Section 6

Deformable Body Mechanics

6.1 Quick reference – mechanical notation

Principal symbols are used to represent mechanical engineering terms. Symbols may have several different meanings – the commonly used ones are shown below.

Table 6.1

Symbol	*Meaning*
a	Acceleration
	Crack length
	Strain hardening constant
	Bore radius of cylinder
A	Cross-sectional area
	Creep constant
A_1	Eutectoid temperature
b	Rim radius of a cylinder
B	A general constant
c	Maximum distance from neutral axis
C	A general constant
CE	Carbon equivalent
CVN	Charpy V-notch energy
d	Diameter
	Depth
e	Misalignment radial
E	Young's modulus
f	Force
	Frequency
f_{cr}	Critical whirling speed
F	Force
F_{cr}	Buckling load (Euler)
g	Acceleration due to gravity
G	Shear modulus
G_c	Toughness (critical strain energy release rate)
G_{1c}	Toughness (plane)
h	Height
	Depth
HAZ	Heat affected zone
HB	Brinel hardness
HRC	Rockwell C hardness

Engineers' Data Book, Fourth Edition. Clifford Matthews.

Table 6.1 (Cont.)

Symbol	*Meaning*
HV	Vickers hardness
I	Second moment of area
Ix	Second moment of area (parallel axis theory)
J	Polar moment of area
k	Spring constant
k_e	Equivalent shear stress (Von Mises)
K	Bulk modulus
K_c	Fracture toughness
K_I	Stress intensity factor
K_{1c}	Plane strain fracture toughness
ΔK	K range in a fatigue cycle
l	Length
m	Mass
	Exponent in crack growth or strain hardening expression
M	Bending moment
	Couple
n	Nominal strain
N	Number of fatigue cycles
N_f	Number of fatigue cycles to failure
p	Pressure
p_{cr}	Critical pressure (external-pressure buckling)
P	Load
Q	Creep activation energy
r	Radius
r_y	Radius of plastic crack-zone tip
R	Reaction force
	Radius
s	Nominal stress
SCF	Stress concentration factor
t	Thickness
	Time
t_f	Time to failure
T	Tension
	Torque
u	Displacement
v	Velocity
V	Volume
	Shear force
w	Uniformly distributed load
W	Weight
	Width of a cracked component
x	Co-ordinate direction
y	Co-ordinate direction
Y	Crack geometry factor
z	Distance from neutral axis
	Co-ordinate direction

6.2 Engineering structures – so where are all the pin joints?

Much of engineering mechanics is based on the assumption that parts of structures are connected by pin joints. Similarly, members are continually assumed to be 'simply supported' and structural members pretend to be infinitely long, compared with their section thickness. The question is: do such members really exist?

They are certainly not immediately apparent – look at a bridge or steel tower and you will struggle to find a single joint containing a pin. The structural members will be channels, I-beams, or box sections surrounded by a clutter of plates, gussets, and flanges, not simple beams of nice prismatic section. So where is the relevance of all those clean theories of statics and vector mechanics?

Fortunately, the answer exists already, hidden in 200 years of engineering experience. Calculations based on simple bending theory, for example, have been validated against actual maximum stresses and deflections experienced in real structures and proved sufficiently accurate (say $\pm$ 10%) to represent reality. Once a factor of safety is introduced (see Section 7.5), then the simplified calculations are as accurate as they need to be. They are, to all intents and purposes, *correct*.

Simply supported assumptions work the same way. The complicated-looking supports of a bridge deck do act like simple supports when you consider the length of the beamlike members they are supporting. Equally, the members themselves dissipate stresses induced by constraint from the 'real' supports within a short distance from the support, so they *act like* long thin members, even though they may not be.

The design of engineering structures is built around findings like this. They have been proven quantitatively, by using strain-gauges and measuring deflections, and by advanced techniques such as FE analysis and photo-elastic models. Complete structures, aeroplanes, ships, and buildings have been investigated to demonstrate the validity of taught theories of statics and mechanics. The results is that all these types of structures in the world are designed using equations which are unerringly similar – proof enough of the validity of the theories behind them. Try to improve theoretical techniques, by all means, but don't ignore what has been found already, including those assumptions about pin joints and simply supported beams.

6.3 Simple stress and strain

$$\text{Stress, } \sigma = \frac{\text{load}}{\text{area}} = \frac{\text{P}}{\text{A}} (\text{units are N/m}^2)$$

$$\text{Strain, } \varepsilon = \frac{\text{change in length}}{\text{original length}} = \left(\frac{\delta l}{l} \text{a ratio, therefore no units}\right)$$

$$\text{Hooke's Law} : \frac{\text{stress}}{\text{strain}} = \text{constant}$$

$$= \text{Young's modulus, } E\ (\text{units are } N/\text{m}^2)$$

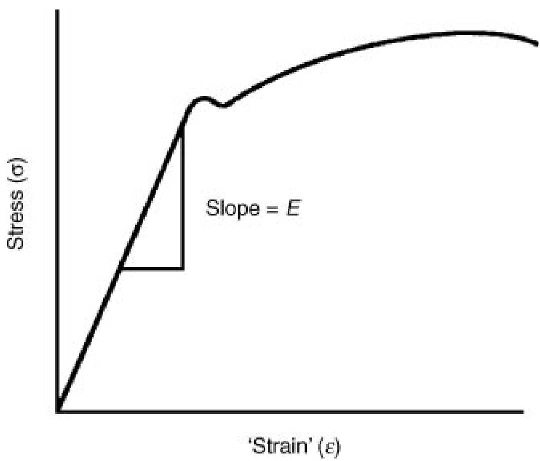


Figure 6.1

$$\text{Poisson's ratio, } \nu = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

$$= \frac{\delta d/d}{\delta l/l} (\text{a ratio, therefore no units})$$

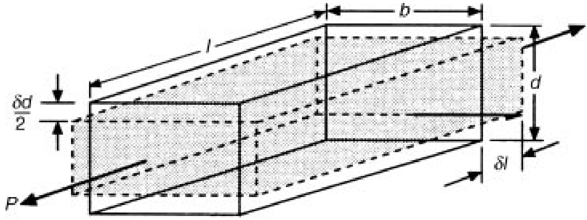


Figure 6.2

$$\text{Shear stress}, \tau = \frac{\text{shear load}}{\text{area}} = \frac{Q}{A} (\text{units are } N/\text{m}^2)$$

$$\text{Shear strain}, \gamma = \text{angle of deformation under shear stress}$$

$$\begin{aligned}\text{Modulus of rigidity}, G &= \frac{\text{shear stress}}{\text{shear strain}} = \frac{\tau}{y} \\ &= \text{Constant}, G \quad (\text{units are N}/m^2)\end{aligned}$$

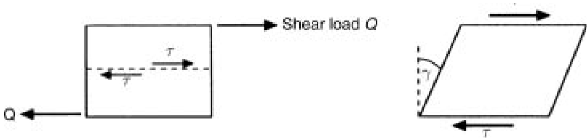


Figure 6.3

$$\text{Thermal stress}, \sigma_t \cong E\varepsilon = E\alpha t$$

where

α = linear coefficient
t = temperature change

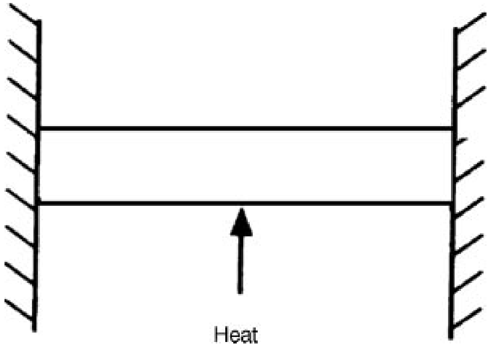


Figure 6.4

6.4 Simple elastic bending

Simple theory of elastic bending is:

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

M = applied bending moment
I = second moment about the neutral axis
R = radius of curvature of neutral axis
E = Young's modulus
σ = stress due to bending at distance y from neutral axis

The second moment of area is defined, for any section, as

$$I = \int y^2 \mathrm{d}A$$

I for common sections is calculated as follows in Fig. 6.5.

Section modulus Z is defined as

$$Z = \frac{l}{y}$$

Strain energy due to bending U is defined as

$$U = \int_0^l \frac{M^2 \mathrm{d}s}{2El}$$

For uniform beams subject to constant bending moment this reduces to

$$U = \frac{M^2 l}{2EI}$$

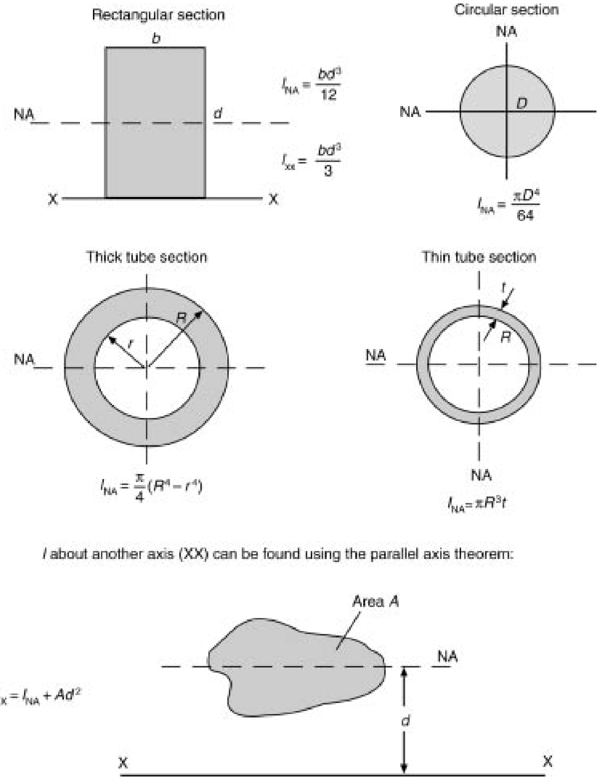


Figure 6.5

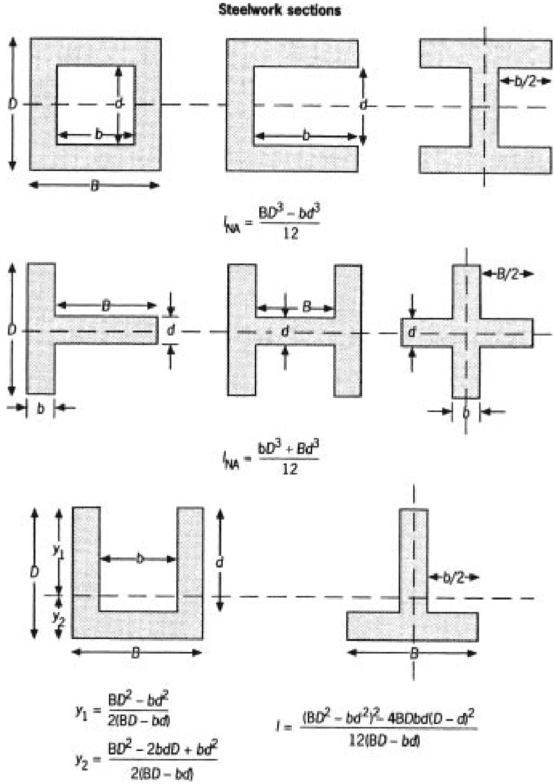


Figure 6.5 (cont.)

6.5 Slope and deflection of beams

Many engineering components can be modelled as simple beams.

The relationships between load W, shear force SF, bending moment M, slope, and deflection are

$$\text{Deflection} = \delta (\text{or } y)$$

$$\text{Slope} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$M = EI\frac{d^2y}{dx^2}$$

$$F = EI\frac{d^3y}{dx^3}$$

$$W = EI\frac{d^4y}{dx^4}$$

Values for common beam configurations are shown in Fig 6.6.

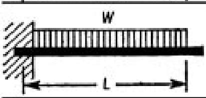
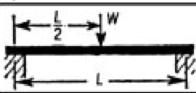
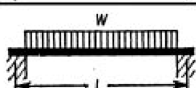
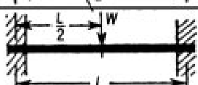
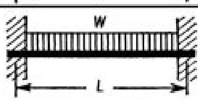
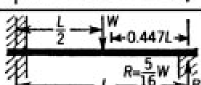
Conditions of support and loading	Bending moment (maximum)	Shearing force maximum)	Safe load W	Deflection (maximum)
	WL	W	$\frac{M}{L}$	$\frac{WL^3}{3EI}$
	$\frac{WL^2}{2}$	W	$\frac{2M}{L}$	$\frac{WL^3}{8EI}$
	$\frac{WL}{4}$	$\frac{W}{2}$	$\frac{4M}{L}$	$\frac{WL^3}{48EI}$
	$\frac{WL^2}{8}$	$\frac{W}{2}$	$\frac{8M}{L}$	$\frac{5WL^3}{384EI}$
	$\frac{WL}{8}$	$\frac{W}{2}$	$\frac{8M}{L}$	$\frac{WL^3}{192EI}$
	$\frac{WL^2}{12}$	$\frac{W}{2}$	$\frac{12M}{L}$	$\frac{WL^3}{384EI}$
	$\frac{3WL}{16}$	$\frac{11W}{16}$	$\frac{16M}{3L}$	$\frac{WL^3}{107EI}$
	$\frac{WL}{8}$	$\frac{5W}{8}$	$\frac{8M}{L}$	$\frac{WL^3}{187EI}$

Figure 6.6

6.6 Torsion

For solid or hollow shafts of uniform cross-section, the torsion formula is

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{l}$$

T = torque applied (N m)
J = polar second moment of area (m^4)
τ = shear stress (N/m^2)
R = radius (m)
G = modulus of rigidity (N/m^2)
θ = angle of twist (rad)

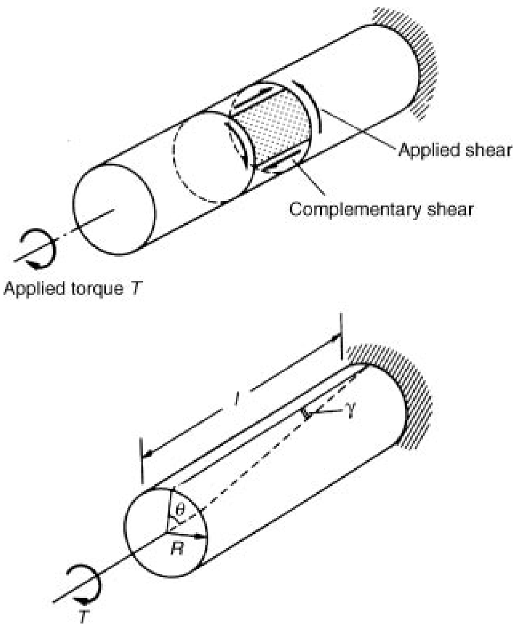
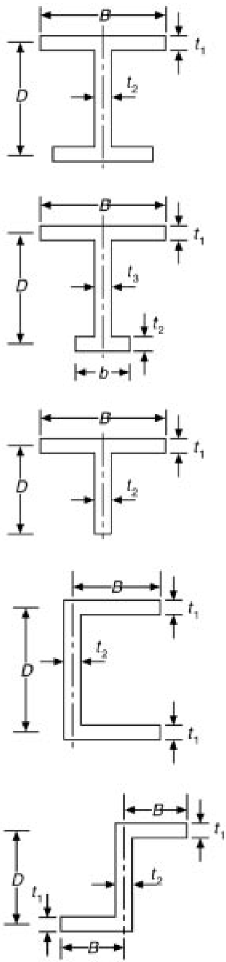


Figure 6.7



$$J = \frac{1}{3}(2Bt_1^3 + Dt_2^3)$$

$$J = \frac{1}{3}(Bt_1^3 + bt_2^3 + Dt_3^3)$$

$$J = \frac{1}{3}(Bt_1^3 + Dt_2^3)$$

$$J = \frac{1}{3}(2Bt_1^3 + Dt_2^3)$$

$$J = \frac{1}{3}(2Bt_1^3 + Dt_2^3)$$

The polar second motion of area (J) m^4 is a measure of the stiffness of a member in pure twisting

Figure 6.8 Torsion Formulae

For solid shafts

$$J = \frac{\pi D^4}{32}$$

For hollow shafts

$$J = \frac{\pi(D^4 - d^4)}{32}$$

For thin-walled hollow shafts

$$J \cong \pi D^3 t$$

where

$r =$ mean radius of shaft wall
$t =$ wall thickness

Strain energy in torsion

$$U = \frac{T^2 l}{2GJ} = \frac{GJ\,\theta^2}{2l}$$

Shaft under combined bending moment, M, and torque, T, from bending

$$\sigma = \frac{MD}{2l}$$

from torsion

$$\tau = \frac{TD}{2J}$$

This results in an 'equivalent' bending moment (M_e) of

$$M_e = \frac{1}{2}(\sqrt{M^2 + T^2})$$

A similar approach can be used to give an equivalent torque T_e

$$T_e = \sqrt{M^2 + T^2}$$

6.7 Thin cylinders

Most pressure vessels have a diameter:wall thickness ratio of >20 and can be modelled using thin cylinder assumptions. The basic equations form the basis of all pressure vessel codes and standards.

Basic equations are

$$\text{Circumferential(hoop)stress, } \sigma_H = \frac{pd}{2t}$$

$$\text{Hoop strain, } \varepsilon_H = \frac{1}{E}(\sigma_H - v\sigma_L)$$

$$\text{Longitudinal(axial)stress, } \sigma_L = \frac{pd}{4t}$$

$$\text{Longitudinal strain, } \varepsilon_L = \frac{1}{E}(\sigma_L - v\sigma_H)$$

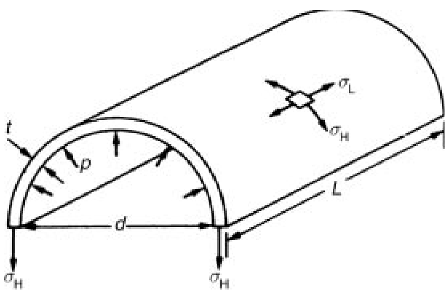


Figure 6.9

6.8 Cylindrical vessels with hemispherical ends

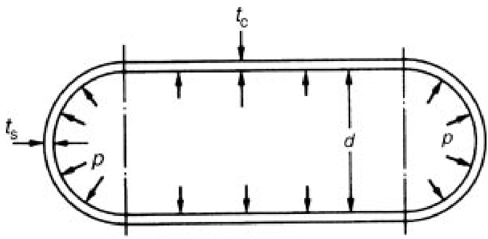


Figure 6.10

For the cylinder

$$\sigma_{HC} = \frac{pd}{2t_c} \text{ and } \sigma_{LC} = \frac{pd}{2t_c}$$

Hoop strain

$$\varepsilon_{HC} = \frac{1}{E}(\sigma_{HC} - \nu\sigma_{LC})$$

For the hemispherical ends

$$\sigma_{HS} = \frac{pd}{4t_s} \text{ and } \varepsilon_{HS} = \frac{pd}{4t_s E}(1-\nu)$$

The differences in strain produce *discontinuity stress* at a vessel head/shell joint.

6.9 Thick cylinders

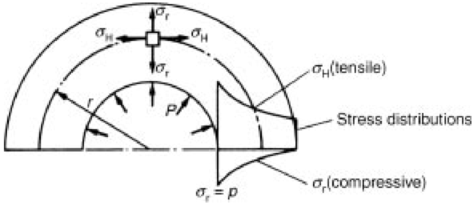


Figure 6.11

Components such as hydraulic rams and boiler headers are designed using thick cylinder assumptions. Hoop and radial stresses vary through the walls, giving rise to the Lamé equations.

$$\sigma = A + \frac{B}{r^2} \text{ and } \sigma_r = A - \frac{B}{r^2}$$

where A and B are 'Lamé' constants

$$\varepsilon_H = \frac{\sigma_H}{E} - \frac{v\sigma_r}{E} - \frac{v\sigma_L}{E}$$

$$\varepsilon_L = \frac{\sigma_L}{E} - \frac{v\sigma_r}{E} - \frac{v\sigma_H}{E}$$

Lamé constant (A) is given by

$$A = \frac{P_1R_1^2 - P_2R_2^2}{R_2^2 - R_1^2}$$

$P_1 =$ internal pressure
$P_2 =$ external pressure
$R_1 =$ internal radius
$R_2 =$ external radius

6.10 Buckling of struts

Long and slender members in compression are termed struts. They fail by buckling before reaching their true compressive yield strength. Buckling loads W_b depend on the loading case.

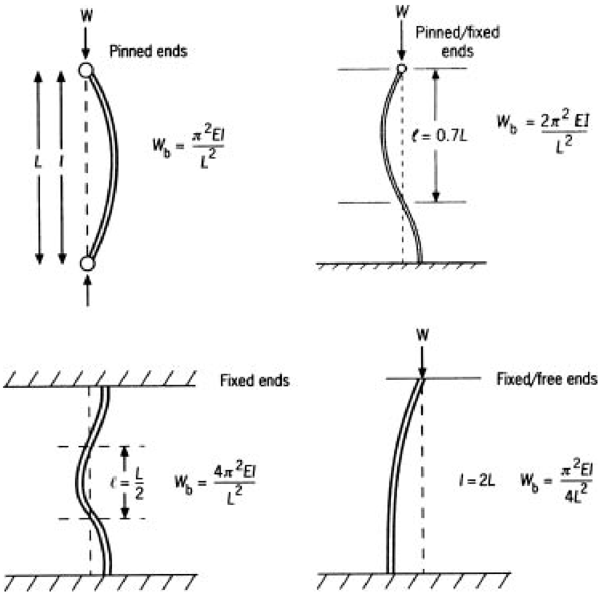


Figure 6.12

The *equivalent length*, *l*, of the strut is the length of a single 'bow' in the deflected condition.

6.11 Flat circular plates

Many parts of engineering assemblies can be analysed by approximating them to flat circular plates or annular rings. The general equation governing slopes and deflections is

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}y}{\mathrm{d}r}\right)\right] = \frac{W}{D}$$

where

$$D = \frac{Et^3}{12(1-\nu^2)}$$

$\hat{y}$ = maximum deflection

$\frac{\mathrm{d}y}{\mathrm{d}r}$ = slope

W = applied load

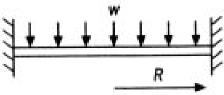
t = thickness

D = flexural stiffness

E = Young's modulus

$\hat{\sigma}_r$ = maximum radial stress

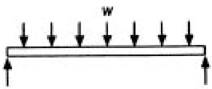
$\hat{\sigma}_z$ = maximum tangential stress



$$\hat{y} = \frac{3wR^4}{16Et^3}(1-\nu^2)$$

$$\hat{\sigma}_r = \frac{3wR^2}{4t^2}$$

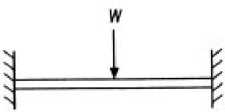
$$\hat{\sigma}_t = \frac{3wR^2}{8t^2}(1+\nu)$$



$$\hat{y} = \frac{3wR^4}{16Et^3}(5+\nu)(1-\nu)$$

$$\hat{\sigma}_r = \frac{3wR^2}{8t^2}(3+\mu)$$

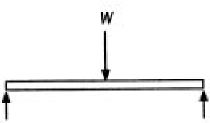
$$\hat{\sigma}_t = \frac{3wR^2}{8t^2}(3+\nu)$$



$$\hat{y} = \frac{3WR^2}{4\pi Et^3}(1-\nu^2)$$

$$\hat{\sigma}_r = \frac{3W}{2\pi t^2}$$

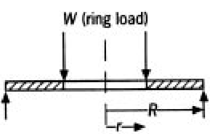
$$\hat{\sigma}_t = \frac{3\nu W}{2\pi t^2}$$



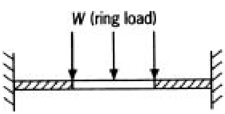
$$\hat{y} = \frac{3WR^2}{4\pi Et^3}(3+\nu)(1-\nu) \quad (1-\nu^2)$$

$$\hat{\sigma}_r = \frac{3W}{2\pi t^2}(1+\nu)\ln\frac{R}{r} \quad \text{at radius } r$$

$$\hat{\sigma}_t = \frac{3W}{2\pi t^2}\left[(1+\nu)\ln\frac{R}{r} + (1-\nu)\right] \quad \text{at radius } r$$



$$\hat{\sigma} = \frac{3W(1+\nu)}{\pi t^2}\left[\frac{R^2}{(R^2-r^2)}\ln\frac{R}{r}\right]$$

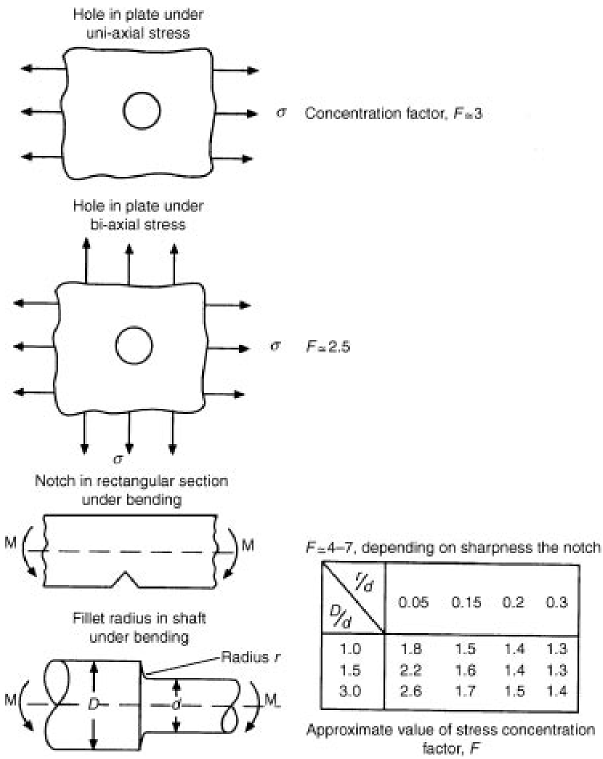


$$\hat{\sigma} = \frac{3W}{2\pi t^2}\left(\frac{R^2-r^2}{R^2}\right)$$

Figure 6.13

6.12 Stress concentration factors

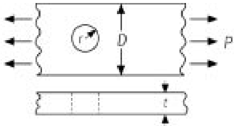
The effective stress in a component can be raised well above its expected levels owing to the existence of geometrical features causing stress concentrations under dynamic elastic conditions. Typical factors are as shown in 6.14.



D/d \ r/d	0.05	0.15	0.2	0.3
1.0	1.8	1.5	1.4	1.3
1.5	2.2	1.6	1.4	1.3
3.0	2.6	1.7	1.5	1.4

Approximate value of stress concentration factor, F

Figure 6.14

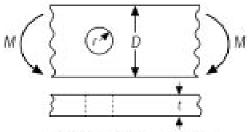


Hole in plate under uni-axial stress

$$\hat{\sigma} \simeq F\left[\frac{P}{t\,(D-2r)}\right]$$

$$F \simeq 3 - 3.13\left(\frac{2r}{D}\right) + 3.66\left(\frac{2r}{D}\right)^2 - 1.53\left(\frac{2r}{D}\right)^3$$

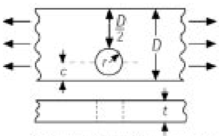
The maximum stress is at the edge of the hole



Hole in plate under in-plane bending

$$\hat{\sigma} \text{ at edge of the hole} \simeq 2\left[\frac{12Mr}{t\,[D^3-(2r)^3]}\right]$$

$$\hat{\sigma} \text{ at edge of the plate} \simeq \frac{6MD}{t\,[D^3-(2r)^3]}$$

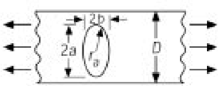


Off-centre hole in plate under uni-axial stress

$$\hat{\sigma} = F\sigma_{nominal}$$

$$\sigma_{nominal} \simeq \frac{P\sqrt{1-(r/c)^2}\,[1-(c/D)]}{Dt[1-(r/c)]\,[1-(c/D)]\,\left[2-\sqrt{1-(r/c)^2}\right]}$$

$$F \simeq 3 - 3.13\,(r/c) + 3.66\,(r/c)^2 - 1.53\,(r/c)^3$$



Elliptical hole in plate under uni-axial stress

$$\hat{\sigma} = F\sigma_{nominal}$$

$$\sigma_{nominal} = \frac{P}{t\,(D-2a)}$$

$$F \simeq F_1 + F_2\left(\frac{2a}{D}\right) + F_3\left(\frac{2a}{D}\right)^2 + F_4\left(\frac{2a}{D}\right)^3$$

for $0.5 \le \frac{a}{b} \le 10$

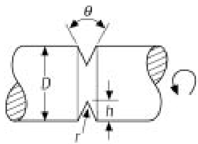
$$F_1 \simeq 1 + \sqrt{\frac{a}{b}} + \frac{2a}{b}$$

$$F_2 \simeq -0.351 - 0.021\sqrt{\frac{a}{b}} - \frac{2.483a}{b}$$

$$F_3 \simeq -3.621 - 5.183\sqrt{\frac{a}{b}} + \frac{4.494a}{b}$$

$$F_4 \simeq -2.27 + 5.2\sqrt{\frac{a}{b}} - \frac{4a}{b}$$

Figure 6.15 Approximate stress concentration factors (Elastic Stresses)



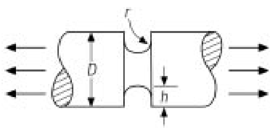
V-notch in circular shaft under torsion

F_v = stress concentration factor for V-notch

$$F_v \simeq F_u - \left[0.02 + 0.14\left(\frac{\theta}{135}\right)^2\right](F_u - 1)\, F_u$$

where F_u = stress concentration factor for U-notch in torsion

for $\frac{r}{D - 2h} \le 0.01$ and $\theta \le 135^\circ$



U-notch in circular shaft under axial tension

F_u = stress concentration for U-notch

$$F_u = F_1 + F_2\left(\frac{2h}{d}\right) + F_3\left(\frac{2h}{d}\right)^2 + F_4\left(\frac{2h}{d}\right)^3$$

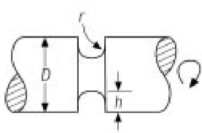
for $0.25 \le \frac{h}{r} \le 2$

$$F_1 = 0.46 + 3.35\sqrt{\frac{h}{r}} - \frac{0.77h}{r}$$

$$F_2 = 3.13 - 16\sqrt{\frac{h}{r}} + \frac{7.4h}{r}$$

$$F_3 = -6.9 + 29.3\sqrt{\frac{h}{r}} + \frac{16.1h}{r}$$

$$F_4 = 4.3 - 16.7\sqrt{\frac{h}{r}} + \frac{9.5h}{r}$$



U-notch in circular shaft under torsion

F_u = stress concentration for U-notch

$$F_u = F_1 + F_2\left(\frac{2h}{D}\right) + F_3\left(\frac{2h}{D}\right)^2 + F_4\left(\frac{2h}{D}\right)^3$$

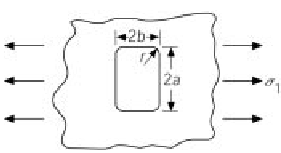
for $0.25 \le \frac{h}{r} \le 2$

$$F_1 = 1.24 + 0.26\sqrt{\frac{h}{r}} + 0.5\frac{h}{r}$$

$$F_2 = -3 + 3.3\sqrt{\frac{h}{r}} + \frac{3.63h}{r}$$

$$F_3 = 7.2 - 11.3\sqrt{\frac{h}{r}} + \frac{8.3h}{r}$$

$$F_4 = -4.4 + 7.75\sqrt{\frac{h}{r}} - \frac{5.17h}{r}$$



Rectangular hole with round corners in "infinite" plate under uniaxial stress

$$\hat{\sigma} = F\sigma_1$$

$$F \simeq F_1 + F_2\left(\frac{b}{a}\right) + F_3\left(\frac{b}{a}\right)^2 + F_4\left(\frac{b}{a}\right)^3$$

for $0.2 \le \frac{r}{b} \le 1$ and $0.3 \le \frac{b}{a} \le 1$

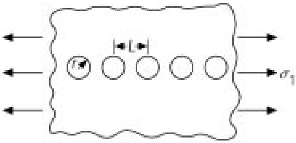
$$F_1 \simeq 14.8 - 15.8\sqrt{\frac{r}{b}} + \frac{8.15r}{b}$$

$$F_2 = -11.2 - 9.7\sqrt{\frac{r}{b}} + \frac{9.6r}{b}$$

$$F_3 = 0.2 + 38.6\sqrt{\frac{r}{b}} - \frac{27.4r}{b}$$

$$F_4 = 3.2 - 23\sqrt{\frac{r}{b}} + \frac{15.5r}{b}$$

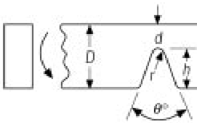
Figure 6.15 (Cont.)



Row of circular holes in "infinite" plate under uniaxial stress

$$\hat{\sigma} = F\sigma_1$$

$$F = 3 - \frac{2r}{L} - 2.1\left(\frac{2r}{L}\right)^2 + 1.9\left(\frac{2r}{L}\right)^3$$

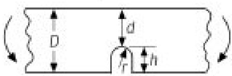


V-notch in rectangular section under bending

F_V = Stress concentration factor for V-notch

$$F_V = 1.11\ F_u - \left[0.03 + 0.11\left(\frac{\theta^\circ}{150}\right)^4\right] F_u^2$$

where F_u = Stress concentration factor for U-notch



U-notch in rectangular section under bending

F_u = Stress concentration factor for u-notch

$$F_u = F_1 + F_2\left(\frac{h}{d}\right) + F_3\left(\frac{h}{d}\right)^2 + F_4\left(\frac{h}{d}\right)^3$$

for $0.5 \le \frac{h}{r} \le 4$

$$F_1 \simeq 0.72 + 2.4\sqrt{\frac{h}{r}} - \frac{0.13h}{r}$$

$$F_2 \simeq 1.98 - 11.5\sqrt{\frac{h}{r}} + \frac{2.2h}{r}$$

$$F_3 \simeq -4.4 + 18.75\sqrt{\frac{h}{r}} + \frac{4.6h}{r}$$

$$F_4 \simeq 2.7 - 9.7\sqrt{\frac{h}{r}} + \frac{2.5h}{r}$$

Figure 6.15 (Cont.)