UNITED ARAB EMIRATES
MINISTRY OF EDUCATION

YEAR OF TOLERANCE

TEACHER EDITION 2018 - 2019

McGraw-Hill Education

Mathematics

8

Elite Stream

United Arab Emirates Edition

Interactive Student Guide

Mc Graw Hill Education

Answer Key

McGraw-Hill Education

Mathematics

Elite Stream

United Arab Emirates Edition

Interactive Student Guide

مجموعات فخر الوطن وعام زايد

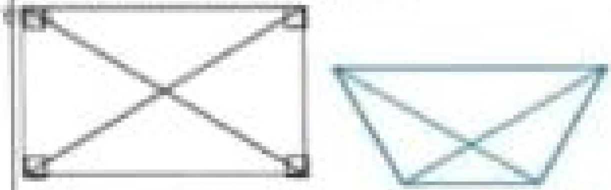
2018 - 2019

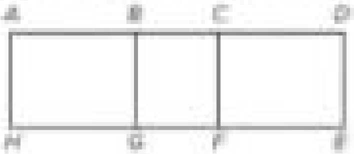
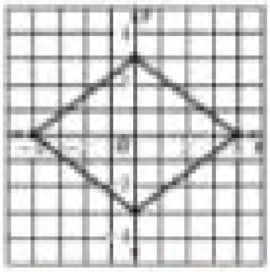
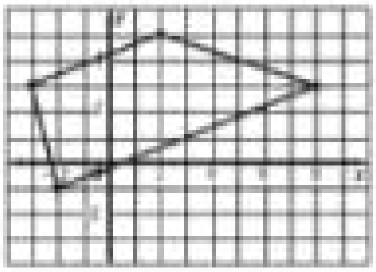
8

Mc
Graw
Hill
Education

11 Quadrilaterals

CHAPTER FOCUS Learn about what you will explore in this chapter. Answer the preview question. As you complete each lesson, return to these pages to check your work.

What You Will Learn	Preview Question
Lesson 11.1: Parallelograms	
Use coordinates to prove simple geometric theorems algebraically. Prove theorems about parallelograms.	Three vertices of a parallelogram are (0, 4), (5, 0), and (10, 4). List all possible locations of the fourth vertex. **(15, 0), (−5, 0), and (5, 8)**
Lesson 11.2: Tests for Parallelograms	
Prove theorems about parallelograms. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Use coordinates to prove simple geometric theorems algebraically.	Reham drew the following figure to prove that if the diagonals of a quadrilateral are congruent, then the quadrilateral is a parallelogram. Draw a counterexample to show that Reham is incorrect. What mistake did Reham make?  **You cannot prove a general statement with an example.** The vertices of quadrilateral *ABCD* are *A*(−2, 3), *B*(1, 6), *C*(7, 3), and *D*(5, 1). Find the slope of each side and determine if *ABCD* is a parallelogram. Explain your reasoning. **Slopes: $\overline{AB}$ has slope $\frac{6-3}{1-(-2)} = 1$, $\overline{BC}$ has slope $\frac{3-6}{7-1} = -\frac{1}{2}$, $\overline{CD}$ has slope $\frac{3-1}{7-5} = 1$, and $\overline{AD}$ has slope $\frac{1-3}{5-(-2)} = -\frac{2}{7}$. *ABCD* is not a parallelogram because it does not have two pairs of parallel opposite sides.** How could point *C* be moved so that *ABCD* is a parallelogram? **Move point *C* to *C*(8, 4), then the slope $\overline{BC}$ is $\frac{4-6}{8-1} = -\frac{2}{7}$ and the slope of $\overline{CD}$ is unchanged. Then *ABCD* has two pairs of parallel opposite sides so it is a parallelogram.**

What You Will Learn	Preview Question
Lesson 11.3: Rectangles	
Prove theorems about parallelograms. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Use coordinates to prove simple geometric theorems algebraically.	ABGH is a rectangle and CDEF is a parallelogram. Can you conclude BCFG is a rectangle? Explain.  No. You know that *BCFG* has two right angles, but you do not know whether *BCFG* is a parallelogram.
Lesson 11.4: Rhombi and Squares	
Prove theorems about parallelograms. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Use coordinates to prove simple geometric theorems algebraically.	Classify the quadrilateral shown on the coordinate grid. Explain.  Rhombus; the side lengths are both 5, so they are congruent. It is not a square because the diagonals are not congruent. The diagonals have lengths 6 and 8.
Lesson 11.5: Trapezoids and Kites	
Use coordinates to prove simple geometric theorems algebraically.	What are the coordinates of the endpoints of the midsegment of the trapezoid?  $\left(-\frac{5}{2}, 1\right)$ and (5, 4)

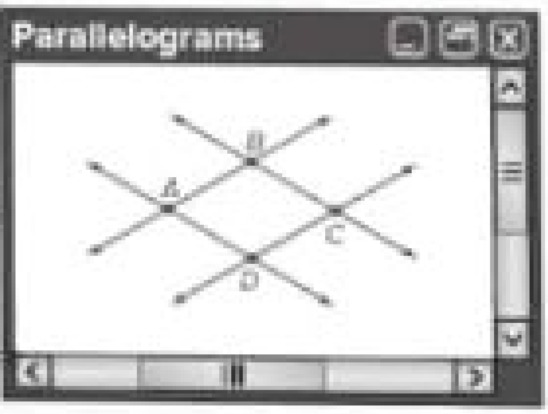
11.1 Parallelograms

Objectives

- Prove theorems about parallelograms using two-column and paragraph proofs.
- Use coordinates to prove theorems about parallelograms.

A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel to one another.

EXAMPLE 1 Investigate the Properties of Parallelograms



EXPLORE Use geometry software to explore parallelograms. As you explore, think about what relationships are true for all parallelograms.

a. **USE TOOLS** Use geometry software to draw two pairs of parallel lines so that one pair intersects the other. Label the points of intersection A, B, C, and D.

b. **USE TOOLS** Use the measurement tools in the softwa to find the measurements listed.

Students' measurements will vary.

AB ________ BC ________ CD ________ DA ________

∠ABC ________ ∠BCD ________ ∠CDA ________ ∠DAB ________

c. **MAKE A CONJECTURE** Make a conjecture about opposite angles and opposite sides in a parallelogram.

Opposite angles are congruent, and opposite sides are congruent.

d. **USE TOOLS** Use geometry software to construct the diagonals of ABCD. Label the point of intersection M. Use the measurement tools to find the measurements listed.

Students' measurements will vary.

AM ________ MC ________ DM ________ MB ________

e. **MAKE A CONJECTURE** Make a conjecture about the diagonals of a parallelogram.

The diagonals bisect each other.

f. FIND A PATTERN Manipulate the parallelogram you constructed in **part a**. Are the relationships that you noticed the same?

Yes; opposite sides and angles remain congruent, and the diagonals bisect each other.

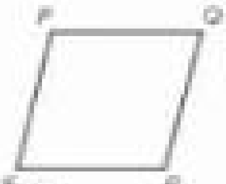
Several properties are true for all parallelograms. All of these properties can be proved using definitions, properties, and theorems that you already know.

KEY CONCEPT

Complete the table by writing the complete theorem that corresponds to each abbreviation.

Theorem	Statement	Abbreviation
11.3	If a quadrilateral is a parallelogram, then its opposite sides are congruent.	Opp. sides of a ▱ are ≅.
11.4	If a quadrilateral is a parallelogram, then its opposite angles are congruent.	Opp. ∠s of a ▱ are ≅.
11.5	If a quadrilateral is a parallelogram, then its consecutive angles are supplementary.	Cons. ∠s in a ▱ are supplementary.
11.6	If a parallelogram has one right angle, then it has four right angles.	If a ▱ has 1 rt. ∠, it has 4 rt. ∠s.
11.7	If a quadrilateral is a parallelogram, then its diagonals bisect each other.	Diag. of a ▱ bisect each other.
11.8	If a quadrilateral is a parallelogram, then each diagonal separates the parallelogram into two congruent triangles.	Diag. separates a ▱ into 2 ≅ △s.

EXAMPLE 1 Prove That Opposite Angles of a Parallelogram Are Congruent



Plan and complete a two-column proof of Theorem 11.4: *If a quadrilateral is a parallelogram, then its opposite angles are congruent.*

a. PLAN A SOLUTION If you wanted to prove that ∠P ≅ ∠R using CPCTC, how could you alter the diagram at the right to assist in your proof? What fact about points and lines justifies your alteration?

Sample answer: I would draw a line from point Q to point S; through any two points, there is exactly one line.

b. **CONSTRUCT ARGUMENTS** Fill in the missing statements and reasons to complete the proof.

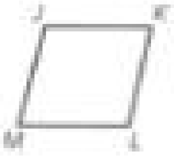
Given: Parallelogram PQRS
Prove: ∠P ≅ ∠R

Statements	Reasons
1. PQRS is a parallelogram.	1. Given
2. $\overline{PQ} \parallel \overline{RS}$ and $\overline{QR} \parallel \overline{SP}$	2. Definition of parallelogram
3. ∠PSQ ≅ ∠RQS and ∠PQS ≅ ∠RSQ	3. Alt. Int. ∠s Thm.
4. $\overline{SQ} \cong \overline{SQ}$	4. Reflexive Property of Congruence
5. △PQS ≅ △RSQ	5. ASA
6. ∠P ≅ ∠R	6. CPCTC

c. **DESCRIBE A METHOD** How could you continue the proof to prove ∠Q ≅ ∠S?

Sample answer: I could draw diagonal $\overline{PR}$ and then prove that △SPR ≅ △QRP. Then I could use CPCTC to prove that ∠Q ≅ ∠S. Corresponding parts of ≅ △s are ≅.

EXAMPLE 3 Prove that Consecutive Angles of a Parallelogram are Supplementary



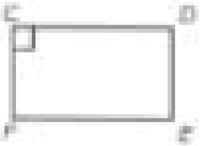
Plan and write a paragraph proof of Theorem 11.5: *If a quadrilateral is a parallelogram, then its consecutive angles are supplementary.*

a. **CONSTRUCT ARGUMENTS** Complete the paragraph proof.

Given: JKLM is a parallelogram.
Prove: ∠J and ∠K, ∠K and ∠L, ∠L and ∠M, and ∠M and ∠J are supplementary.

It is given that JKLM is a parallelogram, so $\overline{JK} \parallel \overline{ML}$ and $\overline{JM} \parallel \overline{KL}$. When two parallel lines are cut by a transversal, consecutive interior angles are supplementary. Therefore ∠J and ∠K, ∠K and ∠L, ∠L and ∠M, and ∠M and ∠J are supplementary.

EXAMPLE 4 Prove Right Angles in Parallelograms



Write a paragraph proof of Theorem 11.6: *If a parallelogram has one right angle, then it has four right angles.*

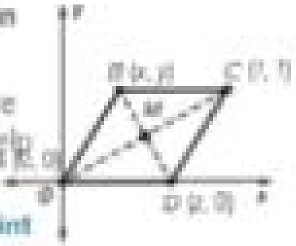
a. **CONSTRUCT ARGUMENTS** Write a paragraph proof.

Given: Parallelogram CDEF; ∠C is a right angle.
Prove: ∠D, ∠E, and ∠F are right angles.

Sample answer: It is given that CDEF is a ▱ and ∠C is a right ∠. Thm. 11.4 says that opp. ∠s of a ▱ are ≅, so ∠C ≅ ∠E; therefore ∠E must also be a rt. ∠. Thm. 11.5 says that consecutive ∠s of a ▱ are supplementary, so ∠C must be supplementary to both ∠F and ∠D. Then $m\angle C + m\angle D = m\angle C + m\angle F = 180$ by the definition of supplementary. So, $m\angle D = m\angle F = 180 - 90 = 90$. Therefore, ∠F and ∠D are also rt. ∠s.

EXAMPLE Prove That Diagonals of a Parallelogram Bisect Each Other

Use algebra to prove Theorem 11.7: *If a quadrilateral is a parallelogram, then its diagonals bisect each other.*



a. **REASON ABSTRACTLY** You know that opposite sides of a parallelogram are parallel and that parallel lines have equal slopes. How can this information help you find the coordinates of point C in parallelogram ABCD?

Sample answer: Point B is y units above and x units to the right of point A. Since $\overline{AB}$ and $\overline{DC}$ have the same slope and $\overline{AD}$ and $\overline{BC}$ have the same slope, point C is also y units above and x units to the right of point D. The coordinates of point C are $(z + x, y)$.

b. **CALCULATE ACCURATELY** What are the midpoints of $\overline{AC}$ and $\overline{BD}$?

The midpoint of $\overline{AC}$ is $\left(\frac{x+z}{2}, \frac{y}{2}\right)$. The midpoint of $\overline{BD}$ is $\left(\frac{x+z}{2}, \frac{y}{2}\right)$.

c. **COMMUNICATE PRECISELY** $\overline{AC}$ and $\overline{BD}$ have the same midpoint, how does that show that the diagonals bisect each other?

Sample answer: By the definition of bisects, any segment, line, or plane that intersects a segment at its midpoint bisects the segment. Since $\overline{AC}$ intersects $\overline{BD}$ at the midpoint of $\overline{BD}$, $\overline{AC}$ bisects $\overline{BD}$. Similarly, since $\overline{BD}$ intersects $\overline{AC}$ at the midpoint of $\overline{AC}$, $\overline{BD}$ bisects $\overline{AC}$. So, the diagonals of a parallelogram bisect each other.

PRACTICE

1. **CONSTRUCT ARGUMENTS** Prove Theorem 11.3: *If a quadrilateral is a parallelogram, then its opposite sides are congruent.*

a. Fill in the missing statements and reasons.

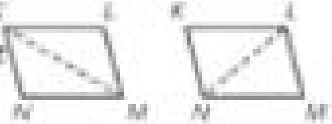
Given: Parallelogram EFGH
Prove: $\overline{EF} \cong \overline{GH}$ and $\overline{HE} \cong \overline{FG}$

Statements	Reasons
1. **$EFGH$ is a parallelogram.**	1. Given
2. $\overline{EF} \parallel \overline{GH}$ and $\overline{HE} \parallel \overline{FG}$	2. **Definition of parallelogram**
3. **$\angle EFH \cong \angle GHF$ and $\angle EHF \cong \angle GFH$**	3. Alt. Int. ∠s Thm.
4. $\overline{FH} \cong \overline{FH}$	4. **Refl. Prop. of ≅**
5. **$\triangle EFH \cong \triangle GHF$**	5. ASA
6. $\overline{EF} \cong \overline{GH}$ and $\overline{HE} \cong \overline{FG}$	6. **CPCTC**

b. Explain why this proof is true for all parallelograms.

Sample answer: All parallelograms have 2 pairs of parallel lines. When a diagonal is drawn, you ca use the ∥ lines to prove that alt. int. ∠s are ≅ and that the △s formed by the diagonal are ≅ △s.

2. **CONSTRUCT ARGUMENTS** Write a paragraph proof of Theorem 11.8. *If a quadrilateral is a parallelogram, then each diagonal separates the parallelogram into two congruent triangles.*

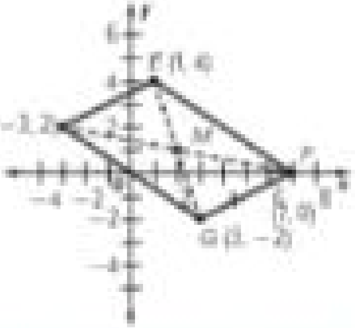


Given: Parallelogram KLMN

Prove: △KLM ≅ △MNK and △NKL ≅ △LMN

Sample answer: Diagonal $\overline{KM}$ divides *KLMN* into 2 △s. It is given that *KLMN* is a parallelogram so $\overline{KL} \parallel \overline{MN}$ and $\overline{KN} \parallel \overline{LM}$. Because they are alt. int. ∠s, ∠*NKM* ≅ ∠*LMK* and ∠*NMK* ≅ ∠*LKM*. By the Refl. Prop. of ≅, $\overline{KM} \cong \overline{KM}$. Using the ASA Theorem, △*NKM* ≅ △*LMK*. Using the diagonal $\overline{LN}$ the same reasoning can be used to prove △*NKL* ≅ △*LMN*.

3. Eiman sketched a parallelogram on a coordinate plane as shown in the diagram.



a. **USE STRUCTURE** Show how she can use algebra to show that the opposite sides of the parallelogram are congruent.

Sample answer: She can use the distance formula.

$DE = \sqrt{(-3-1)^2 + (2-4)^2}\ \sqrt{20} = 2\sqrt{5}$;

$FG = \sqrt{(7-3)^2 + (0-(-2))^2}\ \sqrt{20} = 2\sqrt{5}$;

$EF = \sqrt{(1-7)^2 + (4-0)^2}\ \sqrt{52} = 2\sqrt{13}$; $GD = \sqrt{(3-(-3))^2 + (-2-2)^2}\ \sqrt{52} = 2\sqrt{13}$

b. **USE STRUCTURE** Show how she can use algebra to show that the diagonals bisect each other?

Sample answer: She can use the midpoint formula.

midpoint of $\overline{DF}$ $\left(\frac{-3+7}{2}, \frac{2+0}{2}\right) = (2, 1)$ and midpoint of $\overline{EG}$ $\left(1, \frac{4-2}{2}\right) = (2, 1)$

The diagonals intersect at each other's midpoints, so they bisect each other.

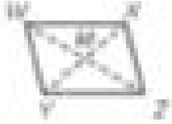
c. **CRITIQUE REASONING** Hala suggests to Eiman that she has found alternative proofs to Theorems 11.3 and 11.7 using algebra. Is she correct? Why or why not?

Sample answer: She is incorrect. Part a does not prove Theorem 11.3 and part b does not prove Theorem 11.7. These demonstrations only verify that the theorem holds for this particular parallelogram. To create a proper proof, Eiman would have to use a general parallelogram.

d. **PLAN A SOLUTION** How could Eiman alter her *DEFG* so that **parts a** and **b** are valid proofs of Theorems 11.3 and 11.7?

Sample answer: She could construct *DEFG* so $\overline{DE} \parallel \overline{FG}$ and $\overline{DG} \parallel \overline{EF}$, and so that the coordinates are in terms of variables instead of any number. This would make *DEFG* a general parallelogram. Eiman could then use the Distance Formula as in part a and the midpoint formula in part b to show that Theorems 11.3 and 11.7 are true for the general parallelogram.

4. **Below is a two-column proof of Theorem 11.7:** ***If a quadrilateral is a parallelogram, then its diagonals bisect each other.***



Given: Parallelogram $WXZY$

Prove: $\overline{WM} \cong \overline{MZ}$ and $\overline{XM} \cong \overline{MY}$

Statements	Reasons
1. $WXZY$ is a parallelogram.	1. Given
2. $\overline{WX} \parallel \overline{ZY}$ and $\overline{WY} \parallel \overline{XZ}$	2. Definition of parallelogram
3. $\angle XWZ \cong \angle YZW$ and $\angle XYZ \cong \angle WXM$	3. Alternate Interior Angles Theorem
4. $\angle WMX \cong \angle YMZ$	4. Vertical angles are congruent.
5. $\triangle WXM \cong \triangle ZYM$	5. AAA
6. $\overline{WM} \cong \overline{MZ}$ and $\overline{XM} \cong \overline{MY}$	6. CPCTC

a. **CRITIQUE REASONING** What is the error in the proof?

AAA is not a valid test for triangle congruence.

b. **CONSTRUCT ARGUMENTS** How would you correct the error?

Sample answer: I would show that opposite sides of the parallelogram are ≅ and then use ASA.

c. Rewrite the proof with your edits.

Statements	Reasons
1. **$WXYZ$ is a parallelogram.**	1. Given
2. **$\overline{WX} \parallel \overline{YZ}$ and $\overline{WY} \parallel \overline{XZ}$**	2. Definition of parallelogram
3. **$\angle XWZ \cong \angle YZW$ and $\angle XYZ \cong \angle WXY$**	3. Alternate Interior Angles Theorem
4. **$\overline{WX} \cong \overline{YZ}$**	4. Theorem 11.3
5. **$\triangle WXM \cong \triangle ZYM$**	5. **ASA**
6. **$\overline{WM} \cong \overline{MZ}$ and $\overline{XM} \cong \overline{MY}$**	6. CPCTC

5. **FIND A PATTERN** Maple Street and Elm Street are parallel. Jefferson Avenue and McKinley Avenue are parallel. Mohammed works at a pizza restaurant on the corner of Jefferson Avenue and Maple Street. He needs to deliver a pizza to a house on the corner of McKinley Avenue and Elm Street. Mohammed is deciding whether to travel on Maple Street and McKinley Avenue or on Jefferson Avenue and Elm Street. If he wants to travel the shortest distance, which route should he choose? Explain your reasoning.

Sample answer: Both routes are the same distance. Because Maple St. is parallel to Elm St. and Jefferson Ave. is parallel to McKinley Ave., the figure formed by the four roads is a parallelogram by the definition of a parallelogram. By Theorem 11.3, opposite sides of a parallelogram are congruent. Therefore, the section of Maple St. has the same length as the section of Elm St. and the section of Jefferson Ave. has the same length as the section of McKinley Ave. Therefore, both routes are the same distance.

11-2 Tests for Parallelograms

Objectives

- Prove theorems about parallelograms by making formal geometric constructions.
- Use coordinates to prove theorems about parallelograms.

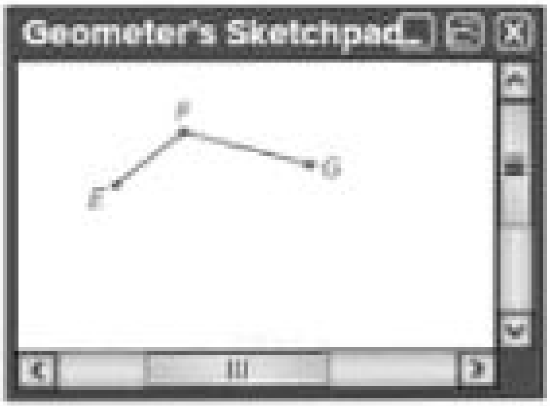
By the definition of a parallelogram, if both pairs of opposite sides of a quadrilateral are parallel, then it is a parallelogram. So, to prove that a quadrilateral is a parallelogram, show that both pairs of opposite sides are parallel.

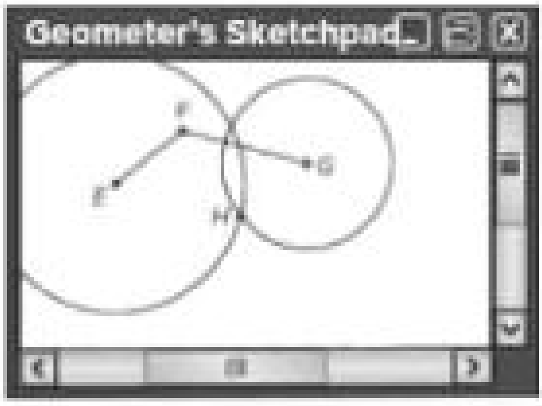
EXAMPLE 1 Investigate Conditions of Parallelograms

EXPLORE Use dynamic geometry software to explore parallelograms. As you do so, think about different ways to use opposite sides to prove that a quadrilateral is a parallelogram.

a. **USE TOOLS** Use The Geometer's Sketchpad to draw two segments that share an endpoint. Label these $\overline{EF}$ and $\overline{FG}$, as shown below on the left.

b. **USE TOOLS** Draw a circle using the "Circle by Center + Radius" tool with center E and radius FG. Draw a second circle in the same way with center G and radius EF, as shown below on the right. Label the point of intersection of the circles H. Construct $\overline{GH}$ and $\overline{HE}$. Then select and hide the circles.





c. **CONSTRUCT ARGUMENTS** Explain why $\overline{EF}$ and $\overline{GH}$ are congruent and why $\overline{FG}$ and $\overline{HE}$ are congruent.

Sample answer: Since the arc of the circle with center G was length $\overline{EF}$, $\overline{EF} \cong \overline{GH}$; similarly, since the arc of the circle with center E was length $\overline{FG}$, $\overline{FG} \cong \overline{HE}$.

d. **USE TOOLS** Use the slope tool to find the slopes of $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, and $\overline{HE}$. What can you conclude about the opposite sides of EFGH?

Sample answer: The slopes of opposite sides are the same, so the opposite sides are parallel.

e. **MAKE A CONJECTURE** What is a reasonable conjecture about parallelograms based on your exploration of quadrilateral *EFGH*?

Sample answer: If both pairs of opposite sides of a quadrilateral are congruent, the quadrilateral is a parallelogram

Showing that opposite sides are parallel is just one way to prove that a quadrilateral is a parallelogram. There are other conditions that ensure a quadrilateral is a parallelogram as well. Remember that only one condition needs to be satisfied to complete a proof.

KEY CONCEPT

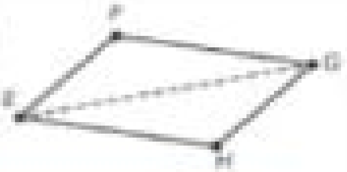
Complete the table by writing the complete theorem that corresponds to each abbreviation.

Theorem	Statement	Abbreviation
11.9	**If both pairs of opposite sides of a quadrilateral are congruent, then it is a parallelogram.**	If both pairs of opp. sides are ≅, then quad. is a ▱.
11.10	**If both pairs of opposite angles of a quadrilateral are congruent, then it is a parallelogram.**	If both pairs of opp. ∠s are ≅, then quad. is a ▱.
11.11	**If the diagonals of a quadrilateral bisect each other, then it is a parallelogram**	If diag. bisect each other, then quad. is a ▱.
11.12	**If one pair of opposite sides of a quadrilateral is both congruent and parallel, then it is a parallelogram**	If one pair of opp. sides is ≅ and ∥, then quad. is a ▱.

EXAMPLE 1 Prove That a Quadrilateral Is a Parallelogram

Complete the two-column proof to show that if both pairs of opposite sides are congruent, then a quadrilateral is a parallelogram.

a. **CONSTRUCT ARGUMENTS** Fill in the missing statements and reasons to complete the proof.



Given: $\overline{EF} \cong \overline{GH}, \overline{FG} \cong \overline{EH}$

Prove: *EFGH* is a parallelogram.

Statements	Reasons
1. Draw $\overline{EG}$.	1. Through any two points there is exactly one line.
2. $\overline{EF} \cong \overline{GH}, \overline{FG} \cong \overline{EH}$	2. Given
3. **$\overline{EG} \cong \overline{GE}$**	3. Reflexive Property of Congruence
4. $\triangle EFG \cong \triangle GHE$	4. **SSS**
5. **$\angle FGE \cong \angle HEG, \angle FEG \cong \angle HGE$**	5. CPCTC
6. $\overline{EF} \parallel \overline{GH}, \overline{FG} \parallel \overline{EH}$	6. **Alternate Interior Angles Converse**
7. *EFGH* is a parallelogram.	7. A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

b. **CRITIQUE REASONING** A student said that because $\angle F \cong \angle H$, it can also be shown that $\triangle EFG \cong \triangle GHE$ by SAS. Do you agree? Justify your answer.

Sample answer: No; it is not given that $\angle F \cong \angle H$ nor has it been proven.

c. **CONSTRUCT ARGUMENTS** Explain why Statement 3 is necessary.

Sample answer: The next step in the proof is to state that $\triangle EFG \cong \triangle GHE$ by SSS. Therefore, it must be explicitly stated that the sides in each corresponding pair, including the shared side represented a$\overline{EG}$ and $\overline{GE}$, are congruent.

d. **CONSTRUCT ARGUMENTS** Describe a general strategy for proving that opposite sides are parallel once it has been shown that the triangles formed by drawing the diagonal are congruent.

Sample answer: Use CPCTC to identify congruent corresponding angles that are also congruent angle pairs when parallel lines are intersected by a transversal. Then, use the converse for that type of angle pair to prove the opposite sides are parallel.

You can solve real-world problems by proving that quadrilaterals are parallelograms.

EXAMPLE 3 Solve a Real-World Problem

Laila is assembling an accordion drying rack that can be folded flat or opened up to various heights as shown. In the figure, E is the midpo$\overline{AE}$ and $\overline{BD}$. Laila wants to show that figure ABCD is a parallelogram.

a. **PLAN A SOLUTION** What must Laila show in order to prove that ABCD is a parallelogram? Explain.

$\overline{AB} \parallel \overline{CD}$, $\overline{BC} \parallel \overline{DA}$; if a quadrilateral has each pair of opposite sides parallel, it is a parallelogram.

b. **USE STRUCTURE** Which segments in the figure have point E as an endpoint? How are these segments related? Explain.

$\overline{AE}$, $\overline{BE}$, $\overline{CE}$, and $\overline{DE}$ $\overline{AE} \cong \overline{CE}$, $\overline{BE} \cong \overline{DE}$; a midpoint divides a line segment into two equal segments.

c. **USE STRUCTURE** Which angles in the figure have point E as a vertex? How are these angles related? Explain.

$\angle AEB$, $\angle BEC$, $\angle CED$, and $\angle DEA$; $\angle AEB \cong \angle CED$, $\angle BEC \cong \angle DEA$; vertical angles are congruent.

You can also prove that a quadrilateral in the coordinate plane is a parallelogram.

EXAMPLE 4 Use Coordinates to Prove a Parallelogram

Point	A	B	C	D
x	?	−2	−1	1
y	?	−2	4	5

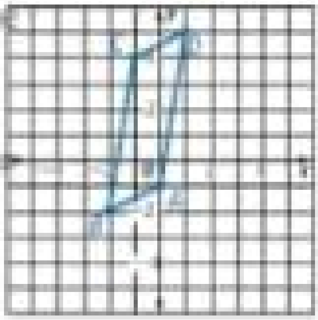
Coordinates for three of the four vertices of parallelogram ABCD are giv the table. Note that the coordinates of point A are missing.

a. **CONSTRUCT ARGUMENTS** What strategy could you use to identify th coordinates of point A? Explain.

Sample answer: Graph B, C, and D. Sketch a ray from D para$\overline{CB}$ and a ray from B parallel to $\overline{CD}$. Label the point of intersection of the rays as A. Use the slope formula to confirm that opposite sides are parallel.

b. USE STRUCTURE Draw parallelogram *ABCD* in the coordinate plane at the right. What are the coordinates of point *A*?

The coordinates of *A* are (0, −1).



c. REASON QUANTITATIVELY Find the slope of each side. The first has been done for you. What does this tell you about the quadrilateral?

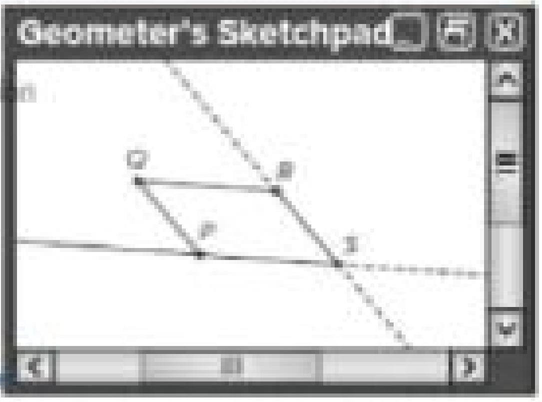
slope of $\overline{AB} = \frac{-2-(-1)}{-2-0}$ or $\frac{1}{2}$ slope of $\overline{DC} = \frac{4-5}{-1-1}$ or $\frac{1}{2}$

slope of $\overline{AD} = \frac{5-(1)}{1-0}$ or 6 slope of $\overline{CB} = \frac{-2-4}{-2-(-1)}$ or 6

Sample answer: Lines that have the same slope are parallel, $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{CB}$. Since opposite sides are parallel, *ABCD* is a parallelogram by definition.

PRACTICE

1. **USE TOOLS** Use dynamic geometry software to construct parallelogram *PQRS* as shown. Remember to select and hide the parallel lines when the construction is complete.



   a. **USE TOOLS** Use the measurement tools in the software to measure ∠*P*, ∠*Q*, ∠*R*, and ∠*S*. What do you notice? Change the shape or location of quadrilateral *PQRS*. Does this relationship remain the same?

   ∠*P* ≅ ∠*R*, ∠*Q* ≅ ∠*S*; these relationships always remain the same.

   b. **MAKE A CONJECTURE** What can you conclude about quadrilateral *PQRS*?

   Sample answer: Both pairs of opposite angles of *PQRS* are congruent, so *PQRS* is a parallelogram

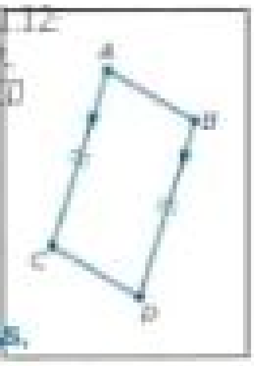
2. **CRITIQUE REASONING** A student wrote the paragraph proof below to prove that *PQRS* is a parallelogram. The proof contains a critical error. Find that error and correct it. Explain.

   a. **Given:** ∠*P* ≅ ∠*R*, ∠*Q* ≅ ∠*S*
   Prove: *PQRS* is a parallelogram.

   Draw $\overline{PR}$ to form two triangles. Because the sum of the angles of one triangle is 180, the sum is 360 for two triangles. So, $m\angle P + m\angle Q + m\angle R + m\angle S = 360$. Since $\angle P \cong \angle R$ and $\angle Q \cong \angle S$, $m\angle P = m\angle R$ and $m\angle Q = m\angle S$. By substitution, $m\angle P + m\angle P + m\angle Q + m\angle Q = 360$. $2(m\angle P) + 2(m\angle Q) = 360$, and dividing by 2 gives $m\angle P + m\angle Q = 180$. Likewise, $2(m\angle P) + 2(m\angle S) = 360$, and dividing by 2 gives $m\angle P + m\angle S = 180$. Consecutive angles are congruent, so $\overline{PS} \parallel \overline{QR}$ and $\overline{PQ} \parallel \overline{SR}$. Opposite sides are parallel, so *PQRS* is a parallelogram.

   Sample answer: The proof should say that consecutive angles are supplementary, not congruent. This proof relies on the condition that if consecutive angles of a quadrilateral are supplementary, then the quadrilateral is a parallelogram.

b. After correcting the proof, the student suggests that you prove Theorem 11.12: If one pair of opposite sides of a quadrilateral is both parallel and congruent, then it is a parallelogram. Complete this proof by drawing quadrilateral ABCD whose opposite sides are congruent and parallel.



Sample answer: $\overline{AC}$ is opposite and congruent $\overline{BD}$. The diagonal $\overline{BC}$, then, is a transversal intersecting $\overline{AC}$ and $\overline{BD}$. Then by the Alternate Interior Angles Theorem, $\angle CBD \cong \angle BCA$. $\overline{BC} \cong \overline{BC}$ by the Reflexive Property of Congruence. Therefore, $\triangle CBD \cong \triangle BCA$ by SAS, and $\overline{AB} \cong \overline{CD}$ by CPCTC. So $ABCD$ is a parallelogram by Theorem 11.9.

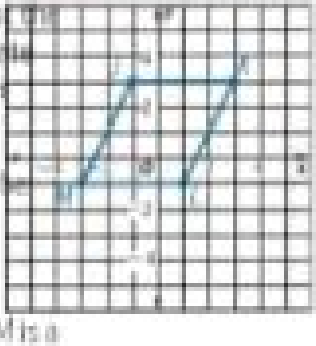
3. **CONSTRUCT ARGUMENTS** Write a paragraph proof of Theorem 11.11: If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

Given: ABCD is a quadrilateral with diagonals that bisect each other.

Prove: ABCD is a parallelogram.

Sample answer: $\overline{AE} \cong \overline{CE}$ and $\overline{BE} \cong \overline{DE}$ because $\overline{AC}$ bisects $\overline{BD}$ and $\overline{BD}$ bisects $\overline{AC}$. Further, $\angle AEB \cong \angle DEC$ and $\angle AED \cong \angle BEC$ because vertical angles are congruent. Therefore $\triangle AEB \cong \triangle CED$ and $\triangle AED \cong \triangle CEB$ by SAS. By CPCTC, $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$.

4. **USE A MODEL** A house lot is in the shape of a parallelogram. To represent the lot in a computer program, the owner draws a quadrilateral in the coordinate plane with vertices at $J(-1, 3)$, $K(2, 3)$, $L(1, -1)$, and $M(-3, -1)$. The owner later discovers that the coordinates for point K were entered incorrectly.



a. **USE STRUCTURE** Identify the correct coordinates for point K. Draw the corresponding parallelogram in the coordinate plane.

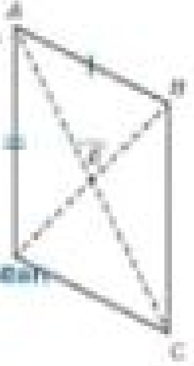
The correct coordinates of K are (3, 3).

b. **CONSTRUCT ARGUMENTS** Use the Slope Formula to prove that $JKLM$ is a parallelogram.

Slopes: $\overline{JK} = 0$; $\overline{LM} = 0$; $\overline{JM} = \frac{-1-3}{-3-(-1)}$, or 2; $\overline{LK} = \frac{3-(-1)}{3-1}$ or 2; opposite sides have the same slope, so $JKLM$ is a parallelogram by definition.

c. **CRITIQUE REASONING** Jamal suggests that it would easier to prove that $JKLM$ is a parallelogram by using Theorem 11.12: *If one pair of opposite sides of a quadrilateral is both parallel and congruent, then the quadrilateral is a parallelogram.* Do you agree? Explain.

Sample answer: Yes; $\overline{JK}$ and $\overline{LM}$ are horizontal, so both have slopes of 0. In addition, it can be determined visually that $JK = 4$ and $LM = 4$; $\overline{JK} \cong \overline{LM}$. One pair of opposite sides is both parallel and congruent, so $JKLM$ is a parallelogram by Theorem 11.12.



5. **USE STRUCTURE** Using your answers from **Example 3 parts b** and **c**, mark the diagram at right for congruency. How are $\triangle AEB$, $\triangle BEC$, $\triangle CED$, and $\triangle DEA$ related? Explain.

$\triangle AEB \cong \triangle CED$, $\triangle BEC \cong \triangle DEA$; the triangles are congruent by SAS.

a. **USE REASONING** How can congruent triangles be used to show that opposite sides of ABCD are parallel? Explain.

Sample answer: BY CPCTC, congruent angles of corresponding triangles can be identified. In $ABCD$, $\angle BAE \cong \angle DCE$ and $\angle DAE \cong \angle BCE$. $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{DA}$ can be shown by the Alternate Interior Angles Converse.

b. **CONSTRUCT AN ARGUMENT** Write a paragraph proof showing that ABCD is a parallelogram.

Sample answer: We are given $\overline{AE} \cong \overline{CE}$ and $\overline{BE} \cong \overline{DE}$ because a midpoint divides a line segment into two equal segments. Also, $\angle AEB \cong \angle CED$ and $\angle BEC \cong \angle DEA$ because vertical angles are congruent. So, $\triangle AEB \cong \triangle CED$ and $\triangle BEC \cong \triangle DEA$ by SAS. By CPCTC, $\angle BAE \cong \angle DCE$ and $\angle DAE \cong \angle BCE$. This means that $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{DA}$ by the Alternate Interior Angles Converse. Opposite sides of $ABCD$ are parallel, so by definition $ABCD$ is a parallelogram.

6. **CRITIQUE REASONING** A student said that another way to prove that the quadrilateral ABCD from **Example 4** is a parallelogram is to use the Distance Formula. Do you agree? Justify your answer. If you agree, complete the proof.

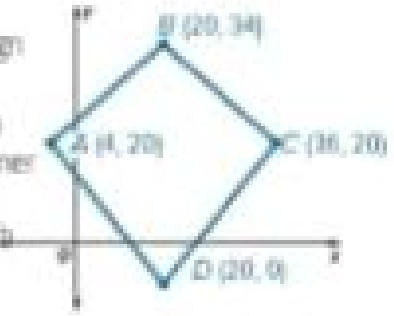
Sample answer: Yes; if opposite sides of a quadrilateral are congruent, then it is a parallelogram. So, use the Distance Formula to show that $AB = DC$ and $AD = CB$.

$AB = \sqrt{((-2) - 0)^2 + ((-2) - (-1))^2} = \sqrt{5}$; $AD = \sqrt{(1 - 0)^2 + (5 - (-1))^2} = \sqrt{37}$;

$CB = \sqrt{((-2) - (-1))^2 + ((-2) - 4)^2} = \sqrt{37}$; $DC = \sqrt{((-1) - 1)^2 + (4 - 5)^2} = \sqrt{5}$;

opposite sides of $ABCD$ are congruent, so $ABCD$ is a parallelogram by Theorem 11.9.

7. **REASON QUANTITATIVELY** A kite manufacturer is experimenting with different designs. The designer wants to modify a current design layout.



a. A current kite design is represented in the coordinate plane with vertices at A(4, 20), B(20, 34), C(36, 20), and D(20, 0). The designer wants to modify the design by shortening the length of the kite. Draw the kite design in the coordinate plane and determine which point should be moved to modify the kite. What are the new coordinates if the kite is to be in the shape of a parallelogram?

Point D should be moved to modify the design; (20, 6).

b. Prove that the new kite design is in fact in the shape of a parallelogram.

Sample answer: The slope $\overline{AB}$ is $\frac{34 - 20}{20 - 4} = \frac{7}{8}$ and the slope $\overline{DC}$ is $\frac{20 - 6}{36 - 20} = \frac{7}{8}$.

Using the distance formula, $AB = \sqrt{(20 - 4)^2 + (34 - 20)^2} = \sqrt{452}$ and

$DC = \sqrt{(36 - 20)^2 + (20 - 6)^2} = \sqrt{452}$. By Theorem 11.12, $ABCD$ is a parallelogram.

11-5 Rectangles

Objectives

- Prove theorems about rectangles using two-column proofs.
- Use coordinates to prove theorems about rectangles.
- Make formal geometric constructions to understand theorems about rectangles.

A **rectangle** is a parallelogram with four right angles. Because a rectangle is a parallelogram, all the properties of parallelograms apply to rectangles.

EXAMPLE 1 Investigate Properties of Rectangles

EXPLORE **Use a compass and straightedge to explore rectangles and their properties.**

a. **USE TOOLS** Construct rectangle *ABCD* using the constructions of parallel and perpendicular lines.

b. **CONSTRUCT ARGUMENTS** Use the definition of a rectangle to explain how you know that *ABCD* is a rectangle.

Sample answer: A rectangle is a parallelogram with 4 rt. ∠s. A parallelogram has 2 pairs of || sides. $\overline{AC}$ || $\overline{BD}$ because both are ⊥ $\overline{AB}$, and if 2 lines are ⊥ to the same line, they are || to each other. $\overline{AB}$ || $\overline{CD}$ since both are $\overline{AC}$. Each ∠ was drawn as a rt. ∠. Therefore, the figure *ABCD* is a rectangle by the definition of rectangle.

c. **MAKE A CONJECTURE** Use a ruler to find *AC* and *BD*. What do you notice? What hypothesis can you make about the diagonals of a rectangle? Can you assume your hypothesis is true based on examples?

Sample answer: *AC* = *BD*; Hypothesis: The diagonals of a rectangle are congruent. No; an example is not a proof. Proofs must be done using logic.

Theorem 11.13: If a parallelogram is a rectangle, then its diagonals are congruent. Because Theorem 11.13 holds for all rectangles, we may add congruent diagonals to the list of properties of a rectangle.

EXAMPLE Prove that the Diagonals of a Rectangle Are Congruent

a. **CONSTRUCT ARGUMENTS** Fill in the missing reasons to complete the proof.

Given: RSTU is a rectangle.

Prove: $\overline{RT} \cong \overline{SU}$

Statement	Reason
1. RSTU is a rectangle	1. Given
2. RSTU is a parallelogram	2. Definition of a rectangle.
3. $\overline{RU} \cong \overline{ST}$	3. Opposite sides of a parallelogram are ≅.
4. $\overline{UT} \cong \overline{UT}$	4. Reflexive Property of Congruence
5. ∠RUT and ∠STU are right angles	5. Definition of a rectangle.
6. ∠RUT ≅ ∠STU	6. All right angles are ≅.
7. △RUT ≅ △STU	7. SAS
8. $\overline{RT} \cong \overline{SU}$	8. CPCTC

b. **REASON ABSTRACTLY** Explain why this proof is true for all rectangles.

Sample answer: The only information that is given is that *RSTU* is a rectangle and you could use the same reasoning for any rectangle no matter how its vertices are labeled.

The converse of Theorem 11.13 is true as well.

Theorem 11.14: If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

Finding congruent diagonals is a valuable tool for proving that a parallelogram is a rectangle.

EXAMPLE Apply Properties of Rectangles

PLAN A SOLUTION **Hana was asked to prove that the figure at the right is a rectangle. She has a ruler but no protractor or other tool to measure angles. How can she prove that the figure is a rectangle?**

a. State the theorem that can be used to prove that the figure above is a parallelogram using only a ruler.

Sample answer: Theorem 11.9 states that if both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

b. State the theorem that can be used to prove that a parallelogram is a rectangle using only a ruler.

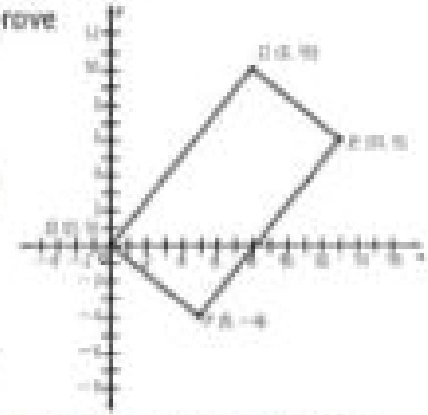
Sample answer: Theorem 11.14 states that if the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

c. Using the theorems found in **part a** and **b**, describe how Hana could show that the figure is a rectangle.

Sample answer: Hana could measure all 4 sides. If opposite sides are ≅, it is a parallelogram. She could then measure the diagonals. If they are ≅, it is a rectangle.

EXAMPLE 4 Proving Rectangles on a Coordinate Plane

The coordinates of a quadrilateral are shown. Use algebra to prove that it is a rectangle.



a. **PLAN A SOLUTION** Describe how you could construct an argument to prove that DEFG is a rectangle?

Sample answer: If opposite sides are equal, *DEFG* is a parallelogram. Find the slopes of the sides to see if consecutive sides are perpendicular.

b. **REASON QUANTITATIVELY** Show that DEFG is a rectangle. Explain.

Sample answer: *DE* $= \sqrt{(8-13)^2+(10-6)^2} = \sqrt{41}$ and *FG* $= \sqrt{(5-0)^2+(-4-0)^2} = \sqrt{41}$;

EF $= \sqrt{(13-5)^2+(6-(-4))^2} = \sqrt{164} = 2\sqrt{41}$ and *GD* $= \sqrt{(8-0)^2+(10-0)^2} = \sqrt{164} = 2\sqrt{41}$.

The lengths of the opposite sides are equal, so *DEFG* is a parallelogram.

EF $= \sqrt{(1-7)^2+(4-0)^2} = \sqrt{52} = 2\sqrt{13}$ and *DF* $= \sqrt{(-3-3)^2+(2-(-2))^2} = \sqrt{52} = 2\sqrt{13}$;

slope $\overline{DE} = -\frac{4}{5}$, slope $\overline{EF} = \frac{5}{4}$, slope $\overline{FG} = -\frac{4}{5}$, and slope $\overline{GD} = \frac{5}{4}$; the slopes of consecutive sides are negative reciprocals, so all angles are right angles. Since *DEFG* is a parallelogram and all of the angles are rt. angles, *DEFG* is a rectangle.

PRACTICE

1. **CRITIQUE REASONING** Abdullah argues that to prove a quadrilateral is a rectangle, it is sufficient to prove that its diagonals are congruent. Do you agree? If so, explain why. If not, explain and draw a counterexample.

No; a rectangle must be a parallelogram in addition to having ≅ diagonals.

Sample answer: An isosceles trapezoid is a counterexample.

a. How can you alter Abdullah's argument to make it correct?

Sample answer: To prove a parallelogram is a rectangle, it is sufficient to prove that its diagonals are congruent

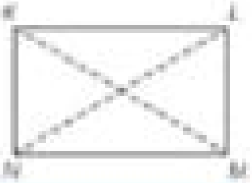
b. Abdullah also says that to show that two diagonals of a quadrilateral are congruent, it is sufficient to show that all four angles of the quadrilateral are right angles. Is he correct? Explain.

Yes; if all four angles of a quadrilateral are right angles, then both pairs of opposite angles are congruent. So, by Theorem 11.10 the quadrilateral is a parallelogram. If a parallelogram has four right angles, then it is a rectangle. By Theorem 11.13, if a parallelogram is a rectangle, then its diagonals are congruent.

2. **CONSTRUCT ARGUMENTS** Fill in the missing parts to complete the proof.

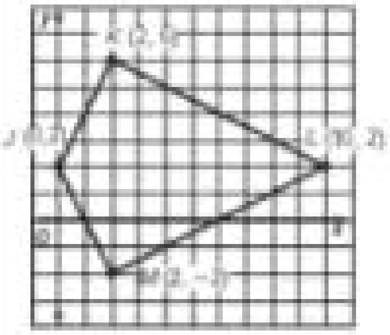
Given: Parallelogram *KLMN*, $\overline{KM} \cong \overline{LN}$

Prove: *KLMN* is a rectangle.



Statement	Reason
1. *KLMN* is a parallelogram, $\overline{KM} \cong \overline{LN}$	1. **Given**
2. $\overline{KN} \cong \overline{LM}$	2. **Opp. sides of a parallelogram are ≅.**
3. $\overline{NM} \cong \overline{NM}$	3. **Reflexive Property of Congruence**
4. **△*KNM* ≅ △*LMN***	4. SSS Theorem
5. ∠*KNM* ≅ ∠*LMN*	5. **Corresponding parts of ≅ △s are ≅.**
6. ∠*KNM* and ∠*LMN* are supplementary.	6. **Cons. ∠s of a parallelogram are supplementary.**
7. ∠*KNM* and ∠*LMN* are right angles.	7. **If 2 ∠s are ≅ and supplementary, they are rt. ∠s.**
8. **∠*NKL* and ∠*MLK* are right angles.**	8. If a parallelogram has 1 rt. ∠, then it has 4 rt. ∠s.
9. *KLMN* is a rectangle.	9. **Definition of rectangle**

3. Students were asked to find whether the quadrilateral formed by connecting *J*(0, 2), *K*(2, 6), *L*(10, 2), and *M*(2, −2) is a rectangle. Two students' solutions are shown below.

Asma	Badria
I found the lengths of the sides. *JK* = *MJ* = $\sqrt{20}$ and *KL* = *LM* = $4\sqrt{5}$, so pairs of sides are equal, and it is a parallelogram. I found the slope of each side: $\overline{JK}$ = 2, $\overline{KL} = -\frac{1}{2}$, $\overline{LM} = \frac{1}{2}$, $\overline{MJ}$ = −2. That shows that $\overline{JK} \perp \overline{KL}$ and $\overline{LM} \perp \overline{MJ}$. If 1 ∠ of a parallelogram is a rt. ∠, then all 4 angles are rt. ∠s. That makes it a rectangle.	I graphed it, and I could see that it is not a rectangle. 

a. **REASON ABSTRACTLY** Evaluate each students' solution.

Sample answer: Asma: Incorrect; consecutive sides (not opposite sides) are congruent, so it is not necessarily a parallelogram. Badria: correct, although she did not prove her answer.

b. **CONSTRUCT ARGUMENTS** Explain how you would solve the problem.

Sample answer: I would find the slopes and lengths like Asma but would analyze the results with a graph like Badria used so I could visualize the sides and angles I was analyzing.

11.4 Rhombi and Squares

Objectives

- Determine whether a figure defined by four points on a coordinate plane is a rhombus or square.
- Prove theorems about rhombi and squares.
- Construct rhombi and squares.

A rhombus is a quadrilateral with four congruent sides. Since the opposite sides are congruent, a rhombus is also a parallelogram and has all of the properties of a parallelogram. Additionally, the diagonals of rhombi have the following properties:

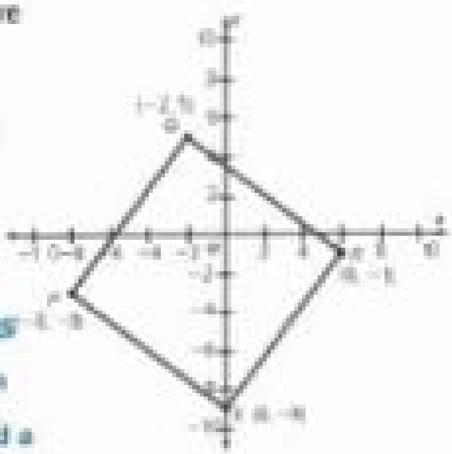
If a parallelogram is a rhombus, then its diagonals are perpendicular.

If a parallelogram is a rhombus, then each diagonal bisects a pair of opposite angles.

A square is a parallelogram with four congruent sides and four congruent angles. This makes a square a rectangle and a rhombus. All of the properties of parallelograms, rectangles, and rhombi also apply to squares.

EXAMPLE 1 Classify a Quadrilateral

The coordinates of a quadrilateral are shown. Use algebra to prove that *PQRS* is a square.



a. **PLAN A SOLUTION** How can you show that PQRS is a square? Include how you can use the Distance Formula and the slopes of perpendicular lines.

Sample answer: If opposite sides are equal, *PQRS* is a parallelogram. If the slopes of the diagonals are perpendicular, then *PQRS* is a rhombus. If the length of the diagonals are equal, then *PQRS* is a rectangle. A quadrilateral that is a rectangle and a rhombus is a square.

b. **REASON QUANTITATIVELY** Prove that PQRS is a square. Explain.

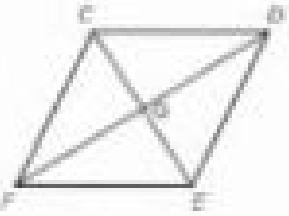
Sample answer: *QR* $\sqrt{(-2-6)^2+(5+1)^2} = 10$, *PS* $\sqrt{(0+8)^2+(-9+3)^2} = 10$, *RS* $= \sqrt{(6-0)^2+(-1+9)^2} = 10$, and *PQ* $\sqrt{(-8+2)^2+(-3-5)^2} = 10$. Opposite sides are equal, so *PQRS* is a parallelogram. The slope $\overline{QS}$ is $\frac{5-(-9)}{-2-0} = -7$ and the slope $\overline{PR}$ is $\frac{-1-(-3)}{6-(-8)} = \frac{1}{7}$, so the slopes of the diagonals are ⊥. This shows that *PQRS* is a rhombus. $\sqrt{(-2-0)^2+(5-(-9))^2} = 10\sqrt{2}$ and *PR* $= \sqrt{(-8-6)^2+(-3-(-1))^2} = \sqrt{[illegible]}$. Therefore, the lengths of diagonals are equal, so *PQRS* is a rectangle. Since *PQRS* is a rectangle and a rhombus, *PQRS* is a square.

c. **CRITIQUE REASONING** Hareb believes quadrilateral PQRS is a square if the diagonals are congruent and perpendicular, and a rhombus if the diagonals are perpendicular but not congruent. Ismail believes this information is not sufficient to classify the quadrilateral. Who is correct? Explain your answer.

Sample answer: Ismail is correct. Hareb's claims are only true if *PQRS* is a parallelogram. *PQRS* coul

be a kite or isosceles trapezoid with perpendicular or perpendicular and congruent diagonals.

EXAMPLE Proving a Parallelogram is a Rhombus



CONSTRUCT ARGUMENTS Prove that if the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

Given: CDEF is a parallelogram, $\overline{CE} \perp \overline{DF}$

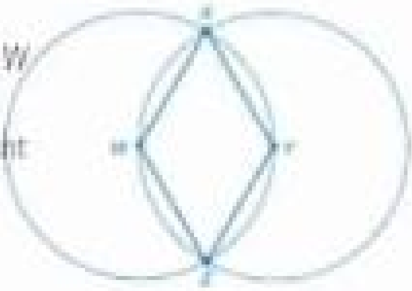
Prove: CDEF is a rhombus

Statements	Reasons
1. CDEF is a parallelogram, $\overline{CE} \perp \overline{DF}$	1. **Given**
2. $\overline{DG} \cong \overline{FG}$	2. **Diagonals of a parallelogram bisect each other.**
3. $\angle CGF$ and $\angle CGD$ are right angles.	**Definition of ⊥**
4. **$\angle CGF \cong \angle CGD$**	4. All right angles are congruent.
5. $\triangle CGF \cong \triangle CGD$	5. **SAS**
6. $\overline{CF} \cong \overline{CD}$	6. **CPCTC**
7. **$\overline{CF} \cong \overline{DE}$, $\overline{CD} \cong \overline{EF}$**	7. Opposite sides of a parallelogram are congruent.
8. **$\overline{CD} \cong \overline{DE} \cong \overline{EF} \cong \overline{CF}$**	8. **Transitive Property of ≅**
9. **CDEF is a rhombus**	9. **Definition of a rhombus**

EXAMPLE Constructing a Rhombus

a. **USE TOOLS** Follow these steps to construct rhombus WXYZ.

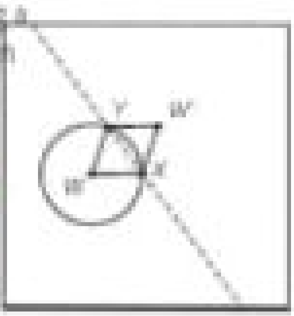
Sample answer:



- In the space to the right, use your compass to construct circle W containing point Y.
- With the compass at point Y, construct circle Y containing point W.
- Label the points of intersection X and Z.
- Draw $\overline{WX}$, $\overline{XY}$, $\overline{YZ}$, and $\overline{WZ}$.

b. **COMMUNICATE PRECISELY** Write a paragraph proof to prove WXYZ is a rhombus.

Sample answer: Circle *W* and circle *Y* are congruent circles because they both have a radius of length *WY*. The four sides of *WXYZ* are each radii of the two congruent circles so these sides are congruent to each other. This makes quadrilateral *WXYZ* a rhombus.



c. **CRITIQUE REASONING** Jassim says that he can construct a rhombus using a circle drawn on patty paper. He constructs circle W through point X and then draws chord $\overline{XY}$, where Y is a point on the circle. He then folds the paper to reflect W across $\overline{XY}$. Is $WXW'Y$ a rhombus? Explain.

Sample answer: Yes. $\overline{WX}$ and $\overline{WY}$ are radii of the same circle, so they are congruent. A reflection is a rigid transformation, so $\triangle WXY \cong \triangle W'XY$. Therefore $\overline{WX} \cong \overline{W'X}$ and $\overline{WY} \cong \overline{W'Y}$. The four sides of $WXW'Y$ are congruent to each other, so $WXYZ$ a rhombus.

d. **USE TOOLS** Use Jassim's process to construct a square. Explain.

Sample answer: Construct circle B. Draw diameter AC through B and construct a perpendicular bisector to AC. Label the intersection of this line and the circle point D. Draw a line that passes through points D and C. Reflect $\triangle BCD$ through $\overline{DC}$ to form square $BCB'D$.

PRACTICE

1\. a. **CRITIQUE REASONING** Buthaina is using coordinate geometry to classify quadrilateral $ABCD$. She finds $AB = BC = CD = AD$ and decides $ABCD$ is a rhombus, but not a square. Do you agree with her conclusion? Explain your answer.

Sample answer: No. Buthaina is correct that $ABCD$ is a rhombus because it has four congruent sides, but $ABCD$ could also be a square. She needs to compare the slopes of a pair of adjacent sides or the lengths of the diagonals.

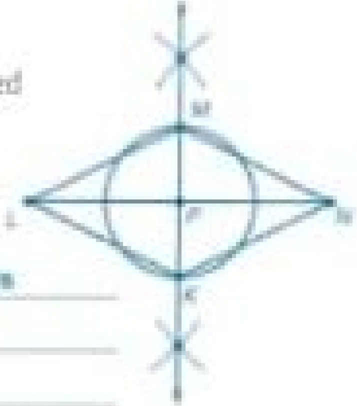
b. **CRITIQUE REASONING** Buthaina is attempting to classify another quadrilateral $EFGH$. She finds that the diagonals $EG = FH = \overline{5}$. Is it possible for a quadrilateral to be both a rectangle and a rhombus?

Yes it is possible. If $EFGH$ is a parallelogram, then by Theorem 11.14 it is a rectangle because its diagonals are congruent. If the diagonals bisect each other, then $EFGH$ is also a rhombus.

2\. **COMMUNICATE PRECISELY** The vertices of parallelogram $QRST$ are $Q(-4, 7)$, $R(1, 9)$, $S(6, 7)$, and $T(1, 5)$. Determine whether $QRST$ is rectangle, rhombus, or square. List all that apply and explain your answer.

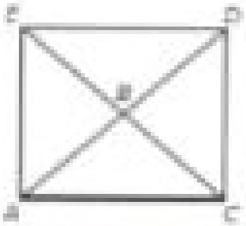
Sample answer: $QRST$ is a rhombus: $QR = RS = ST = QT$ so the figure has four congruent sides. However, $QRST$ is not a rectangle or a square because the diagonals are not congruent: $QS = 10$ and $RT = 4$.

3. **COMMUNICATE PRECISELY** Majed drew segment $\overline{LN}$ and constructed its perpendicular bisector. He labeled the intersection *P*. Then he constructed circle *PM*, where *M* is on the perpendicular line. He labeled the other intersection of the circle with the line point *K*. Write a paragraph proof that quadrilateral *KLMN* is a rhombus. Following Majed's method, make your own construction using a compass and straightedge.



Sample answer: *KLMN* is a parallelogram because the diagonals bisect each other. *P* is the midpoint of $\overline{LN}$ by construction and the midpoint of $\overline{MK}$ because $\overline{MK}$ is a diameter of circle *P*. By construction $\overline{MK}$ is perpendicular to $\overline{LN}$; therefore, *KLMN* is a rhombus.

4. **CONSTRUCT ARGUMENTS** Prove that if the triangle formed by the diagonals and a side of a parallelogram is isosceles, then the parallelogram is a rectangle.



Given: ACDE is a parallelogram, and △ACB is an isosceles triangle with $\overline{AC}$ base

Prove: ACDE is a rectangle

Statements	Reasons
1. **ACDE is a parallelogram; △ACB is isosc.**	1. Given
2. $\overline{AB} \cong \overline{CB}$	2. Def of Isosceles
3. **$\overline{EB} \cong \overline{BC}$ and $\overline{AB} \cong \overline{BD}$**	3. Diags. of a parallelogram bisect each other
4. $AB = CB = EB = BD$	4. Definition of congruence and Transitive Prop.
5. **$AB + BD = EB + CB$**	5. Addition Property of Equality
6. ACDE is a rectangle	6. If diagonals of par. are equal, it is a rectangle

5. **USE STRUCTURE** If the diagonals of quadrilateral *LMNP* form congruent triangles, prove that the quadrilateral is a square. Draw and label a figure and write a paragraph proof.

Sample answer: Given *LMNP* and △*PQN* ≅ △*NQM* ≅ △*MQL* ≅ △*LQP*, then $\overline{PN} \cong \overline{NM} \cong \overline{ML} \cong \overline{LP}$ by CPCTC. *LMNP* is a parallelogram since both pairs of opposite sides are congruent. Then *LMNP* is a rhombus because you can pick any one pair of consecutive sides and they are congruent. $\overline{PQ} \cong \overline{NQ} \cong \overline{MQ} \cong \overline{LQ}$ by CPCTC, which means these segments have the same length. Therefore, $PQ + MQ = NQ + LQ$, so $PM = NL$ and $\overline{PM} \cong \overline{NL}$. Thus, *LMNP* is a rectangle, and, therefore, a square.

Scanned by CamScanner

11.5 Trapezoids and Kites

Objectives

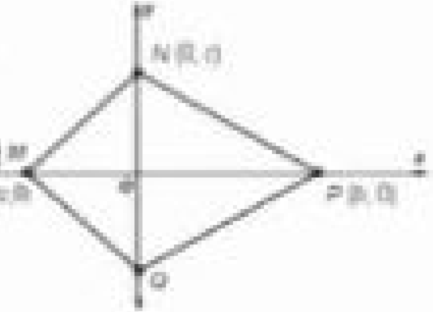
- Determine whether a figure defined by four points is a trapezoid or kite.
- Prove theorems about trapezoids and kites using coordinates.

A trapezoid is a quadrilateral with exactly one pair of parallel sides called bases. The nonparallel sides are called legs. The midsegment of a trapezoid is the segment that connects the midpoints of the legs of a trapezoid.

If the legs of a trapezoid are congruent, then it is an isosceles trapezoid.

A kite is a quadrilateral with exactly two pairs of consecutive congruent sides.

EXAMPLE 1 Using Coordinate Geometry to Explore Kites



a. **INTERPRET PROBLEMS** Without introducing new variables, state the coordinates of point Q, assuming that MNPQ is a kite.

Q (0, −c)

b. **USE STRUCTURE** Rana notices that the figure may be analyzed as two triangles, △MNP and △MQP. What can we reason about the opposite angles ∠N and ∠Q? Explain.

Sample answer: ∠N ≅ ∠Q. If MN = MQ and NP = PQ, △MNP ≅ △MQP by SSS and ∠N ≅ ∠Q by CPCTC.

c. **CONSTRUCT ARGUMENTS** Given kite MNPQ, show that ∠NMQ ≅ ∠NPQ.

Sample answer: From part b, ∠N ≅ ∠Q. If ∠NMQ ≅ ∠NPQ, then MNPQ is a parallelogram by the definition of a parallelogram. This cannot be true since MNPQ is a kite, so ∠NMQ ≇ ∠NPQ.

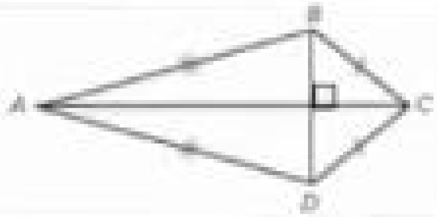
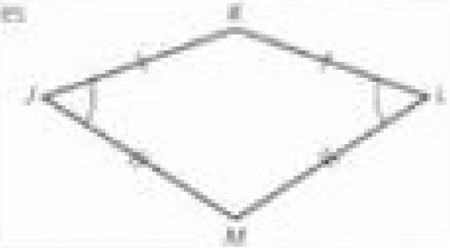
d. **CONSTRUCT ARGUMENTS** Given kite MNPQ, show $\overline{MP}$ is perpendicular to $\overline{NQ}$.

Sample answer: The slope $\overline{MP}$ is $\frac{0-0}{b--a}$ or 0. So $\overline{MP}$ is a horizontal line. The slope $\overline{NQ}$ is $\frac{-c-c}{0-0}$. So, the slope of $\overline{NQ}$ is undefined, and it is a vertical line. Because $\overline{MP}$ is horizontal and $\overline{NQ}$ is vertical, they are perpendicular.

e. **REASON ABSTRACTLY** If a = b = c, is MNPQ still a kite? Justify your answer. Categorize the quadrilateral as specifically as you can.

Sample answer: No; If a = b = c, then MO = OP = NO = QO, so MNPQ is a parallelogram. Since the diagonals are on the x- and y-axes, they are perpendicular; so MNPQ is a rhombus, and since the diagonals are congruent, MNPQ is a rectangle. Therefore, MNPQ is a square.

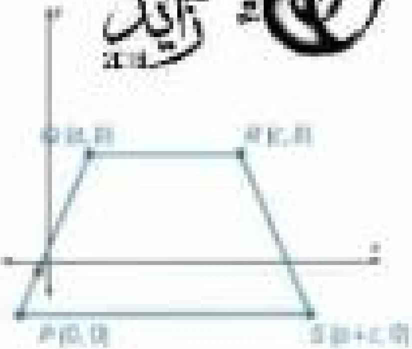
KEY CONCEPT Kites

11.25	If a quadrilateral is a kite, then its diagonals are perpendicular. **Example** If quadrilateral $ABCD$ is a kite, then $\overline{AC} \perp \overline{BD}$.	
11.26	If a quadrilateral is a kite, then exactly one pair of opposite angles is congruent. **Example** If quadrilateral $JKLM$ is a kite, $\overline{JK} \cong \overline{KL}$, and $\overline{JM} \cong \overline{LM}$, then $\angle J \cong \angle L$ and $\angle K \not\cong \angle M$.	

EXAMPLE 7 Using Coordinate Geometry to Classify and Prove Theorems About Trapezoids

a. PLAN A SOLUTION Plot quadrilateral $PQRS$ with vertices $P(0, 0)$, $Q(a, b)$, $R(c, b)$, and $S(a + c, 0)$, where $a > 0$, $b > 0$, and $c > 0$, on the axes to the right.

Sample answer:



b. CALCULATE ACCURATELY Khalid says $PQRS$ is an isosceles trapezoid with bases $\overline{QR}$ and $\overline{PS}$. Do you agree with Khalid? Justify your answer.

Sample answer: Yes; I can show that $\overline{QR}$ and $\overline{PS}$ are parallel (same slope) but have different lengths, while $\overline{PQ}$ and $\overline{RS}$ have different slopes but equal lengths.

$PQ = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$, slope of $\overline{PQ} = \frac{b - 0}{a - 0} = \frac{b}{a}$

$QR = \sqrt{(c - a)^2 + (b - b)^2} = c - a$, slope of $\overline{QR} = \frac{b - b}{c - a} = \frac{0}{c - a} = 0$

$RS = \sqrt{((a + c) - c)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$, slope of $\overline{RS} = \frac{0 - b}{(a + c) - c} = -\frac{b}{a}$

$PS = \sqrt{((a + c) - (0))^2 + (0 - 0)^2} = a + c$, slope of $\overline{PS} = \frac{0 - 0}{(a + c) - 0} = \frac{0}{a + c} = 0$

c. CONSTRUCT ARGUMENTS Show that if a trapezoid is isosceles, then its diagonals are congruent.

Given: $PQRS$ is an isosceles trapezoid with bases $\overline{QR}$ and $\overline{PS}$.

Prove: $\overline{QS} \cong \overline{PR}$

$\overline{QP} \cong \overline{RS}$, since $PQRS$ is an isosceles trapezoid. $\overline{PS} \cong \overline{PS}$ by the Reflexive Property. $\angle QPS \cong \angle RSP$ since if a trapezoid is isosceles, then each pair of base angles is congruent. Therefore, $\triangle QPS \cong \triangle RSP$ by SAS. $\overline{QS} \cong \overline{RP}$ by CPCTC.

d. CRITIQUE REASONING Halima says quadrilateral *PQRS* is an isosceles trapezoid because the diagonals are congruent. Would this be enough information to classify *PQRS* as an isosceles trapezoid? Explain.

No; Sample answer: It would be enough to classify a trapezoid as isosceles, but it is not enough information to classify any quadrilateral as an isosceles trapezoid: squares, rectangles, and some non-special quadrilaterals also have congruent diagonals.

PRACTICE

1. **REASON QUANTITATIVELY** $W(-1, -11)$ and $X(1, 1)$ are two vertices of quadrilateral *WXYZ*.

   a. Find coordinates *Y* and *Z* that will make *WXYZ* a kite. Justify your answer.

   Sample answer: Let *Z* be (−11, 1) and *Y* be (−1, 13) so $\overline{ZX}$ is horizontal and $\overline{WY}$ is vertical. Make the diagonals perpendicular so $\overline{ZX}$ bisects $\overline{WY}$.

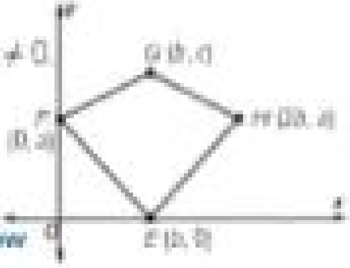
   b. Find coordinates *Y* and *Z* that will make *WXYZ* an isosceles trapezoid. Justify your answer.

   Sample answer: Let $\overline{WX}$ be one leg. The slope $\overline{WX}$ is 6, and the length $\overline{WX}$ is $\sqrt{(-11-1)^2+(-1-1)^2}=\sqrt{148}$. Make the bases parallel to the *x*-axis. Let *Y* be (6, 1). Then *Z* will be (*x*, −11). Find the value of *x* by setting *YZ* equal to *WX* and solving for *x*: $\sqrt{148}=\sqrt{(x-6)^2+(-11-1)^2}$, $\sqrt{148}=\sqrt{(x-6)^2+144}$, so $(x-6)^2=4$. Therefore, $x=8$ or 4. If *Z* is (4, −11), then *WXYZ* will be a parallelogram. So, *Z* must be (8, −11).

   c. If the coordinates of *Y* and *Z* are (4, 1) and (4, −11), identify the shape of *WXYZ*.

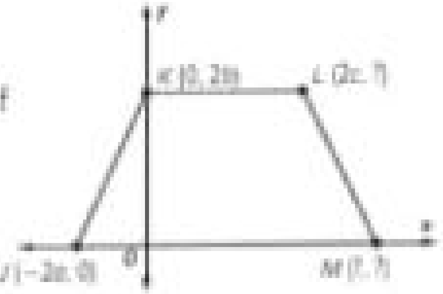
   WXYZ is a trapezoid; it has two parallel sides with legs that are not congruent.

2. **CONSTRUCT ARGUMENTS** *EFGH* is shown to the right, with $a \neq b \neq c \neq 0$. Use coordinate geometry to prove that *EFGH* is a kite.



Sample answer: First note that $\overline{FH}$ is a horizontal line segment $\left(m=\frac{a-a}{2b-b}=\frac{0}{b}=0\right)$, while $\overline{GE}$ is a vertical line segment $\left(m=\frac{c-0}{b-b}=\frac{c}{0}\right)$. Therefore, the two diagonals are perp. Then show that one of the diagonals bisects the other. The line containing $\overline{GE}$ is $x=b$, and the line containing $\overline{FH}$ is $y=a$. Those two lines intersect at (b, a). Call this *M*. Then $FM=\sqrt{(0-b)^2+(a-a)^2}=\sqrt{b^2}=b$ and $MH=\sqrt{(2b-b)^2+(a-a)^2}=\sqrt{b^2}=b$, so $\overline{GE}$ bisects $\overline{FH}$ at *M*. Further, consecutive sides are congruent since $FG=\sqrt{(b-0)^2+(c-a)^2}=\sqrt{b^2+(c-a)^2}=\sqrt{(b-2b)^2+(c-a)^2}=GH$ and $FE=\sqrt{(b-0)^2+(0-a)^2}=\sqrt{b^2+a^2}=\sqrt{(b-2b)^2+(0-a)^2}=EH$.

3. **USE STRUCTURE** Isosceles trapezoid $JKLM$ is shown to the right.



a. Without introducing new variables, state the coordinates of points L and M.

L **$(2c, 2b)$**　　　　M **$(2c + 2a, 0)$**

b. Let point P be the midpoint $\overline{JK}$ and Q be the midpoint of $\overline{LM}$. Use coordinate geometry to show that the midsegment of $JKLM$ is parallel to the bases of $JKLM$ and equal to half the sum of their lengths.

Sample answer: Let $P = (-a, b)$; $Q = (2c + a, b)$.

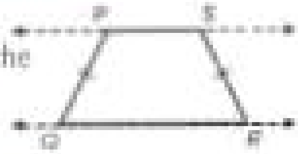
$KL = \sqrt{(2c - 0)^2 + (2b - 2b)^2} = 2c$; slope $\overline{KL} = \frac{2b - 2b}{2c - 0} = \frac{0}{2c} = 0$

$JM = \sqrt{[(2c + 2a) - (-2a)]^2 + (0 - 0)^2} = 2c + 4a$; slope $\overline{JM} = \frac{0 - 0}{(2c + 2a) - (-2a)} = 0$

$PQ = \sqrt{[(2c + a) - (-a)]^2 + (b - b)^2} = 2c + 2a$; slope $\overline{PQ} = \frac{b - b}{(2c + a) - (-a)} = 0$

Since $\overline{PQ}$, $\overline{KL}$, and $\overline{JM}$ have the same slope, they are parallel. Also, since $2c + 2a = \frac{1}{2}[(2c) + (2c + 4a)]$, PQ is equal to half the sum of KL and JM.

c. Let point R be the midpoint $\overline{JM}$ and S be the midpoint $\overline{KL}$. Use coordinate geometry to show that $PSQR$, the quadrilateral connecting the midpoints of $JKLM$, is a rhombus.



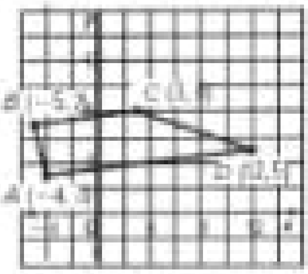
Sample answer: Let $R = (c, 0)$; $S = (c, 2b)$. To show $PSQR$ is a rhombus, we must show that it is a parallelogram, which we can do by showing that $\overline{PS}$ and $\overline{QR}$ are parallel and congruent. We must show that $\overline{PQ}$ and $\overline{RS}$ are perpendicular.

$PS = \sqrt{[c - (-a)]^2 + (2b - b)^2} = \sqrt{(c + a)^2 + b^2}$; slope of $\overline{PS} = \frac{2b - b}{c - (-a)} = \frac{b}{c + a}$;

$QR = \sqrt{(2c + a - c)^2 + (b - 0)^2} = \sqrt{(c + a)^2 + b^2}$; slope of $\overline{QR} = \frac{b - 0}{2c + a - c} = \frac{b}{c + a}$

slope of $\overline{PQ} = \frac{b - b}{2c + a - (-a)} = \frac{0}{2c + 2a} = 0$; and slope of $\overline{RS} = \frac{2b - 0}{c - c} = \frac{2b}{0}$, which is undefined.

4. **CONSTRUCT ARGUMENTS** Quadrilateral $ABCD$ is shown to the right.



a. Show that $ABCD$ is a trapezoid.

The slope of $\overline{BC}$ is $\frac{8 - 7}{3 - (-5)} = \frac{1}{8}$ and the slope of $\overline{AD}$ is $\frac{5 - 3}{12 - (-4)} = \frac{1}{8}$. This means that $ABCD$ has one pair of opposite sides that are parallel. Therefore, it is a trapezoid.

b. Prove that $ABCD$ is not an isosceles trapezoid.

The legs of the trapezoid are $\overline{AB}$ and $\overline{CD}$. Using the distance formula, $AB = \sqrt{17}$ and $CD = \sqrt{90}$. This means that $\overline{AB}$ is not congruent to $\overline{CD}$, so $ABCD$ cannot be an isosceles trapezoid.

Performance Task

Identifying a Quadrilateral

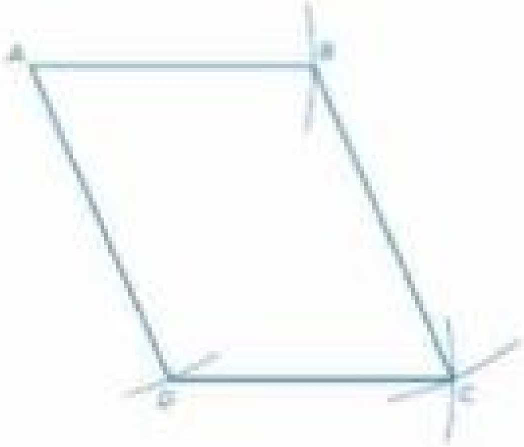
Provide a clear solution to the problem. Be sure to show all of your work, include all relevant drawings, and justify your answers.

You can identify a quadrilateral using the theorems you have learned.

Part A

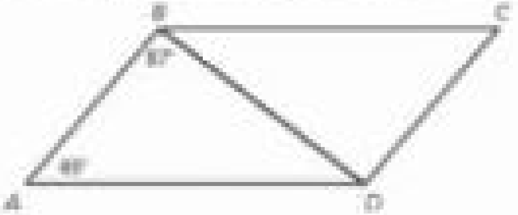
Construct a parallelogram *ABCD* using a compass and straightedge. Explain your construction and prove why the construction resulted in a parallelogram.

Sample answer:



Standardized Test Practice

1. In the diagram below, ABCD is a parallelogram.

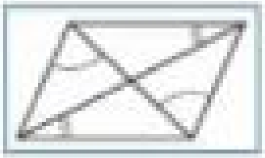


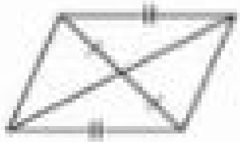
Complete the following.

m∠CDA = 131

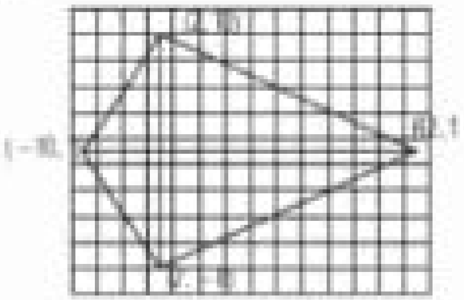
2. Rhombus JKLM has vertices J(−1, −4), K(1, 1), and L(6, 3). The coordinates of M are (4, −2).

3. Circle the figures that are parallelograms.





4. The area of the kite shown below 460 square units.



5. The diagonals and sides of quadrilateral WXYZ form four congruent isosceles triangles. The most specific name that can be given to quadrilateral WXYZ is square.

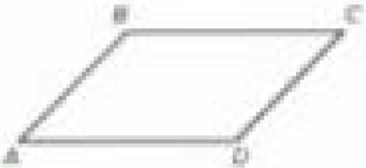
6. Rectangle DEFG has a length that is 2 centimeters longer than its width.

If $FG < EF$, $DF = 58$ centimeters and the perimeter of $\triangle DEF$ is 140 centimeters, the perimeter of $\triangle FHG$ is 98 centimeters.

7. In the table below, the column on the left gives a characteristic of a quadrilateral. Check the columns corresponding to the types of quadrilaterals that have that characteristic.

	Parallelogram	Rectangle	Square	Rhombus	Trapezoid	Kite
Diagonals are congruent.		✓	✓			
Exactly one pair of opposite angles is congruent.						✓
Opposite sides are parallel.	✓	✓	✓	✓		
Diagonals are perpendicular.			✓	✓		✓
Has less than two pairs of opposite congruent sides.					✓	✓
All sides are congruent.			✓	✓		
All angles are right angles.		✓	✓			
Exactly one set of opposite sides are parallel.					✓	

8. Complete the steps and reasons in the following proof.



Given: $\angle A$ is supplementary to $\angle B$

$\angle A$ is supplementary to $\angle D$

Prove: $ABCD$ is a parallelogram

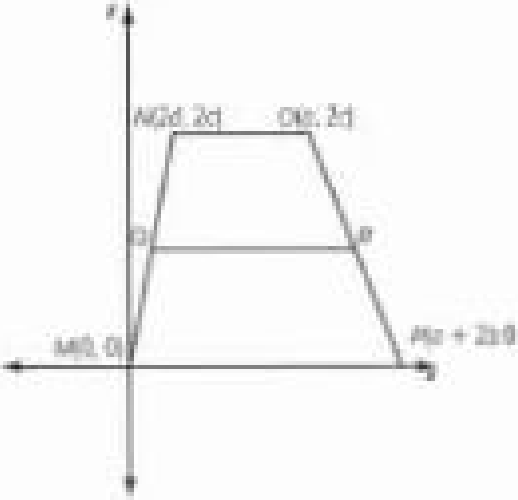
Statement	Reason
$\angle A$ is supplementary to $\angle B$	**Given**
$\overline{AD} \parallel \overline{BC}$	Converse of the Consecutive Interior Angles Theorem
$\angle A$ is supplementary to $\angle D$	Given
$\overline{AB} \parallel \overline{CD}$	**Converse of the Consecutive Interior Angles Theorem**
$ABCD$ is a parallelogram	**Definition of Parallelogram**

9. In the diagram below, $MNOP$ is a trapezoid. Q is the midpoint of $\overline{MN}$ and R is the midpoint $\overline{OP}$.

a. What are the coordinates of Q? Show your work.

(d, c); Using midpoint formula,

$\left(\frac{0+2d}{2}, \frac{0+2c}{2}\right) = (d, c)$



b. What are the coordinates of R? Show your work.

$(a + b, c)$; Using midpoint formula,

$\left(\frac{a+2b+a}{2}, \frac{0+2c}{2}\right) = (a + b, c)$

c. Show that $QR = \frac{NO + MP}{2}$

$NO = a - 2d$, $QR = a + b - d$, $MP = a + 2b$

$\frac{NO + MP}{2} = \frac{a - 2d + a + 2b}{2} = a - d + b =$

$a + b - d = QR$

10. The coordinates of the vertices of quadrilateral $LMNP$ are $L(1, 7)$, $M(4, 3)$, $N(3, 1)$, and $P(1, 2)$.

a. Find the length and slope of each side of $LMNP$.

$LM = 5$ and has slope $-\frac{4}{3}$, $MN = \sqrt{5}$ and has slope 2, NP $\sqrt{5}$ and has slope $-\frac{1}{2}$, $LP = 5$ and has undefined slope.

b. Classify $LMNP$ as a parallelogram, rhombus, trapezoid, kite, or square. Explain your reasoning.

$LMNP$ is a kite because it has exactly two pairs of congruent consecutive sides.

c. Verify that the diagonals of $LMNP$ intersect at right angles by finding the slope of each diagonal and confirming that the two diagonals are perpendicular.

The slope of $\overline{LN}$ is -3 and the slope $\overline{MP}$ is $\frac{1}{3}$. Because the slopes are negative reciprocals, the diagonals are perpendicular, which means they intersect at right angles.

12 Transformations and Symmetry

CHAPTER FOCUS Learn about what you will explore in this chapter. Answer the preview questions. As you complete each lesson, return to these pages to check your work.

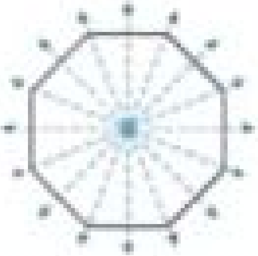
Lesson 12.1: Reflections	
Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).	The point $(-2, 7)$ is reflected in the x-axis. How do you find the coordinates of the image? **Multiply the y-coordinate by -1.** **$(-2, 7) \rightarrow (-2, -7)$**
Lesson 12.2 Translations	
Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.	Describe the image of $(3, 5)$ translated along $\langle a, b \rangle$. **The image is $(3 + a, 5 + b)$.**
Lesson 12.3 Rotations	
Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.	A triangle has vertices $(-2, 1)$, $(1, 2)$, and $(3, -1)$. What are the vertices of its image after a 90° counterclockwise rotation about the origin? **$(-1, -2)$, $(-2, 1)$, and $(1, 3)$**

would

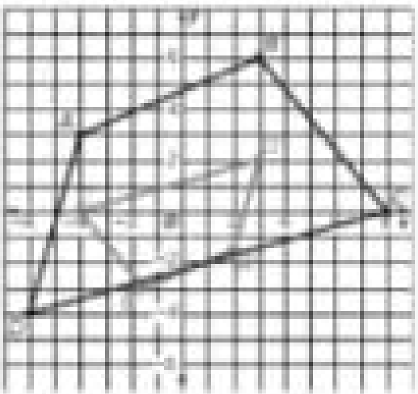
Lesson 12.4 Compositions of Transformations

Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.	A triangle with vertices (0, 4), (8, 2), and (3, −3) is translated along <2, 4> and reflected in the *y*-axis. A student said the vertices of the image are (−2, 8), (−10, 6), and (−5, 1). Is the student correct? Why or why not? **Yes; each point $(x, y) \rightarrow (x + 2, y + 4)$ and then to $(-x, y)$.**

Lesson 12.5 Symmetry

Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.	Draw all lines of symmetry for the regular polygon. 

Lesson 12.6 Dilations

Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch). Verify experimentally the properties of dilations given by a center and a scale factor: **a.** A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged. **b.** The dilation of a line segment is longer or shorter in the ratio given by the scale factor.	What is the rule that describes the transformation of *ABCD* to *A′B′C′D′*? What is the center of dilation? What composition of functions would produce the same image?  **$(x, y) \rightarrow (-0.5x, -0.5y)$; The center of dilation is (0, 0); A dilation by a scale factor of 0.5 followed by a rotation of 180° about (0, 0).**

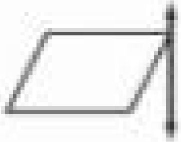
12.1 Reflections

Objectives

- Develop the definition of a reflection in a line.
- Draw reflections using various tools, including geometry software and the coordinate plane.
- Identify transformations that are reflections in a line.

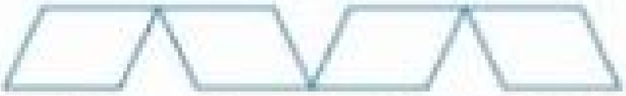
A **transformation** is an operation that maps an original geometric figure onto a new figure. A **reflection** is one type of transformation. When reflecting a shape in a line, the shape before the reflection is known as the **preimage**, and the resulting shape after the reflection is known as the **image**. To obtain the image from the preimage, fold the preimage over the line of reflection.

EXAMPLE 1 Model a Tile Border Using Reflections



EXPLORE Amani wants to create a tile border around her bathroom. She is using a parallelogram for part of her design and will reflect it horizontally in a vertical line of reflection, as shown in the diagram at right.

a. **USE A MODEL** Sketch the first four tiles of the pattern. Fold the shape over the vertical line. Then fold the resulting shape over a vertical line through its rightmost vertex. Continue until you have produced four tiles.



b. **USE TOOLS** Explain how could you use two different tools to verify your answer to **part a**.

Sample answer: I could use tracing paper to flip the parallelogram repeatedly or I could construct the figure in geometry software and use the reflection tool.

c. **USE TOOLS** Using the Geometer's Sketchpad, check your sketch for **part a** by constructing a parallelogram and vertical line as shown above and making a reflection. Use the Geometer's Sketchpad to complete the remaining two reflections. Describe the process you used.

Sample answer: I constructed a line parallel to the vertical line that passed through the bottom right-hand vertex of the image, then reflected the image in the vertical line. Then I constructed another parallel line that passed through the top right-hand vertex of the new image and reflected it in the new vertical line.

d. **FIND A PATTERN** What do you notice about every other image in the pattern? How might you describe reflections in two parallel lines?

Every second image is identical to the preimage. A reflection in two parallel lines could be described as a translation.

EXAMPLE 3 Define a Reflection

EXPLORE **Using the Geometer's Sketchpad, construct a line of reflection, *m*, and a point not on the line, *A*.**

a. Reflect point *A* in the line of reflection and label the image *A'*. Measure the distances between the line and each point. Describe the relationship you find.

Points *A* and *A'* are equidistant from line *m*.

b. **COMMUNICATE PRECISELY** Connect point *A* and image *A'* with a line segment. Would you say that the line of reflection bisects $\overline{AA'}$? How do you know? How would you describe the relationship of the line of reflection to $\overline{AA'}$? How do you know?

Line *m* bisects the segment connecting $\overline{AA'}$; the segment-measuring tool shows these segments are equal. The angle measuring tool shows that the angles formed by the two lines are all 90°. Therefore, the lines are perpendicular to one another.

c. **CONSTRUCT ARGUMENTS** Is there a point *A* for which the observations in **part b** are not true? How would you describe the relationship between the image and preimage for this point?

When point *A* is located on the line, it no longer has any distance from line *m*. Now the image and preimage are the same point.

d. **USE TOOLS** With the Geometer's Sketchpad, construct a triangle and reflect it in the line of reflection. How would you describe the relationship between the image and the preimage? What would you have to do in order to complete this reflection with only a compass and straightedge?

The image and preimage are congruent triangles. Each corresponding vertex of the image and preimage are equidistant from the line of reflection. Using a compass and straightedge, construct line through each vertex of the preimage that is perpendicular to line *m*. Then mark off a point on each of these lines that are equal distances on the opposite side of line *m*. Then connect the three points to form the image triangle.

e. Complete the following definition. A reflection in a line is a function that maps a point to its image such that:

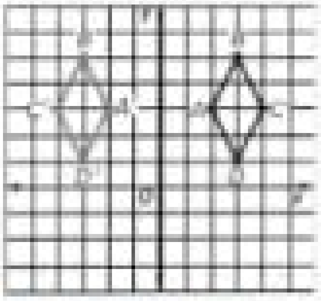
- If the point is not on the line, then the line of reflection is **the perpendicular bisector** of the segment joining the point and its image.
- If a point is on the line, then the image and preimage **the same point**

In addition to constructing reflections using geometry software or using compass and straightedge, reflections can be represented by graphs on a coordinate plane.

EXAMPLE 3 Describe Reflections as Functions

EXPLORE **Complete the following exploration to describe reflections as mapping functions.**

a. *A'B'C'D'* is a reflection of *ABCD* in the *y*-axis. Fill in the chart at right with the ordered pairs that represent the vertices of the image and preimage in the graph below.

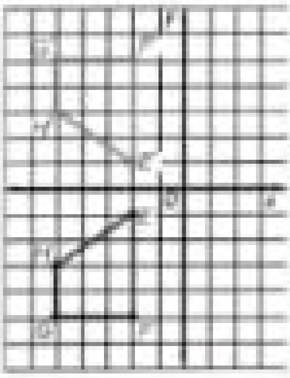


Preimage		Image	
A	(2, 3)	*A'*	(−2, 3)
B	(3, 5)	*B'*	(−3, 5)
C	(4, 3)	*C'*	(−4, 3)
D	(3, 1)	*D'*	(−3, 1)

b. **REASON ABSTRACTLY** Describe the relationship between the coordinates for this reflection in the *y*-axis.

The sign of the *x*-coordinate reversed, but the *y*-coordinate stayed the same.

c. *E'F'G'H'* is a reflection of *EFGH* in the *x*-axis. Fill in the chart at right with the ordered pairs that represent the vertices of the image and preimage in the graph below.

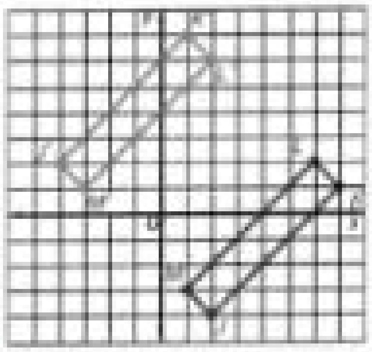


Preimage		Image	
E	(−2, −1)	*E'*	(−2, 1)
F	(−2, −5)	*F'*	(−2, 5)
G	(−5, −5)	*G'*	(−5, 5)
H	(−5, −3)	*H'*	(−5, 3)

d. **REASON ABSTRACTLY** Describe the relationship between the coordinates for this reflection in the *x*-axis.

The sign of the *y*-coordinate reversed, but the *x*-coordinate stayed the same.

e. *J'K'L'M'* is a reflection of *JKLM* in the line $y = x$. Fill in the chart at right with the ordered pairs that represent the vertices of the image and preimage in the graph below.



Preimage		Image	
J	(2, −4)	*J'*	(−4, 2)
K	(7, 1)	*K'*	(1, 7)
L	(6, 2)	*L'*	(2, 6)
M	(1, −3)	*M'*	(−3, 1)

f. REASON ABSTRACTLY Describe the relationship between the coordinates for this reflection in the line $y = x$.

The x-coordinate and the y-coordinate switch places.

KEY CONCEPT

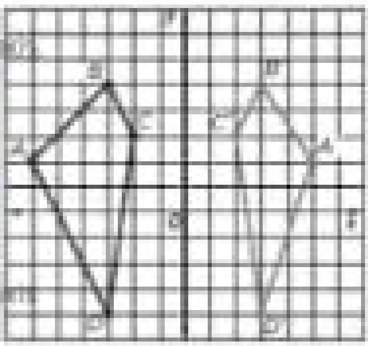
Reflection in the Coordinate Plane

- When a point is reflected in the y-axis, the coordinates of the image can be given by the mapping function $(x, y) \rightarrow (-x, y)$.
- When a point is reflected in the x-axis, the coordinates of the image can be given by the mapping function $(x, y) \rightarrow (x, -y)$.
- When a point is reflected in the line $y = x$, the coordinates of the image can be given by the mapping function $(x, y) \rightarrow (y, x)$.

EXAMPLE 4 Identify Transformations That Are Reflections

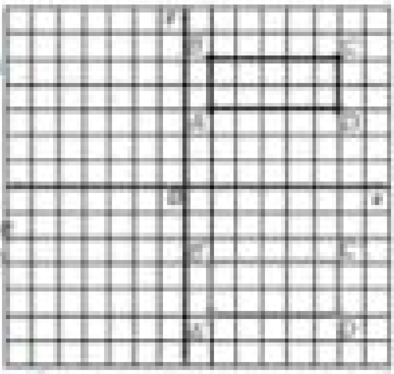
a. CONSTRUCT ARGUMENTS Is $A'B'C'D'$ a reflection of $ABCD$ in the y-axis? Why or why not?

No; Sample answer: Point A is at $(-6, 2)$ and A' is at $(5, 2)$. A' is not a reflection of A in the y-axis, so the figure $A'B'C'D'$ is not a reflection of $ABCD$.



b. CONSTRUCT ARGUMENTS Is $A'B'C'D'$ a reflection of $ABCD$ in the x-axis on the graph to the right? Explain why or why not.

No; Sample answer: A reflection in the x-axis will map each point (x, y) to an image of $(x, -y)$. Point A is at $(1, 3)$ while A' is at $(1, -5)$. A' is not the reflection of A, so $A'B'C'D'$ is not the reflection of $ABCD$ in the x-axis.

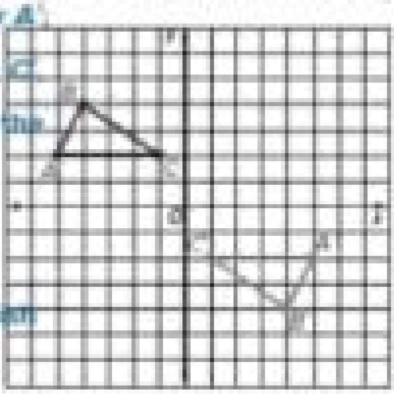


c. COMMUNICATE PRECISELY How could you label the vertices of the image in **part b** so that $A'B'C'D'$ is a reflection of $ABCD$ in the x-axis? What advice can you give to ensure that any reflected polygon is properly labeled?

Sample answer: Label the vertex in the upper left-hand corner A', then, going counterclockwise, label the remaining vertices B', C', and D'. Ensure that corresponding vertices are equidistant to the line of reflection.

d. PLAN A SOLUTION Can one reflection of ABC result in $A'B'C'$? Describe the reflection(s) needed to make this transformation.

No; Sample answer: ABC could be reflected twice to accomplish this transformation: once in the x-axis and once in the y-axis.

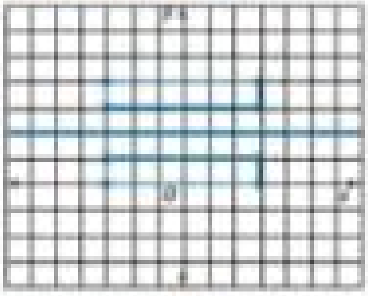


Scanned by CamScanner

PRACTICE

1. **REASON ABSTRACTLY** Make a graph and a chart like the ones in **Example 3** for reflecting a quadrilateral of your choice in the line $y = 2$. Find a mapping function that describes this scenario.

Sample answer:



Vertex	Preimage	Image
1	(−3, 4)	(−3, 0)
2	(3, 4)	(3, 0)
3	(−3, 3)	(−3, 1)
4	(3, 3)	(3, 1)

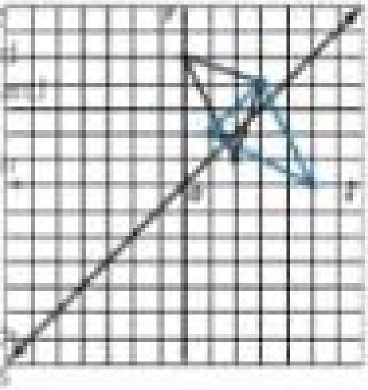
$(x, y) \rightarrow (x, -y + 4)$

2. **USE STRUCTURE** You can graph the **inverse of a function** by reversing the values of x and y for each point of the function. The domain becomes the range and the range becomes the domain.

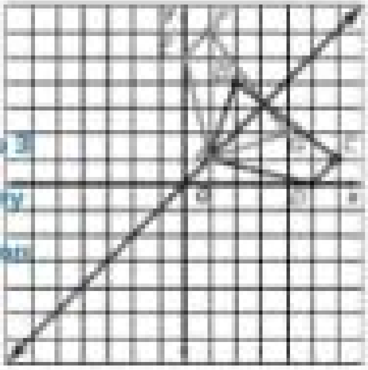
   a. How could you describe the inverse of a function in the context of reflections?

   Sample answer: When the x- and y-coordinates of points are interchanged, this represents a reflection in the line $y = x$.

   b. Draw the triangle with vertices (1, 2), (5, 0), and (3, 4). On the same coordinate plane, draw a triangle using vertices that have the x- and y-coordinates from the first triangle reversed. Draw the line $y = x$ and confirm that the second triangle is a reflection of the first. Finally, draw the line of reflection from your conjecture in **part a** to confirm that the second triangle is the reflection of the first.

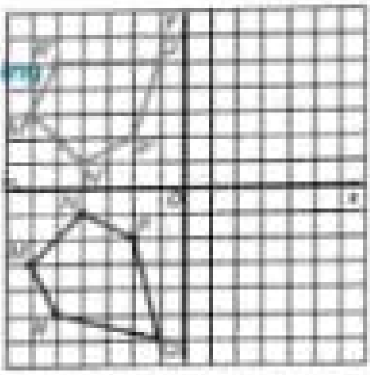


3. **CRITIQUE REASONING** For the graph at the right, Khadija maintains that $AEFG$ is a reflection of $ABCD$ because it fits the definition of a reflection in the line $y = x$. She reasons that A is the same point in each figure because it is on the line of reflection and the remaining vertices are equidistant from that line. Do you agree with Khadija's analysis? Explain.



Khadija is correct that a point on the line of reflection will stay on the line when reflected and she is correct that there are 3 other pairs of points equidistant to the line of reflection, but they are not corresponding points. We cannot say $AEFG$ is a reflection of $ABCD$, but we could say that $AGFE$ is a reflection of $ABCE$.

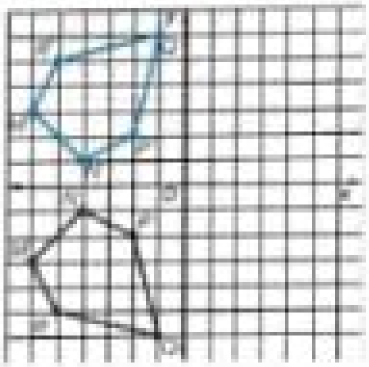
4. **CONSTRUCT ARGUMENTS** Consider the graph at right.



a. Explain why the graph cannot represent a reflection in the *x*-axis.

Since Q is at (−1, −6), Q′ should be at (−1, 6) to fit the mapping function, but Q′ is at (−1, 5).

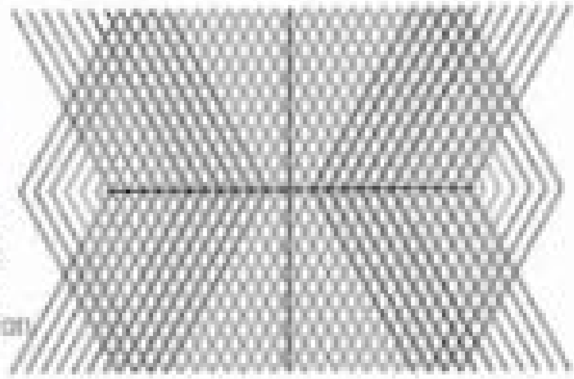
b. Redraw the graph so that it does represent a reflection in the *x*-axis.



USE A MODEL Graphic designers use transformations to create beautiful designs.

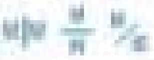
5. Describe the reflections in the design at right.

Sample answer: This has a vertical and a horizontal line of reflection.



6. a. When graphic designers create logos, they often use reflections of letters of the alphabet. Choose a letter to reflect horizontally, vertically and diagonally and sketch an example of each reflection.

Sample answer:



b. Create five points on the coordinate plane to form the letter M. Find their image under a reflection in the *y*-axis, under a reflection in the *x*-axis, and under a reflection in the line $y = x$.

Sample answer: The M can be represented with the points (0, 0), (0, 3), (1, 1), (2, 0), and (2, 3). Reflecting in the *x*-axis gives (0, 0), (0, −3), (1, −1), (2, 0), and (2, −3). Reflecting in the *y*-axis gives (0, 0), (0, 3), (−1, 1), (−2, 0), and (−2, 3). Reflecting in the line $y = x$ gives (0, 0), (3, 0), (1, 1), (0, 2), and (3, 2).

7. **USE A MODEL** Sketch a tile border pattern created by reflecting two different polygons in two vertical lines of reflection.

Sample answer:

12.2 Translations

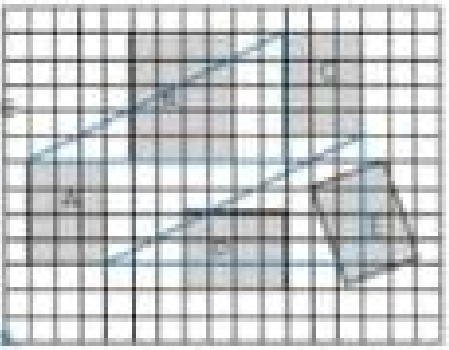
Objectives

- Define, identify, and compare translations on the plane.
- Draw translations given the figure and translation vector.
- Specify a translation that maps one figure onto another.

A **translation** is a function that maps each point to its image along a vector, called the **translation vector**. Each segment joining a point and its image has the same length as the vector, and each of these segments is also parallel to the vector. A translation vector is written as $\langle x, y \rangle$, where x is the number of units translated horizontally and y is the number of units translated vertically. A positive value of x represents a translation to the right, and a negative value of x represents a translation to the left. Similarly, a positive value of y represents a translation up, and a negative value of y represents a translation down. For example, the vector $\langle 4, -3 \rangle$ represents a translation 4 units to the right and 3 units down.

EXAMPLE 1 Identifying Translations

Use the rectangles to answer the questions.



a. **MAKE A CONJECTURE** One of the rectangles shown can be obtained by translating one of the other rectangles. Which of the rectangles shown are translations of each other? Describe the translation in words.

Rectangle A could be a translation of rectangle C down 5 units and to the left 10 units. Rectangle C could be a translation of rectangle A up 5 units and to the right 10 units.

b. **USE TOOLS** Choose two opposite vertices of one of the rectangles you identified in **part a**, and draw segments to the corresponding vertices of the other rectangle you identified. For each segment you drew, form a right triangle with the segment as the hypotenuse. What do you notice about the triangles?

The two triangles look the same. The sides of one are parallel to the sides of the other.

c. **INTERPET PROBLEMS** Write the translation vector that represents the translation from one rectangle to the other. Write the translation vector that represents the translation from the second rectangle to the first.

From Rectangle A to Rectangle C: $\langle 10, 5 \rangle$

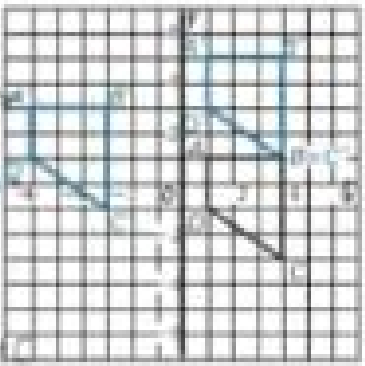
From Rectangle C to Rectangle A: $\langle -10, -5 \rangle$

d. **COMMUNICATE PRECISELY** What do you notice about the translation vectors you wrote in **part c**?

The components of one translation vector are the negatives of the components of the other translation vector.

EXAMPLE 2 Translations on the Coordinate Plane

Use the quadrilateral *ABCD* on the coordinate plane to answer these questions.



a. REASON ABSTRACTLY Predict the effect of translating *ABCD* by the vector ⟨−7, 2⟩. Draw the image *A'B'C'D'*.

The image quadrilateral will be translated 7 units left and 2 units up. It will be congruent to the original quadrilateral.

b. REASON ABSTRACTLY *ABCD* is translated so that the image of point *C* is point *B*. Draw the image quadrilateral *A'B'C'D'* and write the translation vector. Explain how you determined the vector.

⟨0, 4⟩; since point *B* is 4 units above point *C*, the translation is 4 units up.

c. USE STRUCTURE Complete each mapping function to describe the translations from parts a and b.

Translation from part a	**Translation from part b**
$A(x, y) \rightarrow A'(x - 7, y + 2)$	$A(x, y) \rightarrow A'(x, y + 4)$
$B(x, y) \rightarrow B'(x - 7, y + 2)$	$B(x, y) \rightarrow B'(x, y + 4)$
$C(x, y) \rightarrow C'(x - 7, y + 2)$	$C(x, y) \rightarrow C'(x, y + 4)$
$D(x, y) \rightarrow D'(x - 7, y + 2)$	$D(x, y) \rightarrow D'(x, y + 4)$

d. MAKE A CONJECTURE Suppose you have a mapping function that describes what happens to the vertex of a figure for a certain translation. Make a conjecture about the function for all the vertices of the figure. Can you determine the translation vector? Explain.

Sample answer: the same function applies for all vertices. This is because the transformation is a translation, so the same vector moves every point of the original figure the same distance in the same direction, and the vector, therefore, can be determined.

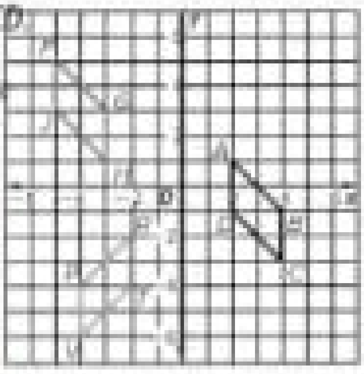
KEY CONCEPT

Complete the table by writing the mapping function for each translation. Assume $a \geq 0$ and $b \geq 0$.

Translation	Mapping Functio
Left a units and up b units	$(x, y) \rightarrow (x - a, y + b)$
Right a units and down b units	$(x, y) \rightarrow (x + a, y - b)$
Left a units and down b units	$(x, y) \rightarrow (x - a, y - b$
Right a units and up b units	$(x, y) \rightarrow (x + a, y + b)$

EXAMPLE 3 Specify a Transformation

In the graph, *FGHJ* and *PRTV* are transformations of parallelogram *ABCD*.



a. Does each figure represent a translation of *ABCD*? If yes, describe the translation in words and give the translation vector. If no, justify why it is not.

FGHJ **Yes; the distances between the corresponding points are equal, and the corresponding sides are parallel; the translation vector is ⟨−7, 4⟩.**

PRVT **No; the distances between corresponding points are not equal, and some corresponding sides are not parallel.**

b. **CRITIQUE REASONING** Jamal says, "I can use the Distance Formula to prove *PRTV* is not a translation of parallelogram *ABCD*." He then calculates that AP = and $DV = \sqrt{61}$. Does this prove his statement? If it does, explain why. If it does not, explain how Jamal could modify what he did so that he does prove his statement.

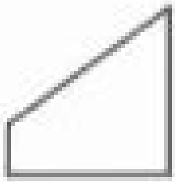
This does not prove the statement. Jamal needed to show that the distance between any two pairs of corresponding points was not equal. Jamal could have calculated another length, such as BR =

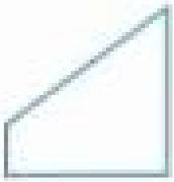
c. **USE REASONING** Khawla draws parallelogram *KLMN* on the coordinate plane. The distances from each vertex of *KLMN* to each corresponding vertex of *ABCD* are the same. Is this sufficient to claim that *KLMN* is a translation of *ABCD*? Justify your answer.

No; sample answer: the vectors that represent the distances may not be parallel. For example, points *K* and *L* could be 4 units right of points *A* and *B*, and points *M* and *N* could be 4 units left of points *C* and *D*, respectively.

PRACTICE

1. **USE TOOLS** Use tracing paper and a centimeter ruler to draw a translation of the figure below. The translation vector is <9, −1>, with units of centimeters. Explain your technique.





I traced the figure, measured 9 cm right and 1 cm down from one vertex, and then slid the paper along the vector. Once the image was in position, I pushed the pencil through to mark each vertex and then used the ruler to connect the vertices and form the image.

2. **CRITIQUE REASONING** Determine if each statement about translations is always, sometimes, or never true. Justify your answer.

a. Lengths and angle measures of the image and preimage are preserved.

Always; translations move each point of a figure along a vector, the same distance in the same direction, so the figure itself looks the same.

b. All lines drawn from the vertices of the preimage to the image are parallel.

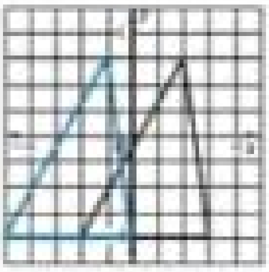
Always; since the points of the preimage all slide along the same vector that represents the translation, all these lines are all parallel.

c. The vector <a, b> will translate each coordinate of a preimage a units right and b units up.

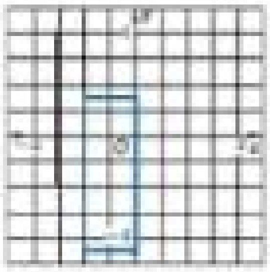
Sometimes; if $a > 0$ and $b > 0$, the statement is true.

USE TOOLS Draw and label the image of each figure after the given translation.

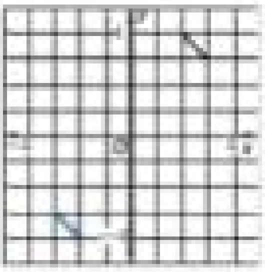
3. 3 units to the left



4. Translation vector <1, −2.5>

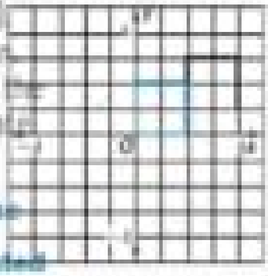


5. Translation vector <−5, −7>



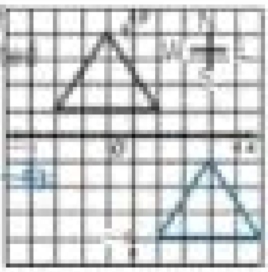
6. **INTEPRET PROBLEMS** A square in the coordinate plane has vertices at (2, 3), (4, 3), (2, 1), and (4, 1). It is translated so that one of the vertices is at the origi Find the coordinates of each vertex of the image, if the translation vector has th least possible length. Explain your reasoning. Draw the image on the coordinat plane.

(0, 2), (2, 2), (0, 0), (2, 0); to minimize the length of the vector, I used the vertex closest to the origin, (2, 1), as the preimage for the point translated to the origin.



7. **USE A MODEL** The triangle represents the area on a map covered by a fleet of fishing ships, where each square represents a kilometer. If this region is translate along the vector <4, −5>, then draw the image and list the coordinates of its vertices. What distance has the coverage area been moved?

(−3 +4, 1 − 5) = (1, −4); (−1 + 4), 4 − 5) = (3, −1); (1 + 4, 1 − 5) = (5, −4); the coverage area has moved + $\sqrt{5}$ + $\sqrt{41}$, or about 6.4 kilometers.



12.3 Rotations

Objectives

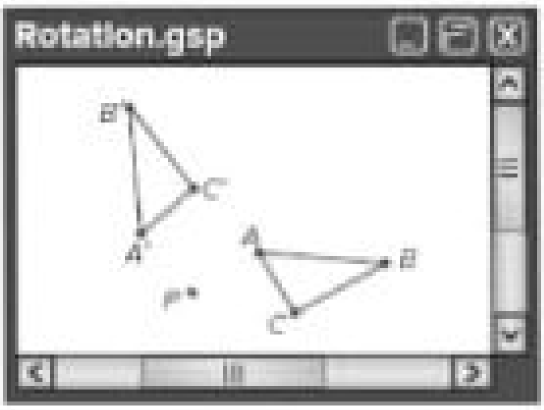
- Develop and understand the definition of rotations.
- Draw the image of a given figure under a rotation.
- Specify the rotation that maps one figure to another.

EXAMPLE 1 Develop the Definition of Rotations

EXPLORE **Use geometry software to explore rotations. As you do so, think about how you could use angles and distances to define a rotation about a point.**

a. **USE TOOLS** Use geometry software to draw a triangle and a point. Label these as △ABC and point P, as shown below on the left.

b. **USE TOOLS** Draw the image of △ABC after a counterclockwise rotation of 100° around point P. Label the image △A′B′C′, as shown below on the right.



c. **USE TOOLS** Use the measurement tools in the software to measure the distance from A to P and the distance from A′ to P. What do you notice? Change the shape or location of △ABC. Does this relationship remain the same?

$AP = A'P$; this relationship remains the same regardless of the shape or location of △ABC.

d. **USE TOOLS** Use the angle measurement tools in the software to measure ∠APA′, ∠BPB′, and ∠CPC′. Change the shape or location of △ABC. What do you notice?

$m\angle APA' = m\angle BPB' = m\angle CPC' = 100$; these angle measures are 100, regardless of the shape or location of △ABC.

e. **MAKE A CONJECTURE** What can you conclude about the distances or angle measures in △ABC and △A′B′C′? Use the software to check your conjecture.

$\overline{AP} \cong \overline{A'P}$; $\overline{BP} \cong \overline{B'P}$; $\overline{CP} \cong \overline{C'P}$; $m\angle APA' = m\angle BPB' = m\angle CPC'$

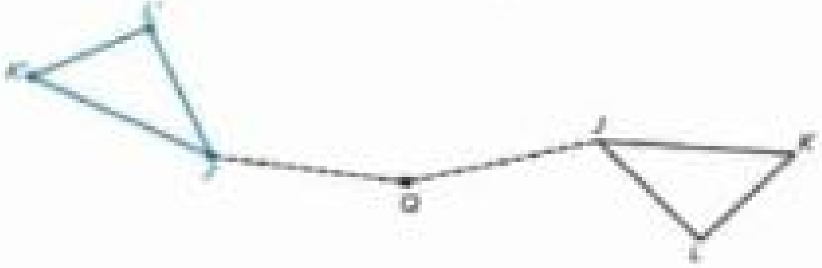
A **rotation** about a fixed point, P, called the **center of rotation**, through an angle of $x°$, is a function that maps a point to its image as follows.

- If the point is the center of rotation, then the image and preimage are the same point.
- If the point is not the center of rotation, then the image and preimage are the same distance from the center of rotation, and the measure of the angle formed by the preimage, the center of rotation, and the image is x.

In the above definition, the **angle of rotation** is $x°$. Unless otherwise stated, you can assume all rotations are counterclockwise.

EXAMPLE 1 Draw a Rotation

Follow the steps below to draw the image of △JKL after a 160° rotation about point Q.



a. **USE TOOLS** Use a straightedge to draw $\overline{JQ}$.

b. **USE TOOLS** Use a protractor to draw a ray that forms a 160° angle with $\overline{JQ}$ as shown.

c. **USE TOOLS** Use a ruler to mark a point J' on the ray so that $J'Q = JQ$.

d. **USE TOOLS** Repeat steps a–c to locate points K' and L'. Then use a straightedge to draw △J'K'L'.

e. **EVALUATE REASONABLENESS** How can you use a piece of tracing paper to check that your drawing is reasonable?

Trace △JKL. Hold down the tracing paper by placing the point of a pencil on point Q. Rotate the paper 160°. The tracing of △JKL should coincide with △J'K'L'.

f. **COMMUNICATE PRECISELY** What would it mean to rotate something −160°?

Rotating 160° is presumed to be a counterclockwise rotation. Therefore, a rotation of −160° would be a rotation in the other direction: 160° clockwise. It is the same as a (360 − 160) = 200° rotation counterclockwise.

Like other transformations, a rotation is a function that takes points of the plane as inputs and gives other points of the plane as outputs. When you perform rotations on a coordinate plane, you can use mapping functions to specify how a point is mapped to its image.

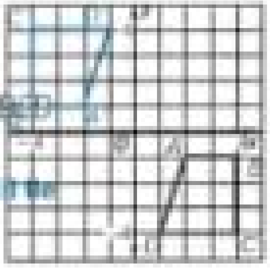
KEY CONCEPT

Complete the table by writing the mapping function for each rotation. The first one has been done for you.

Rotation	Mapping Functio
90° about the origin	$(x, y) \rightarrow (-y, x)$
180° about the origin	$(x, y) \rightarrow (-x, -y)$
270° about the origin	$(x, y) \rightarrow (y, -x)$

EXAMPLE 3 Draw a Rotation

Follow these steps to draw the image of quadrilateral $ABCD$ after a rotation of 180° about the origin.



a. **INTERPRET PROBLEMS** Predict the effect of the rotation on quadrilateral $ABCD$ before drawing the image.

The quadrilateral $A'B'C'D'$ will be "upside down and backwards" and will be entirely in Quadrant II.

o them predict the

b. **INTERPRET PROBLEMS** Complete the table to find the image of each vertex of quadrilateral $ABCD$.

c. **USE TOOLS** Use the table to help you draw the image of quadrilateral $ABCD$ on the coordinate plane above.

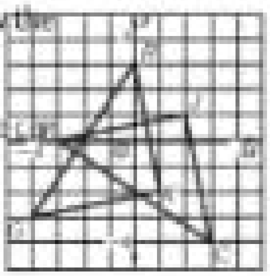
Preimage (x, y)	Image $(-x, -y)$
$A(2, -1)$	$A'(-2, 1)$
$B(4, -1)$	$B'(-4, 1)$
$C(4, -4)$	$C'(-4, 4)$
$D(1, -4)$	$D'(-1, 4)$

d. **CRITIQUE REASONING** A student said that another way to map quadrilateral $ABCD$ to its final image is by first rotating it 90° about the origin and then rotating the image 90° about the origin. Do you agree? Use the mapping functions to support your answer.

Yes; a rotation of 90° maps (x, y) to $(-y, x)$. Another rotation of 90° maps $(-y, x)$ to $(-x, -y)$. Mapping (x, y) to $(-x, -y)$ matches the mapping function for a 180° rotation about the origin.

EXAMPLE 4 Specify a Transformation

Ibrahim is designing a logo for a jewelry store. He used transformations to draw the triangles shown at the right.



a. **COMMUNICATE PRECISELY** Ibrahim's assistant draws △RGX. Explain how he can use a transformation to create △VCJ.

Rotate △RGX 90° counterclockwise about the origin.

b. EVALUATE REASONABLENESS Explain how you can use a mapping function to check that your answer to **part a** is correct.

Use the mapping function for a 90° rotation to find the image of each vertex of △*RGX*. *R*(0, 3) → (−3, 0), which are the coordinates of point *V*; *G*(−4, −3) → (3, −4), which are the coordinates of point *C*; *X*(1, −2) → (2, 1), which are the coordinates of point *J*.

c. COMMUNICATE PRECISELY Is there a different rotation that Ibrahim's assistant can use to create △*VCJ*? Explain.

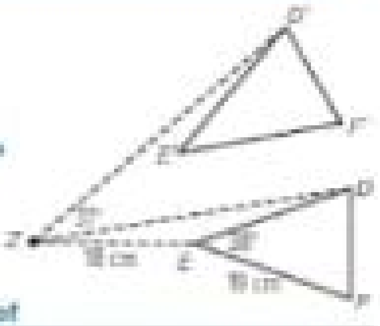
Yes; he can also rotate △*RGX* 270° clockwise about the origin.

d. CONSTRUCT ARGUMENTS Suppose Ibrahim's assistant starts by drawing △*VCJ*. What transformation can he use to create △*RGX*?

Rotate △*VCJ* 270° counterclockwise about the origin.

PRACTICE

1. **COMMUNICATE PRECISELY** In the figure, △*D′E′F′* is the image of ∠*DEF* after a rotation about point *Z*.



   a. What is the distance from *E′* to *Z*? Explain how you know.

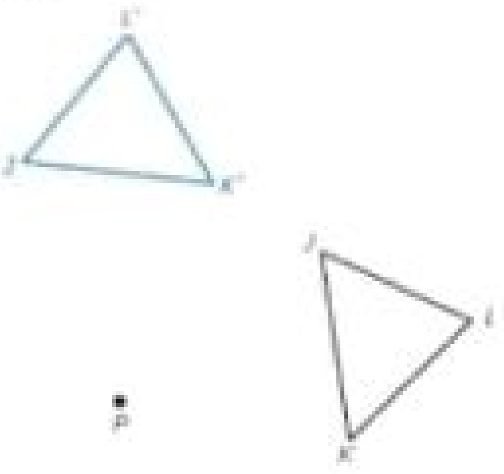
   18 cm; a point and its image are the same distance from the center of rotation.

   b. What is m∠*FZF′*? Explain how you know.

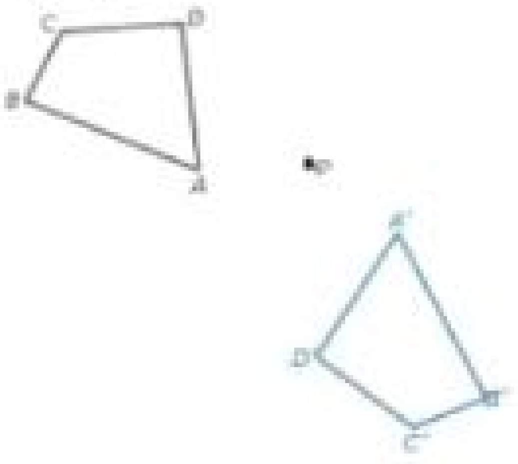
   31; the measure of the angle formed by a point, the center of rotation, and the point's image is equal to the angle of rotation, which is 31°.

USE TOOLS Draw and label the image of each figure after the given rotation about point *P*.

2. 75°

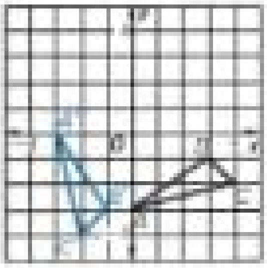


3. 140°

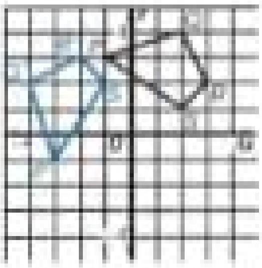


USE TOOLS Draw and label the image of each figure after the given rotation.

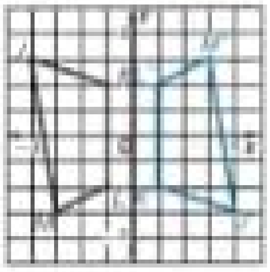
4. 270° about the origin



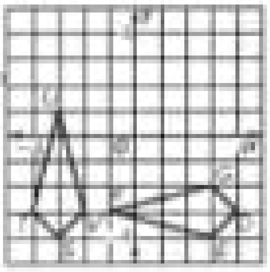
5. 90° about the origin



6. 180° about the origin



7. **COMMUNICATE PRECISELY** Suha is using a coordinate plane to design a video game set in outer space. She draws quadrilaterals DEFG and STUV to represent comets. How can she use a transformation to map DEFG to STUV? Describe the transformation in words and give a mapping function for the transformation.

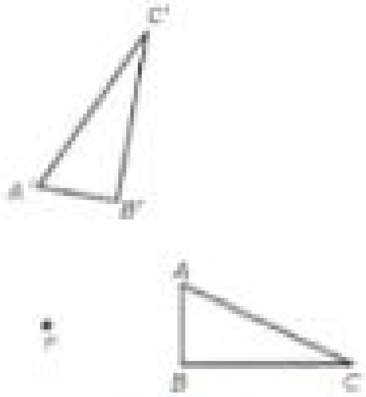


Rotate quadrilateral *DEFG* 270° about the origin:

$(x, y) \to (y, -x)$

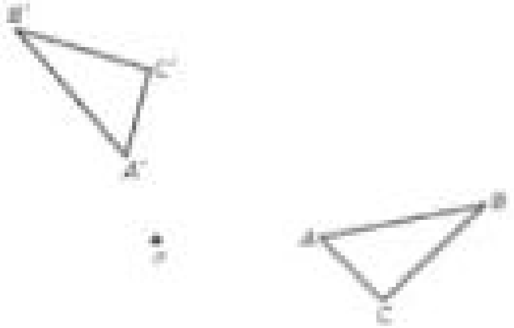
USE TOOLS Determine whether there is a rotation about point *P* that maps △*ABC* to △*A'B'C'*. If so, explain why and use a ruler, protractor, or other tool to help you describe the rotation. If not, explain why not.

8.



Yes; *AP* = *A'P*, *BP* = *B'P*, *CP* = *C'P*, and $m\angle APA' = m\angle BPB' = m\angle CPC' = 80$; 80° counterclockwise rotation about point *P*.

9.

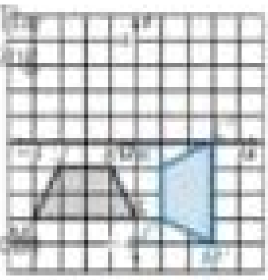


No; since *AP* ≠ *A'P*, any transformation that maps △*ABC* to △*A'B'C'* cannot be a rotation about point *P*.

10. **CONSTRUCT ARGUMENTS** Under a rotation around the origin, the point *A*(5, −1) is mapped to the point *A'*(1, 5). What is the image of the point *B*(−4, 6) under this rotation? Explain.

B'(−6, −4); the rotation maps (x, y) to $(-y, x)$, so it is a 90° rotation about the origin; therefore, the image of (−4, 6) is (−6, −4).

11. An interior designer uses a coordinate plane to position furniture in a room. The designer decides to move a sofa, which is represented by trapezoid *JKLM*, using the rotation $(x, y) \rightarrow (-y, x)$.



a. **USE TOOLS** Draw the new location of the sofa on the coordinate plane.

b. **COMMUNICATE PRECISELY** Write a mapping function that the designer should apply to the sofa in its new location in order to move it back to its original location. Explain.

$(x, y) \rightarrow (y, -x)$; the original rotation is a 90° counterclockwise rotation; to undo this, use a 90° clockwise rotation (which is equivalent to a 270° counterclockwise rotation).

12. a. **COMMUNICATE PRECISELY** What is the result of a rotation followed by another rotation about the same point? Give an example.

A rotation followed by another rotation is still a rotation. For example, a rotation of 30° clockwise followed by a rotation of 20° counterclockwise is the same as a rotation of 10° clockwise.

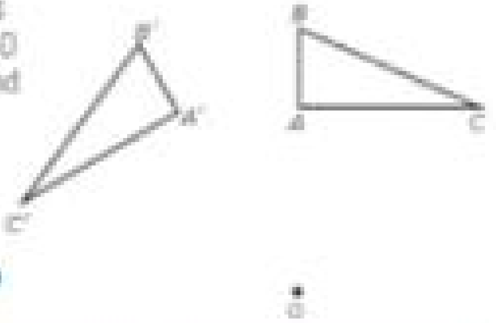
b. **COMMUNICATE PRECISELY** Is a reflection followed by another reflection in a parallel line still a reflection? Explain your reasoning.

No; a reflection must change a figure's orientation. The first reflection changes the preimage's orientation, but the second reflection restores the original orientation.

c. **COMMUNICATE PRECISELY** Saeed claims that a reflection in the *x*-axis followed by a reflection in the *y*-axis is the same thing as a rotation. Is Saeed correct?

Saeed is correct. A reflection in the x-axis followed by a reflection in the y-axis is the same as a rotation of 180° about the origin. This can be seen with the mapping functions. The point (x, y) maps to $(x, -y)$ when reflected in the x-axis. The point $(x, -y)$ maps to $(-x, -y)$ when reflected in the y-axis. So (x, y) maps to $(-x, -y)$, which is the same as a 180° rotation.

13. **CRITIQUE REASONING** Sultan is looking at the figure below, which shows two congruent triangles. He measures the angle that rotates *A* to *A'* around *O* and finds it to be 30 degrees. He measures the angle that rotates *B* to *B'* around *O* and finds it to also be 30 degrees. He then claims that because the two triangles are congruent, a 30 degree rotation has occurred around point *O*. Is Sultan correct? Explain.



No; Sample answer: Point C has not been rotated 30 degrees around O. It appears that a reflection has occurred in addition to a rotation. The two triangles are congruent, but it is not a rotation only that maps one triangle to the other.

12 Compositions of Transformations

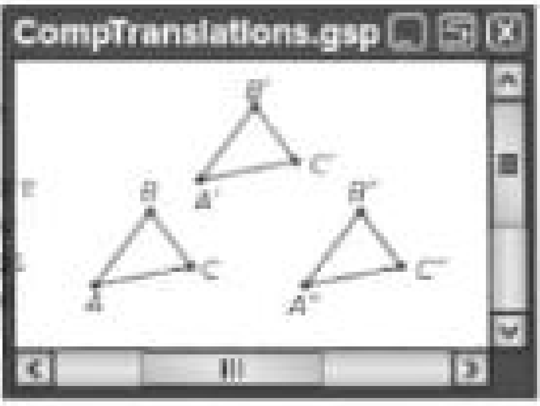
Objectives

- Draw the image of a figure after a composition of transformations.
- Specify a sequence of transformations that maps one figure to another.

A **composition of transformations** is a sequence of transformations in which the first transformation is applied to a given figure and then another transformation is applied to its image.

EXAMPLE 1 Investigate Compositions

EXPLORE Use dynamic geometry software to explore compositions of transformations.

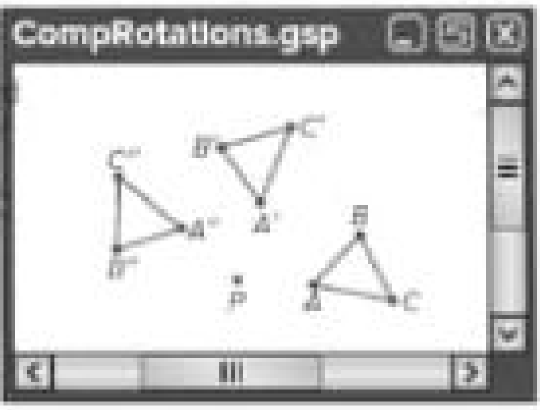


a. USE TOOLS Use geometry software to draw a triangle and label it $\triangle ABC$. Use the software to translate the triangle and label its image $\triangle A'B'C'$. Then use the software to translate $\triangle A'B'C'$ and label its image $\triangle A''B''C''$. Change the shape or location of $\triangle ABC$, and look for relationships among the triangles. Which type of transformation could you use to map $\triangle ABC$ directly to $\triangle A''B''C''$?

translation

b. MAKE A CONJECTURE What conjecture can you make about the composition of two translations?

The composition of two translations is another translation.



c. USE TOOLS Use the software to draw $\triangle ABC$ and a point P. Rotate $\triangle ABC$ 80° about point P and label its image $\triangle A'B'C'$. Then rotate $\triangle A'B'C'$ 70° about point P and label its image $\triangle A''B''C''$. Change the shape or location of $\triangle ABC$, and look for relationships among the triangles. What single transformation could you use to map $\triangle ABC$ directly to $\triangle A''B''C''$?

rotation of 150° about point P

d. MAKE A CONJECTURE What conjecture can you make about the composition of a rotation of m° about point P followed by a rotation of n° about point P?

The composition is a rotation of $(m + n)$° about point P.

e. **USE TOOLS** Use the software to draw △ABC and two parallel lines, *j* and *k*. Reflect △ABC in line *j* and label its image △A'B'C'. Then reflect △A'B'C' in line *k* and label its image △A"B"C". Which type of transformation could you use to map △ABC directly to △A"B"C"?

translation

f. **MAKE A CONJECTURE** What conjecture can you make about the composition of two reflections in parallel lines?

The composition of two reflections in parallel lines is a translation.

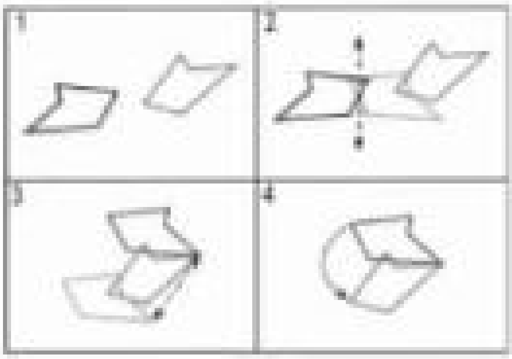
g. **MAKE A CONJECTURE** Repeat **part e**, but this time draw lines *j* and *k* so that they intersect. What conjecture can you make about the composition of two reflections in intersecting lines?

The composition of two reflections in intersecting lines is a rotation.

h. **MAKE A CONJECTURE** Recall that translations, reflections, and rotations are rigid motions because they do not change the size or shape of a figure. Based on your work above, what conjecture can you make about the composition of two rigid motions?

The composition of two rigid motions is another rigid motion.

i. **CRITIQUE REASONING** Abdulaziz claims that translations, reflections, and rotations are all we need to achieve any rigid motion. In other words, every rigid motion can be the composition of translations, reflections, and rotations. Use the following diagram to explain why Abdulaziz is correct.



The image is either the same orientation as the preimage, or it is "flipped." In the case that the image is flipped, first perform a reflection in any line so that the preimage is now the same orientation as the image. Second, pick any point on the new preimage and perform a tranclation that maps the point onto its corresponding point in the image. Finally, rotate the new preimage around that point until the preimage has been fully mapped onto the image. Therefore, the rigid motion is either the composition of a reflection, a translation, and a rotation, or it is the composition of a translation and a rotation.

j. **CRITIQUE REASONING** Abdulaziz now claims that the only rigid motion we need is a reflection. In other words, no matter what rigid motion we want, we can always get it through a series of reflections. Is Abdulaziz correct? Explain.

Abdulaziz is correct. A reflection can obviously be achieved by a reflection. A rotation is the composition of two reflections in intersecting lines. A translation is the composition of two reflections in parallel lines. Therefore, all rigid motions can be achieved by composing reflections.

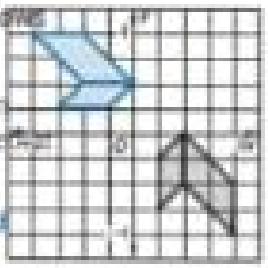
KEY CONCEPT Composition of Transformations

Complete each description in the table.

Composition	Description
Two translations	The composition of two translation **translation**
Two rotations	The composition of a rotation of $m°$ about point P followed by a rotation of $n°$ about point P is a rotatio **$(m° + n)$** about point P.
Two reflections in parallel lines	The composition of two reflections in parallel lines is a **translation**.
Two reflections in intersecting lines	The composition of two reflections in intersecting lines is a **rotation**.
Two rigid motions	The composition of two rigid motio **rigid motion**

EXAMPLE 2 Draw a Composition

Ghaya is using a coordinate plane to design a logo for a company. The figure shows her first design for the logo.



a. **INTERPRET PROBLEMS** Ghaya plans to reflect the logo in the x-axis and then rotate the image 90° about the origin. Before performing the composition, predict the quadrant in which the final image will lie. Explain.

Quadrant II; the reflection in the x-axis maps the logo to Quadrant I and the rotation of 90° maps the image to Quadrant II.

b. **USE TOOLS** Draw the final image of the logo on the coordinate plane shown above.

c. **COMMUNICATE PRECISELY** Could Ghaya have mapped the original logo to its final position using a single transformation? If so, describe the transformation. If not, explain why not.

Yes; she could have reflected the original logo in the line $y = x$.

d. **CONSTRUCT ARGUMENTS** Show how you can use the mapping function for a reflection in the x-axis and for a rotation of 90° about the origin to justify your answer to **part c**.

The mapping function for a reflection in the x-axis is $(x, y) \rightarrow (x, -y)$; applying the mapping function for a 90° rotation about the origin to this result gives $(x, -y) \rightarrow (-(-y), x)$ or (y, x); overall, the mapping is $(x, y) \rightarrow (y, x)$, which is the mapping function for a reflection in the x-axis.

e. **CRITIQUE REASONING** Ghaya claims that she can get the same result by performing the composition in the opposite order: that is, by first rotating the original logo 90° about the origin and then reflecting it in the x-axis. Do you agree? Explain why or why not.

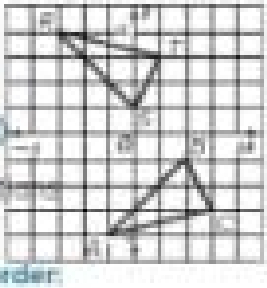
No; if you rotate the original logo by 90° about the origin, the image is in Quadrant I, and reflecting in the x-axis maps it back to Quadrant IV, so in this case the final image cannot be same as when the transformations are performed in the original order.

f. **MAKE A CONJECTURE** In general, does the order in which you perform a sequence of transformations matter? Explain.

Yes; the result of part e shows that performing the transformations in a different order can give a different final image.

EXAMPLE 3 Specify a Sequence of Transformations

Given two figures, specify a sequence of transformations that maps one figure to the other.



a. **COMMUNICATE PRECISELY** It is possible to map $\triangle ABC$ to $\triangle RST$ using a reflection followed by a translation? Describe the reflection and translation.

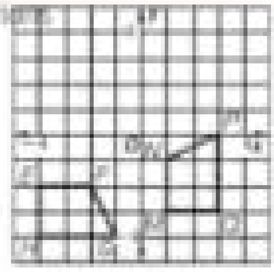
Reflection in the x-axis, followed by the translation along vector $\langle -2, 0 \rangle$.

b. **COMMUNICATE PRECISELY** Is there a different sequence of two transformations that maps $\triangle ABC$ to $\triangle RST$? Explain.

Yes; in this case, you can perform the transformations in the opposite order: translation along vector $\langle -2, 0 \rangle$ followed by a reflection in the x-axis.

c. **COMMUNICATE PRECISELY** Why is your answer in **part b** different from your answer in **Example 2 part e**?

In Example 2 part e, the order in which the two rigid motions were performed mattered. In this example, the order does not matter. It appears that composition is commutative for only certain rigid motions.



d. **COMMUNICATE PRECISELY** Specify a sequence of two or more transformations that maps quadrilateral $EFGH$ to quadrilateral $MNPQ$.

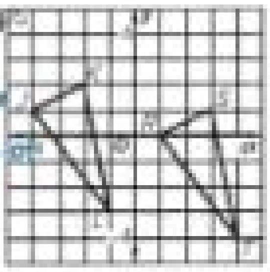
Rotation of 90° about the origin, followed by the translation along vector $\langle -1, 1 \rangle$.

e. **CONSTRUCT ARGUMENTS** Explain how you can use your answer to **part c** to specify a sequence of transformations that maps quadrilateral $MNPQ$ to quadrilateral $EFGH$.

Use transformations that "undo" the transformations in part c: translation along vector $\langle 1, -1 \rangle$, followed by a clockwise rotation of 90° about the origin.

1. **CONSTRUCT ARGUMENTS** Specify a sequence of three translations that maps △JKL to △RST. Is there more than one such sequence? Explain.

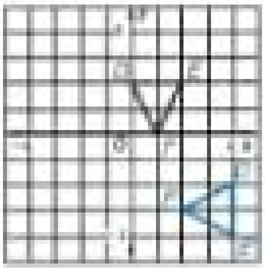
Sample answer: translation along vector ⟨2, 0⟩, translation along vector ⟨3, 0⟩, translation along vector ⟨0, −1⟩; any sequence in which the sum of the vector components is ⟨5, −1⟩ will work.



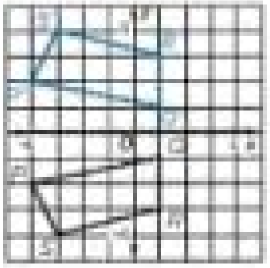
PRACTICE

USE TOOLS Draw and label the image of each figure after the given composition of transformations.

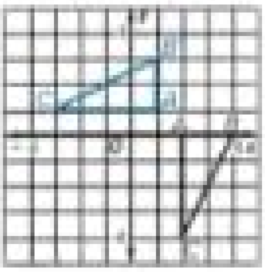
1. 270° rotation about the origin followed by translation along ⟨2, −2⟩



2. reflection in the y-axis followed by 180° rotation about the origin



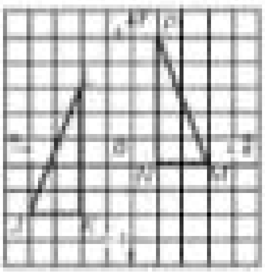
3. translation along ⟨−1, 1⟩ followed by reflection in the line $y = x$



COMMUNICATE PRECISELY Specify a sequence of transformations that maps △JKL to △MNP.

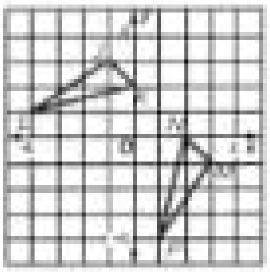
4–6. Sample answers given.

4.



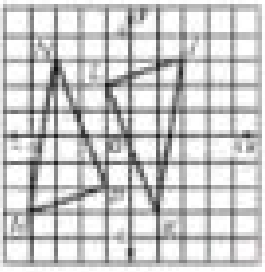
Reflection in y-axis, followed by translation along ⟨−1, 2⟩

5.



Reflection in x-axis, followed by 90° rotation about the origin

6.



Translation along ⟨2, 0⟩ followed by 180° rotation about the origin

7. **REASON ABSTRACTLY** Tarek transforms a figure by applying the transformation $(x, y) \rightarrow (x, -y)$ followed by the transformation $(x, y) \rightarrow (-x, -y)$. Write a single mapping function that has the same effect as the composition. What transformation does your mapping function represent?

$(x, y) \rightarrow (-x, y)$; reflection in the y-axis

Scanned by CamScanner

CONSTRUCT ARGUMENTS **Determine whether each statement is *always*, *sometimes*, or *never* true. Explain.**

8. A composition of two reflections is a rotation.

Sometimes; if the lines of reflection intersect, the composition is a rotation.

9. A composition of two translations is a rotation.

Never; a composition of two translations is always another translation.

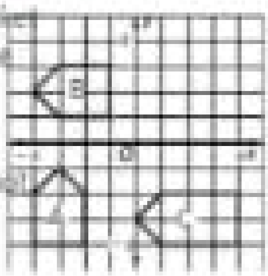
10. A reflection in the *x*-axis followed by a reflection in the *y*-axis leaves a point in its original location.

Sometimes; this is true if the point is the origin.

11. The translation along $\langle a, b\rangle$ followed by the translation along $\langle c, d\rangle$ is the translation along $\langle a + c, b + d\rangle$.

Always; the first translation maps (x, y) to $(x + a, y + b)$, the second translation maps this image to $(x + a + c, y + b + d)$, which is equivalent to the translation along $\langle a + c, b + d\rangle$.

12. **COMMUNICATE PRECISELY** Saleh is a programmer for the video game Rocket Race. He uses a coordinate plane to program the motion of the rockets on the screen.



a. Is it possible for Saleh to use a composition of two rigid motions to map rocket A to rocket B? If so, describe the composition. If not, explain why not.

Yes; sample answer: rotate rocket A 90° about the origin, then translate it along the vector $\langle -5, 5\rangle$.

b. Is it possible for Saleh to use a composition of two rigid motions to map rocket A to rocket C? If so, describe the composition. If not, explain why not.

No; rockets A and C have different shapes, and any composition of rigid motions would be another rigid motion, which would preserve the shape of rocket A.

13. **USE TOOLS** Use dynamic geometry software or other tools to explore the composition of two reflections in parallel lines. What can you say about the translation vector for the translation that is equivalent to this composition?

The length of the vector is twice the distance between the lines; the vector is perpendicular to the lines.

14. **REASON ABSTRACTLY** We have seen that the transformation $(x, y) \rightarrow (y, x)$ is a reflection in the line $y = x$. It is also true that the transformation $(x, y) \rightarrow (-y, -x)$ is a reflection in the line $y = -x$. The examples in this lesson tell us that the composition of these two reflections should be a rotation. Give the mapping function that describes the composition, and describe the resulting rotation.

$(x, y) \rightarrow (-x, -y)$. This is a rotation of 180 degrees about the origin.

12.5 Symmetry

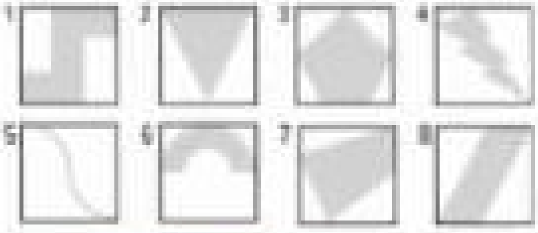
Objectives

- Describe the rotations and reflections that carry a figure onto itself

A figure has **symmetry** if there is a rigid motion (translation, reflection, or rotation) that maps the figure onto itself.

EXAMPLE 1 Identify Symmetry

EXPLORE Shaikha owns a store that sells hand-painted ceramic tiles. The figure shows eight new square tiles that she received today. Shaikha is sorting the tiles for her showroom.



a. **INTERPRET PROBLEMS** Shaikha places tiles 2, 3, and 6 in the same group. Explain what these tiles have in common.

Sample answer: Each of these tiles can be folded along a line so that the two halves of the tile coincide.

b. **USE STRUCTURE** What other set of three tiles could Shaikha group together? Why?

Sample answer: Tiles 1, 5, and 8; each of these tiles can be rotated 180° around a point at the center of the tile so that the image coincides with the preimage.

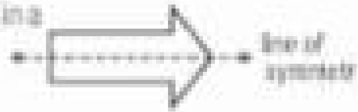
c. **COMMUNICATE PRECISELY** Which of the tiles have symmetry? Justify your answer using the definition of symmetry.

Tiles 1, 2, 3, 5, 6, and 8. For tiles 2, 3, and 6, a rigid motion (reflection) maps each tile to itself. For tiles 1, 5, and 8, a rigid motion (rotation) maps each tile to itself.

d. **COMMUNICATE PRECISELY** Which of the tiles do not have symmetry? Justify your answer using the definition of symmetry.

Tiles 4 and 7. There does not appear to be as rigid motion that maps the preimage back onto itself.

A figure has **line symmetry** if it can be mapped onto itself by a reflection in a line, called the **line of symmetry**.



EXAMPLE 2 Identify Line Symmetry

The figures show a rectangle, a parallelogram, an isosceles trapezoid, and a regular pentagon.

a. **COMMUNICATE PRECISELY** Draw all lines of symmetry on the figures.

b. **COMMUNICATE PRECISELY** Which of the figures have line symmetry? Why?

Rectangle, isosceles trapezoid, regular pentagon; each of these figures can be mapped to itself by a reflection in a line.

c. **MAKE A CONJECTURE** How many lines of symmetry do you think a regular 17-gon has? Explain.

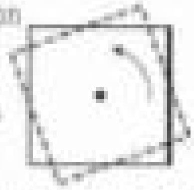
17; there is a line of symmetry through each of the 17 vertices of the polygon.

d. **MAKE A CONJECTURE** How many lines of symmetry do you think a regular *n*-gon has? Explain.

***n*; there is a line of symmetry through each of the *n* vertices of the polygon.**

A figure has **rotational symmetry** if the figure can be mapped onto itself by a rotation between 0° and 360° about the center of the figure, called the **center of symmetry**.

The **order of symmetry** is the number of times a figure maps onto itself as it rotates from 0° to 360°. The **magnitude of symmetry** is the smallest angle through which a figure can be rotated so it maps onto itself.


order of symmetry: 4
magnitude of symmetry: 90°

EXAMPLE 3 Identify Rotational Symmetry

Refer to the rectangle, parallelogram, isosceles trapezoid, and regular pentagon from Example 2.

a. **COMMUNICATE PRECISELY** Which of the figures, if any, do not have rotational symmetry? Explain your choice(s).

Isosceles trapezoid; there is no rotation of less than 360° that will map the trapezoid onto itself.

b. **COMMUNICATE PRECISELY** For each of the figures that have rotational symmetry, give the order of symmetry and magnitude of symmetry.

Rectangle: order of symmetry: 2; magnitude of symmetry: 180°

Parallelogram: order of symmetry: 2; magnitude of symmetry: 180°

Regular pentagon: order of symmetry: 5; magnitude of symmetry: 72°

c. **MAKE A CONJECTURE** What is the order of symmetry of a regular 20-gon? What is the magnitude of symmetry? Explain.

Order of symmetry 20; magnitude of symmetry: 18°; the order of symmetry is equal to the number of sides of the regular *n*-gon; the magnitude of symmetry is 360° divided by the order of symmetry and 360° ÷ 20 = 18°.

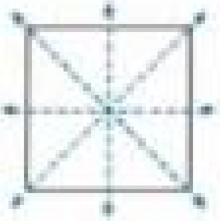
d. **MAKE A CONJECTURE** What is the order of symmetry of a regular *n*-gon? What is the magnitude of symmetry? Explain.

Order of symmetry *n*; magnitude of symmetry: $\left(\frac{360}{n}\right)^\circ$; the order of symmetry is equal to the number of sides of the regular *n*-gon; the magnitude of symmetry is 360° divided by the order of symmetry, or $\left(\frac{360}{n}\right)^\circ$.

PRACTICE

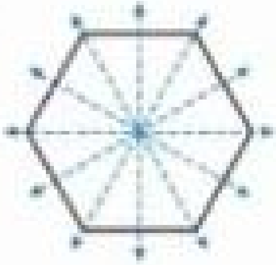
COMMUNICATE PRECISELY State whether each figure has line symmetry and describe the reflections, if any, that map the figure onto itself. Draw any lines of reflection on the figure.

1. Square



Yes; the reflection in any of the 4 lines of symmetry maps the figure onto itself.

2. Regular hexagon



Yes; the reflection in any of the 6 lines of symmetry maps the figure onto itself.

COMMUNICATE PRECISELY State whether each figure has rotational symmetry. If so, describe the rotations that map the figure onto itself by giving the order of symmetry and magnitude of symmetry.

3. Equilateral triangle

yes; order of symmetry: 3; magnitude of symmetry: 120°

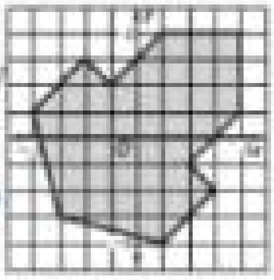
4. Scalene triangle

no rotational symmetry

5. Regular hexagon

yes; order of symmetry: 6; magnitude of symmetry: 60°

6. **COMMUNICATE PRECISELY** The figure shows the floor plan for a new gallery in an art museum. Describe every reflection or rotation that maps the gallery onto itself.



The only transformation that maps the gallery onto itself is a reflection in the line *y* = *x*.

7. a. **CONSTRUCT ARGUMENTS** How many lines of symmetry does a circle have? Explain your reasoning.

Infinitely many; every line through the center of the circle is a line of symmetry, and there are infinitely many such lines.

b. **CONSTRUCT ARGUMENTS** What is the order of rotation for a circle? Explain your reasoning.

Infinity; as the circle rotates all the way around, every degree, no matter how small, maps the circle onto itself.

USE STRUCTURE **Sketch a figure in the space provided with the described symmetry.**

8–11. Sample answers shown

8. No line symmetry; rotational symmetry with order of symmetry 2

9. Exactly one line of symmetry; no rotational symmetry

10. Exactly 3 lines of symmetry; rotational symmetry with order of symmetry 3

11. No line symmetry; rotational symmetry with magnitude of symmetry 120°

12. **CONSTRUCT ARGUMENTS** A regular polygon has magnitude of symmetry 15°. How many sides does the polygon have? Explain.

24; $360^\circ \div 15^\circ = 24$, so the order of symmetry is 24; this means there are 24 sides.

12. Dilations

Objectives

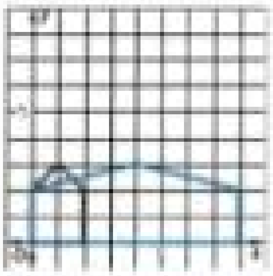
- Compare transformations that preserve distance and angle measures to those that do not.
- Verify the properties of dilations.
- Represent and describe dilations as functions.

EXAMPLE 1 Compare Transformations

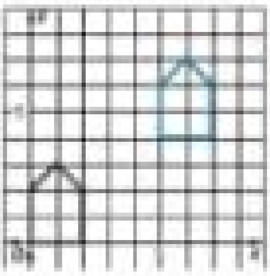
EXPLORE A special effects designer uses function rules to change the size, shape, and/or location of images in a movie. The designer is experimenting with different ways to change the appearance of a tower in a science fiction movie.

a. CALCULATE ACCURATELY Each figure shows the starting position of the tower and the function that describes the transformation that the designer applies to it. Draw the image of the tower under each transformation.

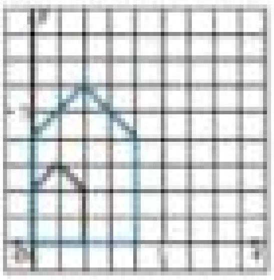
i. $(x, y) \rightarrow (4x, y)$



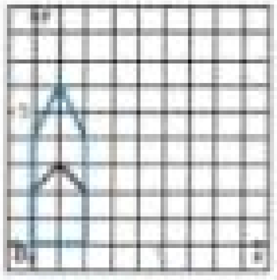
ii. $(x, y) \rightarrow (x + 5, y + 4)$



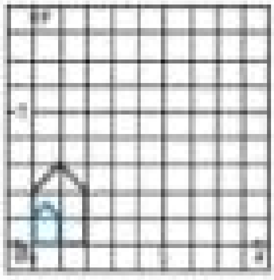
iii. $(x, y) \rightarrow (2x, 2y)$



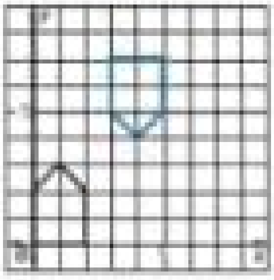
iv. $(x, y) \rightarrow (x, 2y)$



v. $(x, y) \rightarrow (0.5x, 0.5y)$



vi. $(x, y) \rightarrow (x + 3, -y + 7)$



b. USE STRUCTURE Which transformations result in congruent figures? How do you know?

ii and vi; In ii, the image and preimage are related by a translation. In vi, the image and preimage are related by a rotation and a translation. These are rigid transformations that preserve congruence.

c. **COMMUNICATE PRECISELY** Which of the transformations seem to preserve angle measure? How could you be sure?

ii, iii, v, and vi; use a protractor to measure each angle

d. **COMMUNICATE PRECISELY** The designer wants to change the size but not the shape of the tower. Which transformations can the designer use? How would you describe these transformations in everyday language?

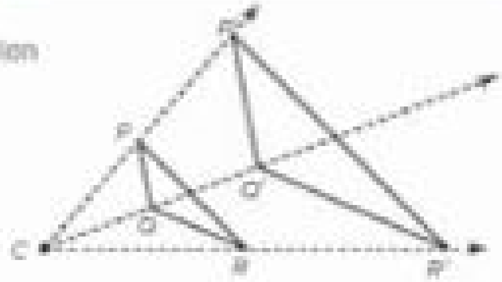
iii and v; Sample answer: Transformation iii is an enlargement. Transformation v is a reduction.

e. **CALCULATE ACCURATELY** In figure iii, what is the relationship of the area of the image to the area of the preimage? Explain how to calculate this.

A(preimage) = 5 units^2. A(image) = 20 units^2. Since $(x, y) \rightarrow (2x, 2y)$, the area of the new image is four times that of the preimage, or $2 \cdot 2 = 4$.

A **dilation** with center C and scale factor k (k > 0) is a function that maps a point P to its image P′ as follows.

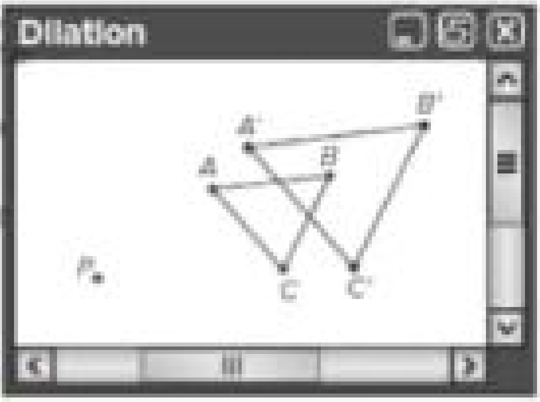
- If P coincides with C, then P′ = P.
- If P does not coincide with C, then P′ lies on $\overrightarrow{CP}$ such that CP′ = k(CP).



EXAMPLE 3 Investigate Properties of Dilations

EXPLORE Use the Geometer's Sketchpad to investigate properties of dilations as follows.

a. **USE TOOLS** Draw a triangle and label the vertices A, B, and C. Then draw a point P. Draw the image of △ABC af a dilation with center P and scale factor k as shown. Change the shape and location of △ABC and notice ho △A′B′C′ is related to △ABC.



b. **FIND A PATTERN** Repeat the process in Step a, but us different values of k, including [illegible], 1, 2, and 3. What can you say about a dilation when the scale factor, k, is less than 1? equal to 1? greater than 1?

When $k < 1$, the image is a reduction. When $k = 1$, the image and preimage are congruent. When $k > 1$, the image is an enlargement.

c. **USE TOOLS** Measure the length of each side of △ABC and the length of each side of △A′B′C′. Change the scale factor and notice how the lengths compare. What conclusions can you make?

The image of the line segment is longer or shorter than the preimage depending on the ratio given by the scale factor of the dilation.

d. **MAKE A CONJECTURE** Draw a point *P* and a line *j*. Then draw the image of line *j* after a dilation with center *P* and scale factor 2, as shown. Label the image *j'*. Change the location of line *j* and the scale factor. Make a conjecture that relates lines *j* and *j'*.

The dilation of line *j* results in a line *j'* which is parallel to *j*.

e. **MAKE A CONJECTURE** Is there ever a situation in which your conjecture in **part d** does not hold? What happens to a line under a dilation in this case?

If the line passes through the center of the dilation, then the dilation maps the line to itself. The line is also mapped to itself if $k = 1$.

f. **FIND A PATTERN** Plot the point $E(-4, 8)$. Find the image of point *E* under a dilation using each scale factor in the table. Complete the table by writing the coordinates of *E'*. Then describe any patterns you notice.

Scale Factor	$\frac{1}{4}$	$\frac{1}{2}$	2	$\frac{5}{2}$	3
Coordinates of *E'*	(−1, 2)	(−2, 4)	(−8, 16)	(−10, 20)	(−12, 24)

To find the coordinates of *E'*, multiply the *x*- and *y*-coordinates of *E* by the scale factor.

g. **FIND A PATTERN** Calculate the coordinates of *E'* in **part f** when the scale factors are increasing integers such as 4, 5, and 6. Describe any patterns you notice in the value of the *x*- and *y*-coordinates?

As the scale factor increases by one, the *x*-coordinate decreases by 4 and the *y*-coordinate increases by 8. When the scale factor is an integer, all the new *x*- and *y*-coordinates are multiples of the corresponding original coordinates.

KEY CONCEPT Dilations in the Coordinate Plane

Complete the description of dilations in the coordinate plane and complete the algebraic rule to show how a dilation with scale factor *k* changes the coordinates of a point. Then complete the example.

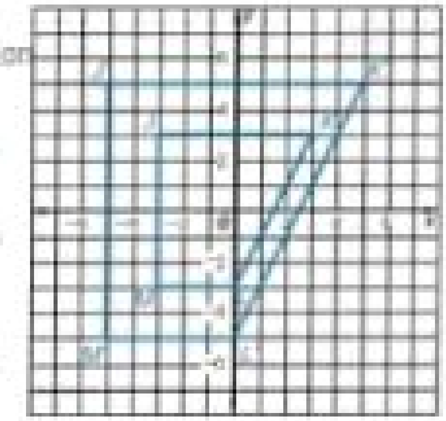
Description	Example
To find the coordinates of a point after a dilation centered at the origin with scale factor *k*, multiply the *x*-coordinate and the *y*-coordinate of the point by *k*. $(x, y) \rightarrow (kx, ky)$	Dilation with scale factor $\frac{3}{2}$  $A(4, 2) \rightarrow A'(6, 3)$

EXAMPLE Use Dilations to Solve a Problem

An architect is using a coordinate plane to design a studio apartment in the shape of a trapezoid. Each unit of the coordinate plane represents one meter. In the original plan, the vertices of the apartment are $J(-3, 3)$, $K(3, 3)$, $L(0, -3)$, and $M(-3, -3)$. The architect wants to enlarge the apartment so that $J'K' = 10$ meters.

a. **INTERPRET PROBLEMS** What scale factor should the architect use to dilate the original apartment so that the image has a side with the required length? Explain.

$\frac{5}{3}$; Sample explanation: the length $\overline{JK}$ is 6 meters; to map $\overline{JK}$ to a segment with a length of 10 meters, the scale factor must be $\frac{10}{6}$ or $\frac{5}{3}$.



b. **COMMUNICATE PRECISELY** Write a function that shows how a point, (x, y), is mapped under this dilation if the center of dilation is the origin.

$(x, y) \rightarrow \left(\frac{5}{3}x, \frac{5}{3}y\right)$

c. **CALCULATE ACCURATELY** Draw and label $JKLM$ and its image, $J'K'L'M'$, on the coordinate plane at the right.

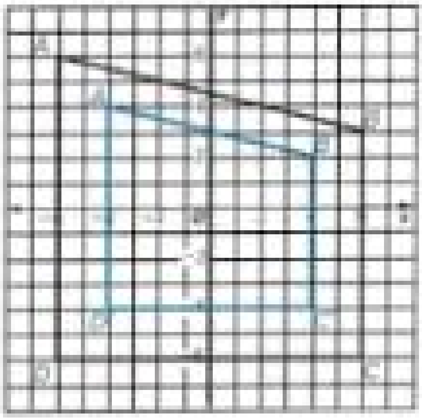
d. **CRITIQUE REASONING** The architect claims that the enlarged apartment will have a perimeter that is more than twice as large as the original perimeter. Do you agree or disagree? Explain.

Disagree; Each side length of the original apartment is multiplied by $\frac{5}{3}$ the perimeter of the enlarged apartment will be $\frac{5}{3}$ the perimeter of the original apartment, and $\frac{5}{3}$.

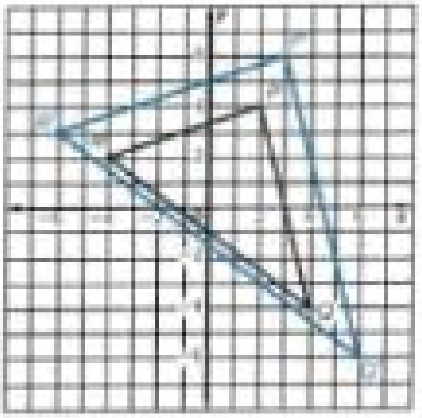
PRACTICE

CALCULATE ACCURATELY Draw and label the image of the figure after a dilation with the given scale factor and center of dilation at the origin. Write a function to describe the transformation.

1. $k = \frac{2}{3}$ $(x, y) \rightarrow \left(\frac{2}{3}x, \frac{2}{3}y\right)$



2. $k = 1.5$ $(x, y) \rightarrow (1.5x, 1.5y)$



Scanned by CamScanner

EVALUATE REASONABLENESS Determine whether each statement is *always*, *sometimes*, or *never* true. Explain.

3. If c is a real number, then a dilation centered at the origin maps the line $y = cx$ to itself.

Always; the line $y = cx$ passes through the origin and a dilation leaves lines through the center of dilation unchanged.

4. If $k > 1$, then a dilation with scale factor k maps $\overline{AB}$ to a segment that is congruent to $\overline{AB}$.

Never; $A'B' = k(AB)$, and $k > 1$, so $A'B' \neq AB$. This means $\overline{AB}$ cannot be congruent to its image.

5. If $0 < k < 1$, then a dilation of parallelogram $ABCD$ with a scale factor of k will result in parallelogram $A'B'C'D'$ with a perimeter equal to that of the original parallelogram.

Never. If the scale factor is between 0 and 1, then it is a rational number between 0 and 1, which means the perimeter of parallelogram $A'B'C'D'$ will always be less than the perimeter of $ABCD$.

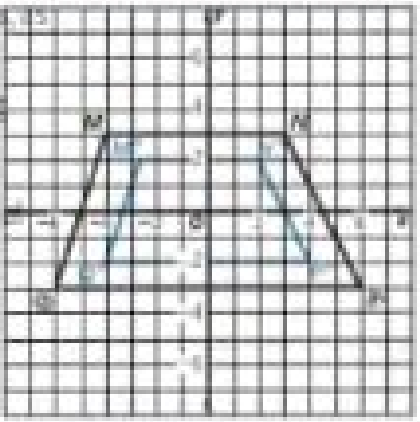
6. The dilation $(x, y) \rightarrow (kx, ky)$ maps a point to a point that is farther from the origin.

Sometimes; this is true when $k > 1$ and the point is not the origin. For any value of k, the origin is mapped to itself, so the statement is not true for the origin.

7. If a is a real number, then a dilation centered at the origin maps the line $x = a$ to a vertical line.

Always; a dilation maps a line to itself (if it passes through the center of dilation) or to a parallel line. Since $x = a$ is a vertical line, its image is also a vertical line.

8. Seif is using a coordinate plane to experiment with quadrilaterals as shown in the figure.



a. **CALCULATE ACCURATELY** Seif creates $M'N'P'Q'$ by enlarging $MNPQ$ with a dilation with scale factor 2. Then he creates $M''N''P''Q''$ by dilating $M'N'P'Q'$ with a scale factor of $\frac{1}{4}$. The center of dilation for each dilation is the origin. Draw and label the fin image, $M''N''P''Q''$.

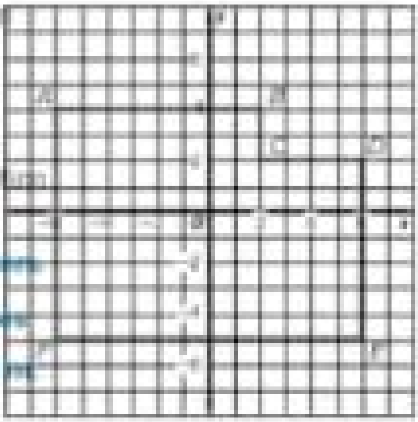
b. **REASON ABSTRACTLY** Can Seif map $MNPQ$ directly to $M''N''P''Q''$ with a single transformation? If so, what transformation should he use?

Yes; use a dilation with scale factor $\frac{1}{2}$.

c. **MAKE A CONJECTURE** In general, what can you say about a dilation with scale factor k_1 that is followed by a dilation with scale factor k_2? State a conjecture and then provide an argument to justify it.

The composition is equivalent to a single dilation with scale factor k_1k_2. The composition maps (x, y) to (k_1x, k_1y), and then this point is mapped to (k_2k_1x, k_2k_1y), which is the algebraic rule for a single dilation with scale factor k_1k_2.

9. **REASON QUANTITATIVELY** The figure shows an architect's pla for a bedroom. Each unit of the coordinate plane represents $\frac{1}{3}$ meter.



a. The architect would like to enlarge or reduce the bedroom as needed so that its perimeter is 21 meters. What transforma should the architect use? Justify your answer.

Dilation with scale factor 1.5; the current perimeter is 14 mete

The dilation with scale factor 1.5 makes each side 1.5 time

longer, so the perimeter of the image will be 1.5 · 14 = 21 m

b. The architect would like to enlarge or reduce the bedroom as needed so that its perimeter is 10.5 meters. What scale factor should the architect use in his dilation? Explain.

The dilation would be a reduction using a scale factor $\frac{3}{4}$. The current perimeter is 14 m, so the new image will be $x \cdot 14 = 10.5$ and, solving for x, $\frac{10.5}{14} = \frac{3}{4} = 0.75$.

c. Based on **parts a** and **b**, write an equation that can be used to find the scale factor (x) of a dilation given any perimeter (y).

$x = \frac{14}{y}$ or $y = 14x$

10. The point P' is the image of point $P(a, b)$ under a dilation centered at the origin with scale factor $k \neq 1$.

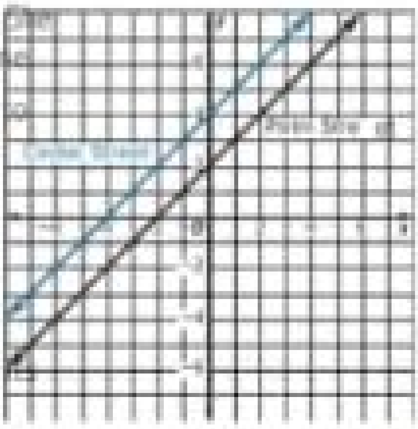
a. **REASON ABSTRACTLY** Assuming that point P does not lie on the y-axis, what is the slope of $\overleftrightarrow{PP'}$? Explain how you know.

The slope is $\frac{b}{a}$. Sample answer: The coordinates of P' are (ka, kb). The slope of $\overleftrightarrow{PP'}$ is $\frac{kb - b}{ka - a} = \frac{b(k - 1)}{a(k - 1)} = \frac{b}{a}$.

b. **USE STRUCTURE** In **part a**, why is it important that P does not lie on the y-axis?

If P lies on the y-axis, then its image P' will also lie on the y-axis. In this case $\overleftrightarrow{PP'}$ will be vertical and the slope will be undefined.

11. A city planner is designing the streets in a new shopping district. has already planned Palm Street, as shown on the coordinate pla



a. **REASON ABSTRACTLY** The city planner will apply a dilation Palm Street to create Cedar Street. Describe how the two streets will be related.

The streets will be parallel.

b. **USE STRUCTURE** The city planner uses a dilation with scale factor 2 and center of dilation at the origin to create Cedar Street. Draw Cedar Street on the coordinate plane and write equation.

$y = x + 4$

Scanned by CamScanner

Part B

For each coordinate of the new triangle, state which coordinate mapped to it through the function you defined in **Part A**.

Part C

If the image in **Part A** is again scaled by the same factor as the original image, what will the new coordinates be? Use a function to relate the new coordinates to the original coordinates.

Part D

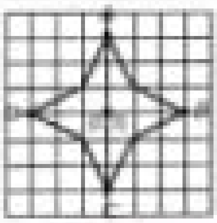
If the original image required 2 mL of ink to print, how much ink would a design that includes the original image, the image in **Part A**, and the image in **Part C** require? Explain your reasoning.

Performance Task

Trendy Transformations

Provide a clear solution to the problem. Be sure to show all of your work, include all relevant drawings, and justify your answers.

Asma is printing shirts for her high school's math club. She starts with the basic design shown, centered on a 20 by 20 unit grid. Each grid measures 0.5 centimeter by 0.5 centimeter. She wants each club member to have a unique shirt, so she instructs them to choose three transformations to perform on the original design. Each transformation will be centered at the highest point of the preimage, and each preimage and each resulting image will be included as part of the design.



Part A

Hessa chooses a translation vector of $\langle 4, -3 \rangle$, a 90° rotation about the highest point in the previous image, and then a reflection in the line $y = 3$. What are the coordinates of A''', B''', C''', and D'''? Justify your answer.

Standardized Test Practice

1. What is the value of x in the following diagram?

$x =$ 5

2. Ahmed connects points $S(5, -2)$ and $T(1, 9)$ to create $\overline{ST}$. He places point M at the midpoint of $\overline{ST}$. The coordinates of M are (3, 3.5).

3. Quadrilateral $ABCD$ has vertices $A(4, 3)$, $B(-2, 7)$, $C(-3, 0)$, and $D(4, -4)$. After quadrilateral $ABCD$ is dilated about the origin, A' is located at $(6, 4.5)$. Find the scale factor and complete the following about $A'B'C'D'$.

Scale factor: 1.5

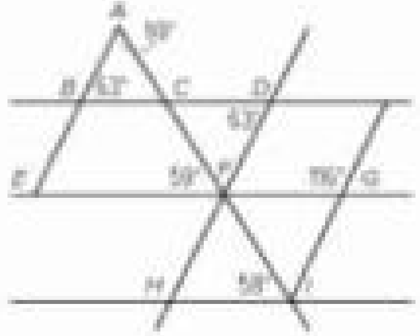
$B'($ −3, 10.5 $)$

$C'($ −4.5, 0 $)$

$D'($ 6, −6 $)$

4. A transformation is performed on $\triangle CDE$ with vertices $C(0, 2)$, $D(-1, 7)$ and $E(2, 5)$ to obtain $\triangle C'D'E'$ with vertices $C'(0, -2)$, $D'(-1, 3)$, and $E'(2, 1)$. The type of transformation is a translation.

5. Consider the following diagram.



Select the triangles that are similar to $\triangle ABC$.

△AFE [✓ △CDF]

[✓ △FIH] △FGI

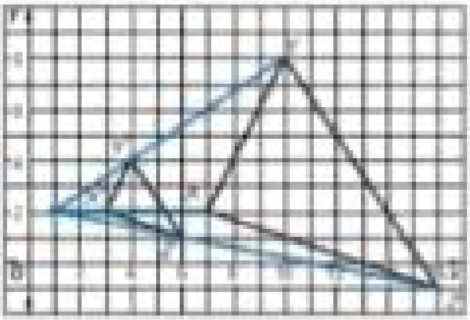
6. Consider the following diagram.

If $AC = 9$, then $AB =$ 6.

7. $\triangle JKL \sim \triangle MNO$. The perimeter of $\triangle JKL$ is 24 and the perimeter of $\triangle MNO$ is 60. If $MO = 15$ and $MN = 20$, then $KL =$ 10.

8. Point D is the midpoint of $\overline{AB}$ and point E is the midpoint of $\overline{BC}$. If the perimeter of $\triangle BDE$ is 15 units, then the perimeter of $\triangle BAC$ is 30 units. If the area of $\triangle ABC$ is 56 square units, then the area of $\triangle BAC$ is 14 square units.

9. $\triangle X'Y'Z'$ is a dilation of $\triangle XYZ$. In the diagram below, draw lines to show how to find the center of dilation. What are the coordinates of the center of dilation?



(1, 2)

10. Consider each figure. Place a check mark under the rotations about the center of the figure that map the figure onto itself.

Figure	60° Rotation	90° Rotation	180° Rotation
Square		✓	✓
Isosceles Trapezoid			
Parallelogram			✓
Regular Hexagon	✓		✓
Rectangle			✓

11. Abeer draws $\triangle XYZ$ with vertices $X(-4, -1)$, $Y(-1, -1)$ and $Z(0, -3)$. After two transformations, the vertices of the image are $X'(-6, 4)$, $Y'(-3, 4)$, and $Z'(-2, 6)$. Describe the transformations that Abeer could have performed on $\triangle XYZ$ to obtain this image.

Sample answer: reflection in the x-axis, followed by translation $(x, y) \rightarrow (x - 2, y + 3)$

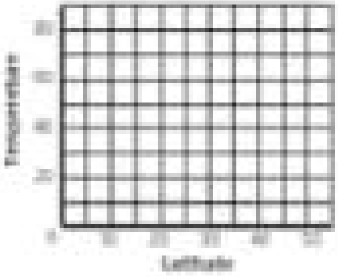
12. $\triangle JKL$ has vertices $J(-9, 4)$, $K(-3, 8)$, and $L(5, -4)$. $\triangle DEF$ has vertices $D(-1, 4)$, $E(2, 2)$, and $F(6, 8)$. Are the triangles similar? Show your work.

Yes; slope of $\overline{JK} = \frac{8-4}{-3-(-9)} = \frac{2}{3}$, slope of $\overline{KL} = \frac{-4-8}{5-(-3)} = -\frac{3}{2}$; $\frac{2}{3}\left(-\frac{3}{2}\right) = -1$, so $\angle K$ is a right angle. Likewise, slope $\overline{DE} = \frac{2-4}{2-(-1)} = -\frac{2}{3}$ slope of $\overline{EF} = \frac{8-2}{6-2} = \frac{3}{2}$; $-\frac{2}{3}\left(\frac{3}{2}\right) = -1$, so $\angle E$ is a right angle. $JK = \sqrt{(8-4)^2 + (-3-(-9))^2} = 2\sqrt{13}$, $KL = \sqrt{(-4-8)^2 + (5-(-3))^2} = 4\sqrt{13}$. $DE = \sqrt{(2-4)^2 + (2-(-1))^2} = \sqrt{13}$, $EF = \sqrt{(8-2)^2 + (6-2)^2} = 2\sqrt{13}$. Since $\frac{JK}{KL} = \frac{DE}{EF} = \frac{1}{2}$, these sides are proportional. By SAS, $\triangle JKL \sim \triangle DEF$.

13 Statistics and Probability

CHAPTER FOCUS Learn about some of the Common Core State Standards that you will explore in this chapter. Answer the preview questions. As you complete each lesson, return to these pages to check your work.

<table>
<tr><th>What You Will Learn</th><th>Preview Question</th></tr>
<tr><td colspan="2">Lesson 13.2: Statistics and Parameters</td></tr>
<tr><td>Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.
Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).</td>
<td>A teacher recorded test scores for students who studied less than two hours with students who studied two or more hours. Compare the means and the standard deviations of each data set.
<table>
<tr><th colspan="6">Studied Less Than Two Hours</th></tr>
<tr><td>72</td><td>76</td><td>81</td><td>90</td><td>65</td><td>65</td></tr>
<tr><td>73</td><td>80</td><td>81</td><td>76</td><td>62</td><td>65</td></tr>
</table>
<table>
<tr><th colspan="6">Studied More Than Two Hours</th></tr>
<tr><td>76</td><td>85</td><td>92</td><td>78</td><td>85</td><td>90</td></tr>
<tr><td>93</td><td>88</td><td>75</td><td>80</td><td>88</td><td>98</td></tr>
</table>
Sample answer: The mean of the first set is 73.8 and the std deviation is 8.5. The mean of the second set is 85.7 and the std deviation is 7.2. Students score better and there is less variation in the scores if they study 2 or more hours.</td></tr>
<tr><td colspan="2">Lesson 13.3: Distributions of Data</td></tr>
<tr><td>Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).
Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.
Represent data with plots on the real number line (dot plots, histograms, and box plots).
Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.</td>
<td>The ages of people in two bowling leagues are shown. Compare and interpret the shape, mean, and standard deviation of the two sets.
<table>
<tr><td>Set 1</td><td>18</td><td>22</td><td>26</td><td>29</td><td>21</td><td>25</td><td>31</td><td>28</td><td>20</td><td>62</td><td>25</td></tr>
<tr><td>Set 2</td><td>22</td><td>18</td><td>24</td><td>28</td><td>25</td><td>32</td><td>30</td><td>21</td><td>35</td><td>24</td><td>25</td></tr>
</table>
Both sets are skewed to the right, even with the exclusion of the outlier in Set 1. Set 1 has a mean of 27 (23 without the outlier); Set 2 has a mean of 25. Set 1 has a std dev of 12 (4.6 without the outlier); Set 2 has a std dev of 4.3. Both sets are clustered about the mid twenties, but Set 2 has a distribution that is closer to normal.</td></tr>
</table>

<table>
<tr><th>What You Will Learn</th><th>Preview Question</th></tr>
<tr><td colspan="2">**Lesson 13.4 Comparing Sets of Data**</td></tr>
<tr><td>Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.
Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).
Represent data with plots on the real number line (dot plots, histograms, and box plots).</td><td>Describe how the mean and range are affected if each number in the data set is halved.
18, 28, 26, 10, 25, 25, 64, 62, 78, 28
The mean of the original data set is 36.4 and the range is 68. The mean of the new data set is 18.2 and the range is 34. The mean and range of the new data set are half the mean and range of the original data set.</td></tr>
<tr><td colspan="2">**Lesson 13.5 Scatter Plots and Lines of Fit**</td></tr>
<tr><td>Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.
Fit a linear function for a scatter plot that suggests a linear association.
Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.</td><td>Temperature is related to latitude. Make a scatter plot using the data in the table. What type of relationship is shown?
<table>
<tr><td>**Latitude**</td><td>35</td><td>33</td><td>30</td><td>25</td><td>40</td></tr>
<tr><td>**Temperature**</td><td>46</td><td>52</td><td>67</td><td>76</td><td>37</td></tr>
</table>


The graph shows a negative correlation between temperature and latitude.</td></tr>
<tr><td colspan="2">**Lesson 13.6 Regression and Median-Fit Lines**</td></tr>
<tr><td>Informally assess the fit of a function by plotting and analyzing residuals.
Compute (using technology) and interpret the correlation coefficient of a linear fit.</td><td>Use the Latitude and Temperature data from the Preview Question for Lesson 13.5 to calculate the equation for the median-fit line. Use a graphing calculator and round to the nearest tenth. Predict the average temperature for the latitude 45°N.
There is a correlation and a causal relationship. Practice improves skill development.</td></tr>
</table>

13-2 Statistics and Parameters

Objectives

- Compare center and spread of different data sets.
- Use statistics from data samples to estimate parameters of populations.
- Explore the effects of outliers in samples.

STANDARDS

Content: S.ID.2, S.ID.3
Practices: 1, 2, 3, 5, 6, 7
Use with Lesson 13-2

A **sample** is a subset of a population chosen to represent the population. A **statistic** is a measure calculated from the sample. Mean, mode, range, and interquartile range are all examples of a statistic. The equivalent measure for the population is called a **parameter** of the population. Parameters are fixed values that can be determined by the entire population but are typically estimated based on the statistics of a carefully chosen random sample.

Statistical inference is the process of drawing conclusions about a population based on information from a sample. For example, the mean height of all 18-year-old women in Abu Dhabi can be inferred from a random sample of 1000 women in Abu Dhabi who are 18 years old.

Making reasonable inferences about parameters of a population requires a different sort of sample statistic than the range or interquartile range.

The **deviation** of a value x_i in a sample data set is its difference from the sample mean $x_i - \bar{x}$.
The **mean absolute deviation** of a sample is the mean of the absolute deviations:
MAD $= \frac{|x_1 - \bar{x}| + |x_2 - \bar{x}| + \cdots + |x_n - \bar{x}|}{n} = \frac{1}{n}\sum |x_i - \bar{x}|$, where the sigma Σ means "sum over all values of i." The standard deviation (σ) of a sample is given by
$\sigma = \sqrt{\frac{|x_1 - \bar{x}|^2 + |x_2 - \bar{x}|^2 + \cdots + |x_n - \bar{x}|^2}{n}} = \sqrt{\frac{1}{n}\sum |x_i - \bar{x}|^2}$.

EXAMPLE 1 Compare Measures of Spread

The following is a table of long jump distances for students in a classroom

Long Jump Distances (m)		
6.1	5.9	6.3
6.3	6.1	5.8

a. **USE TOOLS** Use a calculator to find the mean and standard deviation of the set.

The mean is approximately 6.08 m. The standard deviation is approximately 0.2 m.

b. **CALCULATE ACCURATELY** Find the mean absolute deviation of the jumps.

MAD $= \frac{6.1 - 6.08 + 5.9 - 6.08 + |6.3 - 6.08 + |6.3 - 6.08 + |6.1 - 6.08 + 5.8 - 6.08}{6} = 0.157$

c. **COMMUNICATE PRECISELY** If the data is simply a sample, what might the population be for this example? Use the mean absolute deviation to judge how well the mean represents the data.

Sample answer: The population might be all students of the same age. Because the mean absolute deviation is relatively small compared to the values on the table, the spread is small, and, therefore, the mean adequately represents the data set.

d. **REASON ABSTRACTLY** A different sample of jumps from a second classroo given in the following table. Calculate the mean and standard deviation using calculator, and discuss how this set compares to the first set.

Long Jump Distances (m)		
6.1	5.9	6.5
6.5	6.1	5.4

The mean for this set is 6.08; the standard deviation is 0.4. The two sets have the same mean, but the standard deviation of the second set is larger. This means that the second set is more spread out, and, therefore, the mean is less representative of the sample.

EXAMPLE 3 Identify and Allow for Outliers

Using a GPS device, Ibrahim and his son Badr recorded the length of all their tee shots hit with a driver during a recent round of golf. The results are shown in the table.

Drives (yards)												
Ibrahim	195	203	217	198	193	188	207	200	215	205	210	207
Badr	198	210	225	196	198	14	240	192	221	215	201	210

a. **USE TOOLS** Use technology to find the mean, median, interquartile range, and standard deviation of each sample to the nearest tenth.

Ibrahim: $\bar{x}$ = 203.2 yd, m = 204.0 yd; Q3 − Q1 = 208.5 − 196.5 = 12.0 yd; σ = 8.4 yd

Badr: $\bar{x}$ = 193.3 yd; m = 205.5 yd; Q3 − Q1 = 218 − 197 = 21 yd; σ = 55.7 yd

b. **INTERPRET PROBLEMS** Use the statistics to show in what aspects of driving Ibrahim is better.

Ibrahim's mean (203.2 yd) is higher than Badr's mean (193.3 yd). Therefore, on average, Ibrahim drives further than Badr. Additionally, Ibrahim' standard deviation (8.4 yd) is lower that Badr's (55.7 yd), which means he is a more consistent driver.

c. **INTERPRET PROBLEMS** Use the statistics to show in what aspects of driving Badr is better.

Badr's median (205.5 yd) is higher than Ibrahim' median (204.4 yd). Additionally, Badr's maximum (240 yd) is higher than Ibrahim' maximum drive (217 yd).

d. **INTERPRET PROBLEMS** On one shot, Badr loses his grip and drives the ball only 14 yards. Write what the definition for an outlier is and then use that information to decide if the 14-yard drive is an outlier for Badr's sample drives. If it is, then use technology to recalculate the mean and standard deviation of Badr's sample drives without the outlier and describe how this affects the statistics of the sample.

Sample answer: A statistical outlier is a data point that is at least 1.5 times the interquartile range less than the lower quartile or greater than the upper quartile. The interquartile range is 21 yd. 1.5(21) = 31.5. Since Q1 is 197 yd, anything lower than (197 − 31.5) = 165.5 yd would be an outlier. Therefore, 14 yd is an outlier. Removing it from the data set produced a new mean of 209.6 yd, an a new standard deviation of 14.0 yd. The mean is raised by eliminating the 14, and the standard deviation is reduced.

PRACTICE

1. Two different samples on the shell diameter of a species of snail are shown.

Sample A (mm)		
45	35	37
40	42	40
28	38	31

Sample B (mm)		
20	44	40
27	35	28
26	39	31

a. **INTERPRET PROBLEMS** Use the median and interquartile range to compare the samples.

Sample answer: For sample A, $m = 38$ mm, IQR = Q3 − Q1 = 41 − 33 = 8 mm

For sample B, $m = 31$ mm, IQR = Q3 − Q1 = 39.5 − 26.5 = 13 mm. Sample A has a higher median but a lower IQR. Therefore, sample A tends to be less spread out than B, but also tends to be larger than sample A.

b. **USE STRUCTURE** Based on your findings and on the data points in each sample, which sample appears to be more representative? Explain your reasons.

Sample answer: Sample A is more closely and more evenly grouped around its median. Sample B is skewed toward smaller diameters. So sample A is more representative.

2. **INTERPRET PROBLEMS** Height data samples of 17-year-old male and female students are shown. Use the mean and standard deviation to compare the samples.

Heights of Male Students (inches)		
71	69	67
68	69	70
72	74	68
71	69	72

Heights of Female Students (inches)		
67	62	69
65	71	66
63	65	68
66	63	70

Sample answer: Male students, $\bar{x} = 70.0$ in., $\sigma = 2.0$ in. Female students, $\bar{x} = 66.3$ in., $\sigma = 2.7$ in. Sample answer: For male students, the mean is 70.0 in., and the standard deviation is 2.0 in. For female students, the mean is 66.3 in., and the standard deviation is 2.7 in. On average, males are taller. However, because the standard deviation of males is smaller than that of females, the heights of females are more spread out.

3. Asma is looking at her high jump results from the last two seasons' competitions. She feels that her performance has been skewed by unusual results. Below are her results from the last two seasons.

Season 1 Results (m)			1.69	1.72
1.63	1.66	1.72	1.78	1.69
1.45	1.66	1.57	1.66	1.65

Season 2 Results (m)			1.75	1.68
1.63	1.62	1.71	1.54	1.73
1.59	1.76	1.57	1.61	1.60

a. **REASON QUANTITATIVELY** Consider the two sets of data. Determine whether either contains an outlier. Find the new mean and standard deviations once all outliers are removed from the sets.

In season 1, 1.45 is an outlier. Without 1.45 m, the mean is 1.679 m, and the standard deviation is 0.052 m.

b. **REASON QUANTITATIVELY** Determine which was Asma's better season after the outliers were removed. Explain your answer.

In the second season, the mean is 1.65 m, and the standard deviation is 0.07 m. Once the outlie is removed from season 1, the mean is 1.679 m, and the standard deviation is 0.052. Therefore, on average she jumped better in season one; additionally, because of the smaller standard deviation, she also jumped more consistently in season 1. It seems that season 1 was her better season.

4. Ibrahim is preparing to go golfing again. He is looking for a partner. He wants to pick the best player, so he contacts two friends and obtains their totals on the last six rounds of golf. The data is given below.

Khadija	45	47	46	47	44	49
Rana	49	49	47	47	48	47

a. **CALCULATE ACCURATELY** For each player, find the mean and standard deviation.

Rana's mean score is 46.3. Her standard deviation is 1.60. Rana's mean is 47.8, with a standard deviation of 0.90.

b. **REASON QUANTITATIVELY** What would be the advantages or disadvantages to Ibrahim selecting one player over the other?

Ibrahim could choose Rana for the low mean. However, Rana is a more consistent player; she has a lower standard deviation.

5. Consider the following set of package weights randomly selected by the Post Office during a busy season.

Package Weights		
0.56 lb	1.21 lb	1.03 l
0.78 lb	0.88 lb	0.96 l

a. **CALCULATE ACCURATELY** Find the mean and the mean absolute deviation of the weights.

$\bar{x} = \frac{0.56 + 1.21 + 1.03 + 0.78 + 0.88 + 0.96}{6} = 0.90$ lb.

MAD $= \frac{|0.56 - 0.90| + |1.21 - 0.90| + |1.03 - 0.90| + |0.78 - 0.90| + |0.88 - 0.90| + |0.96 - 0.90|}{6} = 0.16$ lb.

b. **COMMUNICATE PRECISELY** Use the mean absolute deviation to judge how well the mean represents the data.

The mean absolute deviation seems rather large for the values. Therefore, the mean may not represent the values very well.

c. **COMMUNICATE PRECISELY** What could be the population for this sample? Comment on why this data set may or may not be representative of the population?

The population could be all packages sent by the Post Office during the Christmas season. Because the sample was randomly chosen, the data would represent the population better tha it would had the packages not been random. The disadvantage of this set is that the number of data points is fairly small.

13.3 Distributions of Data

Objectives

- Interpret differences in shape, center, and spread for data distributions.
- Select the appropriate statistics to describe samples based on their distributions.

STANDARDS

Content: S.ID.1, S.ID.2, S.ID.3, N.Q.1

Practices: 1, 4, 5, 6, 7, 8

Use with Lesson 13-3

The **distribution** of a sample or population of data describes the observed or theoretical frequencies of the data values, often in a visual way allowing the data to be assessed at a glance.

EXAMPLE 1 Use Statistics to Create a Visual Display of a Distribution

EXPLORE Fifteen expectant women took part in a study of the duration of pregnancy.

Pregnancy Duration (weeks)		
39	41	38
32	39	40
35	39	40
40	41	37
31	40	40

a. **COMMUNICATE PRECISELY** Find the five-number summary of the data, defined as minimum value, first quartile, median, third quartile, and maximum value. Use these statistics and the scale provided to create a box-and-whisker plot of the data.

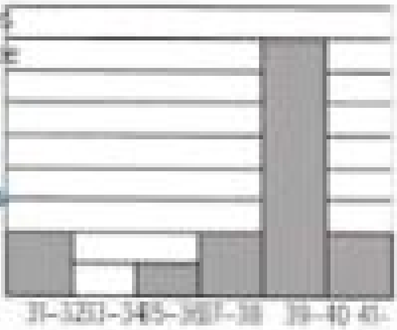
5-number summary:

31, 37, 39, 40, 41

b. **USE STRUCTURE** Based on the box-and-whisker plot, where is the data most concentrated within the overall range of the sample? Explain your answer.

Sample answer: The data is most concentrated in the upper quartile of the sample. One quarter of the data must be in each segment of the plot, so the shorter segments are more concentrated.

c. **INTERPRET PROBLEMS** The histogram represents the same data as part a. How is the position of the histogram's peak related to the shape of your box-and-whisker plot?



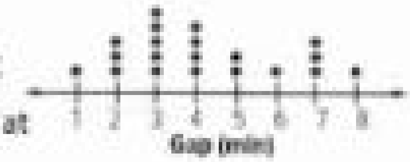
Sample answer: The peak occurs near the median, 39 weeks, which is well to the right within the sample. Most of the data is close to the median but there is a long tail to the left.

d. **EVALUATE REASONABLENESS** Does this data seem reasonable? Why or why not?

Sample answer: The data seems reasonable because the sample has a peak to the right and a long tail to the left. This seems reasonable because pregnancies are normally about 40 weeks, but can sometimes be much shorter, with premature births, but are almost never longer than 41 weeks.

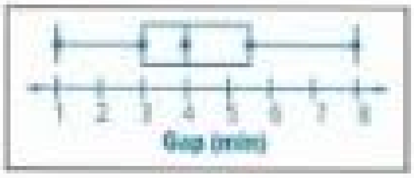
A distribution is **negatively skewed** if it has a peak that is to the right within the distribution, and **positively skewed** if it has a peak to the left. An even distribution around a central peak is **symmetric**.

EXAMPLE Describe the Distribution of a Data Set



The dot plot displays data on the gaps between phone calls received at a company switchboard.

a. **USE TOOLS** Using the dot plot, enter the data into a graphing calculator and create a box-and-whisker plot. Make a sketch of the box-and-whisker plot labeled with the five-number summary data points.



b. **COMMUNICATE PRECISELY** Describe the shape and any other significant features of the distribution.

Sample answer: Both the dot plot and the box-and-whisker plot show that the data set is positiv skewed, with the data concentrated in the lower part of the range.

c. **USE STRUCTURE** A second company records the gaps between phone calls in a chart. Create a histogram from the data provided.

Gaps between calls, in minutes: 4, 3, 5, 4, 1, 2, 4, 4, 7, 6, 5, 3, 4, 5, 5, 3, 6, 1, 2, 6, 3, 2, 7

d. **COMMUNICATE PRECISELY** Contrast the shape of the second data set with the first. What does the difference tell you about the data?

Sample answer: The second set is symmetric about the mean. The peak of the data is in the center. The mean, median, and mode are all the same.

The shape of a sample's distribution affects which statistics best describe the data. Symmetric data sets are well described—or compared if there are two samples—b mean and standard deviation. For skewed distributions, the five-number summary is better description or comparison.

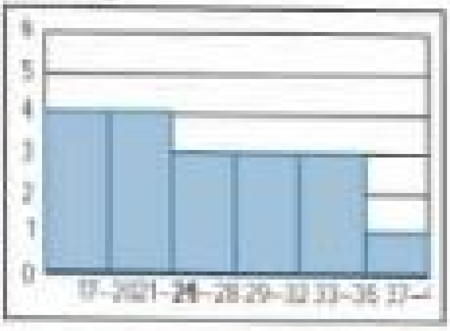
Field Goal Distances (yd)		
17	20	22
23	33	25
28	18	36
24	26	35
32	25	19
38	30	21

EXAMPLE Choose Statistics to Describe a Data Set

During American football practice a coach has his kicker attempt field goals from each yard line between 17 and 40 yards. The coach records the results of each attempt. The kicker's successful field goal distances for the practice are shown at the right.

a. **COMMUNICATE PRECISELY** Choose, create, and sketch a visual representation of the data. Describe its distribution.

Sample answer: The data is positively skewed. There is a fairly even distribution for all the kicks from 17 to 36 yards with no real peaks.



b. **INTERPRET PROBLEMS** Choose and calculate representative statistics for this data set. Justify your choice.

Sample answer: five-number summary, because the distribution is skewed; min = 17 yd, Q1 = 21 yd, median = 25.5 yd, Q3 = 32 yd, IQR = Q3 − Q1 = 11 yd, max = 38 yd

c. **FIND A PATTERN** Discuss the data's distribution based on your sketch and chosen statistics.

Sample answer: The data is concentrated in the lower part of the sample, from 17 yd to about 30 yd. There is also a strong concentration of data between Q1 and Q3. This indicates that the kicker was quite consistent for field goals up to 30 yd, with fewer successful longer kicks.

EXAMPLE 4 Choose Statistics to Compare Data Sets

Daisy is comparing the scores of two different classes on the same algebra test.

Class A			
68	68	32	45
78	98	63	82
90	52	47	67
71	80	69	66
86	75	57	71
60	67	64	64
62	74	77	77

Class B			
60	68	88	84
53	59	70	73
62	68	73	82
82	79	54	93
75	78	73	80
85	64	90	72
72	71	67	78

a. **USE STRUCTURE** What statistics should Hessa use for her comparison? Explain why she should choose these statistics. Mention any outliers that need to be considered.

Sample answer: She should use the mean and standard deviation. The histograms of the data sets show that both distributions are symmetric, particularly after the outlier score of 32 is removed from class A.

b. **INTERPRET PROBLEMS** Calculate your chosen statistics, taking account of outliers. Then, use these statistics to suggest issues for Hessa to discuss in analyzing her comparison.

Sample answer: class A (without outlier): $\mu = 69.6$, $\sigma = 12.0$,

class B: $\mu = 73.4$, $\sigma = 10.2$

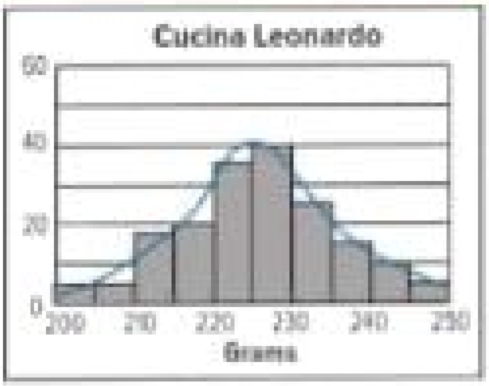
Issues: Why is class A's mean score lower? Why are the class B results less variable?

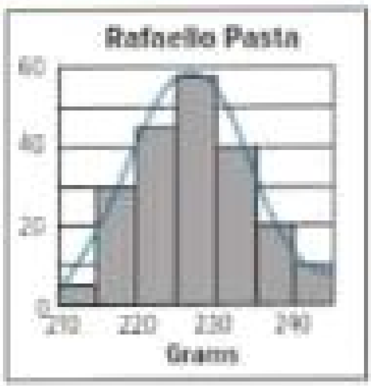
c. **DESCRIBE A METHOD** What are some advantages of comparing the data in **part b** using the other statistics not chosen?

Sample answer: The minimum and maximum give the extreme points of the data set; the median and the interquartile range (=Q3 − Q1) are less affected by outliers than by the mean and standard deviation.

PRACTICE

1. The histograms show the weight of sample boxes of two brands of pasta.





a. **USE A MODEL** Do the two packages of pasta likely have the same advertised weight? Which manufacturer's quantity control appears better? Explain your answers based on the distributions.

Sample answer: 225–230 g would be a reasonable advertised weight for either brand, so it is qu

likely they have the same advertised weight. Rafaello appears to have better control over the exa

quantity in each package, since its distribution is grouped more closely about the mean.

b. **FIND A PATTERN** Infer the two population distribution shapes by sketching smooth curves across the tops of the histograms. Describe the shapes you have sketched.

Sample answer: Both distributions have an inverted, symmetric U-shape with "tails" on either sid

Leonardo's distribution is lower and wider.

2. **CALCULATE ACCURATELY** Consider the data from **Example 3** regarding a kicker's successful field goal attempts. Find the mean and standard deviation for the data and explain what the statistics tell the coach about his kicker.

$\mu = 26.4$, $\sigma = 6.3$; Sample answer: The statistics tell the coach that the kicker makes most of his f

goals from about 20 to 32 yards.

3. **INTERPRET PROBLEMS** The United States has been sending astronauts up in the Space Shuttle since 1981. The table below provides data regarding the duration of Space Shuttle flights from 1981 to 1985 and then from 2005 to 2011.

Length of Flights from 1981–1985 (days)

Days: 2, 2, 8, 7, 5, 5, 6, 6, 10, 8, 7, 6, 8, 8, 3, 7, 7, 7, 8, 7, 4, 7, 7

Length of Flights from 2005–2011 (days)

Days: 14, 13, 12, 13, 14, 13, 15, 13, 16, 14, 15, 13, 13, 16, 14, 11, 14, 15, 12, 13, 16, 13

Choose and calculate the statistics appropriate for the distribution of the data sets. Use the statistics to compare the two sets.

Sample answer: Since the distributions are skewed, compare using five-number summary.

1981–1985: min = 2, Q1 = 5, median = 7, Q3 = 8, IQR = 3, max = 10; 2005–2011: min = 11,

Q1 = 13, median = 13.5, Q3 = 15, IQR = 2, max = 16. From 1981–1985, all the flights were shorte

From 2005–2011, all flights were longer and the durations were more closely and evenly groupe

around the median.

13.4 Comparing Sets of Data

Objectives

- Compare data sets considering measures of center and spread

STANDARDS
Content: S.ID.1, S.ID.2, S.ID.3
Practices: 1, 2, 3, 5, 6, 7, 8
Use with Lesson 13-4

New data sets can be created by performing the same operation on all points in an original set. These operations have predictable effects on measures of spread and center.

EXAMPLE 1 Compare Distributions of Related Sets of Data

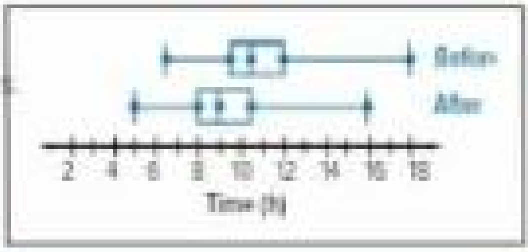
EXPLORE The installation times for Easiflow central air systems are shown.

a. **INTERPRET PROBLEMS** Easiflow develops a new system that will cut installation time on any project by 1.5 hours. The table below shows installation times for a series of projects before the new system was put in place. Fill in the predicted installation times to create a new set of data that would represent the times if the new system were used. Next, compare the two data sets. How do the mean, median, standard deviation, and interquartile range (IQR) change, and why? Verify your answer by finding these measures before and after the reduction.

Installation Times (hours)														
12.0	12.0	9.5	10.0	11.5	10.0	17.5	6.5	9.0	10.5	11.0	13.5	8.0	9.5	15.0
10.5	10.5	8.0	8.5	10.0	8.5	16.0	5.0	7.5	9.0	9.5	12.0	6.5	8.0	13.5

Sample answer: Each data point is reduced by 1.5, so the mean and median will be reduced by 1.5 h. The standard deviation and IQR should not change, since the spread is unchanged. Before: $\mu \approx 11.0$, $m = 10.5$, $\sigma \approx 2.7$, $IQR = 12 - 9.5 = 2.5$; After: $\mu \approx 9.5$, $m = 9.0$, $\sigma \approx 2.7$, $IQR = 10.5 - 8 = 2.5$

b. **USE TOOLS** Predict the effect of implementing the new system on the box-and-whisker plot for the data set. Check your prediction by creating "before" and "after" plots.



Sample answer: The plot should shift left by 1.5 h.

c. **CONSTRUCT ARGUMENTS** Explain what will happen to a histogram of the data as a result of the implementing the new system.

Sample answer: Each interval of the histogram is shifted 1.5 h to the left, but otherwise does not change. So, the histogram will not change shape but will move 1.5 h to the left.

EXAMPLE Compare Data Sets with Symmetric Distributions

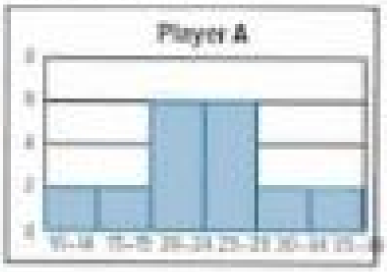
A sports reporter wants to compare two basketball players over the first 20 games of the season. To do so, she collected the points scored by each player for each of the games. The results are shown below.

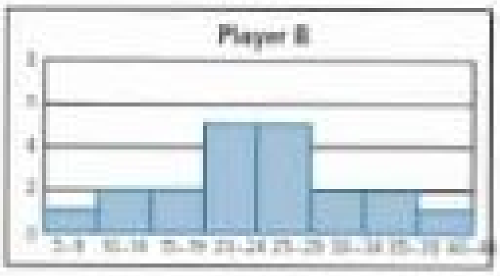
Player A			
24	32	25	29
29	19	27	36
22	35	24	32
13	21	20	17
27	23	11	25

Player B			
32	17	26	39
24	38	25	11
19	31	9	27
24	22	20	21
13	42	28	25

a. **USE STRUCTURE** Choose and sketch a visual representation of the data for each player and describe the shape of each distribution.

Sample answer: Using a histogram to represent the data shows that the distribution for each player is symmetric.





b. **INTERPRET PROBLEMS** Use the appropriate statistics to compare the center and spread of the data for each player. What do they tell you about the scoring tendencies of the two players?

Because the distributions are symmetric, use the mean and standard deviation. For Player A, $\mu = 24.6$, $\sigma = 6.5$. For Player B, $\mu = 24.6$, $\sigma = 8.5$. Sample answer: Both players have the same mean, but Player A is more consistent. Player B seems to have some really high-scoring games a some really low-scoring games.

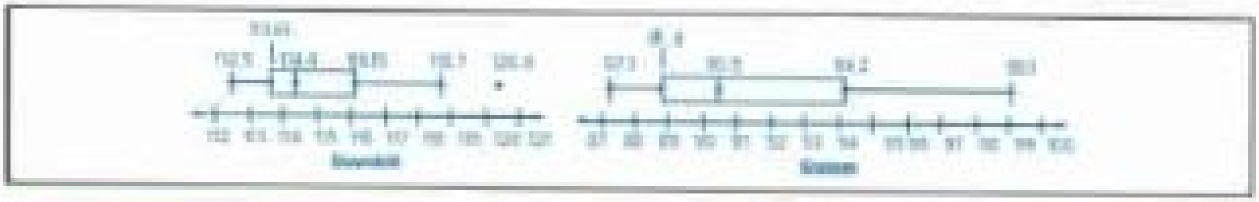
EXAMPLE Compare Distributions in Terms of Shape

The twelve fastest times for two skiing events—a downhill race and a slalom competition—are shown.

Downhill Times (	
114.6	118.7
115.4	113.7
114.0	113.4
113.6	112.5
116.9	114.2
115.1	120.4

Slalom Times (s)	
91.7	90.5
93.5	87.3
96.8	99.1
90.1	94.9
87.7	89.4
88.5	89.1

a. **USE STRUCTURE** Compare the data sets by constructing box-and-whisker plots and using appropriate statistics. Explain why you chose these statistics.



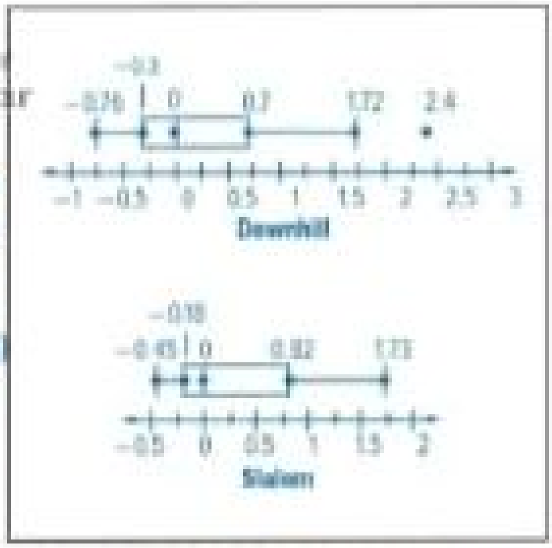
Sample answer: Use five-number summaries because the distributions are skewed. Downhill: min = 112.5, Q1 = 113.65, m = 114.4, Q3 = 116.15, max = 120.4, IQR = 2.5; Slalom: min = 87.3, Q1 = 88.8, m = 90.5, Q3 = 94.2, max = 99.1, IQR = 5.4. The distributions are generally similar, w data clustered toward the lower times, but the spread is greater for the slalom times.

b. DESCRIBE A METHOD How could you transform the two data sets so that you could compare the distributions purely in terms of shape, without differences in center and spread being a factor?

Sample answer: Subtract the median to center each distribution on 0, and divide by the IQR to scale the distributions so both have the same measure of spread (1).

c. COMMUNICATE PRECISELY Using your method, transform both data sets and construct box-and-whisker plots of the transformed data. What would you add to your comparison in **part a**?

Sample answer: The box-and-whisker plots of the new data show that the fastest and slowest times (first and fourth quartiles) are more spread out relative to the middle times (2nd and 3rd quartiles) for the downhill times than for the slalom times.



PRACTICE

1. Saeed owns an electronics store. He is revising his pricing for phone accessories. His current prices for an assortment of accessories is listed at the right. He has also determined that the mean price for the same assortment of accessories at a rival store is AED 10.99.

Saeed's Price Data (AED		
14.99	4.49	9.99
18.49	12.99	6.99
8.49	21.99	13.49
13.99	9.99	10.99
12.49	4.49	12.99

a. REASON QUANTITATIVELY Saeed wants to match his rival's prices. Use the table below to list the new prices. Explain.

New Prices (AED)				
14.19	3.69	9.19	17.69	12.19
6.19	7.69	21.19	12.69	13.19
9.19	10.19	11.69	3.69	12.19

Sample answer: The mean of Saeed's prices is AED 11.79, which is AED 0.80 more than his rival's mean price. The new prices come from subtracting AED 0.80 from each price, which will reduce the mean price to be the same as his rival's.

b. CALCULATE ACCURATELY Compare the mean and standard deviation of the current prices to the new prices.

Current prices: $\mu = 11.79$, $\sigma = 4.60$

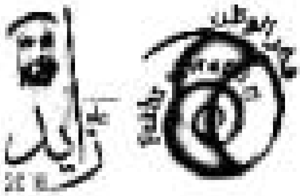
New prices: $\mu = 10.99$, $\sigma = 4.60$

The mean has dropped by 0.8, but the standard deviation has remained constant.

2. At the end of the year, Suha has saved AED 5000 from her wages. She wants to invest the money in an investment vehicle that might earn more than the interest she can make in the bank. She gets information regarding the performance of two investments over the last ten years. The annual growth rates by percent for each investment are shown below.

Investment A	
3.51	3.54
3.52	3.57
3.50	3.51
3.59	3.47
3.52	3.52

Investment B	
3.97	1.67
3.33	5.81
2.31	3.42
3.32	5.14
3.89	3.81



a. **INTERPRET PROBLEMS** Determine the shape of each distribution and use the appropriate statistics to find the center and spread for each set of data.

By creating a histogram, each distribution is symmetric. For Investment A, $\mu = 3.52$ and $\sigma = 0.03$, and for Investment B, $\mu = 3.69$ and $\sigma = 1.14$.

b. **REASON QUANTITATIVELY** What do the measures of center and spread tell you about the two different investment vehicles?

The average rate of return for Investment B is higher than that of Investment A, but it is much more inconsistent.

c. **REASON QUANTITATIVELY** Suha is planning to keep her investment in whichever vehicle she chooses for one year. What advice would you give her?

Sample answer: Although there is risk involved, I would advise Suha to invest in Investment B because there is a chance of a much higher return.

3. Sally is planning a two-week vacation to one of two islands and wants to base her decision on the weather history for the same dates as her vacation. She has collected the number of days that it has rained during this two-week period for each island over the past 10 years. The results are shown to the right.

Island A	
5	0
7	6
5	6
6	6
3	2

Island B	
4	4
6	5
3	7
4	3
5	4

a. **INTERPRET PROBLEMS** Determine the shape of each distribution and use the appropriate statistics to find the center and spread for each set of data.

The distribution for each set of data is skewed. For Island A, $m = 5.5$ and IQR $= 6 - 3 = 3$. For Island B, $m = 4$ and IQR $= 5 - 4 = 1$.

b. **REASON QUANTITATIVELY** What advice would you give Sally regarding which island to visit for her vacation? Explain your reasoning.

Sample answer: I would advise Sally to visit Island B because there is less risk with the numb of rainy days. With the spread of Island A being as large as it is, there is potential for the number of rainy days to be far more than desired for a vacation.

13.5 Scatter Plots and Lines of Fit

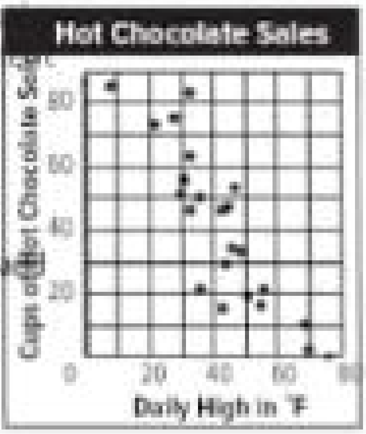
Objectives

- Interpret scatter plots.
- Draw a line of fit for a scatter plot.
- Use lines of fit to solve problems.

Data involving two variables is called **bivariate data**. A **scatter plot** shows relation between a set of bivariate data. If the data clusters in a linear pat the relationship is known as a **correlation**.

EXAMPLE Investigate Scatter Plots

EXPLORE A snack shop owner wondered if there was a relationship between the temperature and his sales of hot chocolate, frozen yogurt, a apples. He recorded data about the sales of each and the temperature. He organized the data using the scatterplots shown.

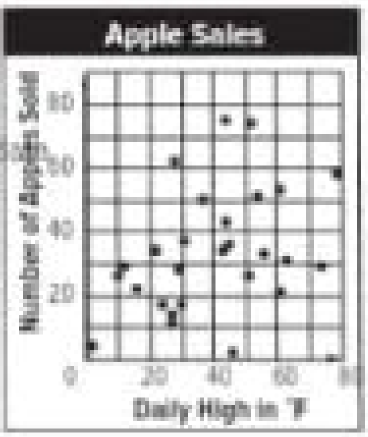


a. **FIND A PATTERN** What trend do you notice in each scatter plot?

Hot Chocolat As the temperature increases, the sales decrease.

Apples The dots do not form a pattern.

Frozen Yogur As the temperature increases, the sales increase.

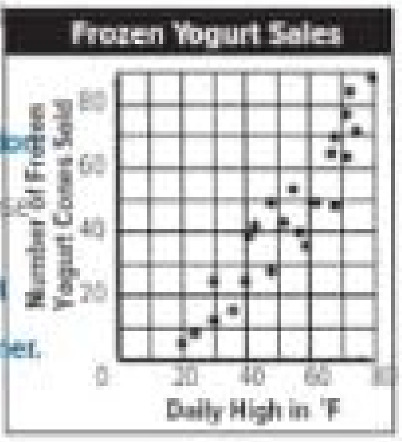


b. **INTERPRET PROBLEMS** When the dots appear to follow a line it, the da has a correlation. When one value increases as the other increases, the data has a **positive correlation**. When one value decreases as the other increases, the data has a **negative correlation**. Describe the type of correlation, if any, shown by each scatter plot.

Hot Chocolat negative correlation

Apple no correlation

Frozen Yogur positive correlation



c. **INTERPRET PROBLEMS** The more closely the data points cluster a a line, the stronger the correlation. Which data shows the strongest correlation?

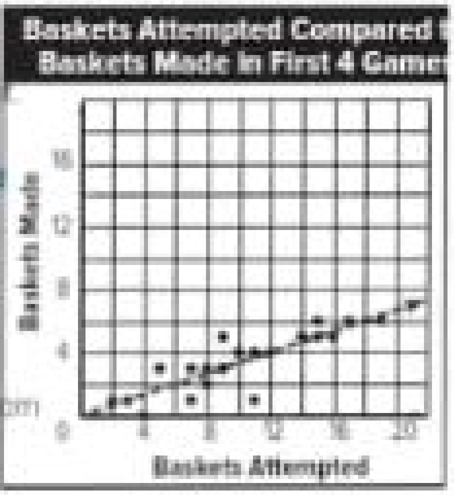
The data about frozen yogurt sales shows the strongest correlatio

d. **REASON QUANTITATIVELY** How can the owner of the snack shop u the scatter plots to run his business more efficiently?

Sample answer: It can help her order more efficiently—more hot chocolate mix in the winter and more frozen yogurt in the summer.

EXAMPLE Lines of Fit

Ghaya's basketball coach kept a record of the number of baskets each player attempted compared to the number made. She recorded the data in a scatter plot. How can she write an equation that models the relationship?



a. **DESCRIBE A METHOD** The coach drew a line that showed the trend of the data. The line is called a **trend line** or a **line of fit**. About half the points are above it and half the points are below i How can the coach write an equation of the line?

Sample answer: Choose two points on or near the line and use them to find the slope and the equation of the line.

b. **USE STRUCTURE** What are two points on the line of fit?

Sample answer: (3, 1) and (12, 4)

c. **USE STRUCTURE** Does a line of fit have to include any points from the collected data?

No. All points can be either above or below the line of fit.

d. **USE STRUCTURE** What is the slope of the line of fit?

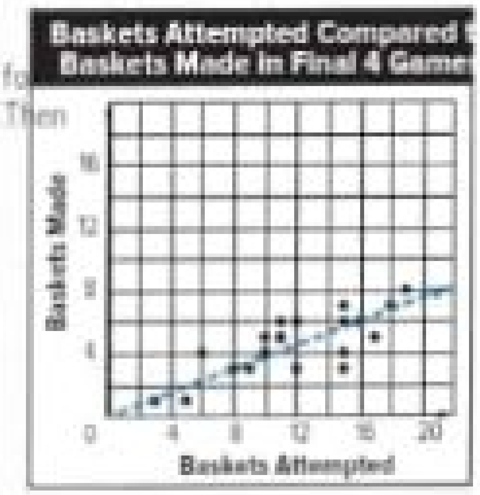
Sample answer: $\frac{1}{3}$

e. **USE STRUCTURE** What is the equation of the line of fit?

Sample answer: $y = \frac{1}{3}x$

f. **USE STRUCTURE** The coach collected and recorded the data fo the final 4 games of the season. Draw a line of fit for the data. Then name two points on the line.

Sample answer: (8, 3) and (16, 6)



g. **USE STRUCTURE** What is the slope of the line of fit?

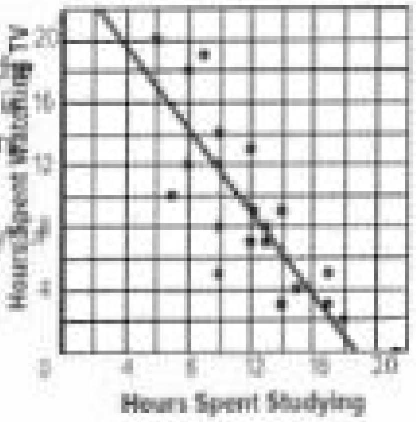
Sample answer: $\frac{2}{8}$

h. **USE STRUCTURE** What is the equation of the line of fit?

Sample answer: $y = \frac{3}{8}x$

EXAMPLE Make Predictions Using a Line of Fit

The scatter plot compares the hours students spend studying with the hours they spend watching television. The equation of the line of fit is $y = -\frac{4}{3}x + 25$ in which y = hours of television watched and x = hours spent studying.



a. **USE STRUCTURE** Is the correlation positive or negative? Explain your reasoning.

Negative; the slope of the line of fit is negative

b. **REASON QUANTITATIVELY** A student says she studies 15 hours a week. Explain how to use the line of fit to predict the number of hours she watches television. How many hours does she spend watching television?

Substitute 15 for x in the equation. $y \frac{4}{3}(15) + 25$; $y = 5$; 5 hours

c. **REASON QUANTITATIVELY** A student says he watches television 15 hours a week. Explain how to use the line of fit to predict the number hours he spends studying. How many hours does he spend studying?

Substitute 15 for y in the equation. $15 = \frac{4}{3}x + 25$; $x = 7.5$; 7.5 hours

d. **REASON ABSTRACTLY** Would you expect to find the data point that represents the number of hours that a particular student spent studying and watching TV on the line of fit? Why or why not?

Sample answer: Not necessarily. The line of fit describes trends in the data, not specific data points.

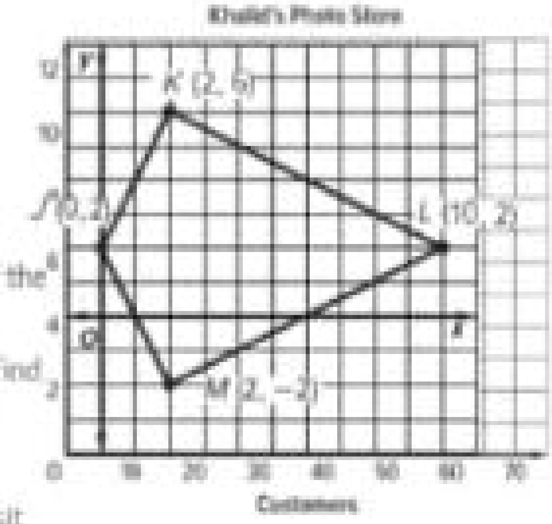
EXAMPLE 4 Solve a Real-World Problem

Khalid kept data on the number of people who visited his photography gallery and the number of sales he made. The data is shown in the table.

Customers	35	34	28	20	43	42	48	53	46	64	12	29	38	24
Sales	2	2	3	1	2	4	3	5	3	6	1	0	3	2

a. **USE A MODEL** Display the data on the scatter plot.

See student's work.



b. **INTERPRET PROBLEMS** Does the scatter plot show correlation? If so, what type of correlation is shown?

Yes, it shows a positive correlation.

c. **USE STRUCTURE** Draw a line of fit to show the trend of the data.

d. **USE STRUCTURE** Choose two points on the line. Then find the equation of the line in slope-intercept form.

Sample answer: (20, 1) and (40, 3); $y = 0.1x - 1$

e. **REASON QUANTITATIVELY** Suppose 50 customers visit Khalid's shop on Saturday. About how many sales can he expect to make? Justify your answer.

$y = 0.1(50) - 1$; $y = 4$ 4 sales

f. **REASON QUANTITATIVELY** Suppose Khalid wants to make 10 sales. About how many customers must visit his gallery? Justify your answer.

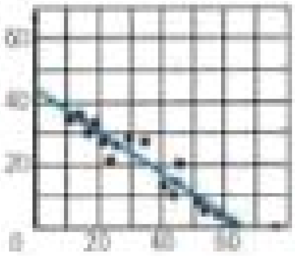
$10 = 0.1(x) - 1$; $x = 110$ 110 customers

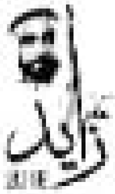
g. **REASON ABSTRACTLY** In parts e and *f* you were able to tell Khalid about how many customers or sales, not exactly. Why could you not provide exact information?

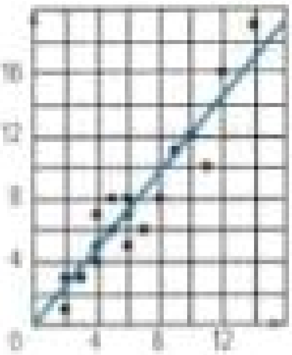
Lines of fit only provide information about trends.

PRACTICE

1. **USE STRUCTURE** Decide if the data in the scatter plot shows positive correlation, negative correlation, or no correlation. Then draw a line of fit that shows the trend of the data. Finally, write the equation of your line of fit.







Correlation **negative**

Equation of Line **Sample answer: $y = -\frac{2}{3}x + 44$**

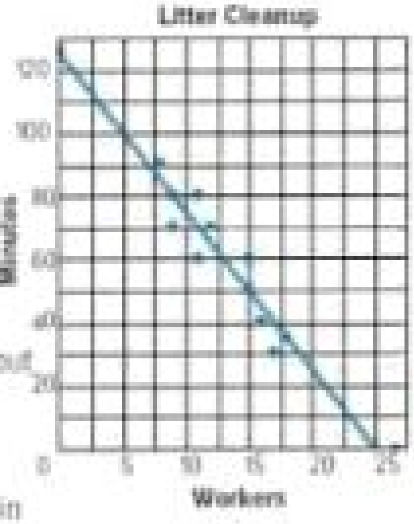
Correlation **positive**

Equation of Line **Sample answer: $y = \frac{4}{3}x$**

2. Several groups volunteered to each clean up litter along a mile of the highway near their town. The table shows how many people were in each group and how long it took each group to finish the job.

Workers	9	16	18	6	15	11	9	17	9	15	11	12
Minutes	80	40	35	90	60	60	70	30	70	50	60	70

a. **USE A MODEL** Display the data on the scatter plot.
See students' work.



b. **USE STRUCTURE** Draw a line of fit to show the trend of the data.

c. **USE STRUCTURE** Choose two points on the line. Then find the equation of the line in slope-intercept form.
Sample answer: (9, 70) and (15, 40); $y = -5x + 115$

d. **REASON QUANTITATIVELY** Another group has 6 workers. About how long can they expect to work? Justify your answer.
$y = -5(6) + 115$; $y = 85$; 85 minutes

e. **REASON QUANTITATIVELY** Another group wants to get done in 45 minutes. About how many workers should they have? Justify your answer.
$45 = -5(x) + 115$; $x = 14$; 14 workers

f. **INTERPRET PROBLEMS** Find the y-intercept of your equations and discuss the problem with its meaning in this situation.
Sample answer: (0, 115); this represents the fact that 0 workers will take 115 minutes to complete job. The problem is that 0 workers will never complete the job because no work is being done.

13.6 Regression and Median-Fit Lines

Objectives

- Find residuals
- Interpret correlation coefficients
- Plot and interpret residuals as a way of assessing a line of fit

EXAMPLE 1 Find Residuals

EXPLORE Abeer kept track of how long she spent reading her history book. She entered the data into her calculator and found $y = 1.3x + 5$ is an equation for the line of best fit.

Pages (x)	12	10	8	11	17	10	26	18	12
Minutes (y)	22	18	14	19	27	20	40	29	20
Expected y	$y = 1.3(12) + 5 = \mathbf{21}$	**18**	**15**	**19**	**27**	**18**	**39**	**28**	**21**
Residual	$22 - 21 = 1$	**0**	**−1**	**0**	**0**	**2**	**1**	**1**	**−1**

a. **USE STRUCTURE** The y-values shown in the table are the actual times it took her to read the given number of pages. The **expected y** is the estimated time given by substituting each number of pages (x) into the equation for the best fit line. Use the equation to find and record the expected time for each number of pages. Round to the nearest minute.

b. **USE STRUCTURE** The **residual** is the difference between the actual value and the expected value. Find and record the residuals for the data in the table.

c. **USE STRUCTURE** Abeer also kept track of how long she spent working math problems. The line of best fit for this data is $y = 1.4x + 2.5$. Use the equation to find the expected time for each number of problems rounded to the nearest minute. Then find each residual.

Problems (x)	27	24	15	16	25	18	16	24	28
Minutes (y)	38	33	33	19	49	21	24	43	34
Expected y	**40**	**36**	**24**	**25**	**38**	**28**	**25**	**36**	**31**
Residual	**−2**	**−3**	**9**	**−6**	**11**	**−7**	**−1**	**7**	**3**

d. **MAKE A CONJECTURE** Why do you think that in each table about half the non-zero residuals are negative and the other half are positive?

About half the data points are above the line of best fit and about half are below it.

e. **REASON QUANTITATIVELY** According to the information provided above, which takes Abeer longer: reading a page from her history book or completing a math problem? How much longer? How do you know?

The slope of the equations tell us that it takes 1.3 min/history page and 1.4 min/math problem. It takes $\frac{1}{10}$ min (6 sec) more for a math problem than a history page.

EXAMPLE Use Residuals to Assess the Fit of a Function

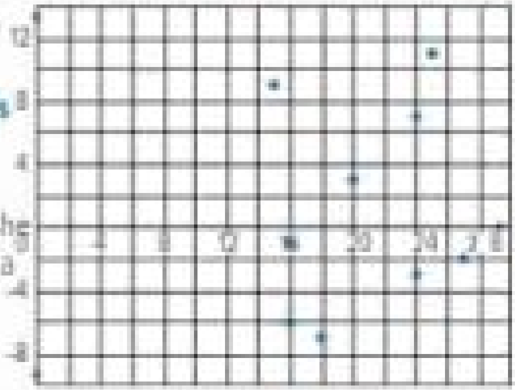
USE STRUCTURE To see if the linear function used to model the data is a good fit, it is possible to use the graph of the residuals. If the graph of (x, residual for the x) shows randomly placed points, the line of fit is accurate. If the graph shows a pattern, the function does not accurately reflect the data. Often it happens because the data is not linear in nature.

a. **USE STRUCTURE** Plot the residuals of Abeer's data about her math homework.

Pages (x)	27	24	15	16	25	18	16	24	20
Minutes (y)	38	33	33	19	49	21	24	43	34
Expected y	40	36	24	25	38	28	25	36	31
Residual	−2	−3	9	−6	11	−7	−1	7	3

b. **INTERPRET PROBLEMS** What does the shape of the data points tell about the line of fit?

The points are scattered randomly so the line of fit fits the data well.

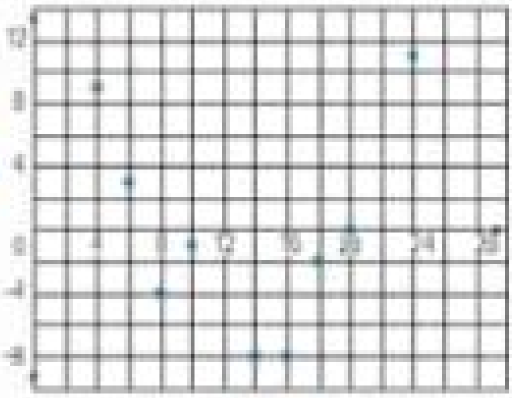


c. **CALCULATE ACCURATELY** Abeer kept track of how long she spent on science homework. The line of best fit for this data is $y = 0.19x + 23$. Complete the table. Then plot the residuals.

Pages (x)	4	6	8	10	14	18	20	24	16
Minutes (y)	33	27	21	24	18	24	27	39	18
Expected y	24	24	25	25	26	26	27	28	26
Residual	9	3	−4	−1	−8	−2	0	11	−8

d. **CONSTRUCT ARGUMENTS** Why does the plot show that Abeer's line of best fit is not accurate?

The points are not scattered randomly; instead they almost form a parabola. A pattern in the residual plot shows that the line of fit is incorrect.



e. **PLAN A SOLUTION** What should Abeer look for when reviewing her work?

Sample response: Abeer should make a scatter plot of the original data and make sure it shows a linear pattern. If it doesn't, she cannot use a linear equation to model it. If it is linear, she should graph the line of best fit and make sure it makes sense for the data. If it does not, she should ma sure she calculated the equation of the line correctly.

EXAMPLE 3 Correlation Coefficients

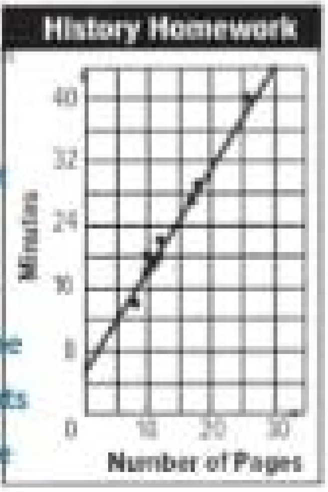
A correlation coefficient tells the strength of the relationship between the two sets of data. If there is no correlation, the correlation coefficient is zero. A perfect positive correlation has a coefficient of +1. The closer a positive correlation is to 1, the stronger the correlation is. A perfect negative correlation has a coefficient of −1. The closer a negative correlation is to −1, the stronger the correlation is.

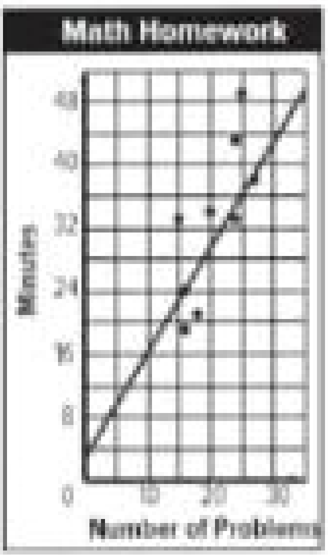
a. **INTERPRET PROBLEMS** Abeer used her calculator to find the correlation coefficients for her data. The correlation coefficient of the history homework data is +0.99. The correlation coefficient for the math homework is +0.75. What do the correlation coefficients mean in terms of the data?

The relationship between the number of pages and the time it takes to read them is almost exact. The relationship between the number of problems and the time it takes to do them is not quite as strong, but it is still a significant correlation.

b. **FIND A PATTERN** Look at the scatter plots of the data. Explain how the correlation coefficient for each set of data can be explained using the scatter plots.

Both have positive slope which shows that the correlation coefficient is positive. The data points on the 1st graph are clustered almost exactly on the line, which means the coefficient is very close to 1. The data points on the 2nd show a linear trend but are more scattered, so the correlation coefficient is much less than 1.

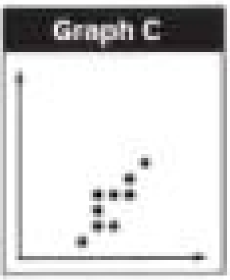


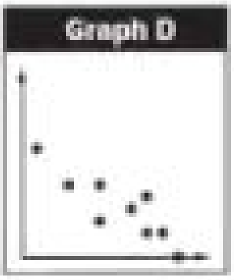


EXAMPLE 4 Interpret Correlation Coefficients

REASON QUANTITATIVELY Fill in the table to explain each correlation coefficient.

Correlation Coefficient	Positive or Negative?	Strong or Moderate?	Graph
−0.88	negative	strong	D
−0.48	negative	moderate	A
+0.59	positive	moderate	B
+0.83	positive	strong	C





PRACTICE

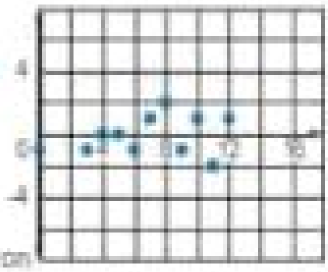
1. **USE STRUCTURE** For a science project Husam measured the effect of light on plant growth. At the end of 3 weeks, he recorded the height of each plant and how many hours of light it received each day.

   a. He used his calculator to find that the equation of a line of best fit was $y = 0.5x + 2.4$. Use the equation to complete the table. Round values to the nearest whole number.

Hours Sunlight (x)	0	3	6	10	4	9	7	8	12	13	5
Height in Inches (y)	1	3	4	8	4	6	7	8	9	6	5
Expected y	2	4	5	7	4	7	6	6	8	8	5
Residual	−1	−1	−1	1	0	−1	1	2	1	−2	0

   b. Plot the residuals. Then use the plot to assess the line of best fit.

   The points on the plot of the residuals are scattered and do not form a pattern. That shows that the line of fit is a good fit.



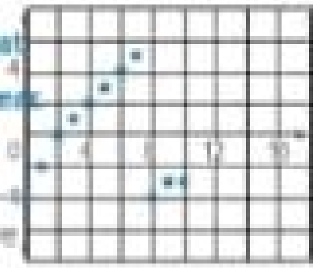
2. **USE STRUCTURE** For her project, Laila measured the effect of fertilizer on plant growth. At the end of 3 weeks, she recorded the height of each plant and how many drops of fertilizer it received each day.

   a. She used her calculator to find that the equation of a line of best fit was $y = -0.6x + 8.5$. Use the equation to complete the table. Round values to the nearest whole number.

Hours Sunlight (x)	0	3	6	10	4	9	7	8	2	1	5
Height in Inches (y)	5	8	9	0	8	0	9	0	7	6	9
Expected y	9	7	5	3	6	3	4	4	7	8	6
Residual	−4	1	4	−3	2	−3	5	−4	0	−2	3

   b. Plot the residuals. Then use the plot to assess the line of best fit.

   The points on the plot of the residuals form 2 lines. That shows that the line of fit is a good fit. The original data may not have been linear.



3. **REASON QUANTITATIVELY** A fertilizer company researched how effective each potential new product was on the yield of potatoes per acre. The correlation coefficient for each product is given. Explain what each means for a potato farmer's crop.

   a. +0.39 Moderate + correlation; Farmer will get more potatoes per acre with the product

   b. −0.1 Weak − correlation; Farmer will get slightly fewer potatoes per acre with the product

   c. +0.80 Strong + correlation; Farmer will get many more potatoes per acre with the product

Performance Task

Choosing a Diver

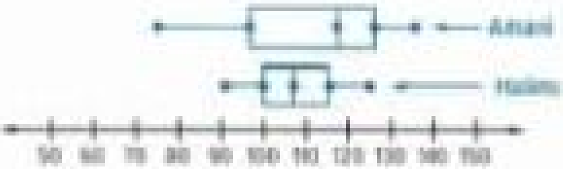
Provide a clear solution to the problem. Be sure to show all of your work, include all relevant drawings, and justify your answers.

Rana is the coach of a diving team. She has been asked to choose one of the members of the team to compete at a state diving meet. Coach Rana is considering two of the team's divers. To help her make the decision she collects the divers' scores from their last 20 diving meets. The scores are shown in the table.

Diver	Scores
Amani	120.1, 135.2, 75.6, 102.3, 120.6, 95.2, 99.3, 125.5, 138.1, 115.3, 124.6, 136.0, 102.3, 126.0, 104.7, 90.3, 130.6, 126.9, 92.4, 80.3
Halima	97.5, 115.3, 104.6, 103.9, 108.3, 97.5, 117.3, 92.6, 125.3, 106.5, 108.3, 103.3, 121.1, 124.0, 91.3, 98.6, 116.3, 111.5, 103.3, 109.3

Part A

Make box plots for the data. For each data set, describe the shape of the distribution and explain what the plots tell you about each diver.



Performance Task

Handedness and Sports

Provide a clear solution to the problem. Be sure to show all of your work, include all relevant drawings, and justify your answers.

A one-on-one sport is a sport in which two people play directly against each other. Some examples of one-on-one sports are shown. Is there a connection between a person's handedness (right-handed or left-handed) and whether or not they play a one-on-one sport? Survey students in your school district. Use the data you have collected to investigate this question as follows.

Examples of One-on-One Sports

Badminton
Fencing
Judo
Ping Pong
Squash
Tennis
Wrestling

Part A

Summarize the data you collected by making a two-way frequency table that shows your results. Describe the categories and subcategories for your table.

Sample answer:

Handedness	Plays	Does Not Play	Totals
Right-Handed	25	175	200
Left-Handed	8	12	20
Totals	33	187	220

Part B

Make a relative frequency two-way table for your data. What are the joint and marginal relative frequencies? What do they represent?

Sample answer:

Handedness	Plays	Does Not Play	Totals
Right-Handed	11.4%	79.5%	90.9%
Left-Handed	3.6%	5.5%	9.1%
Totals	15%	85%	100%

Part C

Based on your data, given that a person is left-handed, what is the probability that he or she plays a one-on-one sport? Given that a person plays a one-on-one sport, what is the probability that he or she is left-handed? Justify your answers.

Part D

It has been suggested that there is an association between being left-handed and playing a one-on-one sport. Use your data to support or refute this claim. Justify your response.

Standardized Test Practice

1. Two data sets are normally distributed, and they have the same mean. Which of the following statements are true?

[✓ The distribution of both data sets is symmetric about the mean.]

The standard deviations of the two data sets are equal.

[✓ The medians of the two data sets are equal.]

The IQR is equal to half the range for both data sets.

2. The table below shows the final score for a basketball team in the first ten games of the season.

43	51	39	48	60
64	56	47	52	48

Complete the following.

Mean: 50.8

Median: 49.5

Mode: 48

Range: 25

IQR: 9

3. Forty six students took a science test. The scores were normally distributed with a mean of 52 and a standard deviation of 7. Which of the following statements are true?

About 95% of the students had a score of between 45 and 59.

[✓ About 23 students had a score that is above 52.]

[✓ The probability that a student randomly selected from students who took the test scored lower than 31 is less than 1%.]

The data is bimodal, with modes of about 45 and 59.

4. Ten students were asked how many minutes it took them to get to school. The responses from nine of the students were 15, 11, 8, 5, 6, 17, 23, 11, and 20 minutes. If the median of the data was 12 minutes, how many minutes did it take the tenth student to get to school?

13 minutes

If the mean of the data was 13 minutes, how many minutes did it take the tenth student to get to school?

14 minutes

If the range of the data was 18 minutes, which of the following could not have been the time it took the tenth student to get to school?

15 minutes

23 minutes

20 minutes

[✓ 25 minutes]

5. The scores for a national standardized test have a mean of 75 with a standard deviation of 9.

Determine the z-score for each of the given test scores. Round your answer to the nearest hundredth.

Test Score	z-score
84	1.00
70	−0.56
90	1.67
80	0.56

The test score that corresponds to a z-score of −3.00 is 48. The test score that corresponds to a z-score of 2.22 is 95.

6. A researcher wants to estimate the average amount of time Emirate of Dubai teenagers play video game He surveys a group of 2300 teens from all over the country and finds that the average weekly time spent playing video games is 2.3 hours. Consider each value listed in the table below. For each value, identify whether it is a parameter or a statistic.

Value	Parameter	Statistic
The average weekly time spent playing video games for the 2300 teens surveyed is 2.3 hours		✓
The average weekly time spent playing video games for all Emirate of Dubai teenagers	✓	

7. The students at a school were asked whether they participate in sports and music. The results show that 45 students participate in sports only, 38 participate in music only, 19 participate in sports and music, and 22 participate in neither sports nor music.

a. Create a two-way frequency table for the data.

	Participate in Sports	Do Not Participate in Sports	Totals
Participate in Music	19	38	57
Do Not Participate in Music	45	22	67
Totals	64	60	124

b. What percent of students surveyed participate in sports?

about 51.6%

c. Of the students who participate in music, what percent do not participate in sports?

about 66.7%

8. The back-to-back stem-and-leaf plot shows the ages of people who attended matinee and evening performances of a play.

Matinee		Evening
3556779	0	9
00349	1	6678999
38	2	0012233446678
25556779	3	0134
00111246	4	38
	5	
037	6	
9	7	

a. Which data set has a higher median? Explain how you can tell.

Matinee; the middle value for matinee data is in the 30s, but for the evening, it is in the 20s.

b. Compare the spreads of the two data sets. Give a possible explanation for why the data sets may hav different spreads.

The matinee data is more spread out than the evening data. The matinee might be attended parents or grandparents with children, giving the ages a bigger spread. The evening show is mostly attended by teens and young adults.

c. Does either data set appear to be normally distributed? Explain how you can tell.

The evening data appear to be normally distributed. The data have a bell-shape, with the middle in the mid-20s and about half above and half below.