ملخص مادة..

معادلات تفاضلية

لجنة الميكانيك

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Differential Equations: (D.E.'s)

A D.E is a mathematical equation for an unknown Function of one or several variables thats relates the values of the function it self and its derivatives

* It can be classified as follows:

$\boxed{1}$ Type: a) Ordinary D.E:
it is a D.E in which the unknown func. is a func of only one independent variables.

b) Partial D.E's:
it is a D.E in which the unknown function is a func. of mor than one independent variable.

(dep. varible) ← $y = F(x)$ → (independent varible)
(dep. varible) ← $\frac{dy}{dx} = F'(x)$
(ind. varible) ← dx
} Ordinary D.E

(dependent) ← $z = F(x,y)$ → (x & y independent)
$\frac{dz}{dx}$, $\frac{dz}{dy}$
} Partial D.E

$\boxed{2}$ Order: it is the order of the highest derivative occurs in the equation.

$\boxed{3}$ Degree: it is the exponent of the highest derivati appears in the equation.

هو الأس الذي يرفع إليه أعلى تفاضل في المعادلة

Ex: ① $\frac{dy}{dx} - \frac{d^2y}{dx^2} + y = 0$ order 2 $\frac{d^2y}{dx^2}$

degree 1 $\left(\frac{d^2y}{dx^2}\right)^1$

② $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{d^4y}{dx^4}\right)^2 = \sin x$ order 4

degree 2

* Ordinary D.E's :

First order ⇒ form $F(x, y, y') = 0$ ---(*)

لـ الأساس في المعادلة يجب أن

تكون موجودة ولو لوحدها

it also can be written in the form

$y' = F(x, y)$ ---- (1)

* the solution of a First order D.E on an open interval $a < x < b$ is a relation between the dep. and indep variables. say $y = h(x)$

Ex: $2x + 2yy' = 0$

$y' = \frac{-2x}{2y} = \frac{-x}{y}$

$\frac{dy}{dx} = \frac{-x}{y} \Rightarrow \int y\,dy = \int -x\,dx$

$\frac{1}{2}y^2 = \frac{-1}{2}x^2 + c$

$y^2 = -x^2 + c_1$

$y = \sqrt{-x^2 + c_1}$

Ex: Find y' — حاصل ضرب اقتران بالنسبة لـ (x)

① $3x^2 - xy^2 = C$

$\to 6x - \cancel{x}(x * 2yy' + y^2) = 0$

$6x - y^2 = 2xyy'$

$$y' = \frac{6x - y^2}{2xy}$$

② $xy^2 - 1 = Cy$

$x(2yy') + y^2 = Cy'$

$2xyy' - Cy' = -y^2$

$y'(2xy - C) = -y^2$

$$y' = \frac{-y^2}{2xy - C}$$

③ $x\sin y + xy^2 = C$

$xy'\cos y + \sin y + x(2yy') + y^2 = 0$

$y'(x\cos y + 2xy) = -\sin y - y^2$

$$y' = \frac{-(\sin y + y^2)}{x\cos y + 2xy}$$

* Seperable D.E's :

* إذا استطعنا أن نجد جميع قيم x مع dx وجميع قيم y مع dy
نستخدم هذه الطريقة (الفصل)

can be written as a product of two continuous func. with respect to (x) and the other with respect to y

i.e : $f(x,y) = g(x)\,h(y)$

$$y' = g(x)\,h(y)$$

$$\frac{dy}{dx} = g(x)\,h(y)$$

$$dy = g(x)\,h(y)\,dx$$

$$\underbrace{\int \frac{1}{h(y)}\,dy}_{c_1} = \underbrace{\int g(x)\,dx}_{c_2}$$

Ex: solve $y' = \frac{-x}{y} \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

$$\int y\,dy = \int -x\,dx \quad \text{---- seperable.}$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$$

$$y^2 = -x^2 + 2c$$

$$y = \sqrt{-x^2+2c} \quad \text{or} \quad y^2 + x^2 = \underbrace{2c = r^2}_{\text{معادلة الدائرة}}$$

Ex: ① $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$$\frac{\sec^2 x \, dx}{\tan x} = \frac{-\sec^2 y \, dy}{\tan y}$$

$$\int \frac{\sec^2 x}{\tan x} dx = \int \frac{-\sec^2 y}{\tan y} dy$$

$$\ln(\tan x) + C = -\ln(\tan y)$$

$$\ln(\tan x) + \ln(\tan y) = C$$

$$\ln(\tan x \tan y) = C$$

$$\tan x \tan y = e^C = A$$

$$\tan y = \frac{A}{\tan x} = A \cot x$$

الحل العام

$y = \tan^{-1}(A \cot x)$ ... general solution.

② $(xy^2 + x)dx + (yx^2 + y)dy = 0$; $y(0) = 0$

$$x(y^2+1)\,dx + y(x^2+1)\,dy = 0$$

$$x(y^2+1)\,dx = -y(x^2+1)\,dy$$

$$\frac{-x}{x^2+1}dx = \frac{y}{y^2+1}dy \Rightarrow -\frac{1}{2}\int \frac{2x}{x^2+1}dx = \frac{1}{2}\int \frac{2y}{y^2+1}dy$$

$$-\frac{1}{2}\ln(x^2+1) + c = \frac{1}{2}\ln(y^2+1)$$

$$C = \ln(y^2+1) + \ln(x^2+1)$$

$$C = \ln\left((y^2+1)(x^2+1)\right)$$

note:
if $\ln x = \text{const.}$
$e^{\text{const.}} = x$
$\therefore \text{const.} = x$

$$(y^2+1)(x^2+1) = C$$

$$y^2+1 = \frac{C}{x^2+1} \Rightarrow y = \sqrt{\frac{C}{x^2+1} - 1}$$

$$y(0) = 0 \Rightarrow 0 = \sqrt{C-1}$$

$$\boxed{C = 1}$$

$$\boxed{y = \sqrt{\frac{1}{x^2+1} - 1}}$$ P.S

Ex: ① $e^{y}(1+x^2)y' = 2x(1+e^{y})$

② $y' = \frac{1+y^2}{\sqrt{1-x^2}}$ ③ $y' = x^2(1+y)$

④ $y' = \frac{6x^5 - 2x+1}{\cos y + e^{y}}$

solution: ① $\int \frac{e^{y}}{1+e^{y}} dy = \int \frac{2x}{1+x^2} dx$

$\ln(1+e^{y}) = \ln(1+x^2) + c$

$\ln(1+e^{y}) - \ln(1+x^2) = c$

$\ln\left(\frac{1+e^{y}}{1+x^2}\right) = c \Rightarrow \frac{1+e^{y}}{1+x^2} = c \Rightarrow 1+e^{y} = (1+x^2)c$

$e^{y} = cx^2 + c - 1 \Rightarrow e^{y} = cx^2 + c$

$e^{y} = (x^2+1)c - 1$

$y = \ln((x^2+1)c - 1)$

② $y' = \frac{1+y^2}{\sqrt{1-x^2}} \Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{\sqrt{1-x^2}}$

$\tan^{-1} y = \sin^{-1} x - c$

$y = \tan(\sin^{-1} x - c)$

③ $y' = x^2(1+y) \Rightarrow \int \frac{dy}{1+y} = \int x^2 dx$

$\ln(1+y) = \frac{1}{3}x^3 + C$

$1+y = e^{\frac{1}{3}x^3 + c} \Rightarrow 1+y = e^{\frac{1}{3}x^3} \cdot \overbrace{e^{c}}^{A \text{ ثابت}} = Ae^{\frac{1}{3}x^3}$

$\boxed{y = Ae^{\frac{1}{3}x^3} - 1}$

④ $\frac{dy}{dx} = \frac{6x^5 - 2x+1}{\cos y + e^{y}} \Rightarrow \int (\cos y + e^{y}) dy = \int (6x^5 - 2x + 1) dx$

$\sin y + e^{y} = x^6 - x^2 + x + C$ //

Reduction to Seperable:
(Homogeneous equations)

A Function $F(x,y)$ is called homogeneous of degre (n) if $f(tx,ty) = t^n * f(x,y)$

ex: $f(x,y) = xy + x^2 + y^2$

$$f(tx,ty) = t^2xy + t^2x^2 + t^2y^2$$
$$= t^2(xy+x^2+y^2)$$

$\therefore f(tx,ty) = t^2 f(tx,ty) \Rightarrow$ Homog. of degree $\underline{\underline{2}}$

ex: $f(x,y) = x^2 e^{\frac{y}{x}} + y^2$

$$f(tx,ty) = t^2x^2 e^{\frac{ty}{tx}} + t^2y^2$$
$$= t^2(x^2 e^{\frac{y}{x}} + y^2) \Rightarrow \text{Homog. of degre } \underline{\underline{2}}$$

إذا وجد في الاقتران $\left.\begin{array}{l} e^x, e^y, \ln x, \ln y \\ \sin x, \sin y \\ \cos x, \cos y \\ \tan x, \tan y \end{array}\right\}$ * يكون الاقتران non-Homog ومش ذالك على جميع الاقترانات المثلثية

أما إذا وجد $\left.\begin{array}{l} \ln\frac{y}{x}, \ln y - \ln x, e^{\frac{y}{x}}, e^{\frac{x}{y}} \\ \sin\frac{x}{y}, \sin\frac{y}{x} \end{array}\right\}$ * يكون الاقتران Homogeneous ومش على جميع الاقترانات المثلثية

Ex: $F(x,y) = \dfrac{x^2+y^2}{xy^2+y^3}$

$$f(tx,ty) = \frac{t^2x^2+t^2y^2}{t^3xy^2+t^3y^3} = \frac{t^2}{t^3}\left(\frac{x^2+y^2}{xy^2+y^3}\right) = t^{-1}(\_\_\_\_)$$

$\therefore$ Homog. of degree -1

Ex: $f(x,y) = 5 \Rightarrow f(tx,ty) = 5 * t^0 \Rightarrow$ H. degree $\underline{\underline{0}}$

* If the R.H.S of ① $y' = f(x,y)$ can be expressed as a function of the ratio $\frac{y}{x}$ then this equ. are called reduction to sep. form. to solve this type we do the following:

$\boxed{1}$ let $u = \frac{y}{x}$

$\boxed{2}$ $y = ux$

$\boxed{3}$ $y' = u + xu'$ (الأول x مشتقة الثاني + الثاني x مشتقة الأول)

and the substitution into the given equ. it becomes seperable D.E.

$$y' = g\left(\frac{y}{x}\right) \qquad \Bigg| \quad \text{let } u = \frac{y}{x}$$

$$y' = g(u) \qquad \Bigg| \quad y = ux$$

$$u + xu' = g(u)$$

$$xu' = g(u) - u$$

$$x\frac{du}{dx} = g(u) - u \Rightarrow \frac{1}{x}\frac{dx}{du} = \frac{1}{g(u)-u}$$

$$\frac{1}{x}dx = \frac{1}{g(u)-u}du \quad \text{.... seperable}$$

وبعدها نعوض قيمة $\left(u = \frac{y}{x}\right)$ في الحل النهائي

* $y' = F(x,y)$ these func. can be written in the form $M(x,y)dx + N(x,y)dy = 0$
Where M & N are two Homogeneous func. of the same degree.

<u>Ex:</u> Solve $2xyy' - y^2 + x^2 = 0$

$$2xyy' = y^2 - x^2 \Rightarrow y' = \frac{y^2 - x^2}{2xy}$$

$u = \frac{y}{x}$, $y = ux$, $y' = u + xu'$

$$u + xu' = \frac{u^2x^2 - x^2}{2ux^2} = \frac{(u^2-1)\cancel{x^2}}{2u\cancel{x^2}}$$

$$xu' = \frac{u^2-1}{2u} - u$$

$$xu' = \frac{u^2 - 1 - 2u^2}{2u}$$

$$xu' = \frac{-u^2-1}{2u}$$

$$\frac{1}{x}\frac{dx}{du} = \frac{2u}{-u^2-1}$$

$$\frac{1}{x}dx = \frac{2u}{-(u^2+1)}du$$

$$\ln x = -\ln(u^2+1) + C$$

$$\ln x + \ln(u^2+1) = C$$

$$\ln(x(u^2+1)) = C$$

$$x(u^2+1) = e^c = A$$

$$x\left(\frac{y^2}{x^2} + 1\right) = A$$

$$\frac{y^2}{x} + x = A \Rightarrow y^2 + x^2 = Ax$$

$$\therefore \underline{y^2 = Ax - x^2}$$

Ex: $(xy'-y)\cos\left(\frac{2y}{x}\right) = -3x^4$

$u = \frac{y}{x}$

$y = ux$

$y' = u + xu'$

$(x(u+xu') - ux)\cos 2u = -3x^4$

$(xu + x^2u' - ux)\cos 2u = -3x^4$

$x^2u'\cos 2u = -3x^4$

$u'\cos 2u = -3x^2$

$\cos 2u\, du = -3x^2 dx$

$\frac{1}{2}\sin 2u = -x^3 + C$

$\frac{1}{2}\sin\frac{2y}{x} = -x^3 + C \Rightarrow$ الحل إلى هنا كافي في الإمتحان

$\sin\frac{2y}{x} = -2x^3 + A$

$\frac{2y}{x} = \sin^{-1}(-2x^3 + A)$

$y = \frac{x}{2}\sin^{-1}(-2x^3 + A)$

Ex: solve
1. $xy' = x\sec\frac{y}{x} + y$
2. $xyy' = 2y^2 + 4x^2$
3. $y^2 dy = x(xdy - ydx)e^{\frac{x}{y}}$
4. $x\sin\left(\frac{y}{x}\right)y' = x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)$

solution: $xy' = x\sec\frac{y}{x} + y \Rightarrow y' = \sec\frac{y}{x} + \frac{y}{x}$

let $u = \frac{y}{x}$

$y = ux$

$y' = u + xu'$

$u + xu' = \sec u + u$

$xu' = \sec u$

$x\frac{du}{dx} = \sec u$

$\frac{1}{x}\frac{dx}{du} = \frac{1}{\sec u} \Rightarrow \int\frac{1}{x}dx = \int\cos u\, du$

$\ln x + C = \sin u \Rightarrow u = \sin^{-1}(\ln x + C)$

$y = x\sin^{-1}(\ln x + c)$

② $xyy' = 2y^2 + 4x^2 \Rightarrow y' = \frac{2y}{x} + \frac{4x}{y}$

$u + xu' = 2u + \frac{4}{\frac{y}{x}}$

$u + xu' = 2u + \frac{4}{u}$

$xu' = u + \frac{4}{u} \Rightarrow \frac{1}{x}dx = \frac{1}{u+\frac{4}{u}}$

$\frac{1}{2}\int \frac{2u}{u^2+u}\,du = \int \frac{1}{x}dx$

$\frac{1}{2}\ln u^2+u = \ln x + C$

$\ln\sqrt{u^2+u} = \ln x + C \Rightarrow \ln\left(\frac{\sqrt{u^2+u}}{x}\right) = C$

$\frac{\sqrt{u^2+u}}{x} = C \Rightarrow u^2+u = x^2C$

$u^2 = x^2c - u \Rightarrow \frac{y^2}{x^2} = x^2C - u$

$\therefore\ y = \sqrt{Cx^4 - 4x^2}$

③ $y^2dy = x(x\,dy - y\,dx) = e^{\frac{x}{y}}$

$y^2dy = x^2e^{\frac{x}{y}}dy - xye^{\frac{x}{y}}dx$

$u = \frac{y}{x}$

$y = ux$

$y' = u + xu'$

$(y^2 - x^2e^{\frac{x}{y}})dy = -xye^{\frac{x}{y}}dx \Rightarrow \frac{dy}{dx} = \frac{-xye^{\frac{x}{y}}}{y^2 - x^2e^{\frac{x}{y}}}$

$u + xu' = \frac{-x(ux)e^{\frac{1}{u}}}{u^2x^2 - x^2e^{\frac{1}{u}}} \Rightarrow u + xu' = \frac{-ux^2e^{\frac{1}{u}}}{u^2x^2 - x^2e^{\frac{1}{u}}}$

$xu' = \frac{-ux^2e^{\frac{1}{u}}}{x^2(u^2 - e^{\frac{1}{u}})} - u$

$xu' = \frac{\cancel{-ue^{\frac{1}{u}}} - u^3 + \cancel{ue^{\frac{1}{u}}}}{u^2 - e^{\frac{1}{u}}}$

تتبع حل السؤال

$$xu' = \frac{-u^3}{u^2 - e^{\frac{1}{u}}} \Rightarrow -\frac{1}{x}dx = \frac{u^2 - e^{\frac{1}{u}}}{u^3}du$$

$$-\frac{1}{x}dx = \frac{u^2 - e^{\frac{1}{u}}}{u^3}du \Rightarrow \frac{-1}{x}dx = \left(\frac{u^2}{u^3} - \frac{e^{\frac{1}{u}}}{u^3}\right)du$$

$$-\frac{1}{x}dx = \left(\frac{1}{u} - u^{-3}e^{\frac{1}{u}}\right)du$$

$$-\ln x = \ln u - \int u^{-3}e^{\frac{1}{u}}du$$

let $k = \frac{1}{u}$, $dk = \frac{-du}{u^2}$, $du = -u^2 dk$

$$= \int u^{-3}e^{k} \cdot -u^2 dk$$

$$= -\int k e^{k} dk$$

$$= -ke^{k} + e^{k} = \frac{-1}{u}e^{\frac{1}{u}} + e^{\frac{1}{u}}$$

$$\begin{array}{ccc} k & & +e^{k} \\ 1 & & -e^{k} \\ 0 & & e^{n} \end{array}$$

$$\therefore -\ln x = \ln u + \frac{1}{u}e^{\frac{1}{u}} - e^{\frac{1}{u}}$$

$$\therefore -\ln x = \ln\frac{y}{x} + \frac{x}{y}e^{\frac{x}{y}} - e^{\frac{x}{y}}$$

(4) $x\sin\left(\frac{y}{x}\right)y' = x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)$

$u = \frac{y}{x}$

$y = ux$

$y' = u + xu'$

$$x\sin u\,(u + xu') = x\cos u + y\sin u$$

$$(xu + x^2u')\sin u = x\cos u + ux\sin u$$

$$\cancel{xu\sin u} + x^2u'\sin u = x\cos u + \cancel{ux\sin u}$$

$$x^2u'\sin u = x\cos u$$

$$-1\int\frac{-\sin u}{\cos u}du = \int\frac{1}{x}dx$$

$$-\ln|\cos u| = \ln x + c$$

$$\ln|(\cos u)x| = c \Rightarrow \cos u = \frac{c}{x}$$

$$u = \cos^{-1}\frac{c}{x} \qquad \boxed{y = x\cos^{-1}\frac{c}{x}}$$

Transform to seperable Form :

$F(x,y) = g(ax+by)$ the substitution $z = ax+by$ will be transform it to be sep.

$$z' = a + by' \Rightarrow y' = \frac{1}{b}(z' - a)$$

$$y' = g(ax+by)$$

$$y' = g(z) \Rightarrow \frac{1}{b}(z' - a) = g(z)$$

$$z' - a = b\,g(z)$$

$$z' = b\,g(z) + a$$

$$\frac{dz}{dx} = b\,g(z) + a$$

$$dx = \frac{1}{b\,g(z) + a}\,dz$$ ---- seperable between x & z

Ex: solve $(x+y)dx + dy = 0$

$$\frac{dy}{dx} = -(x+y) \Rightarrow y' = -(x+y)$$

$z = x+y$
$z' = 1 + y'$
$y' = z' - 1$

$$z' - 1 = -z$$

$$z' = 1 - z$$

$$\frac{dz}{dx} = 1 - z$$

$$\frac{1}{1-z}dz = dx$$

$$\int \frac{-1}{z-1}\,dz = \int dx$$

$$-\ln(z-1) = x + c$$

$$\ln(z-1) = -x - c$$

$$z - 1 = e^{(-x-c)}$$

$$z - 1 = e^{-z} * e^{-c}$$

note:
① not seperable
② $M = x+y \Rightarrow M\,dx + N\,dy = 0$
but M and N has not the same degree
∴ not reduction
③ ∴ ⇒ transform.

$$z-1 = Ae^{-x} \Rightarrow z = Ae^{-x}+1$$
$$x+y = Ae^{-x}+1 \Rightarrow y = Ae^{-x}+1-x \qquad g.s$$

if $y(0)=2$, complete the solution.

$$y(0)=2 = A+1 \Rightarrow \boxed{A=1}$$

$$y = e^{-x}+1-x$$

$$y = e^{-x}-x+1 \quad p.s$$

* Equations with linear coefficients :-
(x,y exponent) المعادلات التفاضلية ذات المعاملات الخطية.

$$(a_1x+b_1y+c_1)dx + (a_2x+b_2y+c_2)dy = 0$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are const.

1. If $c_1 = c_2 = 0 \Rightarrow$ homogeneous.

2. If $a_1b_2 = a_2b_1 \Rightarrow$ the the two lines $a_1x+b_1y+c_1$ and $a_2x+b_2y+c_2$ are parallel which means that $\exists$ k such that

$$a_1x + b_1y = k(a_2x+b_2y)$$

$$\therefore (k(a_2x+b_2y)+c_1)dx + (a_2x+b_2y+c_2)dy = 0$$

then the substitution $z = a_2x+b_2y$ will transform the given equation to be seperable.

$$(a_1x+b_1y+c_1)\,dx+(a_2x+b_2y+c_2)\,dy=0$$

$z=a_2x+b_2y$

$z'=a_2+b_2y'$

$y'=\frac{1}{b_2}(z'-a_2)$

$$\frac{dy}{dx}=\frac{-Kz+C_1}{z+C_2}$$

$$\frac{1}{b_2}(z'-a_2)=\frac{-Kz+C_1}{z+C_2}$$

$$z'=\frac{-b_2(Kz+C_1)}{z+C_2}+a_2$$

$$\frac{dz}{dx}=\frac{-b_2(Kz+C_1)+a_2(z+C_2)}{z+C_2}$$

$$dx=\frac{-(z+C_2)}{b_2(Kz+C_1)+a_2(z+C_2)}\,dz \quad \text{sep.}$$

* ملاحظة: في معادلة الخطين التاليين لمعرفة إذا كانا متقاطعين أم (أو متوازيين) وإيجاد نقطة التقاطع نقوم بما يلي :-

$$-2*(2x+3y+4)=0$$
$$(4x+6y+5)=0$$
$$-4x-6y-8=0$$
$$+4x+6y+5=0$$
$$-3=0$$

$$\boxed{\begin{array}{l} a_1=2,\ b_1=3 \\ a_2=4,\ b_2=6 \\ a_1b_2=a_2b_1 \end{array}}$$

يوجد خطأ. إذا هذان الخطان لا يتقاطعان بل هما خطان متوازيان.

$$-2*(2x+3y+4)=0$$
$$4x+6y+8=0$$
$$-4x-6y-8=0$$
$$4x+6y+8=0$$
$$0=0$$

$$\boxed{\begin{array}{l} a_1=2,\ b_1=3 \\ a_2=4,\ b_2=6 \\ a_1b_2=a_2b_1 \end{array}}$$

معادلة صحيحة

إذا يوجد نقاط مشتركة لا نهائية.
(الخطان متوازيان ومنطبقان على بعضهم).

* نتيجة: إذا كان:

$a_1b_2=a_2b_1 \Leftarrow$ خطان متوازيان

$a_1b_2\neq a_2b_1 \Leftarrow$ متقاطعان بنقطة

يتقاطع

If $a_1b_2 \neq a_2b_1$ then the two lines intersect each other at a point, say (m, n), to solve the given D.E

Note: $x + y + 2 = 0$ $\quad a_1 = 1$, $b_1 = 1$

$-x + 2y + 4 = 0$ $\quad a_2 = -1$, $b_2 = 2$

$a_1 b_2 \neq a_2 b_1$

$$x + y + 2 = 0$$
$$+ \quad -x + 2y + 4 = 0$$
$$3y + 6 = 0 \Rightarrow \boxed{y = -2}$$ يوجد حل

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$$

$$\frac{dy}{dx} = \frac{-(a_1x + b_1y + c_1)}{a_2x + b_2y + c_2}$$

let $x = u + m$, $y = v + n$

$dx = du$ --(1) $\quad dy = dv$ (2)

(2)/(1) $\Rightarrow \frac{dy}{dx} = \frac{dv}{du}$

$$\frac{dv}{du} = \frac{-(a_1u + a_1m + b_1v + b_1u + c_1)}{(a_2u + a_2m + b_2v + b_2u + c_2)}$$

$$\frac{dv}{du} = \frac{-(a_1u + a_1m + b_1v + b_1u + c_1)}{(a_2u + a_2m + b_2v + b_2u + c_2)}$$

الذي عليه دائرة سيبقى والباقي يجب أن تكون قيمة كل واحد منهم صفر

$$\frac{dv}{du} = \frac{-(a_1u + b_1v)}{a_2u + b_2v}$$ two homogeneous function of the same degree

$\therefore \frac{dv}{du} \Rightarrow$ Homog.

نفس الطريقة السابقة

$$w = \frac{v}{u} \Rightarrow v = uw$$

$$v' = w + uw'$$

وبعدها نكمل الحل وبعد نعوض بدل $\Leftarrow$ $v = y - n$, $u = x - m$

Ex: solve:

لاحظ أن الاشارة سالبة لذا لازم ندخلها على القوس

$$(x+2y-4)dx - (2x+y-5)\,dy = 0$$

(يتكرر في الامتحانات)

$a_1 b_2 \neq a_2 b_1$

$1 * -1 \neq 2 * -2$ $\quad (m,n) = (2,1)$ من حل المعادلتين نجد

let $x = u+2$, $y = v+1$ $\quad \Big| \quad \dfrac{dy}{dx} = \dfrac{x+2y-4}{2x+y-5}$

$$\frac{dv}{du} = \frac{u+\cancel{2}+2v+\cancel{2}-\cancel{4}}{2u+\cancel{4}+v+\cancel{1}-\cancel{5}} = \frac{u+2v}{2u+v}$$

$w = \dfrac{v}{u}$, $v = um$, $v' = w+uw'$ $\quad \Big| \quad w+uw' = \dfrac{u+2uw}{2u+uw}$

$$w+uw' = \frac{(1+2w)u}{(2+w)u}$$

$$uw' = \frac{1+2w-2w-w^2}{2+w} \Rightarrow uw' = \frac{1-w^2}{2+w}$$

$$\int \frac{1}{u}\,du = \int \frac{2+w}{1-w^2}\,dw \ \ldots \quad \text{أكمل}$$

Ex: $(2x+3y-1)dx + (4x+6y-2)dy = 0$

$\boxed{y(1) = 3}$

$$\frac{dy}{dx} = \frac{-(2x+3y-1)}{4x+6y+2}$$

$$\frac{dy}{dx} = \frac{-(2x+3y-1)}{2(2x+3y)+2}$$

$a_1 = 2 \quad a_2 = 4$

$b_1 = 3 \quad b_2 = 6$

$a_1 b_2 = a_2 b_1$

متوازيين

$z = 2x+3y$, $z' = 2+3y'$, $y' = \frac{1}{3}(z'-2)$ $\quad \Big| \quad \dfrac{1}{3}(z'-2) = -\dfrac{(z-1)}{2z+2}$

$$z'-2 = \frac{-3(z-1)}{2z+2}$$

$$z' = \frac{-3z+3+4z+4}{2z+2} \Rightarrow z' = \frac{z+7}{2z+2}$$

$$\frac{dz}{dx} = \frac{z+7}{2z+2} \Rightarrow \int dx = \int \left(\frac{2z+2}{z+7}\right) dz \div$$ حل بالقسمة

$$x = \int 2 - \frac{12}{z+7}\, dz$$

$$x = 2z - 12\ln(z+7) \qquad \text{but } z = 3x+3y$$

$$x = 2(3x+3y) - 12\ln(3x+3y+7) + C$$

$$y(1) = 3$$

$$1 = 2(3+9) - 12\ln(3+9+7) + C \Rightarrow C = ??$$

Ex:
1) $(y-2)dx - (x-y-1)dy = 0$
2) $(2x-y)dx + (4x+y-6)dy = 0$
3) $(2x-3y+4)dx + (3x-3)dy = 0 \qquad y(3)=2$
4) $y' = 2(3x+y)^2 - 1 \quad , \quad y(0) = 1$
5) $y' = (x-y+5)^2$

solution:

1) $(y-2)dx + -(x-y-1)dy = 0$

$(y-2)dx + (-x+y+1)dy = 0$

$-y+2=0$

$-x+y+1=0$

$-x+3=0$

$\boxed{x=3}\ \boxed{y=2}$

$$\frac{dy}{dx} = \frac{2-y}{-x+y+1}$$

let $x = 3+u \quad , \quad y = 2+v$

$$\frac{dy}{dx} = \frac{dv}{du}$$

$$\therefore \frac{dv}{du} = \frac{\cancel{2} + \cancel{-2} - v}{\cancel{-3} - u + \cancel{2} + v + \cancel{1}} = \frac{-v}{-u+v} \qquad \text{Homogeneous}$$

let $w = \frac{v}{u}$

$v = wu$

$v' = w + uw'$

$$w + uw' = \frac{-wu}{-u+wu} = \frac{-w\cancel{u}}{\cancel{u}(w-1)} = \frac{-w}{w-1}$$

$$uw' = \frac{-w - w^2 + w}{w-1}$$

$$w' = \frac{-w^2}{u(w-1)} \Rightarrow \frac{du}{dw} = \frac{-u(w-1)}{w^2}$$

$$\int \frac{-1}{u}\,du = \int \frac{w-1}{w^2}\,dw \Rightarrow -\ln u + C = \int \frac{w}{w^2}\,dw - \int \frac{1}{w^2}\,dw$$

$$\int \frac{w}{w^2}\,dw - \int \frac{1}{w^2}\,dw = C - \ln u$$

$$\int \frac{1}{w}\,dw - \int w^{-2}\,dw = C - \ln u$$

$$\Rightarrow \ln w + \frac{1}{w} = C \ln u \Rightarrow \ln \frac{v}{u} + \frac{u}{v} = C - \ln u \text{ -- أكمل}$$

② $(2x - y)\,dx + (4x + y - 6)\,dy = 0$

$2x - y = 0$

$4x + y = 0$

$6x - 6 = 0$

$\boxed{x = 1}\ \boxed{y = 2}$

$$\frac{dy}{dx} = \frac{y - 2x}{4x + y - 6} \qquad \text{let} \quad x = 1 + u, \quad y = 2 + v \qquad \frac{dy}{dx} = \frac{dv}{du}$$

$$\frac{dv}{du} = \frac{2 + v - 2 - 2u}{4 + 4u + 2 + v - 6} = \frac{v - 2u}{4u + v}$$

$$\frac{dv}{du} = \frac{v - 2u}{4u + v} \qquad \text{Homogeneous.}$$

let $w = \frac{v}{u}$

$v = wu$

$v' = w' u + w$

$$w + uw' = \frac{wu - 2u}{4u + wu} = \frac{\cancel{u}(w-2)}{\cancel{u}(4+w)}$$

$$uw' = \frac{w - 2 - 4w - w^2}{4 + w}$$

$$w' = \frac{-w^2 - 3w - 2}{(4+w)u} \Rightarrow \frac{du}{dw} = \frac{-(4+w)\,u}{w^2 + 3w + 2}$$

$$\int \frac{1}{u}\,du = \int \frac{-(4+w)}{w^2 + 3w + 2}\,dw \Rightarrow \ln u + C = \int \frac{-\frac{1}{2}(2w + 8)}{w^2 + 3w + 2}\,dw$$

$$\ln u + C = \int \frac{-\frac{1}{2}(2w+3)\,dw}{w^2+3w+2} + \int \frac{-\frac{5}{2}}{w^2+3w+2}\,dw$$

$$-\frac{1}{2}\ln(w^2+3w+2)$$

$A(w+2) + B(w+1) = -\frac{5}{2}$

if $w=-2 \Rightarrow B = \frac{5}{2}$

$w=-1 \Rightarrow A = \frac{-5}{2}$

أكمل ..

③ $(2x-3y+4)\,dx + (3x-3)\,dy = 0$

$$\frac{dy}{dx} = \frac{-2x+3y-4}{3x-3}$$

$3x-3=0$

$2x-3y+4=0$

$\boxed{x=1}$ $\boxed{y=2}$

let $x = 1+u$, $y = 2+v$ $\Rightarrow \frac{dy}{dx} = \frac{dv}{du}$

$$\frac{dv}{du} = \frac{-2-2u+6+3v-4}{3+3u-3} \Rightarrow \frac{dv}{du} = \frac{-2u+3v}{3u}$$

let $w = \frac{v}{u}$

$v = wu$

$v' = w + uw'$

$$w + uw' = \frac{-2u+3wu}{3u}$$

$$w + uw' = \frac{\cancel{u}(-2+3w)}{3\cancel{u}}$$

$$\cancel{w} + uw' = \frac{-2}{3} + \cancel{w} \Rightarrow uw' = \frac{-2}{3}$$

$$w' = \frac{-2}{3u} \Rightarrow \int dw = \int \frac{-2}{3u}\,du$$

$$w = \frac{-2}{3}\ln u + C \Rightarrow \frac{v}{u} = \frac{-2}{3}\ln u + C$$

أكمل تعويض u, v ...

(4) $y' = 2(3x+y)^2 - 1 \quad , \quad y(0) = 1$

$z = 3x + y$
$z' = 3 + y'$
$y' = z' - 3$

$z' - 3 = 2z^2 - 1$
$z' = 2z^2 + 2$

$\int \frac{dz}{2z^2+2} = \int dx \Rightarrow \frac{1}{2}\int \frac{1}{z^2+2} dz = x + c$

$x + c = \frac{1}{2}\tan^{-1} z$

مهم (5) $y' = (x-y+5)^2$

$1 - z' = (z+5)^2$

$1 - z' = z^2 + 10z + 25$

$z' = -z^2 - 10z - 24$

$\frac{dz}{dx} = -z^2 - 10z - 24 \Rightarrow \frac{dz}{-z^2-10z-24} = dx$

$\int \frac{-1}{z^2+10z+24} dz = x + c \Rightarrow \frac{-1}{z^2+10z+24} = \frac{A}{z+4} + \frac{B}{z+6}$

$\boxed{A = \frac{-1}{2}} \; , \; \boxed{B = \frac{1}{2}}$

$\frac{-1}{2}\ln(z+4) + \frac{1}{2}\ln(z+6) = x + c$ ... أكمل

*Remember :

(1) $\int \frac{g'(x)}{\sqrt{1-g^2(x)}} dx = \sin^{-1} g(x)$

(2) $\int \frac{-g'(x)}{\sqrt{1-g^2(x)}} dx = \cos^{-1} g(x)$

(3) $\int \frac{g'(x)}{1+g^2(x)} dx = \tan^{-1} g(x)$

(4) $\int \frac{-g'(x)}{1+g^2(x)} dx = \cot^{-1}(x)$

(5) $\int \frac{g'(x)}{g(x)\sqrt{g^2(x)-1}} dx = \sec^{-1}(x)$

(6) $\int \frac{-g'(x)}{g(x)\sqrt{g^2(x)-1}} dx = \csc^{-1} g(x)$

Exact D.E's :-

if $M(x,y)\,dx + N(x,y)\,dy = 0$

such that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then we say that the given equation is exact.

* to solve this type do following:

① Integrate $M(x,y)$ with respect to x call it $F(x,y)$ and the arbitrary const. could be a func. of y

$$F(x,y) = \int M(x,y)\,dx + g(y)$$

② differentiate $F(x,y)$ with respect to y

partial differentiate $\longleftarrow$ $\dfrac{\partial F}{\partial y} = \dfrac{d}{dy}\left(\int M(x,y)\,dx\right) + g'(y)$

③ equate $\dfrac{\partial F}{\partial y} = N$

④ Find $g(y)$

يمكن أن نعكس هذه الخطوات بحيث أن اول خطوة نجعل
ونكمل $F(x,y) = \int N(x,y)\,dy + g(x)$

وذلك حسب امكانية التكامل في أول خطوة إذا كان $\int M(x,y)\,dx$ أسهل نستخدمه وهكذا

Ex: $\overbrace{(x^2y+5x)}^{M}dx + \overbrace{(\frac{1}{3}x^3+2y)}^{N}dy = 0$

The condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = x^2 \Rightarrow$ exact.

* ملاحظة: أي معادلة تفاضلية ممكن أن تكون exact أو لا تكون

$$F(x,y) = \int (x^2y+5x)dx + g(y)$$

$$= \frac{1}{3}x^3y + \frac{5}{2}x^2 + g(y)$$

$$\frac{\partial F}{\partial y} = \frac{1}{3}x^3 + g'(y) = N$$

(مهم) هذه الخطوة يجب أن نتخلص من جميع قيم x وإلا فإنه يوجد خطوة خاطئة.

$$\frac{1}{3}x^3 + g'(y) = \frac{1}{3}x^3 + 2y$$

$$g'(y) = 2y \Rightarrow g(y) = y^2$$

g.s $\Rightarrow$ F(x,y) معادلة $\Rightarrow \frac{1}{3}x^3y + \frac{5}{2}x^2 + y^2 = C$

Ex: $(ye^{xy}+2x)dx + (xe^{xy}-2y)dy = 0$

$$\frac{\partial M}{\partial y} = xye^{xy} + e^{xy}$$

$$\frac{\partial N}{\partial x} = xye^{xy} + e^{xy}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y} \quad \therefore \text{exact}$$

$$F(x,y) = g(x) + \int (xe^{xy} - 2y)dy$$

$$F(x,y) = \frac{x}{x}e^{xy} - \frac{2}{2}y^2 + g(x)$$

$$= e^{xy} - y^2 + g(x)$$

$$\frac{\partial F}{\partial x} = ye^{xy} + g'(x) = M$$

$$ye^{xy} + g'(x) = ye^{xy} + 2x$$

$$\int g'(x) \int 2x$$

$$g(x) = x^2$$

$$g.s \Rightarrow e^{xy} - y^2 + x^2 = c \quad \text{if } y(0) = 2$$

$$e^0 - y + 0 = c$$

$$\boxed{c = -3}$$

$$P.S \Rightarrow e^{xy} - y^2 + x^2 = -3$$

Ex: $(\underbrace{1 + ye^x + xe^x y}_{M})dx + (\underbrace{xe^x + 2}_{N})dy = 0$

$$\frac{\partial M}{\partial y} = e^x + xe^x \quad , \quad \frac{\partial N}{\partial x} = xe^x + e^x$$

exact

$$F(x,y) = g(x) + \int (xe^x + 2)dy$$

$$F(x,y) = g(x) + (xe^x + 2)y$$

$$F(x,y) = yxe^x + 2y + g(x)$$

$$\frac{\partial F}{\partial x} = xye^x + ye^x + g'(x) = M$$

$$\cancel{xye^x} + \cancel{ye^x} + g'(x) = 1 + \cancel{e^x y} + \cancel{xe^x y}$$

$$g'(x) = 1 \Rightarrow g(x) = x$$

$$g.s \Rightarrow yxe^x + 2y + x = C$$

Ex: [1] $(1+3x^2\sin y)dx + (x^3\cos y)dy = 0$

[2] $(ye^{xy} - \frac{1}{y})dx + (xe^{xy} + \frac{x}{y^2})dy = 0$

[3] $(\tan y - 2)dx + (x\sec^2 y + \frac{1}{y})dy = 0$

[4] $(4x^3y^2 + \sin x)dx + (2x^4y + \cos y)dy = 0$

solution:

[1] $(1+3x^2\sin y)dx + (x^3\cos y)dy = 0$

$$\frac{\partial M}{\partial y} = 3x^2\cos y \qquad \underbrace{=}_{exact} \qquad \frac{\partial N}{\partial x} = 3x^2\cos y$$

$$F(x,y) = \int (1+3x^2\sin y)\,dx + g(y)$$

$$= x + x^3\sin y + g(y)$$

$$\frac{\partial F}{\partial y} = \cancel{x^3\cos y} + g'(y) = N(x,y) = \cancel{x^3\cos y}$$

$$g'(y) = 0 \Rightarrow g(y) = c$$ ثابت

g.s: $\boxed{x + x^3 + \sin y = C}$

[2] $(ye^{xy} - \frac{1}{y})dx + (xe^{xy} + \frac{x}{y^2})dy = 0$

$$\frac{\partial M}{\partial y} = xye^{xy} + e^{xy} + \frac{1}{y^2} \underbrace{,}_{exact} \frac{\partial N}{\partial x} = xye^{xy} + e^{xy} + \frac{1}{y^2}$$

$$F(x,y) = \int (ye^{xy} - \frac{1}{y})\,dx + g(y)$$

$$= e^{xy} - \frac{1}{y}x + g(y)$$

$$\frac{\partial F}{\partial y} = \cancel{xe^{xy}} + \cancel{\frac{x}{y^2}} + g'(y) = \cancel{xe^{xy}} + \cancel{\frac{x}{y^2}}$$

... أكمل

$\boxed{3}$ $(\tan y - 2)dx + (x\sec^2 y + \frac{1}{y})dy = 0$

$\frac{\partial M}{\partial y} = \sec^2 y \quad , \frac{\partial N}{\partial x} = \sec^2 y \qquad \therefore$ exact.

$F(x,y) = \int(\tan y - 2)\,dx + g(y)$

$F(x,y) = x\tan y - 2x + g(y)$

$\frac{\partial F}{\partial y} = x\cancel{\sec^2 y} + g'(y) = N = x\cancel{\sec^2 y} + \frac{1}{y}$

$g'(y) = \frac{1}{y} \Rightarrow g(y) = \ln|y|$

g.s $\Rightarrow$ $x\tan y - 2x + \ln|y| = C$

$\boxed{4}$ $(4x^3y^2 + \sin x)dx + (2x^4y + \cos y)dy = 0$

$\frac{\partial M}{\partial y} = 8x^3y = \frac{\partial N}{\partial x} = 8x^3y \quad \therefore$ exact.

$F(x,y) = \int(4x^3y^2 + \sin x)\,dx + g(y)$

$F(x,y) = x^4y^2 - \cos x + g(y)$

$\frac{\partial F}{\partial y} = 2x^4y + g'(y) = 2x^4y + \cos y$

$g'(y) = \cos y \Rightarrow g(y) = \sin y$

g.s $\Rightarrow$ $x^4y^2 - \cos x + \sin y = C$

* If $M(x,y)dx + N(x,y)dy = 0$ is both homog. & exact then the g.s. is $(xM + yN) = C$

<u>ex</u>: $(2x+3y)dx + (3x+5y)dy = 0$ Homog. and exact

g.s. $x(2x+3y) + y(3x+5y) = C$

$2x^2 + 3xy + 3xy + 5y^2 = C$

$2x^2 + 6xy + 5y^2 = C$

* Not Exact D.E's :-

an exact of the Form:

$P(x,y)\,dx + Q(x,y)\,dy = 0$ is called not exact if $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$

To solve these equations we have to find the integrating factor (μ) and then multiply this Factor by the original equation which becoms exact

$$\underbrace{\mu P(x,y)}_{M}\,dx + \underbrace{\mu Q(x,y)}_{N}\,dy = 0$$

now, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

How to Find μ :

ملاحظة: قد نستطيع أن نجد قيمة μ من خلال الطرق الخمسة لإيجادها وقد لا نستطيع فإذا لم نستطع إيجادها بأي طريقة عندها لن نستطيع تحويل المعادلة إلى Exact ولن نستطيع حل المعادلة التفاضلية بطريقة Exact بل يمكن حلها بطرق أخرى

① If $\frac{1}{Q}\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) = F(x)$ a function with respect to x only

$$\mu = e^{\int F(x)\,dx}$$

② $\frac{1}{P}\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) = F(y)$

$$\mu = e^{-\int F(y)\,dy}$$

الشرح فيما بعد

③ Ⓐ homogeneous + exact $\rightarrow$ g.s $xM + yN = C$

Ⓑ homogeneous & not exact ~~g.s~~ $\mu = \frac{1}{xP + yQ}$

④ $yF(x,y)\,dx + xg(x,y) = 0 \Rightarrow$

a) $F(x,y) \neq g(x,y)$

$$\mu = \frac{1}{xy(F-g)}$$

b) $F(x,y) = g(x,y)$

g.s $\Rightarrow y = \frac{A}{x}$

Ex: $(2x^2+y)dx + (x^2y-x)dy = 0$

$$\frac{\partial P}{\partial y} = 1 \quad , \frac{\partial Q}{\partial x} = 2xy-1 \Rightarrow \text{not exact}$$

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 1-2xy+1 = 2-2xy$$

$$\frac{1}{Q}(2-2xy) = \frac{2(\cancel{1-xy})^{-1}}{x(\cancel{xy-1})} = \frac{-2}{x}$$

$$\mu = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = \boxed{x^{-2}}$$

$$[(2x^2+y)dx + (x^2y-x)dy]\mu = 0$$

$$(2+yx^{-2})dx + (y-x^{-1})dy = 0$$

$$\underbrace{\frac{\partial M}{\partial y} = x^{-2} \quad , \frac{\partial N}{\partial x} = x^{-2}}_{\text{exact}}$$

$$F = \int (y-\frac{1}{x})dx + g(x)$$

$$F = \frac{1}{2}y^2 - \frac{y}{x} + g(x)$$

$$\frac{\partial F}{\partial x} = \frac{y}{x^2} + g'(x) = M(x,y)$$

$$\frac{y}{x^2} + g'(x) = 2 + \frac{y}{x^2}$$

$$g'(x) = 2 \Rightarrow g(x) = 2x$$

$$g.s \Rightarrow \frac{1}{2}y^2 - \frac{y}{x} + 2x = C$$

Ex: $(3x^2y^4+2xy)dx + (2x^3y^3-x^2)dy = 0$

$$\frac{\partial P}{\partial y} = 12x^2y^3+2x \neq \frac{\partial Q}{\partial x} = 6x^2y^3-2x$$

Not exact

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 12x^2y^3+2x-6x^2y^3+2x$$

$$= 6x^2y^3+4x = 2(3x^2y^3+2x)$$

$$\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{y(3x^2y^3+2x)} * (2(3x^2y^3+2x))$$

$$= \frac{2}{y}$$

$$\mu = e^{-\int \frac{2}{y}dy} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2}$$

$$\mu[(3x^2y^4+2xy)dx + (2x^3y^3-x^2)dy] = 0$$

$$(3x^2y^2+\frac{2x}{y})dx + (2x^3y-\frac{x^2}{y^2})dy = 0 \Rightarrow \text{exact}$$

$$F = \int(3x^2y^2+\frac{2x}{y})dx + g(y)$$

$$F = x^3y^2+\frac{x^2}{y} + g(y)$$

$$\frac{\partial F}{\partial y} = 2x^3y + \frac{-x^2}{y^2} + g'(y) = 2x^3y - \frac{x^2}{y^2}$$

$$g'(y) = 0 \Rightarrow g(y) = C$$

g.s: $x^3y^2 + \frac{x^2}{y} = C$

Ex: ① $y(x+y)dx + (x+2y-1)dy = 0$

تمرين

$$g.s:\ xe^{x}y + y^{2}e^{x} * ye^{x} = c$$

② $(2y - 8x^{2})dx + x\,dy$

$$g.s:\ x^{2}y - 2x^{4} = C$$

③ $y\,dx + (3x - xy + 2)dy = 0$

$$g.s:\ y^{3}xe^{-y} + 2y = C$$

How to Find μ:

③ if $P(x,y)$, $Q(x,y)$ is both homogeneous and not exact such that $xP + yQ \neq 0$

Then $\mu = \dfrac{1}{xP + yQ}$

Ex: $(x^{4} + y^{4})dx - xy^{3}dy = 0$ Homogeneous

$$\frac{\partial P}{\partial y} = 4y^{3},\ \frac{\partial Q}{\partial x} = -y^{3} \quad \text{not exact}$$

$$\mu = \frac{1}{xP + yQ} = \frac{1}{x^{5} + x^{4} + -xy^{4}} = \frac{1}{x^{5}} = x^{-5}$$

$$\mu((x^{4} + y^{4})dx - xy^{3}dy) = 0$$

$$\left(\frac{1}{x} + \frac{y^{4}}{x^{5}}\right)dx - \frac{1}{x^{4}}y^{3}dy = 0 \Rightarrow \text{exact}$$

$$F(x,y) = g(x) - \int \frac{y^{3}}{x^{4}}dy$$

$$F(x,y) = g(x) - \frac{y^{4}}{4x^{4}}$$

$$\frac{\partial F}{\partial x} = g'(x) + \frac{16x^{3}y^{4}}{16x^{8}} = g'(x) + \frac{y^{4}}{x^{5}} = \frac{1}{x} + \frac{y^{4}}{x^{5}}$$

$$g'(x) = \frac{1}{x} \qquad \boxed{g(x) = \ln x}$$

$$g.s:\ \ln x - \frac{y^{4}}{4x^{4}} = C$$

④ If $P(x,y)\,dx + Q(x,y)\,dy = 0$

can be written in the form:

$y\,F(x,y)\,dx + x\,g(x,y)\,dy = 0$ & $F \neq g$

then $\mu = \dfrac{1}{xy(F-g)}$ but if $F = g$

$y\,F(x,y)\,dx + x\,F(x,y)\,dy = 0$

$F(x,y)\,(y\,dx + x\,dy) = 0$

$F(x,y) \neq 0$ because if $F(x,y) = 0$ then we don't have diff. equation.

$\therefore\; y\,dx + x\,dy = 0$

$\frac{1}{x}dx + \frac{1}{y}dy = 0$

$\ln x + \ln y = C$

$\ln(xy) = C$

$xy = A \Rightarrow \boxed{y = \frac{A}{x}}$

Ex: $(2xy^2 + y)\,dx + (x + 2x^2y - x^4y^3)\,dy = 0$

$\frac{\partial P}{\partial y} = 4xy + 1$, $\frac{\partial Q}{\partial x} = 1 + 4xy - 4x^3y^3$

not exact

$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 4x^3y^3$

$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) \rightarrow \neq F(x)$

$\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) \neq F(y)$

and P, Q not homogeneous — الحل على طريقة ④

the solution:

$$y(2xy+1)\,dx + x(1+2xy-x^3y^3)\,dy = 0$$

$$\mu = \frac{1}{xy(F-g)} = \frac{1}{xy(2xy+\cancel{1}-\cancel{1}-2xy+x^3y^3)}$$

$$\mu = \frac{1}{xy(x^3y^3)} = \frac{1}{x^4y^4}$$

$$\left[(2xy^2+y)\,dx + (x+2x^2y-x^4y^3)\,dy = 0\right] * \frac{1}{x^4y^4}$$

$$\left(\frac{2}{x^3y^2}+\frac{1}{x^4y^3}\right)dx + \left(\frac{1}{x^3y^4}+\frac{2}{x^2y^3}-\frac{1}{y}\right)dy = 0$$

$$F(x,y) = \int \frac{2}{y^2}x^{-3}\,dx + \int \frac{1}{y^3}x^{-4}\,dx + g(y) = 0$$

$$F(x,y) = \frac{2}{-2y^2x^2} + \frac{-1}{3y^3x^3} + g(y)$$

$$F(x,y) = \frac{-1}{y^2x^2} - \frac{1}{3y^3x^3} + g(y)$$

$$g.s:\ \frac{-1}{x^2y^2} - \frac{1}{3x^3y^3} - \ln y = C$$

<u>Ex:</u> $(xy^2+x^2y^3)\,dx + (x^2y+x^3y^2)\,dy = 0$

$$y(xy+x^2y^2)\,dx + x(xy+x^2y^2)\,dy = 0$$

$$F = g \Rightarrow g.s:\ \boxed{y = \frac{A}{x}}$$

ممكن كمان exact لكن هذه أسرع

How to Find μ :

(5) If $P(x,y)dx + Q(x,y)dy = 0$

can be written in the form

$$x^{\alpha}y^{\beta}(a_1 y\,dx + a_2 x\,dy) + x^{a}y^{b}(b_1 y\,dx + b_2 x\,dy) = 0$$

where $\alpha, \beta, a_1, b_1, a, b, a_2, b_2$ are const.

then $\mu = x^h y^k$

h, k are two const to be determined.

Ex: $(8y + 4x^2y^4)dx + (8x + 5x^3y^3)dy = 0$

$$8y\,dx + 4x^2y^4\,dx + 8x\,dy + 5x^3y^3\,dy = 0$$

$$(8y\,dx + 8x\,dy) + (4x^2y^4\,dx + 5x^3y^3\,dy) = 0$$

$$(8y\,dx + 8x\,dy) + x^2y^3(4y\,dx + 5x\,dy) = 0$$

$$\mu = x^h y^k$$

$$\left((8y + 4x^2y^4)dx + (8x + 5x^3y^3)dy\right)\mu = 0$$

$$(8y^{k+1}x^h + 4x^{2+h}y^{4+k})dx + (8x^{1+h}y^k + 5x^{3+h}y^{3+k})dy = 0$$

exact

$$\frac{\partial M}{\partial y} = 8(k+1)x^h y^k + 4(k+4)x^{h+2}y^{k+3}$$

$$\frac{\partial N}{\partial x} = 8(h+1)x^h y^k + 5(h+3)x^{h+2}y^{k+3}$$

يجب أن تتساوى Exact لأنه

$$8(k+1)x^h y^k + 4(k+4)x^{h+2}y^{k+3} = 8(h+1)x^h y^k + 5(h+3)x^{h+2}y^{k+3}$$

$8(k+1) = 8(h+1) \Rightarrow h = k$

$4(k+4) = 5(h+3) \rightarrow 4(k+4) = 5(k+3)$

$4k + 16 = 5k + 15$

$\therefore k = 1 = h$

$\mu = xy \Rightarrow$ نعوض قيمة h, k في exact ونكمل

Linear D.E's: the general form of this type of equations is

$$x' + P(y)\,x = Q(y) \quad \text{أو} \quad y' + P(x)\,y = Q(x)$$

to solve this equ do following: لاحظ معامل y' = واحد

$$\frac{dy}{dx} + P(x)\,y = Q(x) \Rightarrow dy + P(x)y\,dx = Q(x)\,dx$$

$$\underset{N}{1} * dy + \underbrace{\left(P(x)\,y - Q(x)\right)}_{M} dx = 0$$

$$\frac{\partial M}{\partial y} = P(x) \qquad \frac{\partial N}{\partial x} = 0 \qquad \text{not exact}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = P(x)$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{P(x)}{1} = P(x)$$

$$\mu = e^{\int P(x)\,dx} = e^{h} \Rightarrow h = \int P(x)\,dx$$

$$\mu\left(y' + P(x)\,y = Q(x)\right)$$

$$\mu y' + \mu P(x)\,y = \mu Q(x)$$

$$\frac{d}{dx}(\mu y) = \mu Q(x) \qquad \text{but } \mu' = (e^{h})'$$

$$= e^{h} \cdot h' = P(x)\,e^{h}$$

$$\int(\mu y) = \int \mu Q(x)\,dx + c$$

$$\therefore\ y = \frac{1}{\mu}\left(\int \mu Q(x)\,dx + c\right)$$

$$y = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} Q(x)\,dx + c\right)$$

$$\boxed{y = e^{-h}\left(\int e^{h} Q(x)\,dx + c\right)}$$

استعمل هذا القانون مباشرة.

Ex: $\frac{1}{x}y' - \frac{2y}{x^2} = x\cos x \qquad x>0$

solution) $y' - \frac{2y}{x} = x^2\cos x \qquad P(x) = \frac{-2}{x}$

g.s: $y = e^{-\int P(x)\,dx}\left[\int e^{\int P(x)\,dx}\,.\,Q(x)\,dx + C\right]$

$y = x^2\left[\int x^{-2}\,x^2\cos x\,dx + C\right]$

$= x^2\left[\int \cos x\,dx + C\right]$

$\boxed{y = x^2\sin x + Cx^2}$

Ex: $\cos^2(x)\,y' + y = \tan x$

sol: $y' + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$

$y' + \underbrace{\sec^2(x)}_{P(x)}y = \sec^2 x\tan x$

g.s: $y = e^{-\tan x}\left[\int e^{\tan x}\sec^2 x\tan x\,dx + C\right]$

$u = \tan x$

$du = \sec^2 x\,dx$

$dx = \frac{du}{\sec^2 x}$

$= e^{-\tan x}$

$= \int e^u \cancel{\sec^2 x}\; u\frac{du}{\cancel{\sec^2 x}} + C$

$= \int u e^u\,du + C$

$= ue^u - e^u + C$

u +	e^u
1	e^u
0	e^u

$= \tan x\, e^{\tan x} - e^{\tan x} + C$

$\therefore$ g.s: $\boxed{y = \tan x - 1 + Ce^{-\tan x}}$

Bernollis equations :

it is of the form : $y' + P(x)y = Q(x)y^n$

& To solve this type of equations, we substitute $u = y^{1-n}$ which will convert the given D.E (non-linear) to be linear then the given equation become

$$y' + P(x)y = Q(x)y^n$$

$$\Downarrow$$

$$u' + \underbrace{(1-n)P(x)}_{P_1(x)}u = \underbrace{(1-n)Q(x)}_{Q_1(x)}$$

$$u' + P_1(x)u = Q_1(x)$$

$$u = e^{-\int P_1(x)dx}\left(\int e^{\int P_1(x)dx} Q_1(x)dx + c\right)$$

Ex: solve $y' - 5y = \frac{-5}{2}xy^3$

let $u = y^{1-3} = y^{-2}$

$$u' - 5(1-n)u = \frac{-5}{2}(1-n)x$$

$$u' + 10u = 5x \quad \text{---- Linear} \quad P(x) = 10$$

$$u = e^{-\int 10dx}\left(\int e^{\int 10dx} \cdot 5x\,dx + C\right)$$

$$= e^{-10x}\left(\int e^{10x} \cdot 5x\,dx + C\right)$$

$$= e^{-10x}\left(\frac{1}{2}xe^{10x} - \frac{1}{20}e^{10x} + C\right)$$

$$= \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}$$

$5x$	+	e^{10x}
5	−	$\frac{1}{10}e^{10x}$
0		$\frac{1}{100}e^{10x}$

$$y^{-2} = \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}$$

$$y^2 = \frac{1}{\frac{1}{2}x - \frac{1}{20} + ce^{-10x}}$$

Ex : $y' - xy = \sqrt{x^3 y}$

$y' - xy = x^{\frac{3}{2}} y^{\frac{1}{2}} \longrightarrow n = \frac{1}{2}$

$u = y^{1-\frac{1}{2}} = y^{\frac{1}{2}}$

$u' - \frac{1}{2} x u = \frac{1}{2} x^{\frac{3}{2}} \xRightarrow{\text{Linear}} \mu = e^{-\int \frac{1}{2} x\,dx} = e^{-\frac{1}{2}\frac{x^2}{2}}$

$= e^{\frac{-x^2}{4}}$

$u = e^{\frac{x^2}{4}} \left[\int e^{\frac{-x^2}{4}} \cdot \frac{1}{2} x^{\frac{3}{2}} dx + c \right] \ldots$

أكمل ؛

Ex : $y(6y^2 - x - 1)dx + 2x\,dy = 0$

$\Rightarrow (6y^3 - y(x+1))dx + 2x\,dy = 0$

$\frac{dy}{dx} = -\frac{(6y^3 - y(x+1))}{2x}$

$y' = \frac{-6y^3}{2x} + \frac{y(x+1)}{2x} \Rightarrow y' = \frac{-6y^3}{2x} + \frac{y(x+1)}{2x}$

$y' - \frac{(x+1)}{2x} y = \frac{-3y^3}{x}$ ...... Bernolly

$\boxed{u = y^{-2}}$

$u' + \frac{x+1}{x} u = \frac{6}{x}$ ---- linear

$\mu = e^{\int (1+\frac{1}{x})dx} = e^{(x+\ln x)} = e^{x} \cdot e^{\ln x}$

$= x e^{x}$

أكمل الحل ....

Clairaut's equations:

إذا كانت درجة المعادلة أكبر من واحد بستخدم هذه الطريقة.

The general form of this type of equations is :-

$$y = xp + F(p) \quad ----(*) \qquad , \quad p = \frac{dy}{dx} = y'$$

where $F(p)$ contains neither (x) nor (y) excepplicity.

to solve $(*)$:-

$$y = xy' + F(y') \Rightarrow \cancel{y'} = xy'' + \cancel{y'} + y'' F'(y')$$

$$(x + F'(y'))\, y'' = 0$$

either $y'' = 0 \longrightarrow y' = c \Rightarrow p = c$

$\therefore \boxed{y = cx + F(c)}$ g.s

or $(x + F'(y')) = 0$

$$x = -F'(y') = -F'(p)$$

$$\therefore y = -pF'(p) + F(p)$$

$\boxed{y = F(p) - pF'(p)}$ singular solution.

if $F(p) = 0$ then $y = xp \Rightarrow y = xy'$ ---- seperable

$$y\,dy = x\,dx$$

Ex: $y = xy' + (y')^3$ معادلة من الدرجة الثالثة لذا نستخدم هذه الطريقة.

g.s: $y = cx + c^3$

$$p = y' \Rightarrow F(p) = p^3 \Rightarrow F'(p) = 3p^2$$

$$x = -F'(p) \Rightarrow x = -3p^2 \Rightarrow p^2 = \frac{-x}{3}$$

$$y = -3p^3 + p^3 = -2p^3 \Rightarrow p^3 = \frac{y}{-2}$$

$$p^2 = \frac{-x}{3} \qquad p^3 = \frac{y}{-2}$$

$$p^6 = \frac{-x^3}{27} \qquad p^6 = \frac{y^2}{4}$$

$$\frac{-x^3}{27} = \frac{y^2}{4} \Rightarrow \boxed{4x^3 + 27y^2 = 0} \text{ singular soln.}$$

Ex: $y = xy' + \cos y'$

$y = cx + \cos c$ (g.s.)

$F(p) = \cos p \Rightarrow F'(p) = -\sin p$

$x = -F'(p) = \sin p \Rightarrow p = \sin^{-1} x$

$y = p\sin p + \cos p$

$\therefore y = \sin^{-1} x \,(\overset{(x)}{\sin(\sin^{-1} x)}) + \cos(\sin^{-1} x)$

$y = x\sin^{-1} x + \cos(\sin^{-1} x)$

تذكر

$y = F(x) \rightarrow x = F^{-1}(y)$

$F(F^{-1}(x)) = x$

Ex: $(x^2-1)p^2 - 2xyp + y^2 - 1 = 0$

* Raicharties equations:-

نحفظ معامل y = واحد

it is of the form $y' = P(x)y^2 + Q(x)y + R(x)$ ...(*)

① If $R(x) = 0$ (*) becomes $y' - Q(x)y = P(x)y^2$

Bernolly $u = y^{1-2}$

② If $P(x) = 0$ (*) becomes $y' - Q(x)y = R(x)$ linear.

③ If $Q(x) = 0$ حالة لم ندرسها

④ If $u(x)$ is a solution of (*) then :-

$u'(x) = P(x)u^2 + Q(x)u + R(x)$ ....(1)

$y' = P(x)y^2 + Q(x) + R(x)$ .....(2)

② - ① $y' - u' = P(x)(y^2 - u^2) + Q(x)(y-u)$

let $w = y - u$

$y = w + u$

$w = \frac{1}{z} \Rightarrow w' = \frac{-z'}{z^2}$

$w' = y' - u'$

$w' = P((w+u)^2 - u^2) + Qw$

$w' = P(w^2 + 2wu + u^2 - u^2) + Qw$

$w' = P(w^2 + 2wu) + Q(w)$

$$w' = P(w^2 + 2wu) + Q(w)$$

$$w = \frac{1}{z} \Rightarrow w' = \frac{-1z'}{z^2}$$

$$\frac{-z'}{z^2} = P\left(\frac{1}{z^2} + \frac{2u}{z}\right) + Q \frac{1}{z}$$

$$-z' = P(1 + 2uz) + Qz \Rightarrow -z' = P + (2uP + Q)z$$

$$z' + \underbrace{(2uP + Q)}_{P_1(x)} z = \underbrace{-P}_{Q_1(x)} \quad \text{--- linear between } x, z$$

نتيجة: للحل بهذه الطريقة يجب أن يعطي السؤال حل للمعادلة التفاضلية مثل u

Ex: $y' = xy^2 + (1 - 2x)y + x - 1$

such that $u = 1$ is a solution. تحقق المعادلة

$$y = w + u \rightarrow y = \frac{1}{z} + u$$

$$w = \frac{1}{z} \qquad y = \frac{1}{z} + 1 \Rightarrow y' = \frac{-z'}{z^2}$$

$$\frac{-z'}{z^2} = x\left(\frac{1}{z} + 1\right)^2 + (1 - 2x)\left(\frac{1}{z} + 1\right) + x - 1$$

$$-\frac{z'}{z^2} = x\left(\frac{1}{z^2} + \frac{2}{z} + 1\right) + \frac{1}{z} + 1 - \frac{2x}{z} - 2x + x - 1$$

$$\frac{-z'}{z^2} = \frac{x}{z^2} + \cancel{\frac{2x}{z}} + \cancel{x} + \frac{1}{z} + \cancel{1} - \cancel{\frac{2x}{z}} - 2x + x - \cancel{1}$$

$$\frac{-z'}{z^2} = \frac{x}{z^2} + \frac{1}{z} \Rightarrow -z' = x + z$$

$$z' + z = -x$$

$$\mu = e^{\int dx} = e^{x}$$

g.s : $z = e^{-x}\left[\int e^{x} \cdot (-x)\, dx + c\right] \Rightarrow = e^{-x}(-xe^{x} + e^{x} + c)$

$$z = -x + 1 + ce^{-x}$$

من الفرض

$$\boxed{z = \frac{1}{y-1}} \qquad \frac{1}{y-1} = -x + 1 + ce^{-x} \Rightarrow y - 1 = \frac{1}{-x + 1 + e^{-x} c}$$

Ex: $y' + xy^2 - 2x^2 y + x^3 = x+1$

$y = x-1$ is a solution.

$y = \frac{1}{z} + x - 1 \Rightarrow y' = \frac{-z'}{z^2} + 1$

$y' = -xy^2 + 2x^2 y - x^3 + x + 1$

$\frac{-z'}{z^2} + 1 = -x\left(\frac{1}{z} + x - 1\right)^2 + 2x^2\left(\frac{1}{z} + x - 1\right) - x^3 + x + 1$

$\frac{-z'}{z^2} + 1 = -x\left(\frac{1}{z^2} + \frac{2(x-1)}{z} + (x-1)^2\right)$

$+ \frac{2x^2}{z} + 2x^3 - 2x^2 - x^3 + x +$

$= -x\left(\frac{1}{z^2} + \frac{2x}{z} - \frac{2}{z} + x^2 - 2x + 1\right) + \frac{2x^2}{z} + 2x^3 - 2x^2 - x^3 + x + 1$

$\frac{-z'}{z^2} + 1 = \frac{-x}{z^2} - \cancel{\frac{2x^2}{z}} + \frac{2x}{z} - \cancel{x^3} + \cancel{2x^2} - \cancel{x} + \cancel{\frac{2x^2}{z}} + \cancel{2x^3} - \cancel{2x^2} - \cancel{x} + x + 1$

$\frac{-z'}{z^2} + 1 = \frac{-x}{z^2} + \frac{2x}{z} + 1$

$-z' = -x + 2xz \Rightarrow z' + 2xz = x$ -- linear

$\mu = e^{\int 2x\,dx} = e^{x^2}$

$z = e^{-x^2}\left(\int x e^{x^2}\,dx + c\right)$

$= e^{-x^2}\left(\int e^{x^2} x\,dx + c\right)$ $\quad u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$= e^{-x^2}\left(\int e^u \, x \frac{du}{2x} + c\right)$

$= e^{-x^2}\left(\frac{1}{2} e^u + c\right)$

$z = e^{-x^2}\left(\frac{1}{2} e^{x^2} + c\right) \Rightarrow z = \frac{1}{2} + ce^{-x^2}$

$y = \frac{1}{z} + x - 1 \Rightarrow \frac{1}{z} = y - x + 1 \Rightarrow z = \frac{1}{y - x + 1}$

$\frac{1}{y - x + 1} = \frac{1}{2} + ce^{-x^2} \Rightarrow y - x + 1 = \frac{1}{\frac{1}{2} + ce^{-x^2}}$

$y = \frac{1}{\frac{1}{2} + ce^{-x^2}} + x - 1$

* Orthognal trajectories: المسارات المتعامدة

If $F(x,y,\overset{\text{const}}{c}) = 0$ ....(1)
is given family of curves and want to find another family of curves $g(x,y,k)=0$ ....(2)

such that at each point of intersections between curves in ① with curves in ② The tangent lines are perpendicular (normal or orthognal) then we say that family ① & ② are orthognal to each other.

* How to find orthognal trajectories of a given family of curves:

① Find the D.E of the given family of curves.
② replace $\frac{dy}{dx}$ in ① by $-\frac{dx}{dy}$

This will give you the D.E of the orthognal traj

$tangent_1 = \frac{-1}{tangent_2}$ المستقيمان متعامدان

③ solve the D.E in ② gives you orthog

Ex: $y = \frac{c}{x} \Rightarrow c = yx$

$y' = \frac{-c}{x^2} \Rightarrow y' = \frac{-yx}{x^2} \Rightarrow y' = \frac{-y}{x}$ ... D.E of the given family

$\frac{dy}{dx} = \frac{-dx}{dy} \Rightarrow y' = \frac{x}{y}$ ... D.E of orthognal

بدل ما نكتب خطوة مكان
نقلب الطرف الأيمن ونضرب (-1)
$\frac{dy}{dx} = \frac{x}{y} \Rightarrow y\,dy = x\,dx$ ... sep.

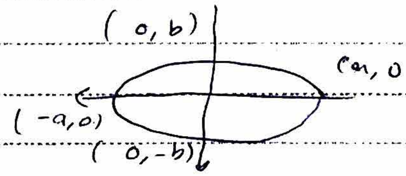
$\frac{1}{2}y^2 = \frac{1}{2}x^2 + c \Rightarrow \frac{1}{2}y^2 - \frac{1}{2}x^2 = c$

$\boxed{y^2 - x^2 = A}$ Hyperbola
قطع زائد

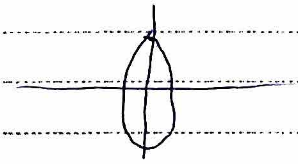
Remember :-

1. $ax^2 + ay^2 = A$ معادلة دائرة
2. $y = Ax^2 + \ldots$ قطع مكافئ
3. $x = Ay^2 + \ldots$ قطع مكافئ
4. $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$ Ellipse

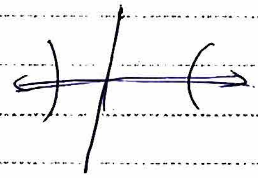
If $a > b \Rightarrow$ Ellipse x-axis



but if $b > a \Rightarrow$ Ellipse y-axis



5. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow$ قطع زائد



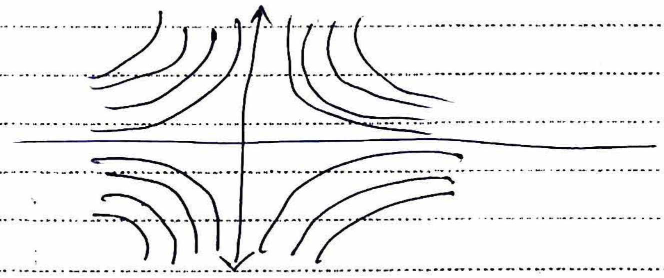
remember

denominator المقام

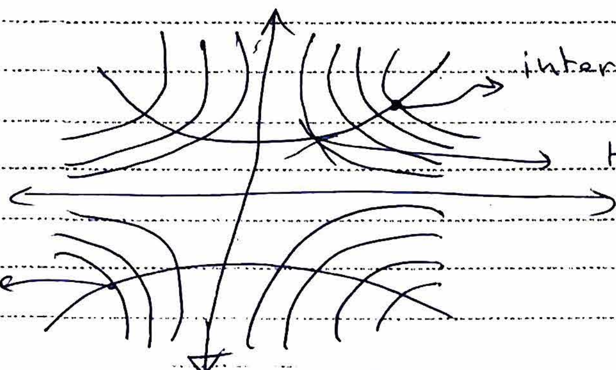
nominator بسط الكسر

The family of curves $y = \frac{c}{x}$

C مجموعة منحنيات متشابهة



The solution diagram



Ex: $y = \dfrac{1+cx}{1-cx}$

solution: $y' = \dfrac{(1-cx)(c) - (1+cx)(-c)}{(1-cx)^2}$

$$y' = \frac{c - c^2x + c + c^2x}{(1-cx)^2} = \frac{2c}{(1-cx)^2}$$

والآن لاستبدال قيمة كل من ⇐ $(1-cx)^2$, c

$$y = \frac{1+cx}{1-cx} \Rightarrow y(1-cx) = 1+cx$$

$$y - cyx = 1+cx$$

$$y - 1 = cx(1+y) \Rightarrow c = \frac{y-1}{x(y+1)}$$

نعوض قيمة c في المقدار $(1-cx)^2$ فنجد:

$$1-cx = 1 - \frac{y-1}{x(y+1)} \cdot x$$

$$1-cx = 1 - \frac{y-1}{y+1} \Rightarrow \frac{y+1-y+1}{y+1} = \frac{2}{y+1}$$

$$1-cx = \frac{2}{y+1} \Rightarrow (1-cx)^2 = \frac{4}{(y+1)^2}$$

$$\therefore y' = \frac{2c}{(1-cx)^2} = \frac{2\left(\frac{y-1}{x(y+1)}\right)}{\frac{4}{(y+1)^2}}$$

$$= 2\left(\frac{y-1}{x(y+1)}\right)\frac{(y+1)^2}{4}$$

$$y' = \frac{y-1}{2x}(y+1) = \frac{y^2-1}{2x}$$ .... D.E of the family

$$y' = \frac{-2x}{y^2-1}$$ .... D.E of the orthognal

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{y^2-1} \Rightarrow (y^2-1)\,dy = -2x\,dx$$

$$\frac{1}{3}y^3 - y = -x^2 + c_1 \Rightarrow y^3 - 3y = -3x^2 + c_2$$

$$\boxed{y^3 - 3y + 3x^2 = c_2}$$

Ex: ① $x^2+y^2=c^2$

② $y^2=cx^3$

③ $y=\ln x(cx)=\ln c+\ln x$ ...... etc

* Higher order Linear D.E :-

The n^{th} L.D.E is $P_n(x)y^{n}+P_{n-1}(x)y^{n-1}+\dots+P_2(x)y''+P_1(x)y'+P_0(x)y=R(x)$ ....(*)

If $R(x)=0$ then (*) is called homog. and otherwise it is called non homog.

* Homogeneous L.D.E with constant coeficient :

Ex: ① $y''+3y'+x=1 \Rightarrow$ 2nd order L.D.E, non homog.

② $yy'''+2yx^2y'+y^2=y^2+1 \Rightarrow y'''+2x^2y'=\frac{1}{y}$

3rd order, non-homog — non-linear.

③ $xyy'''+yxy'=xy+3y$

$\Rightarrow xy'''+xy'=x+3$ 3rd order L.D.E non homog.

④ $y^2x^3y''+y^3=y^2 \Rightarrow$ 2nd order L.D.E non homog.

* If all $P_n(x), P_{n-1}(x), P_0(x)$ in (*) are const. say $a_n, a_{n-1}, \dots a_0$ then (*) becomes

$a_ny^{n}+a_{n-1}y^{n-1}+\dots+a_0y=0$ ....①

it is H.L.D.E with const coeficient

To solve ① we let $y=e^{\lambda x}$ where λ is a parameter that has to be determined

$\Rightarrow a_n \lambda^n e^{\lambda x} + a_{n-1} \lambda^{n-1} e^{\lambda x} + \dots + a_0 e^{\lambda x} = 0$

$e^{\lambda x} [a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_0] = 0$

$e^{\lambda (x)} F(\lambda) = 0$ ----- (2)

either $e^{\lambda x} = 0$ because $e^{\lambda x} > 0$

or $F(\lambda) = 0$ ----- (3)

Remember: $ax^2 + bx + C = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$i = \sqrt{-1}$, $i^2 = 1$

$\Rightarrow \sqrt{-9} = 3i$

$\alpha \pm \beta i \rightarrow$ Imeginary part

$F(\lambda) = 0$ ----- (3) this equ. is called the characteristic polynomial of (1) and it is of degree (n) therefore it has (n) roots and these roots may be distinct repeated or complex conjugates.

case (1): if all roots of the characteristic polynomial and distinct say $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_n$

then the (n) solution are

$y = e^{\lambda_1 x}$, $y = e^{\lambda_2 x}$, ----- , $y_n = e^{\lambda_n x}$

g.s : $y = \underbrace{c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \dots \; c_n e^{\lambda_n x}}_{\text{linear combination of the solu}}$

For example: $y^{(5)} \xrightarrow[\text{is}]{\text{there}} 5c \Rightarrow c_1, c_2, c_3, c_4, c_5$

Ex: solve $y'' - 2y' - 3y = 0$ $\qquad y(0) = 1$, $y'(0) = 2$

soln: $\lambda^2 - 2\lambda - 3 = 0$

$(\lambda-3)(\lambda+1) = 0 \qquad \lambda = 3, -1$

$y_1 = e^{3x}$, $y_2 = e^{-x}$

g.s: $y = c_1 e^{3x} + c_2 e^{-x} \Rightarrow c_1 + c_2 = 1$ ----(1)

$y' = 3c_1 e^{3x} - c_2 e^{-x} \Rightarrow 3c_1 - c_2 = 2$ ----(2)

(1) + (2) $\Rightarrow 4c_1 = 3 \qquad \boxed{c_1 = \frac{3}{4}} \qquad \boxed{c_2 = \frac{1}{4}}$

P.S: $y = \frac{3}{4} e^{3x} + \frac{1}{4} e^{-x}$

case (2): if the roots of the characteristic polyn are repeated (r) times and the remaining roots are distinct $\lambda_1 = \lambda_2 \dots = \lambda_r$

$\lambda_{r+1} \neq \lambda_{r+2} \neq \dots \neq \lambda_n$

~~$y_1 = e^{\lambda_1 x}$, y~~

$y_1 = e^{\lambda_1 x}$, $y_2 = xe^{\lambda_1 x}$ $\quad y_3 = x^2 e^{\lambda_1 x}$ --- $y_r = x^{r-1} e^{\lambda_1 x}$

Ex: $y'' + 6y' + 9y = 0 \Rightarrow \lambda^2 + 6\lambda + 9 = 0$

$(\lambda+3)(\lambda+3) = 0$

$\lambda = -3, -3$

$y_1 = e^{-3x}$, $y_2 = xe^{-3x}$

g.s: $y = c_1 e^{-3x} + c_2 x e^{-3x}$

case (3): if the roots are complex conjugates of the form $\lambda_{\alpha x} = \alpha \mp \beta i$ then

$y_1 = e^{\alpha x} \cos\beta x$, $y_2 = e^{\alpha x} \sin\beta x$

g.s: $y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

Ex: ① $y'' + 4y = 0$

$\lambda^2 + 4 = 0$

$\lambda^2 = -4 \quad , \lambda = \pm\sqrt{-4} = \mp 2i$

$\alpha = 0 \quad \beta = 2$

g.s: $c_1 \cos 2x + c_2 \sin 2x$

② $y''' - 4y'' + y' + 6y = 0$

$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$

$(\lambda+1)(\lambda^2 - 5\lambda + 6) = 0$

$(\lambda+1)(\lambda-3)(\lambda-2) = 0 \qquad \lambda = -1, 3, 2$

g.s: $y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{2x}$

③ $y^{(4)} + 2y^{(3)} + y'' = 0$

$\lambda^4 + 2\lambda^3 + \lambda^2 = 0$

$\lambda^2(\lambda^2 + 2\lambda + 1) = 0$

$\lambda^2(\lambda+1)(\lambda+1) = 0 \qquad \lambda = 0, 0, -1, -1$

g.s: $y = c_1 + c_2 x + c_3 e^{-x} + c_4 x e^{-x}$

④ $y^{(4)} - 2y^{(3)} + 2y' - y = 0$

$\lambda^4 - 2\lambda^3 + 2\lambda - 1 = 0$

$\lambda^4 - 1 - 2\lambda(\lambda^2 - 1) = 0$

$(\lambda^2 - 1)(\lambda^2 + 1) - 2\lambda(\lambda^2 - 1) = 0$

$(\lambda^2 - 1)(\lambda^2 - 2\lambda + 1)$

$(\lambda - 1)(\lambda + 1)(\lambda - 1)(\lambda - 1) = 0 \qquad \lambda = 1, 1, 1, -1$

g.s: $y = c_1 e^{x} + c_2 x e^{x} + c_3 x^2 e^{x} + c_4 e^{-x}$

⑤ $y^{(4)} + 2y'' + y = 0$

$\lambda^4 + 2\lambda^2 + 1 = 0$

$(\lambda^2+1)(\lambda^2+1) = 0 \qquad \lambda = \mp i, \mp i$

g.s: $y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$

⑥ $y^{(4)} - y = 0$

g.s : $y = c_1 \cos x + c_2 \sin x + c_3 e^{x} + c_4 e^{-x}$

⑦ $y^{(3)} - 3y'' + 9y' + 13y = 0$

g.s : $y = c_1 e^{2x} \cos 3x + c_2 e^{2x} \sin 3x + c_3 e^{-x}$

* Existence and uniqeress :-

consider the I.V.P

$P_n(x) y^{(n)} + P_{n-1}(x) y^{(n-1)} + \dots + P_0(x) y = R(x)$

∴ $y(x_0) = y_0$, $y'(x_0) = y_1$, $y''(x_0) = y_2$ — , $y^{n-1}(x_0) = y_{n-1}$

such that $P_n, P_{n-1}, \dots P_0, R(x)$ are ① continuous function at the open interval $I = (a, b)$ & $P_n(x)$ ② never zero in I and ③ $x \in I$, then 1) has auniqe

Ex: Find the largest Possible Interval in which the given I.VP has auniqe solution.

$y'' + 2y' + y = x$ $y(\overset{x_0}{1}) = 2$ $y'(1) = -1$

solution: $P_2 = 1$, $P_1 = 2$, $P_0 = 1$ $R(x) = x$

are const on $R = (-\infty, \infty)$

$P_2(x) = 1 \neq 0$ on $R = (-\infty, \infty)$

$x_0 = 1 \in R = (-\infty, \infty)$

Ex: $x^2 y'' + xy' = 2x + 1$, $y(2) = -1$, $y'(2) = 3$

$P_2 = x^2$, $P_1 = x$, $P_0 = 0$ $R(x) = 2x+1$ are const $R = (-\infty, \infty)$

$P_2(x) = x^2 = 0 \rightarrow x = 0$

$I = (0, \infty)$, $I^* = (-\infty, 0) \longrightarrow$ note :

يجوز تقول $(0,\infty) \cup (-\infty, 0)$

$x_0 = 2 \in (0, \infty)$, $\notin (-\infty, 0)$

$I = (0, \infty)$

Ex: ① $(x-1)y^{(3)} + xy' = 2x \quad y(-1) = 1 \quad y'(-1) = 3$

solu. $I\ (-\infty, 1)$

② $\left(\frac{1}{x-3}\right)y'' + 2y' = 5x+1 \quad y(0) = 1 \quad y'(0) = 2$

solu. $I\ (-\infty, 3)$

③ $\sqrt{x+1}\, y'' + 2y' = 4x$

solu. $I\ (-1, \infty)$

* Linearly independent and dependent functions:

Def: If $F_1, F_2, \ldots F_n$ are functions & $c_1, c_2 \ldots c_n$ are const. not all zeros then we say that these func. are linearly dependent if and only if $\boxed{c_1F_1 + c_2F_2 + \ldots c_nF_n}$
but if $c_1F_1 + c_2F_2 + \ldots c_nF_n = 0$ when
just $c_1 = c_2 = \ldots = c_n = 0$
then $F_1, F_2, \ldots F_n$ linearly independent

Ex: $F_1 = 1,\ F_2 = x,\ F_3 = x^2$
determine whether these functions are linearly indep or dep.

solution: $c_1 * 1 + c_2 * x + c_3 * x^2 = 0 + 0x + 0x^2$
$c_1 = 0,\ c_2 = 0,\ c_3 = 0 \Rightarrow$ linearly indep.

Ex: $F_1 = 0,\ F_2 = 2x,\ F_3 = x^3$

$c_1F_1 + c_2F_2 + c_3F_3 = 0$
$0c_1 + 2c_2 + c_3x^3 = 0 = 0 + 0x + 0x^3$
$c_2 = 0,\ c_3 = 0$ but c_1 arbitrary اختياري
$\Downarrow$ may be $c_1 = 0$ but it may be any number too
linearly dep. $c_1 = 0$ or $c_1 =$ any number.

Ex: $F_1 = \sin 2x \quad F_2 = \sin x \cos x$

Solu: $c_1 \sin 2x + c_2 \sin x \cos x = 0$

$c_1 \sin 2x + \frac{1}{2} c_2 \sin 2x = 0$

$(c_1 + \frac{1}{2} c_2) \sin 2x = 0$

$\Rightarrow c_1 + \frac{1}{2} c_2 = 0$ (non trivial solu. linearly dep)

* notes:-
- trivial solution: all of unknowns are zeros.
- non trivial solution: on of unknown does not equal zero

* في نظام حل المعادلات الخطية $\begin{bmatrix} \ \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

لو اذا كانت كل العناصر صفر $\Leftarrow$ homog. system

* اذا كان عدد المجاهيل أكبر من عدد المعادلات عندها $\Leftarrow$ non trivial soln.

trivial solution always exist in Homog. system *

لكن في نظام Homog نبحث دائماً عن الحل الغير صفري (non trivial)

Ex: $F_1 = \sin^2 x$, $F_2 = \cos^2 x$, $F_3 = \tan^2 x$, $F_4 = \sec^2 x$

$c_1 F_1 + c_2 F_2 + c_3 F_3 + c_4 F_4 = 0$

$c_1 \sin^2 x + c_2 \cos^2 x + c_3 \tan^2 x + c_4 \sec^2 x = 0$

$c_1 \sin^2 x + c_2 (1 - \sin^2 x) + c_3 \tan^2 x + c_4 (1 + \tan^2 x) = 0$

$c_1 \sin^2 x + c_2 - c_2 \sin^2 x + c_3 \tan^2 x + c_4 + c_4 \tan^2 x = 0$

$(c_1 - c_2) \sin^2 x + (c_2 + c_4) + (c_3 + c_4) \tan^2 x = 0$

$$\left.\begin{aligned} c_1 - c_2 &= 0 \\ c_2 + c_4 &= 0 \\ c_3 + c_4 &= 0 \end{aligned}\right\}$$

أربعة مجاهيل و ٣ معادلات

non trivial solution

linearly dep

Ex: $F_1 = \sqrt{x}+1$, $F_2 = \sqrt{x}+5x$, $F_3 = x+5$, $F_4 = x^3$

$c_1F_1 + c_2F_2 + c_3F_3 + c_4F_4 = 0$

$c_1(\sqrt{x}+1) + c_2(\sqrt{x}+5x) + c_3(x+5) + c_4x^3 = 0$

$c_1\sqrt{x} + c_1 + c_2\sqrt{x} + 5c_2x + c_3x + 5c_3 + c_4x^3 = 0$

$(c_1+c_2)\sqrt{x} + c_1 + 5c_3 + (5c_2+c_3)x + c_4x^3 = 0$

$c_1 + c_2 = 0$
$c_1 + 5c_3 = 0$
$5c_2 + c_3 = 0$
$c_4 = 0$

note: non singular matrix
$|A| \neq 0$
singular matrix
$|A| = 0$

$c_1 + c_2 = 0$
$c_1 + 5c_3 = 0$
$5c_2 + c_3 = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 5 \\ 0 & 5 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} =$$

اذا استطعنا التخلص من المصفوفة (A) عندها : $AA^{-1} \leftarrow$

$$1 * \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

ونتخلص من المصفوفة إذا ضربناها بـ (A^{-1}) inverse ويكون لها (A^{-1}) إذا $|A| \neq 0$

$|A| = -25 * -1(0) = -25 - 1 = -26 \neq 0$

اذاً لها inverse :

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 5 \\ 0 & 2 & 1 \end{bmatrix} * A^{-1}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0A^{-1}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$c_1 = 0$
$c_2 = 0$
$c_3 = 0$
$c_4 = 0$

linearly independent.

* Wronskian's:

Def: If $F_1, F_2, \_\_\_ F_n$ are functions, then the wronskians of these functions is denoted by $w(F_1, F_2, --- F_2)$ & is defined as

$$w = \begin{vmatrix} F_1 & F_2 & --- & F_n \\ F_1' & F_2' & --- & F_n' \\ F_1'' & F_2'' & --- & F_n'' \\ F_1''' & F_2''' & --- & F_n''' \\ F_1^{(n-1)} & F_2^{(n-1)} & & F_n^{(n-1)} \end{vmatrix}$$

note:
اذا كانت $w=0$
عند ها لا نستطيع
نقول اي شيء

Theorem: If $w(F_1, F_2, --- F_n) \neq 0$ for at least one point in the interval I then we say that $F_1, F_2 --- F_n$ are linearly independent.

Theorem: If $F_1, F_2 -- F_n$ are linearly dep. then $W(F_1, F_2 --- F_n) = 0$

Ex: $F_1 = 1 \quad F_2 = x \quad F_3 = x^2$

$$W = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

عن طريق الصف الثالث

$W = 0(\quad) + -0(\quad) + 2(1-0) - = 2$

$W \neq 0 \Rightarrow$ linearly independent

Ex: $F_1 = e^{2x} \quad F_2 = 3e^{2x}$

$$W = \begin{vmatrix} e^{2x} & 3e^{2x} \\ 2e^{2x} & 6e^{2x} \end{vmatrix} = 0$$ we can't say any thing, we can't determine.

$c_1 F_1 + c_2 F_2 = 0$

$(c_1 + 3c_2) e^{2x} = 0$

$c_1 + 3c_2 = 0 \Rightarrow$ معادلة بمجهولين
non trivial solution
linearly dep.

* A second solution from a given solution:

$y'' + P(x) y' + q(x) = 0 \quad \text{---} \ (*)$

such that y_1 is a solution.

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$ معامل y'' واحد

Ex: solve $(x^2+1)y'' - 2xy' + 2y = 0$ given that $y = x$ is a solution.

$$y'' - \frac{2x}{(x^2+1)} y' + \frac{2}{(x^2+1)} y = 0$$

$$P(x) = \frac{-2x}{x^2+1} \Rightarrow -\int P(x)dx = \int \frac{2x}{x^2+1} = \ln(x^2+1)$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx = x \int \frac{e^{\ln(x^2+1)}}{x^2} dx = x \int \frac{x^2+1}{x^2} dx$$

$$y_2 = x \int \left(1 + \frac{1}{x^2}\right) dx$$

$$= x\left(x + \frac{x^{-1}}{-1}\right) = x\left(x - \frac{1}{x}\right)$$

$$\boxed{y_2 = x^2 - 1}$$

* لابد للتأكد نقوم بتعويض الحل فإذا حققت المعادلة التفاضلية يكون هذا صحيح.

ملاحظة: في هذا الحل لا نضيف الثابت C لأن اضافته تكون لـ (g.s) فقط.

Ex: $x^2y'' - 6y = 0 \quad x>0 \quad \& \quad y_1 = x^3$ is solution

$$y'' - \frac{6}{x^2}y = 0$$

$-\int P(x)\,dx = c$ arbitrary assume $= 0$

$$y_2 = x^3\int\frac{1}{x^6}\,dx = x^3\int x^{-6}dx = x^3\left(\frac{-1}{5}x^{-5}\right)$$

$$y_2 = \frac{-1}{5}x^{-2} = \frac{-1}{5x^2}$$

* Coushy - Euler D.E's:-

Form: $a_n x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \dots + a_0 y = g(x) \quad --(\alpha)$

is called coushy Euler (C.E) non homogeneous D.E and if $g(x)=0$, then it is called Homog. C.E D.E

$a_n x^n y^{(n)} + \dots a_0 y = 0 \quad --- ①$

Where a_i, $i=1,2,3 --- n$ are const.

to solve ① we let $y = x^m$

$y' = m x^{m-1}$, $y'' = m(m-1)x^{m-2}$ $\quad y''' = m(m-1)(m-2)x^{m-3}$

$y^{(n-1)} = m(m-1)\cdots(m-(m-2))\,x^{m-(n-1)}$

$y^{n} = m(m-1)\cdots(m-(n-1))\,x^{m-n}$

After substitution in① we have:

$a_n m(m-1) --- (m-(n-1))x^m + a_{n-1}\, m(n-1) --- (m-(m-2))x^m$

$+ \dots + a_0 x^m = 0$

$x^m F(m) = 0 \quad --- ② \qquad x^m \neq 0 \Rightarrow F(m) = 0 \quad --- ③$

لان الحل الذي افترضناه $y = x^m$ لايساوي صفر

① If all root are distinct :

$m_1 \neq m_2 \neq m_3 \neq \cdots \neq m_n$

This func. is called the characteristic Polyn. of ① and it has n roots and these roots may be distict, repeated or complex conjugates.

$y_1 = x^{m_1}$ $y_2 = x^{m_2}$ $y_3 = x^{m_3}$, $y_n = x^{m_n}$

Ex: $x^2 y'' - 6y = 0$

كيفية التعويض:

$x^2 y'' \longrightarrow m(m-1)$

$x y' \longrightarrow m$

$x^3 y''' \longrightarrow m(m-1)(m-2)$

$c\, y \longrightarrow c$

$m(m-1) - 6 = 0$

$m^2 - m - 6 = 0$

$(m-3)(m+2) = 0$

$m = 3$, $m = -2$

$y = c_1 x^3 + c_2 x^{-2}$

② If the roots are repeated (r) times and the remaining roots are distinct.

$m_1 = m_2 = \cdots m_r$, $m_{r+1} \neq m_{r+2} \neq \cdots m_n$

$y_1 = x^{m_1}$, $y_2 = x^{m_1} \ln x$, $y_3 = x^{m_1} (\ln x)^2$

Ex: $4x^2 y'' + 8xy' + y = 0$ $y(1) = 1$

$y'(1) = 2$

$4m(m-1) + 8m + 1 = 0$

$4m^2 - 4m + 8m + 1 = 0$

$4m^2 + 4m + 1 = 0$

$(2m+1)(2m+1) = 0$ $m = -\frac{1}{2}, -\frac{1}{2}$

g.s: $y = c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x$

p.s: $y = x^{-\frac{1}{2}} + \frac{5}{2} x^{-\frac{1}{2}}$

أكمل

P.S. عتي يد

③ If the roots are of the form $m = \alpha \pm \beta i$

$y_1 = x^{\alpha}\cos(\beta \ln x)$, $y_2 = x^{\alpha}(\sin\beta \ln x)$

Ex: $x^2 y'' + 3xy' + 3y = 0$

$m(m-1) + 3m + 3 = 0$

$m^2 - m + 3m + 3 = 0 \Rightarrow m^2 + 2m + 3 = 0$

$$m = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \mp \sqrt{-8}}{2} = -1 \mp \sqrt{2}\, i$$

g.s: $y = x^{-1}(c_1 \cos\sqrt{2}\ln x + c_2 \sin\sqrt{2}\ln x)$

Ex: $x^3 y''' + 5x^2 y'' + 7xy' + 8y = 0$

$m(m-1)(m-2) + 5m(m-1) + 7m + 8 = 0$

$m(m^2 - 3m + 2) + 5(m^2 - m) + 7m + 8 = 0$

$m^3 - 3m^2 + 2m + 5m^2 - 5m + 7m + 8 = 0$

$m^3 + 2m^2 + 4m + 8 = 0$

$m = -2$, $m^2 = -4 \Rightarrow m = \mp 2i$

g.s: $y = c_1 x^{-2} + c_2 \cos(2\ln x) + c_3 \sin(2\ln x)$

Ex: If the g.s. of the C.E D.E

$y = x^4\left[c_1\cos(\ln x^5) + c_2\sin(\ln x^5)\right]$ Find the D.E

solution: $y = x^4\left[c_1\cos(5\ln x) + c_2\sin(5\ln x)\right]$

$m = 4 \pm 5i$

$(m-\alpha)^2 + \beta^2 = 0$

$(m-4)^2 + 5^2 = 0$

$m^2 - 8m + 16 + 25 = 0$

$m^2 - 8m + 41 = 0$

$m^2 - m - 7m + 41 = 0$

$x^2 y'' - 7xy' + 41y = 0$

very important
اذا كانت أصفار معادلة هي
$m = \alpha \pm \beta i$
حينها تكون المعادلة $(m-\alpha)^2 + \beta^2 = 0$

Ex: g.s : $y = c_1 x^2 + c_2 x^2 \ln x + c_3$

Solution $m = 2, 2, 0$

$(m-2)(m-2)m = 0 \Rightarrow (m^2 - 4m + 4)\, m = 0$

$m^3 - 4m^2 + 4m = 0$

$m^3 - 3m^2 + 2m - m^2 + 2m = 0$

$-m^2 + m + m = 0$

$x^3 y''' - x^2 y'' + x y' = 0$

$x^3 y''' = m(m-1)(m-2)$
$= m(m^2 - 3m + 2)$
$= m^3 - 3m^2 + 2m$

* Non-Homog. L.D.E's with const. coefficients:

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = R(x)$ ---- (*)

where $a_n, a_{n-1}, \dots, a_0$ const.

- general solution of this equation contains two points:

- the first one is called complementery solution and is denoted by y_c which is the solution of the homogeneous part of (*) and it contains the arbitrary constants ($c_1, c_2, c_3, \dots$)
$\Rightarrow (y_c = c_1 y_1 + c_2 y_2 + c_3 y_3 + \dots)$.

- the second one is called the particular solution of (*) which is a particular solution that satisfies (*) which is denoted by $\Rightarrow y_p$:
has no arbitrary const.

$\therefore$ g.s $\boxed{y = y_c + y_p}$

* How to Find y_p :

المعاملات الغير محددة

1 The method of undetermined coefficient : (U.C)

to use this method the non homogeneous part $(R(x))$ must be one of the following functions:

1. Exponential function $(e^{ax}) = R(x)$
2. Polynomial functions $R(x) = b_n x^n + b_{n-1} x^{n-1} + \dots + b_0$
3. $\sin bx$ or $\cos bx$ or both $\sin bx + \cos bx$
4. Any linear combination or (product) of two or more functions of 1, 2 and 3

ملاحظة :- اذا كان $\sin^2 x$ نستبدلها $\frac{1}{2}(1-\cos 2x)$

اذا كان $\sin x \cos x \Leftarrow \frac{1}{2}\sin 2x$

* To find y_p using this method we do following:

1. Find $y_c = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$
2. Find y_p which will be of the same shape of $R(x)$ if there are common roots between y_c & $R(x)$ but if there are common roots between them, then y_p will be as $R(x)$ but multiplied by $x, x^2, x^3 \dots$

Ex: $R(x) = 2x + 1 + 3e^{-2x}$

0,0 $\quad$ -2

$R(x) = 3xe^{3x} + 2x^2 + 3$ $\quad$ الاصفار التي تعطي هذا الحل $R(x)$

3,3 $\quad$ 0,0,0

Ex: solve $y'' - 3y' + 2y = 5$ $\quad y(0) = 1 \quad y'(0) = 2$

Solution: $\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$\lambda = 1, 2$

$y_1 = e^x \;,\; y_2 = e^{2x} \Rightarrow y_c = c_1 e^x + c_2 e^{2x}$

$R(x) = 5 * 1$

$\lambda' = 0 \Rightarrow R(x), y_c$

لا يوجد أصفار مشتركة بين $R(x), y_c$

لذلك سيكون y_p نفس شكل $R(x)$

$\therefore y_p = A$ substitute in the original equation.

$y_p' = 0$

$y_p'' = 0$

$0 - 3 * 0 + 2A = 5 \Rightarrow 2A = 5 \quad \boxed{A = \frac{5}{2}}$

$\boxed{y_p = \frac{5}{2}}$

g.s: $y = y_c + y_p$

$y = c_1 e^x + c_2 e^{2x} + \frac{5}{2}$

$y(0) = 1 = c_1 + c_2 + \frac{5}{2}$

$c_1 + c_2 = \frac{-3}{2}$ ——— (1)

~~y(0) =~~ $y' = c_1 e^x + 2c_2 e^{2x}$

$y'(0) = 2 = c_1 + 2c_2$ ——— (2)

أكمل الحل لإيجاد قيمة c_1 & c_2

Ex: $y''' - y' = 5$

solution $\lambda^3 - \lambda = 0 \Rightarrow \lambda(\lambda^2 - 1) = 0$

$\lambda(\lambda-1)(\lambda+1) = 0$

$\lambda = 0, 1, -1$

$y_c = c_1 y_1 + c_2 y_2 + c_3 y_3$

$= c_1 + c_2 e^{x} + c_3 e^{-x}$

$R(x) = 5 * 1$

$\lambda' = 0$

$\lambda^* = 0, 0, 1, -1$

الحلول $\Rightarrow$ $1 \quad x \quad e^{x} \quad e^{-x}$

هذا الحل في حال وجود أصفار مشتركة بين $R(x)$, y_c $\Rightarrow$ أصفار y_c وأصفار $R(x)$

الآن نشطب الحلول المشتركة بين λ^* و y_c والذي تبقى هو سيكون y_p

$y_p = Ax$
$y_p' = A$
$y_p'' = 0$
$y_p''' = 0$

عوض

$0 - A = 5 \Rightarrow A = -5$

$y_p = -5x$

$y = c_1 + c_2 e^{x} + c_3 e^{-x} - 5x$

Ex: $y'' - 3y' + 2y = 3e^{x} + 2e^{-2x}$

$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1, 2 \Rightarrow y_1 = e^{x}, y_2 = e^{2x}$

$R(x) = 3e^{x} + 2e^{-2x}$

$\lambda' = 1, -2$

$\lambda^* = 1, 1, 2, -2$

الحلول $e^{x} \quad xe^{x} \quad e^{2x} \quad e^{-2x}$

$y_p = Axe^{x} + Be^{-2x}$ suitable form

$y_p' = Axe^{x} + Ae^{x} - 2Be^{-2x}$

$y_p'' = Axe^{x} + 2Ae^{x} + 4Be^{-2x}$

$$Axe^{x}+2Ae^{x}+4Be^{-2x}-3Axe^{x}-3Ae^{x}+6Be^{-2x}+2Axe^{x}$$
$$+2Be^{-2x}=3e^{x}+2e^{-2x}$$

$$\therefore -Ae^{x}+12Be^{-2x}=3e^{x}+2e^{-2x}$$
$$-A=3 \Rightarrow \boxed{A=-3}$$
$$12B=2 \Rightarrow \boxed{B=1/6}$$
$$y_p=-3xe^{x}+\frac{1}{6}e^{-2x}$$

g.s: ....

Ex: $y''+2y'+2y=x\sin 2x+\sin 2x+2e^{-x}\cos x$

Find the suitable Form of y_p

solution: $\lambda^2+2\lambda+2=0$

مميز $\leftarrow \Delta=b^2-4ac=4-8=-4<0 \therefore$ complex

$$\lambda=\frac{-2\mp\sqrt{-4}}{2}=-1\mp i$$

$y_1=e^{-x}\cos x$, $y_2=e^{-x}\sin x$

$$y_c=e^{-x}(C_7\cos x+C_8\sin x)$$

$$R(x)=(x+1)\sin 2x+2e^{-x}\cos x$$
$$\lambda'=\mp 2i,\ \mp 2i,\ -1\mp i$$

لأنه الصفر مكرر بسبب وجود x

$$\lambda^*=\mp 2i,\ \mp 2i,\ -1\mp i,\ -1\mp i$$

$\cos 2x$, $\sin 2x$ — $x\cos 2x$, $x\sin 2x$ — $e^{-x}\cos x$, $e^{-x}\sin x$ — $xe^{-x}\cos x$, $xe^{-x}\sin x$

$$y_p=C_1\cos 2x+C_2\sin 2x+C_3x\cos 2x+C_4x\sin 2x+C_5xe^{-x}\cos x+C_6e^{-x}\sin x$$

Exersise: (1) $y'' - 3y' + 2y = 2e^{-x} - e^{-2x}$

(2) $y^{(4)} - 2y^{(3)} + 2y' - y = 10$

(3) $y^{(4)} + 2y''' + y'' = 10$

(4) $y'' - 5y' + 6y = xe^{2x} + e^{2x} + 3e^{3x}$

(5) $y''' - 4y' = x + 3\cos x + e^{-2x} + 3$

Theorem: If y_{p_1} is a particular solution of $a_n y^{(n)} + \dots + a_0 y = F_1(x)$ and y_{p_2} is a particular solution of $a_n y^{(n)} + \dots + a_0 y = F_2(x)$ then $c_1 y_{p_1} + c_2 y_{p_2}$ is a particular solution of $a_n y^{(n)} + \dots + a_0 y = c_1 F_1(x) + c_2 F_2(x)$

Ex: If y_1 is a solution of $y'' + P(x)y' + q(x)y = x^2 + 3x$

y_1 لا تكتب y_p and y_2 is a solution of $y'' + P(x)y' + q(x)y = 2x - 3x^2$

نفس الشيء لا y_p اصلاً then the solution of $y'' + P(x)y' + q(x)y = x$

حل المعادلة بالكامل

Solution: $c_1 y_1 + c_2 y_2$

$F_1 = x^2 + 3x \quad , \quad F_2 = 2x - 3x^2$

$c_1 F_1 + c_2 F_2 = x$

$c_1 (x^2 + 3x) + c_2 (2x - 3x^2) = x$

$c_1 x^2 + 3c_1 x + 2c_2 x - 3c_2 x^2 = x$

$(3c_1 + 2c_2)x + x^2(c_1 - 3c_2) = x$

$$\left.\begin{aligned} c_1 - 3c_2 &= 0 \\ 3c_1 + 2c_2 &= 1 \end{aligned}\right\} \quad \begin{aligned} c_2 &= 1/11 \\ c_1 &= 3/11 \end{aligned}$$

$c_1 y_1 + c_2 y_2 = \frac{3}{11} y_1 + \frac{1}{11} y_2 \Rightarrow$ the solution

[2] Variation of parameters :-

The method is more general than the method of U.C because it is applicable for every continuous function $R(x)$ and it can D.E with variable coeff.

to find y_p using this method we find 1st

$$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = R(x) \quad \cdots (*)$$

Then apply on:

$$y_p = \sum_{m=1}^{n} y_m \int \frac{R(x)\, W_m(x)}{W(x)}\, dx$$

$$y_p = y_1 \int \frac{R(x)\, W_1(x)}{W(x)}\, dx + y_2 \int \frac{R(x)\, W_2(x)}{W(x)}\, dx + \cdots + y_n \int \frac{R(x)\, W_n(x)}{W_n(x)}\, dx$$

where $W(x)$ is the wronskain of $y_1, y_2, \cdots y_n$

and $W_i(x)$ $i = 1, \cdots, n$

is the determinent obtained from W by replacing the i^{th} column of W by $(0, 0, 0, \cdots, 1)$

∴ for 2nd order only:

$$y'' + P(x)\, y' + q(x)\, y = R(x)$$

$$y_c = c_1 y_1 + c_2 y_2$$

$$y_p = y_1 u_1 + y_2 u_2$$

$$u_1 = \int \frac{R(x)\, W_1(x)}{W(x)}\, dx$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2$$

$$W = y_1 y_2' - y_2 y_1'$$

∴ $\boxed{u_1 = -\int \frac{R(x)\, y_2}{W(x)}\, dx}$ ∴ $\boxed{u_2 = \int \frac{R(x)\, y_1}{W(x)}\, dx}$

$$u_2 = \int \frac{R(x)\, W_2(x)}{W(x)}\, dx$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1$$

Ex: $y''' + y' = \tan x \qquad 0 < x < \frac{\pi}{2}$

$\lambda^3 + \lambda = 0$

$\lambda(\lambda^2+1) = 0 \qquad \lambda = 0 \,,\ \mp i$

$y_c = c_1 + c_2 \cos x + c_3 \sin x$

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1(\sin^2 x + \cos^2 x) = 1$$

$$W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 1 & -\cos x & -\sin x \end{vmatrix} = 1(\cos^2 x + \sin^2 x = 1$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & 1 & -\sin x \end{vmatrix} = 1(-\cos x) = -\cos x$$

$$W_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & 1 \end{vmatrix} = 1(-\sin x) = -\sin x$$

$y_p = y_1 u_1 + y_2 u_2 + y_3 u_3$

$$u_1 = \int \frac{R(x) W_1(x)}{W(x)}\, dx = \int \frac{\tan x \cdot 1}{1}\, dx = -\ln|\cos x| = \ln(\sec x)$$

$$u_2 = \int \frac{R(x) W_2(x)}{W(x)}\, dx = \int \frac{\tan x * -\cos x}{1}\, dx = \int -\sin x\, dx = \cos x$$

$$u_3 = \int \frac{R(x) W_3(x)\, dx}{W(x)} = \int \frac{\tan x * -\sin x}{1}\, dx = \int \frac{-\sin^2 x}{\cos x}\, dx = \int \frac{1-\cos^2 x}{\cos x}\, dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\, dx = -\int \frac{1-\cos^2 x}{\cos x}\, dx = -\int (\sec x - \cos x)\, dx$$

$$= -(\ln|\sec x + \tan x| - \sin x)$$

$$= \sin x - \ln|\sec x + \tan x|$$

Ex: $y'' + y = \sec x \tan x$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \mp i$

$y_c = \cos x \quad , y_2 = \sin x$

$y_{p_{g}} = y_1 u_1 + y_2 u_2$

$w = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$w_1 = \begin{vmatrix} 0 & \sin x \\ 1 & \cos x \end{vmatrix} = -\sin x$

$w_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & 1 \end{vmatrix} = \cos x$

$u_1 = \int \frac{\sec x \tan x * -\sin x}{1} dx$

$= -\int \frac{\sin^2 x}{\cos^2 x} dx = \int -\tan^2 x dx$

$= -\int (\sec^2 x - 1) dx$

$= -\tan x + x$

$u_2 = \int \frac{\sec x \tan x * \cos x}{1} dx$

$= \int \tan x dx \Rightarrow u_2 = -\ln(\cos x)$

$y_p = \cos x (x - \tan x) + \sin x (-\ln(\cos x))$

$= x \cos x - \tan x \cos x - \sin x \ln \cos x$

$= x \cos x - \sin x - \sin x \ln \cos x$

$= x \cos x - \sin x (1 + \ln \cos x)$

Exersize:

1. $y'' - 3y' + 2y = \frac{e^{-2x}}{e^x + 1}$
2. $y'' - 2y' + y = \frac{1}{x} e^x$
3. $y'' - 12y' + 36y = e^{6x} \ln x$
4. $y'' + y = \tan x$
5. $y'' + 2y' + y = e^{-x} \sec^2 x$

*Power series solution:

method: the power series of $(x-x_0)$ is :-

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 + \cdots +$$

The power series of x is:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

* If $F(x) = \sum_{n=0}^{\infty} a_n x^n$ then $\Rightarrow F'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$F''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

and $\int F(x)dx = \int \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \sum_{n=0}^{\infty} \left(\int a^n x^n \, dx\right)$

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

*If $F(x)$ is a function then the taylor series of F at $x = x_0$ is:-

$$F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(x_0)}{n!} (x-x_0)^n$$

تذكر $\boxed{0! = 1}$

$$= F(x_0) + \frac{F'(x_0)}{1!}(x-x_0) + \frac{F''(x_0)}{2!}(x-x_0)^2 + \cdots$$

* maclaurine series of $x_0 = 0$

the same taylor series but maclaurine at $x_0 = 0$

$$F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} x^n$$

$$= F(0) + \frac{F'(0)}{1!}x + \frac{F''(0)}{2!}x^2 + \cdots$$

<u>Ex:</u> $F(x) = \frac{1}{1-x}$ Find maclaurin series!

$F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} x^n$

$F(x) = 1 + x + x^2 + x^3 + \cdots x^n$

$= \sum_{n=0}^{\infty} x^n$

$F(x) = \frac{1}{1-x} \Rightarrow F(0) = 1$

$F'(x) = \frac{1}{(1-x)^2} \Rightarrow F'(0) = 0$

$F''(x) = \frac{2}{(1-x)^3} \Rightarrow F''(0) = 2$

$F'''(x) = \frac{6}{(1-x)^4} \Rightarrow F'''(0) = 6$

$F^{(4)}(x) = \frac{24}{(1-x)^5} \Rightarrow F^{(4)}(0) = 24$

Result!

هام

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

<u>Ex:</u> $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}$

$F(x) = \frac{1}{(1-x)^2} \Rightarrow \left(\frac{1}{1-x}\right)' = \frac{1}{1-x^2}$

but $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{1-x^2} = \left(\frac{1}{1-x}\right)' = \sum_{n=1}^{\infty} n x^{n-1}$

<u>Ex:</u> $F(x) = \tan^{-1} x$

$\tan^{-1} x = \int \frac{1}{1+x^2} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx$

$= \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

* Maclaurine series for $\boxed{e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$ حفظ

وأيضاً: $e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

* $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

* Maclaurine series for $\sin x$, $\cos x$:

$$\boxed{\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$$

$$\boxed{\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}}$$

وأيضاً:

* $x \cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n!}$

* solutions near an ordinary points :-

Def: A point $x = x_0$ is called an ordinary point of $P_2(x)y'' + P_1(x)y' + P_0(x)y = R(x)$ if $P_2(x_0) \neq 0$ other wise it is called singular point.

Ex: ① $(x^2-1)y'' + 3y = 4$ Find the ordinary point

$x^2 - 1 \neq 0$

$x \in \mathbb{R} \setminus \{-1, 1\}$

بدون

② $\tan x\, y'' + xy' + 1 = 0 \Rightarrow \tan x\, y'' + xy' = -1$ non-homog

$\tan x \neq 0$

$x \in \mathbb{R} \setminus \{\mp n\pi, n = 0, 1, \dots\}$

To solve any D.E of an ordinary points using series method we let $y = \sum_{n=0}^{\infty} a_n x^n$ to be solution

بشكل عام or $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

Ex: $y' + 2xy = 0$ using series method

يمكن حلها بالطريقة السابقة كالتالي

$$y' = -2xy$$
$$\frac{1}{y}dy = -2x\,dx$$
$$\ln y = -x^2 + c$$
$$y = A e^{-x^2}$$

the solution :-

$$\text{let } y = \sum_{n=0}^{\infty} a_n x^n$$
$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
} D.E نعوض في

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$= \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0$$

$\sum_{n=1}^{\infty} n a_n x^{n-1}$ | $\sum_{n=0}^{\infty} 2a_n x^{n+1}$

لجمعهم مجموع واحد

أول خطوة نساوي أسس x

let $m = n-1$ | $m = n+1$

$n = 1 \Rightarrow m = 0$ | $n = 0 \Rightarrow m = 1$

$\boxed{n = m+1}$ | $\boxed{n = m-1}$

الخطوة الثانية: نجعل قاعدة المجموعتين يبدأ من نفس العدد

$$\sum_{m=0}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=1}^{\infty} 2 a_{m-1} x^m = 0$$

$$(1)a_1 + \sum_{m=1}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=1}^{\infty} 2 a_{m-1} x^m = 0$$

نفس البداية لأنه يجوز الجمع

$$= (1)a_1 + \sum_{m=1}^{\infty} \left((m+1) a_{m+1} + 2a_{m-1}\right) x^m = 0$$

$$1 a_1 + \sum_{m=1}^{\infty} \left((m+1) a_{m+1} + 2 a_{m-1}\right) x^m = 1 \cdot 0 + 0 \cdot x + 0 \cdot x^2$$

$$\therefore a_1 = 0$$

$(m+1)\, a_{m+1} + 2a_{m-1} = 0$

$a_{m+1} = \dfrac{-2a_{m-1}}{m+1} \quad \forall\ m \geq 1 \Rightarrow a_2, a_3, a_4 \cdots$

يمكن من خلالها إيجاد أي قيمة

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$m=1 \Rightarrow a_2 = \dfrac{-2a_0}{2} = -a_0$

$m=2 \Rightarrow a_3 = \dfrac{-2a_1}{3} = 0$

$m=3 \Rightarrow a_4 = \dfrac{-2a_2}{4} = \dfrac{-1}{2}a_2 = \dfrac{1}{2}a_0$

$y = a_0 - a_0 x^2 + \dfrac{1}{2} a_0 x^4 + \cdots$

$= a_0\left(1 - x^2 + \dfrac{x^4}{2} + \cdots\right) = a_0 \sum_{n=0}^{\infty} \dfrac{-x^{2n}}{n!} = a_0 \sum_{n=0}^{\infty} \dfrac{(-1)^{n} x^{2n}}{n!}$

$$\boxed{y = a_0 e^{-x^2}}$$

<u>Ex:</u> $y'' + xy' + (x^2+2)y = 0$

let $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

substitute

$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} a_n x^{n-1} + (x^2+2)\sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} + \sum_{n=0}^{\infty} 2a_n x^n = 0$

$$\sum_{n=2}^{\infty} n(n-1)a_n X^{n-2} + \sum_{n=1}^{\infty} n a_n X^{n} + \sum_{n=0}^{\infty} a_n X^{n+2} + \sum_{n=0}^{\infty} 2a_n X^{n} = 0$$

$m = n-2$ $n=2 \Rightarrow m=0$ $n = m+2$	$m = n$	$m = n+2$ $n=0 \Rightarrow m=2$ $n = m-2$	$m = n$

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} X^m + \sum_{m=1}^{\infty} m a_m X^m + \sum_{m=2}^{\infty} a_{m-2} X^m + \sum_{m=0}^{\infty} 2a_m X^m = 0$$

$$(0+2)(0+1)a_2 X^0 + (1+2)(1+1)a_3 X + \sum_{m=2}^{\infty} (m+2)(m+1) a_{m+2} X^m +$$

$$1a_1 + \sum_{m=2}^{\infty} m a_m X^m + \sum_{m=2}^{\infty} a_{m-2} X^m + 2a_0 X^0 + 2a_1 X + \sum_{m=2}^{\infty} 2a_m X^m = 0$$

$$\sum_{m=2}^{\infty} \left((m+2)(m+1)a_{m+2} + m a_m + a_{m-2} + 2a_m \right) X^m + 2a_2 + 6a_3 X + 3a_1 X + 2a_0 = 0$$

عن طريق تساوي المعاملات

$$2a_0 + 2a_2 = 0 \Rightarrow a_2 = -a_0$$

$$6a_3 + 3a_1 = 0 \Rightarrow a_3 = \frac{-1}{2}a_1$$

$$(m+2)(m+1)a_{m+2} + (m+2)a_m + a_{m-2} = 0$$

$$a_{m+2} = \frac{-(m+2)a_m - a_{m-2}}{(m+2)(m+1)}$$

$\boxed{m=2}$ $$a_4 = \frac{-4a_2 - a_0}{4*3} = \frac{4a_0 - a_0}{12} = \frac{1}{3}a_0$$

$\boxed{m=3}$ $$a_5 = \frac{-5a_3 - a_1}{5*4} = \frac{-5 * \frac{-1}{2}a_1 - a_1}{20} = \frac{3}{40}a_1$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

g.S: $$y = a_0 + a_1 X - a_0 X^2 - \frac{1}{2}a_1 X^3 + \frac{1}{4}a_0 X^4 + \frac{3}{40}a_1 X^5 + \dots$$

$$= a_0\left(1 - X^2 + \frac{1}{4}X^4 + \dots\right) + a_1\left(X - \frac{1}{2}X^3 + \frac{3}{40}X^5 + \dots\right)$$

$\Downarrow$ cos( ) $\qquad\qquad$ $\Downarrow$ sin( )

* system of linear D.E's :-

The general form of this system :

$$x' = a_{11}x + a_{12}y + b_1(t)$$

$$y' = a_{21}x + a_{22}y + b_2(t)$$

this system can be written in the matrix Form as follows :-

$$X' = AX + B(t) \quad ---- \quad (*)$$

Where $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}$

عندما تكون حرفا كبير تكون شكل مصفوفة

variable matrix. coefficient matrix.

$$X' = AX + B(t)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times2} \begin{pmatrix} x \\ y \end{pmatrix}_{2\times1}}_{\text{ضرب صحيح}} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}$$

If $B(t) = 0$ then $*)$ is called homog. system of D.E

From $*)$ $X' = AX \quad ---(1)$ and this is so of the system

To solve (1) we let : $X = \underline{u}e^{\lambda t}$ →

where (u) is a column vector and λ is a scalar that have to be determined $\Rightarrow X = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} e^{\lambda t}$

Substitute : $X = \underline{u}e^{\lambda}$ in (1) we get

$\lambda \underline{u} e^{\lambda t} = A \underline{u} e^{\lambda t}$

$A \underline{u} e^{\lambda t} - \lambda u e^{\lambda t} = 0$

$(A \underline{u} - \lambda \underline{u}) e^{\lambda t} = 0 \Rightarrow A \underline{u} - \lambda \underline{u} = 0$

$(A I - \lambda I) \underline{u} = 0 \Rightarrow (A - \lambda I) \underline{u} = 0 \quad \cdots (2)$

This is a homog. system which has a non-trivial solution

if & only if $|A - \lambda I| = 0 \quad --- (3)$

$$\boxed{\text{note : } \lambda I = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}$$

* ملاحظة: اذا أخرجنا عامل مشترك يترك مكانه I (The Identity matrix)

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12} a_{21} = 0$

$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11} a_{22} - a_{12} a_{21} = 0$

The roots of this equation are called eigen values (القيم المميزة) λ_1, λ_2

* for each eigen value we have an eigen vector and we can find these eigen vectors from equation ② we have tree cases:-

case ① If the eigen values of A are distinct, say $\lambda_1 \neq \lambda_2$ then:

$x_1 = A e^{\lambda_1 t}, \quad x_2 = B e^{\lambda_2 t}$

where A & B are two eigen vectors corresponding with λ_1, λ_2 respectively (بالترتيب) ($A = \underline{u}_1 \quad B = \underline{u}_2$)

note : for λ_1 there is $\underline{u}_1 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A$

and λ_2 there is $\underline{u}_2 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = B$

Ex: $x' = Ax$ where $x = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 4 & -1 \\ -4 & 4 \end{pmatrix}$

صيغة السؤال or $x' = 4x - y$

$y' = -4x + 4y$

or $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 4 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

solution: $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 4-\lambda & -1 \\ -4 & 4-\lambda \end{vmatrix} = 0$

$(4-\lambda)(4-\lambda) - 4 = 0 \Rightarrow \lambda^2 - 8\lambda + 16 - 4 = 0$

$\lambda^2 - 8\lambda + 12 = 0 \Rightarrow (\lambda - 6)(\lambda - 2) = 0$

$\lambda = 6, 2 \longrightarrow \lambda_1 = 2, \lambda_2 = 6$ eigen values.

for $\lambda_1 = 2$

$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(A - I\lambda) = \begin{pmatrix} 4-2 & -1 \\ -4 & 4-2 \end{pmatrix}$ $\underline{u}_1$

$2a_1 - a_2 = 0$ ...... (1)

$-4a_1 + 2a_2 = 0$ ...... (2)

2(1) + (2) $\Rightarrow 0 = 0 \Rightarrow 2a_1 - a_2 = 0$ نأخذ معادلة واحدة

$a_1 = \frac{1}{2} a_2$

عليك أخذ أي قيمة لـ (a_2) ما عدا الصفر ثم نجد قيمة a_1

take $a_2 = 2 \rightarrow a_1 = 1$

$A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow$ eigen vector corresponding with $\lambda = 2$

$\therefore X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$ الحل الأول

for $\lambda = 6$

$$\begin{pmatrix} -2 & -1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} -2b_1 - b_2 = 0 \\ -4b_1 - 2b_2 = 0 \end{array}\right\} \text{نفس المعادلة}$$

$-2b_1 - b_2 = 0 \Rightarrow b_1 = \frac{-1}{2} b_2$ لنأخذ ①

take $b_2 = 2 \Rightarrow b_1 = -1$

$\Rightarrow B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$: $x_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{6t}$ الحل الثاني

g.s: $X = c_1 x_1 + c_2 x_2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{6t}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} \\ 2c_1 e^{2t} \end{pmatrix} + \begin{pmatrix} -c_2 e^{6t} \\ 2c_2 e^{6t} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} - c_2 e^{6t} \\ 2c_1 e^{2t} + 2c_2 e^{6t} \end{pmatrix}$$

g.s $\Rightarrow$ $x = c_1 e^{2t} - c_2 e^{6t}$

$y = 2c_1 e^{2t} - 2c_2 e^{6t}$

case $\boxed{2}$ If $\lambda_1 = \lambda_2$ then ~~$x_1 = A\lambda_1$~~ $x_1 = Ae^{\lambda_1 t}$

$x_2 = Ate^{\lambda t} + Be^{\lambda_1 t}$

where B is a vector that can be determined from : $(A - \lambda I)B = A$

Ex: $X' = AX$, $A = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{vmatrix} -\lambda & 1 \\ -4 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4\lambda + 4 = 0$$

$(\lambda - 2)(\lambda - 2) = 0$

$\lambda = 2, 2$

for $\lambda = 2$

$$\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-2a_1 + a_2 = 0$ take $a_2 = 2 \Rightarrow a_1 = 1 \Rightarrow A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

$$X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} te^{2t} + B e^{2t}$$

$$\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$ not homog. system we can take $b_1 = 0$

$(A - \lambda I) B = A$

$-2b_1 + b_2 = 1$ take $b_1 = 0 \Rightarrow b_2 = 1$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X = c_1 X_1 + c_2 X_2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} \right]$$

$$= \begin{pmatrix} c_1 e^{2t} \\ 2c_1 e^{2t} \end{pmatrix} + \begin{pmatrix} c_2 te^{2t} \\ 2c_2 te^{2t} \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} + c_2 te^{2t} \\ 2c_1 e^{2t} + 2c_2 te^{2t} + c_2 e^{2t} \end{pmatrix}$$

g.s.: $x = c_1 e^{2t} + c_2 te^{2t}$

$y = 2c_1 e^{2t} + 2c_2 te^{2t} + c_2 e^{2t}$

Case $\boxed{3}$: If $\lambda = \alpha \pm \beta i$

$$X_1 = e^{\alpha t}(Re\,\underline{v}\cos\beta t - Im\,\underline{v}\sin\beta t)$$

$$X_2 = e^{\alpha t}(Im\,\underline{v}\cos\beta t + Re\,\underline{v}\sin\beta t)$$

for $\lambda = \alpha + \beta i$ the eigen vector will be of the form

$$\underline{V} = \begin{pmatrix} a_1 + b_1 i \\ a_2 + b_2 i \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} i$$

$$= Re\,\underline{v} + Im\,\underline{v}\, i$$

Ex: $X' = AX$ where $A = \begin{pmatrix} 2 & -5 \\ 2 & -4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & -5 \\ 2 & -4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-4-\lambda) + 10 = 0 \Rightarrow \lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-2 \mp \sqrt{-4}}{2} = -1 \mp i$$

for $\lambda = -1 + i$ The eigen vector.

$$\begin{pmatrix} 3-i & -5 \\ 2 & -3-i \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(3-i)a_1 - 5a_2 = 0 \Rightarrow a_1 = \frac{5}{3-i}a_2$$

take $a_2 = 3-i \Rightarrow a_1 = 5$

$$V = \begin{pmatrix} 5 \\ 3-i \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i$$

$$= Re(\underline{v}) + Im(\underline{v})\, i$$

$$\Rightarrow X_1 = e^{\alpha t}(Re\,v\cos\beta t - Im\,\underline{v}\sin\beta t)$$

$$X_1 = e^{-t}\left(\begin{pmatrix} 5 \\ 3 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin t\right)$$

$$X_2 = e^{-t}\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos t - \begin{pmatrix} 5 \\ 3 \end{pmatrix}\sin t\right)$$

$\therefore$ g.s: $X = C_1 X_1 + C_2 X_2$

$$X = e^{-t}\left(C_1\left(\begin{pmatrix} 5 \\ 3 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin t\right) + C_2\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos t + \begin{pmatrix} 5 \\ 3 \end{pmatrix}\sin t\right)\right)$$

* Non - Homogeneous system of D.E's :-

$$X' = AX + B(t) \quad --- \quad (*)$$

The g.s of (*) contains two points,

X_c : completely solution which is the g.s of

$X' = AX$ & contains the arbitrary constant.

X_p : It is a special solution that satisfies (*)

$$X'_p = AX_p + B(t)$$

$$\boxed{X_p = X_1 u_1 + X_2 u_2}$$

$$X' = AX + B(t) \quad , \quad A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \qquad B(t) = \begin{pmatrix} e^t \\ 1 \end{pmatrix}$$

ادهم عبد محمد العطيات
افرضها هكذا

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

The solution for Homog. point

$$\begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0 \qquad \lambda = 2, 1$$

For $\lambda = 1$

$$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -a_1 + a_2 = 0 \Rightarrow a_1 = a_2$$

take $a_2 = 1 \Rightarrow a_1 = 1$

$$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow X_1 = A e^{\lambda_1 t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$$

For $\lambda = 2$

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -2b_1 + b_2 = 0$$

take $b_2 = 2 \longrightarrow b_1 = 1$

$B = \begin{pmatrix}1\\2\end{pmatrix}$ $\quad X_2 = B\, e^{\lambda_2 t} = \begin{pmatrix}1\\2\end{pmatrix} e^{2t}$

$\Rightarrow x_C = c_1 X_1 + c_2 X_2$

for Homog. and non homog. part (all equations)

$X_p = X_1 u_1 + X_2 u_2$

نعوض X_p في $D.E$ فينتج في النهاية بعد الاختصار

$\boxed{X_1 u_1' + X_2 u_2' = B(t)}$ $\rightarrow$ نستخدمها مباشرة

$\begin{pmatrix}1\\1\end{pmatrix} e^t u_1' + \begin{pmatrix}1\\2\end{pmatrix} e^{2t} u_2' = \begin{pmatrix}e^t\\1\end{pmatrix}$

$e^t u_1' + e^{2t} u_2' = e^t$

$e^t u_1' + 2e^{2t} u_2' = 1$

$u_1' + e^t u_2' = 1$

$u_1' + 2e^t u_2' = e^{-t}$

note:
$2\begin{pmatrix}1\\2\end{pmatrix} + 3\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}A\\B\end{pmatrix}$
then
$A = 2*1 + 3*3$
$B = 2*2 + 3*4$

$u_1' = \dfrac{\begin{vmatrix}1 & e^t\\ e^{-t} & 2e^t\end{vmatrix}}{\begin{vmatrix}1 & e^t\\ 1 & e^{2t}\end{vmatrix}} = \dfrac{2e^t - 1}{2e^t - e^t} = \dfrac{2e^t - 1}{e^t}$

$u_1' = \dfrac{2e^t - 1}{e^t} = 2 - e^{-t}$ $\Rightarrow$ $\boxed{u_1 = 2t + e^{-t}}$

$u_2' = \dfrac{\begin{vmatrix}1 & 1\\ 1 & e^{-t}\end{vmatrix}}{e^t} = \dfrac{e^{-t} - 1}{e^t} = e^{-2t} - e^{-t}$

$\Rightarrow \boxed{u_2 = \frac{1}{2} e^{-2t} + e^{-t}}$

$X_p = X_1 u_1 + X_2 u_1$

$= \begin{pmatrix}1\\1\end{pmatrix} e^t (2t + e^{-t}) + \begin{pmatrix}1\\2\end{pmatrix} e^{2t} \left(-\frac{1}{2} e^{-2t} + e^{-t}\right)$

$x_p = \begin{pmatrix}2te^t + 1 - \frac{1}{2} + e^t\\ 2te^t + 1 - 1 + 2e^t\end{pmatrix} = \begin{pmatrix}2te^t + e^t + \frac{1}{2}\\ 2te^t + 2e^t\end{pmatrix}$

$$x_p = \begin{pmatrix} x_p \\ y_p \end{pmatrix} \Rightarrow \begin{array}{l} x_p = 2te^{t} + e^{t} + \frac{1}{2} \\ y_p = 2te^{t} + 2e^{t} \end{array}$$

$$g.s \Rightarrow \quad x = x_c + x_p \quad \ldots$$ أكمل

__Note:__ هذه الطريقة لإيجاد x, y تسمى creamer's Rule

حيث أنه اذا كانت

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

c_1, c_2 const.

then $\quad x = \frac{D_x}{D} \quad , \quad y = \frac{D_y}{D}$

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

__Exer:__ $x' = Ax + B(t) \qquad A = \begin{pmatrix} 2 & 1 \\ -4 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 3e^{2t} \\ te^{2t} \end{pmatrix}$

* Laplace Transforms:

تحويلات لابلاس

(يعامل معاملة التكامل)

Def: If $F(t)$ is defined for all $t \geq 0$ then the Laplace transform of $F(t)$ is denoted by $\mathcal{L}\{f(t)\} = F$ and it is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

__EX:__ find $\mathcal{L}\{2\} = \int_0^\infty e^{-st} * 2\, dt$

$$= 2 * \frac{e^{-st}}{-s} \Big|_0^\infty = \frac{-2}{s} e^{-st} \Big|_0^\infty$$

$$= \frac{-2}{s}\left(e^{-s\infty} - e^{0}\right) = \cancel{\frac{-2}{s} e^{-s\infty}} + \frac{2}{s} e^{0}$$

$$= \frac{2}{s} \qquad s > 0$$

$$* \boxed{\mathcal{L}\{a\} = \frac{a}{s} \qquad s>0}$$

note: يعامل معاملة التكامل

① $\mathcal{L}\{f(t) \mp g(t)\} = \mathcal{L}\{f(t)\} \mp \mathcal{L}\{g(t)\}$

② $\mathcal{L}\{f(t) * g(t)\} \neq \mathcal{L}\{f(t)\} * \mathcal{L}\{g(t)\}$

③ $\mathcal{L}\{c\, f(t)\} = c\, \mathcal{L}\{f(t)\}$

ex: $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\, dt$

$$= \int_0^\infty e^{(a-s)t}\, dt = \frac{1}{a-s} e^{(a-s)t}$$

$$= \frac{1}{a-s}\left(e^{(a-s)\infty} - e^{(a-s)*0}\right)$$

$$= \frac{1}{a-s} \cancel{e^{(a-s)\infty}} - \frac{1}{a-s} = \frac{1}{s-a}, \quad s>a$$

0 if $s>a$

$$\therefore \boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad s>a}$$

$\mathcal{L}\{e^{3t}\} = \frac{1}{s-3}$ $\qquad$ $\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}$

$\boxed{\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}}$ $\qquad$ $\boxed{\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}}$

Ex: ① $\mathcal{L}\{\cos^2 2t\} = \frac{1}{2}\mathcal{L}\{(1+\cos 4t)\}$

$$= \frac{1}{2}\left(\mathcal{L}\{1\} + \mathcal{L}\{\cos 4t\}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$$

$$= \frac{1}{2}\left(\frac{s^2+16+s^2}{s(s^2+16)}\right) = \frac{1}{2}\left(\frac{2s^2+16}{s(s^2+16)}\right) = \frac{s^2+8}{s(s^2+16)}$$

$\boxed{\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}}$ $\qquad$ أوجدها من خلال $\sinh at = \frac{e^{at} - e^{at}}{2}$

$\boxed{\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}}$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}}$$

ex: $\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$

ex: $\mathcal{L}\{t^5\} = \frac{5!}{s^6} =$

ex: $\mathcal{L}\{t^3 - e^t + 2\}$

$$= \frac{3!}{s^4} - \frac{1}{s-1} + \frac{2}{s}$$

$$= \frac{6(s-1) - s^4 + 2s^3(s-1)}{s^4(s-1)}$$

$$= \frac{6s - 6 - s^4 + 2s^4 - 2s^3}{s^4(s-1)}$$

$$= \frac{s^4 - 2s^3 + 6s - 6}{s^4(s-1)}$$

* $\boxed{\mathcal{L}\{e^{at} f(t)\} = F(s-a)}$ where $F(s) = \mathcal{L}\{f(t)\}$

Ex: $\mathcal{L}\{e^{3t}\sin 2t\} = F(s-3)$

$F(s) = \mathcal{L}(\sin 2t) = \frac{2}{s^2+4}$

$F(s-3) = \frac{2}{(s-3)^2+4}$

Ex: $\mathcal{L}\{e^{-2t}\cos 4t\} = F(s+2)$

$F(s) = \mathcal{L}\{\cos 4t\} = \frac{s}{s^2+16}$

$F(s+2) = \frac{s+2}{(s+2)^2+16}$

Ex: $\mathcal{L}\{t^4 e^{3t}\} = F(s-3)$

$F(s) = \mathcal{L}\{t^4\} = \frac{4!}{s^5}$

$F(s-3) = \frac{4!}{(s-3)^5}$

* $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$

$F(s) = \mathcal{L}\{f(t)\}$

Ex: $\mathcal{L}\{t\sin 2t\} = (-1)^1 \frac{dF(s)}{ds}$

where $F(s) = \mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$

$\mathcal{L}\{t\sin 2t\} = -1 * \frac{-2(2s)}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}$

Ex: $\mathcal{L}\{t\cosh 3t\} = (-1) \frac{dF(s)}{ds}$

$F(s) = \mathcal{L}\{\cosh 3t\} = \frac{s}{s^2-9}$

$\therefore \mathcal{L}\{t\cosh 3t\} = -1 * \frac{(s^2-9) * 1 - s(2s)}{(s^2-9)^2}$

$= -1 * \left(\frac{s^2-9-2s^2}{(s^2-9)^2}\right) = \frac{s^2+9}{(s^2-9)^2}$

Ex: $F(t) = \begin{cases} 2 & , 0 \le t < 5 \\ 0 & 5 \le t < 10 \\ e^{2t} & t \ge 10 \end{cases}$

Find $\mathcal{L}\{F(t)\} = \int_0^5 e^{-st} * 2dt + \int_5^{10} 0\,dt + \int_{10}^{\infty} e^{-st} \cdot e^{2t} dt$

$= \frac{-2}{s} e^{-st}\Big|_0^5 + \frac{1}{2-s} e^{(2-s)t}\Big|_{10}^{\infty}$

$= \frac{-2}{s} e^{-5s} + \frac{2}{s} e^{0} + \frac{1}{2-s} e^{(2-s)\infty} - \frac{1}{2-s} e^{(2-s)10}$

$= \frac{-2}{s} e^{-5s} + \frac{2}{s} + \frac{1}{s-2} e^{20-10s}$

$$* \ \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s} \quad , \quad F(s) = \mathcal{L}\{f(t)\}$$

ex: $\mathcal{L}\left\{\int_0^t \sin 3\tau \, d\tau\right\} = \frac{F(s)}{s}$

$$F(s) = \mathcal{L}\{\sin 3t\} \quad t \xleftarrow{\text{بدل}} \tau \quad = \frac{3}{s^2+9}$$

$$\mathcal{L}\left\{\int_0^t \sin 3\tau \, d\tau\right\} = \frac{F(s)}{s} = \frac{3}{s(s^2+9)}$$

another solution:

$$\mathcal{L}\left\{\int_0^t \sin 3\tau \, d\tau\right\} = \mathcal{L}\left\{ -\tfrac{1}{3}\cos 3\tau \Big|_0^t \right\}$$

$$= \mathcal{L}\left\{-\tfrac{1}{3}\cos 3t + \frac{1}{3}\right\}$$

$$= \frac{-1}{3}\left(\mathcal{L}\{\cos 3t\} + \mathcal{L}\{-1\}\right)$$

$$= \frac{-1}{3}\left(\frac{s}{s^2+9} - \frac{1}{s}\right)$$

$$= \frac{-1}{3}\left(\frac{s^2 - s^2 - 9}{s(s^2+9)}\right)$$

$$= \frac{-1}{3}\left(\frac{-9}{s(s^2+9)}\right) = \frac{3}{s(s^2+9)}$$

Ex: $\mathcal{L}\left\{\int_0^t e^{3\tau} \cos 2\tau \, d\tau\right\} = \frac{F(s)}{s}$

$$F(s) = \mathcal{L}\{e^{3t}\cos 2t\} = \frac{s-3}{(s-3)^2+4}$$

$$\therefore \mathcal{L}\left\{\int_0^t e^{3\tau}\cos 2\tau \, d\tau\right\} = \frac{s-3}{s(s-3)^2+4s}$$

Exer: ① $\mathcal{L}\left(\int_0^t \tau \sinh 2\tau \, d\tau\right)$

② $\mathcal{L}\left(t^2 \sin 3t\right)$

③ $\mathcal{L}\left(\int_0^t \sin^2 3\tau \, d\tau\right)$

④ $\mathcal{L}\left\{\int_0^t \tau e^{2\tau} \sin 3\tau \, d\tau\right\}$

$\mathcal{L}\{y(t)\} = Y(s)$

* $\mathcal{L}\{y'(t)\} = s\,\mathcal{L}\{y(t)\} - y(0)$

* $\mathcal{L}\{y''(t)\} = s^2\,\mathcal{L}\{y(t)\} - s\,y(0) - y'(0)$

* $\mathcal{L}\{y'''(t)\} = s^3\,\mathcal{L}\{y(t)\} - s^2 y(0) - s\,y'(0) - y''(0)$

* $\boxed{\mathcal{L}\{y^{(n)}(t)\} = s^n\,\mathcal{L}\{y(t)\} - s^{n-1} y(0) - s^{n-2} y'(0) - s^{n-3} y''(0) - \cdots\ y^{(n-1)}(0)}$

ex: $y'' + 2y' + 3y = 0$

solu: $\mathcal{L}(y'') + 2\mathcal{L}(y') + 3\mathcal{L}(y) = 0$

$$s^2\mathcal{L}(y(t)) - s\,\cancel{y(0)}^{\,0} - y'(0) + 2s\,\mathcal{L}(y(t)) - 2\,\cancel{y(0)}^{\,0} + 3\mathcal{L}(y(t)) = 0$$

$$s^2 Y(s) - 1 + 2s\,Y(s) + 3\,Y(s) = 0$$

$$Y(s)\,(s^2 + 2s + 3) = 1$$

$$Y(s) = \frac{1}{s^2 + 2s + 3}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 2s + 3}\right\} = \mathcal{L}\left\{\frac{1}{(s+2)^2 - 1}\right\}$$

إكمال المربع
بإضافة وطرح 2
$\left(\frac{1}{2} \times \text{معامل } s\right)^2$

$$\rightarrow s^2 + 2s + 3 + 1 - 1 \qquad = \mathcal{L}\left\{\frac{1}{(s+2)^2 - 1}\right\}$$

$$(s+2)^2 - 1$$

$$= e^{-2} \sinh t)$$

* Laplace Inverse Transform:

$\mathcal{L}\{F(t)\} = F(s) \Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\}$

Ex: ① $\mathcal{L}^{-1}\left\{\frac{2}{s}\right\} = 2$

② $\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = t^3$

③ $\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}$

④ $\mathcal{L}^{-1}\left\{\frac{5}{s^3}\right\} = \frac{5}{2} t^2$

⑤ $\mathcal{L}^{-1}\left\{\frac{3}{s+2}\right\} = 3e^{-2t}$

⑥ $\mathcal{L}^{-1}\left\{\frac{s+1}{s^5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^4} + \frac{1}{s^5}\right\} = \frac{1}{6}t^3 + \frac{1}{24} t^4$

⑦ $\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin 2t$

⑧ $\mathcal{L}^{-1}\left\{\frac{s}{s^2+5}\right\} = \cos\sqrt{5}\, t$

⑨ $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-2}\right\} = \cosh\sqrt{2}t + \frac{1}{\sqrt{2}}\sinh\sqrt{2}t$

⑩ $\mathcal{L}^{-1}\left\{\frac{5}{s^2+6}\right\} = \frac{5}{\sqrt{6}} \sin\sqrt{6}\, t$

⑪ $\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} = e^{-t}\cos 2t$

⑫ $\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+9}\right\} = e^{2t}\cos 3t$

⑬ $\mathcal{L}^{-1}\left\{\frac{4}{(s+3)^2+5}\right\} = \frac{4}{\sqrt{5}} e^{-3t}\sin\sqrt{5}\, t$

(14) $\mathcal{L}^{-1}\left\{\frac{s+1}{(s+2)^2+16}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2-1}{(s+2)^2+16}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+16}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+16}\right\} = e^{-2t}\cos 4t - \frac{1}{4}e^{-2t}\sin 4t$

(15) $\mathcal{L}^{-1}\left\{\frac{s-2}{(s-3)^2+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+5}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+5}\right\}$

$= e^{3t}\cos\sqrt{5}t + \frac{1}{\sqrt{5}}e^{3t}\sin\sqrt{5}t$

(16) $\mathcal{L}^{-1}\left\{\frac{6}{(s-1)^4}\right\} = e^{t}t^{3}$

(17) $\mathcal{L}^{-1}\left\{\frac{5}{(s+3)^5}\right\} = \frac{5}{24}e^{-3t}t^{4}$

(18) $\mathcal{L}^{-1}\left\{\frac{s+2}{(s+3)^5}\right\} = \mathcal{L}^{-1}\left\{\frac{s+3-1}{(s+3)^5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^5}\right\}$

$= \left(\frac{e^{-3t}}{6}t^{3} - \frac{1}{24}e^{-3t}t^{4}\right) \rightarrow F(t)$

(19) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+10}\right\} \Rightarrow \quad s^2+6s+10 = s^2+6s+9-9+10$

$(s+3)^2+1 = 0 \quad \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+1}\right\} = e^{-3t}\sin t$

(20) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+6s+25}\right\} \Rightarrow s^2+6s+25+9-9 \Rightarrow (s+3)^2+16$

$\mathcal{L}^{-1}\left\{\frac{s+1}{(s+3)^2+16}\right\}$ - - - -

(21) $\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s+1)}\right\} =$

$$\frac{1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} \Rightarrow 1 = A(s+1) + B(s-3)$$

$$s=-1 \Rightarrow \boxed{B = \frac{-1}{4}}$$

$$s=3 \Rightarrow \boxed{A = \frac{1}{4}}$$

$$\mathcal{L}^{-1}\left\{\frac{\frac{1}{4}}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{-\frac{1}{4}}{s+1}\right\}$$

$$\boxed{\frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}}$$

Laplace تمارين مختلفة كـ

① $\mathcal{L}^{-1}\left\{\frac{4s+5}{s^2+2}\right\} = 4\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{5}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\}$

$$= 4\cos(\sqrt{2}t) + \frac{5}{\sqrt{2}}\sin(\sqrt{2}t)$$

② $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2-4}\right\}$

$$= \cosh 2t + \frac{1}{2}\sinh 2t$$

③ $\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2-s-2}\right\}$ $\quad \frac{3s+1}{s^2-s-2} = \frac{A}{(s+1)} + \frac{B}{(s-2)}$

أكمل

$$= \frac{2}{3}e^{-t} + \frac{7}{3}e^{2t}$$

④ $\mathcal{L}\{\cosh 3t \cos^2 t\} = \mathcal{L}\left\{\left(\frac{e^{3t}+e^{-3t}}{2}\right)\left(\frac{1}{2}(1+\cos 2t)\right)\right\}$

$$= \frac{1}{4}\mathcal{L}\{(e^{3t}+e^{-3t})(1+\cos 2t)\}$$

$$= \frac{1}{4}\mathcal{L}\{e^{3t}+e^{-3t}+e^{3t}\cos 2t+e^{-3t}\cos 2t\}$$

أكمل

(5) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^4+13s^2+36}\right\}$, $\frac{s+1}{s^4+13s^2+36} = \frac{s+1}{(s^2+4)(s^2+9)}$

$$= \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$\Rightarrow s+1 = (As+B)(s^2+9) + (Cs+D)(s^2+4) \quad ---- \ *$$

① $s = 2i \quad \Rightarrow \quad 2i+1 = (-4+9)(2iA+B) + \cancel{(Cs+D)(-4+4)}$

$s^2 = 4i^2$
$= 4 * -1$
$= -4$

$$2i+1 = 10iA + 5B$$

طريقة تساوي المعاملات

$2i = 10iA$ and $1 = 5B$

$2 = 10A$, $\boxed{A = \frac{1}{5}}$, $\boxed{B = \frac{1}{5}}$

② $s = 3i \rightarrow 3i+1 = 0 - 15Ci - 5D$

$-15C = 3 \Rightarrow C = \frac{-1}{5}$

$-5D = 1 \Rightarrow D = \frac{-1}{5}$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^4+13s^2+36}\right\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{5} * \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$-\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{1}{5} * \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= \frac{1}{5}\cos 2t + \frac{1}{10}\sin 2t - \frac{1}{5}\cos 3t - \frac{1}{15}\sin 3t$$

remember: طريقة الكسور الجزئية

$$\frac{1}{(s+3)^3(s^2+4)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{(s+3)^3} + \frac{Dx+E}{(s^2+4)} + \frac{Fx+G}{(s^2+4)^2}$$

(6) $\mathcal{L}\left\{\frac{\cos t}{e^t}\right\} = \mathcal{L}\left\{e^{-t}\cos t\right\} = F(s+1)$

$F(s) = \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$

$F(s+1) = \frac{s+1}{(s+1)^2+1}$

(7) $\mathcal{L}^{-1}\left\{\frac{s-5}{(s-5)^2+9}\right\} = e^{5t}\cos 3t$

(8) $\mathcal{L}^{-1}\left\{\frac{4s+1}{(s+2)^2+25}\right\} = \mathcal{L}^{-1}\left\{\frac{4(s+2-2)+1}{(s+2)^2+25}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{4(s+2)-4*2+1}{(s+2)^2+25}\right\} = \mathcal{L}^{-1}\left\{\frac{4(s+2)-7}{(s+2)^2+25}\right\}$

$= 4\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+25}\right\} - \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^2+25}\right\}$

$= 4e^{-2t}\cos 5t - \frac{7}{5}e^{-2t}\sin 5t$

(9) $\mathcal{L}^{-1}\left\{\frac{10}{(s-3)^5}\right\} = \frac{10}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{(s-3)^5}\right\} = \frac{10}{24}e^{3t}t^4$

(10) $\mathcal{L}^{-1}\left\{\frac{3s+2}{s^2+6s+25}\right\}$ - حلها بالكامل البيت

$\Rightarrow = 3e^{-3t}\cos 4t - \frac{7}{4}e^{-3t}\sin 4t$

Exer: (11) $\mathcal{L}\left\{\frac{s}{s^2+4s+5}\right\}$ (12) $\mathcal{L}^{-1}\left\{\frac{4s+3}{(s^2+4s+4)^2(s^2+9)}\right\}$

(13) $\mathcal{L}^{-1}\left\{\frac{1}{s^3+s^2+s}\right\}$

By using L.T solve D.E:

1) $y' - y = t \qquad y(0) = 1$

Solution: $\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{t\} \Rightarrow sY(s) - y(0) - Y(s) = \frac{1}{s^2}$

$$sY(s) - Y(s) - y(0) = \frac{1}{s^2}$$

$$(s-1)Y(s) = \frac{1}{s^2} + 1$$

$$Y(s) = \frac{1}{s^2(s-1)} + \frac{1}{s-1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\} + e^t$$

$$= e^t - t - 1 + e^t$$

$$y(t) = 2e^t - t - 1$$

$\mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s-1}\right)\right\} = \int_0^t e^{\tau} d\tau = e^{\tau}\Big|_0^t = e^t - 1$

$\mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s}\times\frac{1}{s-1}\right)\right\} = \int_0^t (e^{\tau} - 1) d\tau$

$= e^{\tau} - \tau \Big|_0^t = e^t - t - (1-0)$

$= e^t - t - 1$

Exer: solve by L.T

a) $y'' + 2y' + 2y = 0 \qquad , y(0) = 0 \qquad y'(0) = 1$

b) $y'' + 2y' + y = 4e^t \qquad , y(0) = 2 \qquad , y'(0) = -1$

c) $y'' - y' - 6y = 0 \qquad , y(0) = 1 \qquad y'(0) = -1$

d) $y^{(4)} - 4y = 0 \qquad y(0) = 1 \qquad y'(0) = 0 \qquad y''(0) = -2$
$y'''(0) = 0$

e) $y'' + y = t \qquad y(0) = 1 \qquad y'(0) = 4$

f) $y'' + y = 0 \qquad , y\left(\frac{\pi}{2}\right) = 0 \qquad y\left(\frac{\pi}{3}\right) = 4$

solution:

d) $y^{(iv)} - 4y = 0$

$$\mathcal{L}\{y^{iv}\} - 4\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$s^4 Y(s) - s^3 y(0) - \cancel{s^2 y'(0)} - s y''(0) - \cancel{y'''(0)} - 4Y(s) = 0$$

$$Y(s)(s^2-4) = s^3 - 2s$$

$$Y(s) = \frac{s^3-2s}{s^2-4} = \frac{s\,\cancel{(s^2-2)}}{\cancel{(s^2-2)}(s^2+2)} = \frac{s}{s^2+2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} = \cos\sqrt{2}\,t$$

---

e) $\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{t\}$

$$s^2 Y(s) - \cancel{s\,y(0)} - y'(0) + Y(s) = \frac{1}{s^2}$$

$$(s^2+1)Y(s) = \frac{1}{s^2} + 2 \Rightarrow Y(s) = \frac{1}{s^2(s^2+1)} + \frac{2}{(s^2+1)}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s^2+1}\right\}$$ أكمل.

* في جميع الأمثلة السابقة كانت معطيات السؤال كما يلي
$y(0)$, $y'(0)$, $y''(0)$, $y'''(0)$:
وجميعها لنفس قيمة x وهي $(x=0)$ وهذا النوع يسمى IVP (Initial value problem) أما إذا كانت المعطيات لقيم مختلفة لـ x مثل معطيات السؤال القادم (f) $y'(\frac{\pi}{3})$, $y(\frac{\pi}{2})$ ، يسمى هذا النوع من الأسئلة BVP (Boundary value problem) ويكون الحل كما يلي ...

f) $y'' + y = 0$ $\quad y(\frac{\pi}{2}) = 0$ $\quad y(\frac{\pi}{3}) = 4$

$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{0\}$

$s^2 Y(s) - s y(0) - y'(0) + Y(s) = 0$

let $A = y(0)$

$B = y'(0)$

$s^2 Y(s) - As - B + Y(s) = 0$

$(s^2+1) Y(s) = As + B$

$$Y(s) = \frac{As+B}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{As+B}{s^2+1}\right\}$$

$y(t) = A\cos t + B\sin t$

$y(\frac{\pi}{2}) = 0 \Rightarrow 0 + B = 0 \Rightarrow \boxed{B = 0}$

$y(\frac{\pi}{3}) = 4 \Rightarrow A(\frac{1}{2}) + 0 = 4 \Rightarrow \boxed{A = 8}$

$\therefore y(t) = 8\cos t$