

Chapter 7

Steady-State Heat Conduction

INTRODUCTION

As the temperature of a solid increases, the molecules that make up the solid experience an increase in vibrational kinetic energy. Since every molecule is bonded in some way to neighboring molecules, usually by electrical force fields, this energy can be passed through the solid. Thus, the transfer of heat may be viewed as thermal energy in transit due to the presence of a temperature difference. Heating a wire at one end eventually results in raising the temperature at the other. This type of heat transfer is called conduction—the subject title of this chapter—and is the principle mechanism by which solids transfer heat.

Fluids are capable of transporting heat in a similar fashion. Conduction in a stagnant liquid, for example, occurs by the movement of not only vibrational kinetic energy but also translational kinetic energy as the molecules move throughout the body of the stationary liquid. It is also important to note that there is *no* bulk motion with conduction. The ability of a fluid to flow, mix, and form internal currents on a macroscopic level (as opposed to the molecular mixing just described) allows fluids to carry heat energy by convection as well, a topic that receives treatment in Chapters 9 and 10.

Finally, a microscopic analysis of heat conduction is presented in the last section with additional development material (based on the Monte Carlo method) in Chapter 26.

FOURIER'S LAW

The rate of heat flow by conduction is given by Fourier's law⁽¹⁾

$$\begin{aligned}\dot{Q} &= -kA \frac{dT}{dx} = kA \left(-\frac{dT}{dx} \right) \\ \dot{Q}' &= \frac{\dot{Q}}{A} = k \left(-\frac{dT}{dx} \right)\end{aligned}\tag{7.1}$$

Heat Transfer Applications for the Practicing Engineer. Louis Theodore
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where (in English units) \dot{Q} is the heat flow rate (Btu/h), x is the direction of heat flow (ft), k is the thermal conductivity (Btu/h · ft · °F), (see also Chapter 3), A is the heat transfer area perpendicular to the flow of heat (ft²), and T is the temperature (°F). The term $(-dT/dx)$ is defined as the temperature gradient and carries a negative sign since T is assumed to decrease with increasing x for \dot{Q} being transferred in the positive x direction. Fourier derived his law from experimental evidence and/or physical phenomena, not from basic principles (i.e., theory). The reader should also note that Equation (7.1) is the basis for the definition of thermal conductivity as given in Chapter 3 (i.e., the amount of heat (Btu) that flows in a unit of time (1 h) through a unit area of surface (1 ft²) of unit thickness (1 ft) by virtue of a difference in temperature (1°F)).

Refer to Figure 7.1. Once again, the negative sign in Equation (7.1) reflects the fact that heat flow is from a high to low temperature and therefore the sign of the derivative (dT/dx) is opposite to that of the heat flow. If the thermal conductivity (k) can be considered constant over a limited temperature range (ΔT), Equation (7.1) can be integrated to give

$$\dot{Q} = \frac{kA(T_H - T_C)}{(x_H - x_C)} = \frac{kA\Delta T}{\Delta x} = \frac{kA\Delta T}{L} \quad (7.2)$$

where T_H is the higher temperature at point x_H (°F), T_C is the lower temperature at point x_C (°F), ΔT is the difference between T_H and T_C , L is the distance between points x_H and x_C (ft) = Δx , and A (ft²) is the area across which the heat is flowing.

It should be noted that the thermal conductivity (k) is a property of the material through which the heat is passing and, as such, does vary somewhat with temperature.

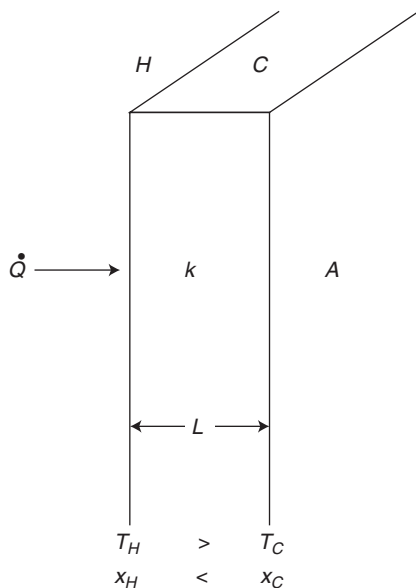


Figure 7.1 Heat flow through a solid wall.

Table 7.1 Thermal Conductivities of Three Common Insulating Materials

	k (Btu/h · ft · °F)
Asbestos–cement boards	0.430
Fiber, insulating board	0.028
Glass wool, 1.5 lb/ft ³	0.022

Equation (7.2) should therefore be strictly employed only for small values of ΔT with an average value of the conductivity used for k .

Values of k for three insulating solids are given in Table 7.1. (A separate chapter on insulation is provided in Part Three.) Approximate values of k for materials at different states are provided in Table 7.2 employing SI units.

As can be seen from Table 7.2, the thermal conductivities of liquids and of gases are very small in comparison with the thermal conductivities of most solids. For example, at 212°F the thermal conductivity (in English units) of silver is 240 Btu · ft/h · ft² · °F; that of a building brick is about 0.8, that of water is about 0.35, and of air is 0.017.

Thermal conductivities, as well as other properties, of various liquids and gases are listed in the Appendix (see Tables B.2–B.8). Properties of metals are provided in Chapter 6.

In general, the variation of thermal conductivity with temperature is small, but a fair approximation is to assume that this variation is linear, i.e.,

$$k = a + bT \quad (7.3)$$

where a and b are constants and T is the temperature. The reader is left the option of determining appropriate and consistent units for a and b . The effect of pressure is almost always negligible, particularly for gases approaching ideal gas behavior.

If a plane wall of thickness L has steady-state temperatures of T_H and T_C on the hot and cold faces, respectively, one can derive an expression for the steady-state one-dimensional heat transfer rate per unit area, \dot{Q}/A , in terms of T_H , T_C , a , b , and L . Substituting Equation (7.3) into Equation (7.1) gives

$$\dot{Q} = (a + bT)A \frac{dT}{dx} \quad (7.4)$$

Table 7.2 Thermal Conductivities for Materials of Different States

	K(W/m · °C)
Solid metals	15–400
Liquids	0.0–100
Gases	0.01–0.2

Note that 1 Btu/h · ft · °F = 1.7307 J/s · m · °C = 1.7307 W/m · °C.

or

$$\frac{\dot{Q}}{A} dx = (a + bT) dT \quad (7.5)$$

This equation may be integrated to give

$$\left(\frac{\dot{Q}}{A}\right)L = a(T_H - T_C) + \frac{b}{2}(T_H^2 - T_C^2) \quad (7.6)$$

Another thermal quantity useful in heat transfer calculations is the thermal diffusivity, α . It is defined as

$$\alpha = \frac{k}{\rho c_P} \quad (7.7)$$

where ρ is the material density and c_P is the heat capacity of the material. The term α has the units of m^2/s or ft^2/s , which is the same units as the kinematic viscosity, ν . When the thermal conductivity is constant and independent of position, the material is termed *homogeneous*. Materials with properties such as thermal conductivity that are independent of the direction of heat transfer are termed *isotropic* (i.e., in rectangular coordinates, $k_x = k_y = k_z = k$). (Refer to the last section of this chapter for additional details.)

CONDUCTIVITY RESISTANCES

Equation (7.2) may be written in the form of the general rate equation:

$$\text{rate} = \frac{(\text{driving force})(\text{area})}{\text{resistance}} = \frac{\Delta T}{L/kA} \quad (7.8)$$

Note that the area term in Equation (7.8) can be combined with the *resistance*, L/k , to form the term L/kA . Since Q in Equation (7.2) is the heat flow rate and ΔT the driving force, the L/kA term may be viewed as the resistance to heat flow. This approach is useful when heat is flowing by conduction through different materials in sequence and of varying heat transfer areas.

Consider, for example, a flat incinerator wall made up of three different layers: an inside insulating layer, a ; a steel plate, b ; and an outside insulating layer, c . Refer to Figure 7.2.

The total resistance to heat flow through the incinerator wall is the sum of the three individual resistances:

$$R = R_a + R_b + R_c; \quad R_i = \frac{L_i}{k_i A_i} \quad (7.9)$$

Note that it is assumed that the heat transferred across these resistances experiences negligible resistance at each interface, an assumption that may not always be true.

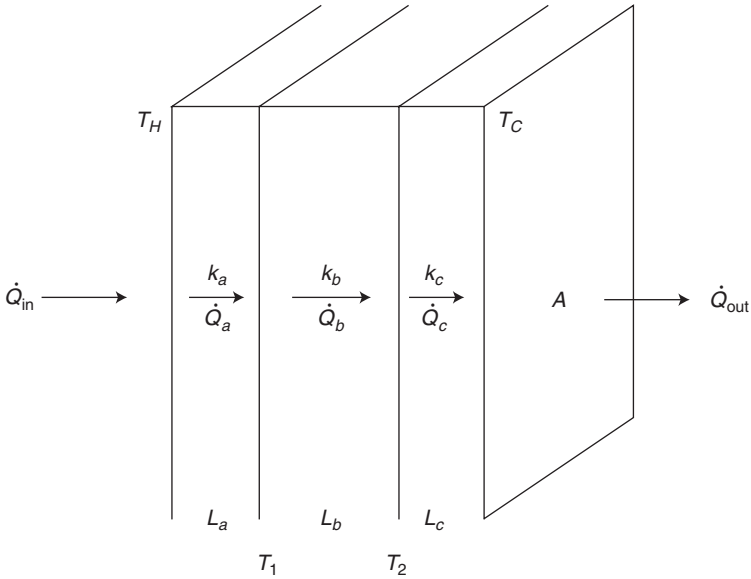


Figure 7.2 Heat transfer through an insulated incinerator wall.

For steady-state conditions

$$\dot{Q}_{in} = \dot{Q}_a = \dot{Q}_b = \dot{Q}_c = \dot{Q}_{out} = \dot{Q} \quad (7.10)$$

with

$$\dot{Q} = k_a \frac{A(T_H - T_1)}{L_a} = k_b \frac{A(T_1 - T_2)}{L_b} = k_c \frac{A(T_2 - T_C)}{L_c} \quad (7.11)$$

Since the overall temperature difference ($T_H - T_C$) is easily measured and usually known, one can solve for it by noting

$$(T_H - T_1) = \dot{Q} \frac{L_a}{k_a A} \quad (7.12)$$

$$(T_1 - T_2) = \dot{Q} \frac{L_b}{k_b A} \quad (7.13)$$

$$(T_2 - T_C) = \dot{Q} \frac{L_c}{k_c A} \quad (7.14)$$

Adding and solving yields

$$(T_H - T_C) = \dot{Q} \left(\sum_i \left(\frac{L_i}{k_i A} \right) \right) \quad (7.15)$$

or

$$\dot{Q} = (T_H - T_C) / \sum_i \left(\frac{L_i}{k_i A} \right) \quad (7.16)$$

Thus, at steady-state, the rate of heat flow through the wall is given by

$$\dot{Q} = \frac{T_H - T_C}{(L_a/k_a A_a) + (L_b/k_b A_b) + (L_c/k_c A_c)} \quad (7.17)$$

where for engineering units k_a, k_b, k_c are the thermal conductivities of each section (Btu/h · ft · °F), A_a, A_b, A_c are the areas of heat transfer of each section (ft²) (these are equal for a constant cross-section of heat conduction), L_a, L_b, L_c are the thickness of each layer (ft), T_H is the temperature at the inside surface of the insulating wall a (°F), and T_C is the temperature at the outside surface of the insulating wall c (°F).

In the above example, the heat is flowing through a slab of constant cross-section (i.e., all the areas available for heat transfer are equal). In many cases of industrial importance, however, this is not the case. For example, in heat flow through the walls of a cylindrical vessel (see Figure 7.3), such as a pipe or tube, the heat transfer area increases with radial distance from the center of the cylinder. The heat flow in this case is given by

$$\dot{Q} = \frac{k A_{\text{lm}} \Delta T}{L} \quad (7.18)$$

The term A_{lm} in this equation represents an average heat transfer area, or more accurately, the log-mean average heat transfer area. This log-mean average is calculated by

$$A_{\text{av}} = A_{\text{lm}} = \frac{A_2 - A_1}{\ln(A_2/A_1)} \quad (7.19)$$

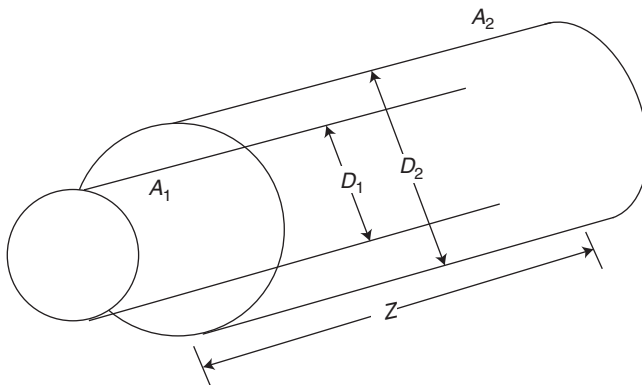


Figure 7.3 Log mean area figure.

where A_2 is the outer surface area of the cylinder (ft^2) and A_1 is the inner surface area of the cylinder (ft^2). The log-mean averaging of terms will be discussed again later in Part Three. When the value of A_2/A_1 does not exceed 2.5, the arithmetic-mean area may be employed; that is, $A_{\text{av}} = (A_1 + A_2)/2$. The geometric mean area may also be used; that is, $A_{\text{av}} = (A_1 A_2)^{0.5}$. Since

$$A = \pi DZ; \quad Z = \text{length of pipe} \quad (7.20)$$

Equation (7.19) may be rewritten as

$$A_{\text{lm}} = \frac{\pi Z(D_2 - D_1)}{\ln(D_2/D_1)} \quad (7.21)$$

For heat flowing through different materials in sequence, the companion to Equation (7.9) is

$$R = R_a + R_b + R_c = \frac{L_i}{k_i A_{i,\text{lm}}} \quad (7.22)$$

with (once again)

$$\dot{Q} = \frac{\Delta T}{\sum R_i} \quad (7.23)$$

$$= \frac{T_H - T_C}{\frac{L_a}{(k_a)(A_{a,\text{lm}})} + \frac{L_b}{(k_b)(A_{b,\text{lm}})} + \frac{L_c}{(k_c)(A_{c,\text{lm}})}} \quad (7.24)$$

ILLUSTRATIVE EXAMPLE 7.1

A glass window is 1 cm thick and has an area of 3 m^2 . The temperature at the outer surface is 10°C . The glass has a conductivity of $1.4 \text{ W/m} \cdot \text{K}$. The heat transfer rate is 3 kW. Calculate the temperature at the inner surface.

SOLUTION: Write the equation for one-dimensional heat conduction:

$$\dot{Q} = \frac{-kA(T_C - T_H)}{L}; \quad T_H > T_C \quad (7.2)$$

Solve for the unknown temperature T_1 at the wall,

$$T_1 = T_2 + \frac{\dot{Q}L}{kA}$$

For this illustrative example, $\dot{Q} = 3000 \text{ W}$ and $L = 0.01 \text{ m}$.

Substitute the values provided and compute T_H .

$$T_H = (273 + 10) + \frac{(3000)(0.01)}{(1.4)(3)} = 290.1 \text{ K} = 17.1^\circ\text{C}$$

Note that there are two kinds of SI temperatures used in this solution: Celsius and Kelvin (absolute). To be safe, one should compute using the absolute temperature, K ($K = ^\circ\text{C} + 273$). It is also important to remember that a change in temperature in Celsius is equal to the change in Kelvin, that is, $\Delta T(\text{K}) = \Delta T(^{\circ}\text{C})$. ■

ILLUSTRATIVE EXAMPLE 7.2

A new 1 ft thick insulating material was recently tested for heat resistant properties. The data recorded temperatures of 70°F and 210°F on the cold and hot sides, respectively. If the thermal conductivity of the insulating material is 0.026 Btu/ft · h · °F, calculate the rate of the heat flux, Q/A , through the wall in Btu/ft² · h. Resolve the problem in SI units.

SOLUTION: For this test,

$$\Delta T = 210 - 70 = 140^{\circ}\text{F}; \quad \Delta x = L = 1 \text{ ft}$$

Substitution into Equation (7.2) leads to

$$\begin{aligned} \dot{Q} &= kA \frac{\Delta T}{\Delta x} \\ \frac{\dot{Q}}{A} &= (0.026) \left(\frac{140}{1.0} \right) = 3.64 \text{ Btu/ft}^2 \cdot \text{h} \end{aligned}$$

In SI units,

$$\frac{\dot{Q}}{A} = (3.64) \left(\frac{252 \text{ cal/h}}{1 \text{ Btu/h}} \right) \left(\frac{1 \text{ ft}^2}{0.093 \text{ m}^2} \right) = 9863 \text{ cal/m}^2 \cdot \text{h} = 9.863 \text{ kcal/m}^2 \cdot \text{h} \quad \blacksquare$$

ILLUSTRATIVE EXAMPLE 7.3

The following information is provided. A rectangular furnace wall is made of fire clay (height, $H = 3$ m, width, $W = 1.2$ m, thickness, $L = 0.17$ m). Temperature of inside surface (area = $H \times W$), T_H , is 1592 K, and of the outside surface (area = $H \times W$), T_C , is 1364 K. Determine the temperature gradient, the heat transfer rate, and the heat transfer flux.

SOLUTION: Determine the temperature at which the wall properties should be calculated. Thermophysical properties are usually calculated at the average wall temperature,

$$\begin{aligned} T_{\text{av}} &= (T_H + T_C)/2 \\ &= (1592 + 1364)/2 \\ &= 1478 \text{ K} \end{aligned}$$

Obtain the fire clay properties (see Table in the Appendix). For $T = 1478$ K,

$$\begin{aligned} \rho &= 2645 \text{ kg/m}^3 \\ k &= 1.8 \text{ W/m} \cdot \text{K} \\ c_p &= 960 \text{ J/kg} \cdot \text{K} \end{aligned}$$

Calculate the thermal diffusivity noting that

$$1 \text{ ft}^2/\text{s} = 0.0929 \text{ m}^2/\text{s}$$

and

$$\alpha = \frac{k}{\rho c_p} \quad (7.7)$$

Substituting

$$\begin{aligned} \alpha &= 1.8 / [(2645)(960)] \\ &= 7.09 \times 10^{-7} \text{ m}^2/\text{s} \\ &= 7.09 \times 10^{-7} / 0.0929 \\ &= 7.63 \times 10^{-6} \text{ ft}^2/\text{s} \end{aligned}$$

Since the temperature gradient may be approximated by

$$\frac{dT}{dx} = \frac{T_H - T_C}{\chi_C - \chi_H} = \frac{T_H - T_C}{L}$$

calculate the temperature gradient.

$$\begin{aligned} \frac{dT}{dx} &= \frac{1364 - 1592}{0.17} \\ &= -1341 \text{ K/m} \\ &= -1341^\circ\text{C/m} \end{aligned}$$

The heat transfer area is

$$\begin{aligned} A &= (H)(W) \\ &= (3)(1.2) \\ &= 3.6 \text{ m}^2 \end{aligned}$$

Calculate the heat transfer rate using Equation (7.2):

$$\dot{Q} = \frac{kA(T_H - T_C)}{L}$$

Substituting

$$\begin{aligned} \dot{Q} &= \frac{(1.8)(3.6)(1592 - 1364)}{0.17} \\ &= 8691 \text{ W} \\ &= 8.7 \text{ kW}; \text{ note that } 1 \text{ W} = 3.412 \text{ Btu/h} \\ &= (8691)(3.412) \\ &= 29,653 \text{ Btu/h} \end{aligned}$$

Calculate the heat transfer flux, \dot{Q}' , where

$$\dot{Q}' = \dot{Q}/A$$

Substituting

$$\begin{aligned}\dot{Q}' &= 8691/3.6 \\ &= 2414 \text{ W/m}^2; \text{ note that } 1 \text{ W/m}^2 = 0.3171 \text{ Btu/ft}^2 \cdot \text{h} \\ &= (2414)(0.3171) \\ &= 765.5 \text{ Btu/h} \cdot \text{ft}^2\end{aligned}$$

Calculate the thermal resistance. The thermal resistance, R , is given by Equation (7.9),

$$R = \frac{L}{kA}$$

Substituting

$$\begin{aligned}R &= \frac{0.17}{(1.8)(3.6)} \\ &= 0.0262 \text{ K/W} \\ &= 0.0262^\circ\text{C/W}\end{aligned}$$

■

ILLUSTRATIVE EXAMPLE 7.4

A concrete wall has a surface area of 30 m^2 and is 0.30 m thick. It separates warm room air from cold ambient air. The inner surface of the wall is known to be at a temperature of 25°C , while the outer surface is at -15°C . The thermal conductivity of the concrete is $1.0 \text{ W/m} \cdot \text{K}$.

1. Describe the conditions that must be satisfied in order for the temperature distribution in the wall to be linear.
2. What is the driving force for the transfer of heat?
3. What is the heat loss through the wall?

SOLUTION:

1. The conditions are:
 - a. steady-state heat transfer
 - b. a homogeneous isotropic medium
2. The driving force for heat transfer is

$$\Delta T = T_H - T_C = 25 - (-15) = 40^\circ\text{C}$$

3. Finally, calculate the heat loss, \dot{Q} , through the wall. The thermal resistance due to conduction is

$$R_t = \frac{L}{kA} \quad (7.9)$$

Substituting,

$$R = \frac{0.3}{(1.0)(30)} = 0.01^\circ\text{C/W} = 0.0005275^\circ\text{F} \cdot \text{h/Btu}$$

Apply Equation (7.8) and substitute.

$$\begin{aligned} \dot{Q} &= \frac{\text{thermal driving force}}{\text{thermal resistance}} \\ &= \frac{40^\circ\text{C}}{0.01^\circ\text{C/W}} = 4000 \text{ W} = 4 \text{ kW} \quad \blacksquare \end{aligned}$$

ILLUSTRATIVE EXAMPLE 7.5

The inside and outside temperatures of the walls of a room are 27°C and 68.7°C , respectively. The room walls consist of 6-inch concrete (C), 8-inch cork-board (B), and 1-inch wood (W) with corresponding thermal conductivities of 0.762, 0.0433, and $0.151 \text{ W/m} \cdot \text{K}$, respectively. Calculate the heat transfer rate across the wall and determine the temperature at the interface between the wood and the cork-board.

SOLUTION: Based on the data provided

$$\begin{aligned} T_C &= T_1 = 27^\circ\text{C} \\ T_H &= T_4 = 68.7^\circ\text{C} \\ L_C &= \Delta x_C = (6)(0.0254) = 0.1524 \text{ m} \\ L_B &= \Delta x_B = (8)(0.0254) = 0.2032 \text{ m} \\ L_W &= \Delta x_W = (1)(0.0254) = 0.0254 \text{ m} \end{aligned}$$

Perform the calculations to follow on 1.0 m^2 of area, so that $A = 1.0$. From Equation (7.9),

$$\begin{aligned} R_C &= \frac{\Delta x_C}{k_C} = \frac{L_C}{k_C} = \frac{0.1524}{0.762} = 0.200 \text{ K/W} \\ R_B &= \frac{\Delta x_B}{k_B} = \frac{L_B}{k_B} = \frac{0.2032}{0.0433} = 4.693 \text{ K/W} \\ R_W &= \frac{\Delta x_W}{k_W} = \frac{L_W}{k_W} = \frac{0.0254}{0.151} = 0.1682 \text{ K/W} \end{aligned}$$

Equation (7.17) is employed to calculate the heat transfer rate

$$\dot{Q} = \frac{T_1 - T_4}{R_C + R_B + R_W} = \frac{27 - 68.7}{0.200 + 4.693 + 0.1682} = -8.239 \text{ W}$$

The negative sign indicates that the heat flows from the hot to the cold surface. If the wall area is 4.0 m^2 , the heat rate is 32.96 W .

The interface temperature between the wood and cork-board, T_2 , can now be calculated,

$$\begin{aligned}\dot{Q} &= \frac{T_1 - T_2}{R_w} \\ -8.239 &= \frac{27 - T_2}{0.1682} \\ T_2 &= 28.4^\circ\text{C} = 301.4 \text{ K}\end{aligned}$$

ILLUSTRATIVE EXAMPLE 7.6

A 3-inch outside diameter steel pipe is covered with a $\frac{1}{2}$ -inch layer of asbestos, (a), which in turn is covered with a 2-inch layer of glass wool, (b). Determine the steady-state heat transfer per foot of pipe, Z , if the pipe outer surface temperature is 500°F and the glass wool outer temperature is 100°F .

Assume an asbestos–glass wool interfacial temperature of 300°F and an average asbestos (a) temperature of 200°F and glass wool (b) temperature of 400°F . Based on a literature review, asbestos and glass wool thermal conductivity values have been estimated to be:

$$\begin{aligned}k_{(a)}(200^\circ\text{F}) &= 0.120 \text{ Btu/h} \cdot \text{ft} \cdot ^\circ\text{F} \\ k_{(b)}(400^\circ\text{F}) &= 0.0317 \text{ Btu/h} \cdot \text{ft} \cdot ^\circ\text{F}\end{aligned}$$

SOLUTION: Apply Equation (7.24) to the materials, asbestos (a) and glass wool (b),

$$\dot{Q} = \frac{T_H - T_C}{\frac{L_a}{(k_a)(A_{a,\text{lm}})} + \frac{L_b}{(k_b)(A_{b,\text{lm}})}}$$

For the asbestos (a), apply Equation (7.21) while maintaining dimensional consistency:

$$A_{a,\text{lm}} = \frac{\pi Z(D_2 - D_1)/12}{\ln(D_2/D_1)}; \quad Z = \text{pipe length}$$

Substituting

$$\begin{aligned}A_{a,\text{lm}} &= \frac{\pi Z(4.0 - 3.0)/12}{\ln(4.0/3.0)} \\ &= 0.910Z; \text{ ft}^2\end{aligned}$$

Similarly, for glass wool (b)

$$\begin{aligned}A_{b,\text{lm}} &= \frac{\pi Z(8.0 - 4.0)/12}{\ln(8.0/4.0)} \\ &= 1.51Z; \text{ ft}^2\end{aligned}$$

Substituting into the equation above gives

$$\begin{aligned}\dot{Q} &= \frac{(500 - 100)}{\frac{(0.5/12)}{(0.120)(0.910Z)} + \frac{(2.0/12)}{(0.0317)(1.51Z)}} \\ &= \frac{400}{\frac{0.382}{Z} + \frac{3.48}{Z}}\end{aligned}$$

Factoring out Z gives

$$\frac{\dot{Q}}{Z} = \frac{400}{3.864} = 103.5 \text{ Btu/h} \cdot \text{ft} \quad \blacksquare$$

ILLUSTRATIVE EXAMPLE 7.7

Refer to the previous example. Calculate the outer asbestos temperature, T_i (i.e., the interfacial temperature between the asbestos and glass wool).

SOLUTION: Since the heat transfer rate is now known, a single-layer equation can be used to determine the interfacial temperature. For the glass wool layer, Equation (7.24) is then

$$\begin{aligned}500 - T_i &= \frac{(2.0/12)(103.5)}{(0.0317)(1.51)} \\ T_i &= 500 - 360 = 140^\circ\text{F}\end{aligned}$$

Alternatively, for the asbestos layer,

$$\begin{aligned}T_i - 100 &= \frac{(103.5)(0.5/12)}{(0.120)(0.910)} \\ T_i &= 100 + 39.5 = 139.5^\circ\text{F}\end{aligned}$$

The interfacial temperatures are in agreement. \blacksquare

MICROSCOPIC APPROACH

As noted in the Introduction to Part Two, this “Microscopic Approach” section has been included to complement the material in this chapter, particularly for those readers interested in a more theoretical approach and treatment of conduction. This section also provides the derivations and/or justifications for the describing equations employed in the earlier macroscopic treatment. Readers are encouraged to compare both approaches while noting that several of the Illustrative Examples to follow were drawn from the literature.^(2,3)

The analysis for Fourier’s law was presented previously for a rather simple system. Generally, a fluid will possess three temperature gradients with corresponding heat-flux components in a given coordinate system. When this is the case, three heat-flux

Table 7.3 Heat-Flux Components

Component	Rectangular coordinates	Cylindrical coordinates	Spherical coordinates
\dot{Q}'_1	$\dot{Q}'_x = -k \frac{\partial T}{\partial x}$	$\dot{Q}'_r = -k \frac{\partial T}{\partial r}$	$\dot{Q}'_r = -k \frac{\partial T}{\partial r}$
\dot{Q}'_2	$\dot{Q}'_y = -k \frac{\partial T}{\partial y}$	$\dot{Q}'_\phi = -k \frac{1}{r} \frac{\partial T}{\partial \phi}$	$\dot{Q}'_\theta = -k \frac{1}{r} \frac{\partial T}{\partial \theta}$
\dot{Q}'_3	$\dot{Q}'_z = -k \frac{\partial T}{\partial z}$	$\dot{Q}'_z = -k \frac{\partial T}{\partial z}$	$\dot{Q}'_\phi = -k \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi}$

terms, \dot{Q}'_x , \dot{Q}'_y , \dot{Q}'_z , arise at every point in a system described in rectangular coordinates. These \dot{Q}'/H flux terms are given by

$$\dot{Q}'_x = -k \frac{dT}{dx} \quad (7.25)$$

$$\dot{Q}'_y = -k \frac{dT}{dy} \quad (7.26)$$

$$\dot{Q}'_z = -k \frac{dT}{dz} \quad (7.27)$$

All components of the heat flux in a given coordinate system can be expressed in terms of the thermal conductivity of the fluid and a temperature gradient. These are presented in Table 7.3 for rectangular, cylindrical, and spherical coordinates.^(2,3) The equations are applicable to all fluids obeying Fourier's law.

ILLUSTRATIVE EXAMPLE 7.8

The temperature of the solid slab pictured in Figure 7.4 is given by the expression

$$T = 100 \cos\left(\frac{\pi z}{2h}\right)$$

Calculate the heat flux at $z = 0$ and $z = h$ in terms of the thermal conductivity and h . Comment on the results.

SOLUTION: This problem is solved using rectangular coordinates. First note that \dot{Q}'_x and \dot{Q}'_y are zero. From Table 7.3,

$$\begin{aligned} \dot{Q}'_z &= -k \frac{\partial T}{\partial z} \\ &= -k \frac{dT}{dz}, \text{ since } T \text{ is solely a function of } z. \end{aligned}$$

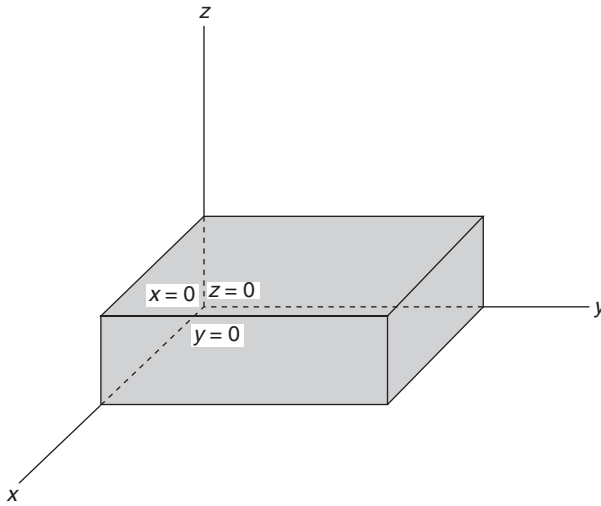


Figure 7.4 Heat-flux components in a rectangular slab.

If

$$T = 100 \cos\left(\frac{\pi z}{2h}\right)$$

$$\frac{dT}{dz} = -100 \left(\frac{\pi}{2h}\right) \sin\left(\frac{\pi z}{2h}\right)$$

$$\dot{Q}'_z = -k \frac{dT}{dz} = + \frac{50\pi k}{h} \sin\left(\frac{\pi z}{2h}\right)$$

so that

$$\dot{Q}'_z = + \frac{50\pi k}{h} \sin\left(\frac{\pi z}{2h}\right)$$

For the two locations,

$$\dot{Q}'_z|_{z=0} = 0$$

$$\dot{Q}'_z|_{z=h} = + \frac{50\pi k}{h}; \quad \text{with units Btu/s} \cdot \text{ft}^2$$

The above results indicate that energy is generated within the solid since *no* energy enters the solid from the bottom surface (it is insulated) and *finite* energy leaves from the top surface. ■

The equation describing energy transfer in *stationary* solids serves as an excellent starting point for the microscopic presentation of the general equation for energy transfer. The energy-transfer equation for solids is developed by applying the conservation law for energy on a time-rate basis to a fixed-volume element. The derivation is available in the literature.^(3,4)

The resulting equation describes the temperature variation in solids due to energy transfer, subject to the assumptions in the development. It applies to either steady-state or unsteady-state systems and is independent of any particular coordinate system.

Table 7.4 Steady-State Energy-Transfer Equation for Stationary Solids

 Rectangular coordinates:

$$\alpha \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] + \frac{A}{\rho c_p} = 0; \quad A = \text{source term} \quad (1)$$

Cylindrical coordinates:

$$\alpha \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \right] + \frac{A}{\rho c_p} = 0 \quad (2)$$

Spherical coordinates:

$$\alpha \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \right] + \frac{A}{\rho c_p} = 0 \quad (3)$$

$A = \text{rate of heat generated in solid; energy/time} \cdot \text{volume}$

The describing equations can be expanded into rectangular, cylindrical, and spherical coordinates for *steady-state conduction*. (The *unsteady-state* conduction equation can be found in the next chapter.) The results are presented in Table 7.4 for a solid that behaves in accordance with Fourier's law.

Table 7.4 provides various differential equations that describe steady-state energy transfer in solids. Integration constants must be specified before a complete solution to the differential equations may be obtained. These integration constants are usually obtained from information at specified locations at the boundary and/or knowledge of initial conditions. The most common conditions specified for these energy transfer problems are:

1. The temperature is specified at a solid boundary or surface.
2. The heat flux is specified at a boundary.
3. The temperatures on either side of an interface are equal.
4. The heat flux at either side of an interface are equal. The gradients on either side of an interface may also be specified.
5. The value of the temperature at time zero is given (for unsteady-state systems, see next chapter).

APPLICATIONS

This section contains six illustrative examples adopted from the earlier work of Theodore.⁽³⁾ The last two are rather lengthy. One solution involves the separation of variables method. Another example introduces the reader to Bessel functions.

ILLUSTRATIVE EXAMPLE 7.9

A long hollow *cylinder* has its inner and outer surfaces maintained at temperatures T_b and T_a , respectively. The inner and outer radii are b and a , respectively. Calculate the temperature profile in the solid section of the cylinder and determine the flux at both surfaces. Comment on the results. Assume steady-state conditions.

SOLUTION: This system is represented pictorially in Figure 7.5. This problem is solved using cylindrical coordinates. If end effects are neglected, T is not a function of ϕ and z . Equation (2) in Table 7.4 reduces to

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = 0$$

or

$$\frac{d}{dr} \left(r \frac{dT}{dr} \right) = 0, \quad \text{since } T \text{ depends solely on } r$$

Integrating once yields

$$r \frac{dT}{dr} = C$$

Integrating again gives

$$T = C \ln r + B \tag{1}$$

The boundary conditions (BC) are

$$\text{BC(1): } T = T_b \quad \text{at } r = b$$

$$\text{BC(2): } T = T_a \quad \text{at } r = a$$

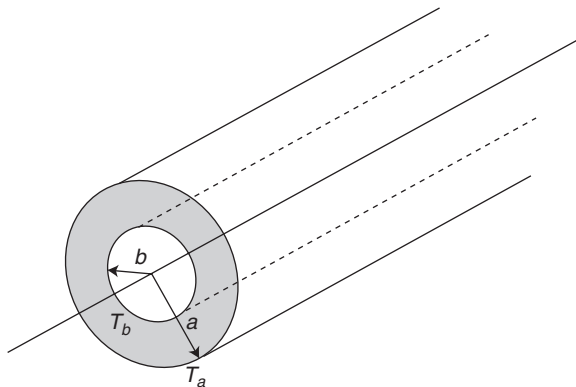


Figure 7.5 Heat conduction, hollow cylinder.

Apply the BC to Equation (1).

$$T_b = C \ln b + B \quad (2)$$

$$T_a = C \ln a + B \quad (3)$$

Solving Equations (2) and (3) simultaneously gives

$$C = \frac{T_a - T_b}{\ln\left(\frac{a}{b}\right)}$$

$$B = T_b - \frac{(T_a - T_b)}{\ln\left(\frac{a}{b}\right)} \ln b$$

Therefore

$$T = T_b + \frac{(T_a - T_b)}{\ln\left(\frac{a}{b}\right)} \ln\left(\frac{r}{b}\right) \quad (4)$$

The heat flux is given by

$$\dot{Q}'_r = -k \frac{dT}{dr}$$

But

$$\frac{dT}{dr} = \frac{C}{r}$$

so that

$$\begin{aligned} \dot{Q}'_r &= -\frac{kC}{r} \\ &= -\frac{k(T_a - T_b)}{\ln\left(\frac{a}{b}\right)} \left(\frac{1}{r}\right) \end{aligned}$$

Therefore

$$\dot{Q}'_{r=a} = -\frac{k(T_a - T_b)}{\ln\left(\frac{a}{b}\right)} \left(\frac{1}{a}\right)$$

$$\dot{Q}'_{r=b} = -\frac{k(T_a - T_b)}{\ln\left(\frac{a}{b}\right)} \left(\frac{1}{b}\right)$$

Although the fluxes are not equal, one can easily show that the rate of energy transfer, \dot{Q} , at both surfaces are equal. If the reader so desires, he/she is invited to review Illustrative Examples 7.6 and 7.7 in order to examine and better appreciate how the microscopic approach can be employed in generating the describing equations for this system. ■

ILLUSTRATIVE EXAMPLE 7.10

A constant rate of energy per unit volume A in the form of heat is uniformly generated in a long solid cylinder. The temperature at the outer surface of the cylinder ($r = a$) is maintained at T_a . Calculate the temperature profile and the heat flux at the outer wall under steady-state conditions.

SOLUTION: This problem is solved once again using cylindrical coordinates. Based on the problem statement and physical grounds, it is concluded that the temperature is solely a function of r . Equation (2) in Table 7.4 reduces to

$$\frac{\alpha}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{A}{\rho c_P} = 0$$

Integrating this equation once and rearranging the coefficients gives

$$r \frac{dT}{dr} = -\frac{Ar^2}{2k} + B$$

Integrating again yields

$$T = -\frac{Ar^2}{4k} + B \ln r + C \quad (1)$$

The BC are now written

$$\text{BC(1): } T = T_a \quad \text{at } r = a$$

$$\text{BC(2): } T \neq -\infty \quad \text{at } r = 0$$

or

$$\frac{dT}{dr} = 0 \quad \text{at } r = 0, \text{ due to symmetry}$$

Substituting the BC into Equation (1) above gives

$$B = 0$$

$$C = T_a + \left(\frac{A}{4k} \right) a^2$$

so that

$$T = T_a + \frac{A}{4k} (a^2 - r^2) \quad (2)$$

The heat flux at the wall is given by

$$\dot{Q}'_r|_{r=a} = -k \left. \frac{dT}{dr} \right|_{r=a}$$

However,

$$\frac{dT}{dr} = -\frac{Ar}{2k}$$

so that

$$\dot{Q}'_{r|r=a} = \frac{Aa}{2}$$

The maximum temperature is obtained at $r = 0$, i.e.,

$$T_{\max} = T_a + \frac{Aa^2}{4k}$$

ILLUSTRATIVE EXAMPLE 7.11

A hollow solid *sphere* has its inner ($r = b$) and outer ($r = a$) surfaces maintained at temperature T_b and T_a , respectively. Calculate the temperature profile in the solid at steady-state conditions. Determine the heat flux at both surfaces and comment on the results.

SOLUTION: This system is represented pictorially in Figure 7.6. This problem is solved using spherical coordinates. Based on the problem statement and symmetry, it is concluded that the temperature is not a function of θ and ϕ . Equation (3) in Table 7.4 reduces to

$$\frac{d}{dr} \left(r^2 \frac{dT}{dr} \right) = 0$$

Integrating once yields

$$r^2 \frac{dT}{dr} = B$$

Integrating again yields

$$T = -\frac{B}{r} + C \quad (1)$$

The BC are

$$\text{BC(1): } T = T_a \quad \text{at } r = a$$

$$\text{BC(2): } T = T_b \quad \text{at } r = b$$

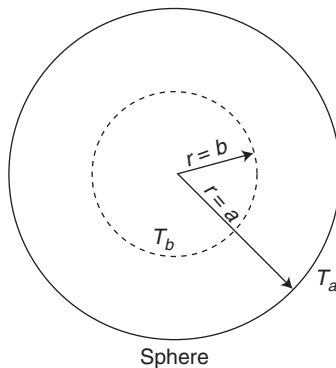


Figure 7.6 Hollow sphere.

Substituting the BC into Equation (1) above gives

$$T_a = -\frac{B}{a} + C \quad (2)$$

$$T_b = -\frac{B}{b} + C \quad (3)$$

Solving Equations (2) and (3) simultaneously results in

$$B = (T_b - T_a) \left(\frac{ba}{b-a} \right)$$

$$C = T_a + (T_b - T_a) \left(\frac{b}{b-a} \right)$$

so that

$$T = T_a + (T_b - T_a) \left(\frac{b}{b-a} \right) \left(1 - \frac{a}{r} \right)$$

The heat flux is given by

$$\dot{Q}'_r = -k \frac{dT}{dr}$$

But,

$$\frac{dT}{dr} = \frac{B}{r^2}$$

so that

$$\dot{Q}'_r = -k \left(\frac{T_a - T_b}{a-b} \right) \left(\frac{ab}{r^2} \right)$$

Therefore

$$\dot{Q}'_{r=a} = -k \left(\frac{T_a - T_b}{a-b} \right) \left(\frac{b}{a} \right)$$

$$\dot{Q}'_{r=b} = -k \left(\frac{T_a - T_b}{a-b} \right) \left(\frac{a}{b} \right)$$

The reader can easily show that the rate of heat transfer across both surfaces are equal by multiplying the flux by the appropriate area. ■

ILLUSTRATIVE EXAMPLE 7.12

An infinite horizontal slab of uniform width h has its lower surface placed in direct contact with the top surface of another slab of the same uniform width. The thermal conductivity of one slab is twice the other. The temperature of the bottom surface of the lower slab is zero, while the temperature of the top surface of the upper slab is T_h . Obtain the temperature profile in both solids at steady-state conditions. Comment on the temperature profile.

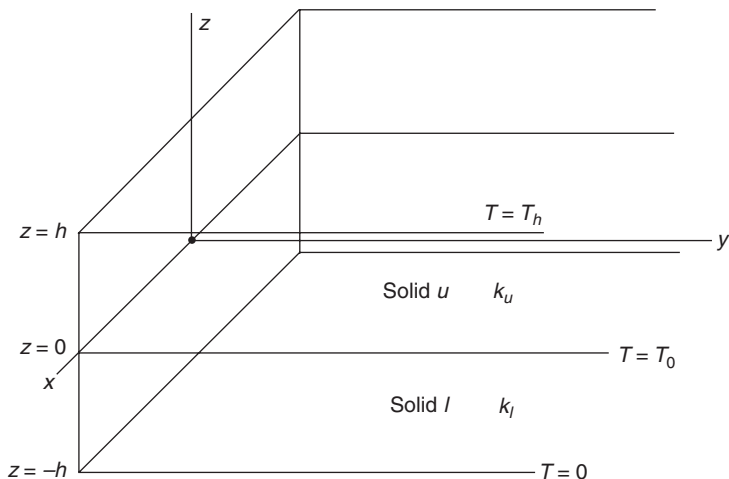


Figure 7.7 Heat conduction, two slabs.

SOLUTION: This system is pictured in Figure 7.7. Select rectangular coordinates with the origin of the z -axis located at the interface of the two solids. A differential equation (DE) is obtained for both solid u and solid l from Table 7.4. The describing equations are given by

$$\frac{d^2 T_u}{dz^2} = 0; \text{ solid } u$$

$$\frac{d^2 T_l}{dz^2} = 0; \text{ solid } l$$

Integrating once yields

$$\frac{dT_u}{dz} = A_u \tag{1}$$

$$\frac{dT_l}{dz} = A_l \tag{2}$$

The BC are now specified. At $z = 0$, the temperature of u must equal that of l . In addition, the heat flux for both solids must be equal; otherwise, an accumulation of energy at $z = 0$ would result—this is a physical impossibility. Therefore, the BC are

$$\text{BC(1): at } z = -h, T_l = 0$$

$$\text{BC(2): at } z = +h, T_u = T_h$$

$$\text{BC(3): at } z = 0, T_l = T_u = T_0$$

$$\text{BC(4): at } z = 0, q_z|_u = q_z|_l$$

or

$$k_u \frac{dT_u}{dz} = k_l \frac{dT_l}{dz}$$

Substituting BC(4) into Equations (1) and (2) above gives

$$k_u \frac{dT_u}{dz} = k_u A_u$$

$$k_l \frac{dT_l}{dz} = k_l A_l$$

or

$$k_u A_u = k_l A_l$$

Integrating Equations (1) and (2) again gives

$$T_u = A_u z + B_u \quad (3)$$

$$T_l = A_l z + B_l \quad (4)$$

Substituting BC(1), BC(2), and BC(3) into Equations (3) and (4) above yields

$$A_u = \frac{T_h}{h} \left(\frac{k_l}{k_l + k_u} \right)$$

$$A_l = \frac{T_h}{h} \left(\frac{k_u}{k_l + k_u} \right)$$

$$B_u = B_l = T_h \left(\frac{k_u}{k_l + k_u} \right)$$

and

$$T_u = T_h \left(\frac{k_l}{k_l + k_u} \right) \left(\frac{z}{h} + \frac{k_u}{k_l} \right)$$

$$T_l = T_h \left(\frac{k_u}{k_l + k_u} \right) \left(\frac{z}{h} + 1 \right)$$

If $k_u = 2k_l$, then

$$T_u = \frac{T_h}{3} \left(\frac{z}{h} + 2 \right); \quad 0 \leq z \leq h$$

$$T_l = \frac{2T_h}{3} \left(\frac{z}{h} + 1 \right); \quad -h \leq z \leq 0$$

At $z = 0$,

$$T_u = T_l = T_0 = \frac{2}{3} T_h$$

The temperature profile is presented in Figure 7.8. ■

ILLUSTRATIVE EXAMPLE 7.13

Consider the two-dimensional problem of a very thin solid bounded by the y -axis ($z = 0$), the lines $y = 0$ and $y = l$, and extending to infinity in the z -direction. The temperature of the vertical

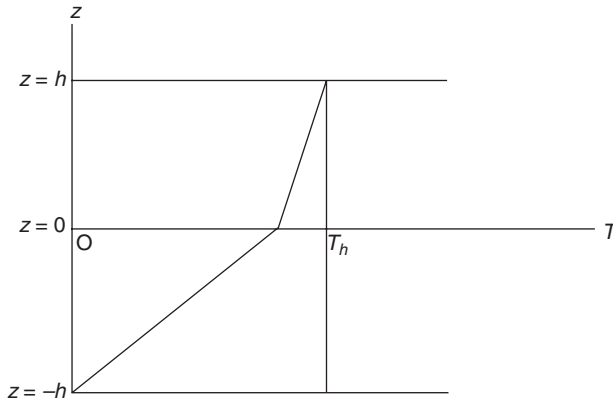


Figure 7.8 Two slab temperature profile for Illustrative Example 7.12.

edge at $y = 0$ and $y = l$ is maintained at zero. The temperature at $z = 0$ is T_0 . Determine the steady-state temperature profile in the solid. Assume both plane surfaces of the solid are insulated.

SOLUTION: A two-dimensional view of this system is presented in Figure 7.9. This problem is solved in rectangular coordinates. Based on the problem statement, T is not a function of x . “Extract” the following equation from Table 7.4.

$$\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{1}$$

This partial differential equation (PDE) is solved using the separation-of-variables method. Since the temperature of the solid is a function of y and z , assume that the solution can be separated into the product of one function $\psi(y)$, which depends solely on y and a second function $\Phi(z)$ that depends only on z .

$$\begin{aligned} T &= T(y, z) \\ &= \psi(y)\Phi(z) \\ &= \psi\Phi \end{aligned} \tag{2}$$

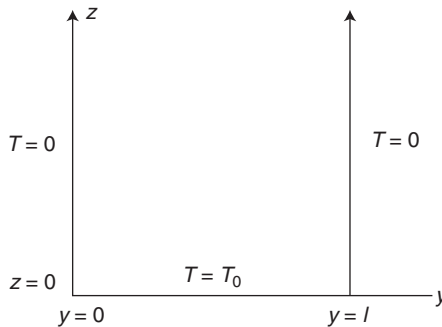


Figure 7.9 Two-dimensional heat transfer; Illustrative Example 7.13.

The first term of Equation (1) above is

$$\frac{\partial^2 T}{\partial y^2} = \Phi \psi'' \quad (3)$$

Note that the double prime represents the second derivative. The second term takes the form

$$\frac{\partial^2 T}{\partial z^2} = \psi \Phi'' \quad (4)$$

Substituting Equations (3) and (4) into Equation (1) gives

$$\Phi \psi'' + \psi \Phi'' = 0$$

or

$$\frac{\psi''}{\psi} = -\frac{\Phi''}{\Phi}$$

$\downarrow \downarrow \downarrow \downarrow$
 function
of y

$\downarrow \downarrow \downarrow \downarrow$
 function
of z

(5)

Since both terms are functions of different independent variables, it is concluded that both terms in Equation (5) are equal to the same constant, which is noted as $-\lambda^2$.

$$\frac{\psi''}{\psi} = -\frac{\Phi''}{\Phi} = -\lambda^2 \quad (6)$$

The reader can verify that a positive constant or zero does not contribute to the solution of the problem. The solution to Equation (6) takes the form

$$\psi = a'_\lambda \sin \lambda y + b'_\lambda \cos \lambda y; \quad a'_\lambda \text{ and } b'_\lambda \text{ are constants that}$$

depend on the value of λ

$$\Phi = Ce^{\lambda z} + De^{-\lambda z}$$

Since

$$T = \psi \Phi$$

$$T = (a'_\lambda \sin \lambda y + b'_\lambda \cos \lambda y)(Ce^{\lambda z} + De^{-\lambda z}) \quad (7)$$

The BC for the solid are:

$$\text{BC(1): } T = 0 \quad \text{at } y = 0$$

$$\text{BC(2): } T = 0 \quad \text{at } y = l$$

$$\text{BC(3): } T = T_0 \quad \text{at } z = 0$$

$$\text{BC(4): } T = 0 \quad \text{at } z = \infty$$

Based on physical grounds, BC(4) gives

$$C = 0$$

Therefore

$$\begin{aligned} T &= (a'_\lambda \sin \lambda y + b'_\lambda \cos \lambda y) D e^{-\lambda z} \\ &= (a_\lambda \sin \lambda y + b_\lambda \cos \lambda y) e^{-\lambda z}; \quad a_\lambda = a'_\lambda D \end{aligned}$$

BC(1) gives

$$b_\lambda = 0$$

and

$$T = a_\lambda (\sin \lambda y) e^{-\lambda z} \tag{8}$$

BC(2) gives

$$0 = a_\lambda (\sin \lambda l) e^{-\lambda z}$$

The RHS of this equation is zero if $a_\lambda = 0$ or $\sin \lambda l = 0$. If a_λ is zero, no solution results. Therefore

$$\sin \lambda l = 0$$

and

$$\begin{aligned} \lambda l &= n\pi; \quad n = 1, 2, 3, 4, \dots \\ \lambda &= \frac{n\pi}{l} \end{aligned}$$

Equation (8) now becomes

$$T = a_n e^{-(n\pi/l)z} \sin\left(\frac{n\pi y}{l}\right)$$

The constant a_n is evaluated from BC(3),

$$T_0 = a_n \sin\left(\frac{n\pi y}{l}\right)$$

A Fourier series analysis^(3,4) gives

$$a_n = \frac{2T_0}{n\pi} [(-1)^{n+1} + 1]$$

Conforming all of the above gives

$$T = \sum_{n=1}^{\infty} \frac{2T_0}{n\pi} [(-1)^{n+1} + 1] e^{-(n\pi/l)z} \sin\left(\frac{n\pi y}{l}\right) \tag{9}$$

ILLUSTRATIVE EXAMPLE 7.14

Energy is being absorbed in a long solid cylinder of radius a . The temperature at the outer surface of the cylinder is maintained at T_a . Calculate the temperature profile in the solid at steady-state conditions. Assume the energy generation term A is a linear function of the temperature (i.e., $A = -ST$, where S is a positive constant).

SOLUTION: This system is presented in Figure 7.10, and the problem is solved in cylindrical coordinates. The appropriate DE is “extracted” from Table 7.4. Since

$$T = T(r) \neq T(\phi, z, t)$$

the following equation is obtained

$$\frac{\alpha}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{A}{\rho c_p} = 0 \quad (1)$$

where

$$A = -ST$$

Substituting and expanding the above DE,

$$\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} - \left(\frac{S}{k} \right) T = 0 \quad (2)$$

This is one form of Bessel’s equation. The solution is given by

$$T = BI_0 \left[r \left(\frac{S}{k} \right)^{1/2} \right] + CK_0 \left[r \left(\frac{S}{k} \right)^{1/2} \right] \quad (3)$$

where I_0 = “modified” Bessel function; first kind, zero order
 K_0 = “modified” Bessel function; second kind, zero order
 B, C = integration constants

The BC are now written

$$\text{BC(1): at } r = a, T = T_a$$

$$\text{BC(2): at } r = 0, \frac{dT}{dr} = 0 \quad \text{due to symmetry}$$

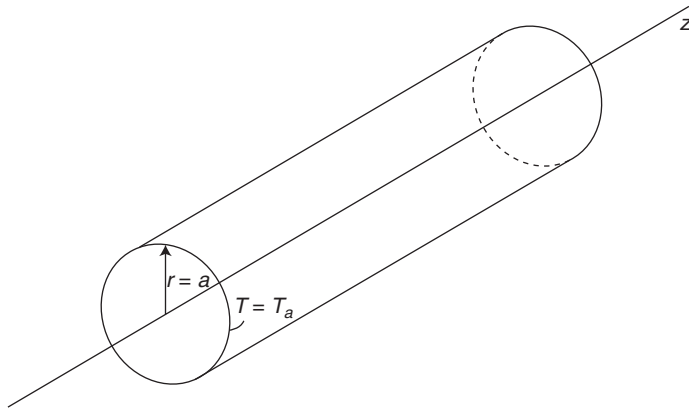


Figure 7.10 Heat absorption in a solid cylinder; Illustrative Example 7.14.

or

$$\text{at } r = 0, T = \text{finite}$$

Taking the derivative of Equation (3) leads to

$$\frac{dT}{dr} = B \left(\frac{S}{k}\right)^{1/2} I_1 \left[r \left(\frac{S}{k}\right)^{1/2} \right] - C \left(\frac{S}{k}\right)^{1/2} K_1 \left[r \left(\frac{S}{k}\right)^{1/2} \right] \quad (4)$$

The above Bessel functions have the property

$$I_0(0) = \text{finite}$$

$$K_0(0) = \infty$$

$$I_1(0) = 0$$

$$K_1(0) = \infty$$

Applying BC(2) gives

$$C = 0$$

Applying BC(1) gives

$$T_a = B I_0 \left[a \left(\frac{S}{k}\right)^{1/2} \right]$$

so that

$$B = \frac{T_a}{I_0 \left[a \left(\frac{S}{k}\right)^{1/2} \right]}$$

Hence, the solution to Equation (1) is

$$T = T_a \frac{I_0 \left[r \left(\frac{S}{k}\right)^{1/2} \right]}{I_0 \left[a \left(\frac{S}{k}\right)^{1/2} \right]} \quad (5)$$

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