C2.7 Category theory: problem sheet 4

1. Let $T\colon \mathrm{Set} \to \mathrm{Set}$ be the endofunctor which sends any set X to the set of all finite words on X.
   (a) Define a monad structure on T.
   (b) Without using the Barr–Beck theorem construct an equivalence between $\mathrm{Alg}_T(\mathrm{Set})$ and the category of monoids.
   (c) Modify the monad T so that the category of algebras will be equivalent to the category of semigroups (non-unital monoids).

2. Let $V \in \mathrm{Vect}$ be a fixed vector space. Recall that $V \otimes -\colon \mathrm{Vect} \to \mathrm{Vect}$ preserves colimits due to the existence of the tensor-Hom adjunction.
   (a) Assume $V \otimes -$ preserves limits. Verify the conditions of the adjoint functor theorem (dual to Theorem 4.18) to conclude that it has a left adjoint F.
   (b) Show that every vector space can be written as a colimit of the ground field k. Conclude that F is given by tensoring with a vector space.
   (c) (*) Identify this vector space and show that $V \otimes -$ preserves limits iff V is finite-dimensional.

3. Let R be a commutative algebra over a field k. In this exercise we explicitly identify the category
$$\mathcal{C} = \mathrm{Fun}_{\mathrm{colim}}(R - \mathrm{mod}, \mathrm{Vect})$$
of colimit-preserving functors from $R - \mathrm{mod}$ to k-vector spaces.
   (a) Let $F\colon \mathcal{C} \to \mathrm{Vect}$ be the functor which sends a functor $G\colon R - \mathrm{mod} \to \mathrm{Vect}$ to $G(R)$. Show that $F^L\colon \mathrm{Vect} \to \mathcal{C}$ where $F^L(V)$ is the functor which sends $M \in R - \mathrm{mod}$ to $M \otimes_k V$ is a left adjoint to F. Hint: $\mathrm{Hom}_{R-\mathrm{mod}}(R, M) \cong M$.
   (b) Show that every R-module is an iterated colimit of R. Deduce that F is conservative.
   (c) Verify the conditions of the Barr–Beck theorem and conclude that $\mathcal{C} \cong R - \mathrm{mod}$.

4. (*) If you have covered some algebraic geometry (for example by taking the course C3.4 or reading 'The geometry of schemes' by D. Eisenbud and J. Harris), then fill in some of the details in §6 of the lecture notes.

5. (*) Find out about 2-categories and ∞-categories (also called quasi-categories), for example by reading
   E. Riehl, Categorical Homotopy theory (CUP, 2014), available online at www.math.jhu.edu/ eriehl/cathtpy.pdf.