2

UNITS AND DIMENSIONAL ANALYSIS

2.1 INTRODUCTION

This chapter is primarily concerned with units. The units used in the text are consistent with those adopted by the engineering profession in the United States. One usually refers to them as the English or engineering units. Since engineers are often concerned with units and conversion of units, both the English and SI system of units are used throughout the book. All the quantities and the physical and chemical properties are expressed using these two systems.

2.1.1 Units and Dimensional Consistency

Equations are generally dimensional and involve several terms. For the equality to hold, each term in the equation must have the same dimensions (i.e., the equation must be dimensionally homogeneous or consistent). This condition can be easily proved. Throughout the text, great care is exercised in maintaining the dimensional formulas of all terms and the dimensional consistency of each equation. The approach employed will often develop equations and terms in equations by first examining each in specific units (feet rather than length), primarily for the English system. Hopefully, this approach will aid the reader and will attach more physical significance to each term and equation.

Consider now the example of calculating the perimeter, P, of a rectangle with length, L, and height, H. Mathematically, this may be expressed as $P = 2L + 2H$.

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This is about as simple as a mathematical equation can be. However, it only applies when P, L, and H are expressed in the same units.

A conversion constant/factor is a term that is used to obtain units in a more convenient form. All conversion constants have magnitude and units in the term, but can also be shown to be equal to 1.0 (unity) with *no* units. An often used conversion constant is

$$12\ \text{inches/foot}$$

This term is obtained from the following defining equation:

$$12\ \text{in} = 1\ \text{ft}$$

If both sides of this equation are divided by 1 ft one obtains

$$12\ \text{in/ft} = 1.0$$

Note that this conversion constant, like all others, is also equal to unity without any units. Another defining equation is

$$1\ \text{lb}_\text{f} = 32.2\ \frac{\text{lb} \cdot \text{ft}}{\text{s}^2}$$

If this equation is divided by lb_f, one obtains

$$1.0 = 32.2 \frac{\text{lb} \cdot \text{ft}}{\text{lb}_\text{f} \cdot \text{s}^2}$$

This serves to define the conversion constant g_c. Other conversion constants are given in Table A.1 of the Appendix.

Illustrative Example 2.1 Convert the following:

1. 8.03 yr to seconds (s)
2. 150 mile/h to yard/h
3. 100.0 m/s^2 to ft/min^2
4. 0.03 g/cm^3 to lb/ft^3

Solution

1. The following conversion factors are needed:
   365 day/yr
   24 h/day
   60 min/h
   60 s/min

The following is obtained by arranging the conversion factors so that units cancel to leave only the desired units.

$$(8.03\,\text{yr})\left(\frac{365\,\text{day}}{\text{yr}}\right)\left(\frac{24\,\text{h}}{\text{day}}\right)\left(\frac{60\,\text{min}}{\text{h}}\right)\left(\frac{60\,\text{s}}{\text{min}}\right) = 2.53 \times 10^8\,\text{s}$$

2. In a similar fashion,

$$\left(\frac{150\,\text{mile}}{\text{h}}\right)\left(\frac{5280\,\text{ft}}{\text{mile}}\right)\left(\frac{\text{yd}}{3\,\text{ft}}\right) = 2.6 \times 10^5\,\text{yd/h}$$

3. $(100.0\,\text{m/s}^2)\left(\frac{100\,\text{cm}}{\text{m}}\right)\left(\frac{\text{ft}}{30.48\,\text{cm}}\right)\left(\frac{60\,\text{s}}{\text{min}}\right)^2 = 1.181 \times 10^6\,\text{ft/min}^2$

4. $(0.03\,\text{g/cm}^3)\left(\frac{\text{lb}}{454\,\text{g}}\right)\left(\frac{30.48\,\text{cm}}{\text{ft}}\right)^3 = 2.0\,\text{lb/ft}^3$

Terms in equations must also be constructed from a "magnitude" viewpoint. Differential terms cannot be equated with finite or integral terms. Care should also be exercised in solving differential equations. In order to solve differential equations to obtain a description of the pressure, temperature, composition, etc., of a system, it is necessary to specify boundary and/or initial conditions for the system. This information arises from a description of the problem or the physical situation. The number of boundary conditions (BC) that must be specified is the sum of the highest-order derivative for each independent differential term. A value of the solution on the boundary of the system is one type of boundary condition. The number of initial conditions (IC) that must be specified is the highest-order time derivative appearing in the differential equation. The value for the solution at time equal to zero constitutes an initial condition. For example, the equation

$$\frac{d^2 v_y}{dz^2} = 0 \tag{2.1}$$

requires 2 BCs (in terms of z). The equation

$$\frac{dT}{dt} = 0; \quad t = \text{time} \tag{2.2}$$

requires 1 IC. And finally, the equation

$$\frac{\partial c_A}{\partial t} = D\frac{\partial^2 c_A}{\partial y^2};\ D = \text{diffusivity} \tag{2.3}$$

requires 1 IC and 2 BCs (in terms of y).

2.2 DIMENSIONAL ANALYSIS

Problems are frequently encountered in fluid flow and other engineering work that involve several variables. Engineers are generally interested in developing functional relationships (equations) between these variables. When these variables can be grouped together in such a manner that they can be used to predict the performance of similar pieces of equipment, independent of the scale or size of the operations, something very valuable has been accomplished.

Consider, for example, the problem of establishing a method of calculating the power requirements for mixing liquids in open tanks. The obvious variables would be the depth of liquid in the tank, the density and viscosity of the liquid, the speed of the agitator, the geometry of the agitator, and the diameter of the tank. There are therefore six variables that affect the power, or a total of seven terms that must be considered. To generate a general equation to describe power variation with these variables, a series of tanks having different diameters would have to be set up in order to gather data for various values of each variable. Assuming that ten different values of each of six variables were imposed on the process, 10^6 runs would be required. Obviously, a mathematical method for handling several variables that requires considerably less than one million runs to establish a *design method* must be available. In fact, such a method is available and it is defined as *dimensional analysis.*(1)

Dimensional analysis is a powerful tool that is employed in planning experiments, presenting data compactly, and making practical predictions from models without detailed mathematical analysis. The first step in an analysis of this nature is to write down the units of each variable. The end result of a dimensional analysis is a list of pertinent dimensionless numbers. A partial list of common dimensionless numbers used in fluid flow analyses is given in Table 2.1.

Dimensional analysis is a relatively "compact" technique for reducing the number and the complexity of the variables affecting a given phenomenon, process or calculation. It can help obtain not only the most out of experimental data but also scale-up data from a model to a prototype. To do this, one must achieve similarity between the prototype and the model. This similarity may be achieved through dimensional analysis by determining the important dimensionless numbers, and then designing the model and prototype such that the important dimensionless numbers are the same in both.

There are three steps in dimensional analysis. These are:

1. List all parameters and their primary units.
2. Formulate dimensionless numbers (or ratios).
3. Develop the relation between the dimensionless numbers experimentally.

Further details on this approach are provided in the next section.

Table 2.1 Dimensionless numbers

Parameter	Definition	Importance	Qualitative Ratio
Cavitation number	$Ca = \frac{P - p'}{\rho v^2/2}$	Cavitation	Pressure / Inertia
Eckert number	$Ec = \frac{v^2}{C_p \Delta T}$	Dissipation	Kinetic energy / Inertia
Euler number	$Eu = \frac{\Delta P}{\rho v^2/2}$	Pressure drop	Pressure / Inertia
Froude number	$Fr = \frac{v^2}{gL}$	Free surface flow	Inertia / Gravity
Mach number	$Ma = \frac{v}{c}$	Compressible flow	Flow speed / Sound speed
Poiseuille number	$Po = \frac{D^2 \Delta P}{\mu L v}$	Laminar flow in pipes	Pressure / Viscous forces
Relative roughness	$\frac{k}{D}$	Turbulent flow, rough walls	Wall roughness / Body length
Reynolds number	$Re = \frac{\rho v D}{\mu} = \frac{vD}{\nu}$	Various uses	Inertia forces / Viscous forces
Strouhal number	$St = \frac{\varpi L}{v}$	Oscillating flow	Oscillation speed / Mean speed
Weber number	$We = \frac{\rho v^2 L}{\sigma}$	Surface forces effect	Inertia / Surface tension

Note: p' = vapor pressure, C_p = heat capacity.

2.3 BUCKINGHAM Pi (π) THEOREM

This theorem provides a simple method to obtain dimensionless numbers (or ratios) termed π parameters. The steps employed in obtaining the dimensionless π parameters are given below(2):

1. List all parameters. Define the number of parameters as n.
2. Select a set of primary dimensions, e.g., kg, m, s, K (English units may also be employed). Let r = the number of primary dimensions.
3. List the units of all parameters in terms of the primary dimensions, e.g., L [=] m, where "[=]" means "has the units of." This is a critical step and often requires some creativity and ingenuity on the part of the individual performing the analysis.

4. Select a number of variables from the list of parameters (equal to r). These are called repeating variables. The selected repeating parameters must include all r independent primary dimensions. The remaining parameters are called "non-repeating" variables.
5. Set up dimensional equations by combing the repeating parameters with each of the other non-repeating parameters in turn to form the dimensionless parameters, π. There will be $(n-r)$ dimensionless groups of (πs).
6. Check that each resulting π group is in fact dimensionless.

Note that it is permissible to form a different π group from the product or division of other πs, e.g.,

$$\pi_5 = \frac{2\pi_1\pi_2}{\pi_3{}^2} \quad \text{or} \quad \pi_6 = \frac{1}{\pi_4} \tag{2.4}$$

Note, however, that a dimensional analysis approach will fail if the fundamental variables are not correctly chosen. The Buckingham Pi theorem approach to dimensionless numbers is given in the Illustrative Example that follows.

Illustrative Example 2.2 When a fluid flows through a horizontal circular pipe, it undergoes a pressure drop, $\Delta P = (P_2 - P_1)$. For a rough pipe, ΔP will be higher than a smooth pipe. The extent of non-smoothness of a material is expressed in terms of the roughness, k. For steady state incompressible Newtonian (see Chapter 5) fluid flow, the pressure drop is believed to be a function of the fluid average velocity v, viscosity μ, density ρ, pipe diameter D, length L, and roughness k (discussed in more detail in Chapter 14), and the speed of sound in fluid (an important variable if the flow is compressible) c, i.e.,

$$\Delta P = f(v,\ \mu,\ \rho,\ D,\ L,\ k,\ c)$$

Determine the dimensionless numbers of importance for this flow system.

Solution A pictorial representation of the system in question is provided in Fig. 2.1.

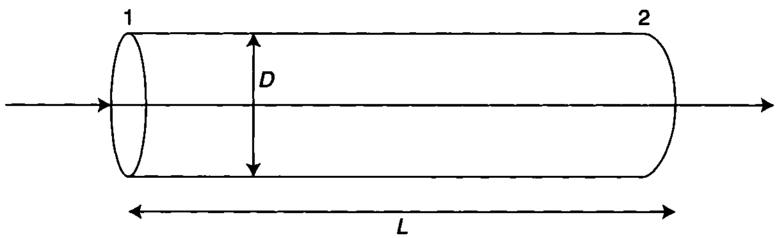

Figure 2.1 Pipe.

List all parameters and find the value of n:

$$\Delta P,\ v,\ \mu,\ \rho,\ D,\ L,\ k,\ c$$

Therefore $n = 8$.

Choose primary units (employ SI)

$$\text{m, s, kg, K}$$

List the primary units of each parameter:

$$\Delta P\,[=]\,\text{Pa} = \text{kg}\,\text{m}^{-1}\,\text{s}^{-2}$$
$$v\,[=]\,\text{m}\,\text{s}^{-1}$$
$$\mu\,[=]\,\text{kg}\,\text{m}^{-1}\,\text{s}^{-1}$$
$$D\,[=]\,\text{m}$$
$$L\,[=]\,\text{m}$$
$$\rho\,[=]\,\text{kg}\,\text{m}^{-3}$$
$$k\,[=]\,\text{m}$$
$$c\,[=]\,\text{m}\,\text{s}^{-1}$$

Therefore $r = 3$ with primary units m, s, kg.

Select three parameters from the list of eight parameters. These are the repeating variables:

$$D\,[=]\,\text{m}$$
$$\rho\,[=]\,\text{kg}\,\text{m}^{-3}$$
$$v\,[=]\,\text{m}\,\text{s}^{-1}$$

The non-repeating parameters are then ΔP, μ, k, c, and L.

Determine the number of πs:

$$n - r = 8 - 3 = 5$$

Formulate the first π, π_1, employing ΔP as the non-repeating parameter

$$\pi_1 = \Delta P v^a \rho^b D^f$$

Determine a, b, and f by comparing the units on both sides of the following equation:

$$0\,[=]\,(\text{kg}\,\text{m}^{-1}\,\text{s}^{-2})(\text{m}\,\text{s}^{-1})^a(\text{kg}\,\text{m}^{-3})^b(\text{m})^f$$

Compare kg:

$$0 = 1 + b. \text{ Therefore } b = -1$$

Compare s:

$$0 = -2 - a. \text{ Therefore } a = -2$$

Compare m:

$$0 = -1 + a - 3b + f. \text{ Therefore } f = 0$$

Substituting back into π_1 leads to:

$$\pi_1 = \Delta P v^{-2} \rho^{-1} = \frac{\Delta P}{\rho v^2}$$

This represents the Euler number (see Table 2.1). Formulate the second π, π_2 as

$$\pi_2 = \mu v^a \rho^b D^f$$

Determine a, b, and f by comparing the units on both sides:

$$0 \, [=] \, (\text{kg m}^{-1}\text{s}^{-1})(\text{m s}^{-1})^a(\text{kg m}^{-3})^b(\text{m})^f$$

Compare kg:

$$0 = 1 + b. \text{ Therefore } b = -1$$

Compare s:

$$0 = -1 - a. \text{ Therefore } a = -1$$

Compare m:

$$0 = -1 + a - 3b + f. \text{ Therefore } f = -1$$

Substituting back into π_2 yields:

$$\pi_2 = \mu v^{-1} \rho^{-1} D^{-1} = \frac{\mu}{v \rho D}$$

Replace π_2 by its reciprocal:

$$\pi_2 = \frac{v \rho D}{\mu} = \text{Re}$$

where Re = Reynolds number (see Chapter 12).

Similarly, the remaining non-repeating variables lead to

$$\pi_3 = k v^a \rho^b D^f \longrightarrow \frac{k}{D}$$

and

$$\pi_4 = cv^a\rho^bD^f \longrightarrow \frac{c}{v} \text{ take inverse}$$

$$\pi_4 = \frac{v}{c} = \text{the Mach number (see Chapter15)}$$

Similarly,

$$\pi_5 = \frac{L}{D}$$

Combine the πs into an equation, expressing π_1 as a function of π_2, π_3, π_4, and π_5:

$$Eu = \frac{\Delta P}{\rho v^2/2} = f\left(\text{Re}, \frac{k}{D}, Ma, \frac{L}{D}\right) = \text{the Euler number}$$

Consider the case of incompressible flow

$$Eu = \frac{\Delta P}{\rho v^2/2} = f\left(\text{Re}, \frac{k}{D}, \frac{L}{D}\right)$$

The result indicates that to achieve similarity between a model (m) and a prototype (p), one must have the following:

$$\text{Re}_\text{m} = \text{Re}_\text{p},$$

$$(k/D)_\text{m} = (k/D)_\text{p}, \text{ and}$$

$$(L/D)_\text{m} = (L/D)_\text{p}$$

Since Eu $= f(\text{Re}, k/D, L/D)$, then it follows that $\text{Eu}_\text{m} = \text{Eu}_\text{p}$ (see Table 2.1).

2.4 SCALE-UP AND SIMILARITY

To scale-up (or scale-down) a process, it is necessary to establish geometric and dynamic similarities between the model and the prototype. These two similarities are discussed below.

Geometric similarity implies using the same geometry of equipment. A circular pipe prototype should be modeled by a tube in the model. Geometric similarity establishes the scale of the model/prototype design. A 1/10th scale model means that the characteristic dimension of the model is 1/10th that of the prototype.

Dynamic similarity implies that the important dimensionless numbers must be the same in the model and the prototype. For a particle settling in a fluid, it has been shown (see Chapter 23) that the drag coefficient, C_D, is a function of the dimensionless Reynolds number, Re, i.e.:

$$C_D = f(\mathrm{Re}) \tag{2.5}$$

By selecting the operating conditions such that Re in the model equals the Re in the prototype, then the drag coefficient (or *friction factor*) in the prototype equals the friction factor in the model.

REFERENCES

1. I. Farag and J. Reynolds, "Fluid Flow", A Theodore Tutorial, East Williston, NY, 1995.
2. W. Badger and J. Banchero, "Introduction to Chemical Engineering", McGraw-Hill, New York, 1955.

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