دفترة..

معادلات تفاضلية

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Differential Equation :-

* $y = F(x)$ — y: dep, x: indepent

dep ← $\dfrac{dy}{dx} = F'(x)$
ind ← dx

* $Z = F(x, y)$ — Z: dep, x, y: ind $\Longrightarrow \dfrac{\partial Z}{\partial x}$ or $\dfrac{\partial Z}{\partial y}$

تصنف
* The DEs can be classified as follows :-

[1] Type :- →

a) ordinary DE (ODE) :- with one indepent variables.

b) partial DE (PDE) :- with several variables.

[2] Order :- It's the highest derivative occurs in the equation.

الأس الذي يرفع إليه أعلى تفاضل
[3] Degree :- It's the experent of the highest derivative appears in the eqn :

Ex :- [1] $y''' + (y'')^4 = x^5 \Longrightarrow$ 3rd order ODE , 1st degree

[2] $y'' + 3y^2 y' = x^2 \sin x \longrightarrow$ 2nd order ODE , 1st degree ,

[4] Linear and non linear :-

[1] يجب أن تكون المشتقات في المعادلة مرفوعة ~~لنفس الأس~~ .

[2] يجب أنه يكون المتغير ~~المستقل~~ التابع اله وحد مرفوع لقوة ١ له

[3] يجب أن لا يكون هناك حاصل ضرب التابع والمشتقة

13, $y\,y''' + 2y = 5x^2 + 2$

3^{rd} order ODE, 1^{st} Degree, nonlinear

14, $y'''' + y'' + (\tan x)\,y = x^5$

4^{th} order ODE, 1^{st} degree, linear.

* First order ODE :-

- These equations contain y' and any function of x and may contain y.
That is :-
$F(y', x, y) = 0$ ... *
or it can be written in the form:
$y' = f(x, y)$ .... [1]

on an open interval $a < x < b$
* The solution of [1] is a relation between the depend and indep variables x and y

i.e: $y = h(x)$ such as it satisfies the given DE.
i.e: $h'(x) = f(x, h(x))$

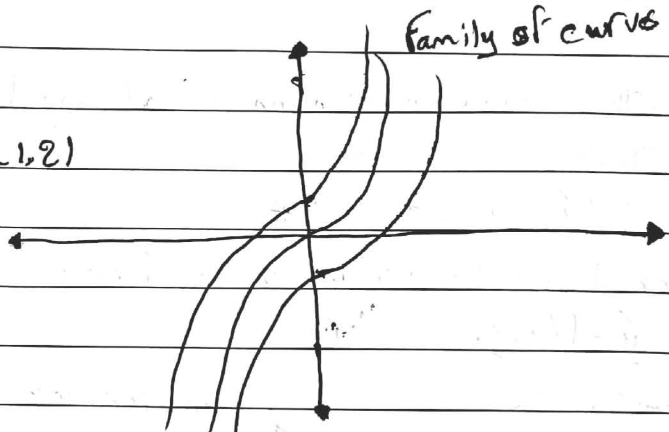
* Any DE could have infinitly many solutions in the form:
$\boxed{y = h(x) + c}$, c is an arbitrary (اعتباري) constant. (because for each value of c we have a solution), this solution is called the general solution. (g.s)

But if the given DE associated (مرتبطة) with an initial condition (particular condition).

in the form:- $y' = f(x, y)$, $\boxed{y(x_0) = y_0}$
which is called initial value problem. (IVP). and in this case the solution we get is called the particular solution (p.s).

Ex:- sketch :-

$y = x^3 + C$



Find the curves that passes throght (1,2)

$y(1) = 2$

$C = \underline{\underline{1}}$

$\therefore y = x^3 + 1$

(3) Seperable ODE:-

If the R.H.S of $y' = f(x,y)$ can be witten as aproduct of tow contiunous function one with respect to x and the other with respect t.y .

i.e :- $f(x,y) = g(x) \cdot h(y)$ then

$y' = g(x) h(y)$

$$\frac{dy}{dx} = g(x) h(y) \Longrightarrow dy = g(x) h(y) dx \Longrightarrow \frac{1}{h(y)} dy = g(x) dx$$

$$\Downarrow$$

$$\int \frac{1}{h(y)} dy = \int g(x) dx$$

Ex:- Solve The IVP:-

$y' = \frac{y}{x}, \; y(1) = 2$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow dy = \frac{y}{x} dx \Rightarrow \frac{1}{y} dy = \frac{1}{x} dx$ sep.

$\ln y = \ln x + c \Rightarrow \ln y - \ln x = c \Rightarrow \ln(\frac{y}{x}) = c \Rightarrow \frac{y}{x} = e^{c} = c \Rightarrow y = cx$

Lines

$2 = c \times 1 \qquad y = 2x$

Ex:- Solve: $y' = \frac{1+y^2}{\sqrt{1-x^2}}$, $y(1) = 1$

$\frac{dy}{dx} = \frac{1+y^2}{\sqrt{1-x^2}} \Rightarrow dy = \frac{1+y^2}{\sqrt{1-x^2}} dx \Rightarrow \frac{1}{1+y^2} dy = \frac{1}{\sqrt{1-x^2}} dx$ sep.

$\tan^{-1} y = \sin^{-1} x + c \Rightarrow y = \tan(\sin^{-1} x + c)$

$\tan^{-1}(1) = \sin^{-1}(1) + c \Rightarrow \frac{\pi}{4} = \frac{\pi}{2} + c \Rightarrow c = -\frac{\pi}{4} \qquad y = \tan(\sin^{-1} x - \frac{\pi}{4})$

Ex:- Solve $y' = \frac{x^3 y + x^3 + y + 1}{x^2 y + x^2}$

$\frac{dy}{dx} = \frac{x^3(y+1) + y + 1}{x^2(y+1)} \Rightarrow \frac{\cancel{x^2(y+1)+1}\ (y+1)(x^3+1)}{x^2(y+1)} = \frac{x^3+1}{x^2}$

$dy = \frac{x^3+1}{x^2} dx \Rightarrow dy = (x + x^{-2}) dx$

$y = \frac{1}{2} x^2 - \frac{1}{x} + c$

Ex:- (1) $y' = \frac{x}{y}$

$\frac{dy}{dx} = \frac{x}{y} \Rightarrow dy = \frac{x}{y}\,dx \Rightarrow dy\,y = x\,dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c$

$y^2 = x^2 + 2c$

(2) $y' = 1 + y^2$

$\frac{dy}{dx} = 1 + y^2 \Rightarrow dy = 1 + y^2\,dx \Rightarrow \frac{1}{1+y^2}\,dy = dx$

$\Rightarrow \tan^{-1} y = x + c \Rightarrow y = \tan(x + c)$

(3) $y' = \sec^2 y$

$\frac{dy}{dx} = \sec^2 y \Rightarrow dy = \sec^2 y\,dx \Rightarrow \frac{1}{\sec^2 y}\,dy = dx$

$\Rightarrow \frac{1}{2}\left(1 + \cos(2y)\right) dy = dx \Rightarrow \frac{1}{2}y - \frac{1}{4}\sin 2y = x + c \Rightarrow \cancel{\frac{1}{2}y = x + \frac{1}{4}\sin(2x) + c}$

$\Rightarrow \cancel{y = 2x + \frac{1}{2}\sin(2x) + c} \Rightarrow$

(4) $y' = (x+1)e^{-x}y^2$

$\frac{dy}{dx} = (x+1)e^{-x}y^2 \Rightarrow \frac{1}{y^2}\,dy = (x+1)e^{-x}\,dx \Rightarrow +\frac{1}{y} = +(x+1)e^{-x} - e^{-x} + c$

$y = \frac{1}{(x+1)}e^{x} - e^{x} + c$

$x+1$	e^{-x}
1	$-e^{-x}$
0	e^{-x}

[5] $(xy+x)dx+(x^2y+x^2)dy=0$, $y\neq 1$ $x\neq 0$

$x(y+1)dx + x^2(y+1)dy=0 \Rightarrow x\cancel{(y+1)}dx = -x^2\cancel{(y+1)}dy = 0$

$\Rightarrow -\frac{1}{x}dx = dy \Rightarrow -\ln x = y+c \Rightarrow \ln x + y = c$

[6] $y'\sin 2\pi x = \pi y \cos 2\pi x$!!!

$\frac{dy}{dx}\sin 2\pi x = \pi y \cos 2\pi x$

$y\,dy = \frac{\cos 2\pi x}{\sin 2\pi x}\,dx$

$\frac{y^2}{2} = \cancel{\cot}\, 2\pi x \;\; \cot^{-} 2\pi x$

[7] $(1+x^2)y' = (1-x)y$, $y(0)=1$

$(1+x^2)\frac{dy}{dx} = (1-x)y \Rightarrow \frac{1}{y}dy = \frac{(1-x)}{(1+x^2)}dx \Rightarrow \ln y = \int\frac{1}{1+x^2}dx - \int\frac{x}{1+x^2}dx$

$\Rightarrow \ln y = \tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}dx \Rightarrow \ln y = \tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c$

[8] $e^{x+y}\,y' = e^{2x}$

$\frac{dy}{dx} \Rightarrow \cancel{e^x}\cdot e^y \cdot y' = \cancel{e^x}\cdot e^x \Rightarrow e^y \cdot y' = e^x \Rightarrow e^y\,dy = e^x\,dx$

$\Rightarrow e^y = e^x + c \Rightarrow \ln e^y = \ln(e^x + c) \Rightarrow$

$\Rightarrow y = \ln(e^x + c)$

$\underline{9)}$ $y(y')^3 = x-1$

$(y')^3 = \frac{x-1}{y} \Rightarrow y' = \sqrt[3]{\frac{x-1}{y}} \Rightarrow y' = \frac{\sqrt[3]{x-1}}{\sqrt[3]{y}} \Rightarrow \sqrt[3]{y}\, dy = \sqrt[3]{x-1}\, dx$

$\Rightarrow \frac{3}{4} y^{4/3} = \frac{3}{4}(x-1)^{4/3} + C$

2, Reducation to sperabale form :- Homogeneous Equation. متجانس

$\Rightarrow$ If the R.H.S of $y' = f(x,y)$ can be written as a function of the ratio $\frac{y}{x}$, i.e $\Rightarrow f(x,y) = g(\frac{y}{x})$

$\therefore y' = g(\frac{y}{x})$, the to solve this type the equation we let $u = \frac{y}{x} \Rightarrow y = ux$

$y' = u + u'x \Rightarrow u + xu' = g(u) \Rightarrow xu' = g(u) - u \Rightarrow x\frac{du}{dx} = g(u) - u$

$\Rightarrow \frac{1}{x}\frac{dx}{du} = \frac{1}{g(u)-u} \Rightarrow \frac{1}{x}dx = \frac{1}{g(u)-u}du$ sep.

Def :- Any function $f(x,y)$ is called homog. of degree n i.e :

$f(tx,ty) = t^n f(x,y)$

Ex :- $f(x,y) = x^2 + y^2$

$\Rightarrow f(tx,ty) = t^2x^2 + t^2y^2 \Rightarrow t^2(x^2+y^2) \Rightarrow t^2 f(x,y)$ homog of degree to.

Ex :- $f(x,y) = x^3 + xy + 5$ non homog.

Ex :- $f(x,y) = xe^x + y$

$f(tx,ty) = txe^{tx} + ty \Rightarrow t(e^{tx} + y)$ non homog.

Ex:- $f(x,y) = x^3 e^{\frac{y}{x}} + y^3$ homog of degree 3

$f(tx,ty) = t^3 x^3 e^{\frac{ty}{tx}} + t^3 y^3 \Rightarrow t^3 (x^3 e^{\frac{y}{x}} + y^3)$

Ex:- $f(x,y) = x \sin x + y$ non homog.

Ex: $f(x,y) = \tan(\frac{y}{x}) + 5$

$f(tx,ty) = \tan(\frac{ty}{tx}) + 5 \Rightarrow \tan(\frac{y}{x}) + 5$ homog of degree 0

$\Rightarrow t^0 (f(x,y))$

Ex:- $f(x,y) = x \sin(\frac{y}{x}) + 5$ non homog.

Ex: $f(x,y) = \frac{x^3 + y^3}{x^3 y} \Rightarrow$ homog of degree 0

Ex: $f(x,y) = \frac{x^2 \cos(\frac{y}{x}) + y^2}{x^3 + y^3} \Rightarrow$ homog of degree -1

Ex: $f(x,y) = \frac{x+y}{x+3} \Rightarrow$ non homog

Def:- Any DE of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ such that M and N are two homog. function of the same degree then it is called homog DE. $\rightarrow \frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$

Ex:- solve:-

$2xyy' = y^2 - x^2$

$y' = \frac{y^2 - x^2}{2xy} = \frac{1}{2}\frac{y}{x} + \frac{1}{2}\frac{x}{y}$

$u = \frac{y}{x} \Rightarrow y = ux \Rightarrow y' = u + xu' \Rightarrow u + xu' = \frac{u^2x^2 - x^2}{2ux^2} = \frac{u^2-1}{2u}$

$\Rightarrow xu' = \frac{u^2-1}{2u} - \frac{u}{1} = \frac{u^2-1-2u^2}{2u} \Rightarrow xu' = \frac{-u^2-1}{2u} \Rightarrow \frac{1}{x}\frac{dx}{du} = \frac{2u}{-(u^2+1)}$

$\Rightarrow \frac{1}{x}dx = \frac{2u}{-(u^2+1)}du \xrightarrow{\text{solve}} \ln x = -\ln(u^2+1) + c \Rightarrow \ln x + \ln(u^2+1) = c$

$\Rightarrow \ln(x(u^2+1)) = c \Rightarrow x(u^2+1) = e^c = c \Rightarrow x\left(\frac{y^2}{x^2}+1\right) = c$

$\Rightarrow \frac{y^2}{x} + x = c \Rightarrow \underline{y^2 + x^2 = cx}$ ... g.c

Ex:- $y' = \frac{y}{x} + e^{\frac{y}{x}} = g\left(\frac{y}{x}\right) \Rightarrow u = \frac{y}{x} \Rightarrow y = ux \Rightarrow y' = u + xu'$

$\Rightarrow \cancel{u} + xu' = \cancel{u} + e^u \Rightarrow xu' = e^u \Rightarrow \frac{1}{x}\frac{dx}{du} = e^{-u}$

$\Rightarrow \ln x = -e^{-u} + c \Rightarrow x = e^{-e^{-u}+c} = e^{-e^{-u}} \cdot e^c$

$\Rightarrow \underline{x = ce^{-e^{-\frac{y}{x}}}}$ g.c

Ex:-

[1]. $xy' = y + x$

$u = \frac{y}{x}$, $y = ux$, $y' = u + xu'$ $\Rightarrow$ $u + xu' = \frac{ux + x}{x}$ $\Rightarrow$ $u + xu' = u + 1$

$\rightarrow xu' = u + 1 - u$ $\rightarrow$ $xu' = 1$ $\Rightarrow$ $\frac{1}{x}dx = du$ $\Rightarrow$ $\ln x = u + c$

$\rightarrow x = e^{u+c}$ $\Rightarrow$ $\underline{\underline{x = e^{\frac{y}{x}} + c}}$

[2] $xy' = y + 2x^3 \sin^2(\frac{y}{x})$

$u = \frac{y}{x}$, $y = ux$, $y' = u + xu'$ $\Rightarrow$ $u + xu' = \frac{ux + 2x^3 \sin^2(u)}{x}$ $\Rightarrow$ $u + xu' = u + 2x^2 \sin(u)$

$\Rightarrow xu' = u + 2x^2 \sin(u) - u$ $\rightarrow$ $xu' = 2x^2 \sin(u)$ $\Rightarrow$ $\frac{\cancel{2x}}{\cancel{x}}\frac{dx}{\cancel{dx}} = \frac{1}{\sin(u)} du$

$\Rightarrow x^2 = \ln|\csc u + \cot u| + c$ $\Rightarrow$ $x^2 = \ln(\csc\frac{y}{x} + \cot\frac{y}{x}) + c$

[3] $(x^2 + 3xy + y^2)dx - x^2 dy = 0$

$x^2 + 3xy + y^2 \, dx = x^2 dy$ $\Rightarrow$ $\frac{x^2 + 3xy + y^2}{x^2} = \frac{dy}{dx}$ $\Rightarrow$ $1 + \frac{3y}{x} + (\frac{y}{x})^2 = y'$

$1 + 3u + u^2 = u + xu'$

[4] $xy' = y + \sqrt{x^2 - y^2}$, $x > 0$

15) $y' = \dfrac{y - 4x}{x - y}$

16) $y' = \dfrac{x^2 + xy + y^2}{x^2}$

$y' = \dfrac{x^2}{x^2} + \dfrac{xy}{x^2} + \dfrac{y^2}{x^2} \Rightarrow y' = 1 + \dfrac{y}{x} + \left(\dfrac{y}{x}\right)^2$

$u = \dfrac{y}{x} \Rightarrow y = ux \Rightarrow y' = u + xu'$

$u + xu' = 1 + u + u^2 \Rightarrow xu' = 1 + u + u^2 - u \Rightarrow x\dfrac{du}{dx} = 1 + u^2$

$\dfrac{1}{x}dx = \dfrac{1}{1+u^2}du$

$\ln x = \tan^{-1} u + c \Rightarrow \ln x = \tan^{-1}\dfrac{y}{x} + c \Rightarrow x = e^{\tan^{-1}\frac{y}{x}} + c$

3:- Transform to seperable form:-

If $f(x,y) = g(ax+by)$ where a and b are constant, then:-
$y' = g(ax+by)$. Then the substitution that transform it to be seperable is: $z = ax+by$.

$z' = a+by' \Rightarrow y' = \frac{1}{b}(z'-a) \Rightarrow \frac{1}{b}(z'-a) = g(z) \Rightarrow z'-a = bg(z)$

$\Rightarrow z' = bg(z) + a$

$\Rightarrow \frac{dz}{dx} = bg(z)+a \Rightarrow \frac{dx}{dz} = \frac{1}{bg(z)+a} \Rightarrow dx = \frac{1}{bg(z)+a}\,dz$ sep.

Ex:- solve:- $y(0)=2$

(1) $(x+y)dx + dy = 0 \Rightarrow y' = -(x+y) \Rightarrow z = x+y \Rightarrow z' = 1+y'$

$\Rightarrow y' = z'-1 \Rightarrow z'-1 = -z \Rightarrow z' = 1-z \Rightarrow \frac{dx}{dz} = \frac{1}{1-z} \Rightarrow dx = \frac{1}{1-z}\,dz$

$\Rightarrow x = -\ln(1-z) + c \Rightarrow 1-z = e^{-x+c} = e^{-x} \cdot e^{c} = ce^{-x}$

$\Rightarrow 1-x-y = ce^{-x}$. $\boxed{g.s} \Rightarrow y = 1-x-ce^{-x}$

$\Rightarrow 2 = 1-0-ce^{0} \Rightarrow c = 1 \Rightarrow y = 1-x-e^{-x}$ P.S

Ex:- $y' = \sin(x-y) \Rightarrow$

$z = x-y \Rightarrow z' = 1-y' \Rightarrow y' = 1-z'$

$\Rightarrow 1-z' = \sin(z) \Rightarrow z' = 1-\sin(z) \Rightarrow \frac{dx}{dz} = \frac{1}{1-\sin(z)} \Rightarrow dx = \frac{1}{1-\sin(z)}dz$

$\Rightarrow x = \int \frac{1}{1-\sin(z)} dz \xRightarrow{\text{ضرب بالمرافق}} x = \int \frac{1+\sin(z)}{1-\sin^2(z)} dz \Rightarrow x = \int \frac{1+\sin(z)}{\cos^2 z} dz$

$\Rightarrow x = \int (\sec^2 z + \sec z \tan z) dz \Rightarrow x = \tan z + \sec z + c$

$\Rightarrow \underline{x = \tan(x-y) + \sec(x-y) + c}$

<u>homework</u>

Ex:- (1) $y' = (x+y-2)^2, \ y(0) = 2$

(2) $y' = (x+y)^2 + 1$

[4] Exact DEs :-

$\underbrace{\qquad\qquad}_{\text{٪١٠ ٪١٠}}$

+ Any equation of the form:- $M(x,y)\,dx + N(x,y)\,dy = 0$ is called exact iff : $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. T solve this Type is equation we do the following:-

[1] Let $F = \int M(x,y) + g(y)$

[2] find $\frac{\partial F}{\partial y} = N(x,y)$ — خطوات الحل *

[3] find $g(y)$

[4] The g.S will be : $F = C$

Ex:- $(2x + 3y)\,dx + (3x + \sin(y))\,dy = 0$

$\frac{\partial M}{\partial y} = 3 = \frac{\partial N}{\partial x}$ exact.

$\Rightarrow F = \int (2x + 3y)\,dx + g(y) \Rightarrow F = x^2 + 3yx + g(y) \Rightarrow \frac{\partial F}{\partial y} = \cancel{3x} + g'(y) = \cancel{3x} + \sin y$

$\Rightarrow g'(y) = \sin y \Rightarrow g(y) = -\cos y \Rightarrow \text{g.S} \Rightarrow x^2 + 3xy - \cos y = C$

Ex:- $(ye^{xy} - 2y^3)\,dx + (xe^{xy} - 6xy^2 - 2y)\,dy = 0$

$\frac{\partial M}{\partial y} = yxe^{xy} + e^{xy} - 6y^2$ $\qquad\qquad$ $\frac{\partial N}{\partial x} = xye^{xy} + e^{xy} - 6y^2$

$F = \int (ye^{xy} - 2y^3)\,dx + g(y) \Rightarrow F = e^{xy} - 2xy^3 + g(y) \Rightarrow \frac{\partial F}{\partial y} = xe^{xy} - 6xy^2 + g'(y) = \cancel{xe^{xy}} \cancel{-6xy^2} - 2y$

$\Rightarrow g'(y) = -2y \Rightarrow g(y) = -y^2$ $\qquad$ g.S $\Rightarrow e^{xy} - 2xy^3 - y^2 = C$

Ex:- $(\cos y \sinh x + 1)\,dx - \sin y \cosh x\,dy = 0$

$\frac{\partial M}{\partial y} = -\sin y \sinh x \qquad \frac{\partial N}{\partial x} = -\sin y \sinh x$

$\Rightarrow F = \int -\sin y \cosh x\,dy + g(x) \Rightarrow f = \cos y \cosh x + g(x) \Rightarrow \frac{\partial f}{\partial x} = \cos y \sinh x + g'(x)$

$\Rightarrow \cancel{\cos y \sinh x} + g'(x) = \cancel{\cos y \sinh x} + 1 \Rightarrow g(x) = x \qquad g.s \Rightarrow \cos y \cosh x + x = C$

Exer:- home work

[1] $\cos(x+y)\,dx + (3y^2 + 2y + \cos(x+y))\,dy = 0$

[2] $(\cos x \cos y - \cot x)\,dx - \sin x \sin y\,dy = 0$

13) $(2x+y\cos(xy))dx + x\cos(xy)\,dy = 0$

14) $(1+y^2+xy^2)\,dx + (x^2y+y+2xy)\,dy = 0$

15) $\frac{1}{(1-xy^2)}\,dx + \left(y^2 + \frac{x^2}{(1-xy)^2}\right)dy = 0$

16, $(2xy - \tan y)\,dx + (x^2 - x\sec^2 y)\,dy = 0$

* If $Mdx + Ndy = 0$ is both homog and exact. then the g.s is:-

  $\underline{\underline{xM + yN = C}}$

Ex: $(3x+y)\,dx + (x+4y)\,dy = 0$

$\Rightarrow 3x^2 + xy + xy + 4y^2 = C \Rightarrow 3x^2 + 2xy + 4y^2 = C \Rightarrow$ g.S

F F

~~[5] Not~~ [5] NoT-Exact DEs :-

Any equation of the form:- " $P(x,y)\,dx + Q(x,y)\,dy = 0$ " is called NoT-exact iff:

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$. To solve it we have to multiply the given DE by an integrating factor (I.F), say "μ".

$\underbrace{\mu P dx}_{M} + \underbrace{\mu Q dy}_{N} = 0 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

* How to find μ :-

[1] If $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = f(x)$ (only) then : $\mu = e^{\int f(x)\,dx}$

[2] If $\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = f(y)$ (only) then : $\mu = e^{-\int f(y)\,dy}$

Ex :- solve :-

$\underbrace{(3x^2+y)}_{P}dx + \underbrace{(x^2y - x)}_{Q}dy = 0 \Rightarrow \frac{\partial P}{\partial y} = 1 \quad , \frac{\partial Q}{\partial x} = 2xy - 1$ not exact

$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 2 - 2xy \Rightarrow \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{2(1-xy)}{x(xy-1)} \Rightarrow -\frac{2}{x}$

$\mu = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$ * نضرب μ بالمعادلة

$\underbrace{(3 + \frac{y}{x^2})}_{M}dx + \underbrace{(y - \frac{1}{x})}_{N}dy = 0 \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{x^2} \quad , \frac{\partial N}{\partial x} = \frac{1}{x^2}$ exact

Ex:- $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0$

$\frac{\partial P}{\partial y}=12x^2y^3+2x \quad , \quad \frac{\partial Q}{\partial x}=6x^2y^3-2x$. not exact

$\Rightarrow \frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}=6x^2y^3+4x \ . \Rightarrow \frac{1}{P}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=\frac{2(3x^2y^3+2x)}{y(3x^2y^3+2x)}$

$M=e^{-\int\frac{2}{y}dy}=y^{-2} \rightarrow (3x^2y^2+\frac{2x}{y})\,dx+(2x^3y-\frac{x^2}{y^2})\,dy=0$

$\frac{\partial M}{\partial y}=6x^2y-\frac{2x}{y^2} \quad , \quad \frac{\partial N}{\partial x}=6x^2y-\frac{2x}{y^2}$ exact.

Ex:- $(xy+y^2)\,dx+(x+2y-1)\,dy=0$

$\frac{\partial P}{\partial y}=\cancel{y}\,x+2y \quad , \quad \frac{\partial Q}{\partial x}=1$ not exact

$\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}=(x+2y)-1 \rightarrow \frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=\frac{(x+2y-1)}{(x+2y-1)}=1$

$\therefore M=e^{\int f(x)dx}=e^{\int 1\,dx}=e^{x} \rightarrow (e^{x}xy+e^{x}y^2)\,dx+(e^{x}x+e^{x}2y-e^{x})\,dy=0$

$\frac{\partial M}{\partial y}=e^{x}x+2e^{x}y \quad , \quad \frac{\partial N}{\partial x}=xe^{x}+e^{x}2y\cancel{-e^{x}}\cancel{+e^{x}}=\frac{\partial N}{\partial x}=xe^{x}+2e^{x}y.$

exact

[3] If $P\,dx + Q\,dy = 0$ is both homog. and not exact and $xP + yQ \neq 0$

$$\Rightarrow \mu = \frac{1}{xP+yQ}.$$

~~Ex:~~ solv: $(x^3+y^3)dx - xy^2 dy = 0$

not exact

$$\frac{\partial P}{\partial y} = 3y^2,\ \frac{\partial Q}{\partial x} = -y^2 \Rightarrow x^4 + \cancel{xy^3} - \cancel{xy^3} = x^4 \Rightarrow \mu = \frac{1}{x^4} \Rightarrow x^{-4}$$

or:- $4y^2 = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \Rightarrow \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{4y^2}{-xy^2} = \frac{-4}{x}$

$$\therefore \mu = e^{\int \frac{-4}{x} dx} = e^{-4\ln x} = x^{-4}.$$

Ex:- $(2x+3y)\,dx + (x-y)\,dy = 0$

$$\Rightarrow 2x^2 + 3xy + xy - y^2 \neq 0 \Rightarrow \therefore \mu = \frac{1}{2x^2+4xy-y^2}.$$

[4] If $P\,dx + Q\,dy = 0$ can be written in the form:

$y\,f(xy)\,dx + x\,g(xy)\,dy = 0$ such that $f \neq g$, then

$\mu = \frac{1}{xy(f-g)}$ but if $f = g$: $y\,f(xy) + x\,f(xy) = 0$

$\Rightarrow f(xy)\,(y\,dx + x\,dy) = 0$, $f(xy) \neq 0$

$$\therefore y\,dx + x\,dy = 0 \Rightarrow \frac{1}{x}dx + \frac{1}{y}dy = 0 \Rightarrow \ln x + \ln y = c \Rightarrow \ln(xy) = c$$

$$xy = e^c = c \Rightarrow y = \frac{c}{x}$$

EX:- solve:-

$$\underbrace{(2xy^2+y)}_{P}dx+\underbrace{(x+2x^2y+x^4y^3)}_{Q}dy=0$$

$\frac{\partial P}{\partial y}=4xy+1$, $\frac{\partial Q}{\partial x}=1+4xy-4x^3y^3$ not exact

$\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}=4x^3y^3$.

$$\rightarrow \underbrace{y(2xy+1)}_{f(xy)}dx+x\underbrace{(1+2xy+x^3y^3)}_{g(xy)}dy=0$$

$$\therefore M=\frac{1}{xy(f-g)}=\frac{1}{xy(\cancel{2xy}+\cancel{1}-\cancel{1}-\cancel{2xy}+x^3y^3)}=\frac{1}{x^4y^4}$$

$\therefore M=x^{-4}y^{-4}$.

EX:- home work.

(IV) $(3y^4+3x^2y^3-x^4y)dx-(xy^3-2x^5)dy=0$!!!

$\rightarrow y(3y^3+3x^2y^2-x^4)-x(y^3-2x^4)dy=0$

(V) $(xy^2+x^2y^3)dx+(x^2y+x^3y^2)dy=0$

$$\underbrace{y(xy+x^2y^2)}_{F(xy)}dx+x\underbrace{(xy+x^2y^2)}_{g(xy)}dy=0$$

$$\therefore M=\frac{1}{xy(f-g)}=\frac{1}{xy(xy+\cancel{x^2y^2}-\cancel{xy}-x^2y^2)}=\frac{1}{xy}$$

$\therefore M=x^{-1}y^{-1}$

[3] $(8y+4x^2y^4)dx + (8x+5x^3y^3)dy = 0$

[6] Linear DEs :-

* Any equ of the form: $y' + p(x)y = Q(x)$ is called Linear DE (LDE).

To solve it we do :-

$$\frac{dy}{dx} + p(x)y - Q(x) = 0 \Rightarrow \underset{N}{1}\,dy + \underbrace{(p(x)y - Q(x))}_{M}\,dx = 0$$

$$\Rightarrow \frac{\partial M}{\partial y} = p(x), \quad \frac{\partial N}{\partial x} = 0.$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = p(x). \rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{p(x)}{1} = p(x)$$

$$\therefore \mu = e^{\int p(x)dx}, \quad \text{let } h = \int p(x)dx$$

$$= e^{h}$$

$$\Rightarrow e^{h}y' + p(x)e^{h}y = e^{h}Q(x) \Rightarrow \frac{d}{dx}(e^{h}y) = e^{h}Q(x) \Rightarrow d(e^{h}y) = e^{h}Q(x)dx$$

تكامل

$$\Rightarrow e^{h}y = \int e^{h}Q(x) + C \Rightarrow y = e^{-h}\int e^{h}Q(x)dx + C$$

$$\Rightarrow y = \frac{1}{\mu}\left[\int \mu \cdot Q(x)\,dx + C\right] \Rightarrow \text{g.s}$$

Ex:- solve:-

$\frac{1}{x}y' - \frac{2y}{x^2} = x\cos x$, $x>0$, $y(\frac{\pi}{2})=0$

$\Rightarrow y' - \frac{2y}{x} = x^2\cos x \Rightarrow \quad P(x) = \frac{-2}{x}$, $Q(x) = x^2\cos x$.

$\Rightarrow \mu = e^{\int \frac{-2}{x}dx} = x^{-2}. \Rightarrow \therefore y = x^2\left[\int x^{-2}\cdot x^2\cos x\,dx + c\right]$

$\Rightarrow y = x^2[\sin x + c] = x^2\sin x + cx^2$

$\Rightarrow 0 = \frac{\pi^2}{4} + c\,\frac{\pi^2}{4} \Rightarrow \underline{c=-1}$ $\quad \therefore \underline{y = x^2\sin x - x^2} \Rightarrow$ P.S

Ex:-

$\cos^2 x\, y' + y = \tan x$

$\Rightarrow y' + \sec^2 x\, y = \sec^2 x \tan x \Rightarrow P(x) = \sec^2 x$, $Q(x) = \sec^2 x \tan x$.

$\Rightarrow \mu = e^{\int \sec^2 x\,dx} = e^{\tan x} \Rightarrow y = e^{-\tan x}\left(\int e^{\tan x}\tan x \sec^2 x\,dx + c\right)$

let $\Rightarrow z = \tan x$

$\Rightarrow dx = \frac{dz}{\sec^2 x} \Rightarrow \int e^z z\,dz \Rightarrow ze^z - e^z$

نشتق	نكامل
z	e^z
$\downarrow$	$\downarrow$
1	e^z
$\downarrow$	$\downarrow$
0	e^z

$\therefore y = e^{-\tan}\left(\tan x\, e^{\tan x} - e^{\tan x} + c\right)$

$\therefore \underline{y = \tan x - 1 + ce^{-\tan x}} \Rightarrow$ g.S exer :- [1] $x^2 y' + 2xy - x + 1 = 0$, $y(1) = 0$

[2] $y' = 1 + 3y\tan x$

[3] Bernoulli's DEs :-

٪ ١٠

Theare of the form :-

$y' + p(x)y = Q(x)y$

non linear ← Bernoull كل

وليس كل non linear ~~non~~ ← Bernoull

→ To solve it we assume $u = y^{1-n}$ to convert it to be linear

→ $u' + (1-n)p(x)u = (1-n)Q(x)$ → $u' + p_1(x)u = Q(x)$

∴ $M = e^{\int p(x)dx} = 0$, $u = \frac{1}{M}\left[\int M \cdot Q(x)\,dx + c\right]$

Ex :- $y' - 5y = -\frac{5}{2}\, x y^3$

Let $u = y^{1-3} = y^{-2}$

أوجد M

→ $u' + 10u = 5x$ → ∴ $M = e^{\int 10 dx} = e^{10x}$

نشتق	تكامل
$5x$	e^{10x}
↓	↓
5	$\frac{e^{10x}}{10}$
↓	↓
0	$\frac{e^{10x}}{100}$

→ $u = e^{-10x}\left[\frac{1}{2}xe^{10x} - \frac{1}{20}e^{10x} + c\right]$

→ $u = \frac{1}{2}x - \frac{1}{20} + ce^{-10x}$ g.s

∴ $y^{-2} = \frac{1}{2}x - \frac{1}{20} + ce^{-10x}$ → $y^2 = \frac{1}{\frac{1}{2}x - \frac{1}{20} + ce^{-10x}}$

Ex:- $y(6y^3 - x - 1)\,dx + 2y\,dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{xy + y - 6y^4}{2x} \Rightarrow y' = \frac{(x-1)y}{2x} - \frac{3}{x}y^4$$

$$\therefore y' - \frac{(x+1)y}{2x} = \frac{-3}{x}y^4 \Rightarrow u = y^{1-4} = y^{-3}$$

$$\Rightarrow u' + \frac{3}{2}\frac{(x+1)}{x}u = \frac{9}{x}$$

$$\therefore \cancel{M = e^{\frac{3}{2}(1+\frac{1}{x})}} \quad \therefore M = e^{\frac{3}{2}\int(1+\frac{1}{x})dx} = e^{\frac{3}{2}(x+\ln x)} = e^{\frac{3}{2}x} \cdot e^{\frac{3}{2}\ln x} = x^{\frac{3}{2}} e^{\frac{3}{2}x}$$

$$u = \frac{e^{-\frac{3}{2}x}}{x^{\frac{3}{2}}}\left[\int x^{\frac{3}{2}} e^{\frac{3}{2}x} \cdot \frac{9}{x}\,dx + c\right]$$

Orthognac Trajecdories :-

If we have a family of curves :-

$f(x, y, c) = 0$ ... [1]

(for each value of curve have a curve) and if we are looking for another family of curve :-

$y(x, y, K) = 0$ --- [2]

such that the tow tangent lines each point to intersection between family [1] and family [2] are perpendiculer (normal) (orthognal)

Then we say that the Two family of curves are orthognal Traj of each other

* How to find orthog. Traj of $f(x,y,c) = 0$:-

[1] find the DE. of the given family of curves $y' = ...$

[2] replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$ in [1] to get the DE of the orth. Traj

[3] solve the DE in [2] to get the orth. Traj you are looking for

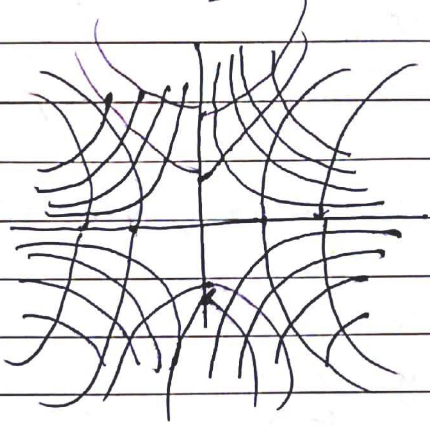
Ex:- $y = \frac{C}{x}$

solu:- $y' = \frac{-C}{x^2} = \frac{-yx}{x^2} = \frac{-y}{x} \Rightarrow y' = \frac{-y}{x} \Rightarrow y' = \frac{x}{y}$

هذه هي المعادلة التفاضلية للمسارات المتعامدة التي نبحث عنها

$\Rightarrow y\,dy = x\,dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$

$\Rightarrow \cancel{2}\,\frac{1}{2}y^2 - \frac{1}{2}x^2 = C \Rightarrow \underline{y^2 - x^2 = C} \rightarrow$

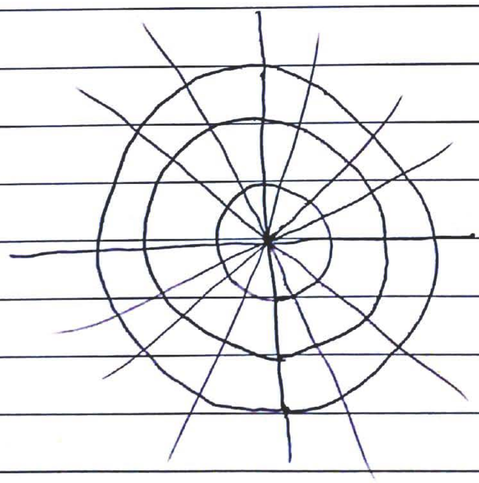


Ex:- $x^2 + y^2 = C^2$

solu:- $2x + 2yy' = 0 \Rightarrow y' = \frac{-2x}{2y} = \frac{-x}{y} \Rightarrow \underline{y' = \frac{y}{x}}$

$\Rightarrow \frac{1}{y}dy = \frac{1}{x}dx \Rightarrow \ln y = \ln x + C \Rightarrow \ln y - \ln x = C$

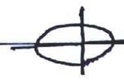
$\Rightarrow \ln\left(\frac{y}{x}\right) = C \Rightarrow \frac{y}{x} = e^{C} = C \Rightarrow \underline{y = Cx} \rightarrow$ Linear



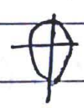
* ملاحظة :-

∴ $x^2 + y^2 = r^2$ circles

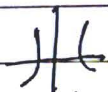
∴ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipes

→ $a > b$  بيضي

قطع ناقص

→ $a < b$  عادي

∴ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

 قطع زائد

hyperbolla

∴ $y = x^2$

 parabolla

Ex:- $y^2 = cx^3$. $\Rightarrow c = \frac{y^2}{x^3}$

solu:- $2yy' = 3cx^2 \Rightarrow 2yy' = 3\frac{y^2}{x^3}x^2 \Rightarrow 2yy' = \frac{3y^2}{x}$

$y' = \frac{3y}{2x}$

$\Rightarrow \frac{2y}{3y^2}dy = \frac{1}{x}dx \Rightarrow \frac{2}{3}\ln y = \ln x + c \Rightarrow \frac{2}{3}\ln y - \ln x = c$

$y' = \frac{-2x}{3y}$

$\Rightarrow \ln y^{2/3} - \ln x = c \Rightarrow \ln\left(\frac{y^{2/3}}{x}\right) = c \Rightarrow \frac{y^{2/3}}{x} = e^c = c$

$3y\,dy = -2x\,dx$

$\Rightarrow y^{2/3} = cx \Rightarrow y = (cx)^{3/2} = c^{3/2} \cdot x^{3/2} = c\,x^{3/2}$

$\frac{3}{2}y^2 + x^2 = c$

* Exer :-

(1) $y = \frac{1+cx}{1-cx}$

(2) $e^x + e^y = c$

(3) $1 + x\sin y = y$

* clairaut's DEs :-

They are the form:- $y = xp + f(p)$, $p = y'$ where $f(p)$ contains neither x nor y explicitly. اشتقاق

$y = xy' + f(y') \Rightarrow y' = xy'' + y' + f'(y')\cdot y'' \Rightarrow xy'' + y'' f'(y') = 0$

$\Rightarrow y''(x + f'(y')) = 0 \Rightarrow y'' = 0 \Rightarrow y' = c \Rightarrow p = c$

$\Rightarrow y = cx + f(c) \Rightarrow$ g.s

or: $x + f'(y') = 0 \Rightarrow x = -f'(y') = -f'(p)$ $\therefore y = -pf'(p) + f(p)$ singular soln.

Ex:- solve:- $y = xy' + (y')^3$

g.s $\Rightarrow y = cx + c^3 \Rightarrow f(p) = p^3 \Rightarrow f'(p) = 3p^2$

$\therefore x = -3p^2 \Rightarrow p^2 = \frac{x}{-3}$ $\therefore y = -3p^3 + p^3 = -2p^3 \Rightarrow p^3 = \frac{y}{2}$

$p^3 = \frac{y}{-2} \Rightarrow p^6 = \frac{y^2}{4}$ $\qquad p^2 = \frac{x}{-3} \Rightarrow p^6 = \frac{x^3}{-27}$

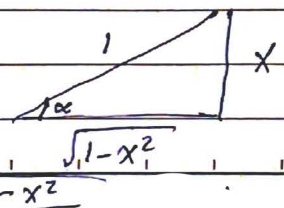
$\therefore -27y^2 = 4x^3 \Rightarrow$ s.s

Ex:- $y = xy' + \cos y' \Rightarrow y = cx + \cos c \Rightarrow$ g.s

$\Rightarrow f(p) = \cos p \Rightarrow f'(p) = -\sin p$ $\qquad \therefore x = +\sin p \Rightarrow p = \sin^{-1} x$

$\therefore y = p\sin p + \cos p \Rightarrow y = (\sin^{-1}x(\sin(\sin^{-1}x)) + \cos(\sin^{-1}x)$

$y = x\sin^{-1}x + \cos(\sin^{-1}x)$ $\qquad \alpha = \sin^{-1}x$, $\sin\alpha = \frac{x}{1}$



$\therefore y = x\sin^{-1}\alpha + \sqrt{1-x^2}$

$\cos\alpha = \frac{\sqrt{1-x^2}}{1}$

Exer :-

1) $xy' - y = \frac{5}{2}\left((y')^3 - y'\right) \Rightarrow$ S.S $\quad y = \frac{(5+2x)^{3/2}}{\sqrt{135}}$

2) $(xy' \pm y)^2 - (y')^2 - 1 = 0 \Rightarrow$ g.s $\quad y = cx \pm \sqrt{c^2 - 1}$

* Higher order DEs:-

The n^{th} order Linear DE is of the form:-

$$P_n(x)\, y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \cdots + P_2(x)\, y'' + P_1(x)\, y' + P_0(x)\, y = R(x) \quad \cdots \; (*)$$

If $(R(x) = 0$, then $(*)$ is called homogeneous.

If we replace $P_i(x)$, $i = 0, 1, 2 \cdots n$ by a_i, $i = 0, 1, 2, \cdots, n$ the $(*)$ become.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0 \quad \cdots \; (I)$$ n^{th} order homog DEs const coff.

To solve (I) we let $y = e^{\lambda x}$, where λ is a parameter that has to be determined.

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}, \; \cdots, \; y^{(n)} = \lambda^n e^{\lambda x}$$

$$a_n \lambda^n e^{\lambda x} + a_{n-1} \lambda^{n-1} e^{\lambda x} + \cdots + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = 0. \Rightarrow e^{\lambda x} \underbrace{(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0)}_{F(\lambda)} = 0$$

$$\Rightarrow e^{\lambda x} F(\lambda) = 0 \Rightarrow \quad \therefore F(\lambda) = 0 \quad \cdots \; (2)$$

$e^{\lambda x} > 0$ always

→ Equation (2) is called the characteristic equ. of (I) which is a polyn.. of degree n with respect to λ, therefore it has n roots, and these roots may be distinct, repeated or complex conjugate.
مختلفة مساوي مركب مرافق

$$\Rightarrow \sqrt{-16} = \sqrt{-1 \cdot 16} = \sqrt{-1} \times 4 = 4i \quad \Rightarrow \quad 0 \pm 4i$$

$$\Rightarrow \quad \alpha \pm \beta$$

real part   imaginary part

* $ax^2 + bx + c = 0$

$\Delta = b^2 - 4ac$ discriment

[1] If $\Delta > 0$, then it has two distinct roots.
[2] If $\Delta = 0$, the it has two repeated roots.
[3] If $\Delta < 0$, then it has two complex conjugate of the form $\alpha \pm \beta i$

*case [1]:- If the roots of the charact equ. are all distinct, say

$$\lambda_1 \neq \lambda_2 \neq \lambda_3 \cdots \neq \lambda_n$$

$y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, ----, $y_n = e^{\lambda_n x}$.

then the g.s is :-

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x}$$

Ex:- solve:- $y'' + 2y' - 3y = 0$, $y'(0) = 2$, $y(0) = 1$

solu:- the charact. equ. is :-

$\lambda^2 + 2\lambda - 3 = 0$

$\Delta = 4 + 12 = 16 > 0$

$(\lambda - 1)(\lambda + 3) = 0$ $\quad \lambda = 1, -3$

$y_1 = e^{x}$, $y_2 = e^{-3x}$

g.s $\Rightarrow y = C_1 e^{x} + C_2 e^{-3x}$

$1 = C_1 + C_2 \cdots [1]$, $y' = C_1 e^{x} - 3C_2 e^{-3x}$

$2 = C_1 - 3C_2 \cdots [2]$

$-1 = -C_1 \mp C_2$

$\Rightarrow 1 = -4c_2 \Rightarrow c_2 = -\frac{1}{4}$, $c_1 = 5/4$

$* \text{P.S} :- \; y = \frac{5}{4}e^{x} - \frac{1}{4}e^{-3x}$. $\quad \therefore y(1) = \frac{5}{4}e^{1} - \frac{1}{4}e^{-3}$

$\text{Ex}:- \; y''' - 2y'' - 3y' + 6y = 0$

* نعوض الصفر في λ ويجب أن تساوي صفر

$\text{solu} :- \; \lambda^3 - 2\lambda^2 - 3\lambda + 6 = 0$

$\hookrightarrow \pm 1, \pm 2, \pm 3, \pm 6$

2	1	-2	-3	6
		2	0	-6
	1	0	-3	0

$(\lambda - 2)(\lambda^2 - 3) = 0$

$\lambda = 2 \qquad \lambda = \pm\sqrt{3}$

$y_1 = e^{2x}$, $y_2 = e^{\sqrt{3}x}$, $y_3 = e^{-\sqrt{3}x}$

$y = c_1 y_1 + c_2 y_2 + c_3 y_3$.

case 3 :- If the roots of the charact. equ. are all repeated $\lambda_1 = \lambda_2 = \cdots \lambda_n$

$y_1 = e^{\lambda x}, \; y_2 = x e^{\lambda x}, \; y_3 = x e^{2\lambda x}, \ldots, \; y_n = x^{n-1} e^{\lambda x}$.

g.s $\Rightarrow$ $y = C_1 y_1 + C_2 y_2 + C_3 y_3 + \cdots + C_n y_n$

Ex :- solve :- $y'' + 6y' + 9y = 0$

solu :-

$\lambda^2 + 6\lambda + 9 = 0$

$(\lambda + 3)(\lambda + 3) = 0$

$\lambda = \pm 3$

$y_1 = e^{-3x}, \; y_2 = x e^{-3x}$

$y = C_1 e^{-3x} + C_2 x e^{-3x} \Rightarrow$ g.s

Ex :- $y''' - 3y' + 2y = 0$

solu :-

$\lambda^3 - 3\lambda + 2 = 0$

$\pm 1, \pm 2 \leftarrow$ قواسم 2

$\lambda = 1 : 1 - 3 + 2 = 0$ ✓

$(\lambda - 1)(\lambda^2 + \lambda - 2) = 0$

$(\lambda - 1)(\lambda - 1)(\lambda + 2) = 0$

$\lambda = 1, 1, -2$

$y_1 = e^{x}, \; y_2 = x e^{x}, \; y_3 = e^{-2x}$

g.s :- $C_1 e^{x} + C_2 x e^{x} + C_3 e^{-2x}$

case [3]:- If the roots are complex conjugate of the form:-

$$\lambda = \alpha \pm \beta i$$

$\cancel{y_1 = e^{\alpha x}}$, $\cancel{y_2 = e}$

$$y_1 = e^{\alpha x} \cos \beta x \quad , \quad y_2 = e^{\alpha x} \sin \beta x$$

g.s:- $c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x \Rightarrow e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Ex:- solve:- $y'' + 9y = 0$

solu:-

$$\lambda^2 + 9 = 0 \Rightarrow \lambda^2 = -9 \Rightarrow \lambda = \sqrt{-9} \Rightarrow \lambda = \pm 3i$$

$$y_1 = \cos 3x \quad , \quad y_2 = \sin 3x$$

g.s $\Rightarrow$ $c_1 \cos 3x + c_2 \sin 3x$

Ex:- $y'' - 2y' + 5y = 0$

solu:-

$$\lambda^2 - 2\lambda + 5 = 0 \qquad \Delta = 4 - 4 \times 5 = \underline{-16}$$

$$\lambda = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

$\cancel{y_1 = e^{x}}$, $y_1 = e^{x} \cos 2x$, $y_2 = e^{x} \sin 2x$

$y = e^{x}(c_1 \cos 2x + c_2 \sin 2x) \Rightarrow$ g.s

EX:- If the roots of the charact. equ of a homog. L. DE with constant coeff. are $2 \pm 3i$ find the DE:-

solu:-

$* \ (\lambda-\alpha)^2+\beta^2=0 \Rightarrow$ قاعدة

$(\lambda-2)^2+9=0 \Rightarrow \lambda^2-4\lambda+13=0 \Rightarrow y''-4y'+13y=0$

$* \ \underbrace{\text{A second solution from A known solution}}$:-

+ If we have: $y''+p(x)y'+q(x)y=0$ such that y_1 is a given solution, then the other solution is:

$* \quad y_2=y_1\int \frac{e^{-\int p(x)dx}}{y_1^2}dx$ يجب أن يكون معامل y'' = 1

Ex:- solve:- $(x^2+1)y''-2xy'+2y=0$ such that $y=x$ is a solution.

solu:- $y''-\frac{2x}{x^2+1}y'+\frac{2}{x^2+1}y=0$, $p(x)=\frac{-2x}{x^2+1}$

$* \ y_2=y_1\int \frac{e^{\int\frac{+2x}{x^2+1}}}{x^2}dx$

$\Rightarrow y_2=y_1\int \frac{e^{\ln(x^2+1)}}{x^2}dx \Rightarrow y_2=y_1\int \frac{x^2+1}{x^2}dx \Rightarrow y_2=x\int(1+x^{-2})dx$

$\Rightarrow y_2=x[x+x^{-1}]=x^2-1$

Ex:- $y''-4y'+4y=0$, $y_1=e^{2x}$

$y_2=e^{2x}\int\frac{e^{\int 4dx}}{e^{4x}}dx \Rightarrow y_2=e^{2x}\int\frac{e^{4x}}{e^{4x}}dx=xe^{2x}$

EX:- $x^2 y'' - 2y = 0$, $y_1 = \frac{1}{x}$

$$y_2 = y_1 \int \frac{e^{\int 0\, dx}}{(\frac{1}{x})^2} dx \Rightarrow y_2 = \frac{1}{x}\int x^2 dx = \frac{1}{x}(\frac{1}{3}x^3)$$

$$\Rightarrow y_2 = \frac{1}{3}x^2$$

* cachy - Eurler DE:-

They are of the form:- $a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_2 x^2 y'' + a_1 x y' + a_0 y = R(x)$ --- [*]

suppose that $R(x) = 0$ --- نفس المعادلة حالة homog. --- [1]

*To solve [1] we let $y = x^m$

$y' = m x^{m-1}$, $y'' = m(m-1) x^{m-2}$, $y''' = m(m-1)(m-2) x^{m-3}$

$y^{(n)} = m(m-1) \dots (m(n-1)) x^{m-n}$

$\Rightarrow a_n m(m-1) \dots (m(n-1)) x^m + \dots + a_0 x^m = 0$

$\Rightarrow x^m \underbrace{[a_n m(m-1) \dots (m-(n-1)) + \dots + a_0]}_{F(m)} = 0$

$\Rightarrow x^m F(m) = 0$

either $x^m = 0$ X

$f(m) = 0$ --- [2]

* $f(m)=0$ ---- [2]

This equ. is a polyn. of degree "n" which is called charact. equ. of [1] and Therefore, it has "n" roots and These roots may be: distinct, repeated or complex conjugate.

* case [1] :- If the roots of the caract. equ. of a C-E.H.L.D.E are all distinct, say:

$m_1 \neq m_2 \neq \cdots \neq m_n$

$y_1 = x^{m_1}, \quad y_2 = x^{m_2}, \ldots, \quad y_n = x^{m_n}$

* g.s:- $y = C_1 x^{m_1} + C_2 x^{m_2} + \cdots + C_n x^{m_n}$

Ex:- solve:- $x^2 y'' - x y' - 3y = 0 \quad , y(1)=0 \quad , y'(1)=1$

solu:- $m(m-1) - m - 3 = 0 \Rightarrow m^2 - 2m - 3 = 0 \Rightarrow (m-3)(m+1) = 0$

$m = 3, -1$

$\Rightarrow y_1 = x^3, \; y_2 = x^{-1} \xrightarrow{g.s} y = C_1 x^3 + \frac{C_2}{x}$

$0 = C_1 + C_2$ --- [1]

$y' = 3C_1 x^2 - \frac{C_2}{x^2} \Rightarrow 1 = 3C_1 - C_2$ -- [2]

$4C_1 = 1 \Rightarrow C_1 = \frac{1}{4} \quad , C_2 = -\frac{1}{4}$

P.S $\Rightarrow y = \frac{1}{4} x^3 - \frac{1}{4} x^{-1}$

Ex:- $x^3 y''' + 2x^2 y'' + 3x y' + y = 0$

solu:- $m(m-1)(m-2) + 2m(m-1) + 3m + 1 = 0 \Rightarrow m^3 - 3m^2 + 2m + m^2 - m + 3m + 1 = 0$

~~$\Rightarrow m^3 - 2m^2 + 4m + 1 = 0$~~

~~$\hookrightarrow \pm 1$~~

* المسألة خاطئة لأنها لا تساوي صفر من عوامله

~~$m^3 - m^2 +$~~

Ex:- $x^3 y''' - 3x^2 y'' + 8x y' - 6y = 0$

solu:- $m(m-1)(m-2) - 3m(m-1) + 8m - 6 = 0 \Rightarrow m^3 - 3m^2 + 2m - 3m^2 + 3m + 8m - 6 = 0$

$\Rightarrow m^3 - 6m^2 + 13m - 6 = 0$

$\hookrightarrow \pm 1, \pm 2, \pm 3, \pm 6$

* نفس إلي فوق . « X »

* case 2 :-

If the roots are repeated, say $m_1 = m_2 = \dots = m_n$

$\Rightarrow y_1 = x^m, \; y_2 = x^m \ln x, \; y_3 = x^m (\ln x)^2, \dots, y_n = x^m (\ln x)^{n-1}$

Ex:- $4x^2 y'' + 8x y' + y = 0$

solu:- $4m(m-1) + 8m + 1 = 0 \Rightarrow 4m^2 - 4m + 8m + 1 = 0 \Rightarrow 4m^2 + 4m + 1 = 0$

$(2m+1)(2m+1) = 0$

$m = \pm \frac{1}{2}$

$\Rightarrow y_1 = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}, \quad y_2 = \frac{\ln x}{\sqrt{x}}$

case B: If the roots are complex conjugate of the form:- $m = \alpha \pm \beta i$.

Then: $y_1 = x^{\alpha} \cos \beta \ln x$, $y_2 = x^{\alpha} \sin \beta \ln x$.

Ex:- solve:- $x^2 y'' + 3x y' + 3y = 0$

solu:-

$m(m-1) + 3m + 3 = 0$

$m^2 + 2m + 3 = 0$ $\qquad \Delta = 4 - 12 = -8$

$m = \frac{-2 \pm \sqrt{-8}}{2} = \frac{-2 \pm 2\sqrt{2}\, i}{2}$ $\qquad \therefore m = -1 \pm \sqrt{2}\, i$

$y_1 = x^{-1} \cos \sqrt{2} \ln x$, $y_2 = x^{-1} \sin \sqrt{2} \ln x$

g.s:- $y = x^{-1} (c_1 \cos \sqrt{2} \ln x + c_2 \sin \sqrt{2} \ln x)$.

Ex:- $x^3 y''' + 5x^2 y'' + 7x y' + 8y = 0$

solu:-

$m^3 - 3m^2 + 2m + 5m(m-1) + 7m + 8 = 0 \Rightarrow m^3 + 2m^2 + 4m + 8 = 0$

$\pm 1, \pm 2, \pm 4, \pm 8$ ← قواسم

$m = -2 \Rightarrow -8 + 8 - 8 + 8 = 0$ ✓

$$\begin{array}{r|rrrr} -2 & 1 & 2 & 4 & 8 \\ & & -2 & 0 & -8 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

$(m+2)(m^2+4) = 0$

$m = -2$ $\qquad m^2 = -4 \Rightarrow m = \pm 2i$

$y_1 = x^{-2}$, $y_2 = \cos 2 \ln x$, $y_3 = \sin 2 \ln x$

Ex:- If the roots of the charact. equ. of the a C-H.H.L-D-E
$0, 0, \pm i, \pm i$ find the solution:-

solu:-

$y_1 = x^0$, $y_2 = x^0 \ln x$, $y_3 = \cos \ln x$, $y_4 = \sin \ln x$

$y_5 = \ln x \cos \ln x$, $y_6 = \ln x \sin \ln x$

Ex: If the roots of a H.C.D.E with the constant coeff. are
$0, 0, 0, 2, 2, 1$, find the solution:-

solu:-

$y_1 = 1$, $y_2 = x$, $y_3 = x^2$, $y_4 = e^{2x}$, $y_5 = x e^{2x}$, $y_6 = e^{x}$

Ex: If the g.s of a D.E is $y = x^3 (c_1 \cos \ln x^5 + c_2 \sin \ln x^5)$
. find the D.E:-

solu:-

$\alpha = 3$, $\beta = 5$, $\underline{(m-\alpha)^2 + \beta^2 = 0}$ ← قاعدة

$(m-3)^2 - 25 = 0$

$m^2 - 6m + 34 = 0$

$\underbrace{m^2 - m}$ $- 5m - 34 = 0$	$m(m-1) \leftrightarrow x^2 y''$
↓	$= m^2 - m$
	$x y' \leftrightarrow m$
$x^2 y'' - 5 x y' + 34 y = 0 \Rightarrow$ D.E	$x^3 y''' \leftrightarrow m^3 + 3m^2 + 2m$

Ex: If the roots are $1, -1, 2$ find C-E. DE. !!

solu:-

$(m-1)(m+1)(m-2) = 0 \Rightarrow (m^2 - 1)(m-2) = 0 \Rightarrow m^3 - 2m^2 - m - 2 = 0$

$\Rightarrow \underline{m^3 - 3m^2 + 2m} + m^2 - 3m + 2 = 0 \Rightarrow x^3 y''' + \underline{m^2 - m} - 2m + 2 = 0$

$\Rightarrow x^3 y''' + x^2 y'' - 2 x y' + 2y = 0$

Ex:- If the charact. equ. of a C-E-DE is $m^3 - m^2 + 3m + 1 = 0$
find the DE:-!!

sol:-

$\underbrace{m^3 - 3m^2 + 2m}_{\downarrow} + 2m^2 + m + 1 = 0$

$x^3 y''' + 2m^2 - 2m + 3m + 1 = 0 \Rightarrow x^3 y''' + 2(m^2 - m) + 3m + 1 = 0$

$\Rightarrow x^3 . y''' + 2x^2 y'' + 3xy' + y = 0$

* non-Homog. L. D. Es :-

They are of the form:

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = R(x)$ --- [*1]

The g.s of [*1] contains two parts, the first one is called the
المتممة complemarlary solution, y_c, which is the g.s of the homog equ. of [*1]
that is:

* $y_c = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$. , i.e :- it contain the arbetrary constants.

. The other part is called the particular solution, y_p, which is any solution that satisfies [*1] and it contain no arbetrary constants.

$g.s = y = y_c + y_p$

* How to find y_p :- غير محددة

[1] the mathod of undetermined coefficients:-

- To use the mathod to find y_p the non-homog. term in [*1] ($R(x)$) must be one of the following function :-

[1] Exponential :- ~~$R(x) = b_n x^n + b_{n-1} x^{n-1} + \dots + b_0$~~ $R(x) = e^{ax}$

[2] polyn. :- $R(x) = b_n x^n + b_{n-1} x^{n-1} + \dots + b_0$

[3] $\left.\begin{matrix} \cos bx \\ \sin bx \end{matrix}\right\}$

[4] Any linear combination (product) of ~~one~~ two or more function from
جمع او طرح
①, ②, and ③

* To find y_p using the mathod we do following :-

[1] find $y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

[2] find y_p which will be the same shape as $(R(x))$ if there are no y_c, but if there are we must to multiply by: $x, x^2, x^3, \ldots$

* $e^x \rightarrow \lambda = 1$

$xe^x \Rightarrow \lambda = 1, 1$

$x^2 e^x \Rightarrow \lambda = 1, 1, 1$

* $1 \Rightarrow \lambda = 0$

$x \Rightarrow \lambda = 0, 0$

$x^2 \Rightarrow \lambda = 0, 0, 0$

* $2 + 3e^x \Rightarrow \lambda = 0, 1$

$2x^2 + 3 + 5e^{-2x} \Rightarrow \lambda = 0, 0, 0, -2$

له وجود x^2 يلغي 3

$(x+3)e^{2x} + 5x + 1 \Rightarrow \lambda = 2, 2, 0, 0$

$xe^{2x} + 3e^{2x} + 5x + 1$

* $\cos 3x \Rightarrow \lambda = \pm 3i$

$\sin 3x \Rightarrow \lambda = \pm 3i$

$3\cos 3x + \sin 3x \Rightarrow \lambda = \pm 3i$

* $3e^x \cos 5x \Rightarrow \lambda = 1 \pm 5i$

$2e^x \sin 2x + \cos 2x \Rightarrow \lambda = 1 \pm 2i, \pm 2i$

* $x \cos x \Rightarrow \lambda = \pm i, \pm i$

$xe^x \cos 2x \Rightarrow \lambda = 1 \pm 2i, 1 \pm 2i$

Ex:- solve:- $y'' + 2y' - 3y = 8$, $y(0) = 1$, $y'(0) = 2$

solu:-

$\lambda^2 + 2\lambda - 3 = 0$

$(\lambda + 3)(\lambda - 1) = 0 \Longrightarrow \lambda = 1, -3$

$y_1 = e^{x}$, $y_2 = e^{-3x}$

$\underline{\underline{y_c}} = c_1 e^{x} + c_2 e^{-3x}$

$R(x) = 8.1 = 8.e^{0x}$

$\lambda' = 0 \longrightarrow y_p = A.1$ $\qquad$ $y_p = \frac{-8}{3}$

$\Longrightarrow y_p' = 0$, $y_p'' = 0$ $\qquad$ $\underline{\underline{g.s}} \Longrightarrow y = c_1 e^{x} + c_2 e^{-3x} - \frac{8}{3}$

$e^{x} + 2e^{x} - 3e^{x} = 0$ $\qquad$ $1 = c_1 + c_2 - \frac{8}{3}$

$0 + 0 - 3A = 8$ $\qquad$ $y' = c_1 e^{x} - 3c_2 e^{-3x}$

$A = \frac{-8}{3}$ $\qquad$ $2 = c_1 - 3c_2$

Ex:- solve:- $y'' + 2y' - 3y = x + 3e^{x}$

solu:-

$\lambda = 1, -3 \Longrightarrow y_c = c_1 e^{x} + c_2 e^{-3x}$

$R(x) = x + 3e^{x}$

$\lambda' = 0, 0, 1$ $\qquad$ موجودة $\lambda^* = 0, 0, 1, 1, -3$

[illegible]

$y_p = A + Bx + cxe^{x}$

$y_p' = B + cxe^{x} + ce^{x}$

$y_p'' = cxe^{x} + 2ce^{x}$

$\Longrightarrow \cancel{cxe^{x}} + 2ce^{x} + 2B + \cancel{2cxe^{x}} + 2ce^{x} - 3A - 3Bx - 3cxe^{x} = x + 3e^{x}$

$\Longrightarrow 4ce^{x} - 3Bx + 2B - 3A = x + 3e^{x}$

$4c = 3$ $\qquad$ $-3B = 1$ $\qquad$ $2B - 3A = 0$

$c = \frac{3}{4}$ $\qquad$ $B = \frac{-1}{3}$ $\qquad$ $A = \frac{-2}{9}$

$y_p = -\frac{2}{9} - \frac{1}{3}x + \frac{3}{4}x e^{x}$ $\quad\quad$ $y_g = c_1 e^{x} + c_2 e^{-3x} + y_p$

Ex:- $y'''' + y''' = 2x + 4 + 5e^{-x}$

sol:-

$\Rightarrow \lambda^4 + \lambda^3 = 0 \Rightarrow \lambda^3(\lambda+1) = 0 \Rightarrow \lambda = 0, 0, 0, -1$

$y_1 = 1, \quad y_2 = x, \quad y_3 = x^2, \quad y_4 = e^{-x}$

$R(x) = 2x + 4 + 5e^{-x}$

$\lambda' = 0, 0, -1$

$\lambda^* = 0, 0, 0, 0, 0, -1, -1$

$y_p = Ax^3 + Bx^4 + Cxe^{-x} \Rightarrow$ The suitable form for y_p

Ex:- $y'' + 4y = 2\sin 2x + \cos x$

$\lambda^2 + 4 = 0 \Rightarrow \quad \lambda = \pm 2i$

$y_1 = \cos 2x, \quad y_2 = \sin 2x, \quad y_c = c_1 \cos 2x + c_2 \sin 2x$

$R(x) = 2\sin 2x + \cos x$

$\lambda' = \pm 2i, \pm i$

$\lambda^* = \pm 2i, \pm 2i, \pm i$

$y_p = Ax\cos 2x + Bx\sin 2x + C\cos + D\sin x$

EX:- $y''' + 4y' = 2\sin 2x + x + 5$

find the <u>suitable</u> form for y_p

solu:- $\lambda^3 + 4\lambda = 0 \Rightarrow \lambda(\lambda^2 + 4) = 0 \Rightarrow \lambda = 0, \pm 2i$

$y_1 = 1 \ , \ y_2 = \cos 2x \ , \ y_3 = \sin 2x$

$R(x) = 2\sin 2x + x + 5$

$\acute{\lambda} = \pm 2i, 0, 0$ — هنا يوجد جذور مشتركة بين λ و $\acute{\lambda}$ لذلك نعمل λ^*

$\lambda^* = 0, 0, 0, \pm 2i, \pm 2i$

لها x^2 لها x لها x — $\cancel{\cos 2x}$, $\cancel{\sin 2x}$ — $x\cos 2x$, $x\sin 2x$

$y_p = Ax + Bx^2 + Cx\cos 2x + Dx\sin 2x$

EX:- $y'' + 6y' + 9y = 3e^{-3x} + xe^{-3x} + 5$

$= (3 + x)e^{-3x} + 5$

solu:-

$\lambda^2 + 6\lambda + 9 = 0 \Rightarrow \lambda = -3, -3$

$y_1 = e^{-3x} \ , \ y_2 = xe^{-3x}$

$R(x) = (3 + x)e^{-3x} + 5$

$\acute{\lambda} = -3, -3, 0$

$\lambda^* = -3, -3, -3, -3, 0$

لها $\cancel{e^{-3x}}$ لها $\cancel{xe^{-3x}}$ لها x^2e^{-3x} لها x^3e^{-3x} لها 1

$y_p = Ax^2e^{-3x} + Bx^3e^{-3x} + C$

$= x^2e^{-3x}(A + B) + C$

Ex:- $y''' - 2y'' - y' + 2y = 3e^{x} + 2e^{2x}$

solution

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

$\hookrightarrow \pm 1, \pm 2$

$\lambda = 1 \qquad 1 - 2 - 1 + 2 = 0$ ✓

1	1	-2	-1	2
		1+	-1+	-2+
	1	-1	-2	0

$$(\lambda - 1)(\lambda^2 - \lambda - 2) = 0$$

$$(\lambda - 1)(\lambda + 1)(\lambda \mp 2) = 0 \implies \lambda = 1, -1, +2$$

$$y_1 = e^{x}, \quad y_2 = e^{-x}, \quad y_3 = e^{2x}$$

$$R(x) = 3e^{x} + 2e^{2x}$$

$$\lambda' = 1, 2$$

$$\lambda^* = 1, 1, -2, 2, 2$$

$\cancel{e^{x}}$ xe^{x} $\cancel{e^{-x}}$ $\cancel{e^{2x}}$ e^{2x}

$$y_p = Axe^{x} + Bxe^{2x} = \underline{\underline{xe^{x}(A+B)}}$$

2) The variation of parameters :-

* If $y_1, y_2, \ldots, y_n$ are function then the workian's of these function's is

$$w(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

* $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

EX :- $\begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} = 8 + 3 = \underline{\underline{11}}$

* $$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(OR)

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$$W_1 = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2$$

$$u_1 = \int \frac{x \cdot (-2)}{2} \, dx = 1/2 \, x^2$$

* Existness and uniqueness Theorem:-

Consider

$$P_n(x)\, y^{(x)} + P_{n-1}(x)\, y^{(n-1)} + \text{-----} + P_0(x)\, y = R(x) \;. \circledast$$

① الشروط

such that: $P_n, P_{n-1}, \text{----}, P_0, R(x)$ are Continuous on an open internal I and ② $P_n(x) \neq 0 \;\; \forall x \in I$ (for all)

منفتوحة الفترة

and if ③ $x_0 \in I$ such that $y(x_0) = y_0$, $y'(x_0) = y_1$, $y''(x_0) = y_2$, ----, $y^{(n-1)}(x) = y_{n-1}$, that $\circledast$ has a unique solution

(single on solution)

* الجذر، غير متصل عند اصفار المقام . اكبر او يساويه

$\sqrt{x - 1}$

$x - 1 \geq 0$

$x \geq 1$

* بسط عام متصل فقط غير متصل عند اصفار المقام

* $\ln(F(x))$ يجب ان تكون $F(x)$ اكبر من صفر — الشرط $f(x) > 0$

Ex: $\ln(0) = -\infty$

Ex: $\ln(x - 5)$

الشرط $x > 5$

Ex:- solve:- $y'' + y = \tan x \sec x$

solu:-

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

$y_1 = \cos x \ , \ y_2 = \sin x$

$y_c = c_1 \cos x + c_2 \sin x$

$y_p = y_1 u_1 + y_2 u_2$

$u_1 = -\int \frac{R(x) \cdot y_2}{w} dx$ | $w = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$= -\int \frac{\tan x \sec x \cdot \sin x}{(1)} dx$

$= -\int \frac{\sin^2 x}{\cos^2 x} dx \Rightarrow = -\int \frac{1-\cos^2 x}{\cos^2 x} dx = -\int (\sec^2 x - 1) dx$

$u_1 = -\tan x + x$

$u_2 = \int \frac{\cancel{\sec x} \tan x \cdot \cancel{\cos x}}{(1)} dx = \int \tan x \, dx = -\ln(\cos x) = \ln 1 - \ln(\cos x) = \ln\left(\frac{1}{\cos x}\right)$

$= \ln(\sec x)$

$\therefore \underline{y_p} = -\cos x \tan x + x \cos x - \sin x \ln(\cos x)$

Ex:- $y'' - 3y' + 2y = \frac{1}{1+e^{-x}}$

solu:-

$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda = 1, 2$

$y_1 = e^{x} , \quad y_2 = e^{2x}$

$y_c = c_1 e^{x} + c_2 e^{2x}$

$\Longrightarrow$

$y_p = y_1 u_1 + y_2 u_2$

$$W = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} \Rightarrow e^{x} \cdot e^{2x} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = e^{3x}$$

$$u_1 = -\int \frac{\frac{1}{1+e^{-x}} \cdot \cancel{e^{2x}}}{\cancel{e^{3x}}\, e^{x}}\, dx = -\int \frac{e^{-x}}{1+e^{-x}}\, dx = \ln\left(1+e^{-x}\right).$$

$$u_2 = \int \frac{\frac{1}{1+e^{-x}} \cdot \cancel{e^{x}}}{\cancel{e^{3x}}\, e^{2x}}\, dx = \int \frac{1}{e^{2x}+e^{x}}\, dx = \int \frac{1}{e^{x}(e^{x}+1)}\, dx$$

$u = e^{x}$

$du = e^{x} dx \Rightarrow dx = \frac{du}{e^{x}}$

$$= \int \frac{1}{u(u+1)} \frac{du}{u} = \int \frac{1}{u^2(u+1)}\, du =$$

$$\frac{1}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1} \Rightarrow \frac{Au(u+1) + B(u+1) + Cu^2}{u^2(u+1)}$$

$1 = Au(u+1) + B(u+1) + Cu^2$

$u=0 \rightarrow B=1 \quad , \quad u=-1 \Rightarrow C=+1 \quad , \quad u=1 \rightarrow A=-1$

Ex:- $y'' - 2y' + y = \frac{1}{x} e^{x}$

$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda-1)(\lambda-1) = 0 \Rightarrow \lambda = 1, 1$

$y_1 = e^{x} \qquad y_2 = xe^{x}$

$\underline{y_c} = c_1 e^{x} + c_2 x e^{x}$

$\Rightarrow$

$y_p = y_1 u_1 + y_2 u_2$

$$W = \begin{vmatrix} e^{x} & xe^{x} \\ e^{x} & (x+1)e^{x} \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & x \\ 1 & x+1 \end{vmatrix} = e^{2x}(x+1-x) = e^{2x}$$

$$u_1 = -\int \frac{\frac{1}{x} \cdot e^{x} \cdot xe^{x}}{e^{2x}} \, dx = -x$$

$y_p = y_1 u_1 + y_2 u_2$

$$w = \begin{vmatrix} e^x & x e^x \\ e^x & (x+1)e^x \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & x \\ 1 & x+1 \end{vmatrix} = e^{2x}(x+1-x) = e^{2x}$$

$$u_1 = -\int \frac{\frac{1}{x} \cdot e^x \cdot x e^x}{e^{2x}} \, dx = \underline{\underline{-x}}$$

Ex:- solve:- $y''' + y' = \tan x \quad , \quad 0 < x < \frac{\pi}{2}$

solu:-

$\lambda^3 + \lambda = 0$

$\lambda(\lambda^2 + 1) = 0 \Rightarrow \lambda = 0, \pm i$

$y_1 = 1 \quad , \quad y_2 = \cos x \quad , \quad y_3 = \sin x$

$y_c = c_1 + c_2 \cos x + c_3 \sin x$

$y_p = y_1 u_1 + y_2 u_2 + y_3 u_3$

$$w = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1 \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} = 1$$

$$w_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 1 & -\cos x & -\sin x \end{vmatrix} = 1 \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$w_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & 1 & -\sin x \end{vmatrix} = 1 \begin{vmatrix} 0 & \cos x \\ 1 & -\sin x \end{vmatrix} = -\cos x$$

$$w_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & 1 \end{vmatrix} = 1 \begin{vmatrix} -\sin x & 0 \\ -\cos x & 1 \end{vmatrix} = -\sin x$$

$$u_1 = \int \frac{R(x) \cdot w_1}{w} dx = \int \frac{\tan x \cdot (1)}{1} dx = -\ln(\cos x) = \ln(\sec x).$$

$$u_2 = \int \frac{\tan x \cdot -\cos x}{(1)} dx = \int -\sin x \, dx = \cos x$$

$$u_3 = \int \frac{\tan x \cdot (-\sin x)}{(1)} dx = \int \frac{\sin^2 x}{\cos} dx = -\int \frac{1-\cos^2 x}{\cos x} dx$$

$$= -\int (\sec x - \cos x) dx = -\ln|\sec x + \tan x| + \sin x$$

Ex: $y''' - y' = x$

$\lambda^3 - \lambda = 0 \Rightarrow \lambda(\lambda^2 - \lambda) = 0$

$\lambda = 0, -1, +1$

$y_1 = 1 \quad , \quad y_2 = e^{-x} \quad , \quad y_3 = e^{x}$

$y_c = c_1 + c_2 e^{-x} + c_3 e^{x}$

$$w = \begin{vmatrix} 1 & e^{x} & e^{-x} \\ 0 & e^{x} & -e^{-x} \\ 0 & e^{x} & e^{-x} \end{vmatrix} = e^{x} \cdot e^{-x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$w_1 = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2$$

$$u_1 = \int \frac{x \cdot (-2)}{2} dx = -\frac{1}{2} x^2$$

* $\underbrace{\text{Existness and uniqueres Theorem :-}}_{\text{o/o o/o}}$

- consider :-

$$P_n(x) y^{(n)} + P_{n-1}(x) y^{(n-1)} + \dots + P_0(x) y = B(x) \quad \dots (*)$$

such that $P_n, P_{n-1}, \dots, P_0, B(x)$ are ~~cont $P_n(x)$~~

Ex:- find the largest possible interval at which the D.E.

$2y'' + x^3 y' - 3xy = \cos x$, $y(2)=1$, $y'(2)=-2$

solu:-

$P_2 = 2$, $P_1 = x^3$, $P_0 = -3x$, $R(x) = \cos x$

(1) are all continu. on $\mathbb{R} = (-\infty, \infty) = I$

(2) $P_2(x) = 2 \neq 0 \; \forall x \in I$

(3) $x_0 = 2 \in I = (-\infty, \infty)$

Largest possible interval is $I = (-\infty, \infty)$

Ex:- $(x+3)y'' + y' \tan x = 5$, $y(-2)=1$, $y'(-2)=3$

solu:-

$P_2 = x+3$, $P_1 = \tan x$, $P_0 = 0$, $R(x) = 5$

$P_2 \downarrow \mathbb{R}$, $P_1 \downarrow \dfrac{\sin x}{\cos x} \downarrow x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \ldots$, $P_0 \downarrow \mathbb{R}$, $R(x) \downarrow \mathbb{R}$

$P_2 = x+3=0 \rightarrow x=-3$

$x_0 = -2 \in (-3, -\frac{\pi}{2}) = I$

Ex:- $x y'' + \csc x \, y' + e^x y = \ln(x+5)$, $y(-4)=2$, $y'(-4)=-1$

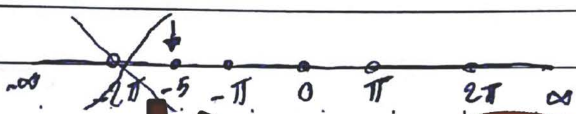
solu. $P_2 = x$, $P_1 = \csc x$, $P_0 = e^x$, $R(x) = \ln(x+5)$

$P_2 \downarrow \mathbb{R}$, $P_1 \downarrow \dfrac{1}{\sin x} \downarrow x = \pm\pi, \pm 2\pi, \pm 3\pi \ldots$, $P_0 \downarrow \mathbb{R}$, $R(x) \downarrow x+5>0$, $x>-5$

(1)

(2) $P_2 = x = 0$

(3) $x_0 = -4 \in I = (-5, -\pi)$



Ex:- $(x^2-1)y'' + 3xy' + \frac{1}{x}y = \sqrt{x+3}$

$y(2) = -1$, $y'(2) = 1$

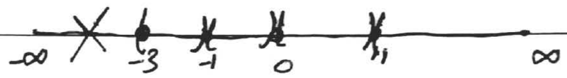
$P_2 = x^2-1$, $P_1 = 3x$, $P_0 = \frac{1}{x}$, $R(x) = \sqrt{x+3}$

$\downarrow$ $\mathbb{R}$, $\downarrow$ $\mathbb{R}$, $\downarrow$ $\mathbb{R}\setminus\{0\}$, $x+3 \geq 0$

$x \geq -3$

(2) $P_2 = x^2-1 = 0$

$x = \pm 1$



(3) $x_0 = 2 \in (1,\infty)$

* Power series Solution :-

* The power series of $x-x_0$ is: $\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 - - -$

* The power series of x is :- $\sum_{n=0}^{\infty} a_n(x)^n = a_0 + a_1 x + a_2 x^2 - - -$

* If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then :

$\therefore f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ $\therefore$, $f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$\therefore \int f(x)dx = \int \left(\sum_{n=0}^{\infty} a_n x^n\right)dx = \sum_{n=0}^{\infty} \int a_n x^n \, dx = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$

* The Tayler series of $f(x)$ at $x = x_0$ is :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \frac{f''(x_0)(x-x_0)^2}{2!} + - - -$$

* The Maclaurine series of $f(x)$ is :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + - - -$$

Ex:-

find Maclaurine series of $f(x) = e^x$.

soln:-

$f(0) = 1$

$f'(x) = e^x \Rightarrow f'(0) = 1$

$f''(x) = e^x \Rightarrow f''(0) = 1$

$f'''(x) = e^x \Rightarrow f'''(0) = 1$

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Ex:- $f(x) = \frac{1}{1-x}$

$f(0) = 1$

$f'(x) = \frac{1}{(1-x)^2}$ $\quad$ $f'(0) = 1$

$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$

$= \sum_{n=0}^{\infty} x^n$

$f''(x) = \frac{1 \times 2\,(1-x)}{(1-x)^4} = \frac{2}{(1-x)^3}$ $\quad$ $f''(0) = 1$

$f'''(x) = \frac{-2 \times 3 (1-x)^2}{(1-x)^6} = \frac{6}{(1-x)^4}$ $\quad$ $f'''(0) = 6$

Ex:- $f(x) = e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (x^2)^n$

$= \sum_{n=0}^{\infty} \frac{1}{n!} (x^{2n})$

Ex: $f(x) = e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x^2)^n$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$

Ex:- $f(x) = e^{3x+1} = e^{3x} \cdot e^{1}$

$= e \sum_{n=0}^{\infty} \frac{1}{n!} (3x)^n = e \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n$

Ex:- $f(x) = \frac{1}{1+x}$

$= \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

Ex:- $\dfrac{1}{2-3x}$

$$= \frac{1}{2(1-\frac{3}{2}x)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3}{2}x\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}\frac{3^n}{2^n}x^n$$

$$= \sum_{n=0}^{\infty}\frac{3^n x^n}{2^{n+1}}$$

Ex:- $f(x) = \dfrac{1}{1-x^2}$

$$= \sum_{n=0}^{\infty}(x^2)^n = \sum_{n=0}^{\infty}x^{2n}$$

Ex:- $f(x) = \dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)}$

$$= \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

* $\cos(x) = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n}}{(2n)!}$

* $\sin(x) = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2x+1}}{(2x+1)!}$

Ex: $\cos(2x) = \sum_{n=0}^{\infty}(-1)^n \dfrac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \dfrac{\overset{4^n}{2^{2n}}\, x^{2n}}{(2n)!}$

Ex: $\sin(x^2) = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{4n+2}}{(2n+1)!}$

EX:- $f(x) = x\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!}$

EX:- $f(x) = \tan^{-1} x$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$$

EX:- $f(x) = \frac{1}{(1-x)^2} =$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

* عند المشتقة تصبح "n=1"

$$= \sum_{n=1}^{\infty} n x^{n-1}$$

* If we have:

$$P_2(x) y'' + P_1(x) y' + P_0(x) y = R(x) \quad --- \ (*)$$

then we say that $x = x_0$ is an ordinary point of $(*)$ if $P_2(x_0) \neq 0$ otherwise it is singular point.

EX:- $(x^2-4)y'' + 3xy' + 5y = \cos x$

$2, -2 \Rightarrow$ are singular point , $R / \{2, -2\}$ ordinary

Ex:- $3y'' + 5xy' + \cos x\, y = 0$

all points are ordinary. non singular point

* To solve it using power series we do the following:

[1] let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ to be the g.s

[2] find $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Ex:- solve using p.s :-

$y' + 2xy = 0 \Rightarrow \frac{1}{y} dy = -2x\,dx \Rightarrow \ln y = -x^2 + c$

$\Rightarrow y = e^{-x^2+c} = e^{-x^2} \cdot e^{c} \Rightarrow y = c e^{-x^2}$

solu:- Let $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0$

$\sum_{m=0}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=1}^{\infty} 2 a_{m-1} x^m = 0$	$m = n-1$	$m = n+1$
	$n=1 \Rightarrow m=0$	$n=0 \Rightarrow m=1$
$(1) a_1 + \sum_{m=1}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=1}^{\infty} 2 a_{m-1} x^m = 0$	$n = m+1$	$n = m-1$

$a_1 + \sum_{m=1}^{\infty} \left((m+1) a_{m+1} + 2 a_{m-1} \right) x^m = 0$ $\qquad \underline{\underline{a_1 = 0}}$

$\therefore (m+1) a_{m+1} + 2 a_{m-1} = 0$

Ex:- $y' + 2xy = 0 \Rightarrow$ g.s is $y = ce^{-x^2}$

solu:-

$$(m+1)a_{m+1} + 2a_{m-1} = 0 \qquad \boxed{a_1 = 0}$$

$$a_{m+1} = \frac{-2a_{m-1}}{(m+1)} \quad \forall\ m \geq 1 \quad \Rightarrow \text{recurrence relation}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$m=1: \ a_2 = \frac{-2a_0}{2} = -a_0$$

$$m=2: \ a_3 = \frac{-2a_1}{3} = 0$$

$$m=3: \ a_4 = \frac{-2a_2}{4} = -\frac{1}{2}a_2 = \frac{1}{2}a_0$$

$$m=4: \ a_5 = 0$$

$$m=5: \ a_6 = \frac{-2a_4}{6} = \frac{-1}{3}a_4 = \frac{-1}{6}a_0$$

$$\therefore \text{g.s} \Rightarrow y = a_0 - a_0 x^2 + \frac{1}{2}a_0 x^4 - \frac{1}{6}a_0 x^6 + \dots$$

$$= a_0\underbrace{\left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \dots\right)}_{e^{-x^2}}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

Ex:- solve $y'' + xy' + (x^2+2)y = 0$

solu:-

Let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$\Longrightarrow$

$m=n-2$	$m=n$	$m=n+2$	$m=n$
$n=2 \rightarrow m=0$		$n=0 \rightarrow m=2$	
$n=m+2$		$n=m-2$	

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m+\sum_{m=1}^{\infty} m a_m x^m + \sum_{m=2}^{\infty} a_{m-2}x^m + \sum_{m=0}^{\infty} 2a_m x^m = 0$$

$$(2)(1)a_2 + (3)(2)a_3 x + (1)a_1 x + 2a_0 + 2a_1 x$$

$$+ \sum_{m=2}^{\infty}\left((m+2)(m+1)a_{m+2} + (m+2)a_m + a_{m-2}\right) x^m = 0$$

$2a_2 + 2a_0 = 0 \Rightarrow \underline{a_2 = -a_0} \Rightarrow$ ثوابت

$6a_3 + 3a_1 = 0 \Rightarrow \underline{a_3 = -\frac{1}{2}a_1} \Rightarrow$ معاملات x

$$(m+2)(m+1)a_{m+2} + (m+2)a_m + a_{m-2} = 0$$

$$a_{m+2} = \frac{-(m+2)a_m - a_{m-2}}{(m+2)(m+1)} \quad \forall\ m \geq 2$$

$$m=2: a_4 = \frac{-4a_2 - a_0}{(4)(3)} = \frac{3a_0}{4\times 3} = \frac{1}{4}a_0$$

$$m=3: a_5 = \frac{-5a_3 - a_1}{(5)(4)} = \frac{\frac{5}{2}a_1 - a_1}{20} = \frac{3a_1}{40}$$

$$g.s \Rightarrow y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$$= a_0 + a_1x - a_0x^2 - \frac{1}{2}a_1x^3 + \frac{1}{4}a_0x^4 + \frac{3}{40}a_1x^5 + \cdots$$

$$= a_0\left(1 - x^2 + \frac{1}{4}x^4 + \cdots\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{3}{40}x^5 + \cdots\right)$$

$$\therefore\ y = a_0y_1 + a_1y_2$$

* Linear system of DEs:-

* If we have:

$$\left.\begin{aligned} x'(t) &= a_{11}x + a_{12}y. \\ y'(t) &= a_{21}x + a_{22}y. \end{aligned}\right\} \text{-- } (*)$$

* we can write $(*)$ in matrix from as:

(1) -- $x' = Ax$, where $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{2\times2}$ is the coefficient matrix and $X = \begin{vmatrix} x \\ y \end{vmatrix}$

$$\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} x \\ y \end{vmatrix}$$

$$\begin{vmatrix} x' \\ y' \end{vmatrix}_{2\times1} = \begin{vmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{vmatrix}_{2\times1}$$

* To solve (1) we let $X = \underline{u}e^{\lambda t}$ where u is a column vector and λ is a parameter to be determined.

$$\underline{u} = \begin{vmatrix} u_1 \\ u_2 \end{vmatrix} \Rightarrow X = \begin{vmatrix} u_1 \\ u_2 \end{vmatrix} e^{\lambda t} = \begin{vmatrix} u_1 e^{\lambda t} \\ u_2 e^{\lambda t} \end{vmatrix}$$

اشتقاق

$$x' = \lambda \underline{u} e^{\lambda t} \Rightarrow \lambda \underline{u} e^{\lambda t} = A\underline{u}e^{\lambda t} \Rightarrow A\underline{u}e^{\lambda t} - \lambda \underline{u} e^{\lambda t} = \underline{0}$$

$e^{\lambda t}(A\underline{u} - \lambda\underline{u}) = \underline{0}$, $\quad e^{\lambda t} > 0$ always

$\therefore (A\underline{u} - \lambda\underline{u}) = 0 \Rightarrow (A - \lambda I)\underline{u} = 0$ --- (2), where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identy matrix

- $A \cdot I = I \cdot A = A$

* Equ. (2) has a nontrivial solution iff $|A - \lambda I| = 0$ -- (3)

$$\left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{---} \boxed{2}$$

$\boxed{3}$... $\therefore \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0 \Rightarrow (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$

$$\Rightarrow a_{11}a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2 - a_{12}a_{21} = 0$$

$$\Rightarrow \lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0 \quad \text{---} \boxed{3}$$

* Equ. $\boxed{3}$ is a polyn. of degree 2, and it's called the characteristic of equ. of A which has tow roots and these roots are called eigenvalues. and the corresponding مناظر vectors to the eigenvalues are called eigenvectors.

$\therefore$ case $\boxed{1}$:-

If ~~the~~ tow eigen values are distinct.

say $\lambda_1 \neq \lambda_2$.

نستخرج من المعادلة $\boxed{3}$ eigen value
" " " $\boxed{2}$ eigenvector

$X_1 = A e^{\lambda_1 t}$, $X_2 = B e^{\lambda_2 t}$

where $A = \begin{vmatrix} a_1 \\ a_2 \end{vmatrix}$, $B = \begin{vmatrix} B_1 \\ B_2 \end{vmatrix}$ are eigen vectors corrosponding to λ_1 and λ_2 respectively

$\therefore$ g.s $\Rightarrow X = c_1 A e^{\lambda_1 t} + c_2 B e^{\lambda_2 t} \Rightarrow \begin{vmatrix} x \\ y \end{vmatrix} = c_1 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{\lambda_2 t}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 a_1 e^{\lambda_1 t} + c_2 b_1 e^{\lambda_2 t} \\ c_1 a_2 e^{\lambda_1 t} + c_2 b_2 e^{\lambda_2 t} \end{pmatrix} \begin{matrix} \rightarrow \text{g.s of } x \\ \rightarrow \text{g.s of } y \end{matrix}$$

* solve the system :-

$x' = Ax$, $A = \begin{pmatrix} 4 & -1 \\ -4 & 4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$

or $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

or $x'(t) = 4x - y$

$y'(t) = -4x + 4y$

sol:- $\begin{vmatrix} 4-\lambda & -1 \\ -4 & 4-\lambda \end{vmatrix}$

$(4-\lambda)(4-\lambda) - 4 = 0 \longrightarrow \lambda^2 - 8\lambda + 12 = 0 \longrightarrow (\lambda-2)(\lambda-6) = 0$

$\lambda = 2, 6$ eigen values

for $\lambda = 2$ $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2a_1 - a_2 = 0$ $\longrightarrow$ بس بنوخذ وحدة

$-4a_1 + 2a_2 = 0$ --- $\times$

$a_1 = \frac{1}{2}a_2$ take $a_2 = 2$, $a = 1$

$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ eigen vectore

$X_1 = A e^{\lambda t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$

for $\lambda = 6$ $\begin{pmatrix} -2 & -1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-2b_1 - b_2 = 0$

$b_1 = -\frac{1}{2}b_2$ take $b_2 = 2$, $b_1 = -1$ $B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ eigen vectore

$$x_2 = B e^{\lambda t} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{6t}$$

$$x = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{6t}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} - c_2 e^{6t} \\ 2c_1 e^{2t} + 2c_2 e^{6t} \end{pmatrix}$$

case $\boxed{2}$:-

* If the eigenvalues are repeated $\lambda_1 = \lambda_2 = \lambda_3$

$x_1 = A e^{\lambda t}$, $x_2 = A t e^{\lambda t} + B e^{\lambda t}$ were B is a vector that can be obtained by:

$$(A - \lambda I) B = A$$

Ex: solve :-

$$x' = AX , \quad A = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} , \quad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

solve :-

$$\begin{vmatrix} -\lambda & 1 \\ -4 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \lambda + 4 = 0$$

$$(\lambda - 2)(\lambda - 2) = 0.$$

$\lambda = 2, 2$

$$\text{for } \lambda = 2 \Rightarrow \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-2a_1 + a_2 = 0$

$a_1 = \frac{1}{2} a_2 \Rightarrow a_2 = 2 , \; a_1 = 1$

$$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

$X_2 = \begin{pmatrix}1\\2\end{pmatrix} e^{2t} + B e^{2t}$

$\begin{pmatrix}-2 & 1\\-4 & 2\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

$-2b_1 + b_2 = 1$, take $b_1 = 0 \Rightarrow b_2 = 1$

$B = \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow X_2 = \begin{pmatrix}1\\2\end{pmatrix} t e^{2t} + \begin{pmatrix}0\\1\end{pmatrix} e^{2t}$

$g.s \Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = c_1 \begin{pmatrix}1\\2\end{pmatrix} e^{2t} + c_2 \begin{pmatrix}1\\2\end{pmatrix} t e^{2t} + \begin{pmatrix}0\\1\end{pmatrix} e^{2t}$

<u>case (3) :-</u>

+ If the eigen value are $\lambda = \alpha \pm \beta i$

for $\lambda = \alpha \pm \beta i$

$A = \begin{pmatrix}a_1 + b_1 i\\a_2 + b_2 i\end{pmatrix} = \begin{pmatrix}a_1\\a_2\end{pmatrix} + \begin{pmatrix}b_1\\b_2\end{pmatrix} i = \underbrace{Re\,A}_{\text{حقيقي}} + \underbrace{Im\,A}_{\text{خيالي}}\, i$

$X_1 = e^{\alpha t}\{Re\,A \cos\beta t - Im\,A \sin\beta t\}$

$X_2 = e^{\alpha t}\{Im\,A \cos\beta t + Re\,A \sin\beta t\}$

Ex: $X' = AX$, $A = \begin{pmatrix}2 & -5\\2 & -4\end{pmatrix}$, $X = \begin{pmatrix}x\\y\end{pmatrix}$

solu:-

$\begin{vmatrix}2-\lambda & -5\\2 & -4-\lambda\end{vmatrix} = 0 \Rightarrow \lambda^2 + 2\lambda + 2 = 0$

$\Delta = 4 - 8 = -4$ $\qquad \lambda = \dfrac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$

for $\lambda = -1 \pm i$

$\begin{pmatrix}3-i & -5\\2 & -3-i\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Longrightarrow$

$(3-i)a_1 - 5a_2 = 0$

$a_1 = \frac{5a_2}{3-i}$, take $a_2 = 3-i$, $a_1 = 5$

$A = \begin{pmatrix} 5 \\ 3-i \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i$

$X_1 = e^{-t} \left\{ \begin{pmatrix} 5 \\ 3 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin t \right\}$

$X_2 = e^{-t} \left\{ \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos t + \begin{pmatrix} 5 \\ 3 \end{pmatrix} \sin t \right\}$

$\begin{pmatrix} x \\ y \end{pmatrix} = e^{-t} \left\{ c_1 \left(\begin{pmatrix} 5 \\ 3 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin t \right) + c_2 \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos t + \begin{pmatrix} 5 \\ 3 \end{pmatrix} \sin t \right) \right\}$

*Laplace Transforms:-

Def:- If $f(t)$ is defined $\forall\ t>0$, then The laplace F.R of $f(t)$ is denoted by $\mathcal{L}\{f(t)\} = f(s)$, and is defined as:-

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\,dt = f(s)$$

Example:-

* Find $\mathcal{L}\{a\} = \int_0^\infty e^{-st} \cdot a\,dt = \frac{-a}{s} e^{-st} \Big]_0^\infty = \frac{-a}{s}\left(e^{-s\infty} - e^{0}\right) = \boxed{\frac{a}{s}}$ → للثابتة

* $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \frac{1}{a-s} e^{(a-s)t} \Big]_0^\infty = \frac{1}{a-s}\left(e^{(a-s)\infty} - e^{0}\right)$

$$= \boxed{\frac{1}{s-a}}, \quad s>a$$

Ex:- $\mathcal{L}\{3 - e^{2t}\} = \mathcal{L}\{3\} - \mathcal{L}\{e^{2t}\} = \frac{3}{s} - \frac{1}{s-2} = \frac{3s-6-s}{(s-2)s} = \frac{2s-6}{s(s-2)} = f(s)$

$$\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}, \quad s>-1$$

$$\mathcal{L}\{e^{2-3t}\} = \mathcal{L}\{\underbrace{e^{2}}_{\text{ثابتة}} \cdot e^{-3t}\} = \frac{e^{2}}{s+3}$$

* $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

* $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$

Example:- $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$

Ex:- $\mathcal{L}\{\sin^2 t\} = \mathcal{L}\{\frac{1}{2}(1-\cos 2t)\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$= \frac{1}{2}\left[\frac{s^2+4-s^2}{(s^2+4)s}\right] = \frac{1}{2}\left[\frac{4}{(s^2+4)s}\right] = \frac{2}{s(s^2+4)}$

Ex:- $\mathcal{L}\{\sin t\cos t\} = \frac{1}{2}\mathcal{L}\{\sin 2t\} = \frac{1}{2} \times \frac{2}{s^2+4} = \frac{1}{s^2+4}$

Ex:- $\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{at}+e^{-at}\} = \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$

$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$

Ex:- $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$

Ex:- $\mathcal{L}\{\sinh^2 t\} = \mathcal{L}\left\{\left(\frac{e^{t}-e^{-t}}{2}\right)^2\right\} = \mathcal{L}\{e^{2t}-2+e^{-2t}\} = \frac{1}{4}\left[\frac{1}{s-2} - \frac{2}{s} + \frac{1}{s+2}\right]$

$= \frac{1}{4}\left[\frac{s(s+2) - 2(s^2-4) + s(s-2)}{s(s^2-4)}\right]$

$= \frac{1}{4}\left[\frac{8}{s(s^2-4)}\right] = \frac{2}{s(s^2-4)}$

* $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\mathcal{L}\{t\} = \frac{1}{s^2}$, $\mathcal{L}\{t^2\} = \frac{2}{s^3}$, $\mathcal{L}\{t^3\} = \frac{6}{s^4}$, $\mathcal{L}\{t^5\} = \frac{120}{s^6}$

Ex:- $\mathcal{L}\left\{\frac{t^3+t^2+1}{t} - \frac{1}{t}\right\} = \mathcal{L}\{t^2+t+\frac{1}{t}-\frac{1}{t}\} = \frac{2}{s^3} + \frac{1}{s^2} = \frac{2+s}{s^3}$

* If $\mathcal{L}\{y(t)\} = Y(s)$, then

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0) = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2 Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}\{y'''(t)\} = s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)$$

$$\therefore \mathcal{L}\{y^{(n)}(t)\} = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \dots - y^{(n-1)}(0)$$

Ex:-

If $\mathcal{L}\{y(t)\} = Y(s)$, such that $y'' - y = 0$, $y(0) = 0$, $y'(0) = -1$ find $Y(s) = ??$

solu:-

$$\mathcal{L}\{y'' - y\} = 0 \Rightarrow \mathcal{L}\{y''\} - \mathcal{L}\{y\} = 0$$

$$s^2 Y(s) - sy(0) - y'(0) - Y(s) = 0 \Rightarrow Y(s)(s^2 - 1) = -1 \Rightarrow Y(s) = \frac{-1}{s^2 - 1}$$

Ex:-

$y'' - y' + y = t^2$, $y(0) = 0$, $y'(0) = 1$

solu:-

$$s^2 Y(s) - sy(0) - y'(0) - sY(0) + y(0) + Y(s) = \frac{2}{s^3}$$

$$\Rightarrow Y(s)(s^2 - s + 1) = \frac{2}{s^3} + 1 = \frac{2 + s^3}{s^3}$$

$$\Rightarrow Y(s) = \frac{s^3 + 2}{s^3(s^2 - s + 1)} \qquad Y(2) = \frac{10}{24}$$

first shifting Thereom :-

* If $\mathcal{L}\{f(t)\} = F(s)$, then :- $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$ ⇒ قاعدة

Ex:- $\mathcal{L}\{e^{2t} \sin 3t\} = F(s-2)$

$F(s) = \mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$ $F(s-2) = \frac{3}{(s-2)^2+9}$

Ex:- $\mathcal{L}\{e^{-3t} \cosh 5t\} = F(s+3)$

$F(s) = \mathcal{L}\{\cosh 5t\} = \frac{s}{s^2-25}$ $F(s+3) = \frac{5}{(s+3)^2-25}$

* $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n} = (-1)^n F^{(n)}(s)$ ⇒ قاعدة

Ex:- $\mathcal{L}\{t \sinh 2t\} = -1 \cdot F'(s)$

$F(s) = \mathcal{L}\{\sinh 2t\} = \frac{2}{s^2-4}$, $F'(s) = \frac{-2 \times 2s}{(s^2-4)^2} = \frac{-4s}{(s^2-4)^2}$

$F_1(3) = \frac{12}{25}$

Ex:- $\mathcal{L}\{t \cos t\} = -1 \cdot F'(s)$

$F(s) = \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$, $F'(s) = \frac{(s^2+1)(1) - 2s^2}{(s^2+1)^2} = \frac{1-s^2}{(s^2+1)^2}$

* If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{\int_0^t f(\tau)\,d\tau\} = \frac{F(s)}{s}$ → قاعدة

Ex:- $\mathcal{L}\{\int_0^t \cos(\tau)\,d\tau\} = \frac{F(s)}{s}$

$F(s) = \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$

$\mathcal{L}\{\int_0^t \cos(\tau)\,d\tau\} = \frac{1}{s^2+1}$

Ex:- $\mathcal{L}\{\int_0^t e^{2\tau}\cos\tau\,d\tau\} = \frac{F(s)}{s}$

$F(s) = \mathcal{L}\{e^{2t}\cos t\} = F_1(s-2)$

$F_1(s) = \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$ $\qquad F_1(s-2) = \frac{s-2}{(s-2)^2+1}$

Ex:- $\mathcal{L}\{e^{3t}t^4\} = F(s-3)$ نعوض

$F(s) = \mathcal{L}\{t^4\} = \frac{24}{s^5}$ $\qquad F(s-3) = \frac{24}{(s-3)^5}$

Ex:- $\mathcal{L}\{\int_0^t e^{-\tau}\tau^2\,d\tau\} = \frac{F(s)}{s}$ نعوض

$F(s) = \mathcal{L}\{e^{-t}t^2\} = F_1(s+1) = \frac{2}{(s+1)^3}$

$F_1(s) = \mathcal{L}\{t^2\} = \frac{2}{s^3}$

* If $\mathcal{L}\{f(t)\} = F(s)$
$f(t) = \mathcal{L}^{-1}\{F(s)\}$

Ex:-

- $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$
- $\mathcal{L}^{-1}\{\frac{3}{s}\} = 3$
- $\mathcal{L}^{-1}\{\frac{1}{s-1}\} = e^{t}$
- $\mathcal{L}^{-1}\{\frac{2}{s+3}\} = 2e^{-3t}$
- $\mathcal{L}^{-1}\{\frac{s+1}{s^2-1}\} = \mathcal{L}^{-1}\{\frac{s+1}{(s+1)(s-1)}\} = e^{t}$
- $\mathcal{L}^{-1}\{\frac{s+1}{s^5}\} = \mathcal{L}^{-1}\{\frac{1}{s^4}\} + \mathcal{L}^{-1}\{\frac{1}{s^5}\}$
$= \frac{1}{6}t^3 + \frac{1}{24}t^4$

$f(1) = \frac{1}{6} + \frac{1}{24} = \frac{4}{24} + \frac{1}{24} = \underline{\underline{\frac{5}{24}}}$

- $\mathcal{L}^{-1}\{\frac{1}{s^2+1}\} = \sin t$
- $\mathcal{L}^{-1}\{\frac{s}{s^2-4}\} = \cosh 2t$
- $\mathcal{L}^{-1}\{\frac{2}{s^2+3}\} = \frac{2}{\sqrt{3}}\mathcal{L}^{-1}\{\frac{\sqrt{3}}{s^2+3}\} = \frac{2}{\sqrt{3}}\sin\sqrt{3}t$
- $\mathcal{L}^{-1}\{\frac{2}{s^3}\} = t^2$
- $\mathcal{L}^{-1}\{\frac{3}{s^4}\} = \frac{3}{6}t^3 = \frac{1}{2}t^3$
- $\mathcal{L}^{-1}\{\frac{1}{(s+2)^2+1}\} = e^{-2t}\sin t$

- $\mathcal{L}^{-1}\{\frac{5}{(s-3)^2-4}\} = \frac{5}{2}e^{3t}\sinh 2t$
- $\mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2+5}\} = e^{-t}\cos\sqrt{5}t$
- $\mathcal{L}^{-1}\{\frac{s-3}{(s-3)^2-16}\} = e^{3t}\cosh 4t$
- $\mathcal{L}^{-1}\{\frac{s-1}{(s-2)^2+9}\} = \mathcal{L}^{-1}\{\frac{s-1-1+1}{(s-2)^2+9}\} =$
$= \mathcal{L}^{-1}\{\frac{s-2}{(s-2)^2+9}\} + \mathcal{L}^{-1}\{\frac{1}{(s-2)^2+9}\} = e^{2t}\cos 3t + \frac{1}{3}e^{2t}\sin 3t$
- $\mathcal{L}^{-1}\{\frac{2}{(s-1)^3}\} = e^{t}t^2$
- $\mathcal{L}^{-1}\{\frac{s+3}{(s+1)^5}\} = \mathcal{L}^{-1}\{\frac{s+3-2+2}{(s+1)^5}\} = \mathcal{L}^{-1}\{\frac{s+1}{(s+1)^5}\} + \mathcal{L}^{-1}\{\frac{2}{(s+1)^5}\}$
$= \frac{1}{6}e^{-t}t^3 + \frac{2}{24}e^{-t}t^4$

- $\mathcal{L}^{-1}\{\frac{s-1}{(s+2)^2-5}\} = \mathcal{L}^{-1}\{\frac{s-1+3-3}{(s+2)^2-5}\}$
$= \mathcal{L}^{-1}\{\frac{s+2}{(s+2)^2-5}\} - \mathcal{L}^{-1}\{\frac{3}{(s+2)^2-5}\}$
$= e^{-2t}\cosh\sqrt{5}t - \frac{3}{\sqrt{5}}e^{-2t}\sinh\sqrt{5}t$
- $\mathcal{L}^{-1}\{\frac{1}{s^2+6s+9}\} = \mathcal{L}^{-1}\{\frac{1}{(s+3)^2}\}$
$= e^{-3t}t$
- $\mathcal{L}^{-1}\{\frac{1}{s^2+2s-3}\} = \mathcal{L}^{-1}\{\frac{1}{(s+3)(s-1)}\}$
$= \frac{A}{s+3} + \frac{B}{s-1} = \frac{A(s-1)+B(s+3)}{(s+3)(s-1)}$
$1 = A(s-1) + B(s+3)$
$s=1 \quad B = \frac{1}{4}, \quad s=-3 \quad A = \frac{-1}{4}$
$= -\frac{1}{4}e^{-3t} + \frac{1}{4}e^{t}$