Chapter 4

Discrete Mathematics: Sets, Relations, and Functions

4.1 INTRODUCTION

Chapter 4 introduces material from the field of discrete mathematics. Much of this chapter will be review material (e.g., sets and functions) for most readers. The concepts of sets, relations, and functions are defined, discussed, and illustrated. A function, with which almost everyone is familiar, is shown to be a specialization of a relation, which in turn is a specialization of a set.

There are some key concepts introduced here that will be referred to in many of the succeeding chapters. For example, we will be discussing requirements and requirements documents in Chapter 6. Many system-level requirements documents are very large, larger than they need to be. These large system-level requirements documents can contain thousands and even tens of thousands of requirements. Examples might include:

- The system shall be able survive attacks from another computer system.
- The system shall be able survive buffer overflow attacks from another computer system.
- The system shall be able to survive stack-based buffer overflow attacks from another computer system.
- The system shall be able to survive stack-based buffer overflow attacks from an internal employee.
- The system shall be able to survive buffer overflow attacks against its operating system.

The Engineering Design of Systems: Models and Methods, Second Edition. By Dennis M. Buede

- The system shall be able to survive buffer overflow attacks against its application programs.
- The system shall be able to survive buffer overflow attacks originating in emails.
- The system shall be able to survive buffer overflow attacks while connected to web sites on the Internet.
- And more of the same.

In Chapter 6 we will present an approach to writing such requirements and make the point that only one or a few of the above requirements should be in the system-level requirements document. We will use the concept of a partition, introduced and defined here in Chapter 4, to make this case. A partition, based on the set theory introduced in this chapter, ensures that the requirements are not overlapping and are complete. Satisfying the non-overlapping part will be relatively easy, but it is amazing how often it happens in practice. Achieving the completeness is a goal that is seldom, if ever, achieved. But there are approaches based on a partition that can help. Many requirements documents contain duplicate, triplicate, and higher copies of requirements. Over time some of these copies of requirements get changed while others do not, resulting in inconsistent requirements such as happened on the Space Shuttle for operations in ambient temperatures, resulting in part in the explosion of the Challenger in 1986. Getting the concept of a partition of a set is key to many aspects of systems engineering.

In Chapter 7 we will discuss functions that systems perform in transforming their inputs into their outputs. When we have this discussion, you should remember the definition of a mathematical function, which we cover here in Chapter 4. What you may not have learned previously is the concept of a mathematical relation, which is a weaker concept than that of a mathematical function. In order to perform mathematical analyses of our system's functional architecture we will need eventually to be able to satisfy the mathematical definition of a function, not simply a relation, provided in this chapter. We will also need to recognize that we are dealing with relations when we are dealing with higher level functions of a system. Ensuring that our functional decomposition is a partition will arise again and again.

As part of the discussion of functional architectures in Chapter 7, we will be talking about decomposing higher level functions into sets of lower level functions. (Note the word *set* has been used again.) The mathematical concept of composition is defined here in Chapter 4 and discussed relative to hierarchical decomposition; mathematical composition will be shown to be a very limited representation of the functional modeling described in Chapter 7.

Two advanced concepts, power set and partial ordering, are introduced in this chapter. These concepts have great usefulness to the theoretical development of the engineering of systems, most of which is beyond the scope of this book but elements of which are discussed in Chapters 6, 7, and 9. The interested reader is referred to Mott et al. [1986] and Rosen [1995] for more details on set theory. Larsen and Buede [2002] provide a mathematical structure for performing early validation of requirements using many of the set theory concepts presented in this chapter.

Section 4.2 introduces the general concept of a set and then discusses special characteristics of sets, including operations on sets, the partition of a set, and the power set of a set. Section 4.3 defines relations in terms of sets. In particular, important characteristics of relations are defined. The partial ordering on a set is introduced and illustrated. Section 4.4 discusses functions and the composition of functions.

There are no models introduced in this chapter, but all of this material is critical in understanding the development of models, as well as the power and limitations of models. Software engineers often make much more use of the discrete mathematics presented here than do the engineers of systems, but the material has the same richness and importance to engineers of systems and should be utilized to a fuller degree in the future. In addition, having a grasp of this material is essential to carrying on a conversation about architectures with many software engineers. I have seen systems engineers lose important and valid arguments to software engineers because the systems engineers were not equipped to understand what the software engineers were saying.

4.2 SETS

A *set* is a collection of well-defined objects, called elements or members. These *elements* or *members* are said to belong to the set. Sidebar 4.1 defines the mathematical symbols used in these and other definitions.

SIDEBAR 4.1 GLOSSARY OF MATHEMATICAL SYMBOLS

$\in$	is an element of
$\notin$	is not an element of
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\not\subset$	is not a subset of
$\supseteq$	is a superset of
$\supset$	is a proper superset of
$\cap$	intersection
$\cup$	union
$\rightarrow, \Rightarrow$	implies
$\Leftrightarrow$	if and only if

$\neq$	is not equal to
$\boldsymbol{\Phi}$	the null set
$\boldsymbol{U}$	the universal set
$\bar{A}$	the complement of A
$\forall$	for all
$\exists$	there exists
$\ni$	such that
$\vert$	given that
$\sim, \neg$	not (negation)
$\wedge$	and
$\vee$	or

Examples of sets are:

- An interval of numbers [7, 21]
- The students in SYST 520 at George Mason University during the spring semester of 1996
- The categories of inputs to elevator
- The possible states or outcomes that a particular input to the elevator can take
- The functions of an ATM (automated teller machine)

4.2.1 Writing Set Membership

A set is denoted by *capital* letter A, B, X, Y, with the exception of sets that are functions, which will be denoted by a lowercase italic, letter. Members are also denoted by *lowercase* letters: a, b, x, y. The mathematical expression of set membership is

$$x \in A : x \text{ is an element of } A$$
$$x \notin A \text{ or } \neg(x \in A) : x \text{ is not an element of } A$$

4.2.2 Describing Members of a Set

There are at least five ways to describe the members of a set.

1. A is the set of elements, x, that satisfies the property (or predicate), $p(x)$. $A = \{x|p(x) \text{ is true}\}$ (braces are the common delimiter of a set's definition). The property $p(x)$ must be *well-defined*, that is, able to be determined by means of rules. One test of such a property is called the

clairvoyant's test — a clairvoyant is able to predict the future or describe the past/present perfectly. Is the property or rule defined sufficiently well that the clairvoyant can answer the question? For example, the property "Is tall" does not meet the clairvoyant's test, but the property "is taller than 6 feet 3 inches" does.

2. Complete enumeration is the listing of all of the members of the set.

$$A_1 = \{0, 1, 2, 3, 4\}$$

$$A_2 = \{\text{student}_1, \text{student}_2, \ldots \text{student}_{31}\}$$

3. Use the characteristic function of the

$$\mu_A(x) = \begin{cases} 1 & \text{for } x = 0, 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

where $\mu_A(x)$ is the characteristic function of set A for elements, x, in the set, U, of all elements. For conventional (crisp, nonfuzzy) sets, $\mu_A(x)$ may only take the values 0 for nonmembers or 1 for members.

4. Use recursive definition: $A = \{x_{i+1} = x_i + 1,\ i = 0,\ 1,\ 2,\ 3;\ \text{where } x_0 = 0\}$. Here A is defined by a recursive formula.
5. Use one or more set operators such as union, intersection, and complement. These operations should be familiar to most readers and will be defined shortly.

4.2.3 Special Sets

***U*:** the universal set or set of all possible members.

***Φ*:** the null set, a set with no elements. Φ and $\{\Phi\}$ are not the same. Φ has no elements, while $\{\Phi\}$ has one.) We can write $\Phi = \{x \in U \mid x \neq x\}$.

Singleton set: a set with only one element.

Finite set: a set with a finite number of distinct elements.

Infinite set: a set with an infinite number of distinct elements.

For example: $A_1 = \{1, 2, 3, 4, \ldots, 101\}$ is finite, $A_2 = \{1, 2, 3, 4, \ldots,\}$ is infinite, and $A_3 = \{x, \{1, 2\}, y, \{z\}\}$ may be finite or infinite. The finiteness of A_3 depends on whether x and y are finite or infinite. (Note $\{1,2\}$ and $\{z\}$ are sets, but each is only one element of A_3. Also note that z is *not* an element of A_3, but $\{z\}$ is.)

Subsets or set inclusion: if A and B are two sets, and if every element of A is an element of B, then A is a *subset* of B, $A \subseteq B$. If A is a subset of B, and if B has

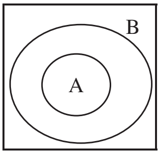

FIGURE 4.1 Set inclusion.

at least one element that is *not* in A, then A is a *proper* subset of B, $A \subset B$. See Figure 4.1.

Equality of sets: if A and B are sets, and A and B have precisely the same elements, then A and B are *equal*, $A = B$.

The following properties follow from the above definitions:

$A \subseteq A$; a set is a subset of itself.

$\Phi \subseteq A$, $A \subseteq U$. The null set is a subset of every set; every set is a subset of the universal set.

If $\Phi \neq A$, then $\Phi \subset A$. If a set is not the null set, then the null set is a proper subset of the set.

If $A \subseteq B$ and $B \subseteq A$, then $A = B$. If two sets are subsets of each other, then they are equal.

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. Set inclusion is transitive, a property that we will formally define later.

4.2.4 Operations on Sets

The following operations are performed on sets:

Absolute complement, $\bar{A}$: Let $A \subseteq U$. $\bar{A} = \{x \mid x \notin A\}$ (Note $\bar{\Phi} = U$, $\bar{U} = \Phi$, $\bar{\bar{A}} = A$) See Figure 4.2.

Relative complement of A with respect to B, $B - A$: Let A and B be sets, $B - A = \{x \mid x \in B$ and $x \notin A\}$. The relative complement is also called *set difference*. See Figure 4.3.

Union of A and B, $A \cup B$: $A \cup B = \{x \mid x \in A$ or $x \in B$ or both$\}$.

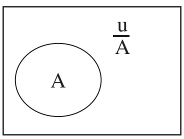

FIGURE 4.2 Absolute complement.

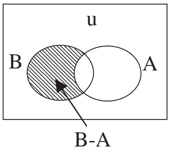

FIGURE 4.3 Relative complement.

Intersection of A and B, $A \cap B$: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. (Note A and B are called *disjoint* if $A \cap B = \Phi$. See Figure 4.4.

Boolean sum (symmetrical difference), $A + B$ or $A \Delta B$:

$$A + B = \{x | x \in A \text{ or } x \in B, \text{but not both}\} = (A - B) \cup (B - A)$$

The following properties of the above set operations can be easily derived:

1. $A \cup \Phi = A$, and $A \cap \Phi = \Phi$.
2. $A \cup U = U$, and $A \cap U = A$.
3. Idempotent: $A \cup A = A$, and $A \cap A = A$
4. Associative:

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

5. Commutative: $A \cup B = B \cup A$, and $A \cap B = B \cap A$
6. Distributive:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

7. DeMorgan's Laws: $(\overline{A \cup B}) = \bar{A} \cap \bar{B}$, and $(\overline{A \cap B}) = \bar{A} \cup \bar{B}$

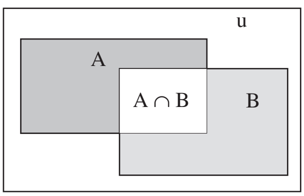

FIGURE 4.4 Set intersection.

Example Use DeMorgan's laws to prove that the complement of $(\bar{A} \cap B) \cap (A \cup \bar{B}) \cap (A \cup C)$ is $(A \cup \bar{B}) \cup (\bar{A} \cap (B \cup \bar{C}))$.

Solution: Starting with $(\bar{A} \cap B) \cap (A \cup \bar{B}) \cap (A \cup C)$, note that $(A \cup \bar{B}) \cap (A \cup C)$ is the same as $A \cup (\bar{B} \cap C)$.

Step 1: Making this substitution, we want to find the complement $\overline{(\bar{A} \cap B) \cap (A \cup (\bar{B} \cap C))}$.

Step 2: By DeMorgan's law, the complement of an intersection is the union of set complements. So this can be written as $\overline{(\bar{A} \cap B)} \cup \overline{(A \cup (\bar{B} \cap C))}$.

Step 3: Again, the complement of an intersection is the union of the set complements. So this can be written as $(A \cup \bar{B}) \cup \overline{(A \cup (\bar{B} \cap C))}$.

Step 4: Also by DeMorgan's law, the complement of a union is the intersection of the set complements. So this can be written as $(A \cup \bar{B}) \cup (\bar{A} \cap \overline{(\bar{B} \cap C)})$.

Step 5: Again, the complement of an intersection is the union of the set complements. This yields $(A \cup \bar{B}) \cup (\bar{A} \cap (B \cup \bar{C}))$. QED

4.2.5 Partitions

A *partition* on a set A is a collection P of disjoint subsets of A whose union is A. For a collection B_i ($i = 1, 2, \ldots, n$) to be a partition P of A:

1. $B_i \subseteq A$ for $i = 1, 2, \ldots, n$.
2. $B_i \cap B_j = \Phi$ for $i \neq j$.
3. for any $x \in A$, $x \in B_i$ for some i; (alternatively $B_1 \cup B_2 \cup \ldots \cup B_n$)

The concept of a partition (Fig. 4.5) is the most basic and *far-reaching mathematical concept* to our development of systems engineering. We will talk

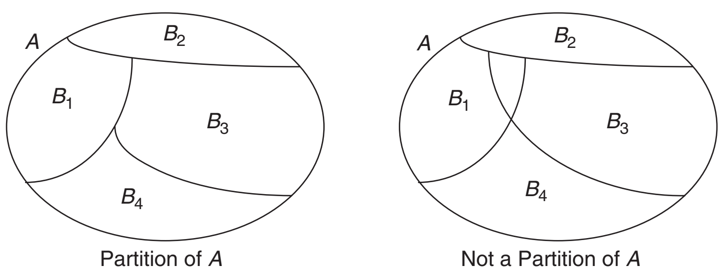

FIGURE 4.5 Set partition.

about the importance of creating a partition of the system's requirements, and a partition of the system's function, and a partition of the system's physical resources. This is just the beginning.

4.2.6 Power Set

The power set of a set A is denoted, $\mathbf{P}(A)$. The *power set* is the set of all sets that are subsets of A. Mathematically, the power set is the family (or set) of sets such that $X \subseteq A \Leftrightarrow X \in \mathbf{P}(A)$, or $\mathbf{P}(A) = \{X \mid X \subseteq A\}$.

1. Let $A_0 = \Phi$, $\mathbf{P}(\Phi) = \{\Phi\}$, [where A_0 is a set with ***zero*** elements and $\mathbf{P}(A_0)$ has one element].
2. Let $A_1 = \{a\}$; $\mathbf{P}(A_1) = \{\Phi, A_1\} = \{\Phi, \{a\}\}$ [where A_1 is a set with ***one*** element and $\mathbf{P}(A_1)$ has two elements].
3. Let $A_2 = \{a, b\}$; $\mathbf{P}(A_2) = \{\Phi, \{a\}, \{b\}, \{a, b\}\}$ [where A_2 is a set with ***two*** elements and $\mathbf{P}(A_2)$ has four elements].

How many elements does the power set of a set of A_{n} have?

Theorem If A_{n} is a set with n elements, then $\mathbf{P}(A_{\text{n}})$ has 2^n elements.

Proof We will use mathematical induction. For n = 0, 1, 2, 3, …, let $S(n)$ be the statement: If A_{n} is a set with n elements, then $\mathbf{P}(A_{\text{n}})$ has 2^{n} elements.

i. First show that if A_0 has 0 elements, then $\mathbf{P}(A_0)$ has $2^0 = 1$ element.

$$A = \Phi,\ \mathbf{P}(A) = \{\Phi\}$$

ii. Assume $S(k)$ is true and then show that $S(k+1)$ is true. Let A_{k+1} be a set with $k+1$ elements. Define B to be a proper subset of A_{k+1} with k of A_{k+1}'s elements:

$$A_{k+1} = \{a_1, a_2, \ldots, a_k, a_{k+1}\}$$

$$B = A_k = \{a_1, a_2, \ldots, a_k\}$$

$$\text{So } A_{k+1} = \{a_{k+1}\} \cup B.$$

Therefore, every subset of A_{k+1} either contains a_{k+1}, or it does not.

1. If a subset does not contain a_{k+1}, then it is a subset of B, and we know there are 2^k subsets of B, by induction.
2. If a subset does contain a_{k+1}, then it is the union of a subset of B and a_{k+1}. There must be 2^k of these since there are 2^k subsets of B.
   So there are $2^k + 2^k = 2^k(1+1) = 2^k\ 2 = 2^{k+1}$ subsets of A_{k+1} or 2^{k+1} elements of $\mathbf{P}(A_{k+1})$.

The concept of a power set has many potential uses in systems engineering. For example, the power set of system inputs is an upper bound on the test sequences required to test the system exhaustively.

4.3 RELATIONS

This section defines relations using the concepts of ordered pairs and Cartesian products. Important properties of relations are defined, followed by definitions of partial orderings and equivalence relations.

4.3.1 Ordered Pairs and Cartesian Products

An *ordered pair* is (x, y) if $x \in A$, $y \in B$. A *Cartesian product*, $A \times B$, is defined over two sets, A and B, such that $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$. That is, the Cartesian product of two sets is the set of all possible ordered pairs of those two sets. The following are examples of Cartesian products:

1. $A = \{1\}$, $B = \{2\}$: $A \times B = \{(1, 2)\}$ and $B \times A = \{(2, 1)\} \neq A \times B$.
2. $X = \{\text{students of SYST 520 during the spring semester of 1996}\} = \{S_1, S_2, \ldots, S_{31}\}$, $Y = \{A, B, C\}$: $X \times Y = \{(S_1, A), (S_1, B), (S_1, C), \ldots, (S_{31}, A), (S_{31}, B), (S_{31}, C)\}$

An ordered n-tuple is defined to be $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, i = 1, 2, \ldots, n\}$, where $(a_1, a_2, \ldots, a_n)$.

4.3.2 Unary and Binary Relations

A *unary relation* on a set A relates elements of A to itself and is a subset, R, of $A \times A$. R is usually described by a predicate that defines the relation. Examples are $\leq, =, >$, "taller than," and "older than." If a_1 and $a_2 \in A$, we write $(a_1, a_2) \in R$, which means that $a_1 \; R \; a_2$ or a_1 "is related to" a_2.

A binary *relation* is a relation R that relates elements of A to elements of B and is a subset of $A \times B$. The *domain* of R, written as "dom R," is defined as: dom $R = \{x \mid x \in A \text{ and } (x, y) \in R \text{ for some } y \in B\}$. The *range* of R, written as "ran R," is defined as: ran $R = \{y \mid y \in B \text{ and } (x, y) \in R \text{ for some } x \in A\}$. Again $(a_1, b_1) \in R \Leftrightarrow a_1 \; R \; b_1$.

Example Let R be the relation from $A = \{1, 3, 5, 7\}$ to $B = \{1, 3, 5\}$, which is defined by "x is less than y." Write R as a set of ordered pairs.

Solution:

$R = \{(x, y) \mid x \in A, y \in B, x < y\}$
$R = \{(1, 3), (1, 5), (3, 5)\}$

Recall the relations within and between systems engineering classes that were discussed in Chapter 2. The hierarchy of requirements was defined by the relation "incorporates" in moving from the top of the requirements hierarchy to the bottom; "incorporated in" was the relation that moved from bottom to top. The relation "is decomposed by" moved from the top of the functional decomposition to the bottom; "decomposes" moves in the opposite direction. The physical hierarchy of a system and its components used the relation "is built from" in moving from top to bottom and "is built in" for moving from bottom to top.

Binary relations included the tracing from requirements to functions or the system, the performance of functions by the system and its components, and inputs and outputs of items for functions. The relation "is traced to" was used for the binary relations of input/output stakeholders' requirements being mapped to functions and for system-wide/technology requirements being mapped to the system. The binary relation for the system and components being related to functions used the relation "pertains." The relations "inputs" and "outputs" addressed functions being related to items.

To discuss the properties of unary relations, some additional information is needed concerning the possible ways to prove an implication. An implication is an "If …, then …" statement, which is commonly written as "If p is true, then q is true" or "$p \rightarrow q$." There are eight common methods for proving implications of this form.

1. *Trivial proof*: Show that q is true independently of the truth of p.
2. *Vacuous proof*: By mathematical convention, whenever p is false, $p \rightarrow q$ is true. The vacuous proof involves showing that p is false. *This method is key to understanding the full implications of the properties of unary* relations *that are discussed below*.
3. *Direct proof*: Assume that p is true and use arguments based upon other known facts and logic to show that q must be true.
4. *Indirect proof*: Use direct proof of the contrapositive of $p \rightarrow q$. The contrapositive of a true implication is known to be true; the contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$ (or q is false implies p is false). Here we assume q is false and prove via logic and known facts that p must be false.
5. *Contradiction-based proof*: DeMorgan's laws can be used to show that $p \rightarrow q$ is equivalent to $\sim(p \wedge (\sim q))$, that is, the statement "p is true and q is false" is false. Proof by contradiction starts by assuming that $(p \wedge (\sim q))$ is true and then proving, based on this assumption, that some known truth must be false. If the only weak link in the argument is the assumption of $(p \wedge (\sim q))$, then this assumption must be wrong.
6. *Proof by cases*: If p can be written in the form of p_1 or p_2 or … or p_n $(p_1 \vee p_2 \vee \ldots \vee p_n)$, then $p \rightarrow q$ can be proven by proving $p_1 \rightarrow q$, $p_2 \rightarrow q$, …, $p_n \rightarrow q$ as separate arguments.
7. *Proof by elimination of cases is an extension of the method above*: Recall from the second method that $p \rightarrow q$ is equivalent to $[\,(p \vee q) \wedge (\sim p)]$, that is

(p and q are true) or (p is false). Now p can be partitioned into a set of cases as done in 6 and attacked one at a time.

8. *Conditional proof*: If we are to prove $p \rightarrow (q \rightarrow r)$, we can prove the equivalent $(p \wedge q) \rightarrow r$.

4.3.3 Properties of Unary Relations on *A*

The seven properties discussed here are reflexive, irreflexive, symmetric, antisymmetric, asymmetric, transitive, and intransitive.

1. *Reflexive*: $x\ R\ x$ for all $x \in A$, e.g., equality, $\leq$, $\geq$.
2. *Irreflexive*: $x \not R\, x$ for all $x \in A$, for example, greater than, is the father of.
3. *Symmetric*: If $x\ R\ y$, then $y\ R\ x\ \forall x, y \in A$, for example, equality, is spouse of. Note if $x \not R\ y$ for all x and y in A, then the relation is symmetric by a vacuous proof.
4. *Antisymmetric*: If $x\ R\ y$ and $y\ R\ x$, then $x = y\ \forall x, y \in A$, for example, equality, $\leq$, $\geq$. Note if there is no situation in which "$x\ R\ y$ and $y\ R\ x$" is true, then the relation is antisymmetric by vacuous proof.
5. *Asymmetric*: If $x\ R\ y$, then $y \not R\ x\ \forall x, y \in A$, e.g., $<$, $>$.
6. *Transitive*: If $x\ R\ y$ and $y\ R\ z$, then $x\ R\ z\ \forall x, y, z \in A$, for example, $\leq$, $\geq, =, >$. This property is the most difficult to grasp. If there is no situation in which "$x\ R\ y$ and $y\ R\ z$," then the relation is transitive by vacuous proof.
7. *Intransitive*: If for some $x, y, z \in A$, it is true that $x\ R\ y$, $y\ R\ z$, but $x \not R\ z$, the relation is considered intransitive.

Example Let L be the set of lines in the Euclidean plane and let R be the relation on L defined by "x is parallel to y." Is R a reflexive relation? Why? Is R a symmetric relation? Why? Is R a transitive relation?

Solution:

1. This question reduces to whether a line is parallel to itself. If the definition of parallel is having no points in common (everywhere equidistant), then a line cannot be parallel to itself because the two lines have every point in common. So R is not a reflexive relation.
2. R is a symmetric relation. Consider each $x \in L$. x will have an infinite number of $y \in L$ which satisfy the parallel relationship. Each such y is in turn parallel to x. Thus, $(x, y) \in R$ for all x and y that are parallel, and $(y, x) \in R$, so the relation is symmetric.
3. R is a transitive relation. Again, consider $(x, y) \in R$ and $(y, z) \in R$; x will be parallel to z, so $x\ R\ z$ and R is transitive for all $x, y, z \in L$.

Example Let F be the set of functions in the functional decomposition for a system. Let R be the relation on F defined by "is decomposed by." Is R a reflexive relation? Why? Is R a symmetric relation? Why? Is R a transitive relation?

Solution:

1. R is not a reflexive relation because a function does not decompose itself.
2. R is not a symmetric relation because if f_1 decomposes f_0, then f_0 cannot decompose f_1.
3. R is not a transitive relation. The function f_0 is decomposed by f_1, f_2 and f_3, and f_1 is decomposed by f_{11}, f_{12} and f_{13}. However f_0 is not decomposed by f_{11}, f_{12} or f_{13}.

4.3.4 Partial Ordering

A relation R on A is a *partial ordering* if R is reflexive, antisymmetric, and transitive. Examples of partial orderings are $\geq$ or $\leq$ on the real number line, or $\supseteq$ or $\subseteq$ on $\mathbf{P}(A)$. Examples of nonpartial orderings are $<$ or $>$ on the real number line, $\subset$ or $\supset$ on $\mathbf{P}(A)$. (Both of these are asymmetric and antisymmetric.)

4.3.5 Equivalence Relations

A relation R on a set A is an *equivalence relation* if R is reflexive, symmetric, and transitive. An example of an equivalence relation is equality.

4.4 FUNCTIONS

This section defines functions and discusses the composition of functions.

4.4.1 Definitions

Let A and B be two nonempty sets. We write a function f as $f: A \rightarrow B$ and say that f maps every element of A (the domain) to one and only one element of B (the range). If $(a, b) \in f$, then element b is the *image* of element a under f. Note that a function can map elements of A onto itself, $f: A \rightarrow A$. A *function* f from A to B is a *relation* such that

(a) dom $f = A$
   (i) f is defined for each element of A, $a \in A$.
   (ii) $\exists(a, b)$ where $b \in B$ for each element of A, $a \in A$.

(b) if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$; that is, f is single-valued, or no element of A is related to two elements of B.

A function is called *one-to-one* or *injective* if $(a, b) \in f$ and $(c, b) \in f$ implies $a = c$. That is, no two elements of A can be mapped into the same element of B by f.

A function $f: A \rightarrow B$ is *onto* or *surjective* if and only if the range of $f = B$, that is, f is defined for every $b \in B$.

If a function is both one-to-one and onto (or *bijective*), then the relation f^{-1} is single-valued and maps every element of B onto some element of A; f^{-1} is therefore a function, called the *inverse* function.

Example If $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$, determine if the following functions are one-to-one or onto.

(a) $f = \{(1, a), (2, a), (3, b), (4, d)\}$
(b) $g = \{(1, d), (2, b), (3, a), (4, a)\}$
(c) $h = \{(1, d), (2, b), (3, a), (4, c)\}$

Solution:

(a) f is NOT one-to-one since $f^{-1}(a) = \{1, 2\}$. f is NOT onto since $f^{-1}(c) = \Phi$.
(b) g is NOT one-to-one since $g^{-1}(a) = \{3, 4\}$. g is NOT onto since $g^{-1}(c) = \Phi$.
(c) h is *one-to-one* since all elements of B correspond to unique elements in A. h is *onto* since every element of B has some pre-image in A.

So we have progressed mathematically from sets to relations to functions.

Functions $\subseteq$ Relations $\subseteq$ Sets, or a function is a relation is a set.

As systems engineers we will focus on functional architectures. We will represent the functions of the system as relations or functions in graph-like structures. The underlying theory is set theory.

4.4.2 Composition

Let R be a relation from A to B, and S be a relation from B to C. (a, c) is an element of the *composition* of R and S, (denoted $R \bullet S$ or $R\ S$) if and only if there is an element $b \in B$ such that $a\ R\ b$ and $b\ S\ c$. That is, a and c must be linked together by b; a is mapped to b and b is mapped to c. (Note that some authors write the composition of R and S as $S \bullet R$ so be careful.)

The composition of functions is defined in the same way as the composition of relations.

Example Assume R and S are relations from A to A. If $A = \{1, 2, 3, 4\}$, $R = \{(1, 2), (2, 3), (3, 4), (4, 2)\}$, and $S = \{(1, 3), (2, 4), (4, 2), (4, 3)\}$, then compute $R \bullet S$, $S \bullet R$ and $R \bullet R$.

Solution:

$R \bullet S = \{(1, 4), (3, 2), (3, 3), (4, 4)\}$.

(1, 2) from R is composed with (2, 4) from S (this is written $(1, 2) \bullet (2, 4)$) and yields (1, 4).

(1, 2) from R cannot be composed with any of the other elements of S because they do not begin with a 2.

$(3, 4) \bullet (4, 2) = (3, 2)$.
$(3, 4) \bullet (4, 3) = (3, 3)$.
$(4, 2) \bullet (2, 4) = (4, 4)$.
$S \bullet R = \{(1, 4), (2, 2), (4, 3), (4, 4)\}$, which is not equal to $R \bullet S$.
$R \bullet R = \{(1, 3), (2, 4), (3, 2), (4, 3)\}$.

As systems engineers we will employ functional decomposition to develop the functional architecture. Composition is the mathematical property from which decomposition derives its name. However, as discussed in Chapter 7, composition is only applicable to functional decomposition in limited situations.

4.5 SUMMARY

This chapter began with the introduction of a set, the foundation of a branch of mathematics called discrete mathematics. A great deal of terminology was introduced to define special sets such as the universal and null sets and operations on sets.

During the discussion of sets, the concept of partition was defined. The partition is perhaps the most important mathematical concept introduced in this chapter for application in this book. A partition is a subdivision of a set into subsets, which contain no common members, and yet the union of the subsets contains every element of the original set. In future chapters requirements will be partitioned, functional decompositions will be defined to be partitions, and the physical decomposition will be defined to be a partition.

The power set of a set is the set of all subsets of that set. This notion of a power set is not exploited fully in this book but will become key to the future development and application of mathematics to the engineering of systems.

The next major section of this chapter dealt with relations and the key properties associated with relations. A relation is a set of ordered pairs; the elements of the ordered pairs come from one or two sets. If the functions of a system are not fully defined in terms of inputs, then these system functions are, in fact, mathematical relations. Functions are relations that satisfy certain properties; a function maps every element of the domain of the function to some element of the range, but does not map any element of its domain to more than one element of the range. One-to-one and onto properties of functions were also discussed. Finally the composition of functions was defined.

PROBLEMS

4.1 Define the students enrolled in this class during this semester as a set, S.

a. Specify a partition of S into 2 subsets.

b. Specify a partition of S into 3 subsets.

c. Specify a partition of S into 5 subsets.

4.2 Let $A_1 = \{1, 3, 5, 7, 9, 11\}$, $A_2 = \{-2, 6, 9, 11\}$, $A_3 = \{-2, 4, 6, 9, 11\}$. Show that:

a. $A_1 + A_2 = (A_1 - A_2) \cup (A_2 - A_1)$

b. $A_1 \cup (A_2 \cap A_3) = (A_1 \cup A_2) \cap (A_1 \cup A_3)$

4.3 Prove that the following relations are true in general:

a. $A_1 + A_2 = (A_1 - A_2) \cup (A_2 - A_1)$

b. $A_1 \cup (A_2 \cap A_3) = (A_1 \cup A_2) \cap (A_1 \cup A_3)$

4.4 Let R be a relation from A to B and defined "x is at least twice as big as y." Write R as a set of ordered pairs for

a. $A = \{1, 3, 5, 7\}$ and $B = \{2, 3, 4, 6\}$

b. $A = \{0, 1\}$ and $B = \{0, 1\}$

c. $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{3, 6\}$

4.5 Let R be relation from A to B where "x is greater than or equal to y squared."

Then define R as a set of ordered pairs for the following:

a. $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3, 4, 5\}$

b. $A = \{25\}$, $B = \{5, 6, 7\}$

4.6 There are three families defined by the sets A, B, and C; each family has a dad, mom and three kids:

$A = \{$Dad, Mom, Doris, Bill, Tom$\}$

$B = \{$Dad, Mom, Doris, Daisy, Debbie$\}$

$C = \{$Dad, Mom, Bill, Bob, Biff$\}$

Consider the relations "is the spouse of," "is the brother of," and "is the blood relative of." (Hints: I am not the brother of myself. Two people are blood relatives if they share the blood of a common ancestor, who may or may not be part of sets A, B, or C. I am the blood relative of myself. Biff is a male.)

Identify which of these relations satisfy which of the seven properties of unary relations for each of the three sets by placing a yes or no in the empty cells of the following table.

	Reflexive	Irreflexive	Symmetric	Anti-symmetric	Asymmetric	Transitive	Intransitive
"is the spouse of" on A							
"is the brother of" on A							
"is the blood relative of" on A							
"is the spouse of" on B							
"is the brother of" on B							
"is the blood relative of" on B							
"is the spouse of" on C							
"is the brother of" on C							
"is the blood relative of" on C							

4.7 Let R be a relation from A to B and S be a relation from B to C.

a. Find $R \bullet S$ for $A = \{1, 3, 5, 7\}$, $B = \{1, 2, 4, 5, 7\}$, $C = \{1, 2, 3, 4, 5, 6\}$, $R = \{(1, 2), (3, 4), (5, 2), (7, 4)\}$ and $S = \{(1, 2), (2, 4), (4, 3), (7, 5)\}$.

b. Are any of these relations R, S, $R \bullet S$ functions? One-to-one functions? One-to-one and onto functions?

4.8 If $A_1 = \{1, 2, 3, 4\}$ and $A_2 = \{1, 4, 9, 25\}$, determine if the following functions that map A_1 onto A_2 are one-to-one, onto, or both one-to-one and onto.

a. $f_1 = \{(1, 1), (2, 4), (3, 4), (4, 25)\}$

b. $f_2 = \{(1, 1), (2, 4), (3, 25), (4, 25)\}$

c. $f_3 = \{(1, 1), (2, 4), (3, 9), (4, 25)\}$

4.9 Develop two relations R (from A to B) and S (from B to C) that have to do with people. Show the result of $R \bullet S$.

4.10 Let R and S be relations from $A \to A$, where $A = \{1, 2, 3, 4\}$ and:

$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (2, 1), (3, 1), (3, 2)\}$

$S = \{(2, 3), (1, 2), (2, 1), (3, 1), (1, 3)\}$

a. Find if these relations are symmetric, reflexive, and transitive.

b. Find $R \bullet S$, $S \bullet R$ and $R \bullet R$.

4.11 Let A be a set of three colors: {red, blue, green}. What are the elements of the power set of A?

4.12 Let $SIBLINGS = \{$Andrea, Bobby, Catherine, David, Eric$\}$. Find the elements of the power set of $SIBLINGS$, $\mathbf{P}(SIBLINGS)$.

4.13 Show that the $\mathbf{P}\{$Andrea, Bobby$\}$ is a subset of the $\mathbf{P}(SIBLINGS)$ from Problem 4.12.

4.14 Prove that for any two sets A and B, $(\mathbf{P}(A) \cap \mathbf{P}(B)) = \mathbf{P}(A \cap B)$.

4.15 Find two sets A and B that show $(\mathbf{P}(A) \cup \mathbf{P}(B)) \neq \mathbf{P}(A \cup B)$.

4.16 Prove that for any two sets A and B, $(\mathbf{P}(A) \cup \mathbf{P}(B)) \subseteq \mathbf{P}(A \cup B)$.

4.17 Prove that the seven properties of set operations in Section 4.2.4 are true.