C2.7 Category theory: problem sheet 1

Starred questions are optional.

1. (a) Let $(X, \leq)$ be a poset (i.e. a partially ordered set). Show that one can make it into a category such that for any $x_1, x_2 \in X$ the set $\mathrm{Hom}(x_1, x_2)$ has a single element if $x_1 \leq x_2$ and is empty otherwise.
   (b) Let $[n]$ be the category associated to the poset of integers $\{0, 1, ..., n\}$ with the usual ordering. Given another category $\mathcal{C}$ describe the functor category $\mathrm{Fun}([n], \mathcal{C})$.
2. Given a group G consider the category $*/G$ with one object $*$ and $\mathrm{Hom}(*, *) = G$.
   (a) How can we describe functors $*/G_1 \to */G_2$ and natural transformations between such functors in group-theoretic terms?
   (b) Describe functors $*/\mathbb{Z} \to */G$. What are natural transformations between two such functors (for fixed G)?
3. Recall that an equivalence between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms relating their compositions to identity functors on $\mathcal{C}$ and $\mathcal{D}$.
   Show that the linear duality functor $\mathrm{Hom}_{\mathrm{Vect}_k^{fd}}(-, k)$ defines an equivalence
   $$\mathrm{Vect}_k^{fd} \to (\mathrm{Vect}_k^{fd})^{op}$$
   from the category Vect_k^{fd} of finite-dimensional vector spaces over a field k to its opposite.
4. Recall that a groupoid is a category in which every morphism is invertible. A contractible category is a groupoid $\mathcal{C}$ such that any two objects of $\mathcal{C}$ are isomorphic and $\mathrm{End}(c)$ is the trivial group for some $c \in \mathcal{C}$. Show that for any two objects x, y of a contractible category $\mathcal{C}$ the set $\mathrm{Hom}(x, y)$ consists of a single element, and that $\mathcal{C}$ is equivalent to a discrete category with a single object.
5. (*) Given a set of groups $\{G_i\}_{i \in I}$ indexed by I one has the category $\sqcup_{i \in I} */G_i$ with I its set of objects, endomorphisms of $i \in I$ given by G_i, and no other morphisms. Show that any (small) groupoid is equivalent to a category of this form.
6. (*) Show that associating to a group its centre cannot be made into a functor $\mathrm{Grp} \to \mathrm{Ab}$.