Chapter 8

Unsteady-State Heat Conduction

INTRODUCTION

Engineers and scientists spend a good deal of time working with systems that are operating under steady-state conditions. However, there are many processes that are transient in nature, and it is necessary to be able to predict how process variables will change with time, as well as how these effects will impact the design and performance of these systems. The prediction of the unsteady-state temperature distribution in solids is an example of one such process. It can be accomplished very effectively using conduction equations; the energy balance equations can usually be easily solved to calculate the spatial and time variation of the temperature within the solid.

In a very real sense, this material is an extension of that presented in the previous chapter. The relationships developed in the preceding chapter applied only to the steady-state conditions in which the heat flow and spatial-temperature profile were constant with time. *Unsteady-state* processes are those in which the heat flow, the temperature, or both, vary with time at a fixed point in space. Batch heat-transfer processes are typical unsteady-state processes. For example, heating reactants in a tank or the startup of a cold furnace are two unsteady-state applications. Still other common examples include the rate at which heat is conducted through a solid while the temperature of the heat source varies, the daily periodic variations of the heat of the Sun on various solids, the quenching of steel in an oil or cold water bath, cleaning or regeneration processes, or, in general, any process that can be classified as intermittent.

A good number of heat transfer conduction problems are time dependent. These *unsteady*, or *transient*, situations usually arise when the boundary conditions of a system are changed. For example, if the surface temperature of a solid is changed, the temperature at each point in the solid will also change. For some cases, the changes will continue to occur until a *steady-state* temperature distribution is reached.

Heat Transfer Applications for the Practicing Engineer. Louis Theodore

Transient effects also occur in many industrial heating and cooling processes involving solids. The solids are generally designated in one of three physical categories:

1. Finite
2. Semi-infinite
3. Infinite

Each of the above three geometries receives treatment later in this chapter.

The remaining three sections of this chapter address a general all-purpose topic that classifies a host of application categories. The chapter concludes with a treatment of heat conduction from a microscopic viewpoint and is followed, in turn, by applications that are based on these microscopic equations.

The reader should note that the solution of heat conduction problems using Monte Carlo methods is treated in Chapter 26.

CLASSIFICATION OF UNSTEADY-STATE HEAT CONDUCTION PROCESSES

In order to treat common applications of batch and unsteady-state heat transfer, Kern[1] defined processes as either liquid (fluid) heating or cooling and solid heating or cooling. These real world examples are outlined below.

1. Heating and cooling liquids
   a. Liquid batches
   b. Batch reactors
   c. Batch distillation
2. Heating and cooling solids
   a. Constant solid temperature
   b. Periodically varying temperature
   c. Regenerators
   d. Granular solids in stationary beds
   e. Granular solids in fluidized beds

The *physical* representation of several solid systems is provided below.

1. Finite wall (or slab or plate)
2. Semi-infinite solid
3. Semi-infinite flat wall
4. Infinite flat wall
5. Finite rectangular parallelepiped
6. Finite hollow rectangular parallelepiped
7. Semi-infinite rectangular parallelepiped
8. Infinite rectangular parallelepiped

9. Short finite cylinder
10. Long finite cylinder
11. Short finite hollow cylinder
12. Long finite hollow cylinder
13. Semi-infinite cylinder
14. Semi-infinite hollow cylinder
15. Infinite cylinder
16. Infinite hollow cylinder
17. Sphere
18. Hollow sphere

The reader should also note that most of these geometric systems are employed to describe not only conduction systems but also forced convection (Chapter 9), free convection (Chapter 10), and radiation (Chapter 11) systems. Although a comprehensive treatment of all of the above is beyond the scope of this text, the reader is referred to the classic work of Carslaw and Jaeger(2) for a truly all-encompassing treatment of nearly all of these systems.

The above categories can be further classified to include specific applications involving unsteady-state heat conduction in *solids*. These include:

1. Walls of furnaces
2. Structural supports
3. Mixing elements
4. Cylindrical catalysts
5. Spherical catalysts
6. Fins (Chapter 17)
7. Extended surfaces (Chapter 17)
8. Insulating materials (Chapter 19)

Transient heat transfer in infinite plates, infinite cylinders, finite cylinders, spheres, bricks, and other composite shapes has been studied extensively in the literature. Farag and Reynolds(3) provide an excellent review that is supplemented with numerous worked illustrative examples; a semi-theoretical approach to describing the time-position variations in these systems has been simplified by use of a host of figures and tables.

MICROSCOPIC EQUATIONS

As noted in the Introduction to Part Two, this "Microscopic Equations" section has been included to complement the qualitative material presented earlier in this chapter. It should serve the needs of those readers interested in a more theoretical approach and treatment of conduction. This section also provides the results of the derivations

Table 8.1 Unsteady-State Energy-Transfer Equation for Stationary Solids

Rectangular coordinates:

$$\frac{\partial T}{\partial t} = \alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{A}{\rho c_p} \tag{1}$$

Cylindrical coordinates:

$$\frac{\partial T}{\partial t} = \alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{A}{\rho c_p} \tag{2}$$

Spherical coordinates:

$$\frac{\partial T}{\partial t} = \alpha\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right] + \frac{A}{\rho c_p} \tag{3}$$

and/or justifications for the describing equations employed in the "Applications" section to follow.

The microscopic equations describing transient heat conduction can be found in Table 8.1.[4,5] The equations are provided for rectangular, cylindrical, and spherical coordinates, and are valid subject to the assumptions specified for Table 7.4 in the previous chapter.

APPLICATIONS

This section contains six illustrative examples adopted from the earlier work of Theodore.[4] The last example is rather lengthy. One solution re-examines the separation of variables method. Another example introduces the reader to the error function.

ILLUSTRATIVE EXAMPLE 8.1

A constant rate of energy per unit volume (A) is *uniformly* liberated in a solid of arbitrary shape. The solid is insulated. Obtain the temperature of the solid as a function of position and time if the initial temperature of the solid is everywhere zero.

***SOLUTION*:** Based on the problem statement and physical grounds, it is concluded that the temperature of the solid is *not* a function of position. Therefore, the describing equation(s) are independent of the coordinate system. The equation(s) in Table 8.1 reduce to

$$\frac{\partial T}{\partial t} = \frac{A}{\rho c_p}$$

or

$$\frac{dT}{dt} = \frac{A}{\rho c_p}$$

since T is solely a function of t. Integrating this equation gives

$$T = \frac{A}{\rho c_P} t + B$$

The initial condition (IC) is

$$T = 0 \quad \text{at } t = 0$$

Therefore, $B = 0$ and

$$T = \left(\frac{A}{\rho c_p}\right) t$$

The temperature of the solid will increase linearly with time. ■

ILLUSTRATIVE EXAMPLE 8.2

The temperature of an infinite horizontal slab of uniform width h is everywhere zero. The temperature at the top of the slab is then set and maintained at T_h, while the bottom surface is maintained at zero. Determine the temperature profile in the slab as a function of position and time.

***SOLUTION*:** The "initial" one-dimensional profile of this system is described in Figure 8.1(a)–(b). The steady-state solution takes the form (see Chapter 7). See also Fig. 8.1(d).

$$T = T_h\left(\frac{z}{h}\right)$$

During the transient period (see Figure 8.1(c)), T is a function of t as well as z, that is,

$$T = T(z, t)$$

This problem is solved in rectangular coordinates. Now proceed to "extract" the following equation from Table 8.2:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} \tag{1}$$

Note that the partial derivative signs may *not* be replaced by ordinary derivatives since T is a function of two variables, z and t. The separation-of-variables method is again applied. Assume that the solution for T may be separated into the product of one function $\psi(z)$ that depends solely on z, and by a second function $\theta(t)$ that depends only on t,

$$\begin{aligned} T &= T(z, t) \\ &= \psi(z)\theta(t) \\ &= \psi\theta \end{aligned} \tag{2}$$

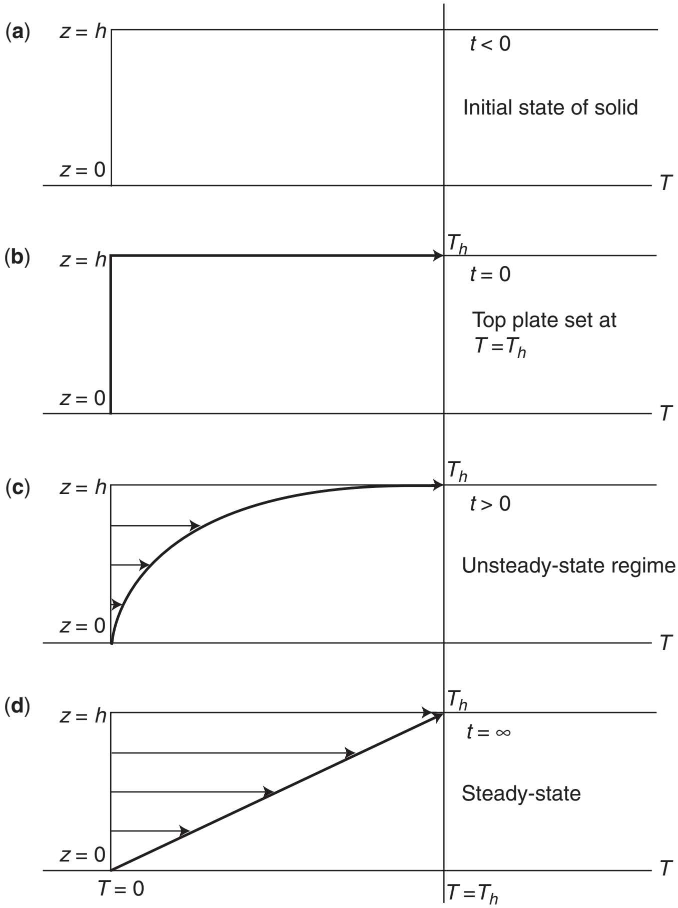

Figure 8.1 Temperature profile; Illustrative Example 8.2.

The left-hand side of Equation (1) above becomes

$$\begin{aligned} \frac{\partial T}{\partial t} &= \frac{\partial}{\partial t}(\psi\theta) \\ &= \psi\theta' \end{aligned} \tag{3}$$

The right-hand side of Equation (1) is given by

$$\alpha \frac{\partial^2 T}{\partial z^2} = \alpha\theta\psi'' \tag{4}$$

Combining Equations (3) and (4)

$$\frac{\theta'}{\alpha\theta} = \frac{\psi''}{\psi} = -\lambda^2 \tag{5}$$

where $-\lambda^2$ is a constant. The BC and/or IC are now written:

$$\text{BC(1): at } z = 0,\ T = 0 \ \text{ for } t \geq 0$$

$$\text{BC(2): at } z = h,\ T = T_h \ \text{ for } t > 0$$

$$\text{IC: at } t = 0,\ T = 0 \ \text{ for } 0 \leq z \leq h$$

The following solution results if $-\lambda^2$ is zero:

$$T_0 = C_1 + C_2 z \tag{6}$$

Substituting BC(1) and BC(2) into Equation (5) gives

$$T_0 = T_h\left(\frac{z}{h}\right) \tag{7}$$

Note that Equation (7) is the steady-state solution to the partial differential equation (PDE). If the constant is nonzero, one obtains

$$T_{-\lambda^2} = e^{-\alpha\lambda^2 t}(a_\lambda \sin \lambda z + b_\lambda \cos \lambda z) \tag{8}$$

Equations (7) and (8) are both solutions to Equation (1). Since Equation (1) is a linear PDE, the sum of Equations (7) and (8) also is a solution, that is,

$$\begin{aligned} T &= T_0 + T_{-\lambda^2} \\ &= T_h\left(\frac{z}{h}\right) + e^{-\alpha\lambda^2 t}(a_\lambda \sin \lambda z + b_\lambda \cos \lambda z) \end{aligned} \tag{9}$$

Resubstitution of the BC gives

$$T = T_h\left(\frac{z}{h}\right) + a_n e^{-\alpha(n\pi/h)^2 t} \sin\left(\frac{n\pi z}{h}\right) \tag{10}$$

The constant a_n is evaluated using the IC.

$$a_n = \frac{2T_h}{n\pi}(-1)^n$$

Combining the above gives

$$T = T_h\left(\frac{z}{h}\right) + \frac{2T_h}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} e^{-\alpha(n\pi/h)^2 t} \sin\left(\frac{n\pi z}{h}\right) \tag{11}$$

↓ ↓ ↓ steady-state solution ↓ ↓ ↓ ↓ ↓ ↓ ↓ transient solution ■

ILLUSTRATIVE EXAMPLE 8.3

Another system, somewhat similar but slightly different from the previous application is examined in this illustrative example.

Consider the insulated cylindrical copper rod pictured in Figure 8.2. If the rod is initially ($t = 0$) at T_A and the ends of the rod are maintained at T_S at $t \geq 0$, provide an equation that describes the temperature (profile) in the rod as a function of both position and time.

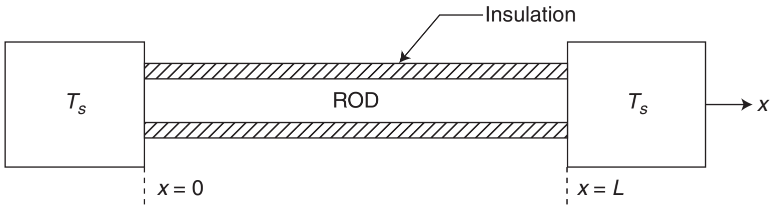

Figure 8.2 Transient rod temperature system; Illustrative Example 8.3.

SOLUTION: Although the rod is of cylindrical form, the geometry of the system is best described in rectangular coordinates. The describing equation once again takes the form

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

However, the BC and/or IC are different and given by

$$\begin{aligned} T &= T_A \quad \text{at } t = 0 \quad \text{(IC)} \\ T &= T_S \quad \text{at } x = 0 \quad \text{(BC)} \\ T &= T_S \quad \text{at } x = L \quad \text{(BC)} \end{aligned}$$

The solution to this equation can again be obtained via the separation of variables technique. The solution is given by[4]

$$T = T_S + (T_A - T_S)\left[\sum_{n=1}^{\infty} 2\left[\frac{(-1)^{n+1} + 1}{n\pi}\right] e^{-\alpha(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right)\right] \tag{2}$$

Additional details are provided in Chapter 26. ■

ILLUSTRATIVE EXAMPLE 8.4

Refer to Illustrative Example 8.3. If T_S represents saturated steam at 15 psig, the initial temperature T_A is 71°F, and the rod is 20 inches in length of stainless steel with $k = 9.1$ Btu/h · ft · °F, $\rho = 0.29$ lb/in^3, and $c_p = 0.12$ Btu/lb · °F, calculate the temperature 0.875 inches from one of the ends after 30 minutes. Note: This unit was designed by the author and is located in the Unit Operations Laboratory of Manhattan College.

SOLUTION: First calculate α

$$\alpha = \frac{9.1}{(0.29)(17.28)(0.12)} = 0.151 \approx 0.15$$

From the steam tables in the Appendix, at P = 15 + 14.7 = 29.7 psig, $T_S = 249.7$°F. Assuming only the first term (i.e., $n = 1$), in the infinite series in Equation (2) above contributes significantly to the solution, one obtains

$$\begin{aligned} T &= 249.7 + (71 - 249.7)\left[\frac{(2)(-1)^2 + 1}{(1)(\pi)}\right] e^{-0.15[(1)(\pi)/(20/12)]^2 0.5} \sin\left[\frac{(1)(\pi)(0.875)}{20}\right] \\ &= 232^\circ\text{F} \end{aligned}$$

The reader is left the exercise of determining the effect on the calculation by including more terms in the infinite series.

The finite difference method of solving the differential equation describing this system can be found in Chapter 26. As indicated earlier, Chapter 26 also details how the Monte Carlo method can be applied in the solution to some partial differential equations. ■

ILLUSTRATIVE EXAMPLE 8.5

Outline how to verify that Equation (2) in Illustrative Example 8.3 describes the system.

***SOLUTION*:** Equation (2) may be verified if it satisfies the boundary and initial conditions *and* if the equation can be differentiated in order to satisfy Equation (1). For example, for BC(1),

$$T = T_S \quad \text{at} \quad x = L$$

$$T = T_S + (T_A - T_S)\left[\sum_{n=1}^{\infty} \frac{4}{n\pi} e^{-\alpha(n\pi/L)^2 t} \sin\left(\frac{n\pi L}{L}\right)\right]$$

$$= T_S + (T_A - T_S)(0)$$

$$= T_S$$

For BC(2),

$$T = T_S \quad \text{at } x = 0$$

so that

$$T = T_S + (T - T_S)\left[\sum_{n=1}^{\infty} \frac{4}{n\pi} e^{-\alpha(n\pi/L)^2 t} \sin(0)\right]$$

$$= T_S + (T - T_S)(0)$$

$$= T_S$$

Differentiating Equation (2) to determine if it satisfies Equation (1) is left as an exercise for the reader. ■

ILLUSTRATIVE EXAMPLE 8.6

A plane membrane, impervious to the transfer of heat, separates an infinite solid into two equal parts. One half of the solid's temperature is initially at T_0 while the other half is at zero. At time $t = 0$, the membrane is removed and the solids brought into direct contact with each other. Calculate the temperature in the solid as a function of position and time.

***SOLUTION*:** The initial temperature profile in the system is shown in Figure 8.3. The PDE describing this system can easily be shown to be

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{1}$$

where $T = T(y, t)$.

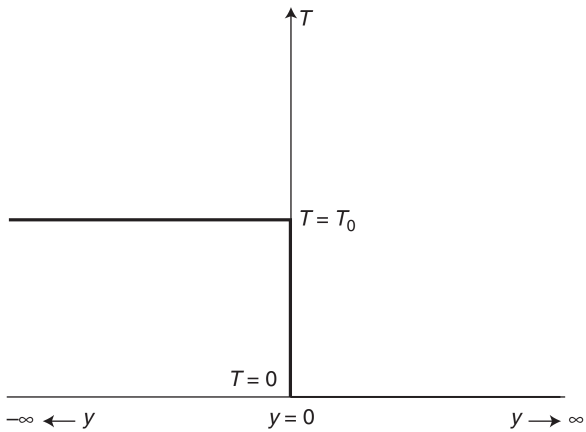

Figure 8.3 Infinite solid separated by a membrane: initial profile; Illustrative Example 8.6.

The BC and IC are

BC(1): $T = \text{finite}$ at $y = \infty$, for all t and approaching $\dfrac{T_0}{2}$ as $t \to \infty$

BC(2): $T = \text{finite}$ at $y = -\infty$, for all t and approaching $\dfrac{T_0}{2}$ as $t \to \infty$

IC: $T = T_0$ at $t = 0$, for $-\infty \leq y \leq 0$

$T = 0$ at $t = 0$, for $0 < y \leq +\infty$

The separation-of-variables method is usually applicable to finite-media systems. However, this technique often produces meaningless results when used for semi-infinite or infinite media. Equation (1) can be solved by using Laplace transforms or Fourier integrals. An outline of the Laplace transform method of solution is presented below.

Begin by multiplying both sides of Equation (1) by $e^{-pt}\,dt$ and integrating from 0 to ∞

$$\int_0^\infty e^{-pt}\frac{\partial T}{\partial t}\,dt = \int_0^\infty \alpha\frac{\partial^2 T}{\partial y^2}e^{-pt}\,dt \tag{2}$$

The left-hand side (LHS) of Equation (2) is integrated by parts to give

$$\begin{aligned} \text{LHS} &= e^{-pt}T|_0^\infty + p\int_0^\infty e^{-pt}T\,dt \\ &= -T(0) + p\overline{T};\ T(0) = \text{initial temperature} \end{aligned} \tag{3}$$

where the integral

$$\int_0^\infty e^{-pt}\,T\,dt \tag{4}$$

is represented by $\overline{T}$ and defined as the Laplace transform of T. $\overline{T}$ is a function of p and y but not of t. The right-hand side (RHS) of Equation (2) becomes

$$\text{RHS} = \alpha \frac{\partial^2}{\partial y^2} \int_0^\infty T e^{-pt}\, dt \tag{5}$$

$$= \alpha \frac{\partial^2 \overline{T}}{\partial y^2} = \alpha \frac{d^2 \overline{T}}{dy^2} \tag{6}$$

The resulting equation is

$$-T(0) + p\overline{T} = \alpha \frac{d^2 \overline{T}}{dy^2} \tag{7}$$

Note that this operation has converted the PDE(1) to an ODE(7) and eliminated time as a variable. It is an order of magnitude easier to integrate. This ordinary DE can now be solved subject to a revised BC. The remainder of this solution is not presented. Instead, the following simpler method of solution is discussed. First note that Laplace's source solution satisfies[2,4] the above PDE(1), that is,

$$T(y, t) = \frac{1}{\sqrt{t}} e^{-(y^2/4\alpha t)} \tag{8}$$

One can easily verify that this is a solution to the equation, since

$$\frac{\partial T}{\partial t} = -\frac{1}{2t^{3/2}} e^{-\xi^2} + \frac{1}{t^{1/2}} e^{-\xi^2} \left(\frac{y^2}{4\alpha t^2} \right) \tag{9}$$

$$\frac{\partial T}{\partial y} = -\frac{1}{t^{1/2}} e^{-\xi^2} \left(\frac{y}{2\alpha t} \right) \tag{10}$$

$$\frac{\partial^2 T}{\partial y^2} = -\frac{1}{2\alpha t^{3/2}} e^{-\xi^2} + \frac{y^2}{4\alpha t^{5/2}} e^{-\xi^2} \tag{11}$$

where

$$\xi^2 = \frac{y^2}{4\alpha t}$$

The solution

$$T(y, t) = \frac{B}{\sqrt{4\pi\alpha t}} e^{-(y^2/4\alpha t)} \tag{12}$$

where

$$B = \text{constant} = \frac{Q}{\rho c_p A} \tag{13}$$

also satisfies Equation (1). Q/A is a source of heat (Q per unit area A) located at the point $y = 0$ (origin) at $t = 0$. If the location of the source is changed from point $y = 0$ to $y = y'$, the solution becomes

$$T(y, t) = \frac{Q}{\rho c_p A\sqrt{4\pi\alpha t}} e^{-[(y-y')^2/4\alpha t]} \tag{14}$$

The quantity of heat located at $y = y'$ must be differential, dQ, since it is located at a point, the dimensions of which are differential. If the rest of the system is initially at zero temperature, this heat will produce a differential temperature at any other y and t, that is,

$$dT(y, t) = \frac{dQ}{\rho c_p A\sqrt{4\pi\alpha t}} e^{-[(y-y')^2/4\alpha t]} \tag{15}$$

The volume element at y' is $A\,dy'$. The amount of heat present in the volume element at y' is initially

$$dQ = T(y')A\rho c_P\, dy' \tag{16}$$

where $T(y')$ is the initial temperature at y'. Therefore,

$$dT(y, t) = \frac{T(y')}{\sqrt{4\pi\alpha t}} e^{-[(y-y')^2/4\alpha t]}\, dy' \tag{17}$$

This is the equation describing the temperature at any y and t for a source initially located at y'. Therefore, to calculate $T(y, t)$ for all sources between $-\infty$ and $+\infty$, sum the above equation (integrate) over all values of y', that is,

$$T(y, t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi\alpha t}} e^{-[(y-y')^2/4\alpha t]} T(y')\, dy' \tag{18}$$

Note that y and t are constants and y' varies in the integration of Equation (18). The reader can verify that Equation (18) is also a solution to Equation (1). Equation (18) is a solution to systems with infinite media providing $T(y')$ is the *initial* temperature profile in the system. Physically, the above solution represents the temperature at any t and any y due to the presence of an initial temperature at y'. The summation, or integral, of all the initial contributions gives rise to $T(y, t)$.

The initial temperature distribution in this problem is

$$T = T_0 \quad \text{for } -\infty \le y < 0$$
$$T = 0 \quad \text{for } 0 < y \le \infty$$

Therefore

$$T = \frac{1}{2\sqrt{\alpha\pi t}} \int_{y'=-\infty}^{y'=0} T_0 e^{-[(y-y')^2/4\alpha t]}\, dy' + \frac{1}{2\sqrt{\alpha\pi t}} \int_{y'=0}^{y'=\infty} (0) e^{-[(y-y')^2/4\alpha t]}\, dy' \tag{19}$$

$$= \frac{T_0}{2\sqrt{\alpha\pi t}} \int_{-\infty}^{0} e^{-[(y-y')^2/4\alpha t]}\, dy' \tag{20}$$

Introducing a new variable,

$$\xi = \frac{y - y'}{\sqrt{4\alpha t}} \tag{21}$$

so that

$$y' = y - \sqrt{4\alpha t}\,\xi \tag{22}$$

$$dy' = -\sqrt{4\alpha t}\,d\xi \tag{23}$$

and

$$\text{at } y' = 0, \quad \xi = \frac{y}{\sqrt{4\alpha t}}$$

$$\text{at } y' = -\infty, \quad \xi = +\infty$$

Therefore, (after reversing limits),

$$\begin{aligned} T &= -\frac{T_0}{\sqrt{4\pi\alpha t}} \int_{\xi=\infty}^{\xi=y/\sqrt{4\alpha t}} e^{-\xi^2} \sqrt{4\alpha t}\,d\xi \\ &= -\frac{T_0}{\sqrt{\pi}} \int_{\infty}^{y/\sqrt{4\alpha t}} e^{-\xi^2}\,d\xi \\ &= -\frac{T_0}{\sqrt{\pi}} \int_{y/\sqrt{4\alpha t}}^{\infty} e^{-\xi^2}\,d\xi \\ &= \frac{T_0}{\sqrt{\pi}} \left[\underbrace{\int_0^{\infty} e^{-\xi^2}\,d\xi}_{I(1)} - \underbrace{\int_0^{y/\sqrt{4\alpha t}} e^{-\xi^2}\,d\xi}_{I(2)} \right] \end{aligned} \tag{24}$$

For the first integral, $I(1)$, proceed as follows. Set

$$\xi^2 = x; \quad \xi = x^{1/2} \tag{25}$$

so that

$$2\xi\,d\xi = dx$$

$$d\xi = \frac{dx}{2\xi}$$

$$d\xi = \tfrac{1}{2}x^{-1/2}\,dx$$

and

$$I(1) = \frac{1}{2}\int_0^{\infty} e^{-x} x^{-1/2}\,dx \tag{26}$$

$$= \frac{\Gamma(\frac{1}{2})}{2} = \frac{\sqrt{\pi}}{2} \tag{27}$$

where Γ = gamma function.

For $I(2)$, first define the error function, erf,

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\xi^2}\, d\xi \tag{28}$$

Therefore,

$$\begin{aligned} T &= \frac{T_0}{2} - \frac{T_0}{2} \operatorname{erf} z \\ &= \frac{T_0}{2}\left[1 - \operatorname{erf}\left(\frac{y}{\sqrt{4\alpha t}}\right)\right] \end{aligned} \tag{29}$$

The error function is tabulated in most advanced mathematics texts.(6,7) A plot of erf z against z is presented in Figure 8.4. Note that this function has the properties

$$\operatorname{erf}(\infty) = 1.0$$
$$\operatorname{erf}(0) = 0$$
$$\operatorname{erf}(-\infty) = -1.0$$

and

$$\operatorname{erf}(-z) = -\operatorname{erf}(z)$$

It can also be evaluated numerically by expanding $e^{-\xi^2}$ in a power series,

$$e^{-\xi^2} = 1 - \xi^2 + \tfrac{1}{2}\xi^4 - \cdots \tag{30}$$

The RHS of this equation can then integrated term by term to give

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \tag{31}$$

This power series converges for all values of z. However, it converges very slowly for large z.

The reader can verify that Equation (29) is a solution to Equation (1). A check of the BC and/or IC yields

BC: at $t = \infty$ and any y, $T = \dfrac{T_0}{2}$

IC: at $t = 0$ and $y > 0$,

$$\begin{aligned} T &= \frac{T_0}{2}[1 - \operatorname{erf}(\infty)] \\ &= \frac{T_0}{2}[1 - 1] \\ &= 0 \end{aligned}$$

at $t = 0$ and $y < 0$,

$$\begin{aligned} T &= \frac{T_0}{2}[1 - \operatorname{erf}(-\infty)] \\ &= \frac{T_0}{2}[1 - (-1)] \\ &= T_0 \end{aligned}$$

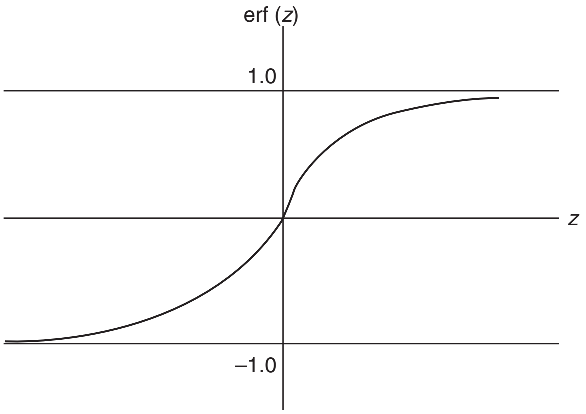

Figure 8.4 Error function.

The reader should note that the analysis above can be extended to include systems with *continuous* sources of heat. The equation describing the temperature due to an *instantaneous* source of heat at $t = 0$ and $y = 0$ was given by [see also Equation (15)].

$$dT(y, t) = \frac{dQ}{\rho c_p A\sqrt{4\pi\alpha t}} e^{-[(y^2)/4\alpha t]} \tag{32}$$

If the source is liberated at $t = t'$ rather than $t = 0$, the solution can be shown to be

$$dT(y, t) = \frac{dQ}{\rho c_p A\sqrt{4\pi\alpha(t - t')}} e^{-[(y^2)/4\alpha(t-t')]} \tag{33}$$

The quantity of heat liberated at the instant dt' is given by

$$dQ = \dot{Q}(t')dt' \tag{34}$$

where $\dot{Q}(t')$ is the rate of heat liberated at t'. The temperature arising due to a *continuous* source of heat from $t = 0$ to $t = t'$ is obtained by adding (integrating) the contributions from each instantaneous source, i.e.,

$$T(y, t) = \int_{t'=0}^{t'=t'} \frac{\dot{Q}(t')e^{-[(y^2)/4\alpha(t-t')]}}{\rho c_p A\sqrt{4\pi\alpha(t - t')}} dt' \tag{35}$$

The term y^2 is replaced by $(y - y')^2$ in the above solution if the source is located at y' rather than the origin. A more detailed presentation is provided by Carslaw and Jaeger(2).

Most of the material in the literature on this subject is concerned with semi-infinite media. The analysis presented above can be applied to these systems as well. ■

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