqrt{n}$ and $\bar{\bar{X}}$ and $\bar{R}/d_2$ are substituted for $\hat{\mu}$ and $\hat{\sigma}$, respectively, the resulting control limits can be determined as follows:

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_2\bar{R}$$

$$CL_{\bar{X}} = \bar{\bar{X}}$$

$$LCL_{\bar{X}} = \bar{\bar{X}} - A_2\bar{R}$$

For the R chart:

$$UCL_{\bar{R}} = D_3\bar{R}$$

$$CL_{\bar{R}} = \bar{R}$$

$$LCL_{\bar{R}} = D_4\bar{R}$$

where A_2, D_3, and D_4 are tabled values for subgroup size n (see Appendix).

The AT&T runs rules should always be employed for the $\bar{X}$ chart and should be used for the R chart when the sample size is five or more (AT&T 1985). The first rule of the AT&T runs rules should always be used for both charts.

In practice, the initial estimation of the control limits may require an iterative process in securing control limits that are statistically valid. In particular, it may be necessary to estimate the limits initially. If a given $\bar{X}$ or R plots beyond the initial control limits, it is recommended that the cause of the extraordinary point be resolved and then removed from the calculation. That is, the control limits should be recalculated with the out-of-control point(s) removed. In some extreme cases, this iterative process may be repeated several times. Once the observed data are contained within the trial limits, continued production should be used to collect more data to further validate the limits. Once process stability is obtained, that is control is maintained for an extended period, the new control limits should become the operational standard. Furthermore, once the trial limits are established, the AT&T Runs Rules should be deployed in totality to insure protection against process disturbances.

Example 8 demonstrates the construction of $\bar{X}$ and R charts when standards are not known.

Example 8. Consider the process data of Example 7, except do not assume that the standards are known. Determine the appropriate trial control limits for this process.

i	X_{i1}	X_{i2}	X_{i3}	X_{i4}	X_{i5}	$\bar{X}_i$	R_i
1	51.69	49.59	52.51	51.07	50.01	50.97	2.92
2	49.37	49.13	49.77	49.61	49.89	49.55	0.77
3	50.59	50.08	50.4	47.82	48.67	49.51	2.77
4	51.24	49.79	50	49.89	51.98	50.58	2.19
5	50.78	48.82	50.59	51.48	49.12	50.16	2.66
6	49.86	48.43	49.1	48.76	49.45	49.12	1.43
7	50.86	50.78	50.43	52.12	49.7	50.78	2.42
8	50.94	51.61	50.2	49.08	51.63	50.70	2.55
9	49.25	49.46	51.71	49.08	51.55	50.21	2.63
10	50.52	50.92	49.06	50.43	51.46	50.48	2.40
11	49.45	49.02	51.23	49.96	49.9	49.91	2.20
12	50.12	51.06	50.99	50.98	50.09	50.65	0.97
13	49.32	49.93	51	50.68	48.14	49.81	2.86
14	50.04	50.1	49.08	49.14	50.3	49.73	1.22
15	50.25	49.78	50.51	49.37	48.21	49.62	2.30
16	50.07	49.69	48.81	50.72	50.25	49.91	1.91
17	49	50.25	49.93	48.87	49.32	49.47	1.38
18	49.64	51.22	49.84	50.51	50.68	50.38	1.58
19	51.5	49.49	48.24	48.02	50.86	49.62	3.48
20	51	49.58	49.57	50.08	51.75	50.39	2.19
21	49.66	49.41	47.43	49.68	49.39	49.11	2.25
22	48.64	48.53	50.03	49.85	48.77	49.16	1.50
23	50.35	49.34	51.2	50.82	49.72	50.29	1.87
24	51.49	51.44	50.78	48.91	49.86	50.50	2.57
25	49.81	49.45	49.43	51.46	49.76	49.98	2.03
26	49.82	49.99	51.3	49.4	50.28	50.15	1.90
27	48.59	49.31	49.95	50.63	48.78	49.45	2.04
28	51.67	51.13	49.33	50.93	48.61	50.33	3.05
29	50	50.85	49.53	50.7	49.3	50.08	1.55
30	50.2	48.74	50.42	51.27	48.57	49.84	2.71
						50.02	2.14
						$\bar{\bar{X}}$	$\bar{R}$
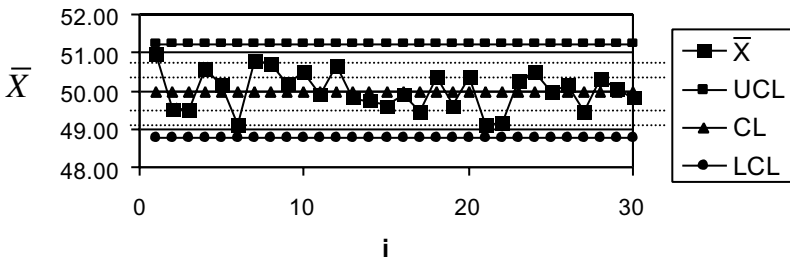
Figure 10 $\overline{X}$ Chart.

The resulting control limits:
For the $\overline{X}$ chart:

$$\text{UCL}_{\overline{x}} = 50.02 + 0.577(2.14) = 51.255$$

$$\text{CL}_{\overline{x}} = 50.02$$

$$\text{LCL}_{\overline{x}} = 50.02 - 0.577(2.14) = 48.785$$

For the R chart:

$$\text{UCL}_{\overline{R}} = 2.115(2.14) = 4.526$$

$$\text{CL}_{\overline{R}} = 2.14$$

$$\text{LCL}_{\overline{R}} = 0(2.14) = 0$$

The resulting control charts are given in Figures 10 and 11.

As demonstrated, all points plot within the control limits; therefore, these trial limits should be employed for production. Furthermore, it is highly recommended that the production also utilize the AT&T runs rules on the $\overline{X}$ chart (see zones on Figure 10) for subsequent production to monitor the state of the process.

4.2. Other Charts

There are many other control charts to monitor variable data. In each case, there should be a control chart for monitoring the mean and a control chart for monitoring the variance. These additional charts include the cumulative sum charts, exponentially weighted moving average charts, modified control charts, trend charts, control charts for short production times, individuals charts, moving average charts, moving range charts, control charts for variable sample sizes, S chart, variance charts, and acceptance control charts. The reader is referred to comprehensive textbooks such as Montgomery (1996) for a complete treatise on these more advanced techniques.
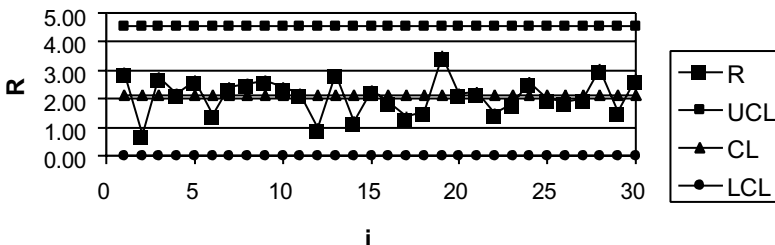

Figure 11 R Chart.

4.3.   Process Capability Analysis

Once the process has generated sufficient data to support the conclusion that it is stable or in a state of statistical process control, it is desirable to judge the process's ability to meet customer specifications, that is, the upper specification limit (USL) and lower specification limit (LSL). Within most industries, process capability ratios are used to measure a given process's ability to meet specifications. The effort employed to make this judgment is known as process capability analysis (PCA).

The PCA ratios common to industry are C_p and C_{pk}. The ratios provide different perspectives to PCA. Generally, it is desirable that the ratios exceed 1 to ensure that the process is capable of meeting specifications.

4.3.1.   Capability Ratios

The first ratio considered is C_p. This capability ratio estimates the potential that a process has to meet specifications. As demonstrated in the formulation below, the mean of the process is not accounted for.

$$C_p = \frac{\text{USL} - \text{LSL}}{6\hat{\sigma}}$$

4.3.2. C_{pk}

With the C_p calculation above, it is clear that a very good C_p could be determined while the process is generating practically all-bad product. As a result, C_{pk} has been formulated to embed the concern of process centering. C_{pk} is calculated as follows:

$$C_{pk} = \text{minimum} \left(\frac{\text{USL} - \hat{\mu}}{3\hat{\sigma}}, \frac{\hat{\mu} - \text{LSL}}{3\hat{\sigma}} \right)$$

In the formulas above for C_p and C_{pk}, $\hat{\sigma}$ may be determined with almost any known technique. Since the process must be assumed to be in a state of statistical control, it is usually found that $\overline{R}/d_2$ is the most convenient estimate of σ.

It is important to note that the now-famous Motorola 6σ program is based upon PCA. This highly effective program simply requires that C_p and C_{pk} be greater than 2.

Example 9 presents the use of the PCA ratios.

Example 9. For the data presented in Example 8, compute both C_p and C_{pk} and make the appropriate PCA conclusions. The USL is 54 and the LSL is 47.

For the data collected,

$$\hat{\mu} = 50.02$$

$$\hat{\sigma} = \frac{\overline{R}}{d_2} = \frac{2.14}{2.326} = 0.920$$

$$C_p = \frac{54 - 47}{6(0.920)} = 1.268$$

$$C_{pk} = \text{minimum} \left(\frac{54 - 50.02}{3(0.920)}, \frac{50.02 - 47}{3(0.920)} \right) = \text{minimum}(1.442, 1.094) = 1.094$$

As a result, it appears that the process is capable of meeting specifications, but the performance is not at the 6σ level, since C_p and C_{pk} are both less than 2.

4.3.3.   Inferential Approaches

In the previous section, the statistical inference of PCA using the capability ratios, C_p and C_{pk}, is questionable. This stems from the fact that only a single point estimate is made and compared to a standard of 1. Clearly, due to the random nature of any process, sample estimates or observed values of C_p and C_{pk} can be greater than 1 when, in fact, the true process values are less than 1.

Chou et al. (1990) provide an inferential approach to PCA using C_p and C_{pk}. Their work necessarily assumes the observations collected from the process in question to be independent and to follow the normal distribution. They establish minimum critical values for both of the parameters that provide a statistical basis to concluding the capability of the process. Essentially, they establish

a critical value for both C_p and C_{pk} for the appropriate test of hypothesis. The test of hypotheses for the ratios is as follows:

For C_p,

$$H_o : C_p < C_0$$
$$H_A : C_p \geq C_0$$

where C_0 is the desired minimum value of C_p such that the process is deemed capable.

For C_{pk},

$$H_o : C_{pk} < C_k$$
$$H_A : C_{pk} \geq C_k$$

where C_k is the desired minimum value of C_{pk} such that the process is deemed capable.

For the estimates of C_p and C_{pk}, the unbiased estimate of σ^2, S^2, should be utilized for the estimates of the capability ratios. Furthermore, the estimate of the mean, μ, should be the sample average, commonly known as $\bar{X}$. In Chou et al. (1990), as long as the calculated or estimated value of the specific capability ratio is greater than the critical value, as defined in Table 1 for various sample sizes ($n = 10, 20, 30,$ and 50), it is concluded that the "process is considered capable 95% of the time." Table 1 presents a small yet practical set of the values presented in Chou et al. (1990).

It is interesting to note that if it is desired that C_p and C_{pk} be both greater than 1, the calculated values of C_p and C_{pk} should be considerably larger than 1 for both cases for the practical sample sizes shown in Table 1. The Motorola 6σ performance criteria ($C_p = C_{pk} = 2.0$) are even greater. Example 10 presents applications of the inferential approach to PCA.

Example 10. Thirty observations have been collected from the in-control process of Examples 8 and 9.

1. If minimum values of one are required for C_p and C_{pk}, is the process capable.
2. Is the process of 6σ quality?
Data:

TABLE 1   Critical Values for Estimated C_p and C_{pk}

C_p:				
C_o	$n = 10$	20	30	50
1.0	1.65	1.37	1.28	1.20
1.2	1.97	1.64	1.54	1.44
1.4	2.30	1.92	1.79	1.68
1.6	2.63	2.19	2.05	1.92
1.8	2.96	2.47	2.30	2.16
2.0	3.29	2.74	2.56	2.40
C_{pk}:				
C_k	$n = 10$	20	30	50
1.0	1.80	1.46	1.35	1.25
1.2	2.12	1.73	1.61	1.49
1.4	2.45	2.01	1.86	1.73
1.6	2.78	2.28	2.12	1.97
1.8	3.11	2.55	2.37	2.21
2.0	3.44	2.83	2.63	2.45

i	Observation (X_i)
1	50.13
2	49.65
3	52.41
4	49.19
5	51.16
6	50.01
7	49.99
8	48.76
9	49.59
10	49.73
11	50.36
12	49.19
13	50.02
14	49.46
15	49.47
16	49.75
17	50.42
18	51.84
19	51.14
20	51.74
21	48.07
22	48.9
23	49.95
24	48.85
25	50.24
26	48.87
27	50.5
28	53.13
29	48.82
30	48.39

$S^2 = 1.401$
$\overline{X} = 49.99$

1. The resulting calculated values of C_p and C_{pk} are as follows (assuming the USL of 54 and LSL of 47 from Example 9):

$$C_p = \frac{54 - 47}{6\sqrt{1.401}} = 0.986$$

$$C_{pk} = \text{minimum}\left(\frac{54 - 49.99}{3\sqrt{1.401}}, \frac{49.99 - 47}{3\sqrt{1.401}}\right) = \text{minimum}(1{,}129,\ 0.842) = 0.842$$

In both cases, the calculated values are less than the critical values. That is, the critical value for C_p when C_o is 1 is 1.28, and the critical value for C_{pk} when C_k is 1 is 1.35. As a result, a strong, statistically based conclusion can be made that the process is not capable.

2. Obviously, if the process is not capable at values of 1 for the PCA ratios, it is not capable to meet the 6σ performance criteria. For this sample of 30, minimum critical values for C_p and C_{pk} would have been 2.56 and 2.63, respectively, if C_o and C_k values of 2 were required (which is implied by the 6σ program).

5.   CONTROL CHARTS FOR ATTRIBUTE DATA

5.1.   Attribute Data

At this point, we have reviewed $\overline{X}$ and R control charts. Both of these charts are used for variable data. Many more control charts exist for varying conditions for variable data, as described above. In this section, two control charts are presented that are most useful in monitoring attribute data. In particular, a control chart known as the p chart is presented to provide a tool for monitoring the

percent nonconforming from a process. Secondly, the *c* chart is used to monitor the number of defects being produced in a process.

5.2.   Control Chart for Percent Nonconforming: *p* Chart

The percentage of nonconforming units being produced is of utmost concern to all organizations. A nonconforming unit represents the reality of a process's failure to produce product that meets customers' expectations or specifications. As such, the percent nonconforming control chart can also be thought of as an ongoing estimate of the process's ability to meet customer expectations.

For the *p* chart, the percent nonconforming is estimated with sampled data observations. Typically, 25–100 observations are collected for a given subgroup. The current percent nonconforming, *p*, is estimated as follows:

$$\hat{p} = \frac{D_i}{n}$$

where D_i = the number of nonconforming observations in the current subgroup *i*
    n = the number of observations or samples in each subgroup

It is assumed that the number of observations that are defective in a subgroup of *n* samples follows the binomial distribution with parameters *p* (the true population percent nonconforming) and *n*. For the formulation presented here, *n* is assumed to be fixed. This assumption can be relaxed, but a modified approach for the varying sample size must be employed as demonstrated in most standard quality control textbooks (i.e., Montgomery 1996). The *p* chart is essentially the plotting of the observed values of $\hat{p}$. These values are individually compared to the control limits (shown below) to determine the current status of the process (i.e., state of statistical control or out of control). Furthermore, the AT&T runs rules should be deployed when possible.

Obviously, *p*, the true population percent nonconforming, is typically estimated by averaging the observed values of *p*. As a rule of thumb, it is suggested that 25–30 subgroups be collected before the estimate of *p* is made. This parameter, known as $\bar{p}$, is estimated as follows:

$$\bar{p} = \frac{\sum_{i=1}^{m} \hat{p}_i}{m}$$

where *m* is the number of subgroups collected (recommended: 25–30).

Both $\hat{p}$ and $\bar{p}$ are known to be unbiased estimators of *p*. Once $\bar{p}$ is determined, the control limits can be determined as follows:

$$\text{UCL}_p = \bar{p} + 3 \sqrt{\frac{\bar{p}(1 - \bar{p})}{n}}$$

$$\text{CL}_p = \bar{p}$$

$$\text{LCL}_p = \text{maximum} \left(0, \bar{p} - 3 \sqrt{\frac{\bar{p}(1 - \bar{p})}{n}} \right)$$

The LCL is truncated to 0 when the calculated value falls below 0. The AT&T runs rules can be employed in situations where the control limits are reasonably symmetric (AT&T 1985).

The establishment of the trial control limits for the *p* chart follows the same practice as described for the $\bar{X}$ and *R* charts. That is, if a given point plots beyond the initial control limits, it is recommended that the cause of the extraordinary point be resolved and then removed from the calculation. That is, the control limits should be recalculated with the out-of-control point(s) removed. In some extreme cases, this process may have to be repeated. Once the observed data are contained within the trial limits, continued production should be used to collect more data in an effort to validate the limits. Once process stability is obtained, that is, control is maintained for an extended period, the new control limits should become the operational standard. Furthermore, once the trial limits are established, the AT&T runs rules should be deployed in totality to ensure protection against process disturbances.

Example 11 demonstrates the use of the *p* chart.

Example 11. Thirty subgroups of 100 observations each have been collected from your process of producing the toy presented earlier in this chapter. The data collected and calculated $\hat{p}_i s$ are as follows:

i	D_i	p_i
1	1	0.01
2	0	0
3	2	0.02
4	1	0.01
5	2	0.02
6	1	0.01
7	0	0
8	0	0
9	1	0.01
10	1	0.01
11	1	0.01
12	1	0.01
13	2	0.02
14	2	0.02
15	1	0.01
16	1	0.01
17	1	0.01
18	1	0.01
19	0	0
20	4	0.04
21	0	0
22	0	0
23	1	0.01
24	3	0.03
25	2	0.02
26	0	0
27	0	0
28	1	0.01
29	1	0.01
30	3	0.03
	$\bar{p}$	0.011333

Computing the trial limits:

Toy data
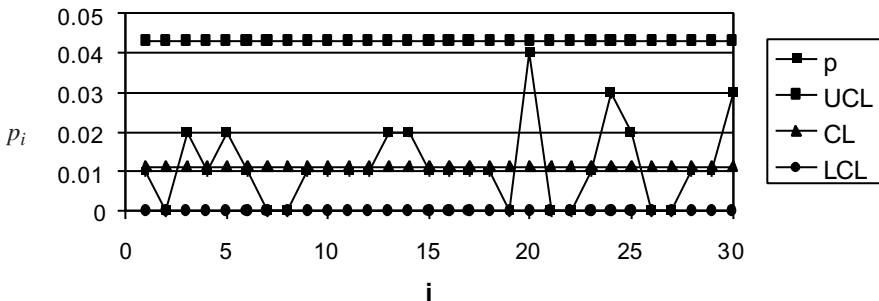
Figure 12 *p* Chart.

$$\text{UCL}_p = 0.011333 + 3 \sqrt{\frac{0.011333(1 - 0.011333)}{100}} = 0.0431$$

$$\text{CL}_p = \bar{p} = 0.011333$$

$$\text{LCL}_p = \text{maximum} \left(0, 0.011333 + 3 \sqrt{\frac{0.011333(1 - 0.11333)}{100}} \right) = 0$$

As demonstrated, all points plot within the control limits; therefore, these trial limits should be employed for production. Since the control limits are not symmetric, only rule number one of the AT&T runs rules should be used. AT&T (1985) describes an approach that may be used for customizing runs rules to improve the detection strength of a p chart.

5.3.   Control Chart for Number of Defective Units: c Chart

An equally important concern in production is the number of defects that are produced. Examples of defects include the type of data included on the check sheet described in Example 2 (i.e., paint drips, scratches). This type of data collected and monitored on a consistent basis in an organization gives rise to the organizational understanding of the defect types and clearly document sustained performance. For this type of data, the sample should be a fixed amount of product, that is, linear feet, number of units, pounds of product. Once the sample is collected, the number of defects, c_i, is determined through the careful analysis of the sampled product. Typically, although not necessarily, all defects are grouped together to construct one control chart.

It is assumed that the number of defects in a sample follows the Poisson distribution with parameter λ (the average defect rate for this application). The c chart is essentially the plotting of the observed values of c_i. These values are individually compared to the control limits (shown below) to determine the current status of the process (i.e., state of statistical control or out of control). Furthermore, the AT&T runs rules should be deployed when the resulting control limits are symmetric (AT&T 1985).

Obviously, c (or λ for the classic Poisson distribution), the true population average defect rate, is never known with certainty. As a result, it is estimated by averaging m c_i's. As a rule of thumb, it is suggested that 25–30 subgroups be collected before the estimate of c, called $\bar{c}$, is made. This parameter is estimated as follows:

$$\bar{c} = \frac{\sum_{i=1}^{m} c_i}{m}$$

where m is the number of samples collected

With an accurate estimate of the average defect rate in hand, the control limits for the c chart are computed as follows:

$$\text{UCL}_c = \bar{c} + 3\sqrt{\bar{c}}$$

$$\text{CL}_c = \bar{c}$$

$$\text{LCL}_c = \text{maximum}(0, \bar{c} - 3\sqrt{\bar{c}})$$

The LCL is truncated to 0 when the calculated value falls below 0. The AT&T runs rules should be employed when the control limits are reasonably symmetric (AT&T 1985). Rule number one should always be used. AT&T (1985) describes an approach that may be used for customizing runs rules to improve the detection strength of a p chart.

The establishment of the trial control limits for the c chart follows the same practice as described for the $\bar{X}$ and R and the p charts. Once the observed data are contained within the trial limits, continued production should be used to collect more data in an effort to validate the limits. Once process stability is obtained, that is, control is maintained for an extended period, the new control limits should become the operational standard. Furthermore, once the trial limits are established, the AT&T runs rules should be deployed in totality to ensure protection against process disturbances when appropriate.

The following example demonstrates the use of the c chart.

Example 12. c Chart: 30 samples of five toys have been collected from your process of producing the toy presented earlier in this chapter. The data collected and calculated c_is are as follows:

i	c_i
1	2
2	4
3	9
4	2
5	1
6	4
7	5
8	1
9	3
10	6
11	1
12	8
13	7
14	2
15	4
16	3
17	5
18	4
19	3
20	5
21	7
22	6
23	2
24	2
25	6
26	7
27	3
28	4
29	5
30	7
$\bar{c}$	4.266667

The resulting trial control limits are as follows:

$$\text{UCL}_c = 4.266667 + 3\sqrt{4.266667} = 10.463$$

$$\text{CL}_c = 4.266667$$

$$\text{LCL}_c = \text{maximum}(0,\ 4.266667 - 3\sqrt{4.266667}) = 0$$

The resulting control chart is as follows:
The process clearly exhibits stability.
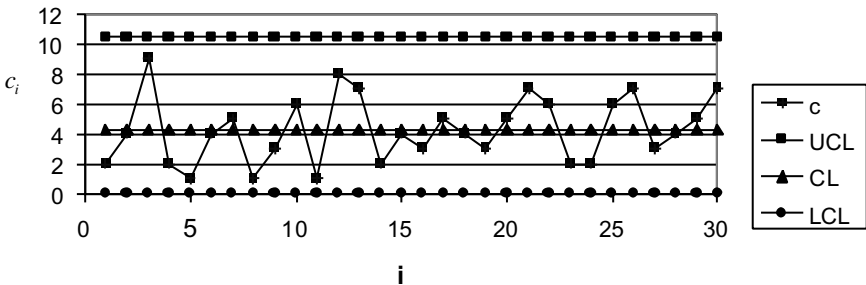
Toy Process

Figure 13 *c* Chart.

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APPENDIX
Table Values for the $\overline{X}$ and R Charts

The table values of statistical constants in this appendix are a portion of the values that can be found in any basic quality control text (i.e., Grant and Leavenworth 1980; Montgomery 1996).

Subgroup size, n	A_2	d_2	D_1	D_2	D_3	D_4
2	1.880	1.128	0	3.686	0	3.267
3	1.023	1.693	0	4.358	0	2.575
4	0.729	2.059	0	4.698	0	2.282
5	0.577	2.326	0	4.918	0	2.115
6	0.483	2.534	0	5.078	0	2.004
7	0.419	2.704	0.204	5.204	0.076	1.924
8	0.373	2.847	0.388	5.306	0.136	1.864
9	0.337	2.970	0.547	5.393	0.184	1.816
10	0.308	3.078	0.687	5.469	0.223	1.777