

6

Similitude, Dimensional Analysis and Modelling

Although many practical engineering problems involving momentum, heat and mass transport can be modelled and solved using the equations and procedures described in the preceding chapters, an important number of them can be solved only by relating a mathematical model to experimentally obtained data.

In fact, it is probably fair to say that very few problems involving real momentum, heat, and mass flow can be solved by mathematical analysis alone. The solution to many practical problems is achieved using a combination of theoretical analysis and experimental data. Thus engineers working on chemical and biochemical engineering problems should be familiar with the experimental approach to these problems. They have to interpret and make use of the data obtained from others and have to be able to plan and execute the strictly necessary experiments in their own laboratories. In this chapter, we show some techniques and ideas which are important in the planning and execution of chemical and biochemical experimental research. The basic considerations of dimensional analysis and similitude theory are also used in order to help the engineer to understand and correlate the data that have been obtained by other researchers.

One of the goals of the experimental research is to analyze the systems in order to make them as widely applicable as possible. To achieve this, the concept of similitude is often used. For example, the measurements taken on one system (for example in a laboratory unit) could be used to describe the behaviour of other similar systems (e.g. industrial units). The laboratory systems are usually thought of as *models* and are used to study the phenomenon of interest under carefully controlled conditions. Empirical formulations can be developed, or specific predictions of one or more characteristics of some other similar systems can be made from the study of these models. The establishment of systematic and well-defined relationships between the laboratory model and the “other” systems is necessary to succeed with this approach. The correlation of experimental data based on dimensional analysis and similitude produces models, which have the same qualities as the transfer based, stochastic or statistical models described in the previous chapters. However, dimensional analysis and similitude do not have a theoretical basis, as is the case for the models studied previously.

6.1

Dimensional Analysis in Chemical Engineering

In order to explain dimensional analysis in chemical engineering, we present a typical problem of chemical engineering that requires an experimental approach. Consider the steady flow of an incompressible Newtonian fluid through a long, smooth-walled, horizontal and circular pipe which is heated from the outside.

In this system two important characteristics are of interest to an engineer designing the pipeline:

1. the pressure drop per unit length along the pipe as a result of friction,
2. the heat transfer coefficient that shows the kinetics of heat transfer from the pipe wall to the bulk fluid.

The first step in planning an experiment to study this problem would take into consideration the choice of factors, or variables that affect the pressure drop ($\Delta p/l$) and the heat transfer coefficient (α). As a first approach, we can consider the effects of temperature and pressure separately. In fact, the temperature variation has no direct effect on the pressure drop but has an effect on the fluid's physical properties.

We can formulate that the pressure drop is a function of the pipe diameter, d , the fluid density, ρ , the fluid viscosity, η , and the mean velocity at which the fluid is flowing in the pipe (w). Thus, we can express this relationship as:

$$\Delta p/l = f(d, \rho, \eta, w) \quad (6.1)$$

The heat transfer coefficient is considered as a function of the parameters previously described and of the two thermal properties of the liquid: the heat capacity, c_p , and the thermal conductivity, λ :

$$\alpha = f(d, \rho, \eta, c_p, \lambda, w) \quad (6.2)$$

To carry out the experiments in a meaningful and systematic way, it will be necessary, first, to consider one of the parameters as a variable while keeping the others constant and then to measure the corresponding pressure drop. The same type of experiment is carried out for the measurement of the heat transfer coefficient. Contrary to the mass transport pressure drop, which could be measured directly, the heat transfer coefficient is obtained indirectly by measuring the temperature of the wall and of the fluid at the entrance and exit of the pipe. The determination of the functional relationship between $\Delta p/l$, α and the various parameters that influence the process is illustrated in Fig. 6.1.

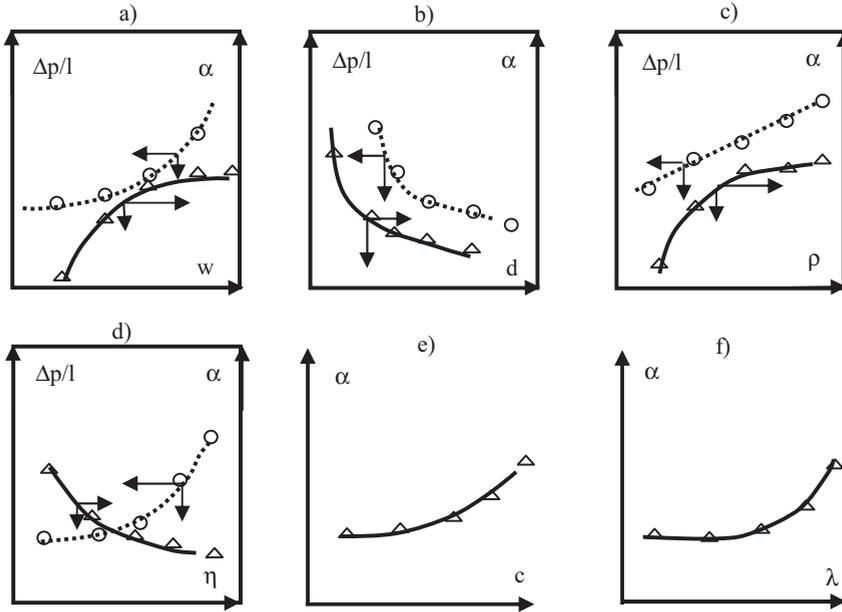


Figure 6.1 Illustrative plots showing the dependence of $\Delta p/l$ and α on the state of different process factors. (a) $d, \rho, \eta, c_p, \lambda$ constant, (b) $w, \rho, \eta, c_p, \lambda$ constant, (c) w, d, η, c_p, λ constant, (d) w, d, ρ, c_p, λ constant, (e) $w, d, \rho, \eta, \lambda$ constant, (f) w, d, η, c_p, ρ constant.

Some of the results shown in this figure have to be obtained from experiments that are very difficult to carry out. For example, to obtain the data illustrated in Fig. 6.1(c) we must vary the liquid density while keeping the viscosity constant. For the data needed in Fig. 6.1(e), the thermal conductivity has to be varied while the density, the thermal capacity and viscosity are kept constant. These curves are actually almost impossible to obtain experimentally because the majority of the studied parameters are dependent on each other. This problem could be solved using a much simpler approach with the dimensionless variables that are described below. In fact, we can combine the different parameters described in Eqs. (6.1) and (6.2) in non-dimensional combinations of variables (called dimensionless groups, products' criteria)

$$\frac{\Delta p}{\rho w^2} \frac{d}{l} = \Phi \left(\frac{wd\rho}{\eta} \right) \quad (6.3)$$

and

$$\frac{\alpha d}{\lambda} = \Phi \left(\frac{wd\rho}{\eta}, \frac{c_p \eta}{\lambda} \right) \quad (6.4)$$

Thus, instead of working with five parameters for the estimation of $\Delta p/l$, we have only two. In the case of α , which depends on seven parameters, this has been reduced to three dimensionless variables.

In the first case, the experimental work will simply consist of variation of the dimensionless product $wd\rho/\eta$ and determination of the corresponding value of $\Delta p/(\rho w^2)(d/l)$. The results of the experiments can then be represented by a single universal curve, as illustrated in Fig. 6.2(a). Varying the dimensionless product $wd\rho/\eta$ and determining, for the dimensionless group $c_p\eta/\lambda$, the corresponding value of $\alpha d/\lambda$, makes it possible to obtain the results shown in Fig. 6.2(b) for the pipe heat transfer. From these results we can conclude that carrying out the experimental work will be much simpler, easier, and cheaper. The basis of these simplifications lies in consideration of the involved variables' dimensions. It is known that the physical quantities can be given in terms of basic dimensions such as mass, M, length, L, time, T, temperature, θ , quantity of substance N and light intensity, Λ . The derivation systems of basic dimensions also coexist between them, F L T θ N Λ are the most common.

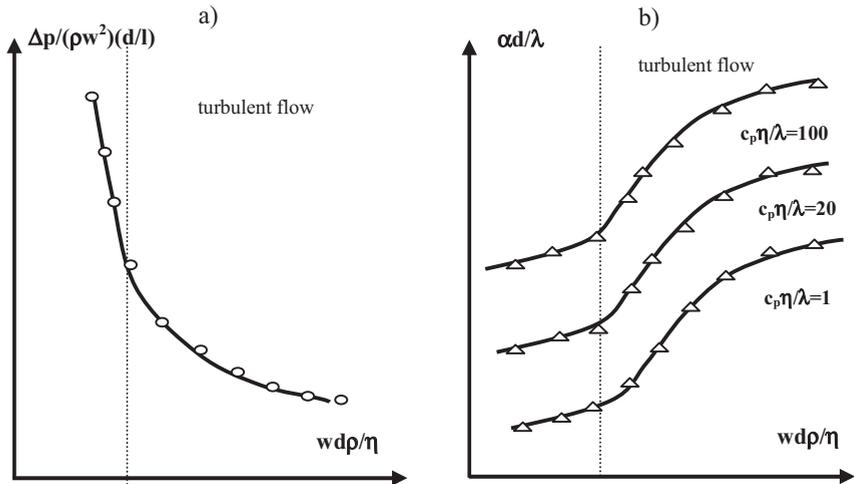


Figure 6.2 An illustrative example for pressure drop and heat transfer coefficient evaluation using dimensionless groups: (a) dimensionless pressure drop, (b) dimensionless heat transfer coefficient.

For example Newton's second law, $F = ma$, can be written as:

$$[F] = [m][a] = M L T^{-2} \tag{6.5}$$

Here, the brackets are used to indicate an operation using the basic dimension of the variables. It is not difficult to obtain the dimension formulae for the variables presented in the previously discussed examples; these are:

$$[\Delta p/L] = ML^{-2}T^{-2}, [w] = LT^{-1}, [d] = L, [\rho] = ML^{-3}, [\eta] = ML^{-1}T^{-1}, [c_p] = L^2T^{-2}\theta^{-1},$$

$$[\lambda] = MLT^{-3}\theta^{-1}, [\alpha] = MT^{-3}\theta^{-1}.$$

A rapid check of the groups' dimension, which appears in relationships (6.3) and (6.4), shows that they actually are dimensionless criteria:

$$\left[\frac{\Delta p}{\rho w^2} \frac{d}{L} \right] = \frac{ML^{-2}T^{-2}L}{ML^{-3}(LT^{-1})^2} = M^0L^0T^0, \quad \left[\frac{wd\rho}{\eta} \right] = \frac{(LT^{-1})L(ML^{-3})}{ML^{-1}T^{-1}} = M^0L^0T^0$$

and

$$\left[\frac{\alpha d}{\lambda} \right] = \frac{(MT^{-3}\theta^{-1})L}{MLT^{-3}\theta^{-1}} = M^0L^0T^0\theta^0, \quad \left[\frac{c_p\eta}{\lambda} \right] = \frac{(L^2T^{-2}\theta^{-1})(ML^{-1}T^{-1})}{MLT^{-3}\theta^{-1}} = M^0L^0T^0\theta^0$$

With this methodology, not only has the number of variables been reduced, but also the new groups are dimensionless combinations of variables, which means that the results presented in Fig. 6.2 will be independent of the system of units used. This type of analysis is called dimensional analysis. The basis for its application to a wide variety of problems is found in the Buckingham Pi Theorem described in the next section. Dimensional analysis is also used for other applications such as:

- establishing the dimensional formula for the derived physical variables,
- verifying the dimensional homogeneity of the physical relationships and equations used for the characterization of a process,
- verifying whether the units of measurement used for process variables are correct.

6.2

Vaschy–Buckingham Pi Theorem

When researchers want to use dimensional analysis of a process, the first and fundamental question they have to answer concerns the number of dimensionless groups that are required to replace the original list of process variables. The answer to this question is given by the basic theorem of dimensional analysis, which is stated as follows:

“If a process is characterized by an equation involving m physical variables, then this equation can be reduced to a relationship between $m - n$ independent dimensionless groups, where n represents the number of basic dimensions used to describe the variable”.

The dimensionless groups are frequently called “pi terms” due to the symbol used by Buckingham [6.1] to define the fact that the dimensionless group is a product. Their first modern presentation was given by Vaschy [6.2], even though several early investigators, including Rayleigh, contributed to the development of

the pi theorem. In spite of the simplicity of the pi Theorem, its improvement is not simple. This will not, however, be presented here, because the detailed mathematical improvement is beyond the scope of this chapter. Many books give a more detailed treatment of the pi theorem and dimensional analysis [6.3–6.15].

The pi theorem is based on the idea of the dimensional homogeneity of the process equations or on the relationships that characterize one particular process. From this point of view, all the coefficients of statistical models that have already been discussed in Chapter 5 have a physical dimension, because the dependent and the independent process variables have a physical dimension. Essentially, we assume that any physically meaningful equation, which characterizes one process and which involves m variables, such as $y_1 = f(x_1, x_2, \dots, x_m)$ presents, for each term contained on the right-hand side, the same dimension as for the left-hand side. This equation could be transformed into a set of dimensionless products (pi terms):

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \dots, \Pi_{m-n}) \quad (6.6)$$

The required number of pi terms is lower than the number of original n variables, where n is determined by the minimum number of basic dimensions required to describe the original list of variables. For common momentum and mass transfer, the basic dimensions are usually represented by M , L , and T . For heat transfer processes, four basic dimensions – M , L , T , θ – have to be used. Moreover, in a few rare cases, the variables could be described by a combination of basic dimensions such as, for any flow processes, M/T^2 and L . The use of the pi theorem may appear to be mysterious and complicated, although there are systematic and relatively simple procedures to develop the pi theorem for a given problem.

6.2.1

Determination of Pi Groups

Several methods can be used to form the dimensionless pi terms in a dimensional analysis. The most important are those applying a systematic determination of the pi terms, but they can be used only when the terms are dimensionless and independent. These methods, which will be described in detail later, are called “method of base non-complete group” [6.16] or “method of repeating variables” [6.17]. The determination of pi groups must be considered as the beginning of modelling for a process using dimensional analysis. We can consider that a model is completely established if a general characteristic process function, obtained after the application of this method, can be particularized by experimental data.

One of the simplest analyses consists in dividing the method into a series of distinct steps that can be followed for any given problem. The description given below is very similar to the methodology generally applied in the production of a mathematical model process as previously presented in Chapter 2.

Step 1: List all the variables that are involved in the problem (process)

This step is one of the most difficult and is, of course, extremely important because all pertinent variables have to be included in the analysis. The term variable includes any physical quantity, dimensional and apparently non-dimensional constant that plays a role in the phenomenon under investigation. The determination of the variables must take into account practical knowledge of the problem as well as the physical laws governing the phenomenon. Variables typically include the parameters that are necessary not only to describe the geometry of the system (such as the diameter of the pipe in the example below), but also to define the fluid properties (such as the density, viscosity, thermal capacity, thermal conductivity of the fluid, the diffusion coefficient for one species in the working fluid, etc.) as well as to indicate the external effects that influence the system (such as the driving pressure drop in the further discussed cases).

These general types of variables are intended to be as broad as possible in order to be helpful in identification. However, in some cases, the variables may not easily fit into one of these categories. This is why each problem has to be carefully analyzed.

Two conditions are very important during this analysis. First, generally, the researchers wish to have a minimum number of variables in order to minimize the experimental work. Secondly, these variables have to be independent. For example, for a problem of flow in a pipe, the geometric dimensions such as the pipe diameter and the section flow, could both be considered as variables. However, only the pipe diameter will be considered in the list of variables because the section flow already contains the basic geometric dimension.

Step 2: Establishment of the dimensional formula for each variable from the selected list

For a typical chemical engineering problem, the dimensions considered are generally M, L, T and θ . The dimensions F, L, T, θ can also be used but, in this case, especially for heat transfer problems and for coupled heat and mass transfer processes, complicated dimensional formulae are derived. To establish a dimensional formula for a variable, it is necessary to have a relationship containing this variable. This relationship can be independent of the process to which the dimensional analysis is applied. The use of tables containing dimensional formulae for physical variables can also be effective.

Step 3: Determination of the required number of pi terms

This step can be accomplished by means of the pi theorem which indicates that the number of pi terms is equal to $m - n$, where m (determined in step 1) is the number of selected variables and n (determined in step 2) is the number of basic dimensions required to describe these variables. The reference dimensions usually correspond to the basic dimensions and can be determined by a careful inspection of the variables' dimensions obtained in step 2. As previously noted, the basic dimensions rarely appear combined, which results in a lower number of reference dimensions than the number of basic ones.

Step 4: Selection of a non-complete group containing the same number of variables and basic dimensions

Here we select some variables from the original list in order to combine them with the remaining variables to form the pi term. The variables contained in the non-complete group do not change during the process of pi term production. All the required reference (basic) dimensions must be included within the non-complete group of repeating variables. Each repeating variable must be dimensionally independent of the others (a similar consideration is taken into account when the dimensions of one repeating variable cannot be reproduced by any combination of the exponent product of the remaining repeating variables). In fact, we can conclude that the repeating variables cannot be combined with other repeating variables to form dimensionless criteria.

For any given problem, we are usually interested in determining how one particular variable influences (and is influenced by) other variables. A one-dimensional analysis accepts only one dependent variable. It is recommended not to choose the dependent variable as one of the repeating variables, since the repeating variable will generally appear in more than one pi group term and then the variable separation cannot be carried out easily.

Step 5: Development of the pi terms one at a time by multiplying a non-repeating variable by a non-complete group which has the repeating variable necessary to obtain the arbitrary different exponents

Essentially, each pi term will be of the form $x_i x_1^\alpha x_2^\beta x_3^\gamma$ where x_i is a non-repeating variable and x_1, x_2, x_3 represent the repeating variables of the non-complete group. The exponents α, β, γ are determined in order to give a dimensionless combination. The case presented here corresponds to a process where variables are introduced with three basic dimensions (M, L, T). For heat transfer and the coupling of heat and mass transfer processes, the form used for a pi term is $x_i x_1^\alpha x_2^\beta x_3^\gamma x_4^\delta$. The values of the exponents α, β, γ are determined in this step by generating a system of linear algebraic equations containing these exponents. The basis for the development of the system is represented by the condition of the dimensionless pi group.

Step 6: Checking all the resulting pi terms to make sure they are dimensionless

In order to prove that the pi terms are correctly formulated, their dimensionless condition should be confirmed by replacing the variables in the dimensional formula by the pi terms. This step can be carried out by writing the variables in terms of M, L, T, θ . If the dimensional analysis has been produced using F, L, T, θ as basic dimensions, then check the formula to make sure that the pi terms are dimensionless.

Step 7: Establishment of the final form as a relationship among the pi terms

The most frequently used form of the final dimensional analysis is written as Eq. (6.6) where Π_1 will contain the dependent variables in the numerator. It should be emphasized that, if you have started out with a good list of variables (and the other

steps of the analysis have been completed correctly), the relationship in terms of the pi groups can be used as a basis to describe the investigated problem. All we need to do is work with the pi groups and not with the individual variables. However, it should be clearly noted that the functional relationship between the pi groups has to be determined experimentally. The result is a relationship criterion able to show the main behaviour of the analyzed system or process. The chemical engineering research methodologies can also result in obtaining a theoretical relationship criterion using various theoretical bases [6.18–6.20]

To illustrate the steps described above, we will consider the problem already introduced at the beginning of this chapter, which was concerned with the pressure drop and heat transfer of an incompressible Newtonian fluid flowing in a pipe.

The first problem is the classical example used to show the scientific force of the dimensional analysis – and especially of the pi theorem. Remember that we are interested in the pressure drop per unit length ($\Delta p/l$) along the pipe. According to the experimenter's knowledge of the problem and to step 1, we must list all the pertinent variables that are involved; in this problem, it was assumed that:

$$\Delta p/l = f(d, \rho, \eta, w)$$

where d is the pipe diameter, ρ and η are the fluid density and viscosity, and w is the mean fluid velocity.

In step 2, we express all the variables in terms of basic dimensions. Using M, L, T as basic dimensions, it follows that:

$$[\Delta p/l] = \frac{[F/S]}{[l]} = \frac{MLT^{-2}L^{-2}}{L} = ML^{-2}T^{-2}$$

$$[d] = L$$

$$[\rho] = ML^{-3}$$

$$[\eta] = ML^{-1}T^{-1}$$

$$[w] = LT^{-1}$$

We could also use F, L, and T as basic dimensions. Now, we can apply the pi theorem to determine the required number of pi terms (step 3). An inspection of the variable dimensions obtained in step 2 reveals that the three basic dimensions are all required to describe the variables. Since there are five ($m = 5$) variables (do not forget to count the dependent variable, $\Delta p/l$) and three required reference dimensions ($n = 3$), then, according to the pi theorem, two pi groups ($5 - 3$) will be required.

We need to select three out of the four variables (d, ρ, η, w) in the list of the incomplete group with repeating variables (step 4) to be used to form the pi terms. Remember that we do not want to use the dependent variable as one of the repeat-

ing variables. Generally, we will try to select the dimensionally simplest repeating variables. For example, if one of the variables has a length dimension, we can choose it as one of the repeating variables. We can note that this incomplete group has to contain all the basic dimensions established by step 2. For this step, we use d , ρ and w as repeating variables in the incomplete group.

We are now ready to form the two π groups and to identify the exponents associated with the repeating variables from the incomplete group (step 5). Typically, we will start with the dependent variable and combine it with the repeating variables to form the first π term:

$$\Pi_1 = (\Delta p/l)d^\alpha \rho^\beta w^\gamma \quad (6.7)$$

This combination has to be dimensionless and, in the particular example, only M , L and T are presented:

$$[\Pi_1] = [(\Delta p/l)d^\alpha \rho^\beta w^\gamma] \quad (6.8)$$

The dimensional relationship (6.8) is developed into Eq. (6.9). Then, exponents α , β , γ must be determined so that the resulting exponent of each of the basic dimensions M , L and T , is zero (it gives a dimensionless combination). Thus, we can also write the relationship (6.10):

$$M^0 L^0 T^0 = M L^{-2} T^{-2} (L)^\alpha (M L^{-3})^\beta (L T^{-1})^\gamma = M^{(1+\beta)} L^{(-2+\alpha-3\beta+\gamma)} T^{(-2-\gamma)} \quad (6.9)$$

$$\begin{cases} 1 + \beta = 0 \\ -2 + \alpha - 3\beta + \gamma = 0 \\ -2 - \gamma = 0 \end{cases} \quad (6.10)$$

Solution of the equation system (6.10) gives the desired values for α , β , γ . It is easy to observe that the following solution is obtained: $\alpha = 1$, $\beta = -1$, $\gamma = -2$. Therefore, the π group is:

$$\Pi_1 = \frac{\Delta p}{\rho w^2} \frac{d}{l}$$

This procedure is now repeated for the remaining non-repeating variables. In this example, there is only one additional variable (η):

$$\Pi_2 = \eta d^\alpha \rho^\beta w^\gamma \quad (6.11)$$

By analogy with Eqs. (6.8) and (6.9), we can write Eqs. (6.12) and (6.13) which allow one to build a system of linear algebraic equations (6.14). This system gives the values of α , β , γ associated with Π_2 .

$$[\Pi_2] = [\eta d^\alpha \rho^\beta w^\lambda] \quad (6.12)$$

$$M^0 L^0 T^0 = M L^{-1} T^{-1} (L)^\alpha (M L^{-3})^\beta (L T^{-1})^\gamma = M^{(1+\beta)} L^{(-1+\alpha-3\beta+\gamma)} T^{(-1-\gamma)} \quad (6.13)$$

$$\begin{cases} 1 + \beta = 0 \\ -1 + \alpha - 3\beta + \gamma = 0 \\ -1 - \gamma = 0 \end{cases} \quad (6.14)$$

Solving Eq. (6.14), it follows that $\alpha = -1$, $\beta = -1$, $\gamma = -1$ and:

$$\Pi_2 = \frac{\eta}{w d \rho} \quad (6.15)$$

At this point, we can check the dimensionless condition of the pi groups (step 6). However, before checking, we have to write the dimensional formulae for the variables contained in the selected list using the basic dimensions F, L, T. To obtain this transformation in the dimensional formulae used in step 2, the relationship $F = M L T^{-2}$ is used to replace the mass (M). The result obtained is:

$$[\Delta p/l] = F L^{-3}, [d] = L, [\rho] = F L^{-4} T^2, [\eta] = F L^{-2} T, [w] = L T^{-1}$$

Now we can check whether the obtained pi groups are dimensionless:

$$[\Pi_1] = \left[\frac{\Delta p}{l} \frac{d}{\rho w^2} \right] = \frac{(F L^{-3})(L)}{(F L^{-4} T^2)(L T^{-1})^2} = F^0 L^0 T^0$$

$$[\Pi_2] = \left[\frac{\eta}{w d \rho} \right] = \frac{F L^{-2} T}{(L T^{-1})(L)(F L^{-4} T^2)} = F^0 L^0 T^0$$

or alternatively,

$$[\Pi_1] = \left[\frac{\Delta p}{l} \frac{d}{\rho w^2} \right] = \frac{M L^{-2} T^{-2} (L)}{(M L^{-3})(L T^{-1})^2} = M^0 L^0 T^0$$

$$[\Pi_2] = \left[\frac{\eta}{w d \rho} \right] = \frac{M L^{-1} T^{-1}}{(L T^{-1})(L)(M L^{-3})} = M^0 L^0 T^0$$

Finally (step 7), we can express the result of dimensional analysis as:

$$\frac{\Delta p}{\rho w^2} \frac{d}{l} = \phi \left(\frac{\eta}{\rho w d} \right) \quad (6.16)$$

This result indicates that this problem can be studied in terms of these two pi terms, rather than in terms of the original five variables. Nevertheless, the dimensional analysis will not provide the form of the function ϕ . This can be obtained from a suitable set of experiments. The power form for ϕ has been successfully

used in chemical engineering literature. Thus, Eq. (6.16) can be particularized into Eq. (6.17) and, after the introduction of the Reynolds number, Eqs. (6.18) and (6.19) are obtained. Eq. (6.19) is the famous Fanning expression for the fluid pressure drop in the pipe. We can also derive the friction factor, λ_f , from Eqs. (6.18) and (6.19):

$$\frac{\Delta p}{\rho w^2} \frac{d}{l} = c \left(\frac{\eta}{\rho w d} \right)^p \quad (6.17)$$

$$\Delta p = c \text{Re}^{-p} \frac{1}{d} \frac{w^2}{2} \rho \quad (6.18)$$

$$\Delta p = \lambda_f \frac{1}{d} \frac{w^2}{2} \rho \quad (6.19)$$

The second problem, introduced at the beginning of this chapter and discussed here, is meant to show how – with the presented 7-step algorithm – we can obtain a simple dimensionless relationship between the various process variables affecting the heat transfer between the wall and the fluid.

Step 1 is rapidly resolved, based on the discussion of these problems at the beginning of this chapter (Fig. 6.1). The list of variables considers that:

$$\alpha = F(d, \rho, \eta, c_p, \lambda, w)$$

where the definition of each variable has been presented above.

Step 2 requires expressing all variables in terms of the basic dimensions. Using M, L, T, and θ as basic dimensions, the process variables show the dimensional formulae:

$$[\alpha] = \text{MT}^{-3}\theta^{-1}$$

$$[d] = \text{L}$$

$$[\rho] = \text{ML}^{-3}$$

$$[\eta] = \text{ML}^{-1}\text{T}^{-1}$$

$$[c_p] = \text{L}^2\text{T}^{-2}\theta^{-1}$$

$$[\lambda] = \text{MLT}^{-3}\theta^{-1}$$

$$[w] = \text{LT}^{-1}$$

A similar result is obtained if we use F, L, T, θ as basic dimensions. As previously described, the M dimension is replaced by F. In the case of $[\rho]$, the basic dimensions of M, L, T, and θ are replaced by F, L, T, and θ . From $F = \text{MLT}^{-2}$ we obtain

$M = FL^{-1}T^2$ which is used in the $[\rho]$ formula to finally obtain: $[\rho] = FL^{-1}T^2L^{-3} = FL^{-4}T^2$.

Step 3 begins with determining the number of basic dimensions (M, L, T, θ). In this case $n = 4$ and with $m = 7$ (the number of variables considered in the first step) we conclude that the number of pi groups, n_π , is: $n_\pi = m - n = 7 - 4 = 3$.

In order to start step 4, we need to choose an incomplete group composed of n variables; the variables of this incomplete group will be coupled one by one with the remaining variables. Remember that we do not want to use the dependent variable as one of the repeating variables. At the same time, the incomplete group of repeating variables has to include all basic dimensions. We have chosen an incomplete group which includes d, ρ, η and λ because it has a very high number of variables with simple dimensional formulae. We are now ready to form three pi terms (step 5). To do so, we have to begin with the dependent variable and combine it with the repeating variables. Therefore, the first pi term is:

$$\Pi_1 = ad^\beta \rho^\gamma \eta^\delta \lambda^\epsilon \tag{6.20}$$

Since this combination has to be dimensionless, we can write:

$$[\Pi_1] = [ad^\beta \rho^\gamma \eta^\delta \lambda^\epsilon] \tag{6.21}$$

or:

$$M^0 L^0 T^0 \theta^0 = M T^{-3} \theta^{-1} (L)^\beta (ML^{-3})^\gamma (ML^{-1}T^{-1})^\delta (MLT^{-3}\theta^{-1})^\epsilon \tag{6.22}$$

Respectively:

$$M^{0(1+\gamma+\delta+\epsilon)} L^{(\beta-3\gamma-\delta+\epsilon)} T^{(-3\beta-\delta-3\epsilon)} \theta^{(-\epsilon-1)} \tag{6.23}$$

Now we can identify the exponents β, γ, δ and ϵ of the basic dimensions using the equality between the exponents of the basic dimensions on the left-hand side and the exponents on the right-hand side (Eq. (6.23)). Then we obtain the next system of linear equations:

$$\begin{cases} 1 + \gamma + \delta + \epsilon = 0 \\ \beta - 3\gamma - \delta + \epsilon = 0 \\ -3\beta - \delta - 3\epsilon = 0 \\ 1 - \epsilon = 0 \end{cases} \tag{6.24}$$

The solution of this system of algebraic equations gives the desired values for β, γ, δ and ϵ . It is simple to obtain $\epsilon = -1, \beta = 1, \gamma = 0, \delta = 0$ and therefore to write:

$$\Pi_1 = Nu = \frac{ad}{\lambda} \tag{6.25}$$

which is the classical Nusselt dimensionless number (Nu), currently used in heat transfer processes. Step 5 must be repeated in order to obtain the dimensionless groups Π_2 and Π_3 . We still have some variables able to be coupled with the incomplete groups, which are the flow rate (w) and the liquid thermal capacity (c_p).

If the selected variable is the flow rate, we can write the dimensionless expression:

$$\Pi_2 = wd^\beta \rho^\gamma \eta^\delta \lambda^\varepsilon \quad (6.26)$$

In this case, Eqs. (6.21)–(6.24) used for complete identification of this group, show the following particularizations:

$$[\Pi_2] = [wd^\beta \rho^\gamma \eta^\delta \lambda^\varepsilon] \quad (6.27)$$

$$M^0 L^0 T^0 \theta^0 = L T^{-1} (L)^\beta (ML^{-3})^\gamma (ML^{-1} T^{-1})^\delta (MLT^{-3} \theta^{-1})^\varepsilon \quad (6.28)$$

$$M^{(\gamma+\delta+\varepsilon)} L^{(1+\beta-3\gamma-\delta+\varepsilon)} T^{(-1-\delta-3\varepsilon)} \theta^{(-\varepsilon)} \quad (6.29)$$

$$\begin{cases} \gamma + \delta + \varepsilon = 0 \\ 1 + \beta - 3\gamma - \delta + \varepsilon = 0 \\ -1 - \delta - 3\varepsilon = 0 \\ \varepsilon = 0 \end{cases} \quad (6.30)$$

The solution of this system of algebraic equations gives the new values for β , γ , δ , and ε adapted to the Π_2 group. It is then simple to obtain $\varepsilon = -0$, $\beta = 1$, $\gamma = 1$, $\delta = -1$ and therefore:

$$\Pi_2 = Re = \frac{wd\rho}{\eta} \quad (6.31)$$

If we carry out step 5 again, we obtain the group formed by coupling the liquid thermal capacity with the incomplete group of repeating variables. After the usual procedure Π_3 is written as:

$$\Pi_2 = c_p d^\beta \rho^\gamma \eta^\delta \lambda^\varepsilon \quad (6.32)$$

The new values needed for β , γ , δ and ε , will be obtained by applying the algorithm to the Π_3 group in Eq. (6.32). The next relationships show the following particularization:

$$[\Pi_3] = [c_p d^\beta \rho^\gamma \eta^\delta \lambda^\varepsilon] \quad (6.33)$$

$$M^0 L^0 T^0 \theta^0 = L^2 T^{-2} \theta^{-1} (L)^\beta (ML^{-3})^\gamma (ML^{-1} T^{-1})^\delta (MLT^{-3} \theta^{-1})^\varepsilon \quad (6.34)$$

$$M^0 L^0 T^0 \theta^0 = M^{(\gamma+\delta+\varepsilon)} L^{(2+\beta-3\gamma-\delta+\varepsilon)} T^{(-2-\delta-3\varepsilon)} \theta^{(-1-\varepsilon)} \quad (6.35)$$

$$\begin{cases} \gamma + \delta + \varepsilon = 0 \\ 2 + \beta - 3\gamma - \delta + \varepsilon = 0 \\ -2 - \delta - 3\varepsilon = 0 \\ -1 - \varepsilon = 0 \end{cases} \quad (6.36)$$

It is then simple to obtain the new values of β , γ , δ and ε adapted to the Π_3 group. These are:

$\varepsilon = -1$, $\delta = 1$, $\gamma = 0$ and $\beta = 0$ and therefore:

$$\Pi_3 = Pr = \frac{c_p \eta}{\lambda} \quad (6.37)$$

This dimensionless group is recognized as the Prandtl number, which is currently used in heat transfer processes. This number is very important when the boundary layer theory is applied because it shows the relationship between the corresponding thickness of the heat transfer boundary layer and the hydrodynamic boundary layer [6.12].

The next step consists in making sure that the pi groups obtained are dimensionless (step 6). As explained above, the dimensional formulae for the variables contained in this selected list will be produced in the case of basic dimensions F, L, T and θ . Therefore, in the dimensional equations used in step 2, mass M will be replaced by force F using the relationship $F = MLT^{-2}$:

$$\begin{aligned} [\alpha] &= FL^{-1}T^{-1}\theta^{-1}, \quad [d] = L, \quad [\rho] = FL^{-4}T^2, \quad [\eta] = FL^{-2}T, \quad [w] = LT^{-1}, \\ [c_p] &= L^2T^{-2}\theta^{-1}, \quad [\lambda] = FT^{-1}\theta^{-1} \end{aligned}$$

Now, let us check whether the obtained pi groups (Nu, Re, Pr) are dimensionless:

$$[\Pi_1] = [Nu] = \left[\frac{\alpha d}{\lambda} \right] = \frac{(FL^{-1}T^{-1}\theta^{-1})L}{FT^{-1}\theta^{-1}} = F^0 L^0 T^0 \theta^0$$

$$[\Pi_2] = [Re] = \left[\frac{wd\rho}{\eta} \right] = \frac{(LT^{-1})(L)(FL^{-4}T^2)}{FL^{-2}T} = F^0 L^0 T^0 \theta^0$$

$$[\Pi_3] = [Pr] = \left[\frac{c_p \eta}{\lambda} \right] = \frac{(L^2T^{-2}\theta^{-1})(FL^{-2}T)}{FT^{-1}\theta^{-1}} = F^0 L^0 T^0 \theta^0$$

or alternatively,

$$[\Pi_1] = [Nu] = \left[\frac{\alpha d}{\lambda} \right] = \frac{(MT^{-3}\theta^{-1})L}{MLT^{-3}\theta^{-1}} = M^0 L^0 T^0 \theta^0$$

$$[\Pi_2] = [\text{Re}] = \left[\frac{wd\rho}{\eta} \right] = \frac{(LT^{-1})(L)(ML^{-3})}{ML^{-1}T^{-1}} = M^0L^0T^0\theta^0$$

$$[\Pi_3] = [\text{Pr}] = \left[\frac{c_p\eta}{\lambda} \right] = \frac{(L^2T^{-2}\theta^{-1})(ML^{-1}T^{-1})}{MLT^{-3}\theta^{-1}} = M^0L^0T^0\theta^0$$

Finally (step 7), we can express the result of dimensional analysis as:

$$\frac{\alpha d}{\lambda} = \Phi \left(\frac{wd\rho}{\eta}, \frac{c_p\eta}{\lambda} \right) \quad (6.38)$$

Equation (6.38), which contains function Φ , has already been proved theoretically [6.12] and experimentally [6.18]. The famous relationship (6.39), which is applicable when pipe flow is fully developed, is currently used to characterize the heat transfer kinetics in other similar examples:

$$\text{Nu} = 0.023 \text{Re}^{0.8} \text{Pr}^{0.33} \quad (6.39)$$

To summarize, the methodology to be followed in performing a dimensional analysis using the method of incomplete groups of repeating variables, consists in following this series of steps:

- Step 1: List all variables that are involved in the investigated phenomenon. This step needs a very good knowledge of these variables.
- Step 2: Each variable has to be described by its dimensional formula.
- Step 3: Establish the required number of pi groups.
- Step 4: Select the incomplete group of repeating variables. The number of repeating variables and basic dimensions involved in the problem are identical.
- Step 5: Form the pi term by multiplying one of the non-repeating variables by the incomplete group where the repeating variables have arbitrary powers and identify the actual pi expression. Repeat this step for all non-repeating variables.
- Step 6: Check all the resulting pi terms to make sure they are dimensionless.
- Step 7: Express the final form as a relationship among pi terms and add supplementary commentaries if necessary.

6.3

Chemical Engineering Problems Particularized by Dimensional Analysis

The two cases analyzed above give a model used to produce a particularization of the dimensional analysis to a chemical engineering problem. It has been observed that dimensional analysis is a good tool to rapidly elaborate a dimensionless frame for a system on which the experiments could be carried out by measuring the suggested variables. A further advantage lies in the “scale invariance” of dimensionless groups, thus enabling the only reliable scaling-up of the analyzed phenomena.

In the description of the various steps, it has been established that there are only two real problems in dealing with dimensional analysis. The first problem is the listing of all the relevant parameters that describe the process. Because chemical engineering processes are influenced by a high number of parameters, it is not easy to establish a good list of variables. The second problem is the determination of the process characteristics and of the real operational numbers, particularly in the case of large-scale factors. From the viewpoint of dimensional analysis, the descriptive chemical engineering model based on graphic representations is frequently effective in obtaining the correct interpretation of a process. We shall develop this problem in the following examples.

6.3.1

Dimensional Analysis for Mass Transfer by Natural Convection in Finite Space

We introduce this problem with two particular examples. The first is the etching of a metal placket immersed in a large specifically formulated liquid, with no gas production. The second is the drying of a recently built wall. In both cases, we have a non-observable flow and a particularization of the dimensional analysis is required.

These two examples do not appear to have any similarities. Nevertheless, after a deep analysis, we can conclude that both cases consist of a natural convection process produced by a concentration gradient.

This is presented schematically in Fig. 6.3, which also shows that the kinetics of these processes is described by the transport rate of A from the wall to the adjacent media. Using Fig. 6.3, we can establish that two elementary processes are presented in this system. The first is the flow induced by the concentration gradient and the second is the mass transfer sustained by the processes on the surface (a chemical reaction in the case of the metal placket immersed in a specifically formulated liquid and the transport through the porosity in the case of the drying wall). The case presented here corresponds to the situation when, in respect of the bulk density, the fluid density begins to decrease near the wall. This generates the displacement of the media and the specific ascension force, which is equivalent to the density difference. This phenomenon depends on the concentration difference in fluid A $\Delta c_A = (c_{Ap} - c_{A\infty})$. From Fig. 6.3 we can write a list of process variables:

$$k = f(H, \rho, \eta, D_A, g\beta_d \Delta c_A) \quad (6.40)$$

where g represents the gravitational acceleration, β_d is the density coefficient of the density–concentration dependence and Δc_A is the gradient for the natural convection. It is then easy to observe that the product $\beta_d \Delta c_A$ is dimensionless.

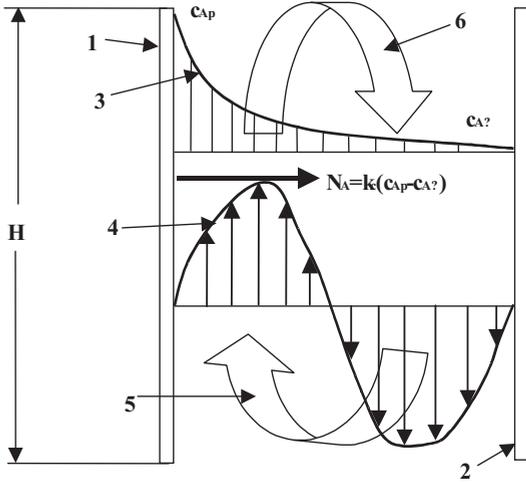


Figure 6.3 Mass transfer mechanism of natural convection between a placket and an adjacent medium. 1: placket or drying wall, 2: limit of adjacent medium, 3: concentration of A, 4: fluid velocity, 5 and 6: fluid global displacement. System properties: Geometric properties: H : height of placket. Fluid properties: C_A : concentration of A, density (ρ), viscosity (η), diffusion coefficient of A (D_A). Displacement properties: specific ascension force ($g\beta_c \Delta c_A$). Interaction properties: mass transfer coefficient (k_c).

We can now complete the first step of the dimensional analysis. The dimensions of the variables, using the MLT system for basic dimensions, are:

$$[k] = LT^{-1}$$

$$[H] = L$$

$$[\rho] = ML^{-3}$$

$$[\eta] = ML^{-1}T^{-1}$$

$$[D_A] = L^2T^{-1}$$

$$[g\beta_c \Delta c_A] = LT^{-2}$$

We can observe that all the basic dimensions (also specific to moment and mass transfer) are required to define the six variables, taking into consideration that, according to the Buckingham pi theorem, three pi terms will be needed (six variables minus three basic dimensions, $m - n = 6 - 3$).

The next step is the selection of three repeating variables such as H , ρ , and η to form the incomplete group of repeating variables. A quick inspection of these reveals that they are dimensionally independent, since each of them contains a basic dimension not included in the others. Starting with the dependent variable k_c , the first pi term can be formed combining k_c with the repeating variables so that:

$$\Pi_1 = k_c H^\alpha \rho^\beta \eta^\gamma \quad (6.41)$$

in terms of dimensions we have:

$$[\Pi_1] = [k_c H^\alpha \rho^\beta \eta^\gamma] \quad (6.42)$$

$$M^0 L^0 T^0 = L T^{-1} (L)^\alpha (M L^{-3})^\beta (M L^{-1} T^{-1})^\gamma \quad (6.43)$$

or

$$M^0 L^0 T^0 = M^{(\beta+\gamma)} L^{(1+\alpha-3\beta-\gamma)} T^{(-1-\gamma)} \quad (6.44)$$

the dimensionless condition of Π_1 implies:

$$\begin{cases} \beta + \gamma = 0 \\ 1 + \alpha - 3\beta - \gamma = 0 \\ -1 - \gamma = 0 \end{cases} \quad (6.45)$$

and, therefore $\alpha = 1$, $\beta = 1$ and $\gamma = -1$, the pi term then becomes:

$$\Pi_1 = \frac{k_c H \rho}{\eta} \quad (6.46)$$

The procedure is then repeated with the second non-repeating variable, D_A :

$$\Pi_2 = D_A H^\alpha \rho^\beta \eta^\gamma \quad (6.47)$$

It follows that

$$[\Pi_2] = [D_A H^\alpha \rho^\beta \eta^\gamma] \quad (6.48)$$

$$M^0 L^0 T^0 = L^2 T^{-1} (L)^\alpha (M L^{-3})^\beta (M L^{-1} T^{-1})^\gamma \quad (6.49)$$

$$M^0 L^0 T^0 = M^{(\beta+\gamma)} L^{(2+\alpha-3\beta-\gamma)} T^{(-1-\gamma)} \quad (6.50)$$

and

$$\begin{cases} \beta + \gamma = 0 \\ 2 + \alpha - 3\beta - \gamma = 0 \\ -1 - \gamma = 0 \end{cases} \quad (6.51)$$

The solution of this system is: $\alpha = 0$, $\beta = 1$ and $\gamma = -1$, and therefore:

$$\Pi_2 = Sc = \frac{D_{AP}}{\eta} \quad (6.52)$$

where symbol Sc introduces the Schmidt criterion which is frequently used in mass transfer problems. In the theory of boundary layers, the Schmidt criterion gives the relationship between the diffusion and hydrodynamic boundary layers. Figure 6.3 can be completed considering the additional thickness of the boundary layers formed at the placket wall and adjacent medium. The remaining non-repeating variable is $g\beta_c\Delta c_A$, where the third pi term is:

$$\Pi_3 = g\beta_c\Delta c_A H^\alpha \rho^\beta \eta^\gamma \quad (6.53)$$

and

$$[\Pi_3] = [g\beta_c\Delta c_A H^\alpha \rho^\beta \eta^\gamma] \quad (6.54)$$

$$M^0 L^0 T^0 = L T^{-2} (L)^{\alpha} (M L^{-3})^{\beta} (M L^{-1} T^{-1})^{\gamma} \quad (6.55)$$

$$M^0 L^0 T^0 = M^{(\beta+\gamma)} L^{(1+\alpha-3\beta-\gamma)} T^{(-2-\gamma)} \quad (6.56)$$

and, therefore,

$$\begin{cases} \beta + \gamma = 0 \\ 1 + \alpha - 3\beta - \gamma = 0 \\ -2 - \gamma = 0 \end{cases} \quad (6.57)$$

Solving this system, we obtain $\alpha = 3$, $\beta = 2$ and $\gamma = -2$ and we can write:

$$\Pi_3 = Gr_d = \frac{g\beta_c\Delta c_A H^2 \rho^2}{\eta^2} \quad (6.58)$$

Here Gr_d is the diffusion Grassy number. It represents the natural convection displacement based on the concentration difference.

We have obtained the three required pi terms, which have to be checked in order to make sure that they are dimensionless. To do so, we use F , L and T , which will also verify the correctness of the original dimensions used for the variables. As explained earlier, we first have to replace M by F in the dimensional variable formula. Then the result is:

$$[k] = LT^{-1}, [H] = L, [\rho] = FL^{-4}T^2, [\eta] = FL^{-2}T, [D_A] = L^2T^{-1}, [g\beta_c\Delta c_A] = LT^{-2}$$

The dimensionless verification gives:

$$[\Pi_1] = \left[\frac{k_c H \rho}{\eta} \right] = \frac{(LT^{-1})(L)(FL^{-4}T^2)}{FL^{-2}T} = F^0L^0T^0$$

$$[\Pi_2] = \left[\frac{D_A \rho}{\eta} \right] = \frac{(L^2T^{-1})(FL^{-4}T^2)}{FL^{-2}T} = F^0L^0T^0$$

$$[\Pi_3] = \left[\frac{g\beta_c\Delta c_A H^3 \rho^2}{\eta^2} \right] = \frac{(LT^{-2})(L)^3(FL^{-4}T^2)^2}{(FL^{-2}T)^2} = F^0L^0T^0$$

If this analysis results in a bad agreement with the dimensionless condition, we have to go back to the original list of variables and check the dimensional formula of each variable as well as the algebra used to obtain the exponents α , β and γ .

Before finishing the application, we show that each pi group obtained can be replaced by a combination between this pi number and others. So, if we divide Π_1 by Π_3 , we obtain:

$$\Pi_4 = Sh = \frac{\Pi_1}{\Pi_2} = \frac{k_c H \rho}{\eta} \frac{\eta}{D_A \rho} = \frac{k_c H}{D_A} \quad (6.59)$$

where Sh represents the Sherwood number which encrypts the mass transfer kinetics of the investigated process. Finally, we can represent the results of the dimensional analysis particularization in the form of:

$$Sh = f(Gr_d, Sc) \quad (6.60)$$

However, at this stage of the analysis, the form and nature of the function f are unknown. To continue, we will have to perform a set of experiments or we can use one theoretical method able to show this function.

6.3.2

Dimensional Analysis Applied to Mixing Liquids

Mixing various components in a liquid medium is a chemical engineering operation with large industrial applications. Some examples of these applications are: paint production, resin and pigment mixing, gas-liquid transfer or reaction by bubbling in liquid, solid dissolution and solid crystallization in mixed liquid media, homogenous and heterogeneous chemical reactions involving liquid agitated media, aerobic and anaerobic biochemical reactions with molecular transformations in the liquid phase. These examples show the importance of the optimization of mixing liquids for the chemical industry.

An important number of factors having a key influence on this unit operation [6.19, 6.20], together with the examples described above, show how difficult it is to formulate a complete and unitary mixing theory responding to the various technical questions such as mixing time, distribution of the residence time, power consumption, heat and mass transfer kinetics in mixed media, scaling-up of a laboratory mixing plant etc.

In order to simplify the problem, we will apply dimensional analysis to liquid mixing in a particular case. The studied example will take into account the interactions showing:

- dependence of the power consumption with respect to process factors,
- dependence of the mixing time with respect to process factors,
- dependence of the mass transfer kinetics with respect to process factors in the case of dissolving suspended solids,
- dependence of the heat transfer kinetics with respect to process factors in the case of a wall heated by an agitated liquid.

The first necessary condition [6.21] to be taken into account in all particularization cases is the use of general mixing parameters (factors related to the geometry of agitation, the properties of liquid media, the type of agitators and rotation speed) as well as the use of the specific factors of the studied application. For example, in the case of suspended solid dissolution, we can consider the mass transfer coefficient for dissolving suspended solids, the mean dimension of the suspended solid particles, and the diffusion coefficient of the dissolved species in the liquid.

In this chapter, we present two particularizations: the first concerns the dependence of the power consumption on the considered influencing factors; the second shows the relationship between the mixing time and its affecting factors.

In order to establish the list of variables, we use the explicative Fig. 6.4 in both cases. It especially shows the geometry of agitation, allowing the introduction of geometric, material and dynamic factors.

In the first example, we considered that the power consumed by an agitator N , depends on the agitator diameter d , on the geometric position of the agitator in the liquid tank – expressed by the coordinates H , D , h , as well as on the rotation speed of the agitator n , and on the liquid physical properties (density ρ , viscosity η , and superficial tension σ). The interest here consists in formulating a relationship between the power consumption and the different affecting factors.

Considering Fig. 6.4, we can write (step 1 of the application procedure of the dimensional analysis) the following list of variables:

$$N = f(d, D, H, h, b, n, \rho, \eta, \sigma) \quad (6.61)$$

Now we can write (step 2) all variables in terms of basic dimensions. Using M , L and T it follows that:

$$\begin{aligned} [N] &= ML^2T^{-3}, [d] = L, [D] = L, [H] = L, [h] = L, [b] = L, [n] = T^{-1}, [\rho] = ML^{-3}, \\ [\eta] &= ML^{-1}T^{-1}, [\sigma] = MT^{-2} \end{aligned}$$

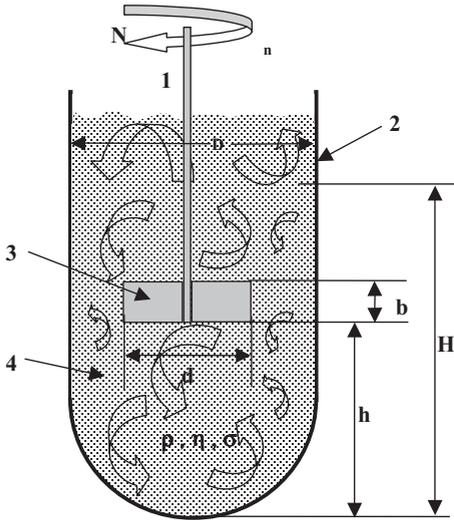


Figure 6.4 Schematic representation of mixing in liquid media. 1: Axis of paddle agitator, 2: tank of mixing system, 3: paddle of mixing system, 4: mixed liquid medium. System properties: geometric: agitator diameter (d) tank diameter (D), liquid height (H), paddle width (b), bottom paddle position (h); fluid: density (ρ), viscosity (η), superficial tension (σ); displacement: rotation speed (n); interaction: power consumption (N).

By applying the pi theorem (step 3), we obtain that the number of pi groups required is 7 because $m = 10$ (process variables) and $n = 3$ (basic dimensions). The repeating variables of the incomplete group have been selected according to d , ρ and n and to the considerations of step 4. We can now form all the pi groups one at a time. Typically, we begin with the coupling of the dependent variable (power consumption, N) with the incomplete group. The formulation of the first pi term is:

$$\Pi_1 = Nd^\alpha \rho^\beta n^\gamma \quad (6.62)$$

By applying the dimensional formulation to this relationship we have:

$$M^0 L^0 T^0 = M L^2 T^{-3} (L)^\alpha (M L^{-3})^\beta (T^{-1})^\gamma \quad (6.63)$$

and

$$M^0 L^0 T^0 = M^{(1+\beta)} L^{(2+\alpha-3\beta)} T^{(-3-\gamma)} \quad (6.64)$$

and, consequently, the system of equations obtained with the exponents is:

$$\begin{cases} 1 + \beta = 0 \\ 2 + \alpha - 3\beta = 0 \\ -3 - \gamma = 0 \end{cases} \quad (6.65)$$

The solution of this system gives the desired values for α , β and γ . It follows that $\alpha = -5$, $\beta = -1$ and $\gamma = -3$ and therefore:

$$\Pi_1 = K_N = \frac{N}{d^5 n^3 \rho} \quad (6.66)$$

By repeating this calculation for the first independent variable, which has not been used as repeating variables in the incomplete group (the diameter of the vessel D), we have:

$$\Pi_2 = D d^\alpha \rho^\beta n^\gamma \quad (6.67)$$

$$M^0 L^0 T^0 = L(L)^\alpha (ML^{-3})^\beta (T^{-1})^\gamma \quad (6.68)$$

$$M^0 L^0 T^0 = M^0 L^{(1+\alpha-3\beta)} T^{(-\gamma)} \quad (6.69)$$

$$\begin{cases} \beta = 0 \\ 1 + \alpha - 3\beta = 0 \\ \gamma = 0 \end{cases} \quad (6.70)$$

With these values for α , β and γ ($\alpha = -1$, $\beta = 0$ and $\gamma = 0$) the second pi group is:

$$\Pi_2 = \frac{D}{d} \quad (6.71)$$

For the other geometric factors, we obtain the next dimensionless relationships:

$$\Pi_3 = \frac{H}{d}, \Pi_4 = \frac{h}{d}, \Pi_5 = \frac{b}{d}$$

The remaining two pi groups are now identified. For the non-repeating variable η , the dimensional analysis calculation shows that:

$$\Pi_6 = \eta d^\alpha \rho^\beta n^\gamma \quad (6.72)$$

$$M^0 L^0 T^0 = ML^{-1} T^{-1} (L)^\alpha (ML^{-3})^\beta (T^{-1})^\gamma \quad (6.73)$$

$$M^0 L^0 T^0 = M^{(1+\beta)} L^{(-1+\alpha-3\beta)} T^{(-1-\gamma)} \quad (6.74)$$

$$\begin{cases} 1 + \beta = 0 \\ -1 + \alpha - 3\beta = 0 \\ -1 - \gamma = 0 \end{cases} \quad (6.75)$$

The solution to this system is $\alpha = -2$, $\beta = -1$ and $\gamma = -1$ and the sixth pi group can be written as:

$$\Pi_6 = \text{Re} = \frac{n d^2 \rho}{\eta} \quad (6.76)$$

This criterion is recognized as the Reynolds number for mixing in a fluid.

The last non-repeating independent variable included in the list of variables gives the next formulation for the seventh pi group and generates all the calculation procedures for the identification of α , β and γ :

$$\Pi_7 = \sigma d^\alpha \rho^\beta n^\gamma \quad (6.77)$$

$$\text{M}^0 \text{L}^0 \text{T}^0 = \text{M} \text{T}^{-2} (\text{L})^\alpha (\text{M} \text{L}^{-3})^\beta (\text{T}^{-1})^\gamma \quad (6.78)$$

$$\text{M}^0 \text{L}^0 \text{T}^0 = \text{M}^{(1+\beta)} \text{L}^{(\alpha-3\beta)} \text{T}^{(-2-\gamma)} \quad (6.79)$$

$$\begin{cases} 1 + \beta = 0 \\ \alpha - 3\beta = 0 \\ -2 - \gamma = 0 \end{cases} \quad (6.80)$$

The group identified by the introduction of α , β and γ values ($\alpha = 3$, $\beta = -1$ and $\gamma = -2$) into Eq. (6.77) is called the Weber number for mixing in a fluid. We can observe that, in this case, as in the previous one for the Re number, the original pi groups are transformed by the inversion of the terms of their algebraic fraction:

$$\Pi_7 = \text{We} = \frac{n^2 \rho d^3}{\sigma} \quad (6.81)$$

As in the previous examples, the next step (step 7) of the dimensional analysis procedure (which is not presented here) allows one to confirm that the obtained criteria are dimensionless. Now, finally, we can state the result of the dimensional analysis as:

$$K_N = f\left(\frac{D}{d}, \frac{H}{d}, \frac{h}{d}, \frac{b}{d}, \text{Re}, \text{We}\right) \quad (6.82)$$

The transformation of this relationship into the frequently used relationship for the theoretical power consumption for mixing in a fluid (Eq. (6.83)) is easily obtained. The We group relationship with K_N and the geometry dependence of the mixing constants a and b are needed for this transformation:

$$N = a d^{5-b} n^{3-b} \rho^{1-b} \eta^b \quad (6.83)$$

When the mixing time τ_M represents the dependent variable of the mixing in the fluid, all the independent variables used for the power consumption remain as variables affecting the mixing time. We also have to introduce a specific indepen-

dent variable which is the A diffusion coefficient (D_A). The totality of the variables for this case will be:

$$\tau_M = f(d, D, H, h, b, n, \rho, \eta, \sigma, D_A) \quad (6.84)$$

Because the dimensional formula of D_A does not introduce a new basic dimension, we can establish that, in this case, the number of pi groups is 8 (eleven physical variables and three basic dimensions). If we use the incomplete group of the repeating variables, as in the case of the dependence of the power consumption factors, then we have to replace the K_N group by a group introduced by the new dependent variable (τ_M) and complete the established seven with a new group which includes the D_A factor. In this case, the formulation of the first pi group is given by Eq. (6.85):

$$\Pi_1 = \tau_M d^\alpha \rho^\beta n^\gamma \quad (6.85)$$

Applying the dimensional analysis procedure, we identify $\alpha = 0$, $\beta = 0$ and $\gamma = -1$ thus:

$$\Pi_1 = \tau_M n \quad (6.86)$$

As far as the Π_2 – Π_7 groups are the same as those identified in the case of the power-factor dependence, we can identify the eighth pi group:

$$\Pi_8 = D_A d^\alpha \rho^\beta n^\gamma \quad (6.87)$$

Exponents α , β and γ have been identified by the following relationships:

$$M^0 L^0 T^0 = L^2 T^{-1} (L)^\alpha (ML^{-3})^\beta (T^{-1})^\gamma \quad (6.88)$$

$$M^0 L^0 T^0 = M^\beta L^{(2+\alpha-3\beta)} T^{(-1-\gamma)} \quad (6.89)$$

$$\begin{cases} \beta = 0 \\ 2 + \alpha - 3\beta = 0 \\ -1 - \gamma = 0 \end{cases} \quad (6.90)$$

from Eq. (6.90) we get that $\alpha = -2$, $\beta = 0$ and $\gamma = -1$ and that the Π_8 expression could be written as:

$$\Pi_8 = \frac{D_A}{nd^2} \quad (6.91)$$

This criterion is recognized as the Fourier number for mixing time in liquid media. Finally, for this case, we can express the result of dimensional analysis as:

$$\tau_M n = f\left(\frac{D}{d}, \frac{H}{d}, \frac{h}{d}, \frac{b}{d}, \frac{nd^2 \rho}{\eta}, \frac{n^2 \rho d^3}{\sigma}, \frac{D_A}{nd^2}\right) \quad (6.92)$$

This result indicates that this problem can generally be studied in terms of eight pi terms, or – for a fixed geometry – in terms of four pi terms, instead of the original eleven variables we started with. It also shows the complexity of this currently used chemical engineering operation.

6.4 Supplementary Comments about Dimensional Analysis

Despite the fact that other methods can be used to identify pi groups [6.22], we think that the method of the incomplete group of repeating variables explained in the preceding section, provides a systematic procedure for performing a dimensional analysis that can be easy enough for beginners. Pi terms can also be formed by inspection, as will be briefly discussed in the next sections. Regardless of the basis of dimensional analysis application for a concrete case, certain aspects of this important tool must seem a little baffling and mysterious to beginners and sometimes to experienced researchers as well.

In this section, we will show some of the guidelines required for a logical good start in a particular dimensional analysis. First, we need to have a good knowledge of the case being studied; this condition is one of the most important for successful application of this method. Some methodology guidelines will also be presented to establish a mathematical model (see, for example, the case of the conditions of univocity for the mathematical model of a particular process.)

6.4.1 Selection of Variables

One of the most important and difficult steps when applying dimensional analysis to any given problem, is the selection of the variables that are involved (see for example the introduction in the natural convection application presented in the preceding section). No simple procedure allows the variables to be easily identified. Generally, one must rely on a good understanding of the phenomena involved and of their governing physical laws. If extraneous variables are included, too many pi terms appear in the final solution, and it may then be difficult, and time and money consuming, to eliminate them experimentally. However, when important variables are omitted, an incorrect result will be produced.

These two aspects (introduction of extraneous variables and omission of important variables) show that enough time and attention has to be given when the variables are determined. Most chemical engineering problems involve certain simplifying assumptions that have an influence on the variables to be considered. Usually, a suitable balance between simplicity and accuracy is a required goal. The accuracy of the solution to be chosen depends on the objective of the study. For example, if we are only concerned with the general trends of the process, some variables that are thought to have a minor influence could be neglected for simplicity.

For all the engineering branches that use dimensional analysis as a methodology, the pertinent variables of one process can be classified into four groups:

- the variables describing the geometry of the system when the process occurs,
- the variables showing the properties of the materials involved in the evolution of the process being analyzed,
- the variables showing the internal dynamics of the process,
- the variables imposed by the external effects and having an important influence on the process dynamics.

6.4.1.1 Variables Imposed by the Geometry of the System

The geometric characteristics can usually be described by a series of lengths and angles. The application related to the mixing in a liquid medium (described above) shows the importance of geometry variables in a dimensional analysis problem. As in the above-mentioned case, the geometry of the system plays an important role in the majority of chemical engineering problems. Thus, a sufficient number of geometric variables must be included to describe the system. These variables can usually be identified quickly.

6.4.1.2 Variables Imposed by the Properties of the Materials

Fluid flow, heating and composition, which change by reaction or by transfer at one interface, represent the specificity of the chemical engineering processes. The response of a system to the applied effects that generate the mentioned cases depends on the nature of the materials involved in the process. All the properties of the materials such as density, viscosity, thermal capacity, conductivity, species diffusivity or others relating the external effects to the process response must be included as variables. The identification of these variables is not always an easy task. A typical case concerns the variation of the properties of the materials, in a nonlinear dependence with the operation variables. For example, when studying the flow of complex non-Newtonian fluids such as melted polymers in an externally heated conduct, their non-classical properties and their state regarding the effect of temperature make it difficult to select the properties of the materials.

6.4.1.3 Dynamic Internal Effects

Variables, such as the heat or mass transfer coefficients from or to the interface or the flow friction coefficient for a given geometry, represent variables that can be included in this group. They have a dynamic effect on the process state and generally represent the dependent variables of the process.

6.4.1.4 Dynamic External Effects

This definition is used to identify any visible variables that produce or tend to produce a change in the process. Pressure, velocity, gravity and external heating are some of the most frequently used variables from this group.

Since we wish to keep the number of variables to a minimum, it is important to have a selected list which contains only independent variables. For example, in the case of a flow problem, if we introduce the equivalent flow diameter (d_e), we do not have to introduce the flow area (A) nor the wetted perimeter (P) into the list of variables, because both variables have already been taken into consideration by the equivalent flow diameter ($d_e = 4A/P$). Generally, if we have a problem in which the variables are:

$$f(y, x_1, x_2, x_3, x_4, \dots, x_n) \quad (6.93)$$

and it is known that an additional relationship exists among some of the variables, for example:

$$x_3 = f(x_4, \dots, x_n) \quad (6.94)$$

then x_3 is not required and can be omitted. Conversely, if it is known that the variables x_4, x_5, \dots, x_n can only be taken into account through the relationship expressed by the functional dependence (6.94), then the variables x_4, x_5, \dots, x_n can be replaced by the single variable x_3 , thus reducing the number of variables.

In addition to these supplementary comments about dimensional analysis, we can also discuss the following points, which are necessary to establish the list of variables. To do so, indeed, we have to:

1. Define the problem clearly using a descriptive model and auxiliary graphic presentation. Establish the main variable of interest (which is the dependent variable of the process).
2. Consider the basic laws that govern the phenomenon or accept an empirical theory describing the essential aspects of the investigated process as an open procedure for identifying independent variables.
3. Start the identification of the variables process by grouping them into the four groups of variables presented above (geometry, material properties, internal dynamic effects and external dynamic effects).
4. Verify whether other variables not included in the four groups of variables are important and must be considered and ensure that the dimensional constant, which can be introduced in the list of variables, has been accepted.
5. Make sure that all variables are independent and, to this end, the relationships among the subsets of the variables must be carefully observed.

6.5

Uniqueness of Pi Terms

A review of the method of an incomplete group of repeating variables used for identifying pi terms reveals that the specific pi terms obtained depend on the somewhat arbitrary selection of this incomplete group. For example, in the problem of studying the heat transfer from a wall to a fluid flowing in the pipe, we have selected d , ρ , η , and λ as repeating variables. This has led to the formulation of the problem in terms of pi terms:

$$\frac{\alpha d}{\lambda} = F\left(\frac{wd\rho}{\eta}, \frac{c_p\eta}{\lambda}\right) \quad (6.95)$$

What will the result be if we select d , ρ , η , and c_p as repeating variables? A quick check will reveal that the pi term involving the heat transfer coefficient (α) becomes:

$$\Pi_1 = \frac{\alpha d}{\eta c_p}$$

and the next pi terms remain the same. Thus, we can express the second result as:

$$\frac{\alpha d}{\eta c_p} = F_1\left(\frac{wd\rho}{\eta}, \frac{c_p\eta}{\lambda}\right) \quad (6.96)$$

Both results are correct, and will lead to the same final equation for α . Note, however, that the functions F and F_1 in Eqs. (6.95) and (6.96) will be different because the dependent pi terms are different for both relationships. From this example, we can conclude that there is no unique set of pi terms arising from a dimensional analysis. Nevertheless, the required number of pi terms has been fixed, and once a correct set has been determined, other possible sets can be developed by a combination of the products of the powers of the original set. This is a classical algebra problem, which shows that, if we have n independent variables (the pi terms obtained by the incomplete group method *are* independent variables), then each of these can be modified by a combination of the others and the resulting new set has n independent variables. For example, if we have a problem involving three pi terms: $\Pi_1 = F(\Pi_2, \Pi_3)$, we could form a new set from this initial one by combining the pi terms in order to form the new pi term Π_2' , and to give $\Pi_2' = \Pi_2^\alpha \Pi_3^\beta$, where α and β are arbitrary exponents. For $\alpha = -1$ and $\beta = 0$ we obtain the inversion of the Π_2 expression group. Then the relationship between the dimensionless groups could be expressed as:

$$\Pi_1 = F_1(\Pi_2', \Pi_3) \quad \text{or} \quad \Pi_1 = F_2(\Pi_2, \Pi_2')$$

It must be emphasized, however, that the required number of pi terms cannot be reduced by this manipulation; only their form is altered. Thanks to this technique,

we can see that the pi terms in Eq. (6.96) can be obtained from those presented in Eq. (6.95); then, if we multiply Π_1 from Eq. (6.95) by Π_3^{-1} we have:

$$\left(\frac{\alpha d}{\lambda}\right) \left(\frac{c_p \eta}{\lambda}\right)^{-1} = \frac{\alpha d}{\eta c_p}$$

which is the Π_1 of Eq. (6.96).

One may ask: which form is the best for the pi groups? Usually, we recommend keeping pi terms as simple as possible. In addition, it is easier to use the pi terms that could be improved by the experimental methodology used. The final choice remains arbitrary and generally depends on the researcher's background and experience.

6.6 Identification of Pi Groups Using the Inspection Method

The previously presented method of the incomplete group of repeating variables, provides a systematic procedure which, when properly executed, provides a correct, complete and unique set of pi terms. In other words, this method offers an excellent algorithm for the calculus. In this case, only the list of variables has to be determined by the researcher. Since the only restrictions for the pi terms are to be (a) correct in number, (b) dimensionless, and (c) independent, it is possible to produce other identifying procedures. One of them is the production of pi terms by inspection, without resorting to a more formal methodology.

To illustrate this approach, we will consider a new example: the case of a simple tubular membrane reactor for which we wish to show the dependence between the conversion (η_r) of the main reactant and other variables which influence the process.

The membrane reactor shown in Fig. 6.5 consists of a tubular shell containing a tubular porous membrane. It defines two compartments, the inner and the outer (shell) compartments. The reactants are fed into the inner compartment where the reaction takes place. We can observe that when the reactants flow along the reactor, one or more of the reaction participants can diffuse through the porous membrane to the outer side. In this case, we assume that only one participant presents a radial diffusion. This process affects the local concentration state and the reaction rate that determine the state of the main reactant conversion. The rate of reaction of the wall diffusing species is influenced by the transfer resistance of the boundary layer ($1/k_c$) and by the wall thickness resistance (δ/D_p).

As geometric variables, we can consider the diameter (d) and the length of the tubular reactor (l). The apparent constant rate of the chemical reaction (k_r) and the diffusion coefficient (D_m) of the species diffusing through the wall could be chosen as the internal dynamic variables of the process. The variables showing the properties of the materials (density and viscosity) as well as the variables characterizing the flow and the velocity (w) for example, can be considered, but these

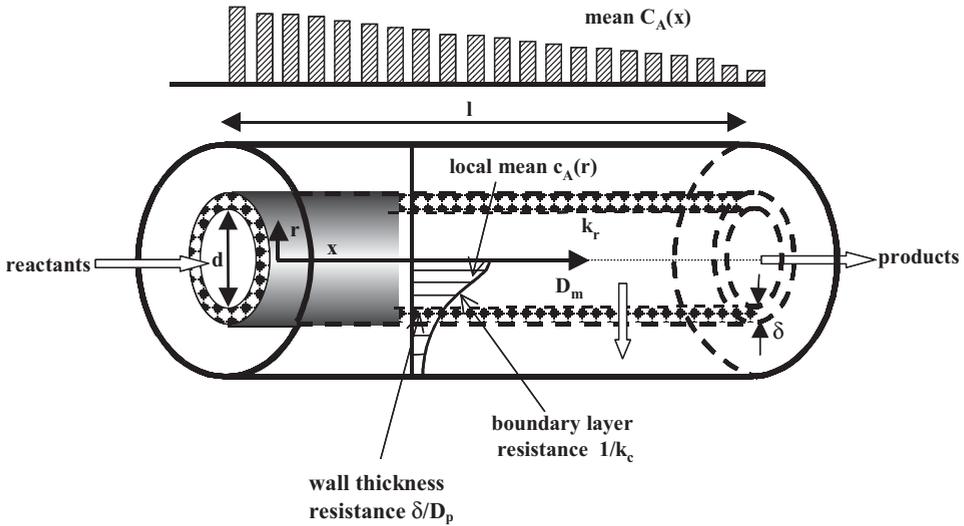


Figure 6.5 Principle of the simple tubular membrane reactor.

are already introduced by the mass transfer coefficient (k_c). With this descriptive introduction, we can appreciate that, in this case, the variables are:

$$\eta_r = f(d, l, k_r, k_c, D_m, D_p/\delta) \quad (6.97)$$

Using M, L and T as basic dimensions, the following dimensional formulae of the variables are obtained:

$$[\eta_r] = M^0 L^0 T^0$$

$$[d] = L$$

$$[l] = L$$

$$[k_r] = T^{-1}$$

$$[k_c] = LT^{-1}$$

$$[D_m] = L^2 T^{-1}$$

$$\left[D_p/\delta \right] = LT^{-1}$$

In this dimensional analysis problem, five pi terms are needed because we have seven variables and two reference dimensions. The first pi term is represented by the conversion of the main reactant because this variable is dimensionless. The construction of the second pi group begins with variable d . This has a length

dimension (L) and to form a dimensionless group, it must be multiplied by a variable the dimension of which is L^{-1} :

$$\Pi_2 = \frac{d}{l}$$

Inspecting the remaining variable, we observe that $[k_r d^2] = L^2 T^{-1}$. Then, by multiplying $k_r d^2$ by $1/D_m$, we obtain the third pi group:

$$\Pi_3 = \frac{D_m}{k_r d^2}$$

The formulation of the fourth pi group (Π_4) takes into account the observation that k_c and D_p/δ present the same dimensional formula and that their ratio is therefore dimensionless:

$$\Pi_4 = \frac{k_c \delta}{D_m}$$

The last pi group (Π_5) can be obtained by multiplying $k_c d$ (which has $L^2 T^{-1}$ dimension) by $1/D_m$, which also results in a dimensionless formula:

$$\Pi_5 = \frac{k_c d}{D_m}$$

The last three pi groups are well known in chemical engineering (Π_3 is recognized as the Fourier reaction number (Fo_r), Π_4 is the famous Biot diffusion number (Bi_d) and Π_5 is the Sherwood number (Sh)).

Relationship (6.98) shows the last result of this particularized case of dimensional analysis.

$$\eta_r = f(d/l, Fo_r, Bi_d, Sh) \quad (6.98)$$

It is important to note that when pi terms are formulated by inspection, we have to be certain that they are all independent. In this case or in any other general case, no pi group could result from the combination of two or more formulated pi groups. The inspection procedure of forming pi groups is essentially equivalent to the incomplete group method but it is less structured.

6.7

Common Dimensionless Groups and Their Relationships

Approximately three hundred dimensionless groups [6.23] are used to describe the most important problems that characterize chemical engineering processes. Out of these, only a limited number is frequently used and can be classified according to the flow involved in the investigated process, the transport and interface transfer of one property (species, enthalpy, pressure) and the interactions of the transport mechanisms of the properties. In order to be considered in this anal-

ysis, the dimensionless groups have to present the following general characteristics:

- Each dimensionless group provides a physical interpretation, which can be helpful in assessing its influence in a particular application.
- The dimensionless groups that characterize a particular application are correlated with others by the dimensional analysis relationship.
- When the process involves a transfer through an interface, some of the relationships between the dimensionless groups can be considered as relationships between the kinetic transfer and the interface properties.

6.7.1

Physical Significance of Dimensionless Groups

The physical interpretation of each dimensionless group is not an easy task. Because each dimensionless group can present various physical interpretations, the study of each particular dimensionless pi term has to be carefully carried out.

To illustrate this, we will discuss the example shown in Fig. 6.6, which presents one deformable fluid particle moving along a streamline. We can describe this system taking into account inertia, resistive (viscous) force and weight force. The magnitude of the inertia force along the streamline can be written as:

$$F_i = ma_s = m \frac{dw_s}{d\tau} = m \frac{dw_s}{ds} \frac{ds}{d\tau} = mw_s \frac{dw_s}{ds} \quad (6.99)$$

where ds is measured along the streamline and m is the particle mass. Based on the fact that a streamline is representative of a flow geometry when a mean flow rate, w , and a characteristic length are known, we can produce the dimensionless transformation for w_s and dw_s/ds . The dimensionless velocity and streamline position are respectively $w_{as} = w_s/w$ and $s_a = s/l$. Then Eq. (6.99) becomes:

$$F_i = m \frac{w^2}{l} w_{as} \frac{dw_{as}}{ds_a} \quad (6.100)$$

The weight force is described by $F_g = mg$, then the ratio between the inertia and the gravitational force is:

$$\frac{F_i}{F_g} = \frac{w^2}{gl} w_{as} \frac{dw_{as}}{ds_a} \quad (6.101)$$

The ratio between forces F_i/F_g is proportional to w^2/gl and its square root (w/\sqrt{gl}) is recognized as the Froude number. Its physical interpretation is the index of the relative importance of the inertial forces acting on the fluid particles with respect to the weight of these particles.

Now we consider the resistive force characterizing the movement of the particle along the streamline expressed as the product between tensor τ_{ss} and its normal surface A ($A = m/\rho \cdot s_d$, where s_d is the apparent height of the deformed particle)

$$F_{rs} = \tau_{ss}A = \eta \frac{dw_s}{ds} \frac{m}{\rho s_d} \tag{6.102}$$

Using the dimensionless velocity and streamline position, and completing these values with the dimensionless height of the deformable particle $s_{da} = s_d/l$, Eq. (6.102) can be written as:

$$F_{rs} = \frac{\eta}{\rho l^2} \frac{m}{s_{da}} \frac{dw_{as}}{ds_a} \tag{6.103}$$

Then, the ratio between the inertia and the resistive forces is:

$$\frac{F_i}{F_{rs}} = \frac{wl\rho}{\eta} \frac{w_{as}}{s_{da}} \frac{dw_{as}}{ds_a} = Re \frac{w_{as}}{s_{da}} \frac{dw_{as}}{ds_a} \tag{6.104}$$

Here we can identify the Reynolds number (Re), which is a measure or an index of the relative importance of the inertial and resistive (viscous) forces acting on the fluid. If we write the general expression for the s direction rate of one property when the transport is molecular and convective, we have:

$$\vec{J}_{tAs} = -D_{\Gamma A} \frac{d\vec{\Gamma}_A}{ds} + \vec{w}_s \Gamma_A$$

we can obtain another physical interpretation for the Reynolds number, after particularizing the momentum transfer and replacing the corresponding terms ($D_{\Gamma A} = \nu = \eta/\rho$, $\Gamma_A = \rho w_s$, $\vec{J}_{tAs} = \vec{\tau}_{tsy}$). This particularization gives:

$$\vec{\tau}_{tsy} = -\eta \frac{d\vec{w}_s}{dy} + \vec{w}_s (\rho w_s) \tag{6.105}$$

Using the dimensionless velocity, we can write Eq. (6.106), which presents the ratio between the right-hand side terms of Eq. (6.105).

$$\frac{w_s (\rho w_s)}{\eta \frac{dw_s}{dy}} = cw_{as} \frac{\rho wl}{\eta} = cw_{as} Re \tag{6.106}$$

Equation (6.106) shows that the Reynolds number expresses the relationship between the momentum quantity supplied by the convection and the momentum quantity supplied by the molecular movement. At the same time, because the convective mechanism can be associated with the presence of the turbulence, we can consider the following ratio:

$$Re = \frac{\text{Momentum quantity transferred by turbulent mechanism}}{\text{Momentum quantity transferred by molecular mechanism}}$$

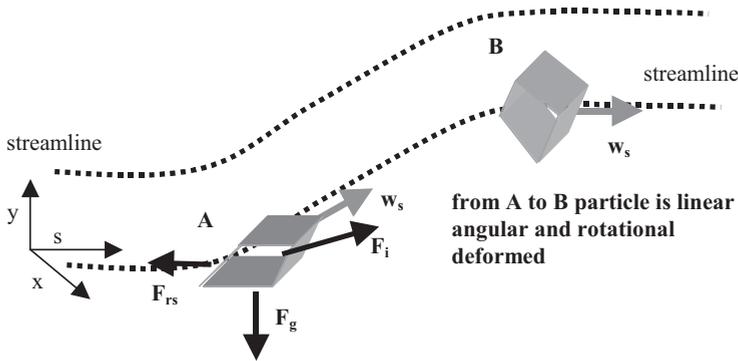


Figure 6.6 Particle moving along a streamline.

The case of the Reynolds number discussed above shows that the physical interpretation of one dimensionless group is not unique. Generally, the interpretation of dimensionless groups used in the flow area in terms of different energies involved in the process, can be obtained starting with the Bernoulli flow equation. The relationship existing between the terms of this equation introduces one dimensionless group.

6.6.2

The Dimensionless Relationship as Kinetic Interface Property Transfer Relationship

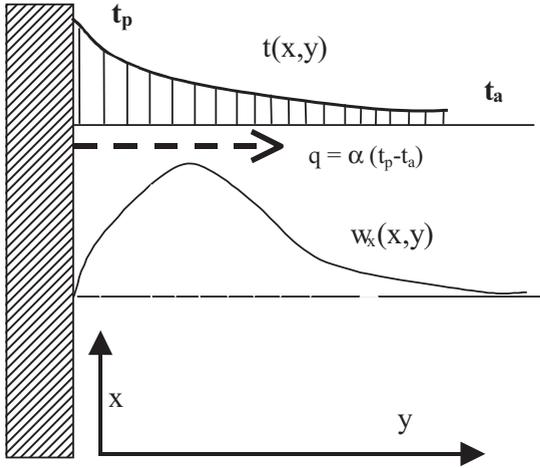
We begin this section by analyzing the case of free convection in an infinite medium. The example chosen is shown in Fig. 6.7. A two-dimensional surface with constant temperature t_p transfers heat to the adjacent infinite media. As a result of the temperature difference between the surface and the media, a natural convection flow is induced. A dimensional analysis applied to this problem shows that:

$$Nu = f(Gr_t)$$

where the Nusselt number (Nu) and the Grashof number for thermal convection (Gr_t) are given by:

$$Nu = \frac{\alpha d}{\lambda}, \quad Gr_t = \frac{g\beta_t \Delta t H^3 \rho^2}{\eta^2} \quad (6.107)$$

The goal of this analysis is to obtain a relationship describing the kinetics of the heat transfer from the heated two-dimensional plate to the adjacent medium. This relationship is one dimensionless pi group. Moreover, we can use this example as a guide for the introduction of the relationships existing between dimensionless groups such as the relationships for the property transfer kinetics. To write the mathematical model for the problem of infinite medium natural heat convection, we use the particularization of the property transport equations. The corresponding equations were previously established in Chapter 3.



Variables:

-Geometry:
height of the heated plate H

-Material:
Fluid density ρ ,
Fluid viscosity η ,
Fluid thermal conductivity λ ,
Fluid thermal capacity c_p .

-Internal dynamics :
Specific ascension force ; $g\beta_t\Delta t$
Heat transfer coefficient α

Figure 6.7 Heat transfer by natural convection from a plate to an infinite medium.

The momentum and energy transfer equation for the presented case may be written as:

$$w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y} = \nu \frac{\partial^2 w_x}{\partial y^2} + \beta_t g(t - t_\infty) \tag{6.108}$$

$$w_x \frac{\partial t}{\partial x} + w_y \frac{\partial t}{\partial y} = a \frac{\partial^2 t}{\partial y^2} \tag{6.109}$$

$$\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} = 0 \tag{6.110}$$

where $\nu = \eta/\rho$ is the kinematic viscosity and $a = \lambda/\rho c_p$ is the thermal diffusivity of the medium.

The boundary conditions attached to the problem are:

$$y = 0, 0 < x < H, w_x = 0, w_y = 0, t = t_p \tag{6.111}$$

$$y = \infty, 0 < x < H, w_x = 0, t = t_\infty \tag{6.112}$$

$$x = 0, y = 0, w_x = w_y = 0, t = t_p \tag{6.113}$$

We can now introduce the following dimensionless notation:

$$Gr_x = \frac{g\beta_t(t_p - t_\infty)x^3}{\nu^2}, \quad \xi = \frac{y}{x} \left(\frac{Gr_x}{4} \right)^{1/4} \tag{6.114}$$

where Gr_x is the local Grashof number and ξ is a combination of the cartesian coordinate. Therefore, if we use the method of the stream function ψ for the transformation of the original model, we can write:

$$\psi = 4\nu \left(\frac{Gr_x}{4} \right)^{1/4} \varphi(\xi) \quad (6.115)$$

and then:

$$w_x = \frac{d\psi}{d\xi} \frac{d\xi}{dy} = \left(\frac{g\beta_t(t_p - t_\infty)}{4\nu^2} \right) 4\nu x^{1/2} \varphi'(\xi) \quad (6.116)$$

$$w_y = - \frac{d\psi}{d\xi} \frac{d\xi}{dx} = \nu x^{-1/4} \left(\frac{g\beta_t(t_p - t_\infty)}{4\nu^2} \right)^{1/4} [\xi \varphi'(\xi) - 3\varphi(\xi)] \quad (6.117)$$

With these conditions, the equations of the original model can be written as:

$$\varphi''(\xi) + 3\varphi(\xi)\varphi'(\xi) - 2[\varphi'(\xi)]^2 + \theta(\xi) = 0 \quad (6.118)$$

$$\theta''(\xi) + Pr \varphi(\xi)\theta'(\xi) = 0 \quad (6.119)$$

where $\theta(\xi) = \frac{t(\xi) - t_\infty}{t_p - t_\infty}$ is the dimensionless temperature and Pr represents the Prandtl number.

The boundary conditions are now the following:

$$\xi = 0 \quad \varphi = \varphi' = 0 ; \theta = 1 \quad (6.120)$$

$$\xi = \infty \quad \varphi' = 0 ; \theta = 0 \quad (6.121)$$

With the approximation of $\varphi''(0)$ and $\theta'(0)$, the model represented by assembling Eqs. (6.118)–(6.121) can be readily solved by an adequate numeric method. The Prandtl number is, in this case, a parameter of numerical integration.

The heat transfer kinetics is represented by the heat flux produced and transferred by the plate.

$$q = \lambda \left(\frac{dt}{dy} \right)_{y=0} = \alpha_x (t_p - t_\infty) \quad (6.122)$$

The preceding expression can also be written as:

$$Nu_x = \frac{\alpha_x x}{\lambda} = \frac{\lambda \left(\frac{dt}{dy} \right)_{y=0} x}{\lambda (t_p - t_\infty)} = \frac{x \left(\frac{dt}{dy} \right)_{y=0}}{(t_p - t_\infty)} \quad (6.123)$$

When we introduce the combined variable ξ and the dimensionless temperature $\theta(\xi)$ into Eq. (6.123) we have:

$$Nu_x = \frac{x \left[\frac{d\theta(\xi)}{d\xi} \frac{d\xi}{dy} \frac{dy}{dx} \right]_{\xi=0}}{(t_p - t_\infty)} = -\theta'(0) \left(\frac{Gr_x}{4} \right)^{1/4} \tag{6.124}$$

and

$$Nu = \frac{1}{H_0} \int_0^H Nu_x dx = -0.404\theta(0)Gr^{1/4} \tag{6.125}$$

The values used to calculate the temperature gradient and the velocity gradient near the heated vertical plate are given in Table 6.1.

Table 6.1 Some values of the temperature and velocity gradients at the surface in the case of natural convection heat transfer.

$Pr = c_p \eta / \lambda$	0.01	0.793	1	2	10	100	1000
$-\theta'(0)$	0.0812	0.5080	0.5671	0.7165	1.1694	2.191	3.966
$\varphi''(0)$	0.9862	0.6741	0.6421	0.5713	0.4192	0.2517	0.1450

When the heated medium is air ($Pr \approx 0.793$) Eq. (6.125) takes the value of 0.508 for $-\theta'(0)$, then we have:

$$Nu = 0.205Gr_i^{1/4} \tag{6.126}$$

In the Nusselt and Grashof numbers, the height of the heated plate is the characteristic length.

The example presented here allows these important conclusions:

- when we solve the transport property equation, we obtain a dimensionless relationship which characterizes the kinetics of the transfer for the property near the interface,
- the form of the obtained relationship can be simplified to a power type dependence.

Now we will consider the case of a transferable property for the contact between two phases. The transfer kinetics is characterized by the two transfer coefficients of the property given by Eq. (3.15). If we analyze the transport process with reference only to one phase, then we can write:

$$k_\Gamma = \frac{-D_\Gamma \left[\frac{d\Gamma}{dx} \right]_{x=x_i}}{(\Gamma_\infty - \Gamma_{i_i})} \tag{6.127}$$

where the index of the phase definition has been omitted and index i indicates the interface position. If the preceding relationship is multiplied by the ratio between

the characteristic length and the diffusion coefficient of the property (l/D_Γ), the relationship becomes:

$$\frac{k_\Gamma l}{D_\Gamma} = \frac{\left[\frac{d\Gamma}{d(x/l)} \right]_{x=x_i}}{(\Gamma_i - \Gamma_{\infty_i})} \quad (6.128)$$

This equation can also be written as:

$$Nu_\Gamma = \left(\frac{d\Gamma_a}{dx_a} \right)_{x_a=x_{ai}} \quad (6.129)$$

where Γ_a is the concentration of the dimensionless property and x_a represents the dimensionless transport coordinate. Nu_Γ is used here as a generalized Nusselt number and gives the transport kinetics for any kind of property (heat, species, etc.). The dimensionless groups' relationship that is able to explain the property gradient near the interface, in terms of other dimensionless groups characterizing the process, can be obtained from Eq. (6.129) if we consider that the interface is a plane given by the equation $x = x_i$. For this separated phase (for example, the left side of the interface), the flow is considered as two-dimensional with a normal and parallel direction with respect to the interface. We consider that the steady state flow and the participating natural convection are not excluded. The considered flow is similar to that shown in Fig. 6.6. The continuity of x and y could be written using the Navier–Stokes equations:

$$\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} = 0 \quad (6.130)$$

$$w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y} = \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 w_x}{\partial x^2} + \frac{\partial^2 w_x}{\partial y^2} \right) \quad (6.131)$$

$$w_x \frac{\partial w_y}{\partial x} + w_y \frac{\partial w_y}{\partial y} = g_y \beta_\Gamma \Delta \Gamma + \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 w_y}{\partial x^2} + \frac{\partial^2 w_y}{\partial y^2} \right) \quad (6.132)$$

The flow equations are completed with the corresponding boundary conditions, which, for example, show:

- a maximum velocity at the interface coupled with a constant pressure:

$$x = x_i, \quad -y_v \leq y \leq y_v; \quad w_x = 0, \quad \frac{dw_y}{dx} = 0, \quad p = p_i \quad (6.133)$$

- the absence of the velocity component for planes $y = y_v$ and $y = -y_v$ (normal planes at the interface) coupled with the linear pressure state:

$$0 \leq x \leq x_i, \quad y = -y_v, \quad y = y_v; \quad w_x = w_y = 0, \quad p = p_i + \rho g(x_i - x) \quad (6.134)$$

- a particular velocity state for plane $x = 0$:

$$x = 0, \quad -y_v \leq y \leq y_v; \quad w_y = f(y), \quad w_x = 0 \quad (6.135)$$

The convective-diffusion equation characterizes the transport of the property for the fluid placed on the left of the interface. Here, the property participates in a reaction process, which is described by simple kinetics. Then, the particularization of the convective property transport equation becomes:

$$w_x \frac{\partial \Gamma}{\partial x} + w_y \frac{\partial \Gamma}{\partial y} = D_\Gamma \left(\frac{\partial^2 \Gamma}{\partial x^2} + \frac{\partial^2 \Gamma}{\partial y^2} \right) - k_r \Gamma \quad (6.136)$$

The first boundary condition for the convective-diffusion equation shows that, at the interface, the property flux is written using the transfer property coefficient:

$$x = x_i, \quad -y_v \leq y \leq y_v; \quad k_\Gamma (\Gamma_i - \Gamma_0) = D_\Gamma \frac{d\Gamma}{dx} \quad (6.137)$$

If, for the second boundary condition, we consider a constant concentration (Γ_0) of the property at plane $x = 0$, we can write:

$$x = 0, \quad -y_v \leq y \leq y_v; \quad \Gamma = \Gamma_0 \quad (6.138)$$

The third boundary condition considers that planes $ys = s - y_v$ and $ys = sy_v$ are impermeable to the transferred property:

$$0 \leq x \leq x_i, \quad y = -y_v, \quad y = y_v; \quad \frac{d\Gamma}{dy} = 0 \quad (6.139)$$

The equations described above could be written in a dimensionless form taking into account different dimensionless parameters. They include a geometrical dimension such as the dimensionless coordinates $x_a = x/l$ and $y_a = y/l$; the dimensionless velocity, the pressure and property concentration:

$$w_x^a = \frac{w_x}{w}, \quad w_y^a = \frac{w_y}{w}, \quad p_a = \frac{p}{\Delta p}, \quad \Gamma_a = \frac{\Gamma - \Gamma_0}{\Gamma_i - \Gamma_0} \quad (6.140)$$

where w is a stable, real or computed velocity, characteristic of the system and Δp is the differential pressure ($p_0 - p_i$). With these dimensionless definitions, the basic model equations become:

$$\frac{\partial w_x^a}{\partial x_a} + \frac{\partial w_y^a}{\partial y_a} = 0 \quad (6.141)$$

$$\frac{w^2}{l} w_x^a \frac{\partial w_x^a}{\partial x_a} + \frac{w^2}{l} w_y^a \frac{\partial w_x^a}{\partial y} = \frac{\Delta p}{\rho l} \frac{\partial p_a}{\partial x_a} + \frac{v w}{l^2} \left(\frac{\partial^2 w_x^a}{\partial x_a^2} + \frac{\partial^2 w_x^a}{\partial y_a^2} \right) \quad (6.142)$$

$$\frac{w^2}{l} w_x^a \frac{\partial w_y^a}{\partial x_a} + \frac{w^2}{l} w_y^a \frac{\partial w_y^a}{\partial y_a} = g_y \beta_\Gamma \Delta \Gamma + \frac{\Delta p}{\rho l} \frac{\partial p_a}{\partial y} + \frac{v w}{l^2} \left(\frac{\partial^2 w_y^a}{\partial x_a^2} + \frac{\partial^2 w_y^a}{\partial y_a^2} \right) \quad (6.143)$$

$$\frac{w(\Gamma_i - \Gamma_0)}{1} w_x^a \frac{\partial \Gamma_a}{\partial x_a} + \frac{w(\Gamma_i - \Gamma_0)}{1} w_y^a \frac{\partial \Gamma_a}{\partial y_a} = \frac{D_\Gamma(\Gamma_i - \Gamma_0)}{l^2} \left(\frac{\partial^2 \Gamma_a}{\partial x_a^2} + \frac{\partial^2 \Gamma_a}{\partial y_a^2} \right) - k_{r\Gamma}[\Gamma_0 + \Gamma_a(\Gamma_i - \Gamma_0)] \quad (6.144)$$

If we multiply Eqs. (6.142) and (6.143) by $\frac{l^2}{uw}$ and Eq. (6.144) by $\frac{l^2}{D_\Gamma(\Gamma_i - \Gamma_0)}$, we obtain the new forms of this set of equations. These are dimensionless and consequently include coefficients, which are dimensionless groups or combinations of the dimensionless groups. Assembling Eqs. (6.145) and (6.154) shows the initial model in its dimensionless form:

$$\frac{\partial w_x^a}{\partial x_a} + \frac{\partial w_y^a}{\partial y_a} = 0 \quad (6.145)$$

$$\text{Re} \left(w_x^a \frac{\partial w_x^a}{\partial x_a} + w_y^a \frac{\partial w_x^a}{\partial y} \right) = \text{Eu. Re.} \frac{\partial p_a}{\partial x_a} + \left(\frac{\partial^2 w_x^a}{\partial x_a^2} + \frac{\partial^2 w_x^a}{\partial y_a^2} \right) \quad (6.146)$$

$$\text{Re} \left(w_x^a \frac{\partial w_y^a}{\partial x_a} + w_y^a \frac{\partial w_y^a}{\partial y_a} \right) = \text{Gr}_\Gamma \text{Re}^{-1} + \text{Eu. Re.} \frac{\partial p_a}{\partial y} + \left(\frac{\partial^2 w_y^a}{\partial x_a^2} + \frac{\partial^2 w_y^a}{\partial y_a^2} \right) \quad (6.147)$$

$$\text{Re.Pr}_\Gamma \left(w_x^a \frac{\partial \Gamma_a}{\partial x_a} + w_y^a \frac{\partial \Gamma_a}{\partial y_a} \right) = \left(\frac{\partial^2 \Gamma_a}{\partial x_a^2} + \frac{\partial^2 \Gamma_a}{\partial y_a^2} \right) - \text{Fo}_{r\Gamma} \Gamma_a \quad (6.148)$$

$$x_a = x_1^a, \quad -y_v^a \leq y_a \leq y_v^a; \quad w_x^a = 0, \quad \frac{dw_y^a}{dx_a} = 0, \quad p = p_1^a \quad (6.149)$$

$$0 \leq x_a \leq x_1^a, \quad y_a = -y_v^a, \quad y_a = y_v^a; \quad w_x^a = w_y^a = 0, \quad p_a = p_1^a + \rho g(x_1^a - x_a).l/\Delta p \quad (6.150)$$

$$x_a = 0, \quad -y_v^a \leq y_a \leq y_v^a; \quad w_y^a = f(y_a), \quad w_x^a = 0 \quad (6.151)$$

$$x_a = x_1^a, \quad -y_v^a \leq y_a \leq y_v^a; \quad \text{Nu}_\Gamma = \frac{d\Gamma_a}{dx_a} \quad (6.152)$$

$$x_a = 0, \quad -y_v^a \leq y_a \leq y_v^a; \quad \Gamma_a = 0 \quad (6.153)$$

$$0 \leq x_a \leq x_1^a, \quad y_a = -y_v^a, \quad y_a = y_v^a; \quad \frac{d\Gamma_a}{dy_a} = 0 \quad (6.154)$$

The formal solution for this complete dimensionless model can be obtained when the flow equations can be resolved separately. Then we obtain:

$$w_x^a = f(\text{Eu}, \text{Re}, \text{Gr}_\Gamma, x_a, y_a) \quad (6.155)$$

$$w_y^a = g(\text{Eu}, \text{Re}, \text{Gr}_\Gamma, x_a, y_a) \quad (6.156)$$

where f and g define any particular function.

The solution for the concentration state of the transferable property can be written as:

$$\Gamma_a = h(\text{Re}, \text{Pr}_\Gamma, \text{Fo}_{r\Gamma}, w_x^a, w_y^a, x_a, y_a) \quad (6.157)$$

The substitution of Eqs. (6.155) and (6.156) into Eq. (6.157) gives a new form to the concentration state of the transferable property:

$$\Gamma_a = F(\text{Eu}, \text{Gr}_\Gamma, \text{Re}, \text{Pr}_\Gamma, \text{Fo}_{r\Gamma}, x_a, y_a) \quad (6.158)$$

Using this last relationship, we can now appreciate the value of the concentration dimensionless gradient of the transferable property near the interface. Then, the result is:

$$\left(\frac{d\Gamma_a}{dx_a} \right)_{x_a=x_a^i} = G(\text{Eu}, \text{Gr}_\Gamma, \text{Re}, \text{Pr}_\Gamma, \text{Fo}_{r\Gamma}, i_x, i_y) \quad (6.159)$$

where G is the F function derivative and i_x and i_y are the geometric simplex (ratio between the interface coordinates and the characteristic geometrical length).

The combination of Eqs. (6.130) and (6.159) gives a relationship between the general dimensionless groups characterizing the interface kinetic transfer of one property:

$$\text{Nu}_\Gamma = G(\text{Eu}, \text{Gr}_\Gamma, \text{Re}, \text{Pr}_\Gamma, \text{Fo}_{r\Gamma}, i_x, i_y) \quad (6.160)$$

We can conclude that the transfer intensity is determined by pressure (introduced by the Euler number (Eu)) as well as by natural convection (expressed by the non-particularized Grashof number (Gr_Γ)), by controlled convection (given by the Reynolds number (Re)), by chemical reaction (expressed by the reaction Fourier number ($\text{Fo}_{r\Gamma}$)), by the transport properties of the medium (assigned by the Prandtl number (Pr_Γ)) and finally by the geometry of the system (shown by the geometrical simplex i_x, i_y). Moreover, some of these actions are over represented because they cannot be used together. For example, the pressure action produces a controlled flow, which is characterized by the Reynolds number. However, in Eq. (6.160), the Euler number is not an independent parameter and can consequently be eliminated. Another example shows that, in the case of an important convective action (turbulent flow), the effect of the natural convection can be neglected. The same consideration shows that, in the cases of pure natural convection flow, the Reynolds number is not important. Finally, in the case of gas transfer at moderate pressures and temperatures, the generalized Prandtl number presents a constant value and, consequently, its influence in the kinetic relationship is not important.

Table 6.2 sums up these considerations in the case of transfer with no chemical reaction. We observe, for example, that the geometry of the system, the Reynolds number and the generalized Prandtl number, determine the intensity of the property transfer in a liquid medium with a turbulent flow.

Table 6.2 Relationships of the kinetic transfer dimensionless groups.

Fluid	Type of flow	Particularization of Eq. (6.160)	Particularization of heat transfer	Particularization of mass transfer
Liquid	natural convection	$Nu_{\Gamma} = G(Gr_{\Gamma}, Pr_{\Gamma}, i_x, i_y)$	$Nu = G(Gr_t, Pr, i_x, i_y)$	$Sh = G(Gr_d, Sc, i_x, i_y)$
	forced convection	$Nu_{\Gamma} = G(Re, Pr_{\Gamma}, i_x, i_y)$	$Nu = G(Gr_t, Pr, i_x, i_y)$	$Sh = G(Re, Sc, i_x, i_y)$
Gas	natural convection	$Nu_{\Gamma} = G(Gr_{\Gamma}, i_x, i_y)$	$Nu = G(Gr_t, i_x, i_y)$	$Sh = G(Gr_d, i_x, i_y)$
	forced convection	$Nu_{\Gamma} = G(Re, i_x, i_y)$	$Nu = G(Re, i_x, i_y)$	$Sh = G(Re, i_x, i_y)$

It is important to note that the classification presented above is not unique. Indeed, each particular case has its G function. For example, when a chemical reaction occurs, the generalized Fourier number (Fo_{Γ}) and its particularization for heat and mass transfer can be introduced as a G function argument.

6.6.3

Physical Interpretation of the Nu, Pr, Sh and Sc Numbers

This section will present one of the possible physical interpretations of these important dimensionless numbers. First, to show the meaning of Nusselt number, we consider the heat transfer flux in the x direction in the case of a pure molecular mechanism compared with the heat transfer characterizing the process when convection is important. The corresponding fluxes are then written as:

$$q_m = -\lambda \left(\frac{dt}{dx} \right)_{x=x_i} \quad (6.161)$$

$$q_c = \alpha(t_i - t_{\infty}) \quad (6.162)$$

where α , λ and t have been defined above (for instance, see Fig. 6.7). Index i indicates the position of the interface. By analogy with the model already used to determine the significance of the Reynolds number, we calculate the ratio between both heat fluxes which is represented by the following relationship:

$$\frac{q_c}{q_m} = \frac{\alpha l}{\lambda} \frac{(t_\infty - t_i)}{\left(\frac{dt}{d(x/l)}\right)_{x=x_i}} = \text{Nu} \left(\frac{dt_a}{dx_a}\right)_{x_a=x_i^*} \quad (6.163)$$

The result shows that the physical significance of the Nusselt number is:

$$\text{Nu} = \frac{\text{quantity of heat transferred by the convective mechanism}}{\text{quantity of heat transferred by the molecular mechanism}}$$

As for the Prandtl number, we consider the heat transfer flux which can be written with the use of the fluid enthalpy (Eq. (6.164)) and the molecular momentum flux given by Eq. (6.165):

$$q_m = -\frac{\lambda}{\rho c_p} \left(\frac{d(\rho c_p t)}{dx}\right)_{x=x_i} \quad (6.164)$$

$$\tau_{myx} = -\frac{\eta}{\rho} \left(\frac{d(\rho w_y)}{dx}\right)_{x=x_i} \quad (6.165)$$

The ratio between both fluxes shows that the Prandtl number is an index giving the relative quantity of the momentum transported by the molecular mechanism and of the heat transported by the same mechanism at the interface:

$$\frac{\tau_{myx}}{q_m} = \frac{c_p \eta}{\lambda} \left(\frac{d(\rho w_y)}{d(\rho c_p t)}\right)_{x=x_i} = \text{Pr} \left(\frac{d(\rho w_y)}{d(\rho c_p t)}\right)_{x=x_i} \quad (6.166)$$

As for the significance of the Sherwood number, the following mass transfer fluxes for species A are used:

- the flux of component A transported to the interface by pure diffusion (molecular mechanism):

$$N_{Am} = D_A \left(\frac{d(\rho \omega_A)}{dx}\right)_{x=x_i} \quad (6.167)$$

- the flux of component A transported to the interface by natural and provoked/induced/forced convection;

$$N_{Ac} = k_c ((\rho \omega_A)_i - (\rho \omega_A)_\infty) \quad (6.168)$$

The ratio between both fluxes shows that the Sherwood number can be considered as an index of the relative participation of the convective and molecular mechanisms to the transport process.

The next relationship gives the mathematical form of this physical interpretation:

$$\frac{N_{Ac}}{N_{Am}} = \frac{k_c ((\rho \omega_A)_i - (\rho \omega_A)_\infty)}{D_A \left(\frac{d(\rho \omega_A)}{dx}\right)_{x=x_i}} = \frac{k_c l}{D_A} \frac{((\rho \omega_A)_i - (\rho \omega_A)_\infty)}{\left(\frac{d(\rho \omega_A)}{d(x/l)}\right)_{x=x_i}} = \text{Sh} \frac{1}{\left(\frac{d\omega_A^3}{dx_a}\right)_{x_a=x_i^*}} \quad (6.169)$$

$$\text{Sh} = \frac{\text{Quantity of the species A transported to the interface by convective mechanism}}{\text{Quantity of the species A transported to the interface by molecular mechanism}}$$

The Schmidt number is the mass transfer analogue of the Prandtl number. Indeed, by analogy, we can note that the Schmidt number is a measure characterizing the ratio between the quantity of the momentum transported to the interface by the molecular mechanism and the quantity of species A transported to the interface by the same mechanism. Equation (6.171) shows this statement:

$$\frac{\tau_{mxy}}{N_{Am}} = \frac{\eta}{\rho D} \left(\frac{d(\rho w_y)}{d(\rho \omega_A)} \right)_{x=x_i} = \text{Sc} \left(\frac{d(w_y)}{d(\omega_A)} \right)_{x=x_i} \quad (6.170)$$

6.6.4

Dimensionless Groups for Interactive Processes

When a process is produced under the divergent or convergent action of two different forces, the ratio between them represents a dimensionless number. The heat and mass transfer enhanced by the supplementary action of a pulsating field (vibration of apparatus, pulsation of one (or two) phase flow(s), ultrasound action etc.) has been experimented and applied in some cases [6.25–6.27]. Then, the new dimensionless number $I_\omega = \frac{g}{\omega^2 A}$ has to be added to the list of dimensionless groups presented above in this chapter.

As an example of an interactive process, we can mention the rapid drying of a porous material when heat and mass transport occur simultaneously. This case corresponds to very intensive drying such as high frequency or conductive drying. In this case, a rapid transfer of humidity from the liquid to the vapour state, associated with local change in pressure, induces a rapid vapour flow in the porous structure. To establish the equations of water transport, we assume that the gradient is established from the material matrix to the outside. It results in additional moisture and heat transfer induced by the hydrodynamic (filtration) motion of liquid and vapours. The total pressure gradient within the material appears as the result of evaporation and of the resistance of the porous skeleton during vapour motion. The air from the adjacent medium flows by molecular and slip diffusion in the capillarity of the system.

In the case of a high-rate heat- and mass-transfer process, heat and mass flows are not described by the classical Onsager equations (as, for instance, in Eq. (3.12)):

$$\vec{j}_k = \sum_i L_{ki} \vec{X}_i \quad (6.171)$$

but by the following generalized equation:

$$\vec{j}_k = \sum_i L_{ki} \vec{X}_i + L_k^{(r)} \frac{\partial \vec{j}_k}{\partial \tau} \quad (6.172)$$

for example, the Fourier heat-conduction equation $\vec{q}_m = -\lambda \vec{\nabla} t$ will be replaced by:

$$\vec{j}_q = \vec{q} = \lambda \vec{\nabla} t - \tau_{rq} \frac{\partial \vec{q}}{\partial \tau} \quad (6.173)$$

Equation (6.173) is valid only for one-dimensional problems. For a multidimensional study, it can be used as an approximation where the relaxation period of the thermal stress τ_{rq} is defined as one experimental constant. A similar relationship is used for moisture diffusion. The term $L_k^{(r)} \frac{\partial \vec{j}_k}{\partial \tau}$ corresponds to the finite propagation velocity of a certain substance. The stress relaxation period τ_{rk} of substance k (mass, heat, etc.) is defined by:

$$\tau_{rk} = \frac{D_k}{v_k^2} \quad (6.174)$$

where v_k is the finite propagation velocity and D_k the diffusivity of substance k . From Eq. (6.172), we can now describe the flux of property with the local Γ_k concentration as:

$$\vec{j}_{tk} = \sum_{i=1}^n \left(D_{\Gamma_{ki}} \vec{n}_i \vec{\nabla} \Gamma_{ki} \right) + \vec{w} \vec{\nabla} \Gamma_k - \tau_{rk} \frac{\partial \vec{j}_k}{\partial \tau} \quad (6.175)$$

When we particularize this relationship for the mass transport of the humidity into a porous medium ($\vec{w} = 0$, because there is no microscopic displacement), we can observe the superposition of the thermo-diffusion and of the diffusive filtration (where p is the humidity flowing by filtration) over the pure diffusion process:

$$\vec{j}_{tu} = D_m \vec{\nabla} u + D_m \delta \vec{\nabla} t + D_m \delta_p \vec{\nabla} p \quad (6.176)$$

Based on Eq. (6.176), we obtain the next particularization for the general conservation relationship (see, for instance, Eq. (3.6)), which was established in Chapter 3:

$$\tau_{rk} \frac{\partial^2 \Gamma_k}{\partial \tau^2} + \left(\frac{\partial \Gamma_k}{\partial \tau} + \vec{w} \vec{\nabla} \Gamma_k \right) = \text{div} \left(\sum_{i=1}^n L_{ki} \vec{n}_i \vec{\nabla} \Gamma_{ki} \right) + j_{vk} \quad (6.177)$$

It is not difficult to particularize this relationship for the three simultaneous processes occurring in the porous medium. The result is the Luikov [6.28] complete relationships:

$$\tau_{ru} \frac{\partial^2 u}{\partial \tau^2} + \frac{\partial u}{\partial \tau} = K_{11} \nabla^2 u + K_{12} \nabla^2 t + K_{13} \nabla^2 p \quad (6.178)$$

$$\tau_{rt} \frac{\partial^2 t}{\partial \tau^2} + \frac{\partial t}{\partial \tau} = K_{21} \nabla^2 u + K_{22} \nabla^2 t + K_{23} \nabla^2 p \quad (6.179)$$

$$\tau_{rp} \frac{\partial^2 p}{\partial \tau^2} + \frac{\partial p}{\partial \tau} = K_{31} \nabla^2 u + K_{32} \nabla^2 t + K_{33} \nabla^2 p \quad (6.180)$$

where the coefficients K_{ij} ($i, j = 1, 2, 3$) correspond to:

$$K_{11} = D_m, K_{12} = D_m \delta = (D_{m1}^t + D_{m2}^t), K_{13} = D_m \delta_p = k_p / \rho_0 \quad (6.181)$$

$$K_{21} = D_m \frac{r\varepsilon}{c}, K_{22} = a, K_{32} = D_m \frac{\varepsilon r \delta_p}{c} \quad (6.182)$$

$$K_{31} = -D_m \frac{\varepsilon}{c_{ph}}, K_{32} = -D_m \frac{\varepsilon \delta}{c_{ph}}, K_{33} = D_p - D_m \frac{\varepsilon \delta_p}{c_{ph}} \quad (6.183)$$

where k_p is the filtration moisture-coefficient defined by the equation $\vec{J}_p = -k_p \vec{\nabla} p$; δ_p is the dimensionless filtration moisture flow, $\delta_p = k_p / D_m \rho_0$; a_p is the convective filtration diffusion coefficient, $a_p = \frac{k_p}{c_{ph} \rho_0}$; c_{ph} is the coefficient of humid air capacity in a porous material defined by the relationship $d(u_1 + u_2) = c_{ph} dp$; u_1 is the material moisture in the vapour state, u_2 is the moisture in the liquid state; ε is the dimensionless fraction defined by $\varepsilon = \frac{D_{m1}}{D_{m1} + D_{m2}} = \frac{D_{m1}^t}{D_m}$; a is the thermal diffusivity, and D_{m1}^t and D_{m2}^t are respectively the thermo-diffusion coefficients of vapour and liquid humidity.

Taking into account the above description, the mass transfer similarity numbers, which characterize this process, can be formulated. The following similarity numbers can then be formulated from the differential moisture transfer equations ((6.178) – (6.183)):

1. The homochronism of the transfer numbers of the field potential referred to as Fourier numbers:

$$Fo_q = \frac{a\tau}{l^2}, Fo_m = \frac{D_m \tau}{l^2}, Fo_p = \frac{a_p \tau}{l^2} \quad (6.184)$$

These dimensionless groups are related by the criteria Lu and Lu_p (drying Luikov dimensionless groups).

2. The mass transfer relaxation Fourier number:

$$Fo_{rm} = \frac{D_m \tau_{rm}}{l^2} \quad (6.185)$$

It is then important to specify that this number is formed by known magnitudes. The relaxation period of mass stress is about 10^4 times the thermal stress relaxation. The Fourier mass transfer number is, therefore, many times greater than the Fourier heat transfer relaxation number: $Fo_{rq} = a\tau_{rq}/l^2$.

3. The diffusion moisture-transfer number, with respect to the heat diffusion or the moisture-flow diffusion number (drying Luikov number [6.23, 6.28]):

$$Lu = \frac{D_m}{a}, Lu_p = \frac{D_p}{a} \quad (6.186)$$

The Lu number is the ratio between the mass diffusion coefficient and the heat diffusion coefficient. It can be interpreted as the ratio between the propagation velocity of the iso-concentration surface and the isothermal surface. In other words, it characterizes the inertia of the temperature field inertia, with respect to the moisture content field (the heat and moisture transfers inertia number). The Lu_p diffusive filtration number is the ratio between the diffusive filtration field potential (internal pressure field potential) and the temperature field propagation.

For some moist materials, the Lu number increases with the moisture content following a slow linear dependence. From Eq. (6.184), we can appreciate that, for $Lu > 1$, the propagation velocity of the mass transfer potential is greater than the propagation velocity of the temperature field potential. The value of the diffusive filtration number of moisture Lu_p is normally much higher than one. The total internal pressure relaxation of the vapour–gas mixture in a capillary porous body is 2–3 orders of magnitude higher than the relaxation of the temperature field. The relationships between Fourier numbers may be expressed in terms of Lu and Lu_p :

$$Fo_m = Fo_q Lu, Fo_p = Fo_q Lu_p \quad (6.187)$$

4. The Kossovich (Ko) and Posnov (Pn and Pn_p) numbers [(6.23), (6.28)] defined by Eqs. (6.188)–(6.190) are obtained from the drying model (Eqs. (6.178)–(6.183) completed with specific initial and boundary conditions). Moreover, they are converted into a dimensionless form by applying the pi theorem:

$$Ko = \frac{r\Delta u}{c_q \Delta t} \quad (6.188)$$

$$Pn = \frac{\delta \Delta t}{\Delta u} = \frac{D_m^t \rho_0 \Delta t}{\rho_0 D_m \Delta u} \quad (6.189)$$

$$Pn_p = \frac{\delta_p \Delta p}{\Delta u} = \frac{k_p \Delta p}{\rho_0 D_m \Delta u} \quad (6.190)$$

The Ko number shows the relationship between the heat consumed by the liquid evaporation ($r\Delta u$) and the heating of the moist body ($c_q \Delta t$). The Pn number is an index of the

ratio between the quantity of the humidity transported by the thermal-diffusive mechanism and the pure diffusive mechanism. A similar consideration can be advanced with respect to the Pn_p number.

5. Using the property of the dimensionless groups which show that a mathematical combination of such groups gives a dimensionless group, we introduce the Feodorov number (Fe) [6.23, 6.28] which is as a dimensionless number describing the drying process of a porous material:

$$Fe = \varepsilon Ko Pn = \frac{\varepsilon \delta r}{c_q} \quad (6.191)$$

This number is independent of the heat and mass transfer potentials because it is defined by coefficients ε , δ , r and c_q (the last two are, respectively, the vaporization latent heat of moisture and the specific heat capacity of the moist body).

6. The Rebinder (Rb) number is formulated using the methodology described above in item 5. This dimensionless number is given as the ratio between the dimensionless temperature coefficient of drying and the Kossovich number:

$$Rb = \frac{B}{Ko} = \frac{b \frac{\Delta u}{\Delta t}}{\frac{r \Delta u}{c_q \Delta t}} = \frac{c_q b}{r} \quad (6.192)$$

As in the case of the Fedorov number, the Rebinder number is independent of the choice of the heat and mass transfer potentials. This number is part of the fundamental heat balance of the drying process. Unlike the Pn number, the temperature-drying coefficient describes the changes occurring in the integral mean temperature (\bar{t}) and in the mean moisture content (\bar{u}). In other words, it relates the kinetic properties of integral heat with moisture transfer properties, whereas, the Pn number is concerned with local changes in u and t .

7. The Biot numbers of heat and mass transfer could be obtained from the boundary conditions of a third kind:
 - the heat transfer Biot number:

$$Bi_q = \frac{\frac{\alpha_{ex} - 1}{\rho_{ex} c_{pex}}}{a} \cong \frac{\alpha_{ex} l}{\lambda_m} \quad (6.193)$$

- the mass transfer Biot number:

$$Bi_m = \frac{k_{cex} l}{D_m} \quad (6.194)$$

Considering these Biot numbers, we can observe that they are similar to the Nusselt and Sherwood numbers. The only difference between these dimensionless numbers is the transfer coefficient property characterizing the Biot numbers' transfer kinetics for the external phase (α_{ex} : heat transfer coefficient for the external phase, k_{cex} : mass transfer coefficient for the external phase). We can conclude that the Biot number is an index of the transfer resistances of the contacting phases.

When the boundary conditions of the third kind cannot be established for heat ($q_{int}(\tau)$) and mass flux ($N_{Aint}(\tau)$) flows, then Bi_q and Bi_m are substituted by two Kirpichev (Ki) [6.23, 6.28] numbers:

$$Ki_q = \frac{q_{int}(\tau)l}{\lambda_o \Delta t}, \quad Ki_m = \frac{N_{Aint}(\tau)l}{D_m \Delta u} \quad (6.195)$$

If fluxes $q_{int}(\tau)$ and $N_{Aint}(\tau)$ are defined by the Newton laws, these numbers (Bi and Ki) are related by simple equations:

$$Ki_q = Bi_q \frac{t_c - t_{int}}{\Delta t}, \quad Ki_m = Bi_m \frac{u_c - t_{int}}{\Delta u} \quad (6.196)$$

where indexes c and int indicate the central and interface position of the moist body. Quantities Δt , Δu and Δp , appearing in the heat and mass transfer similarity numbers, are chosen taking the conditions of the problem into account.

The problem here discussed can be considered as an example, which can be generalized when the different elementary processes interact. At the same time, it shows the large potentialities of the chemical engineering methodologies in defining and using dimensionless groups for process characterization. All newly introduced dimensionless groups can also be obtained through an adequate dimensional analysis using the pi theorem procedure described earlier in this chapter.

6.6.5

Common Dimensionless Groups in Chemical Engineering

It is not easy to produce a basic list of the dimensionless groups frequently utilized in chemical engineering problems. This is due to the very large number of dimensionless groups that characterize the totality of chemical engineering processes. Table 6.3 gives a list of variables, which are commonly encountered in this type of analysis. Obviously, the list is not exhaustive but indicates a broad range of variables typically found in chemical engineering problems. Moreover, we can combine these variables with some of the common dimensionless groups given in Table 6.4.

Table 6.3 Some common variables typically used in chemical engineering.

Symbol	Definition
l	characteristic length
β_d	concentration convection coefficient
ρ	density
D	diffusion coefficient
ΔH_r	enthalpy of reaction
g	gravity acceleration
	heat transfer coefficient
k_c	mass transfer coefficient
ω	oscillation frequency
$p, \Delta p$	pressure or pressure difference
k_r	reaction kinetics constant
c	speed of sound
σ	surface tension
c_p or c_v	thermal capacity
λ	thermal conductivity
β_t	thermal convection coefficient
w	velocity
η	viscosity
R	universal constant of gases

Table 6.4 Some dimensionless groups typically used in chemical engineering.

Name and symbol	Definition formula	Physical interpretation	Type of application
Reynolds number Re	$\frac{wl\rho}{\eta}$	$\frac{\text{momentum quantity transferred by turbulent mechanism}}{\text{momentum quantity transferred by molecular mechanism}}$	all types of momentum, heat and mass transfer with forced convection
Froude number Fr	$\frac{w}{\sqrt{gl}}$	$\frac{\text{inertia force}}{\text{gravitational force}}$	flow with a free surface, pipe and packed bed two phase flow
Euler number Eu	$\frac{\Delta p}{\rho w^2}$	$\frac{\text{energy involved by the surface forces}}{\text{energy involved by the inertia forces}}$	problems in which pressure or pressure differences are of interest (jets from nozzles, injectors etc.)

Name and symbol	Definition formula	Physical interpretation	Type of application
Mach number Ma	$\frac{w}{c}$	$\frac{\text{inertia force}}{\text{compressibility force}}$	flow in which the compressibility of the fluid is important
Strouhal number St	$\frac{\omega l}{w}$	$\frac{\text{local inertia force}}{\text{global inertia force}}$	rotational steady and unsteady flow
Weber number We	$\frac{\rho w^2 l}{\sigma}$	$\frac{\text{energy involved by the inertia forces}}{\text{energy involved by the surface tension forces}}$	problems in which surface tension is important (bubbles, drops and particles)
Archimedes number Ar	$\frac{g l^3 \rho \Delta \rho}{\eta^2}$	$\frac{\text{archimedian force}}{\text{viscous force}}$	flow of the particles, drops and bubbles in liquid and gaseous media, fluidized and spurted bed
Grashof thermal number Gr _t	$\frac{g l^3 \beta_t \Delta t \rho^2}{\eta^2}$	$\frac{\text{thermal convection force}}{\text{viscous force}}$	flow and heat transfer by natural thermal convection
Grashof diffusion number Gr _t	$\frac{g l^3 \beta_d \Delta C \rho^2}{\eta^2}$	$\frac{\text{concentration convection forces}}{\text{viscous forces}}$	flow and mass transfer by natural convection
Nusselt number Nu	$\frac{\alpha l}{\lambda}$	$\frac{\text{quantity of the heat transferred by convection}}{\text{quantity of heat transferred by the molecular mechanism}}$	all heat transfer problems
Prandtl number Pr	$\frac{c_p \eta}{\lambda}$	$\frac{\text{momentum quantity transferred by molecular mechanism}}{\text{heat quantity transferred by molecular mechanism}}$	all heat transfer problems in forced convection
Biot number Bi	$\frac{\delta / \lambda}{1 / \alpha}$	$\frac{\text{conductive resistance media}}{\text{convective resistance media}}$	all heat transfer problems with interface flux condition
Fourier reaction number Fo _r	$\frac{k_r l^2}{D^2}$	$\frac{\text{quantity of species consumed by reaction}}{\text{quantity of species transported by molecular mechanism}}$	mass transfer problems with chemical reaction
Schmidt number Sc	$\frac{\eta}{\rho D}$	$\frac{\text{momentum quantity transferred by molecular mechanism}}{\text{species quantity transferred by molecular mechanism}}$	all mass transfer problems with forced convection
Sherwood number Sh	$\frac{k_c l}{D}$	$\frac{\text{species quantity transported by convective mechanism}}{\text{species quantity transported by molecular mechanism}}$	all mass transfer problems

Some additional details or commentaries about these important dimensionless groups are discussed in the next sections.

Reynolds Number (Re)

This dimensionless number is undoubtedly the most famous parameter in chemical engineering and fluid mechanics. It was named after Osborne Reynolds (1842–1912), a British engineer, who, with his famous experiment called the “Reynolds experiment” (1892) showed for the first time that this combination of variables could be used as a criterion to characterize laminar and turbulent flows. The Reynolds number is the measure of the ratio between the inertia and the viscous forces of a fluid element. The flow occurring in different systems such as mono-phase, two-phase and three-phase flows in a packed bed, two-phase and three-phase flows in trays can also be characterized by different Reynolds numbers. In addition, in chemical engineering, all the kinetic relationships, where forced convection is present (see for instance Table 6.2), can be described according to the Reynolds number. For example, the general kinetic relationship $Nu_T = G(\text{Re}, Pr_T, i_x, i_y)$ will show a particular form of the function G for a laminar flow and another particular form of the same function when turbulent flow occurs. The small values of the Reynolds number indicate that the viscous forces are dominant in the system and that we can consequently eliminate the participation of the forced convective mechanism in the flux property equation. It results in simplified forms of the equations related to flow field and to property field. In some cases, this type of simplification allows an analytical solution. Another example can be shown in the case of flow occurring over an immersed body (packed bed, sedimentation, etc.). In this example, for very large Reynolds numbers, inertial effects predominate over viscous effects and it may be possible to neglect the effect of viscosity and consider the problem as if it involved a ‘non-viscous’ fluid. In this case the Navier–Stokes equations can easily be reduced to the Euler equations for flow.

Froude Number (Fr)

The Froude number is named after William Froude (1810–1879), a British civil engineer, mathematician, and naval expert who pioneered the use of towing tanks of ship design. In some scientific papers the Froude number is defined as the square of the mathematical equation considered here, (for instance, see Table 6.8). This dimensionless number shows the importance of the gravitational force in some chemical engineering processes. This is typically the case for natural convection and surface flows. Figure 6.8 shows two examples of chemical engineering processes, which are used to separate or put into contact two different phase fluids. In case A, where a two-phase flow occurs in a contacting tray device, the Froude number is used to characterize the hydrodynamics and stability of the flow, whereas, in case B, where we have a co-current two-phase pipe flow, it defines the different states of the flow.

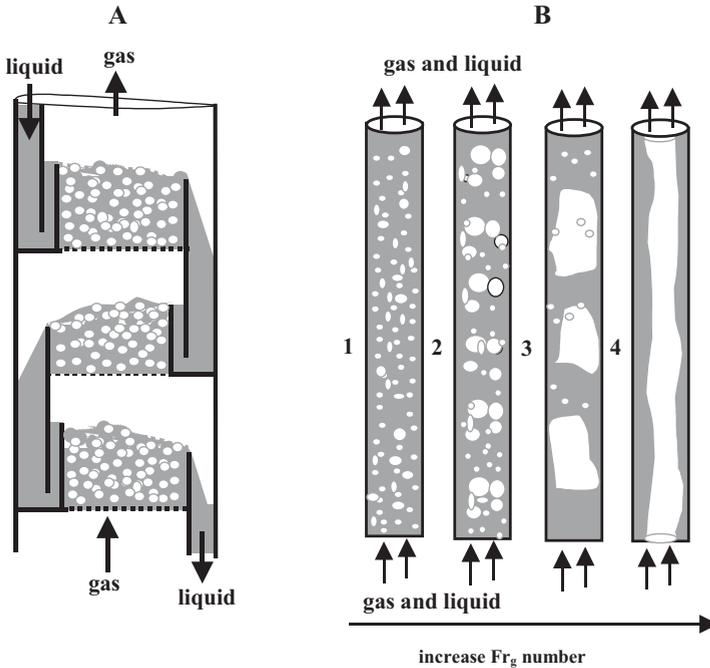


Figure 6.8 Two specific chemical engineering processes where the Fr number is applied. A: device with trays for contacting two phases; B: two-phase pipe flow in co-current configuration. 1: bubble flow; 2: aggregated flow; 3: plug flow; 4: annular flow.

Euler Number (Eu)

The Froude number described above is frequently used for the description of radial and axial flows in liquid media when the pressure difference along a mixing device is important. When cavitation problems are present, the dimensionless group $(p_r - p_v)/\rho w^2$ – called the Euler number – is commonly used. Here p_v is the liquid vapour saturation pressure and p_r is a reference pressure. This number is named after the Swiss mathematician Leonhard Euler (1707–1783) who performed the pioneering work showing the relationship between pressure and flow (basic static fluid equations and ideal fluid flow equations, which are recognized as Euler equations).

Mach Number (Ma)

The Austrian physicist E. Mach (1838–1916) is the recognized founder of this dimensionless group. This number is not very useful for most of the chemical engineering flow problems because it considers that the flowing fluid density is not affected by the field flow. In chemical engineering processes the Mach number takes values lower than 0.3. This means that this type of process is placed on the boundary between flows without and with compressibility effects. However, in

some cases where the sound field has been introduced as the active factor to enhance a specific unit operation (for example absorption) characterization using this number could be useful. This number is the more commonly used parameter in the fields of gas dynamics and aerodynamics.

Strouhal Number (St)

This number is used to characterize the stationary and unsteady oscillatory flow when the oscillatory field frequency presents a significant value. This type of flow can be generated for example, when a fluid is transported by piston pumps. In this case, the frequency flow parameters could be described by a combination of Strouhal number and Reynolds number:

$$Ff = \text{St. Re} = \frac{\rho\omega l^2}{\eta}$$

A second example can be generated when an intensive flow over a body produces closed field lines (called a vortex) at variable distances. This effect was observed by Strouhal (1850–1912) when some flow of air over wires produced a song. The Strouhal's 'singing wires' give the measure of the frequency that characterizes the vortex flow. This type of flow has been used to produce the so-called grid turbulence, which has various applications in the forced cooling of electronic devices [6.29].

Weber Number (We)

The Weber number is used when the surface tension forces acting on a fluid element are important. The Weber number for this special flow case is introduced by applying the pi theorem particularized to mixing in a liquid medium. In this case, it characterizes the ratio between the surface forces along the paddle that retain the flowing fluid element and the inertial forces that displace the flowing fluid element. This dimensionless number may be useful for the characterization of thin film flow and for the formation and breaking of droplets and bubbles. However, not all the problems involving a flow with an interface will require the inclusion of the surface tension.

When surface tension differences appear or are produced between some points or some small regions of an interface, the flow produced is called the Marangoni flow or flow with Marangoni effect. The Marangoni number, used to characterize the flow shown on Fig. 6.9, is a combination of the Reynolds number, the Weber number and the Schmidt number:

$$\text{Mn} = \text{Re}^a \text{We}^b \text{Sc}^c$$

if we make the assumption that the Mn number is dimensionless we obtain

$$a = 2, b = -1, c = 1$$

Then the Marangoni number could be written as:

$$\text{Mn} = \frac{\Delta\sigma l}{\rho D}$$

where $\Delta\sigma$ will be $\partial\sigma/\partial t * \Delta t$ when the Marangoni flow is caused by a temperature gradient or where $\Delta\sigma$ will be $\partial\sigma/\partial c * \Delta c$ when it is produced by the concentration gradient.

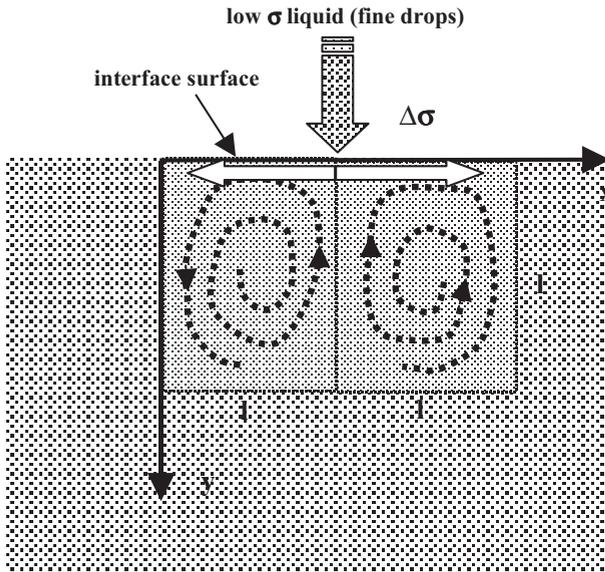


Figure 6.9 Flow and surface forces produced by surface tension gradients ($\Delta\sigma$).

Grashof Numbers (Gr_t , Gr_d)

These two numbers (diffusion Grashof number and thermal Grashof number) are used to characterize the natural convection produced by a thermal or concentration gradient. At moderate temperature or concentration gradient values, the natural convection flow keeps the properties of a laminar flow. However, higher gradient values of temperature and/or concentration can be caused by a turbulent natural convection flow. All the flows associated with heat and mass transfer processes, which occur without an important external action, are characterized with the Grashof number. Grashof numbers are also used in meteorology. Grashof numbers and the Froude number are the most used dimensionless groups that include gravitational acceleration as a physical parameter.

Nusselt Number (Nu)

This number is the main dimensionless group for heat transfer problems. With the partial heat transfer coefficient as physical parameter, it characterizes the kinetics of interface heat transfer. Unfortunately we cannot generally appreciate

the intensity of the heat transfer from the Nu number values because, in some cases, the intensity of the heat transfer process is not directly related to the Nu value. For example, the Nusselt number computed for liquid boiling, which is a very intensive heat transfer operation, can be lower than the Nusselt number computed for heat transfer from a heating device to adjacent media. This is caused by the differences between the characteristic lengths. Table 6.2 shows that the Nusselt number is coupled with other dimensionless groups by various relationships. Different ways are used in chemical engineering to particularize these relationships: the analytical and numerical solution of the equations of heat and mass transfer at steady state, particularized for a system of simple geometry; the particularizations of the boundary layer theory for a heat transfer case; the particularization of the transfer analogies for an actual heat transfer case and finally the experimental data correlation.

Prandtl Number (Pr)

The significance of the Prandtl number has been given earlier in this chapter. Another meaning is given by the boundary layer heat transfer theory and it shows that we can consider the Prandtl number as a relationship between the heat boundary layer and the hydrodynamics boundary layer associated in a concrete case.

We should also note that the boundary layer is the region where the solid interacts mechanically and thermally with the surrounding flow. A practical spin-off of Prandtl's recognition of the boundary layer is the understanding of the mechanisms of skin friction and heat transfer. This number is named after Ludwig Prandtl (1875–1953). Indeed, his discovery of the boundary layer is regarded as one of the most important breakthroughs of all time in fluid mechanics and has earned Prandtl the title of “Father of Modern Fluid Mechanics”. The heat and mass transfer analogies frequently used in chemical engineering are based on the Prandtl theory (Prandtl and Prandtl–Taylor boundary layer analogies). For heat transfer in gaseous media at moderate pressures, the Prandtl number can be neglected since, in this case, its values are between 0.7 and 1.

Schmidt Number (Sc)

As explained earlier, with respect to the heat and mass transfer analogies, the Schmidt number is the Prandtl number analogue. Both dimensionless numbers can be appreciated as dimensionless material properties (they only contain transport media properties). For gases, the Sc number is unity, for normal liquids it is 600–1800. The refined metals and salts can have a Sc number over 10 000.

Sherwood Number (Sh)

Initially called the diffusion Nusselt number, this number characterizes the mass transfer kinetics when expressed in dimensionless terms.

All the statements given before for the Nusselt number have the same significance for the Sherwood number if we change the words “couple heat transfer” to “couple mass transfer”. We have to specify that, as far as the Sherwood number is

concerned, we can use the mass transfer coefficient apparent values. This is the case for tray columns when the mass transfer coefficients are reported for the geometric active tray area, not for the real mass transfer tray area. This is very useful in problems when a precise value of the real mass transfer area cannot be precisely established.

6.7

Particularization of the Relationship of Dimensionless Groups Using Experimental Data

After establishment of the general dimensional analysis relationship for a concrete case, we have to formulate an adequate theory or an experimental investigation that will show the specific relationships between dimensionless groups. At the same time, the obtained relationships have to be justified as usable. The beginning of this chapter shows that dimensional analysis is an aid in the efficient interpretation of experimental data. As previously shown, a dimensional analysis cannot provide a complete answer to any specific relationship among the groups which are unknown. The general methods used to produce groups of relationships have been mentioned earlier; they are:

1. the analytical solution of the equations of all transfer properties for a particular example,
2. the numerical solution of the equations of all transfer properties supplemented with a correlation and regression analysis with respect to the relationships between the dimensionless groups,
3. the particularization of the boundary layer theory if, for the studied case, a stable boundary layer can be defined,
4. the particularization of the transfer analogies and their experimental validation,
5. the development of a consistent experimental research programme supplemented with a correlation and regression analysis with respect to the relationships between the dimensionless groups.

In the last case, when the determination of the relationships between the dimensionless groups is based on suitable experimental data, all the methods presented here can be used successfully. The degree of difficulty involved in this process depends on the number of pi terms and the nature of the experiments (for example experiments that require a change in the geometric simplex cannot be accepted because they require new experimental plants).

For a chemical engineering problem, the concrete activity of the correlation of experimental data shows the following two particularities:

1. No more than three, exceptionally four, pi non-geometric groups characterize the majority of analyzed problems.

2. The power form of the dimensionless pi groups is the one most used from among the possible relationships.

The simplest problem concerns only one pi term. The complexity of the analysis increases rapidly with increasing number of pi terms because then the choice of the experimental plan related to the proposed relationship for the dimensionless pi groups cannot be solved by an automatic procedure.

6.7.1

One Dimensionless Group Problem

When a one-dimensional analysis problem shows that the difference $m - n$ ($m =$ number of process variables, $n =$ number of basic dimensions associated with the process variables) is unitary, then only one pi term is required to describe the process. The functional relationship that can be used for one pi term is:

$$\Pi_1 = C$$

where C is a constant. The value of the constant, however, has to be determined experimentally. Normally, only one experiment is needed for the identification of C .

The case of Stokes settling velocity is considered as an illustrative example. If we assume that the stationary settling velocity, w_0 , of a small particle flowing into a liquid or gaseous medium is a function of its diameter, d , specific weight, $g\Delta\rho$, and the viscosity of the gaseous or liquid medium, it follows that:

$$w_0 = f(d, g\Delta\rho, \eta)$$

and the dimensions of the variables are:

$$[w_0] = LT^{-1} \quad [d] = L \quad [g\Delta\rho] = [g(\rho_p - \rho)] = ML^{-1}T^{-2} \quad [\eta] = ML^{-1}T^{-1}.$$

We observe that four variables and three basic dimensions (M , L , T) are required to describe the variables. For this problem, one pi term (group) can be produced according to the pi theorem. This pi group can easily be expressed as:

$$\Pi_1 = \frac{w_0\eta}{g\Delta\rho d^2}$$

Since there is only one pi group, it follows that:

$$\frac{w_0\eta}{g\Delta\rho d^2} = C$$

or

$$w_0 = Cgd^2 \frac{\Delta\rho}{\eta}$$

Thus, for a given particle and fluid, the gravitational settling velocity varies directly with d^2 , $\Delta\rho$, and with $1/\eta$. However, we cannot predict the value of the settling velocity since the constant C is unknown. In this case, we have to carry out experiments to measure the particle velocity and diameter, the density difference and the fluid viscosity. We can run a single experimental test but we will certainly have to repeat it several times in order to obtain a reliable value for C . It should be emphasized that once the value of C is determined, it is not necessary to run similar tests using different spherical particles and fluids because C is a universal constant. Indeed, the settling velocity of the small particles is a function only of the diameter and the specific weight of the particles and the fluid viscosity.

An approximate solution to this problem can also be obtained with the particularization of the Hadamard–Rybczynski problem [6.30, 6.31] from which it is found that $C = 1/18$ so that:

$$w_0 = \frac{1}{18}gd^2 \frac{(\rho_p - \rho)}{\eta} \quad (6.197)$$

This relationship is commonly called the “Stokes settling velocity” and is applicable for $Re = w_0 d \rho / \eta < 1$ and when particle interactions are not present during the settling process.

6.7.2

Data Correlation for Problems with Two Dimensionless Groups

The chemical engineering processes which can be characterized by two dimensionless groups are important, especially for heat and mass transfer with gaseous media, as shown in Table 6.4. If the phenomenon can be described with two pi terms we have:

$$\Pi_1 = \Phi(\Pi_2)$$

the form of the Φ function can be identified by varying Π_2 experimentally and measuring the corresponding Π_1 value. The results can be conveniently presented in graphical form as in Fig. 6.10. Here the uniqueness of the relationship between Π_1 and Π_2 , is shown.

Nevertheless, since it is an empirical relationship, we can only conclude that it is valid over the range of Π_2 dealt with by the experiments. It would be unwise to extrapolate beyond this range since, as illustrated with the dashed lines in the figure, the nature of the phenomenon can dramatically change if the range of Π_2 is extended.

For the valid range, we clearly obtain a curve with a break (B), then for each part, we will produce a particularization of the function Φ from the general Π_1 and Π_2 relationship.

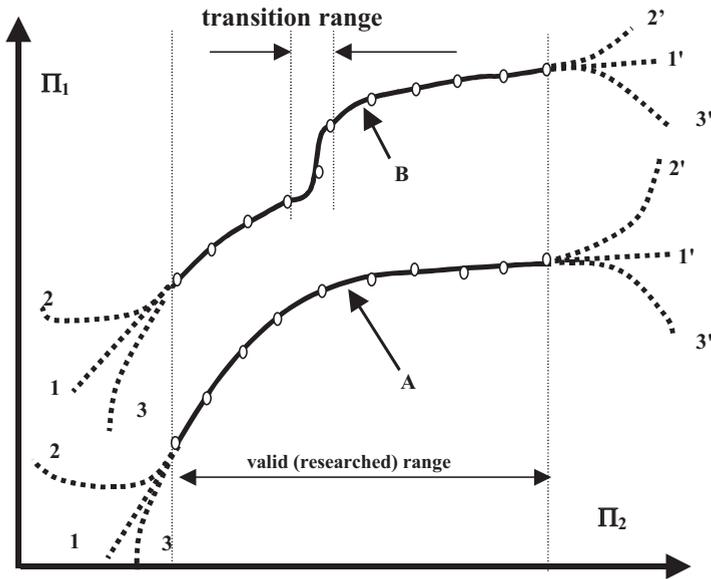


Figure 6.10 Π_1 versus Π_2 and illustration of the effect of extrapolation data over the valid range (A: continuous curve, B: curve with a break; 1, 1', 2, 2', 3, 3' possible extrapolations).

The break in the curve can be associated with a fundamental change in the process mechanism (such as, for example: a change from laminar to turbulent flow, a change from moderate natural convection to turbulent natural convection, etc.)

If we assume that the function Φ from the general Π_1 and Π_2 relationship is a power expression, then the relationship $\Pi_1 = a\Pi_2^b$ will be obtained. If we apply the logarithm to this relationship, we can identify a and b using a normalized linear system Eq. (5.15).

To illustrate this methodology, we show the case of pressure drop per unit length for one-phase flow in a packed bed. In laboratories, the pressure drop is measured over a 0.1 m length of packed bed using an apparatus as shown in Fig. 6.11. The fluid used is water at 20 °C ($\rho = 1000 \text{ kg/m}^3$, $\eta = 10^{-3} \text{ kg/ms}$). While the tests are carried out the velocity is varied and the corresponding pressure drop is measured. Table 6.5 shows the results of these tests.

Table 6.5 Measurements of packed bed pressure drop for the experimental device from Fig. 6.11.

Fictive water velocity (m/s)	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
Pressure drop for 1m bed height (N/m ²)	10930	21350	32760	43736	142727	198217	261759	332893	417838	497497

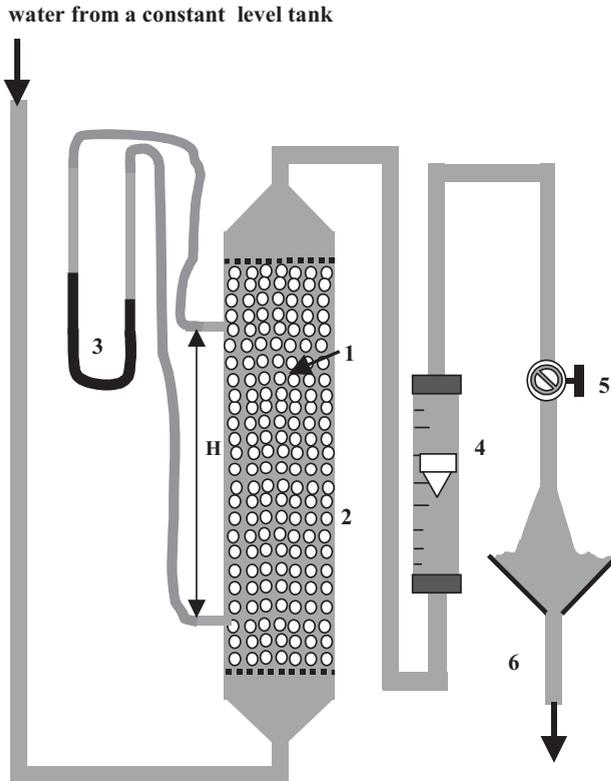


Figure 6.11 Device with a packed bed one-phase flow for the measurement of the pressure drop. 1: glass spherical particles (diameter = 1.5 mm), 2: glass column (diameter = 30 mm, $H = 0.1$ m), 3: differential manometer, 4: flow meter, 5: control valve, 6: water collector.

We will use these data to obtain a general relationship between the pressure drop per unit height of packed bed and the other variables. To search for an actual solution to this problem, we begin by performing a dimensional analysis, which can be realized without any experiment. We will assume that the pressure drop per unit height of packed bed, $\Delta p/H$, is a function of the equivalent packed body diameter, d_e , the fluid density, ρ , the fluid viscosity, η , and the mean packed bed fluid velocity, w .

The equivalent packed body diameter, d_e , is related to the bed holdup, ε , and specific packed surface, s , as well as via the relationship $d_e = 4\varepsilon/\sigma$. For the packed bed from spherical bodies ($\varepsilon = 0.44$ and $\sigma = 6/d_p$), the equivalent packed body diameter depends only on the sphere diameter. The mean internal packed bed fluid velocity represents the ratio between the fictive velocity and the packed bed porosity (ε). According to these data, we can write:

$$\frac{\Delta p}{H} = f(d_e, \rho, \eta, w)$$

the application of the pi theorem yields:

$$\frac{\Delta p}{\rho w^2} \frac{d_e}{H} = \Phi \left(\frac{w d_e \rho}{\eta} \right)$$

The simplest way to obtain $\frac{\Delta p}{\rho w^2} \frac{d_e}{H}$ for various $Re = \frac{w d_e \rho}{\eta} = \frac{4 w_f \rho}{\sigma \eta}$ is to vary the fictive velocity.

Based on the data given in Table 6.5, we can calculate the values for both pi terms. The results obtained are given in Table 6.6. A plot of these pi terms can now be made as a function of the Reynolds number. The results are shown in Fig. 6.12.

Table 6.6 Packed bed pressure drop in dimensionless terms.

$\frac{\Delta p}{\rho w^2} \frac{d_e}{H}$	7	3.5	2.33	1.75	3.65	3.50	3.42	3.33	3.25	3.18
$\frac{w d_e \rho}{\eta}$	10	20	30	40	50	60	70	80	90	100

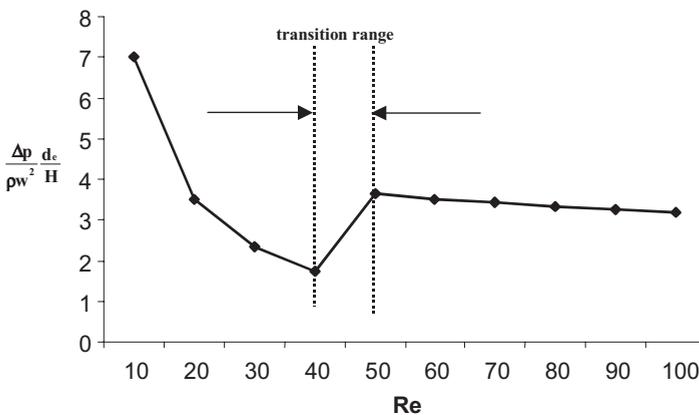


Figure 6.12 Dimensionless packed bed pressure drop versus Reynolds number.

The correlation appears to be quite good and shows that the valid range is divided into two parts. The first part corresponds to a Reynolds number lower than 40, which corresponds to the laminar flow range in the packed bed. The apparent turbulent flow range in the packed bed is obtained in the second part, corresponding to Reynolds numbers greater than 50 [6.32].

A wrong correlation may be due either to important experimental errors or to the omission of an important variable. The curve shown in Fig. 6.12 represents the general relationship between the pressure drop and the other factors for Reynolds numbers between 10 and 100. For this range of Reynolds number, as far as

the provided independent variables (d_e , ρ , η , w) are the only important parameters, it is not necessary to repeat the test for other packed beds or fluids. In order to determine the power of the relationships of pi terms, the data from Table 6.5 allow the identification of the next equations:

$$\frac{\Delta p}{\rho w^2} \frac{d_e}{H} = \frac{70}{Re} \quad \text{for } Re < 40$$

and:

$$\frac{\Delta p}{\rho w^2} \frac{d_e}{H} = \frac{8}{(Re)^{0.2}} \quad \text{for } Re > 40$$

It is then not difficult to write:

$$\Delta p = \zeta \frac{H\sigma}{8\varepsilon^3} w_f^2 \rho, \quad \zeta = 140/Re \quad \text{for } Re < 40; \quad \zeta = 16/(Re)^{0.2} \quad \text{for } Re > 40;$$

$$Re = (4w_f \rho)/(\sigma \eta)$$

which represents the Javoronkov procedure for packed bed pressure drop on phase flow calculation [6.32]. This so-called Javoronkov procedure is based on numerous experimental results similar to the type used in this example.

6.7.3

Data Correlation for Problems with More than Two Dimensionless Groups

When the number of pi groups involved in the dimensional problems increases, it becomes more difficult to organize the experimental research, to display the results in a convenient graphical form and to determine a specific empirical equation describing the phenomenon. If we accept that a power relationship between the pi groups is validated for all experimental ranges or for clearly identified portions of a range, we can easily identify the coefficients that characterize this relationship.

In this case, we can use a special experiment planning characterized by the repetition of a classical second order planning, where the centre of the plan is changed to cover a large range of each pi group and to discover the possible breaks in the state of dependent pi groups. In the previous chapter, it was shown that the majority of the functional dependences can be reduced to a multiple linear regression. If we propose for the relationships between the pi groups equations different from powers then they can be identified by using the particularized system of equations of the multiple regression. For most of the problems involving heat and mass transfer, the dependent pi groups are represented by Nusselt and Sherwood numbers, the independent pi groups characterizing the flow by the Reynolds or Grashof numbers and the media properties by the Prandtl and Schmidt numbers.

Changes in the fundamental flow mechanism are expected if large ranges of the pi groups characterizing the flow are considered. Consequently, a transition zone in the state of pi dependent groups has to be observed. This case is illustrat-

ed in Fig. 6.13. We have to specify that the general dependences considered in this figure cannot be extended to heat and mass transfers associated with phase transformations, because this case is more complicated. In complicated systems it is often more feasible to use models to predict specific characteristics of the system rather than to try to develop general correlations. If we extend the situation shown by the figure below to more than three terms, we obtain a very complicated problem where a graphical representation and a suitable empirical equation become intractable.

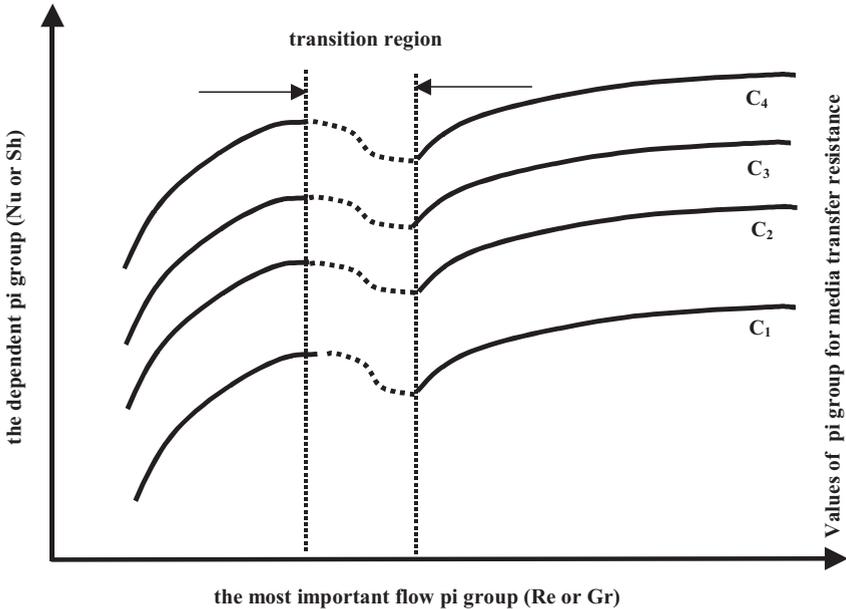


Figure 6.13 The graphical presentation of data for chemical engineering problems involving three pi dimensionless groups.

6.8

Physical Models and Similitude

To validate models based on transfer equations or stochastic models and, especially, to develop a coherent and planned experimental investigation of the studied process, the researcher has to imagine and build up a reduced scale experimental installation (laboratory device or model, LM). The goal using this reduced scale pilot plant is to obtain the experimental data necessary to validate the models.

Major chemical engineering projects involving structures, tray or packed columns, reactors, separators, heat exchangers and heaters, reservoirs and special deposits, fluid pumping as well as compressing devices, frequently involve the use of small scale studies using laboratory scale devices. According to the context,

the term “laboratory model”, “engineering model”, “physical model”, “laboratory pilot unit” or “small prototype” may be used. It is important to note that, as explained above, the term LM concerns one small pilot unit, which is different from the term “mathematical model” currently used throughout this book.

The pilot laboratory units, which are generally a different size, may involve different fluids, and often operate under different conditions (temperature, pressure, velocity, etc.). These units are frequently designed so that the parameters can be varied independently. The idea is not only to facilitate the study of the influence of the different process parameters but also to include the complexity of industrial scale units in the study.

Until now, the classical way to scale up industrial plants using laboratory scale units was very tedious and laborious: it consisted in systematically studying all the influencing parameters and operating conditions. In addition, many works were based on the use of different increasing scale. Consequently, methods concerning the characterization by a more rapid jump from laboratory small pilot to larger scale plants began to be developed [6.22, 6.33, 6.34]. These methods are therefore based on mathematical model simulations with incomplete laboratory experimental data for one actual problem [6.35]. There is, of course, an inherent danger in the use of models if the predictions realized are not correctly validated, because they can be erroneous and it may not be possible to detect errors until the industrial size plant is found not to perform as predicted. It is, therefore, imperative to have a properly designed and tested model, as well as correctly interpreted results. This is the basic question of the similitude theory: “To what extent can experimental data be relied upon when the dimensions of the experimental devices increase or decrease?”

In the following sections we will present some procedures and examples which show how LM can be designed in order to have a similar behaviour evolution for different device scales such as laboratory device (small scale) and prototype units (medium scale).

6.8.1

The Basis of the Similitude Theory

Which mathematical models or designing procedures were used to build the Egyptian pyramids or the gothic cathedrals? Since the ancient times, geometric scaling-up procedures have been used. These rules are based on similitude laws, which are still used today.

It is recognized that a phenomenon which occurs in an apparatus or a plant at different scales (various geometrical dimensions) presents the same evolution for all scales only if the conditions of the geometric similarity (geometric similitude), material similarity (material similitude), dynamic similarity (dynamic similitude) are respected and if the phenomenon shows the same initial state in all cases. The parametric description of a phenomenon occurring at laboratory and prototype scales is given in Table 6.7. In this case, we consider that the initial state of the phenomenon is identical for both scales.

Table 6.7 Determinant general parameters for the phenomenon evolution (two dimension scales).

Name of parameters	Symbolic notations for the laboratory model	Symbolic notations for the prototype
1 Characteristic geometric parameters (geometric dimensions of apparatus or plant)	$l_0, l_1, l_2, \dots, l_n$	$L_0, L_1, L_2, \dots, L_n$
2 Characteristic material parameters (specific properties of the materials used in experiments: density, viscosity, etc.)	$c_0, c_1, c_2, \dots, c_r$	$C_0, C_1, C_2, \dots, C_r$
3 Characteristic parameters for the phenomenon dynamics (flow velocities, mixing flow, heating rate, mass transfer rate, reaction rate, etc.)	$\tau_0, \tau_1, \tau_2, \dots, \tau_q$	$T_0, T_1, T_2, \dots, T_q$

With respect to the evolution of the phenomenon considered in Table 6.7, if a_g , a_c and a_τ are the scaling factors (the coefficients that multiply the laboratory model parameters in order to obtain the value of the prototype's parameters) then these can be written as:

$$a_g = \frac{L_0}{l_0} = \frac{L_1}{l_1} = \frac{L_2}{l_2} = \dots = \frac{L_n}{l_n} \quad (6.198)$$

$$a_c = \frac{C_0}{c_0} = \frac{C_1}{c_1} = \frac{C_2}{c_2} = \dots = \frac{C_r}{c_r} \quad (6.199)$$

$$a_\tau = \frac{T_0}{\tau_0} = \frac{T_1}{\tau_1} = \frac{T_2}{\tau_2} = \dots = \frac{T_q}{\tau_q} \quad (6.200)$$

It is obvious that $a_c = 1$ or $c_0 = C_0$, $c_1 = C_1$, $c_2 = C_2$, $c_r = C_r$ when the same materials are used for the LM and for the prototype unit.

Equations (6.198)–(6.200) can be arranged to show dimensionless or dimensional ratios, which express only the LM or the prototype. These dimensionless or dimensional relations are called similitude simplexes when they result from the same type of parameters and similitude multiplexes when they are composed of different types of parameters.

Using l_0 and L_0 , c_0 and C_0 and τ_0 and T_0 as the characteristic parameters for the laboratory device and prototype geometry, used materials and phenomenon dynamics respectively we transform the preceding relationships as:

$$i_{g1} = \frac{l_1}{l_0} = \frac{L_1}{L_0}, \quad i_{g2} = \frac{l_2}{l_0} = \frac{L_2}{L_0}, \quad \dots, \quad i_{gn} = \frac{l_n}{l_0} = \frac{L_n}{L_0} \quad (6.201)$$

$$i_{c1} = \frac{c_1}{c_0} = \frac{C_1}{C_0}, \quad i_{c2} = \frac{c_2}{c_0} = \frac{C_2}{C_0}, \quad \dots, \quad i_{cr} = \frac{c_r}{c_0} = \frac{C_r}{C_0} \quad (6.202)$$

$$i_{\tau_1} = \frac{\tau_1}{\tau_0} = \frac{T_1}{T_0}, \quad i_{\tau_2} = \frac{\tau_2}{\tau_0} = \frac{T_2}{T_0}, \quad \dots, \quad i_{\tau_q} = \frac{\tau_q}{\tau_0} = \frac{T_q}{T_0} \quad (6.203)$$

where $i_{g_1}, i_{g_2}, \dots, i_{g_n}$ are the geometric simplexes, $i_{c_1}, i_{c_2}, \dots, i_{c_r}$ are the dimensional material simplexes or multiplexes and $i_{\tau_1}, i_{\tau_2}, \dots, i_{\tau_q}$ are the dynamic dimensional multiplexes. Then we can postulate that “the phenomenon occurring in the laboratory device and in the prototype present the same evolution only when all $i_{g_1}, i_{g_2}, \dots, i_{g_n}, i_{c_1}, i_{c_2}, \dots, i_{c_r}, i_{\tau_1}, i_{\tau_2}, \dots, i_{\tau_q}$ are the same ($i_{g_1}, i_{g_2}, \dots, i_{g_n}, i_{c_1}, i_{c_2}, \dots, i_{c_r}, i_{\tau_1}, i_{\tau_2}, \dots, i_{\tau_q}$ stay unchanged when the dimensions of the model increase or decrease)”

Because all the dimensional material and dynamic multiplexes are reported only for the LM or for the prototype, we can combine these $r \cdot q$ dimensional multiplexes and the characteristic geometric parameters (so we have $r \cdot q + 1$ dimensional terms) to formulate new dimensionless independent multiplexes. It is not difficult to observe that these independent dimensionless multiplexes are the dimensionless pi groups that characterize the evolution of the phenomenon. If $\Pi_1, \Pi_2, \dots, \Pi_s$ represent the dimensionless groups that characterize the evolution of the phenomenon in the laboratory device or in the prototype, we transform the similitude postulate into the next new statement:

“One phenomenon occurring in two differently scaled devices (models) presents the same evolution only if the dimensionless pi groups characterizing the phenomenon have the same values. In other words, we have similitude if the dimensionless pi groups characterizing the phenomenon stay unchanged when the dimension of the device (model) changes”.

The theory of the models can be readily developed using the principles of dimensional analysis. It has been shown that any given problem can be described in terms of a set of pi terms as:

$$\Pi_1 = \Phi(\Pi_2, \Pi_3, \dots, \Pi_s) \quad (6.204)$$

Once this relationship is formulated, all we need to know is the general nature of the physical phenomenon and variables. Specific values for variables (size of components, fluid properties, etc.) are not needed to perform the dimensional analysis. This relationship could be applied to any system, if it is governed by the same variables and laws. If Eq. (6.204) describes the behaviour of a laboratory device, a similar relationship can be written for evolution of the phenomenon in the prototype:

$$\Pi_{1p} = \Phi(\Pi_{2p}, \Pi_{3p}, \dots, \Pi_{sp}) \quad (6.205)$$

where the subscript p shows that this is the case of the evolution of the phenomenon in the model prototype.

The pi terms can be developed so that Π_{1p} contains the variables that have to be predicted from the observation made on the laboratory apparatus. Therefore, if

the prototype is designed and operated, in relationship to the LM, under the following conditions:

$$\Pi_{2p} = \Pi_2, \Pi_{3p} = \Pi_3, \dots, \Pi_{sp} = \Pi_s \quad (6.206)$$

then, with the presumption that the form of function Φ is the same for the LM and for the prototype, it follows that:

$$\Pi_{1p} = \Pi_1 \quad (6.207)$$

Equation (6.207) indicates that the measured value of Π_1 for the LM will be identical to the corresponding Π_1 for the prototype as long as the other pi terms are similar.

The conditions specified by Eq. (6.206) provide the conditions required to design the model, also called similarity requirements or modeling laws. The same analysis could be carried out for the governing differential equations or the partial differential equation system that characterize the evolution of the phenomenon (the conservation and transfer equations for the momentum). In this case the basic theorem of the similitude can be stipulated as: "A phenomenon or a group of phenomena which characterizes one process evolution, presents the same time and spatial state for all different scales of the plant only if, in the case of identical dimensionless initial state and boundary conditions, the solution of the dimensionless characteristic equations shows the same values for the internal dimensionless parameters as well as for the dimensionless process exits".

As an example of this methodology, we can consider the problem of determining the heat loss of a rectification column which is placed perpendicularly to a fluid flowing at the velocity w . (see Fig. 6.14). The dimensional analysis of this problem shows that:

$$\alpha = f(d, H, \eta, \rho, \lambda, w)$$

where α represents the heat transfer coefficient, d , the column diameter, H , the column height, λ , the fluid thermal, while η and ρ show the fluid viscosity and density conductivity and w , represents the incident air velocity. Application of the pi theorem gives:

$$\frac{\alpha d}{\lambda} = \Phi\left(\frac{H}{d}, \frac{wd\rho}{\eta}\right) \quad (6.208)$$

We are now concerned with the design of a laboratory device which can be used to predict the heat loss on a different-sized prototype.

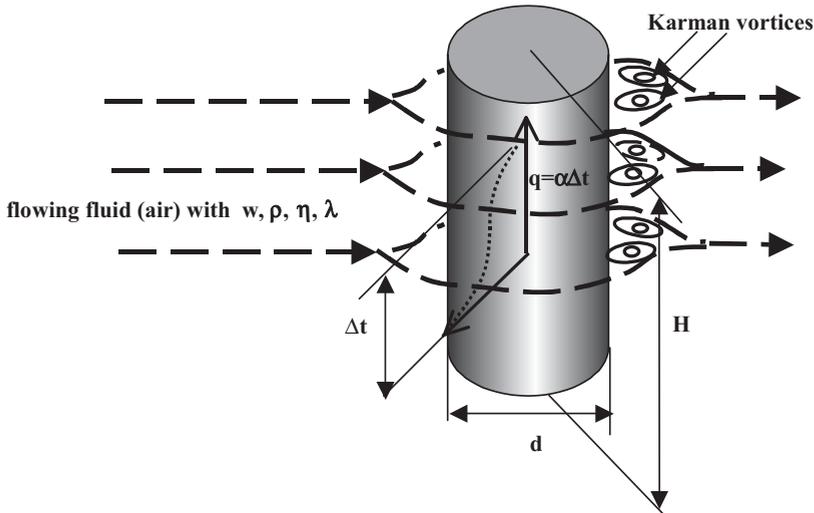


Figure 6.14 Heat loss in a rectification column (flow around the column).
 Fluid characteristics: w : velocity, η : viscosity, ρ : density, λ : thermal conductivity.
 Column characteristics: d : diameter, H : height. Δt : differential temperature between the air and the column surface.

Since the relationship expressed by Eq. (6.208) applies to both prototype and laboratory models, we can assume that for the prototype, a similar relationship could be written:

$$\frac{\alpha_p d_p}{\lambda_p} = \Phi \left(\frac{H_p}{d_p}, \frac{w_p d_p \rho_p}{\eta_p} \right) \quad (6.209)$$

the design conditions, or similarity requirements are therefore:

$$\frac{H}{d} = \frac{H_p}{d_p}, \quad \frac{wd\rho}{\eta} = \frac{w_p d_p \rho_p}{\eta_p} \quad (6.210)$$

the size of the laboratory device is obtained from the first requirement which indicates that:

$$d = \frac{H}{H_p} d_p \quad (6.211)$$

we can then establish the height ratio H/H_p , and the diameter of the laboratory device d is fixed in accordance with Eq. (6.210).

The second similarity requirement indicates that the LM and the prototype must be operated at the same Reynolds number. The required velocity for the laboratory model is obtained from the relationship:

$$w = \frac{\eta}{\eta_p} \frac{\rho_p}{\rho} \frac{d_p}{d} w_p \quad (6.212)$$

Note that this model design requires not only geometric scaling, as specified in Eq. (6.211), but also the correct scaling of the velocity in accordance with Eq. (6.212). This result is typical of most design procedures, where the scaling up is more difficult than simply scaling the geometry.

With the foregoing similarity requirement satisfied, the equation for the prediction of the column heat loss is:

$$\frac{\alpha_p d_p}{\lambda_p} = \frac{\alpha d}{\lambda}$$

or

$$\alpha_p = \alpha \frac{d}{d_p} \frac{\lambda_p}{\lambda}$$

or

$$Q_p = \alpha_p \pi d_p H_p \Delta t = \alpha \frac{d}{d_p} \frac{\lambda_p}{\lambda} \pi d_p H_p \Delta t = Q \left(\frac{H_p}{H} \right) \left(\frac{\lambda_p}{\lambda} \right) \quad (6.213)$$

where Q represents the heat loss of the laboratory size column that is operated with a temperature difference (Δt) between its surface and the incident flowing fluid. Once the heat loss is measured on the laboratory device, Q has to be multiplied by the ratios corresponding to the height of the columns and the conductivity of the flowing fluids in order to obtain the predicted value of the heat loss for a real column.

If we analyze the case of the rectification column which loses heat by natural convection, then we change the list of variables by considering the specific ascension force, $g\beta_t \Delta t$, as an important variable and by removing the fluid velocity, w . In this case, the application of the pi theorem shows that:

$$\frac{\alpha H}{\lambda} = \Phi \left(\frac{d}{H}, \frac{g\beta_t \Delta t H^3 \rho^2}{\eta^2} \right) \quad (6.214)$$

the design conditions, or similarity requirements are therefore:

$$\frac{d}{H} = \frac{d_p}{H_p}, \quad \frac{g\beta_t \Delta t H^3 \rho^2}{\eta^2} = \frac{g\beta_t \Delta t H_p^3 \rho_p^2}{\eta_p^2} \quad (6.215)$$

if the same fluid is used for both geometric scales, then the size of LM cannot be established because we obtain:

$$d = d_p \frac{H}{H_p}, \quad H = H_p$$

the same result will be obtained when we remove the diameter (d), from the list of variables.

This very simple example shows that sometimes it is not possible to use this scaling-up procedure. Fortunately, the majority of the problems presenting scaling-up impossibilities can be solved using other methodologies. As illustrated in this example, to achieve the similarity between the behaviour of both the laborato-

ry model and the prototype, all the corresponding pi terms must be equated between these two scales. Usually, one or more pi terms involve ratios of important lengths (such as H/d in the foregoing example). Thus, when we equate the pi terms involving length ratios we are requiring “a complete geometric similarity” to exist between the laboratory device and the prototype. Geometric scaling could be extended to the finest feature of the system, such as surface roughness, or small protuberances on a structure, since the surface state also determines the flow pattern. When a deviation from the complete geometric scaling must be considered, a careful analysis has to be carried out. For example, the design of one new packed body, or of a new sort of tray, cannot be produced without a complete geometric scaling.

Other groups of characteristic pi terms (such as $Re = \frac{wd\rho}{\eta}$ in the foregoing example) involve mechanisms’ ratio or forces’ ratio as noted in Table 6.4. The equality of these pi terms requires the same mechanisms’ or forces’ ratios in laboratory devices and prototypes. Thus, for similar Reynolds numbers, the values defining the turbulent and laminar flow mechanisms have to be the same for both devices. If other pi groups are involved, such as the Froude, Weber, Archimede or Grashof numbers, a similar conclusion can be drawn; that is, the equality of these pi groups requires identical ratios of identical forces for the laboratory apparatus and for the prototype. In the case of similar pi terms, we can say that we have a “hydrodynamic similarity”.

Other similarities used in chemical engineering concern the “field concentration similarity” and the “property transfer similarity”. The field concentration of the property is characterized by the Prandtl, Schmidt or reaction Fourier numbers (Table 6.4). It has been shown here that if the geometric and hydrodynamic similarities are respected, the transfer similarities exist between the laboratory device and the prototype. These last two similarity conditions represent the conditions of “transfer kinetic similarity”. In order to have complete similarity, we need to maintain the similarity between the geometry, dynamics and transfer kinetics between both units. If scaling up is found to be impossible, we have to ascertain whether all important variables are included in the dimensional analysis, and whether all the similarity requirements based on the resulting pi groups are satisfied.

6.8.2

Design Aspects: Role of CSD in Compensating for Significant Model Uncertainties

The experimental studies with one laboratory device, generally involve simplifying assumptions concerning the variables to be considered. In spite of the fact that the number of assumptions is less restrictive than required for mathematical models they introduce some uncertainty into the design of the device. It is, therefore, desirable to check the design experimentally whenever possible. Generally, the purpose of the LM is to predict the effects of certain changes proposed in a given prototype or in a larger-scale device. The LM has to be designed, constructed, and tested and then the predictions can be compared with the available

data from larger-scale devices. If the agreement is satisfactory, then the changes allowing one to build a bigger model can be accepted with increased confidence. Another useful procedure is to run tests with a series of LM having different sizes, where one of the devices can be considered as the prototype and the others as models of this prototype. With the devices designed and operated on this basis, we also need to improve other conditions for the validity of the LM design: accurate predictions have to be made between any pair of devices, since each of them can always be considered as the model of another one.

It is important to note that a good agreement in the validation tests described above does not unequivocally indicate that the scaling up is correct, especially when the dimensions of the scales between the different LMs are significantly different from those required by the basic laboratory model. However, if the agreement between the various models is not good, it is impossible to use the same model design to predict the behaviour of the basic laboratory model.

Some designing cases show that the general ideas, which establish similarity conditions for models when we use simple corresponding pi terms, are not always able to satisfy all the known requirements. To illustrate such a case: if, for a relationship $\Pi_1 = \Phi(\Pi_2, \Pi_3, \dots, \Pi_s)$ one or more similarity requirements are not respected, such as, for example: $\Pi_2 \neq \Pi_{2p}$, then it follows that the prediction relationship $\Pi_1 = \Pi_{1p}$ is not true. The models designed without satisfying all the requirements are called limited models or distorted models.

The classic example of a distorted model occurs in the study of liquid media which are mixed mechanically as described earlier in this chapter. The dimensional analysis shows that:

$$K_N = \Phi\left(\frac{H}{d}, \frac{h}{d}, \frac{b}{d}, \text{Re}, \text{We}\right)$$

where:

$$K_N = \frac{N}{d^5 n^3 \rho}, \quad \text{Re} = \frac{nd^2 \rho}{\eta}, \quad \text{We} = \frac{\rho n^2 d^3}{\sigma}$$

If we consider that heat transfer occurs during the mixing, the dimensional analysis shows that:

$$\text{Nu} = \Phi\left(\frac{H}{d}, \frac{h}{d}, \frac{b}{d}, K_N, \text{Re}, \text{We}\right) \quad (6.216)$$

where the Nusselt number is $\text{Nu} = (ad)/\lambda$ where a is the heat transfer coefficient to the wall of the mixing unit and λ the thermal conductivity of the mixed fluid. If a_g is the geometric scaling factor the geometric similarity requires:

$$H_p = Ha_g, \quad h_p = ha_g, \quad b_p = ba_g$$

The similarity of the Euler number (the group K_N is a form of the Euler number for mixing) requires:

$$\frac{N_p}{\rho_p n_p^3} = \frac{N}{\rho n^3} a_g^5$$

or

$$N_p = N \frac{\rho_p}{\rho} \left(\frac{n_p}{n} \right)^3 a_g^5$$

The similarity of the Reynolds number requires:

$$\frac{n_p^2 \rho_p}{\eta_p} = \frac{n^2 \rho}{\eta} \frac{1}{a_g^2} \quad \text{or} \quad \frac{n_p}{n} = \left(\frac{\eta_p}{\eta} \frac{\rho}{\rho_p} \right)^{1/2} \frac{1}{a_g}$$

Whereas the similarity of the Weber number requires:

$$\frac{\rho_p n_p^2}{\sigma_p} = \frac{\rho n^2}{\sigma} \frac{1}{a_g^3} \quad \text{or} \quad \frac{n_p}{n} = \left(\frac{\rho}{\rho_p} \frac{\sigma_p}{\sigma} \right)^{1/2} \frac{1}{a_g^{3/2}}$$

Since the scale of the speed of rotation $\left(\frac{n_p}{n} \right)$ is expressed by two relationships, their combination could be written as:

$$\left(\frac{\eta_p}{\eta} \frac{\sigma}{\sigma_p} \right)^{1/2} = a_g^{1/2} \quad (6.217)$$

When we use the same fluid for the LM and the prototype, we obtain $a_g=1$ which is unacceptable. Apparently, with a different fluid, it may be possible to satisfy this design condition but it may be quite difficult, if not impossible, to find a suitable model fluid, particularly for a small scale unit. When the identity requirement of the Weber numbers is eliminated, we obtain a distorted model, which gives a good approach to intensive mixing (a large Reynolds number shows that inertia forces are dominant in the mixing process). A distorted model could also be obtained when the identity requirements of the Reynolds numbers are eliminated. Then these models are good for the description of cases in which the mechanical mixing is slow but intensive because the forces at the surface are important.

Distorted models can be used successfully, but the interpretation of the results obtained using this type of model is obviously difficult compared to the true models for which all similarity requirements are obtained. The success of using distorted models is dependent on the skill and experience of the investigator responsible for the design of the model and on the interpretation of the experimental data produced with the model. In many cases, the distorted models are associated with one or more uncertainties and the use of their data in the scaling-up design of the complex processes must be appreciated and compensated using an adequate control system design (CSD).

6.8.2.1 Impact of Uncertainties and the Necessity for a Control System Design

While designing complex systems, we can basically encounter two types of uncertainties. In the first, we know that the system will work but it is difficult to deter-

mine its scale. Typical examples of these are the uncertainties in physical properties such as heat transfer coefficients, mass transfer coefficients, tray efficiency etc. [6.36–6.39]. A judicious over design will solve the uncertainties. If we extend the observation, we can appreciate that their main impact concerns the capacity of the plant. Under-sized equipment will bottleneck the process. The important point is that a judicious over-size design is relatively cheap, while retrofitting to relieve bottlenecks is expensive in both low-cost capacity and capital cost. This is the more critical type of uncertainty, which, if not adequately addressed, can lead to a total process failure. We can illustrate this case with important examples extracted from the experience of some chemical industries [6.40] when the scaling-up of one plant ended up in a great fiasco. Even if the plant was not entirely abandoned, extensive and expensive modifications were required to operate it.

However, this problem can be easily avoided through straightforward concurrent design. One might also argue that building a large pilot plant or a small demonstration plant could minimize some of these difficulties and risk involved. For example, the Exxon case, in which an expensive demonstration plant was built to demonstrate the new Fischer-Tropsch process, is well known [6.40]. This type of expensive unit can be justified in some special cases but will not always be necessary.

Pilot plants are not often designed to provide the essential information for a scaling up. Instead, they are operated to demonstrate a single steady state that gives acceptable results. They are seldom meant to investigate the impact of the process variables, which is essential for safe scaling up and control design. There is no great difference between designing a large pilot plant or a commercial plant. In both cases we have to make certain that the design can deal with the risks of scaling up.

To avoid any misunderstanding, we would like to emphasize that our main concern here includes such critical parts of the plant as new chemical reactors, processes or some complex separations that cannot be reliably modelled from a limited set of experiments carried out in a laboratory or on a small-scale pilot plant. For the remaining and more classical plant devices we can use modern simulators that provide all sorts of mathematical models and that have had a tremendous impact on the modern designing of plants. At the same time, we must note that the scaling-up based on combining complete complex mathematical models with experimental data from small-sized laboratory units [6.41–6.47] begins to be frequently used for the needs of the design and also for better operation of existing plants.

Nevertheless, we have to specify that the scaling up of one process must be considered as a complex problem with not only controlled goals determined by the dominant process variables but also independent degrees of freedom. Here, the efficiency of the process, the process modelling and design procedure have to be identified, computed, changed if necessary and used in order to perform an efficient process control. For a process scaling-up or design, the following objectives of control have to be considered:

1. Allow the system to meet the specifications. Allow on-line change of the specifications.
2. Stabilize the system.
3. Compensate for changes in feedstock, in properties of the catalyst (if the core process is a catalytic reaction) and unknown dynamic and persistent perturbations.
4. Allow compensation for uncertainties in the design and scaling-up of the unit.

To achieve these goals, both dynamic and steady state controls are required. For most chemical plants, the control to meet specifications is the primary objective. The capacity to stabilize and reject perturbations is essential to achieve the goals mentioned above.

Before proceeding to the designing methodology itself, it will be helpful to review and define some of the principles and concepts of partial control. We are concerned with the control of a system in which the number of process variables to be controlled is higher than the number of variables which are manipulated to realize this control. If all the process variables have to be controlled according to exact set points, the process has to be abandoned or the design modified in order to provide the requisite number of manipulated variables. However, it is quite often the case that many of these variables need only be controlled within prescribed limits, hence the terminology of partial control.

Dominant variables are characterized by observation, which shows that they exert a strong influence on many of the other interesting process variables. The operating temperature of the reactor is a typical example, because changes in this parameter generally modify the reactant conversion and product composition. Thus, by controlling the dominant variables, we can maintain the other process variables of concern within their prescribed limits. A variable is dominant for the stability if, by controlling it, we exert an effect on the system stability. This is particularly important if the system is an unstable open loop. It is very important to identify which variables are dominant in the laboratory. Strictly speaking, each apparatus presents its dominant variables. For example, potential dominant variable candidates for a catalytic reactor are: the temperature, pressure, space velocity, catalyst activity, and the properties of the reactants.

While the impact of dominant variables on process outputs can be identified and measured in the laboratory, this is not always true for stability. In this case, we have to rely on the availability of used models to identify possible dominant variables. It is clear that all dominant variables for stability are also dominant variables for outputs.

In conventional control (design control, evolution control), the number of degrees of freedom is considered as the number of available manipulated variables. We define the practical degrees of freedom as the number of dominant variables that can be controlled independently. The ability to have an impact on the outputs and stability of one chemical system is limited by the number of independently controllable dominant variables available in the design. The design of a

control system is one part of the whole design and is completely dependent on the scaling-up process. In the situation with one system of control, the choice of the dominant variables and the independent degrees of freedom is deemed to be sufficient if they provide the management with adequate constraints and also result in a good stabilization of the system.

For the design of any complex system, including or not a chemical reactor, we do not need a complete model but rather minimal information of the model, which strongly depends on the design itself. The laboratory identification of all dominant variables is essential, together with sufficient data on their impact on the most important outputs of the process. This is essential for a safe scaling-up and to produce a preliminary model. It is also important to know how the mini-

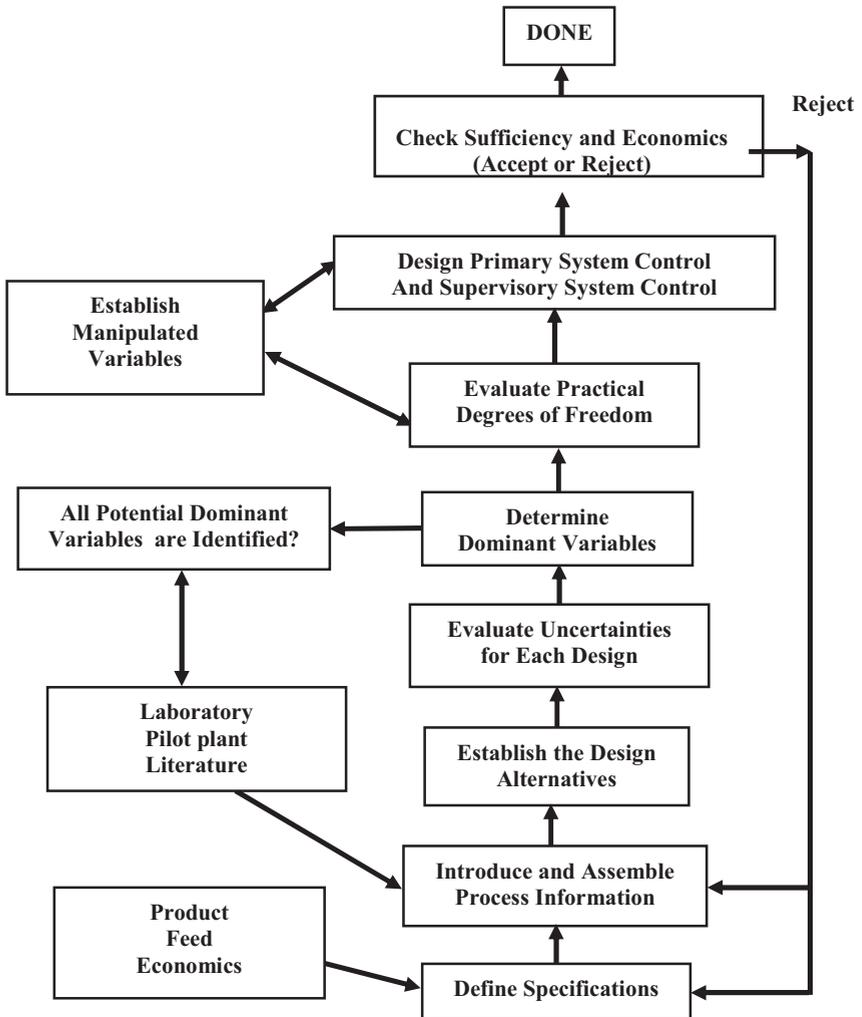


Figure 6.15 Possible structure for the design of a chemical fabrication.

mal information of the model determines the choice of the basic design and the control design.

In all cases the design key features are:

- the identification of the control system goals (specifications),
- the generation of the model information that adequately characterizes the system,
- the identification of the dominant process variables,
- the determination of the effective degrees of freedom,
- the determination of the control structure.

Figure 6.15 shows the general detailed structure of the working steps for the design of a scaling-up and control unit for a chemical fabrication. It is observable that we can generate a number of alternate process designs, either sequentially or in parallel. Then, one must also develop and evaluate the best partial structure control for each of these designs. The design which represents the best compromise between cost and controllability in the face of uncertainty could be considered as the final design.

6.9

Some Important Particularities of Chemical Engineering Laboratory Models

Generally, the chemical engineering processes include steps where interface transfer with or without a chemical reaction is dominant. In these cases the surface of transfer is one the parameters which controls the transfer efficiency. Some of the various technical solutions which have been developed to increase the surface of transfer are:

- the use of packed beds of small bodies for the differential transfer apparatus involved in the phase contacting procedure,
- the development of the highly efficient tray equipped with devices that produce small bubbles or drops with dense and uniform spatial distribution.
- the use of catalytic fixed beds with catalytic pellets with a diameter not exceeding 10–15 mm as well as fluidized beds with catalysts in powder form.

For all the examples given above, the analysis of the characteristic geometric length shows that this dimension is very small (diameter of packed body, bubble or drop diameter, catalyst particle diameter). In this section, we have shown that both the laboratory plant and the scaling-up apparatus (plant) have to be designed to work with the same characteristic geometric lengths. We can conclude that the relationship $l/l_p = 1$ is necessary in the case of the geometric scaling-up of laboratory models. In addition, since the majority of the laboratory models are designed to use real fluids (those that will be used in the extended model) in their tests, we can appreciate that we have scales of the same unitary value for all materials.

If we analyze the fundamental dynamics aspects of the fluids contacting in the laboratory device, we can easily observe that the velocities and other dynamic parameters of the specific phases, could be identical to those of the extended model. If we neglect the wall effects, which, in the case of LM could be important, we can easily conclude that the dimensionless pi terms that characterize the dynamics of the process present the same values for the laboratory plant and for the extended model. Taking these observations into consideration, we can see that LMs do not require a scaling up of the data and information obtained when we want to use them on experimental investigations of a physico-chemical process. This means that the relationships, the curves and the qualitative observations obtained with an LM could be directly applicable to larger devices.

We cannot finish without presenting some uncertainties, which show the differences between the LMs and their real homologue:

- the chemical processes are frequently studied at laboratory scale taking into account only the critical parts. This means that the elements are considered as new or unknown. Then, the studied problems may be presented when computed and experimented parts are assembled.
- The stationary time of laboratory models is quite small when compared with the corresponding time for extended models. This fact introduces important uncertainties in the capacity of the experimental data to predict the states of the system not covered by experimentation (in, for example, concentration and temperature fields).
- Frequently the experimental models are tested with boundary conditions that are not identical to those of the extended model. This is sustained by the different evolution of the temperature and concentration fields described above.

If we can ensure the control and compensation of these uncertainties, then we can appreciate the enormous importance of the experimental chemical engineering research for developing new processes or for modernizing those that are already in use.

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