Chapter **8**

System Reliability

EDWARD POHL, Ph.D.

> Failure should be our teacher, not our undertaker. Failure is delay, not defeat. It is a temporary detour, not a dead end. Failure is something we can avoid only by saying nothing, doing nothing, and being nothing.
>
> —Denis Waitley

> Would you like me to give you a formula for success? It's quite simple, really. Double your rate of failure. You are thinking of failure as the enemy of success. But it isn't at all. You can be discouraged by failure—or you can learn from it. So go ahead and make mistakes. Make all you can. Because, remember, that's where you will find success.
>
> —Thomas J. Watson

8.1 INTRODUCTION TO SYSTEM EFFECTIVENESS

Modeling and measuring the effectiveness of a proposed solution to a problem is a necessary component of the systems decision process. When analyzing potential solutions to system problems, we generally speak about system effectiveness. To understand a system's effectiveness, we break apart the term and investigate its components. There are many definitions of what constitutes a system across a variety of fields and disciplines. In the first chapter we defined a system using the definition adopted by INCOSE:

Decision Making in Systems Engineering and Management, Second Edition
Edited by Gregory S. Parnell, Patrick J. Driscoll, Dale L. Henderson

> A system is an integrated set of elements that accomplish a defined objective. These elements include products (hardware, software, firmware), processes, people, information, techniques, facilities, services, and other support elements [1].

Webster's Dictionary defines "effective" as "producing a decided, decisive, or desired effect" [2].

Soban and Marvis [3] combine these two components to form a definition that captures the need to quantify the intended effects of the system under study. They define *system effectiveness* as follows:

> System effectiveness is a quantification, represented by system level measure, of the intended or expected effect of the system achieved through functional analysis [3].

When studying complex systems, we are concerned with how well the system, performs its "mission." Blanchard [4] specifies several of the common system level measures that Soban and Marvis [3] allude to in their definition. The key system level measures that appear in value hierarchies for most complex systems are reliability, availability, and capability. Capability is a system specific measure that captures the overall performance objectives associated with the system. Reliability and availability are measures that apply to all types and levels of system analysis. Because of their importance in analyzing system effectiveness, we spend the remainder of the chapter discussing how to model and analyze the reliability and availability of a system.

8.2 RELIABILITY MODELING

Reliability is one of the core performance measures for any system under study. Today, more than ever, reliability is not only expected, but is in demand in the market. The current global economic environment is forcing system designers to find creative ways to make cost-effective, reliable systems that meet or exceed the performance expectations of their consumers and users. To maintain their competitiveness, system designers must design for reliability and make those system level trades early in the designing process. Attempting to improve reliability after the system has been designed is a costly approach as illustrated in Figure 8.1.

Reliability is the *probability* that an item (component, subsystem, or system) or process operates properly for a *specified amount of time* (design life) under *stated use conditions* (both environmental and operational conditions) without failure. What constitutes failure for a component, subsystem, system, or process must be clearly defined as the item or process is developed. In addition, proper operating and environmental conditions must be adequately defined so that the designers and operators have a common understanding of how, when, and where the item or process should be used. In simple terms, reliability is nothing more than the

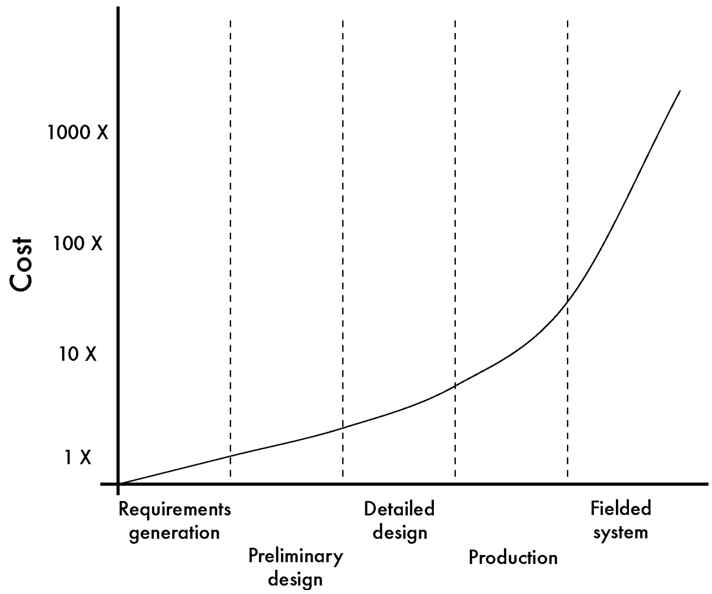

Figure 8.1 Cost of reliability improvement versus time. (Adapted from reference 5.)

probability that the system under study operates properly for a specified period of time.

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8.3 MATHEMATICAL MODELS IN RELIABILITY

As discussed previously, the definition of reliability has to be precisely constructed. In this section we are going to develop basic mathematical models of reliability that focus on items that can be in one of the two states:

- working ($X(t) = 1$) and
- not working ($X(t) = 0$).

Basic mathematical models of reliability are often constructed by analyzing test data and using these data in conjunction with the probability theory to characterize the items' reliability. Suppose we put N_0 identical items on test at time $t = 0$. We will assume that each item that is put on test at time $t = 0$ is functional (i.e., $X_i(0) = 1, i = 1$ to N_0).

Let $N_s(t)$ be the number of items that have survived to time t. Let $N_f(t)$ be a random variable representing the number of items that have failed by time t, where $N_f(t) = N_0 - N_s(t)$. Thus, the reliability at time t can be expressed as

$$R(t) = \frac{E\left[N_s(t)\right]}{N_0} \tag{8.1}$$

Remember that reliability is a probability and it represents the probability that the item is working at time t. If we let T be a random variable that represents the time to failure of an item, then the reliability can be written as

$$R(t) = P(T > t) \tag{8.2}$$

and the unreliability of an item $F(t)$ can be expressed as

$$F(t) = P(T \le t) \tag{8.3}$$

$F(t)$ is commonly referred to as the *cumulative distribution function* (CDF) of failure. $F(t)$ can also be expressed as

$$F(t) = \frac{E\left[N_f(t)\right]}{N_0} \tag{8.4}$$

Using this information, we can establish the following relationship:

$$R(t) = \frac{E\left[N_s(t)\right]}{N_0} = \frac{E\left[N_0 - N_f(t)\right]}{N_0} = \frac{N_0 - E\left[N_f(t)\right]}{N_0} = 1 - F(t). \tag{8.5}$$

Given the cumulative distribution function of failure $F(t)$, the probability density function (PDF) of failure $f(t)$ is given by

$$f(t) = \frac{d}{dt}F(t) \tag{8.6}$$

Thus

$$f(t) = \frac{d}{dt}(1 - R(t)) = -\frac{d}{dt}R(t) \tag{8.7}$$

Another important measure is the hazard function. The hazard function represents the probability that a unit fails in the next increment of time and is represented mathematically by the equation below:

$$h(t) = \lim_{\Delta t \to 0}\left\{\frac{1}{\Delta t}\Pr\left[T \le t + \Delta t | T > t\right]\right\} \tag{8.8}$$

$$= \lim_{\Delta t \to 0}\left\{\frac{1}{\Delta t}\frac{F(t + \Delta t) - F(t)}{R(t)}\right\} \tag{8.9}$$

$$= \frac{f(t)}{R(t)} \tag{8.10}$$

The hazard function provides another way of characterizing the failure distribution of an item. Hazard functions are often classified as increasing failure rate (IFR), decreasing failure rate (DFR), or constant failure rate (CFR), depending on the behavior of the hazard function over time. Items that have an increasing failure rate exhibit "wear-out" behavior. Items that have a CFR exhibit behavior characteristic of random failures.

A general relationship between the reliability function $R(t)$ and the hazard function $h(t)$ can be established as follows:

$$h(t) = \frac{1}{R(t)}\left[\frac{-dR(t)}{dt}\right] \tag{8.11a}$$

$$h(t)\,dt = \frac{-dR(t)}{R(t)} \tag{8.11b}$$

Integrating both sides, we obtain

$$\int_0^t h(u)\,du = \int_{R(0)}^{R(t)} \left[\frac{-dR(t)}{dt}\right] \tag{8.12a}$$

$$\int_0^t h(u)\,du = \int_1^{R(t)} \frac{-dR(t)}{dt} \tag{8.12b}$$

$$-\int_0^t h(u)\,du = \ln(R(t)) \tag{8.12c}$$

$$R(t) = \exp\left[-\int_0^t h(u)\,du\right] \tag{8.12d}$$

and therefore

$$F(t) = 1 - \exp\left[-\int_0^t h(z)\,dz\right] \tag{8.13}$$

The most common form of the hazard function is the bathtub curve (see Figure 8.2). The bathtub curve is most often used as a conceptual model of a population of items rather than a mathematical model of a specific item. Early on, during the development of an item, initial items produced will oftentimes be subject to manufacturing defects. Over time, the manufacturing process is improved and these defective units are fewer in number, so the overall hazard function for the remaining population decreases. This portion of the bathtub curve is referred to as the "infant mortality" period. Once the manufacturing system matures, fewer items will fail early in their lifetime and items in the field will exhibit a constant hazard function. This period is known as the "useful life" period. During this time, failures are purely random and usually caused by some form of unexpected or random stress placed on the item. At the end of its useful life, items in the field may begin to "wear out" and as a result the hazard function for the population of items remaining in the field will exhibit a rapidly increasing rate of failure.

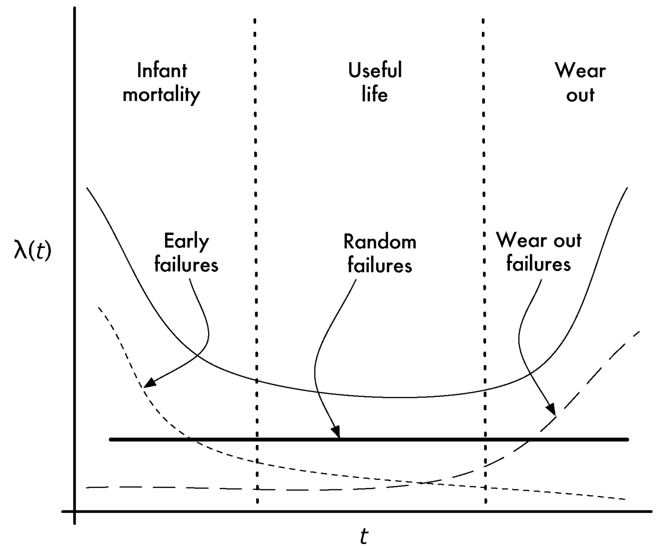


Figure 8.2 General bathtub-shaped hazard function. (Adapted from reference 6.)

To remain competitive, most product developers attempt to reduce or eliminate the "infant mortality" period and the associated "quality" related failures by using a variety of quality control tools and initiatives. For high-reliability items, many manufacturers may use environmental stress screening (ESS) or "burn-in" to enhance their quality initiatives and further reduce or eliminate this infant mortality period. ESS subjects the items to a variety of stresses (i.e., temperature, vibration, voltage, humidity, etc.) to cause the "weak" components to fail before they enter the field. Burn-in is used to filter out the defective items by having each item operate for some predefined period of time, often at an increased temperature. Ideally, to maximize reliability, we would like items to operate in the useful life period. Many organizations develop maintenance strategies that allow their products to remain in this period. The goal of these maintenance strategies is to replace the items in the field before they enter the wear-out period and fail in the field.

One final measure that is often used to characterize the reliability of an item is the items' mean time to failure (MTTF). The MTTF is nothing more than the expected value of the failure time of an item. Mathematically, the MTTF is calculated as follows:

$$\text{MTTF} = E\,[T] = \int_0^\infty tf(t)\,dt \tag{8.14}$$

or

$$\text{MTTF} = E\,[T] = \int_0^\infty R(t)\,dt \tag{8.15}$$

The variance for the time to failure T is given by the following relationship:

$$\mathrm{Var}[T] = E\left[T^2\right] - E[T]^2 = \int_0^\infty t^2 f(t)\,dt - \left[\int_0^\infty t f(t)\,dt\right]^2 \tag{8.16}$$

8.3.1 Common Continuous Reliability Distributions

As was mentioned previously, reliability is a function of time, and time is a continuous variable. Thus, most items that operate over continuous periods should be modeled using a continuous time-to-failure distribution. There are many continuous time-to-failure distributions that can be used to model the reliability of an item. Some of the well-known distributions include those listed in Table 8.1.

Most reliability books (see references 7–11) cover in detail many of these distributions. Leemis [12] has done an exceptional job by describing each of the above distributions and their relationships with each other. Despite this large number of possible failure distributions, the two most widely used continuous failure distributions are the exponential and Weibull distributions. We will discuss each in detail.

Exponential Failure Distribution The exponential distribution is probably the most used and often abused failure distribution. It is a single parameter distribution that is easily estimated for a variety of data collection methods. Its mathematical tractability for modeling complex combinations of components, subsystems, and systems make it attractive for modeling large-scale systems and system of systems. Empirical evidence has shown that systems made up of large numbers of components exhibit exponential behavior at the system level. The exponential distribution is a CFR distribution. It is most useful for modeling the "useful life" period for an item. The exponential distribution possesses a unique property called the memoryless property. It is the possession of this property that often results in this distribution being used in inappropriate situations. The memoryless property states that if an item has survived until a specific time t, the probability that it will survive for the next time period $t + s$ is independent of t and only dependent

TABLE 8.1 Common Continuous Reliability Distributions

• Exponential	• Extreme value
• Weibull	• Logistic
• Normal	• Log logistic
• Lognormal	• Pareto
• Beta	• Inverse Gaussian
• Gamma	• Makeham
• Rayleigh	• Hyperexponential
• Uniform	• Muth

on s. Therefore, this distribution should not be used to model components that have wear-out failure mechanisms. The PDF, CDF, reliability function, and hazard function for the exponential distribution are

$$f(t) = \lambda e^{-\lambda t} \tag{8.17}$$

$$F(t) = 1 - e^{-\lambda t} \tag{8.18}$$

$$R(t) = e^{-\lambda t} \tag{8.19}$$

$$h(t) = \lambda \tag{8.20}$$

Figures 8.3–8.5 illustrate the PDF, reliability function, and hazard function for an exponential distribution with $\lambda = 0.0001$. The MTTF for this distribution is

$$\text{MTTF} = E[T] = \int_0^\infty t\lambda e^{-\lambda t} = \frac{1}{\lambda} \tag{8.21}$$

There are several methods available to estimate the hazard rate of an exponential distribution. Techniques include the method of moments, maximum likelihood, and rank regression. We will explore the use of the first two techniques for the exponential distribution. The method of moments estimator is found by equating the population moments with the sample moments. Let $T_1, T_2, \ldots, T_n$ denote a

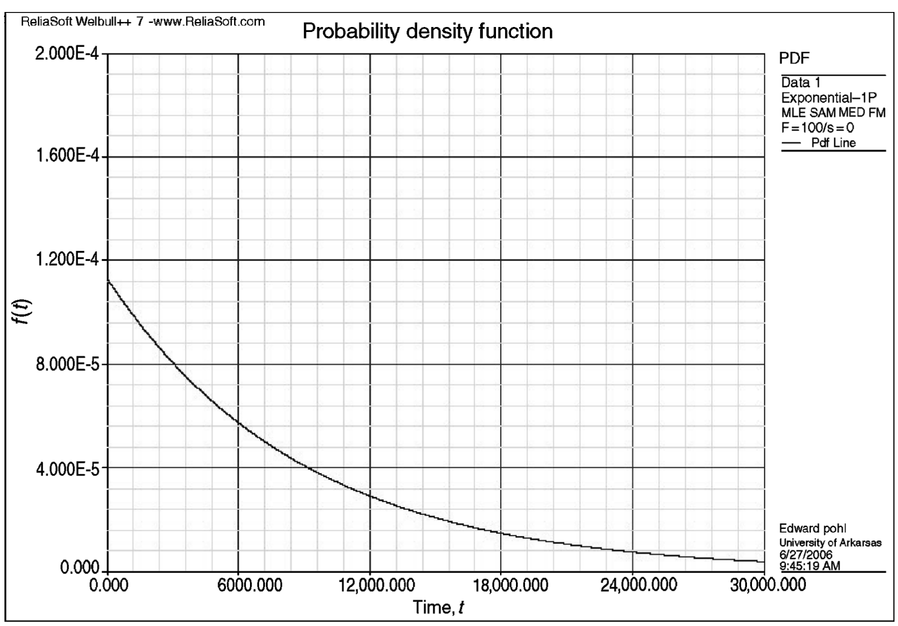


Figure 8.3 Probability density function for an exponential distribution $\lambda = 0.0001$.

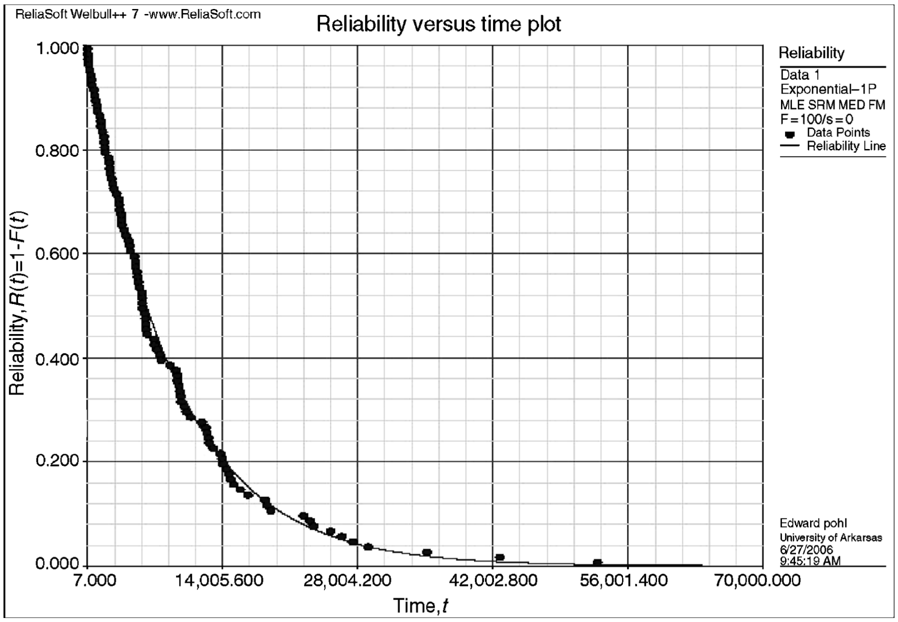


Figure 8.4 Reliability function for an exponential distribution $\lambda = 0.0001$.

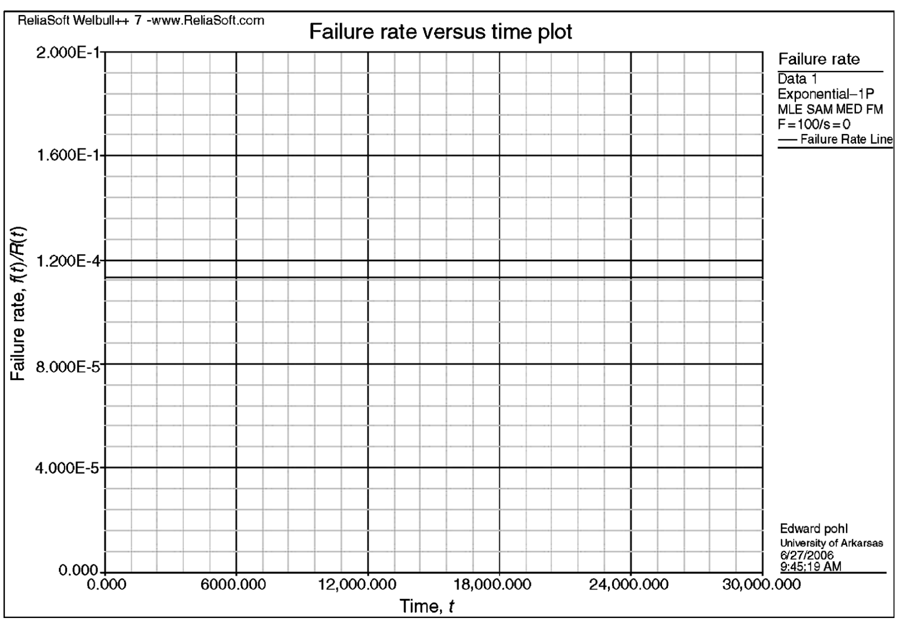


Figure 8.5 Hazard function for the exponential distribution $\lambda = 0.0001$.

random sample of failure times for n items from the total population of items. We can construct the sample mean, and sample variance as follows:

$$\bar{T} = \sum_{i=1}^{n} \frac{T_i}{n} \tag{8.22}$$

$$S^2 = \sum_{i=1}^{n} \frac{\left(T_i - \bar{T}\right)^2}{n-1} \tag{8.23}$$

We can then equate the population mean, the MTTF, to the sample mean and solve for the parameter. Thus,

$$\text{MTTF} = \frac{1}{\lambda} = \bar{T} = \sum_{i=1}^{n} \frac{T_i}{n} \tag{8.24}$$

$$\hat{\lambda} = \frac{1}{\bar{T}} = \frac{n}{\sum_{i=1}^{n} T_i} \tag{8.25}$$

A second approach and the one that is most often used, especially for medium to large samples, is the method of maximum likelihood. Caution should be exercised for small samples as the maximum likelihood estimation (MLE) has been shown to be biased for small samples. Maximum likelihood estimators are found by maximizing the likelihood function. The likelihood function is derived by observing the status of all the items in the sample. This technique allows us to account for censored data. Censoring occurs when the exact failure time of an item on test is unknown. The most common situation occurs when n items have been put on test for s hours, p items have failed, and the remaining $n - p$ items have not failed, by time s. These $n - p$ items have been censored (i.e., the test has been terminated before they where allowed to fail). The general form of the likelihood function is given as

$$L(T;\lambda) = \prod_{i \in \text{failed}} f(T;\lambda) \prod_{j \in \text{censored}} R(s;\lambda) \tag{8.26}$$

$$L(T;\lambda) = \prod_{i=1}^{p} \lambda e^{-\lambda T_i} \prod_{j=1}^{n-p} e^{-\lambda s} \tag{8.27}$$

For the case where no censoring has occurred (i.e., every item was run to failure in the sample), the following likelihood function is obtained:

$$L(T;\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda T_i} \tag{8.28}$$

It is often easier to maximize the log-likelihood function. This is accomplished by setting the first partial derivatives equal to zero and solving for the parameter.

$$\ln L(T;\lambda) = n \ln \lambda - \lambda \sum_{i=1}^{n} T_i \tag{8.29}$$

$$\frac{\partial}{\partial \lambda} \ln L(T;\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} T_i = 0 \tag{8.30}$$

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} T_i} \tag{8.31}$$

Example. The next generation over the horizon radar system is currently under development. As part of the development process, seven systems have been tested for 2016 hours. Two systems failed, one at 1700 hours and the other at 2000 hours. The remaining five systems were still operating at the end of the test period. Given your test data, what is the probability that the system will operate 24 hours a day, seven days a week for 30 days? Assume that the time to failure is adequately modeled by the exponential failure distribution.

Solution. First, we need to estimate the parameter for the exponential distribution. We will first derive the likelihood function and then construct the MLE.

$$L(T;\lambda) = \prod_{i=1}^{2} \lambda e^{-\lambda T_i} \prod_{j=1}^{5} e^{-\lambda s} = \lambda^2 e^{\sum_{i=1}^{2} T_i} e^{-\lambda \sum_{j=1}^{5} s}$$

$$\ln L(T;\lambda) = 2 \ln \lambda - \lambda \sum_{i=1}^{2} T_i - 5\lambda s$$

$$\begin{aligned} \frac{\partial \ln L(T;\lambda)}{\partial \lambda} &= \frac{2}{\lambda} - \left[\sum_{i=1}^{2} T_i + 5s\right] \\ &= \frac{2}{\lambda} - [1700 + 2000 + 5(2016)] = 0, \end{aligned}$$

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{2} T_i + 5s} = \frac{2}{13780} = 1.4514 \times 10^{-4}$$

Once we know the parameter we can calculate the probability the system can work continuously for 30 days without failure.

$$R(t) = e^{-\hat{\lambda} t},$$

$$R(5040) = e^{-(1.4514 \times 10^{-4})5040} = 0.6937.$$

Weibull Failure Distribution The Weibull distribution is commonly used because of its flexibility in modeling a variety of situations. It has also been shown to fit a wide variety of empirical data sets, especially for mechanical systems. The most common form of the Weibull distribution is a two-parameter distribution. The two parameters are the scale parameter, η, and the shape parameter, β. When $\beta < 1$, the hazard function for the distribution is DFR. When $\beta > 1$, the hazard function for the distribution is IFR. Finally, when $\beta = 1$, the hazard function is CFR. The PDF, CDF, reliability function, and hazard function for the Weibull distribution are given by the following relationships:

$$f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} e^{-\left(\frac{t}{\eta}\right)^{\beta}} \tag{8.32}$$

$$F(t) = 1 - e^{-\left(\frac{t}{\eta}\right)^{\beta}} \tag{8.33}$$

$$R(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}} \tag{8.34}$$

$$h(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} \tag{8.35}$$

Figures 8.6–8.8 illustrate the various shapes the PDF and hazard function take when the shape parameter is varied. Notice that when $\beta \cong 3$ the PDF takes a shape similar to a normal distribution. The MTTF for this distribution is

$$\text{MTTF} = E[T] = \int_0^{\infty} t\frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} e^{\frac{t}{\eta}^{\beta}} = \eta\Gamma\left(1 + \frac{1}{\beta}\right) \tag{8.36}$$

where the Γ function is

$$\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx \tag{8.37}$$

A three-parameter version of the Weibull distribution is sometimes used and has the following PDF, CDF, reliability function, and hazard function:

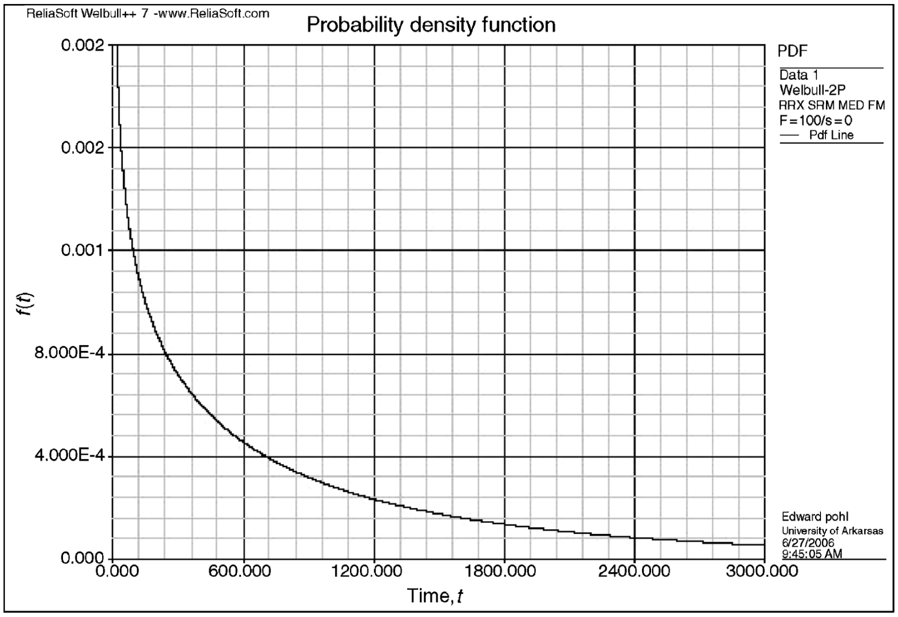
$$f(t) = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}$$

$$F(t) = 1 - e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}$$

$$R(t) = e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}$$

$$h(t) = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1}$$

where γ is the location parameter for the distribution, and is sometimes called the "guaranteed life" parameter because it implies that if $\gamma > 0$, then there is zero



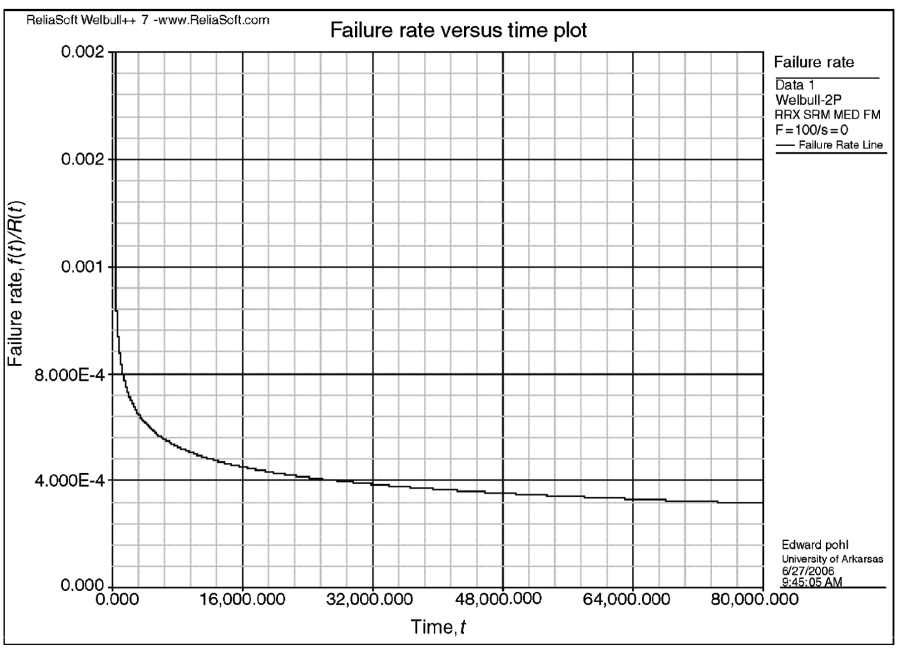
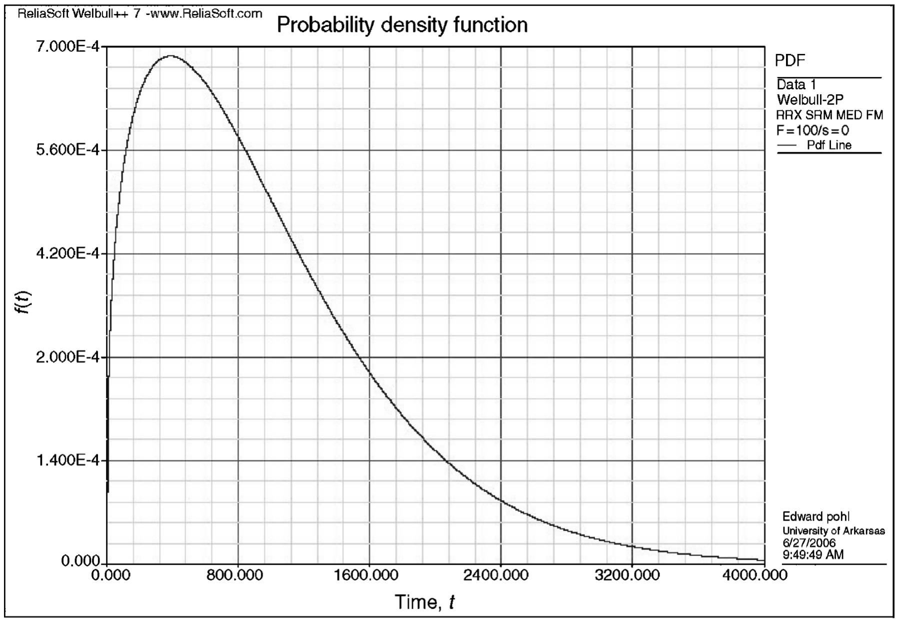


Figure 8.6 Weibull PDF and hazard function, $\beta = 0.77$.



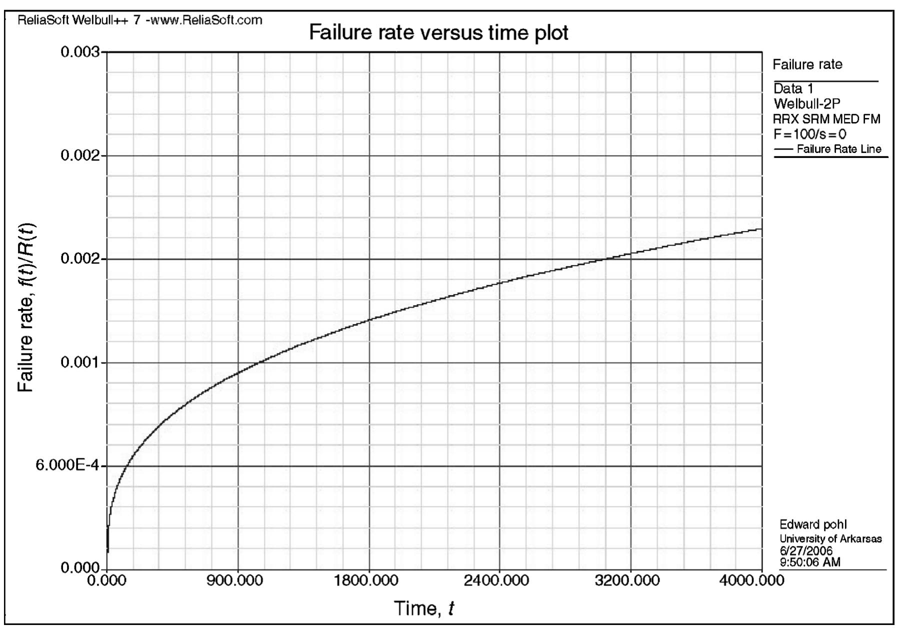


Figure 8.7 Weibull PDF and hazard function for $\beta = 1.36$.

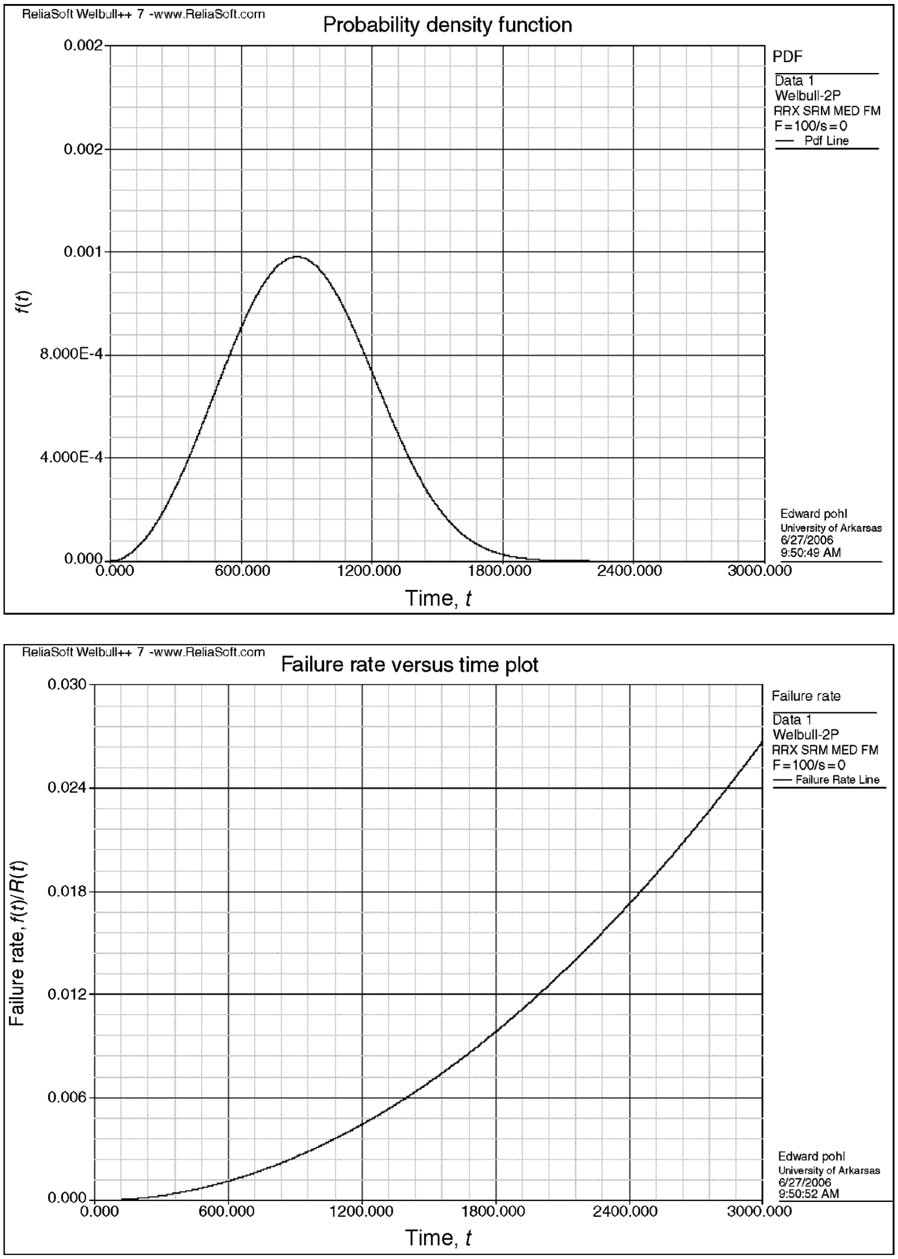


Figure 8.8 Weibull PDF and hazard function, $\beta = 2.96$.

probability of failure prior to γ. This is often a difficult assumption to prove and one of the reasons that the two-parameter model is used more often. The MTTF for the three-parameter Weibull failure distribution is given by the following relationship:

$$\text{MTTF} = E[T] = \int_0^{\infty} t\frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}} = \gamma + \eta\Gamma\left(1+\frac{1}{\beta}\right) \tag{8.38}$$

Like the exponential distribution, there are a variety of techniques available for estimating the distribution parameters. The method of moments and maximum likelihood techniques are both reasonable techniques and constructed in the same manner as was demonstrated earlier on the exponential distribution. Kececioglu [13] provides a detailed description of each of these techniques for the Weibull distribution as well as several others.

8.3.2 Common Discrete Distributions

Certain components or systems may have performance characteristics that require them to be modeled using a discrete distribution. For example, a switch's performance may be better characterized by the number of cycles (on/off) rather than the amount of time it is operated. Another example is a satellite launch vehicle. It either launches successfully or does not. Time to failure is not an adequate measure to describe the performance of the launch vehicle. We will explore three of the common discrete distributions utilized in measuring system performance. Method of moments and maximum likelihood methods are common approaches for estimating the parameters for these distributions.

Binomial Distribution The binomial distribution is a distribution that characterizes the sum of n independent Bernoulli trials. A Bernoulli trial occurs when an item's performance is a random variable that has one of two outcomes; it either works (success) or fails (failure) when needed. The probability of success, p, is constant for each trial. Mathematically, the PDF for a Bernoulli random variable is

$$f(x) = p^x(1-p)^{1-x} \tag{8.39}$$

The mean and variance of the distribution are

$$E[x] = p \tag{8.40}$$

$$\text{Var}\,[x] = p(1-p) \tag{8.41}$$

The PDF for the binomial distribution is given by

$$f(x) = \binom{n}{x} p^x(1-p)^{n-x} \tag{8.42}$$

where

$$\binom{n}{x} = \frac{n!}{(n-x)!x!} \tag{8.43}$$

The mean and variance for the binomial distribution are

$$E[x] = np \tag{8.44}$$

$$\text{Var}\ [x] = np(1-p) \tag{8.45}$$

Example. Suppose the next bomber is designed such that it has three engines. Assume that for the plane to complete its mission, one engine must operate. If the probability that an engine fails during flight is $(1-p)$ and each engine is assumed to fail independently, then what is the probability that a plane returns safely.

Solution. Since each engine is assumed to fail independently, then the number of engines remaining operative can be modeled as a binomial random variable. Hence, the probability that the three-engine next-generation bomber makes a successful flight is

$$= \binom{3}{1} p(1-p)^2 + \binom{3}{2} p^2(1-p)^1 + \binom{3}{3} p^3$$

$$= \frac{3!}{2!1!} p(^1-p)^2 + \frac{3!}{1!2!} p^2(1-p) + \frac{3!}{0!3!} p^3$$

$$= 3p(1-p)^2 + 3p^2(1-p) + p^3$$

Geometric Distribution The geometric distribution is commonly used to model the number of cycles to failure for items that have a fixed probability of failure, p, associated with each cycle. In the testing arena this distribution has been used to model the distribution of the number of trials until the first success. The PDF for this distribution is given by

$$f(x) = (1-p)^{x-1} p. \tag{8.46}$$

The mean and variance for the geometric distribution are given by

$$E[x] = \frac{1}{p} \tag{8.47}$$

$$\text{Var} = \frac{1-p}{p^2} \tag{8.48}$$

Example. A manufacturer of a new dipole light switch has tested 10 switches to failure. The number of on/off cycles until failure for each switch is given below:

Switch	# Cycles	Switch	# Cycles
1	30,000	6	75,000
2	35,000	7	80,000
3	40,000	8	82,000
4	56,000	9	83,000
5	70,000	10	84,500

The marketing department believes they can improve market share if they advertise their switch as a high-reliability switch. They want to advertise that their product has reliability greater than 98% for a 5-year period. They assume that the switch is cycled three times a day, 365 days a year. Using the estimated value of p, calculate the point estimate for the reliability of the switch for a 5-year period.

Solution.

$$\hat{p} = \frac{n}{\sum_{i=1}^{10} C_i} = \frac{10}{635500} = 1.5736\text{E} - 05$$

$$E[C] = \frac{1}{\hat{p}} = 63550$$

$$P[C > (3 \times 365 \times 5)] = P[C > 5475] = 1 - P[C \leq 5475]$$

$$= 1 - \sum_{x=1}^{5475} (1-p)^{x-1} p$$

$$= 1 - 0.0826 = 0.9174$$

8.4 BASIC SYSTEM MODELS

Most systems are composed of many subsystems, each of which can be composed of hundreds or thousands of components. In general, reliability analysis is performed at the lowest levels and the results then aggregated into a system level estimate. This is done to save time and money during the development process. System level testing cannot be accomplished until the entire system is designed and assembled. Waiting to test components and subsystems until the entire system is assembled is not time- or cost-effective. Usually, a system's functional and physical decompositions are used to help construct a system level reliability block diagram. We utilize this structure to compute system level reliability performance in terms of the component and subsystem reliabilities. In the next couple of sections, we will use basic probability concepts to explore some of the common system structures utilized by design engineers. We will begin with the two basic structures, series and parallel systems.

8.4.1 Series System

Assume that a system consists of N functionally independent components, each with individual measures of reliability $R_1, R_2, \ldots, R_N$ for some specified period of performance in a specified environment. The set of components constitute a series system if the success of the system depends on the success of all of the components. If a single component fails, then the system fails. The reliability block diagram for this situation is given in Figure 8.9. The series system reliability, which is the probability of system success, is given by

$$R_s(t) = R_1(t) \cdot R_2(t) \cdot \ldots \cdot R_n(t) = \prod_{i=1}^{N} R_i(t) \qquad (8.49)$$

It should be noted that the system reliability is always smaller than the worst component. Thus, the worst component in a series system provides a loose upper bound on the system reliability.

Example. Four identical components form a series system. The system reliability is supposed to be greater than 0.90 for a specified period. What should the minimum reliability be for each of the components to achieve the desired system reliability for that specified time?

Solution.

$$R_s = R_1 \cdot R_2 \cdot R_3 \cdot R_4 = R^4 \geq 0.90$$
$$R \geq (0.90)^{1/4} \geq 0.974$$

8.4.2 Parallel System

Assume that a system consists of N functionally independent components, each with individual measures of reliability $R_1, R_2, \ldots, R_N$ for some specified period of performance in a specified environment. The set of components constitute a parallel

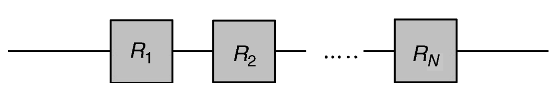


Figure 8.9 Series system.

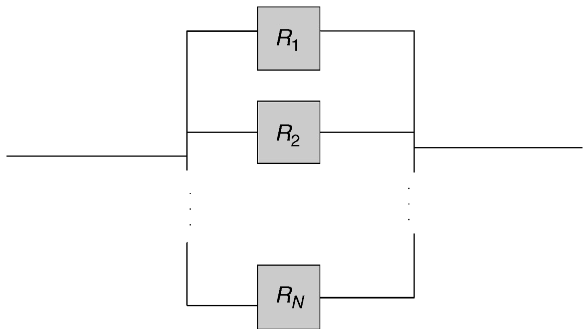


Figure 8.10 Parallel system.

system if the success of the system depends on one or more of the components operating successfully. If a single component survives, then the system succeeds. The reliability block diagram for this situation is given in Figure 8.10. The reliability for a parallel system can be expressed as the probability that at least one component in the system survives. Mathematically, this can be expressed as one minus the probability that all of the components fail. The system reliability for a parallel set of components is given by

$$R_s(t) = 1 - [(1 - R_1(t))(1 - R_2(t)) \ldots (1 - R_N(t))] \tag{8.50}$$

It should be noted that the system reliability for a parallel system is always larger than the reliability of the best component. Thus, the reliability of the best component in a parallel system provides a loose lower bound on the system reliability.

Example. Suppose a system consists of three time-dependent components arranged in parallel, one component has a Weibull failure distribution with a scale parameter of 1000 h, and a shape parameter of 2, the other two components have exponential failure distributions with $\lambda_1 = 0.005$ and $\lambda_2 = 0.0001$. Derive an expression for the system reliability and compute the reliability of the system for the first 1000 h.

Solution.

$$R_s(t) = 1 - \left[\left(1 - e^{-\left(\frac{t}{\eta}\right)^{\beta}}\right)\left(1 - e^{-\lambda_1 t}\right)\left(1 - e^{-\lambda_2 t}\right)\right],$$

$$R_s(1000) = 1 - \left[\left(1 - e^{-\left(\frac{1000}{1000}\right)^{2}}\right)\left(1 - e^{-0.005(1000)}\right)\left(1 - e^{-0.0001(1000)}\right)\right]$$

$$= 0.94025$$

8.4.3 *K*-out-of-*N* Systems

K-out-of-N systems provide a very general modeling structure. It includes both series systems and parallel systems as special cases. In this structure, we assume that a system consists of N functionally independent components each with identical measures of reliability, for some specified period of performance in a specified environment. The set of components constitute a K-out-of-N structure if the success of the system depends on having K or more of the components operating successfully. If less than K components are operating, then the system has failed. The reliability for a K-out-of-N system can be expressed as the probability that at least K components in the system survive. Mathematically, this can be modeled as an application of the binomial distribution. The system reliability for a K-out-of-N system is given by

$$R_s = \sum_{i=k}^{N} \binom{N}{j} R^j (1 - R)^{N-j} \tag{8.51}$$

An N-out-of-N system is equivalent to a series system and a 1-out-of-N system is equivalent to a parallel system.

8.4.4 Complex Systems

Most systems are complex combinations of series and parallel system structures of components and subsystems. Consider the following bridge network of functionally independent components with individual reliabilities R_1, R_2, R_3, R_4, R_5 for a specified period of performance in a specified environment. The components are arranged in what is commonly called a bridge network (Figure 8.11). The system reliability for this structure can be constructed by the method of system decomposition. Decomposing the system around component 3 and using conditional probability, we get the following expression where C_3 is the event that component 3 is working, $\bar{C}_3$ is the event that component 3 has failed, and S is the event that the system is working:

$$R_s = P(S|C_3)P(C_3) + P(S|\bar{C}_3)P(\bar{C}_3) \tag{8.52}$$

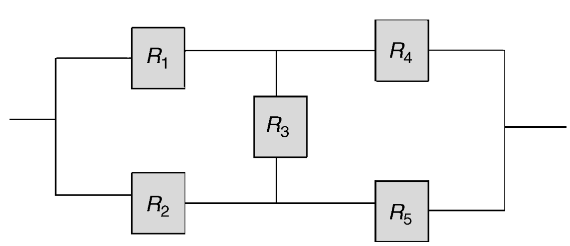


Figure 8.11 Bridge network.

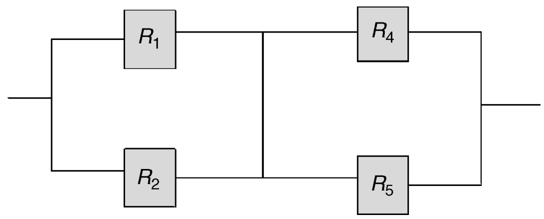


Figure 8.12 System structure when C_3 is working.

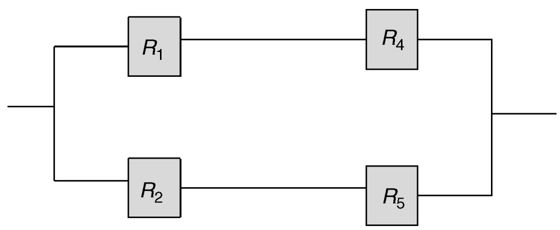


Figure 8.13 System structure when C_3 has failed.

If component 3 is working, then the system structure reduces to the series arrangement of parallel systems as shown in Figure 8.12. The reliability for the system is given by the following:

$$R_s(t)|C_3 = \{1-[(1-R_1(t))(1-R_2(t))]\}\cdot\{1-[(1-R_4(t))(1-R_5(t))]\} \tag{8.53}$$

If component 3 fails, then the system structure reduces to a parallel arrangement of series components as shown in Figure 8.13. The reliability for this system is given by the following:

$$R_s(t)|\bar{C}_3 = 1-[(1-R_1(t)R_4(t))(1-R_2(t)R_5(t))] \tag{8.54}$$

Thus, the unconditional system reliability is given by the following relationship:

$$\begin{aligned} R_s(t) = {} & \{1-[(1-R_1(t))(1-R_2(t))]\}\cdot\{1-[(1-R_4(t))(1-R_5(t))]\}\cdot R_3(t) \\ & + (1-[(1-R_1(t)R_4(t))(1-R_2(t)R_5(t))])\,(1-R_3(t)) \end{aligned} \tag{8.55}$$

Example. Suppose all five components in the bridge structure are identical. Assume that the components have a Weibull distribution with a scale parameter of 10,000 h and a shape parameter of 3. What is the 5000-h reliability for the bridge system?

Solution. The first step is to calculate the component reliability for a 5000-h period.

$$R_1(5000) = e^{-\left(\frac{5000}{10000}\right)^3} = 0.8825$$

Substituting this probability into the system reliability equation, we get the following:

$$\begin{aligned} R_s(5000) = \{&1 - [(1 - 0.8825)(1 - 0.8825)]\} \\ &\cdot \{1 - [(1 - 0.8825)(1 - 0.8825)]\}(0.8825) \\ &+ (1 - [(1 - (0.8825)(0.8825))(1 - (0.8825)(0.8825))]) \\ &\times (1 - 0.8825) = 0.970 \end{aligned}$$

8.5 COMPONENT RELIABILITY IMPORTANCE MEASURES

One of the general concerns associated with a complex system of components is which component is the most important for the success of the system. This question often arises when system designers are trying to decide which component should be improved first if resources become available [14]. To determine which component to improve first, we take the partial derivative of the system reliability function, $R_s(t)$, with respect to each of the component reliabilities, $R_i(t), i = 1, 2, \ldots, N$.

8.5.1 Importance Measure for Series System

Assume that a system consists of N functionally independent components, each with individual measures of reliability $R_1, R_2, \ldots, R_N$ for some specified period of performance in a specified environment, and these components are functionally arranged as a series system as shown in Figure 8.9. The component importance measure is calculated as follows:

$$\frac{\partial R_s(t)}{\partial R_i(t)} = \frac{\partial}{\partial R_i(t)}[R_1(t)R_2(t)\ldots R_i(t)\ldots R_N(t)] = \prod_{\substack{j=1 \\ j \neq i}}^{N} R_j(t) = \frac{R_s(t)}{R_i(t)} \tag{8.56}$$

This suggests that the most important component R_{i^*} is the component that has the largest importance measure. Thus,

$$\frac{R_s(t)}{R_{i^*}} = \max \frac{R_s(t)}{R_i(t)} \tag{8.57}$$

$$R_{i^*} = \min R_i(t) \tag{8.58}$$

This tells us that for a series system we should improve the design of the least reliable component.

8.5.2 Importance Measure for Parallel System

Assume that a system consists of N functionally independent components, each with individual measures of reliability $R_1, R_2, \ldots, R_N$ for some specified period of performance in a specified environment, and these components are functionally arranged as a parallel system as shown in Figure 8.10. The component importance measure is calculated as follows:

$$\frac{\partial R_s(t)}{\partial R_i(t)} = \frac{\partial}{\partial_i(t)}\left[1 - \prod_{i-1}^{N}[1 - R_i(t)]\right] = \prod_{\substack{j=1 \\ j \neq i}}^{N}(1 - R_j(t)) = \frac{1 - R_s(t)}{1 - R_i(t)} \tag{8.59}$$

The most important component, R_{i^*}, is the component that has the largest importance measure. Thus,

$$\frac{(1 - R_s(t))}{(1 - R_{i^*})} = \max\left(\frac{1 - R_s(t)}{1 - R_i(t)}\right) \tag{8.60}$$

$$R_{i^*} = \max R_i(t) \tag{8.61}$$

This tells us that for a parallel system we should improve the design of the most reliable component.

8.6 RELIABILITY ALLOCATION AND IMPROVEMENT [5]

In Section 8.5, we saw that the reliability importance measure can be used to identify which component should be improved to maximize the system reliability. Unfortunately, system designers generally do not operate in an unconstrained environment. Often, depending on the system, there are costs associated with improving a component's reliability. Costs can take a variety of forms such as dollars, weight, volume, quantity, and so on. If we let the cost per unit reliability of the ith component be C_i, then the incremental cost to improve the reliability of component i is given by

$$R_i(t) + \Delta_i = C_i \Delta_i \tag{8.62}$$

Now, if we assume that the system under study is a series system, then the improvement in system reliability as a result of the improvement in component i is given by

$$R_s^*(t) = \prod_{\substack{j=1 \\ j \neq i}}^{N} R_j(t)[R_i(t) + \Delta_i] = R_s(t) + \frac{R_s(t)}{R_i(t)}\Delta_i \tag{8.63}$$

If we assume that $R_s^*(t)$ can also be obtained by increasing the reliability of one of the other components by an incremental amount Δ_j at a cost of $C_j \Delta_j$, then the reliability of the improved system is given by

$$R_s^*(t) = R_s(t) + \frac{R_s(t)}{R_j(t)}\Delta_j = R_s(t) + \frac{R_s(t)}{R_i(t)}\Delta_i \tag{8.64}$$

Therefore,

$$\frac{\Delta_j}{R_j(t)} = \frac{\Delta_i}{R_i(t)} \tag{8.65}$$

Multiplying both sides by the associated costs for the incremental improvement in reliability and rearranging terms, we get

$$C_i \Delta_i = \frac{C_i}{C_j}\frac{R_i(t)}{R_j(t)} C_j \Delta_j \tag{8.66}$$

Now, for $C_i \Delta_i < C_j \Delta_j$ to be true, the following relationship must hold:

$$\frac{C_i}{C_j}\frac{R_i(t)}{R_j(t)} < 1 \tag{8.67}$$

$$C_i R_i(t) < C_j R_j(t) \tag{8.68}$$

Therefore, to improve the system reliability for a series system to $R_s^*(t)$ at minimum cost, the component that should be improved is the component that satisfies the following relationship:

$$C_{i^*} R_{i^*}(t) = \min\ C_i R_i(t) \tag{8.69}$$

Example. An advanced optical package has been designed for the next generation weather satellite for the National Weather Service. The basic optical package can be modeled functionally as a three-component series system. Each component has a 5-year mission reliability of 0.99, 0.995, and 0.98, respectively. Due to the design constraints, there is a weight constraint for the optical package of 1000 lb. Suppose that the reliability of the system can be improved by adding redundant components to the optical package. Your task is to determine the optimal combination of components that maximize reliability at minimal cost subject to the 1000-lb weight constraint. Assume the weights of the components are 150, 200, and 300 lb, respectively. The initial reliability of the system is given by

$$R_s = (0.99)(0.995)(0.98) = 0.9653$$

Solution. Let us assume that the effectiveness of the satellite is measured by its reliability. Let n_1, n_2, and n_3 represent the number of each type of component used in the recommended optical package. The initial weight for the system is given by $(150 + 200 + 300) = 650$ lb. To proceed, we should investigate the cost in terms of weight per unit of reliability improvement for each of the components. We will start with component 1 by investigating the improvement in reliability for the function associated with component 1 if a redundant component is added:

$$R_1^* = 1 - (1 - 0.99)(1 - 0.99) = 0.9999$$

Thus, the improvement in the contribution to the reliability for component 1 is 0.0099 at a cost of 150 lb. Therefore,

$$C_1 = \frac{150}{0.0099} = \frac{15151.51 \text{ lb}}{\text{unit of reliability}}$$

Similarly, we can calculate the same costs for components 2 and 3.

$$C_2 = \frac{40201 \text{ lb}}{\text{unit of reliability}}$$

$$C_3 = \frac{15306.12 \text{ lb}}{\text{unit of reliability}}$$

Thus, we should add a redundant component 1. Adding an additional component 1 increases the weight of the optical package to 800 lb. Now, we can compare the cost of adding a third component 1 or a second component 2. Since we only have 200 1b available, we cannot consider adding an additional component 3.

$$R_i^{**} = 1 - (1 - 0.99)(1 - 0.99)(1 - 0.99) = 0.999999$$

$$C_1^{**} = \frac{150}{0.000099} \frac{1515151.51 \text{ lb}}{\text{unit of reliability}}$$

Given this enormous cost associated with adding a third component 1, the best solution is to add an additional component 2. Thus, the reliability for the final configuration is given by

$$R_s^* = (0.9999)(0.999975)(0.98) = 0.979877$$

8.7 MARKOV MODELS OF REPAIRABLE SYSTEMS [15]

In this section we focus our modeling efforts on using continuous-time Markov chains (CTMCs) to model *repairable systems*. A *repairable system* (RS) is a system that, after failure, can be restored to a functioning condition by some maintenance action other than replacement of the entire system [16]. A CTMC is a stochastic process that moves from state to state in accordance with a discrete-time Markov chain (DTMC). It differs from a DTMC in that the amount of time it spends in each state before it transitions to another state is exponentially distributed [17]. Like a DTMC, it has the Markovian property whereby the "future is independent of the past, given the present." In this section, we assume that the CTMC has stationary (homogeneous) transition probabilities (i.e., $P[X(t+s)=j|X(s)=i]$ is independent of s). Ross [18] formally defines a CTMC as a stochastic process where each time it enters state i,

1. The amount of time it spends in state i before it transitions into a different state is exponentially distributed with a rate v_i.
2. When the process leaves state i, it will enter state j with some probability p_{ij}, where $\sum_{j \neq i} P_{ij} = 1$.

8.7.1 Kolmogorov Differential Equations

In the discrete-time case, $p_{ij}(n)$ represents the probability of going from state i to j in n transitions. In the continuous case we are interested in $p_{ij}(t)$, which represents the probability that a process currently in state i will be in state j in t time units from the present. Mathematically, we denote this by

$$p_{ij}(t) = P[X(t+s) = j|X(s) = i] \tag{8.70}$$

In the continuous-time case, we can define the intensity at which transitions occur by examining the *infinitesimal transition rates*:

$$-\frac{d}{dt}p_{ij}(0) = \lim_{t\to 0}\frac{1-p_{ij}(t)}{t} = v_i \tag{8.71}$$

$$-\frac{d}{dt}p_{ij}(0) = \lim_{t\to 0}\frac{p_{ij}(t)}{t} = q_{ij} \tag{8.72}$$

where v_i represents the rate at which we leave state i, and q_{ij} represents the rate at which we move from state i to state j. However, for small $\Delta t, q_{ij}\Delta t$ can be interpreted as the probability of going from state i to state j in some small increment of time Δt, given we started in state i. Using the transition intensities, as well as making use of the Markovian property, one can derive the Kolmogorov differential equations for $p_{ij}(t)$. The backward and forward Kolmogorov equations are given by Equations (8.73) and (8.74). These equations can be used to derive the transient probabilities of a CTMC. This is best illustrated through the use of an example.

$$\frac{d}{dt}p_{ij}(t) = \sum_{k \neq i} q_{ik} p_{kj}(t) - v_i p_{ij}(t) \tag{8.73}$$

$$\frac{d}{dt}p_{ij}(t) = \sum_{k \neq j} q_{kj} p_{ik}(t) - v_i P_{ij}(t) \tag{8.74}$$

8.7.2 Transient Analysis

Consider a single-component system that fails according to an exponential failure distribution with rate λ and whose repair time is exponentially distributed with rate μ. This system can be in one of the two states. It can be working (state 0) or can fail and be undergoing repair (state 1). A state transition diagram for this system is given in Figure 8.14. This diagram shows the states and the associated transition rates between the states. Using the state transition diagram and the Kolmogorov forward equation, Equation (8.74), we can derive the transition probabilities for the CTMC.

$$\frac{d}{dt}p_{ij}(t) = \sum_{k \neq j} q_{ik} p_{ik}(t) - v_i p_{ij}(t) \tag{8.75}$$

$$\frac{d}{dt}p_{00}(t) = \sum_{k \neq j} q_{10} p_{01}(t) - v_0 p_{00}(t) \tag{8.76}$$

$$\frac{d}{dt}p_{00}(t) = \mu p_{01}(t) - \lambda p_{00}(t) \tag{8.77}$$

$$\frac{d}{dt}p_{00}(t) = \mu[1 - p_{00}(t)] - \lambda p_{00}(t) \tag{8.78}$$

$$\frac{d}{dt}p_{00}(t) = \mu - (\mu + \lambda) p_{00}(t) \tag{8.79}$$

$$\frac{d}{dt}p_{00}(t) + (\mu + \lambda) p_{00}(t) = \mu \tag{8.80}$$

Solving this differential equation, we obtain

$$e^{(\lambda+\mu)t}\left[\frac{d}{dt}p_{00}(t) + (\mu + \lambda) p_{00}(t)\right] = \mu e^{(\lambda+\mu)t} \tag{8.81}$$

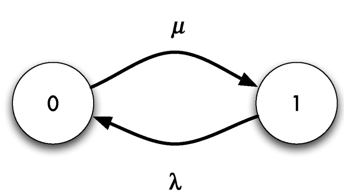


Figure 8.14 Single-component state transition diagram.

$$\frac{d}{dt}\left[e^{(\lambda+\mu)t}p_{00}(t)\right] = \mu e^{(\lambda+\mu)t} \tag{8.82}$$

$$e^{(\lambda+\mu)t}p_{00}(t) = \frac{\mu}{\lambda+\mu}e^{(\lambda+\mu)t} + c \tag{8.83}$$

$$\text{since } p_{00}(t) = 1, \quad c = \frac{\mu}{\lambda+\mu} \tag{8.84}$$

Therefore, $p_{ij}(t)$ for $i = j$ are given below:

$$p_{00}(t) = \frac{\lambda}{\lambda+\mu}e^{(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu} \tag{8.85}$$

$$p_{11}(t) = \frac{\mu}{\lambda+\mu}e^{(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu} \tag{8.86}$$

Note that $p_{00}(t)$ represents the probability that the system is operating at time t. This is also known as the *system availability* $A(t)$. If we take the limit of $p_{00}(t)$ as t goes to infinity, we get the limiting or steady state availability. The limiting availability is given below:

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} p_{00}(t) = \frac{\mu}{\lambda+\mu} \tag{8.87}$$

In general, we can establish a set of N first-order differential equations that characterize the probability of being in each state in terms of the transition probabilities to and from each state. Mathematically, the set of N first-order differential equations is summarized in matrix form in Equation (8.88), and the general form of the solution to this set of differential equations is given by Equation (8.89) [19].

$$\frac{d\underline{P}(t)}{dt} = [T_R]\underline{P}(t) \tag{8.88}$$

$$\underline{P} = \exp[T_R]t \cdot \underline{P}(0) \tag{8.89}$$

In Equation (8.89), T_R is the rate matrix. For our simple single-system example, using Figure 8.14, we get the following rate matrix.

$$T_R = \begin{pmatrix} -\lambda & \mu \\ \lambda & -\mu \end{pmatrix} \tag{8.90}$$

To solve the set of differential equations one must compute the matrix exponential. There are several different approaches to computing the matrix exponential. Two such methods include the infinite series method and the eigenvalue/eigenvector approach. Such routines are readily available in many of the commercially available mathematical analysis packages (Maple™, Mathematica®, and MATLAB®). In many instances, as the problem complexity increases, the Kolmogorov differential equations cannot be solved explicitly for the transition probabilities. In such cases, we will use numerical solution techniques; we might use simulation (see reference 20), or for a variety of reasons, focus our attention on the steady-state performance of the system.

8.7.3 Steady-State Analysis

For many systems, it is the limiting availability (aka, steady-state availability), $A(\infty)$, that is of interest. Another common name for the steady state availability is the uptime ratio. For example, the uptime ratio is of critical importance in a production facility. Similarly, for a communication system, the average message transfer rate will be the design transfer rate times the uptime ratio. So knowing the uptime ratio is essential for analyzing the performance of many systems.

We can compute the steady-state probabilities by making use of the following:

$$\text{let } \rho_j = \lim_{t\to\infty} p_{ij}(t) \tag{8.91}$$

We can then state the following:

$$v_j \rho_j = \sum_j p_i q_{ij} \; \forall j = 0, 1, 2, \ldots, N \tag{8.92}$$

$$\sum_j \rho_j = 1 \tag{8.93}$$

Expression 8.92 is called the "balance" equations. The balance equations state that the rate into each state must be equal to the rate out of each state for the system to be in equilibrium. Equation (8.93) states that we must be in some state, and the sum of the probabilities associated with each state must be equal to 1. Using $(N-1)$ of the balance equations and Equation (8.93), we can easily derive the steady-state probabilities for each state.

8.7.4 CTMC Models of Repairable Systems

In this section we illustrate how to model and analyze a variety of repairable systems using continuous-time Markov chains. We focus specifically on the single machine cases. Consider a single repairable machine. Let T_i denote the duration of the ith interval of machine function, and assume that $T_1, T_2, \ldots$ is a sequence of independent identically distributed exponential random variables having failure rate λ. Upon failure, the machine is repaired. Let D_i denote the duration of the ith machine repair, and assume $D_1, D_2, \ldots$ is a sequence of independent identically distributed exponential random variables having repair rate μ. Assume that no preventive maintenance is performed on the machine.

Recall that $X(t)$ denotes the state of the machine at time t. Under these assumptions, $\{X(t), t \geq 0\}$ transitions among two states, and the time between transitions is exponentially distributed. Thus, $\{X(t), t \geq 0\}$ is a CTMC having the rate diagram shown in Figure 8.15. We can easily analyze the "steady-state" behavior of the CTMC. Let ρ_j denote the long-run probability that the CTMC is in state j. We use balance equations to identify these probabilities. Each state of the CTMC has

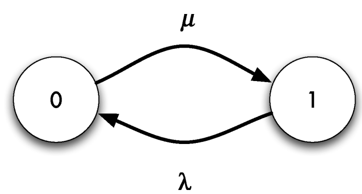


Figure 8.15 Single-machine rate diagram.

a balance equation that corresponds to the identity “rate in” = “rate out.” For the rate diagram in Figure 8.15, the balance equations are

$$\text{state 0:} \quad \lambda\rho_1 = \mu\rho_0 \tag{8.94}$$

$$\text{state 1:} \quad \mu\rho_0 = \lambda\rho_1 \tag{8.95}$$

These balance equations are equivalent, so we need an additional equation to solve for ρ_0 and ρ_1. We use the fact that the steady-state probabilities must sum to 1.

We then use the two equations to solve for the two unknowns.

$$\rho_1 = \frac{\mu}{\lambda + \mu} \tag{8.96}$$

$$\rho_2 = \frac{\lambda}{\lambda + \mu} \tag{8.97}$$

Note that ρ_1 is equivalent to the steady-state availability found from taking the limit of the transient probabilities in Equation (8.87).

Let us consider another single-machine example. Just like the first example, let T_i denote the duration of the ith interval of machine function, and assume $T_1, T_2, \ldots$ is a sequence of independent identically distributed exponential random variables having failure rate λ. Upon failure, the machine is repaired. But this time, each repair requires two distinct repair operations, A and B. Assume that the duration of repair is exponentially distributed with rate μ_j where $j = (A, B)$. For this example, assume that there are enough resources available so that the repairs can be done concurrently.

This problem differs significantly from the first in that we now have four different states. State 0 is when the machine is operating; State 1 is when the machine is down and we are awaiting the completion of repair process A; State 2 is when the machine is down and we are awaiting the completion of repair process B; and State 3 is when the machine is down and we are awaiting the completion of both repair processes. The rate diagram for this model is shown in Figure 8.16. Using the rate diagram, the set of balance equations can be written as

$$\mu_A\rho_1 + \mu_B\rho_2 = \lambda\rho_0 \tag{8.98}$$

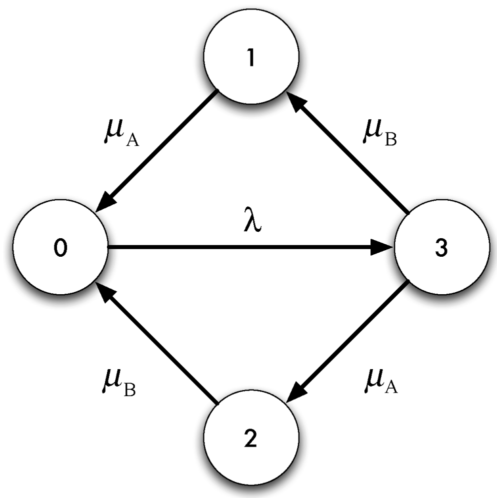


Figure 8.16 Rate diagram for multiple repair process.

$$\mu_B \rho_3 = \mu_A \rho_1 \tag{8.99}$$

$$\mu_A \rho_3 = \mu_B \rho_2 \tag{8.100}$$

$$\lambda \rho_0 = (\mu_A + \mu_B)\rho_3 \tag{8.101}$$

Using the balance equations in conjunction with the total probability equation, we can solve for the individual steady-state values for each of the states. The state of interest is state 0, as it represents the system steady-state availability. Suppose the system described above has a mean time between failure of 100 h, and the mean repair time for process A is 10 hours and for process B it is 5 h. Determine the system steady-state availability. Using the balance equations, we can derive Equation (8.102) and determine that the system has a steady state availability 0.96.

$$\rho_0 = \frac{\mu_A + \mu_B}{\lambda}\left(\frac{\mu_A + \mu_B}{\lambda} + \frac{\mu_A}{\mu_B} + \frac{\mu_B}{\mu_A}\right)^{-1} = \frac{0.1 + 0.2}{0.01}(32.5)^{-1} = 0.9231 \tag{8.102}$$

8.7.5 Modeling Multiple Machine Problems

Suppose the repairable "system" of interest actually consists of m identical machines that correspond to the assumptions of the previous section. To model this situation using a CTMC, we must first modify our definition of the system state X(t). Let $X(t)$ now represent the number of machines functioning at time t. However, $\{X(t), t_0\}$ is still a CTMC because the number of states is discrete and transition times are exponentially distributed.

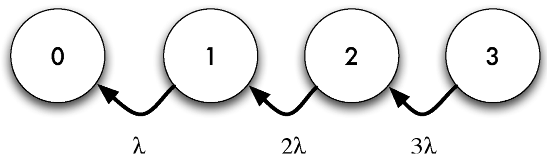


Figure 8.17 Partial rate diagram when $m = 3$.

A partial rate diagram for the case in which $m = 3$ is constructed in Figure 8.17. Note that the repair rates on the diagram depend on s, the number of maintenance technicians in the system. Note that we assume each repair requires exactly one technician.

Suppose $m = 3, s = 2, \lambda = 2$ failures per day, and $\mu = 10$ repairs per day. The completed rate diagram for the resulting CTMC is given in Figure 8.18. Note that the transition rate from state 3 to state 2 is 6. This is because three machines are functioning; each has a failure rate of 2, so the total failure rate is 6. Note that the transition rate from state 1 to state 2 is 20. This is because two machines have failed; this implies that both technicians are repairing at a rate of 10, so the total repair rate is 20. The corresponding balance equations are

$$\text{state 0:} \quad 2\rho_1 = 20\rho_0 \rightarrow \rho_1 = 10\rho_3 \tag{8.103}$$

$$\text{state 1:} \quad 20\rho_0 + 4\rho_2 = 22\rho_1 \rightarrow \rho_2 = 50\rho_0 \tag{8.104}$$

$$\text{state 3:} \quad 10\rho_2 = 6\rho_3 \rightarrow \rho_3 = \frac{250}{3}\rho_0 \tag{8.105}$$

The solution to these equations is

$$\rho_0 = \frac{3}{433}$$
$$\rho_1 = \frac{30}{433}$$
$$\rho_2 = \frac{150}{433}$$
$$\rho_3 = \frac{250}{433}$$

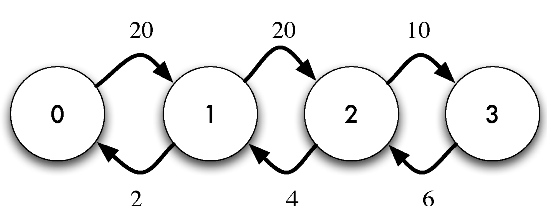


Figure 8.18 Completed rate diagram example.

Note that we can use the steady-state probabilities to obtain both machine and technician utilization. For example, the average number of machines functioning is

$$\text{AVG M } = 0\rho_0 + 1\rho_1 + 2\rho_2 + 3\rho_3 = 2.49 \text{ (83\% utilization)} \tag{8.106}$$

and the average number of busy technicians is

$$\text{AVG S } = 2\rho_0 + 2\rho_1 + 1\rho_2 + 0\rho_3 = 0.50 \text{ (25\% utilization)} \tag{8.107}$$

At this point, a reasonable question is, How many technicians should be assigned to maintain these machines, that is, should s equal 1, 2, or 3? To answer this question, first we modify the CTMC for the cases in which s equals 1 and s equals 3. Then, we compute the steady-state probabilities and utilization measures for each case. Then, we can use an economic model to determine the optimal value of s. Let c_s denote the cost per day of employing a technician, let c_d denote the cost per day of machine downtime, and let C denote the cost per day of system operation. Then

$$E(C) = c_s S + c_d(m - \text{ AVG M}) \tag{8.108}$$

For our example, suppose $c_s = \$200$ and $c_d = \$2500$. Then, $E(C) = \$1664.25$. For $s = 1$, the rate diagram is provided in Figure 8.19. The resulting steady-state probabilities are $\rho_0 = 0.0254, \rho_1 = 0.1271, \rho_2 = 0.3178$, and $\rho_3 = 0.5297$. Furthermore, $\text{AVG M} = 2.3518$ and $E(C) = \$1820.50$. For $s = 3$, the rate diagram is provided in Figure 8.20.

The resulting steady-state probabilities are $\rho_0 = 0.0046, \rho_1 = 0.0694, \rho_2 = 0.3472$, and $\rho_3 = 0.5787$. Furthermore, $\text{AVG M} = 2.4999$ and $E(C) = \$1850.25$. Thus, $s = 2$ is the optimal staffing level.

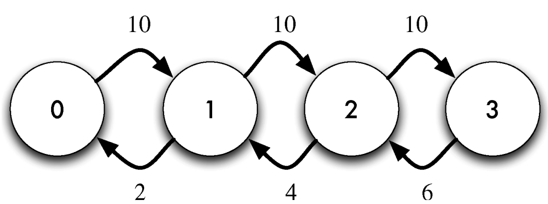


Figure 8.19 Example rate diagram, $s = 1$.

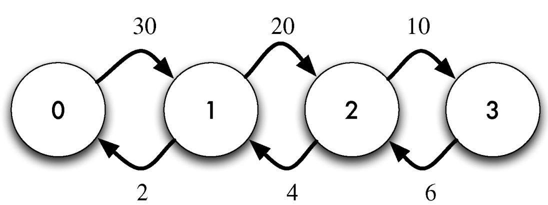


Figure 8.20 Example rate diagram, $s = 3$.

Nonidentical Machine Problems An interesting variation of the multiple-machine problem is the case in which the machines are not identical. For example, suppose a system contains two machines of different types that are repaired upon failure (no PM), and suppose two equally trained technicians maintain these machines. Let λ_i denote the failure rate for machine i, and let μ_i denote the repair rate for machine i. Modeling this problem using a CTMC requires a more complex definition of the system state.

$$X(t) = \begin{cases} 1,1 & \text{both machines are functioning} \\ 1,0 & \text{machine 1 is functioning, machine 2 is down} \\ 0,1 & \text{machine 1 is down, machine 2 is functioning} \\ 0,0 & \text{both machines are down} \end{cases} \tag{8.109}$$

The corresponding rate diagram is provided in Figure 8.21. For example, suppose $\lambda_1 = 1, \mu_1 = 8, \lambda_2 = 2,$ and $\mu_2 = 10$. Construction and solution of the balance equations yields $\rho_{1,1} = 0.7407, \rho_{1,0} = 0.1481, \rho_{0,1} = 0.0926,$ and $\rho_{0,0} = 0.0185$. The steady-state probabilities can then be used to compute machine availability:

$$\text{machine 1:} \quad \rho_{1,1} + \rho_{1,0} = 0.8889 \tag{8.110}$$

$$\text{machine 2:} \quad \rho_{1,1} + \rho_{0,1} = 0.8333 \tag{8.111}$$

and machine and technician utilization

$$\text{AVG M} = 2\rho_{1,1} + \rho_{1,0} + \rho_{0,1} = 1.7222 \ (86\%) \tag{8.112}$$

$$\text{AVG S} = \rho_{1,0} + \rho_{0,1} + 2\rho_{0,0} = 0.2778 \ (14\%) \tag{8.113}$$

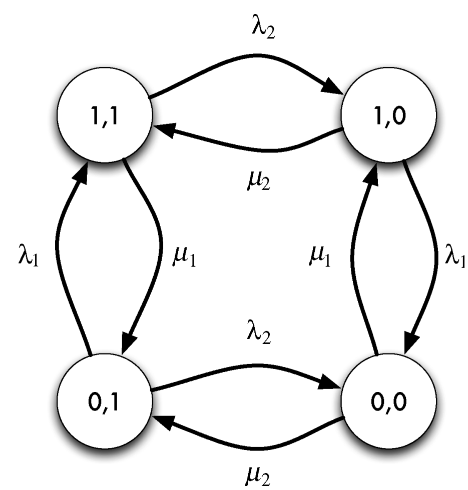


Figure 8.21 Rate diagram for two different machines.

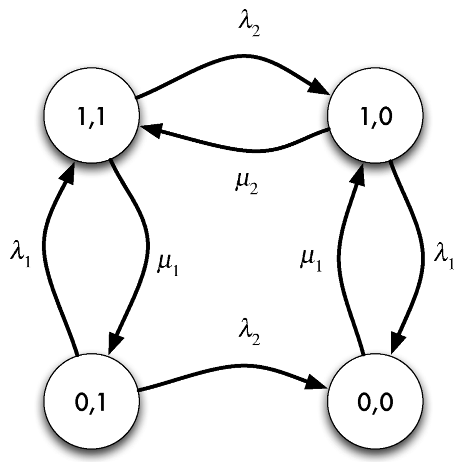


Figure 8.22 Rate diagram, multiple-machine, single technician, prioritized repair.

Another interesting variation occurs if we only have one technician available. Suppose, in addition to only having one technician, we assume that machine 1 has higher priority than machine 2. Thus, if machine 2 is being repaired and machine 1 fails, the technician will leave machine 2 to go work on machine 1. Once machine 1 is repaired, the technician will resume work on machine 2. The rate diagram for this situation is given in Figure 8.22. Notice that the rate diagram for this situation is similar to the previous example. The key difference is that when the system is in a state $(0, 0)$, it can only transition to state $(1, 0)$ because of the fact that machine 1 has a higher priority than machine 2 and therefore the technician must repair machine 1 as soon as it fails. Let us assume that parameters have the same values as in the previous example; therefore $\lambda_1 = 1, \mu_1 = 8, \lambda_2 = 2,$ and $\mu_2 = 10$. Construction and solution of the balance equations yields $\rho_{1,1} = 0.72859, \rho_{1,0} = 0.16029, \rho_{0,1} = 0.07285$, and $\rho_{0,0} = 0.03825$. The steady-state probabilities can then be used to compute machine availability:

$$\text{machine 1:} \quad \rho_{1,1} + \rho_{1,0} = 0.8889 \tag{8.114}$$

$$\text{machine 2:} \quad \rho_{1,1} + \rho_{0,1} = 0.8015 \tag{8.115}$$

and machine and technician utilization

$$\text{AVG M} \ = 2\rho_{1,1} + \rho_{1,0} + \rho_{0,1} = 1.69 \ (84.52\%) \tag{8.116}$$

$$\text{AVG S} \ = \rho_{1,0} + \rho_{0,1} + \rho_{0,0} = 0.2714 \ (27\%) \tag{8.117}$$

Examining the results, we find that despite the fact that we have only one technician, we can keep machine 1's steady-state availability at the same level as if we had two technicians by giving it priority when it fails. As expected, machine 2's steady-state availability is reduced from 0.8333 to 0.8015. Another important point to take note of is that the technician utilization increases from 14% to 27%. Depending on the costs associated with an increase in downtime for machine 2, using a priority maintenance process may be more cost-effective than having two technicians available to perform maintenance.

8.7.6 Conclusions

In this section, we presented some basic techniques for modeling multistate system deterioration using Markov chains. Using Markov chains as an analysis tool for reliability and maintainability has both advantages and disadvantages, depending on the complexity of the system. A key advantage is the modeling flexibility that Markov chains give to an analyst to perform relatively quick analysis. As shown in most of our examples, Markov chains are particularly well-suited for modeling repairable systems. They are also often used to model redundancy (hot and cold standby), system dependencies, and fault tolerance systems. For most of these systems, Markov chain models are mathematically tractable and thus avoid the necessity of using simulation (see reference 21). The biggest disadvantage of Markov chain models is the "curse of dimensionality" [20]. For complex systems, the number of states required can be quite large, resulting in excessively long solution times. Fortunately, there are many commercial software tools available that help with the modeling and analysis of complex systems using Markov chains, such as Relex, ITEM, and SHARPE.

8.8 EXERCISES

8.1. Find the PDF, CDF, reliability function, and MTTF, assuming that the system has the following hazard function

$$h(t) = 0.1t^{-0.5}$$

8.2. The hazard function for a mechanical component is given by

$$h(t) = 0.0005\left(2 + 4t + 2t^{1.5}\right)$$

(a) What is the reliability at $t = 5000$ h?

(b) What is the mean time to failure for this system?

(c) What is the expected number of failures in 1 year of operation?

8.3. Given the following failure-time data in hours:

$$50, 65, 78, 92, 99, 107, 120, 190, 200, 205$$

(a) Assuming that the system under study follows an exponential failure distribution, use the method of moments to estimate the parameter for the exponential distribution.

(b) Calculate the maximum likelihood estimator for the parameter for exponential distribution.

(c) The system under study has a 50-h warranty. What is the probability that a system in the field will fail before the warranty period?

8.4. Twelve fuel pumps are placed on test and run until failure. Their failure times are given below. Prior experience indicates that the fuel pumps follow a two-parameter Weibull failure distribution.

$$201, 1402, 351, 1078, 496, 768, 258, 480, 677, 611, 798, 802$$

(a) Use the method of moments to estimate the parameters for the Weibull distribution.

(b) Plot the PDF, reliability function, and hazard function for this system.

8.5. The failure distribution for an automotive component is modeled with a Weibull distribution with a scale parameter of 30,000 h and a shape parameter of 2.5.

(a) Find the probability that a component in the field will be operational after 2 years.

(b) Find the conditional probability that the component will fail during the third year, given that it has survived the second year of operation.

8.6. The probability that component survives a reliability test is 0.95. If 10 items are put on test, calculate the following probabilities:

(a) Exactly eight survive the test.

(b) All 10 survive the test.

(c) seven or more survive the test.

8.7. The time to failure of a component in a washing machine is known to follow an exponential distribution. The probability that the component fails during the first year is 0.975. How often should the service center expect to replace the component?

8.8. Each of the components in the system below has an exponential failure distribution. The parameters for each of the components are

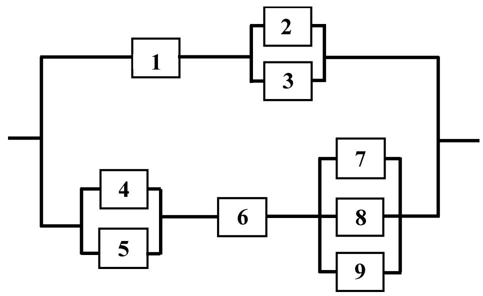
$$\lambda_1 = 0.0001$$

$$\lambda_2 = \lambda_3 = 0.005$$

$$\lambda_4 = \lambda_5 = 0.0075$$

$$\lambda_6 = 0.00005$$

$$\lambda_7 = \lambda_8 = \lambda_9 = 0.025$$



(a) Construct the system reliability function.

(b) Plot the system reliability function.

(c) What is the probability that the system will operate for one year with out failure? (assume 24–7 operation)

(d) Use the component importance measures to determine which component should undergo improvement first?

8.9. The following table is intended to provide the relationships between the probability density functions, cumulative distribution function, reliability function, and hazard function. For each expression in each row of the first column, identify the appropriate relationship using the function identified across the top of the table. Please identify the appropriate relationship for the lettered block in the table.

Expressed by	$F(t)$	$f(t)$	$R(t)$	$\lambda(t)$
$F(t) =$	—	$\int_0^t f(u)du$	$1 - R(t)$	e
$f(t) =$	$\frac{d}{dt}F(t)$	—	c	f
$R(t) =$	$1 - F(t)$	a	—	g
$\lambda(t) =$	$\frac{dF(t)/dt}{1 - F(t)}$	b	d	—

8.10. Identify each of the following statements as true or false. If false correct the statement so it is true.

(a) If a component is IFR, then $R(t + T_0|T_0) < R(t)$.

(b) If a component is CFR, then $R(t + T_0|T_0) > R(t)$.

(c) If a component is DFR, then the mean residual life will be a decreasing function of T_0.

8.11. Compute each of the following quantities assuming that the time to failure of the system is exponential with a mean time to failure of 1000 h and a guaranteed life of 200 h. Be sure to include units where appropriate.

(a) the system's failure rate

(b) the system's 100-h reliability

(c) the system's 300-h reliability

(d) the system's 700-h reliability given survival up to 600 h

(e) the system's design life for a 95% reliability.

8.12. Respond to the following statements assuming that the time to failure of the system is Weibull with a shape parameter of 2.0 and a scale parameter of 100 hrs. = 2.0 and 100 h. Be sure to include units where appropriate.

(a) Classify the system's hazard function as either DFR, CFR, or IFR.

(b) Compute the system's 100-h reliability.

(c) Compute the system's MTTF.

(d) Compute the systems characteristic life.

(e) Compute the system's 150-h reliability given survival up to 100 h.

(f) Compute the systems median time to failure.

(g) Compute the probability that the system fails between 100 and 150 h.

(h) Is this system a candidate for burn-in? Why or why not?

8.13. Consider a system that consists of a collection of four components such that $R_1 = 0.95, R_2 = 0.975, R_3 = 0.99,$ and $R_4 = 0.999$. Assume that the cost to improve each component by one unit of reliability is given by $C_1 = \$100, C_2 = \$200, C_3 = \$75,$ and $C_4 = \$85$. Answer the following questions.

(a) Provide a lower bound on the system's reliability.

(b) Provide an upper bound on the system's reliability.

(c) Assuming that the four components are arranged in series, which component is the most important?

(d) Assuming that the four components are arranged in parallel, taking cost of improvement into consideration, which component should we improve first?

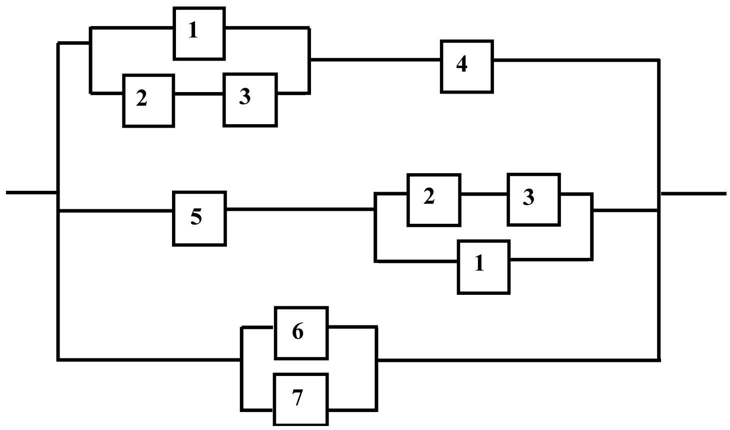
8.14. Assume that the components in the previous problem must be arranged in a series arrangement in order to function. Suppose you can afford to purchase an additional copy of each of the components.

(a) Construct a reliability block diagram assuming system level redundancy.

(b) Construct a reliability block diagram assuming component level redundancy.

(c) Which arrangement has the higher reliability?

8.15. Consider a system represented by the following reliability block diagram. Note that $R_1 = 0.9, R_2 = 0.975, R_3 = 0.999, R_4 = 0.92, R_5 = 0.95, R_6 = R_7 = 0.85$. Note the following subsystem definitions.



(a) Subsystem A Components 2, 3

(b) Subsystem B Component 1 and subsystem A

(c) Subsystem C Component 4 and subsystem B

(d) Subsystem D Components 5 and subsystem B

(e) Subsystem E Components 6,7

Determine the following:

(a) The reliability of subsystem B

(b) The reliability of subsystem C

(c) The reliability of subsystem D

(d) The reliability of subsystem E

(e) The reliability of the system

(f) Identify the most important component in the system.

8.16. An electronic relay has two main failure modes: premature closure (PC) and failure to close (FTC). We can model each failure mode using an exponential distribution with the following failure rates:

$$\lambda_{\text{PC}} = 5 \times 10^{-3} \text{ failures per hour}$$

$$\lambda_{\text{FTC}} = 5 \times 10^{-4} \text{ failures per hour}$$

The mean time to repair a PC failure is assumed to be 2 h, while the mean time to repair FTC failure is 48 hours. The repair times are assumed to be exponentially distributed [10].

(a) Establish a rate diagram for this system.

(b) Write the system balance equations.

(c) Calculate the steady-state availability for this system.

(d) Calculate the mean time to failure for the system.

8.17. Two identical aircraft hydraulic control systems are operated as a parallel system. During normal operation both hydraulic systems are functioning. When the first hydraulic system fails, the other system has to do the entire job alone with a higher load than when both hydraulic systems are in operation. Assume that the hydraulic systems have identical constant failure rates and that when one fails and a single system is performing the job, it has a higher failure rate.

$$\lambda_S = 5 \times 10^{-4} \text{ failures per hour (rate when systems share load)}$$

$$\lambda_{FTC} = 5 \times 10^{-4} \text{ failures per hour}$$

(rate when single system is moving full load)

Assume that both systems may fail at the same time due to some external event (common cause failure). The constant failure rate with respect to common cause failures has been estimated to be $\lambda_{CC} = 9.0 \times 10^{-5}$ common cause failures per hour. This type of external stress can affect the system irrespective of how many units are functioning.

Repair is initiated as soon as one of the hydraulic systems fails. The mean downtime of a system has been estimated to be 20 h. When both systems are in a failed state, the entire process has to be shut down. In this case, the system will not be put into operation again until both pumps have been repaired. The mean downtime, when both pumps have failed, has been estimated to be 40 h (21).

(a) Establish a rate diagram for this system.

(b) Write the balance equations for the system.

(c) Calculate the steady-state probabilities.

(d) Determine the percentage of time when (i) both pumps are functioning; (ii) only one of the pumps is functioning; (iii) both pumps are in a failed state.

(e) Compute the system MTTF.

8.18. You are given a system that has five components in parallel. For the system to work, four of the five components must be working. Assume all components are identical with a failure rate of λ_1 when five components are working and a failure rate of λ_2 when only four components are operating. Assume that the repair rate for all components is μ. Assume a single repairman.

(a) Establish the reliability state diagram for the system.

(b) Write the state equations.

(c) Compute the MTTF for the system.

(d) Establish the availability state diagram

(e) Write the state equations.

(f) Calculate the steady-state availability.

8.19. Given two components in a standby configuration with components that have different failure rates. Let component 1 be the main component and component 2 the standby component. Assume component 1 has an exponential failure rate of λ_1. Let component 2 have an exponential failure rate of λ_{2I} when it is in standby mode and λ_{2O} when it is operating. When component one fails a switch is used to bring component 2 online. Assume that the switch has a probability of success of p. Use a Markov modeling approach to model the system.

(a) Draw the state transition diagram

(b) Derive the system state equations

(c) Compute the reliability of the system.

(d) What is the mean time to failure for the system?

8.20. The following poem by Oliver Wendell Holmes appears in or is referenced by many reliability text books.

(a) Why do you think this poem is found in many reliability textbooks?

(b) Identify in the poem the various components of the systems decision process.

The Deacon's Masterpiece, or the Wonderful "One-Hoss Shay": A Logical Story

—by Oliver Wendell Holmes (1809–1894)

Have you heard of the wonderful one-hoss shay,
That was built in such a logical way
It ran a hundred years to a day,
And then, of a sudden, it ah, but stay,
I'll tell you what happened without delay,
Scaring the parson into fits,
Frightening people out of their wits,
Have you ever heard of that, I say?

Seventeen hundred and fifty-five.
Georgius Secundus was then alive,
Snuffy old drone from the German hive.
That was the year when Lisbon-town
Saw the earth open and gulp her down,
And Braddock's army was done so brown,
Left without a scalp to its crown.
It was on the terrible Earthquake-day
That the Deacon finished the one-hoss shay.

Now in building of chaises,
I tell you what,
There is always somewhere
a weakest spot,
In hub, tire, felloe, in spring or thill,
In panel, or crossbar, or floor, or sill,
In screw, bolt, thoroughbrace, lurking still,
Find it somewhere you must and will,
Above or below, or within or without,
And that's the reason, beyond a doubt,
A chaise breaks down, but doesn't wear out.

But the Deacon swore (as Deacons do,
With an "I dew vum," or an "I tell yeou")
He would build one shay to beat the taown
'N' the keounty 'n' all the kentry raoun';

It should be so built that it couldn' break daown:
"Fur," said the Deacon, "'tis mighty plain
Thut the weakes' place mus' stan' the strain;
'N' the way t' fix it, uz I maintain,
Is only jest
T' make that place uz strong uz the rest."

So the Deacon inquired of the village folk
Where he could find the strongest oak,
That couldn't be split nor bent nor broke,
That was for spokes and floor and sills;
He sent for lancewood to make the thills;
The crossbars were ash, from the straightest trees,
The panels of white-wood, that cuts like cheese,
But lasts like iron for things like these;
The hubs of logs from the "Settler's ellum,"
Last of its timber, they couldn't sell 'em,
Never an axe had seen their chips,
And the wedges flew from between their lips,
Their blunt ends frizzled like celery-tips;
Step and prop-iron, bolt and screw,
Spring, tire, axle, and linchpin too,
Steel of the finest, bright and blue;
Thoroughbrace bison-skin, thick and wide;
Boot, top, dasber, from tough old hide
Found in the pit when the tanner died.
That was the way he "put her through."
"There!" said the Deacon, "naow she'll dew!"

Do! I tell you, I rather guess
She was a wonder, and nothing less!
Colts grew horses, beards turned gray,
Deacon and deaconess dropped away,
Children and grandchildren where were they?
But there stood the stout old one-hoss shay
As fresh as on Lisbon-earthquake-day!

EIGHTEEN HUNDRED; it came and found
The Deacon's masterpiece strong and sound.
Eighteen hundred increased by ten;
"Hahnsum kerridge" they called it then.
Eighteen hundred and twenty came;
Running as usual; much the same.
Thirty and forty at last arrive,
And then come fifty, and FIFTY-FIVE.

Little of all we value here
Wakes on the morn of its hundreth year
Without both feeling and looking queer.
In fact, there's nothing that keeps its youth,
So far as I know, but a tree and truth.
(This is a moral that runs at large;
Take it. You're welcome. No extra charge.)
FIRST OF NOVEMBER, the Earthquake-day,
There are traces of age in the one-hoss shay,
A general flavor of mild decay,
But nothing local, as one may say.
There couldn't be, for the Deacon's art
Had made it so like in every part
That there wasn't a chance for one to start.
For the wheels were just as strong as the thills,
And the floor was just as strong as the sills,
And the panels just as strong as the floor,
And the whipple-tree neither less nor more,
And the back crossbar as strong as the fore,
And spring and axle and hub encore.
And yet, as a whole, it is past a doubt
In another hour it will be worn out!

First of November, 'Fifty-five!
This morning the parson takes a drive.
Now, small boys, get out of the way!
Here comes the wonderful one-horse shay,
Drawn by a rat-tailed, ewe-necked bay.
"Huddup!" said the parson. Off went they.
The parson was working his Sunday's text,
Had got to fifthly, and stopped perplexed
At what the Moses was coming next.
All at once the horse stood still,
Close by the meet'n'-house on the hill.
First a shiver, and then a thrill,
Then something decidedly like a spill,
And the parson was sitting upon a rock,

At half past nine by the meet'n-house clock,
Just the hour of the Earthquake shock!
What do you think the parson found,
When he got up and stared around?
The poor old chaise in a heap or mound,
As if it had been to the mill and ground!

You see, of course, if you're not a dunce,
How it went to pieces all at once,
All at once, and nothing first,
Just as bubbles do when they burst.

End of the wonderful one-hoss shay.
Logic is logic. That's all I say.

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