C2.2 Homological Algebra

Notes by Siddharth Unnithan Kumar

Based on lectures by André Henriques

Autumn 2019

Contents

0.1 Introduction . . . 2
0.2 Prerequisites . . . 2
0.3 Operations on R-modules . . . 3
0.3.1 Construction by elements . . . 3
0.3.2 Construction by universal property . . . 4
0.3.3 Isomorphisms . . . 6
0.3.4 Understanding Tensor and Hom . . . 7

1 Homological algebra in R-mod **8**
1.1 Short exact sequences . . . 8
1.1.1 Split short exact sequences . . . 8
1.1.2 Non-split sequences . . . 9
1.2 Tensor and Hom . . . 11
1.2.1 Hom is left-exact . . . 11
1.2.2 Tensor is right-exact . . . 12
1.2.3 The Tensor-Hom adjunction . . . 13

2 Abelian Categories and Homology **15**
2.1 Abelian categories . . . 15
2.1.1 Categorical framework . . . 15
2.1.2 Defining the abelian category . . . 16
2.1.3 Morphisms in abelian categories . . . 16
2.1.4 Exact functors . . . 17
2.2 Homology . . . 18
2.2.1 Motivation . . . 18
2.2.2 Defining homology . . . 18
2.3 The long exact sequence in homology . . . 19

3 R-mod reprise **21**
3.1 Projective modules . . . 21
3.2 Flat modules . . . 22
3.3 Limits and colimits in R-mod . . . 24
3.3.1 Directed systems . . . 24
3.3.2 Examples of inverse and direct limits . . . 25
3.3.3 Direct limits and flatness . . . 25
3.4 Injective modules I: Baer's Criterion . . . 26
3.5 Injective modules II: Working over $\mathbb{Z}$ . . . 27

4 Resolutions and Derived Functors **31**
4.0.1 Projective and injective objects . . . 31
4.1 Resolutions . . . 32
4.1.1 Existence of resolutions . . . 32
4.1.2 Constructing resolutions in R-mod . . . 34
4.1.3 Preparing for derived functors . . . 34

4.2 Derived functors . . . 35
4.2.1 Derivations of the inverse limit functor . . . 35

5 Tor and Ext **36**
5.1 Motivation . . . 36
5.2 Defining Tor and Ext . . . 36
5.3 Total complexes . . . 36
5.4 Computing Tor and Ext . . . 36

6 Understanding Ext **37**
6.1 Extensions and the Baer sum . . . 37
6.2 The Ext ring . . . 37

A Results in abelian categories **38**

0.1 Introduction

Homological algebra has its origins in the algebraic topology and abstract algebra of the late 19th century, with the involvement of Noether, Hilbert, Poincaré and others. Its central notion is the *chain complex*, from which the concepts of *homology* and *cohomology* are derived. In the context of topology and geometry, these notions can provide the algebraic analogue of topological properties – for example, n-dimensional holes and the Euler characteristic. Indeed, it was in this setting in the early 20th century that Poincaré and other algebraic topologists first developed the theory. But it was in the mid 20th century alongside the birth of category theory that homological algebra emerged as a subject in its own right, landmarked by the influential book [**cartan-eilenberg**].

The central theme of the C3.1 Algebraic Topology course is the homology and cohomology of topological spaces. In contrast, the C2.2 Homological Algebra course studies the notion of (co)homology in its full generality; first in the setting of modules over rings, and then moving into the generality of *abelian categories*. But don't let the categorical language daunt you; a famous theorem in homological algebra (see [**weibel**] Chapter 1.6) essentially states that any result which holds in the category of R-modules alsos hold in any abelian category (for a general ring R, which will always be assumed to have unity). So even if you are working in a categorical setting in which you cannot think of objects as having elements and maps being defined on these elements - in contrast with familiar algebraic objects which have an underlying structure of a set – your intuition from algebra can serve you well. And because of its generality, homological algebra has now reached far beyond its topological origins to become totally central to modern algebraic geometry, and a powerful tool in number theory, quantum physics and beyond.

0.2 Prerequisites

The primary reference for this course will be [**weibel**]. It is comprehensive and thorough, but contains a lot of difficult and high-level material, and so the C2.2 course traces a path through mostly just the first three chapters.

Category Theory: This provides the framework for much of homological algebra. For this course however, only familiarity with a few basic notions of category theory is required (namely, the definition of a category, functor and natural transformation), and we will define any other categorical constructions involved. A suitable reference would be the appendix of [**weibel**], since it covers all such notions used in the course. For those wanting to learn more about categorical concepts like *limits* and *universal property*, [**leinster**] is well-written and an excellent source of examples, and is also available online for free on arXiv.

Algebra: A good understanding of rings and modules will be invaluable. The level required will be roughly of that offered by Oxford's A3 Rings and Modules course but there are also many good introductory books here, for example [**hartley-hawkes**]. Algebraic structures which will appear in many examples include: the ring of integers and its quotients, modules defined over polynomial rings, vector spaces and fields, and direct sums and products of these objects. Which leads us to...

0.3 Operations on R-modules

Since homological algebra works in both the concrete setting of R-modules and in the abstract setting of categories, it is useful to have both the set-theoretic and category-theoretic definitions of rudimentary operations using R-modules. These two ways of thinking each have their own benefits; while your intuition may be to approach a problem by considering the elements of objects involved and their images under functions, it can sometimes be much more natural (and efficient) to think diagramatically and in terms of universal properties. We will prove the equivalence of these two approaches at the end of Section 0.3.2, and the complementary nature of these two approaches will become apparent with the proofs of R-module isomorphisms in Section 0.3.3.

Notation and terminology: For these notes, please keep in mind the choices made as follows. R will always be assumed to be a ring with unity, but not necessarily commutative. The subset relation $\subset$ need not denote a strict containment. For any $n \in \mathbb{N}$, the notation $\mathbb{Z}/n$ will be used to denote the group of integers modulo n.

0.3.1 Construction by elements

Direct sums and products: We begin in the concrete setting. Let $\{M_i\}_{i\in\mathcal{I}}$ be a collection of left R-modules over some indexing set $\mathcal{I}$. We define the *direct sum* $\bigoplus_{i\in\mathcal{I}} M_i$ to be

$$\bigoplus_{i\in\mathcal{I}} M_i := \left\{ f : \mathcal{I} \to \bigcup_{i\in\mathcal{I}} M_i \mid f(i) \in M_i \text{ and } f(i) \neq 0 \text{ for finitely many } i \in \mathcal{I} \right\}.$$

Its R-module structure is given by $(f+g)(i) = f(i) +_{M_i} g(i)$ and $(r \cdot f)(i) = r \cdot_{M_i} (f(i))$. The direct sum is a submodule of the *direct product* $\prod_{i\in\mathcal{I}} M_i$, which differs from the direct sum only in that elements $f : \mathcal{I} \to \bigcup_{i\in\mathcal{I}} M_i$ are allowed to take infinitely many nonzero values.

Tensor products: Given a right R-module M and a left R-module N, the *tensor product* $M \otimes_R N$ over R is the abelian group generated by the symbols $m \otimes n$ where $m \in M$ and $n \in N$, under the equivalence relation given by:

$$\begin{aligned}(m+m') \otimes n &= m \otimes n + m' \otimes n \\ m \otimes (n+n') &= m \otimes n + m \otimes n' \\ mr \otimes n &= m \otimes rn\end{aligned}$$

The tensor product of R-modules will always be an abelian group, but if R is commutative then it becomes an R-module via $r \cdot (\sum m_i \otimes n_i) := \sum m_i \otimes rn_i$.[1]

For two vector spaces V and W over a field k, their tensor product has a straightforward description, and its dimension $\dim_k(V \otimes W)$ is simply given by the product of the dimensions of V and W. However, tensor products of modules can (upon first encounters) be less intuitive to work with than direct sums and products of modules. To illustrate this, consider the abelian groups (equivalently, $\mathbb{Z}$-modules[2]) $\mathbb{Z}/p$ and $\mathbb{Z}/q$ where $p, q \in \mathbb{N}$ are coprime. By the Chinese Remainder Theorem, $\mathbb{Z}/p \oplus \mathbb{Z}/q \simeq \mathbb{Z}/pq$. But we can also deduce from the equivalence relations above that $\mathbb{Z}/p \otimes \mathbb{Z}/q = 0$: use Bézout's Lemma to find $a, b \in \mathbb{Z}$ such that $ap + bq = 1$, and then any $[m] \otimes [n] \in \mathbb{Z}/p \otimes \mathbb{Z}/q$ becomes

$$[m] \otimes [n] = [m.1] \otimes [n] = [amp + bmq] \otimes [n] = [bmq] \otimes [n] = [bm] \otimes [nq] = [bm] \otimes [0] = 0.$$

[1] What happens if R is not commutative? The categorical definition of the tensor product (of which this set-theoretic definition is a special case) requires the map $M \times N \xrightarrow{\theta_r} M \otimes N$ given by $\theta_r(m,n) = m \otimes n$ to be middle-linear. In other words, we require $m \otimes rsn = \theta_r(m, sn) = \theta_r(ms, n) = ms \otimes rn$. By the third tensor product relation, this is equivalent to $m \otimes rsn = m \otimes srn$, and this will be satisfied if R is commutative. This condition is not necessary however, since $M \otimes_R N$ can be endowed with the structure of a left R-module under the weaker condition that M is an *R-bimodule*.

[2] Recall that $\mathbb{Z}$-modules are the same thing as abelian groups. This is because we can interpret the ring action $n \cdot a$ on an abelian group A as simply the sum of n copies of a in A.

Hom: The fourth construction here will be $\mathrm{Hom}_R(M,N)$, the set of R-module homomorphisms $M \to N$ for left R-modules M and N. This is an abelian group, and if R is commutative then it has the structure of a left R-module via $(r \cdot f)(m) := r \cdot (f(m))$. If R is not commutative then $\mathrm{Hom}_R(M,N)$ need not have an R-module structure (for similar reasons to the tensor product - see footnotes).[3]

Kernels and cokernels: Familiar constructions from algebra are that of the kernel and image of a map of R-modules $f : M \to N$. These are defined by $\ker(f) := \{m \in M \mid f(m) = 0\}$ and $\mathrm{im}(f) := \{n \in N \mid n = f(m),\ \text{some } m \in M\}$. Another closely related notion is the *cokernel* of f, which is defined to be $\mathrm{coker}(f) := N/\mathrm{im}(f)$. The kernel and image inherit the R-module structures of M and N respectively, and because $\mathrm{im}(f)$ is a submodule of N, we also have that $\mathrm{coker}(f)$ is a well-defined R-module.

0.3.2 Construction by universal property

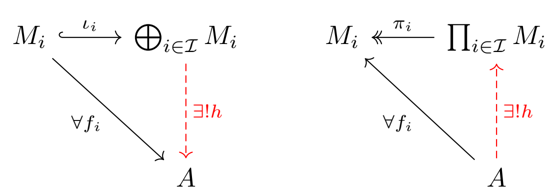
What is meant by the term *universal property*, and why is this concept so ubiquitous in modern mathematics? To understand this, I highly recommend reading the Introduction of [**leinster**] for a very accessible exposition of universal properties which contains many examples. But taking inspiration from Leinster's book, I will try and give brief answers to these two questions here.

Firstly, an object X in a category $\mathcal{A}$ is said to satisfy a universal property if it is the *unique* object in $\mathcal{A}$ which satisfies a certain *condition*, say C. This means that if you have some object $A \in \mathcal{A}$ which satisfies C, then A must be isomorphic in $\mathcal{A}$ to X. Furthermore, for any universal property, it is straightforward to show that this isomorphism must be unique.

In regards to the second question, we now know that a universal property defines an object up to unique isomorphism. This is one reason behind their importance in mathematics. For example, we know from elementary ring theory that for any ring R, there exists a unique ring homomorphism $\mathbb{Z} \to R$ (since we must map $1 \mapsto 1_R$, and then this defines the map for any $n \in \mathbb{Z}$). So if we have an unknown ring A for which know there exists a unique map $A \to R$ for every ring R, then by the universal property of $\mathbb{Z}$ in the category of rings we must have that $A \simeq \mathbb{Z}$. In this course we will meet many other familiar examples of algebraic objects which also satisfy universal properties, and this will become a powerful tool in proving statements which involve these objects.

We now give a second definition of the constructions in **0.3.1** in terms of universal properties, exhibiting them as the unique objects which make the various following diagrams commute. We will conclude **0.3.2** by proving the equivalence of these two constructions. These diagrammatical definitions will be in line with the format given in the Introduction of [**leinster**].

Direct sums and products: We start with the direct sum and product, which in the category of R-modules are the special cases of the categorical notions of *coproduct* and *product*, and are in turn instances of *colimits* and *limits*. Together with their inclusion and projection maps respectively, they satisfy the following universal property:[4]

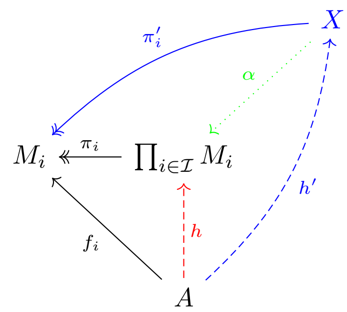


Let's unpack this definition of the direct product for a collection of R-modules $\{M_i\}_{i \in \mathcal{I}}$ to understand what is being said.[5]Its diagrammatical definition states the following two things. Firstly, for *any* R-module A and collection of maps $(f_i : A \to M_i)_{i \in \mathcal{I}}$, there exists a unique map $h : A \to \prod_{i \in \mathcal{I}} M_i$ which makes the above diagram commute (in

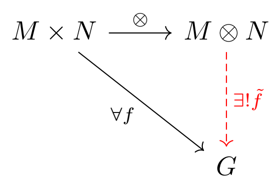
[3]For example, if $R = M_2(k)$ and $M = k^2$ then $\mathrm{Hom}_R(M,M) = k$ and so does not admit an R-module structure. Explicitly, $\mathrm{Hom}_R(M,M)$ is the subset of $\mathrm{Hom}_k(M,M) = M_2(k)$ given by imposing the condition that $r \cdot f(m) = f(rm)$ for $r \in R = M_2(k)$, $m \in M$ and $f \in \mathrm{Hom}_k(M,M) = M_2(k)$. This amounts to requiring that $r \in M_2(k)$ commutes with every matrix in $M_2(k)$ and thus $r = \lambda I$ for some $\lambda \in k$, giving us that $\mathrm{Hom}_R(M,M) = k$. This cannot have an R-module structure because it is 'too small'; let's make this precise. An R-module structure on an abelian group A is equivalent to giving a ring homomorphism $R \to \mathrm{End}(A)$. In our case this means finding a ring homomorphism $M_2(k) \to \mathrm{End(k)} = k$. This will require that $I \mapsto 1$, which cannot happen because $\begin{pmatrix}1&0\\0&0\end{pmatrix}^2 = \begin{pmatrix}0&0\\0&1\end{pmatrix}^2 = 0$ and so we have two elements of $M_2(k)$ which each map to 0 but whose sum maps to 1.

[4]The arrows $\hookrightarrow$ and $\twoheadrightarrow$ denote injections and surjections respectively. Monomorphisms and epimorphisms (which we will meet later in the course) are denoted by $\rightarrowtail$ and $\twoheadrightarrow$.

other words, $f_i = \pi_i \circ h$ for each $i \in \mathcal{I}$). Secondly, the direct product is *universal* with respect to this property, which means that if another pair $(X, \{\pi'_i\}_{i\in\mathcal{I}})$ satisfies the same above property as the direct product, then we have a unique map $\alpha : X \xrightarrow{\cong} \prod_{i\in\mathcal{I}} M_i$ which makes the following diagram commute:

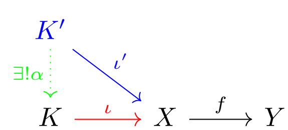


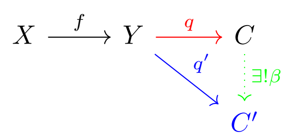
Tensor products: We now give a similar definition of the tensor product of two R-modules M and N. As mentioned in **0.3.1**, $M \otimes_R N$ may not itself be an R-module, so our diagram will instead be in the category of abelian groups. The maps $\otimes$ and f in this definition will be assumed to be *R-balanced*,[6] and $\tilde{f}$ is a group homomorphism:



Kernels and cokernels: You may have noticed that the diagrammatical definitions of the direct sum and product look very similar. In fact, if we were to reverse all the arrows in the diagram on the left, replace inclusions with projections, and change the direct sum into a direct product, we would get the diagram on the right. This is because they are *dual* notions in category theory: see [**leinster**] 1.1.9 and 1.1.10 for the principle of categorical duality.

We can construct a kernel of a map $f : X \to Y$ in terms of a universal property. In this categorical setting, the corresponding cokernel becomes a much more natural object to work with than the image[7]of f, because the cokernel is just the categorical dual of the kernel. The kernel (K, ι) and cokernel (C, q) of f are respectively defined as being universal with respect to $f \circ \iota = 0$ and $q \circ f = 0$. Thus for any other pair (K', ι') satisfying $f \circ \iota' = 0$ (resp. (C', q') satisfying $q' \circ f = 0$), there exists a unique map $\alpha : K' \to K$ (resp. $\beta : C \to C'$) making the following diagrams commute:





Remark on limits and colimits: Kernels and cokernels are respectively the *equaliser* and *coequaliser* of the pair consisting of $f : X \to Y$ and the zero morphism $0 : X \to Y$, which are in turn another instance of limits and colimits in category theory. There is a nice way to think about objects that are (co)limits, which lends intuition for using them in proofs, and is as follows. Look at the definition of the kernel (K, ι): for any other pair (K', ι') which 'wants' to be the kernel (and by this we mean that $f \circ \iota' = 0$), (K, ι) is 'better' in the sense that ι' must factor through K.

[5] The case for the direct sum is similar, and in fact differs only by having all of the arrows reversed. See the diagrammatical definition of the kernel and cokernel for a mention of categorical duality.

[6] For an abelian group G and for M and N right and left R-modules respectively, a map $f : M \times N \to G$ is *R-balanced* if it is additive in each argument and $f(m, r \cdot n) = f(m \cdot r, n)$.

[7] We usually think of the *image* of a map $f : A \to B$ as all the elements of B which come from A. But if we are in a categorical setting in which the objects A and B need not have the structure of underlying sets, we then have to find another way to define the image of f. In order for our categorical definition of the image to agree with the set-theoretic definition, we must *define* the image of f to be $\ker(\operatorname{coker}(f))$, since the image of f is precisely the kernel of the cokernel map $B \to B/\operatorname{im}(f)$.

In this way, we can think about objects which are limits in their category (like products and kernels, and terminal objects which we'll meet in Chapter 2) as being 'optimal targets', and colimits (like direct sums, cokernels and initial objects) as being 'optimal sources'. We will study some of the forms limits and colimits take in R-mod in Section 3.3 of these notes, but those interested can read more about limits and colimits in [**leinster**] Chapter 5.

Proposition 0.1. *In the category R-mod, the element-wise and universality definitions of the above constructions agree.*

Proof. X □

0.3.3 Isomorphisms

We are now ready to prove some rudimentary isomorphisms which bring together some of these R-module constructions. There will be no deep ideas required in proving these statements, so the main challenge will be to understand how the definitions given above can be used to prove the isomorphisms. The maps involved in the following proposition will all be *natual* maps. Roughly speaking, a map is natural if it arises 'naturally' from the construction of the objects involved; see [**leinster**] Section 1.3 for a precise definition of naturality and functoriality.[8]

Proposition 0.2. *Let* $\{A_i\}_{i\in\mathcal{I}}$ *and* B *be R-modules. There exist canonical isomorphisms*

$$\left(\bigoplus_{i\in\mathcal{I}} A_i\right)\otimes B \simeq \bigoplus_{i\in\mathcal{I}}(A_i\otimes B) \tag{1}$$

$$\mathrm{Hom}\left(\bigoplus_{i\in\mathcal{I}} A_i, B\right) \simeq \prod_{i\in\mathcal{I}}\mathrm{Hom}(A_i,B) \tag{2}$$

$$\mathrm{Hom}\left(B, \prod_{i\in\mathcal{I}} A_i\right) \simeq \prod_{i\in\mathcal{I}}\mathrm{Hom}(B,A_i) \tag{3}$$

Proof. These isomorphisms have straightforward universality proofs. We prove (2) and leave the others as an exercise. To proceed, consider an element $f\in\prod_{i\in\mathcal{I}}\mathrm{Hom}(A_i,B)$. This is equivalent to a family of maps $\{f_i\}_{i\in\mathcal{I}}$. By the universal property of $\bigoplus$ (see its diagrammatical definition), we have a unique $h\in\mathrm{Hom}(\bigoplus_{i\in\mathcal{I}}A_i,B)$ such that $h\circ\iota_i=f_i$ for each $i\in\mathcal{I}$.

This proves one direction. For the converse, simply map $f\mapsto\{f_i\}_{i\in\mathcal{I}}$, where $f_i:=f\circ\iota_i$. Now all that is left is the naturality of these mutually inverse maps, and this holds by the commutativity of the diagrams in **0.3.2**. □

Remarks: In the above proposition, (1) is known as the *distributivity* of the tensor product, just like the way in which multiplication distributes over addition. And looking at (2) and (3), it is no coincidence that we find the direct sum in the first argument of Hom and the direct product in the second. We can see why from the definitions in **0.3.1**: firstly, since the direct sum is a free R-module, it has a basis and so defining maps out of it is straightforward (since we just define them on the basis elements); secondly, since the direct product can have elements with infinitely many non-zero entries, it will much more often be able to receive maps than the direct sum. The definitions in **0.3.2** make these notions precise. Finally, you may notice that the Hom construction does not feature explicitly in **0.3.2** as it does in **0.3.1**: this is because any category $\mathcal{C}$ already comes equipped with sets[9]Hom(A,B) for any pair of objects A,B in $\mathcal{C}$.

It is also worth noting why in (2) the direct sum is replaced by the direct product when it moves outside of the Hom-set: let's think about the different between $\mathrm{Hom}(\bigoplus A_i,B)$ and $\bigoplus\mathrm{Hom}(A_i,B)$. Recall the definition of $\bigoplus$ from **0.3.1**: its elements are maps $f:\mathcal{I}\to\bigcup M_i$ of finite support. Thus an element of $\bigoplus\mathrm{Hom}(A_i,B)$ is a collection of maps $\{f_i:A_i\to B\}$ such that cofinitely many of these f_i *must be the zero map*. In contrast, the finiteness condition on $\mathrm{Hom}(\bigoplus A_i,B)$ is only on the domain of a map $g:\bigoplus A_i\to B$ and not on the map itself. This means that g is allowed to be nonzero on all the basis elements e_i of $\bigoplus A_i$ and so defining $g_i:=g\circ\iota_i$ gives an element of the direct

[8]Natural maps are the most general choice of map in the sense of preserving the structure of their domains; they tend to be unique, arise naturally, and avoid arbitrary choices. Because of this, they are often referred to as *canonical* maps, which roughly speaking is map which is the 'most natural choice'. A classic example is that the isomorphism of a finite-dimensional vector space V with its dual V^* is *not* canonical: the isomorphism arises from a *choice* of basis of V instead of from its structure as an abstract vector space. The isomorphism with its double-dual V^{**} however *is* canonical, since it does not depend on a choice of basis for V.

product $\prod \mathrm{Hom}(A_i, B)$ which need not be in $\bigoplus \mathrm{Hom}(A_i, B)$. For example, if we take $A_i = \mathbb{Z}/i$, $B = \bigoplus_{j\in\mathcal{I}} \mathbb{Z}/j$ and $\mathcal{I} = \{2, 3, 4, \ldots\}$, then $\mathrm{id}_B \in \mathrm{Hom}(B, B)$ is an element of $\mathrm{Hom}(B, B)$ of infinite order and thus cannot be an element of $\bigoplus \mathrm{Hom}(A_i, B)$ in which every element must have finite order.

0.3.4 Understanding Tensor and Hom

How shall we compute tensor products and homomorphism groups for R-modules in practice? The following proposition will prove invaluable in this course for calculations involving tensor and Hom with R-modules. But to have a feel of how these isomorphisms work in practice, let us first study a simple example in $\mathbb{Z}$-mod which we can solve with basic reasoning:

For an abelian group A, how do we interpret $\mathrm{Hom}(\mathbb{Z}/2, A)$? Since $\mathbb{Z}/2$ consists of two elements, and a group homomorphism $\phi : \mathbb{Z}/2 \to A$ must map $0 \mapsto 0_A$, we only have to think about what $\phi(1)$ could be. Well, since $2 \cdot 1 = 0$ in $\mathbb{Z}/2$, $\phi(1)$ must be a 2-torsion element of A. Since there are no other restrictions on ϕ for it to be a group homomorphism, we see that $\mathrm{Hom}(\mathbb{Z}_2, A) \simeq \{a \in A \mid 2a = 0\} \leqslant A$.

Now let's think about $A \otimes \mathbb{Z}/2$. Using the third relation of the tensor product defined in **0.3.1**, we see that any element of A which can be expressed as $2a$ for some $a \in A$ becomes zero in $A\otimes\mathbb{Z}/2$ because $2a\otimes[1] = a\otimes[2\cdot1] = a\otimes[0] = 0$. Since the other two tensor product relations do not give us any new information, we have that $A \otimes \mathbb{Z}/2 \simeq A/2A$.

Proposition 0.3. *Let M and N be R-modules, and let I be a left-ideal of R. There exist canonical isomorphisms*

$$M \otimes_R R/I \simeq M/MI \tag{4}$$

$$\mathrm{Hom}_R(R/I, N) \simeq \{n \in N \mid rn = 0 \ \forall r \in I\} \tag{5}$$

Proof. We provide a proof of (4) by constructing two R-module homomorphisms which we show are mutually inverse. The proof of (5) is carried out in a similar manner and is left as an exercise.[10] The naturality of these two isomorphisms is a consequence of the fact that we have not made any arbitrary choices when constructing the maps involved.[11]

We define the map $\Phi : M\otimes_R R/I \to M/MI$ by sending $\sum m_i\otimes[r_i] \mapsto [\sum m_i r_i]$. To verify that Φ is well-defined, we need to check that it is invariant under the three equivalence relations involved in $\otimes$ and also the equivalence relation in R/I. The former invariance is clear, and the latter holds because for any $a \in I$, both $m\otimes[r]$ and $m\otimes[r+a]$ map to the same element $[mr] = [m(r+a)] \in M/MI$. We define the inverse map $\Psi : M/MI \to M\otimes_R R/I$ by sending $[m] \mapsto m\otimes[1]$. This map is well-defined because for any $a \in I$, Ψ maps ma to $ma \otimes [1] = m \otimes [a] = m \otimes [0] = [0] \in M \otimes_R R/I$.

The composition $\Psi \circ \Phi = \mathrm{id}_{M\otimes_R R/I}$ since it maps $[m] \mapsto m \otimes [1] \mapsto [m]$. The composition $\Phi \circ \Psi = \mathrm{id}_{M/MI}$ because it maps $\sum m_i \otimes [r_i] \mapsto [\sum m_i r_i] \mapsto (\sum m_i r_i) \otimes [1]$, which equals $\sum m_i \otimes [r_i]$ by the equivalence properties of the tensor product. ∎

[9] All categories we will meet in this course (and most of which occur in homological algebra) will be *locally small* categories, which means that Hom(A,B) is actually a *set* (as opposed to *small* categories, which requires that obj($\mathcal{C}$) is also a set), even if A and B need not have underlying set structures. Thus we can safely call these objects 'Hom-sets'.

[10] Hint: the map which sends $f \mapsto f(1)$ provides the isomorphism.

[11] To be precise, the naturality of a map $\alpha_M : M \otimes_R R/I \to M/MI$ would require that any map $f : M \to N$ induces a map $\tilde{f} : M \otimes_R R/I \to N \otimes_R R/I$ such that $\tilde{f} \circ \alpha_M = \alpha_N \circ f$. It is straightforward to check that the maps Φ, Ψ indeed satisfy this property.

Chapter 1

Homological algebra in R-mod

The full generality of homological algebra studies *chain complexes* and *exact sequences* in the setting of abelian categories, but Chapter 1 will be devoted to the manifestation of these ideas in the category of R-modules, since much of the course will take place in this setting. For a $\mathbb{Z}$-indexed sequence of R-modules (of which infinitely many may be zero)

$$\ldots \longrightarrow M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \longrightarrow \ldots \tag{1.1}$$

we say this sequence is *exact at* M_n if $\ker(f_n) = \mathrm{im}(f_{n+1})$, and the sequence itself is *exact* if this equality holds at each n. The notion of exactness in homological algebra is so important that we shall begin by studying a useful special case of sequences of R-modules called *short exact sequences*.

1.1 Short exact sequences

Consider a sequence of R-modules:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1.2}$$

We call this a *short exact sequence* of R-modules if f is injective, g is surjective, and $\ker(g) = \mathrm{im}(f)$. This name is justified by the above definition of exactness because the injectivity of f is equivalent to the exactness of the sequence at A, and the surjectivity g is equivalent to exactness at C. An equivalent definition of the sequence (1.2) to be exact would be to require that $(A, f) = \ker(g)$ and $(C, g) = \mathrm{coker}(f)$, since the former is equivalent to the injectivity of f together with exactness at B, and the latter is equivalent to the surjectivity of g together with exactness at B. To develop a familiarity with short exact sequences (which occur throughout homological algebra), let us look at some examples.

1.1.1 Split short exact sequences

An inclusion of R-modules $N \subset M$ is called *split* if there exists $N' \subset M$ such that $M = N \oplus N'$. For example, $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/6$ is split since $\mathbb{Z}/6 \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/3$ via the map $1 \mapsto (1, 1)$. So, our first type of short exact sequence will be those in which B can be thought of as the direct sum $A \oplus C$. In precise terms, the sequence (1.2) is a *split short exact sequence* if there is an isomorphism h which makes the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & A & \hookrightarrow & A \oplus C & \twoheadrightarrow & C & \longrightarrow & 0 \end{array} \tag{1.3}$$

So for any R-modules A and C, we can fit them into a short exact sequence by taking $B = A \oplus C$. However, it need not be obvious to see when a short exact sequence splits, so we will often use the following lemma:[1]

[1] A retraction $r : B \to A$ is a map which satisfies $r \circ f = 1_A$. Dually, a section is a map $s : C \to B$ which satisfies $g \circ s = 1_C$.

Lemma 1.1. *A short exact sequence of R-modules* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is split* $\Leftrightarrow$ *there exists a retraction* $r : B \to A$ $\Leftrightarrow$ *there exists a section* $s : C \to B$.

Proof. We first assume the existence of a retraction $r : B \to A$ to construct a section $s : C \to B$. Consider the map $1_B - fr : B \to B$. This map is zero on $f(A) \subset B$ and thus gives a well-defined map $B/f(A) \to B$. By exactness, $C \simeq B/f(A)$ and so we have a map $s : C \to B$. Now let us show that s is indeed a section by showing that $gs(c) = (c)$ for any $c \in C$. By surjectivity of g, we may choose $b \in B$ such that $g(b) = c$. And since $g \circ f = 0$:

$$gs(c) = g(b - fr(b)) = g(b) - gfr(b) = g(b) = c$$

We similarly show that the existence of a section $s : C \to B$ implies the existence of a retraction $r : B \to A$. The map $1_B - sg : B \to B$ composes with $g : B \to C$ to give zero since s is a section, and thus by exactness its image lands in $ker(g) = f(A) \subset B$. Thus we have a well defined map $r = f^{-1}(1 - sg)$ which is indeed a retraction since $g \circ f = 0$:

$$rf(a) = f^{-1}(1 - sg) \circ f(a) = a$$

We have now established the second equivalence in the lemma. The forward direction of the first equivalence is immediate, since the direct sum is equipped with projection and inclusion maps which are respectively a retraction and section. For the converse, let us assume the existence of a retraction $r : B \to A$ and section $s : C \to B$. It then follows that $B \simeq A \oplus C$ via the mutually inverse maps $b \mapsto (r(b), g(b))$ and $(a, c) \mapsto f(a) + s(c)$. ☐

Remark: This proof makes extensive use of the fact that objects in R-mod have elements on which maps are defined, and so this proof does not generalise to categorical settings in which the objects of a category need not have the structure of an underlying set. However, we can without much work rephrase the proof in categorical terms with kernels and cokernels, which will allow the existence of a section to follow immediately from the existence of a retraction by categorical duality.

Example: Let's use Lemma 1.1 in $\mathbb{Z}$-mod to prove the splitting of the short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} \mathbb{Z} \longrightarrow 0 \tag{1.4}$$

for arbitrary abelian groups A and B. By Lemma 1.1, we need to find a section[2] $s : \mathbb{Z} \to B$. Since g is surjective, there exists $b \in B$ such that $g(b) = 1$. Construct $s : \mathbb{Z} \to B$ by setting $s(1) = b$, which in turn determines $s(n) = n \cdot b$ for each $n \in \mathbb{Z}$. Then $g \circ s(1) = g(b) = 1$ so $g \circ s = id_{\mathbb{Z}}$ and we are done.

1.1.2 Non-split sequences

Examples: By choosing appropriate maps on the basis elements, any short exact sequence of free modules (and thus vector spaces) splits. But not all sequences split! In contrast to the example above with $\mathbb{Z}/6$, there is no isomorphism $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/4$ since the domain has no element of order 4, and so we get our first example in $\mathbb{Z}$-mod of a non-split short exact sequence:

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{1 \mapsto 1} \mathbb{Z}/2 \longrightarrow 0 \tag{1.5}$$

More generally for p prime, the following short exact sequence does not split by the same argument:

$$0 \longrightarrow \mathbb{Z}/p^{n-1} \longrightarrow \mathbb{Z}/p^n \longrightarrow \mathbb{Z}/p \longrightarrow 0 \tag{1.6}$$

Now let us consider the ring $R = \mathbb{R}[x]/x^n$. If the inclusion $\iota : R/x \hookrightarrow R$ were split[3], then we would have $R = R/x \oplus M$ for some R-module M. By Lemma 1.1, this would imply the existence of a retraction $r : R \twoheadrightarrow R/x$ with $r \circ \iota(1) = r(x^{n-1}) = 1$, giving us the contradiction $1 = r(x \cdot x^{n-2}) = x \cdot r(x^{n-2}) = 0$. Thus we have the following short exact sequence which does not split:

$$0 \longrightarrow R/x \xrightarrow{\cdot x^{n-1}} R \xrightarrow{\cdot x} R/x^{n-1} \longrightarrow 0 \tag{1.7}$$

[2] Why do we choose to look for a section instead of trying to find a retraction? Since $\mathbb{Z}$ is a free $\mathbb{Z}$-module, defining a map out of $\mathbb{Z}$ just requires us to define where we send $1 \in \mathbb{Z}$. So defining a section out of $\mathbb{Z}$ should be straightforward.

Next, consider a ring R, which is itself always an R-module with submodules precisely its ideals. For two coprime ideals $I, J \triangleleft R$ such that $I \cap J \neq \emptyset$, we can form the following short exact sequence with maps given by $x \mapsto (x, -x)$ and $(x, y) \mapsto x + y$, which does not split:

$$0 \longrightarrow I \cap J \longrightarrow I \oplus J \longrightarrow R \longrightarrow 0 \tag{1.8}$$

Key example: We now look at a considerably more involved example, the kind of which occurs frequently in homological algebra. Consider the k-algebra[4] $R = k[x, y]$, and R-modules $M_1 := R^2/\langle (x, 0), (y^2, -x), (0, y)\rangle$ and $M_2 := R/\langle x^2, xy, y^3\rangle$. Let us try and find example of an R-module M which fits into a non-split short exact sequence as follows:

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0 \tag{1.9}$$

To find such an M in a general way, let us first examine the R-modules M_1 and M_2. Since these are modules over the ring $k[x, y]$, they admit the structure of a k-vector space. And as k-vector spaces, they have bases $\{(1, 0), (y, 0), (y^2, 0), (0, 1)\}$ and $\{1, x, y, y^2\}$ respectively, where in the basis for M_1 we use the relation $(y^2, -x) = 0 \Rightarrow (y^2, 0) = (0, x)$. Also note that in M_1, $(0, x^2) = 0$, since

$$(0, x^2) = x \cdot (0, x) = x \cdot (y^2, 0) = y \cdot (xy, 0) = 0.$$

So, we know that $\dim_k M_1 = \dim_k M_2 = 4$, and thus by rank-nullity we must have from (1.9) that $\dim_k M = 8$ if such an M is to exist. Now that we have an idea about the size of M, how shall we construct it? To understand the actual structures of the modules which we're working with, we give a visual representation of the ring $k[x, y]$ which can be extremely helpful in these kinds of problems:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
x^2 & \xrightarrow{\cdot y} & x^2y & \xrightarrow{\cdot y} & x^2y^2 & \longrightarrow & \dots \\
\uparrow{\scriptstyle \cdot x} & & \uparrow{\scriptstyle \cdot x} & & \uparrow{\scriptstyle \cdot x} & & \\
x & \xrightarrow{\cdot y} & xy & \xrightarrow{\cdot y} & xy^2 & \longrightarrow & \dots \\
\uparrow{\scriptstyle \cdot x} & & \uparrow{\scriptstyle \cdot x} & & \uparrow{\scriptstyle \cdot x} & & \\
1 & \xrightarrow{\cdot y} & y & \xrightarrow{\cdot y} & y^2 & \longrightarrow & \dots
\end{array} \tag{1.10}$$

In this diagram, the nodes represent the basis elements of $k[x, y]$, with the arrows representing the action of x and y on these basis elements: vertical arrows are multiplication by x and horizontal arrows are multiplication by y. We now depict M_1 and M_2 in this way:

$$\begin{array}{ccccccccccc}
(1,0) & \xrightarrow{\cdot y} & (y,0) & \xrightarrow{\cdot y} & (y^2,0) & \quad & x & & & & \\
 & & & & \uparrow{\scriptstyle \cdot x} & & \uparrow{\scriptstyle \cdot x} & & & & \\
 & & & & (0,1) & & 1 & \xrightarrow{\cdot y} & y & \xrightarrow{\cdot y} & y^2
\end{array} \tag{1.11}$$

The nodes where the arrows stop in (1.11) indicates that the action of the ring has become multiplication by zero. To find an R-module M which fits into the non-split short exact sequence (1.9), we need to combine these two diagrams to construct M in such a way that: (1) $\dim_k M = 8$ by the above argument, (2) M contains M_1 as a submodule since $M_1 \simeq \text{im}(M_1) \leqslant M$, and (3) M is not isomorphic to $M_1 \oplus M_2$. Diagramatically, this means fitting together the two diagrams in (1.11) in such a way that we get a well-defined R-module M which satisfies these three properties. Here is an example:

[3] The inclusion map must be given by $1 \mapsto x^{n-1}$. This is because the action by x sends $x \cdot 1 = 0$ in R/x, and so the image of 1 must also be sent to 0 by x.

[4] So R is both a ring *and* a k-vector space (which in this case is of infinite dimension over the field k).

$$\begin{array}{ccccccc} x & \xrightarrow{\cdot y} & xy & \xrightarrow{\cdot y} & xy^2 & \xrightarrow{\cdot y} & xy^3 \\ {\scriptstyle \cdot x}\uparrow & & {\scriptstyle \cdot x}\uparrow & & {\scriptstyle \cdot x}\uparrow & & {\scriptstyle \cdot x}\uparrow \\ 1 & \xrightarrow{\cdot y} & y & \xrightarrow{\cdot y} & y^2 & \xrightarrow{\cdot y} & y^3 \end{array} \tag{1.12}$$

Let's check that this does indeed satisfy properties (1)-(3) above. Firstly, it does indeed have 8 basis elements as a vector space over k. Secondly, it contains M_1 as a submodule (shown in red) because multiplication by x and y along the arrows within this copy of M_1 does not take us outside of this submodule. Lastly, we could show that no section $s : M_2 \to M$ exists, but it is quicker to observe that M cannot be isomorphic to $M_1 \oplus M_2$ because the elements 1 and x of M are not sent to zero under multiplication by y^3, but there is no element in $M_1 \oplus M_2$ with this property, so their structures cannot be identical.

The above construction of M yields the R-module $M = R/\langle x^2, y^4\rangle$, but there are many more such examples which work here! Simply attaching the diagrams of M_1 and M_2 in different ways can produce different examples of M (and the case where there are *no* green arrows recovers the case $M \simeq M_1 \oplus M_2$). But there's one thing to keep in mind: if we were to remove any of the vertical green arrows in (1.12) then we would no longer have an R-module. These vertical green arrows ensure that the actions of x and y are well defined, since if we were to remove the arrow $y \to xy$ then x would act as zero on the basis element y, which would give that $y \cdot x \cdot 1 = xy \neq 0 = x \cdot y \cdot 1$ and thus multiplication from the ring is no longer well-defined.

1.2 Tensor and Hom

The two functors we will meet in Chapter 1 are the nothing but the tensor product and Hom operations on R-modules viewed as functors $R\text{-mod} \to \mathbf{Ab}$. They are defined as follows:

$$M \otimes_R - : R\text{-mod} \to \mathbf{Ab} \qquad A \mapsto M \otimes_R A \tag{1.13}$$
$$\mathrm{Hom}_R(N, -) : R\text{-mod} \to \mathbf{Ab} \qquad A \mapsto \mathrm{Hom}_R(N, A) \tag{1.14}$$

For this definition to make sense, we take M to be a right R-module and N to be a left R-module, so that the actions of $M \otimes_R -$ and $\mathrm{Hom}_R(N, -)$ are well defined on the category R-mod of left R-modules. Both functors above are *covariant*,[5]but we could equally use the *contravariant* functor $\mathrm{Hom}_R(-, N)$.

Since $M \otimes_R -$ and $\mathrm{Hom}_R(N, -)$ respect identity morphisms and composition of morphisms, they satisfy the properties of being a functor.[6]Tensor and Hom are ubiquitous in modern algebra, and because the notion of exactness is so important in homological algebra we would like to understand the interaction of these two functors with exact sequences.

1.2.1 Hom is left-exact

Let's now examine the relationship between the Hom functors $\mathrm{Hom}_R(N, -) : R\text{-mod} \to \mathbf{Ab}$ and $\mathrm{Hom}_R(-, N) : R\text{-mod}^{op} \to \mathbf{Ab}$ and short exact sequences in the case when $R = \mathbb{Z}$. We'll refer to objects as being in 'positions' A, B or C if they correspond to the positions of A, B or C in the sequence $0 \to A \to B \to C \to 0$.

Consider the short exact sequence of abelian groups:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow\!\!\!\!\rightarrow \mathbb{Z}/2 \longrightarrow 0 \tag{1.15}$$

[5]A functor $F : \mathcal{C} \to \mathcal{D}$ is called *covariant* if morphisms $f : A \to B$ are sent to morphisms $F(f) : F(A) \to F(B)$. $G : \mathcal{C} \to \mathcal{D}$ is a *contravariant* functors if morphisms $f : A \to B$ are sent to morphisms $G(f) : G(B) \to G(A)$. In other words, we just reverse the arrows. Contravariant functors $\mathcal{C} \to \mathcal{D}$ are usually denoted as functors $\mathcal{C}^{op} \to \mathcal{D}$, and we drop the terms covariant and contravariant.

[6]Recall that functors are defined on the level of both objects *and* morphisms, though often the effect on morphisms can be inferred from seeing where objects are sent to. Thus a morphism $f : A \to B$ is sent under $M \otimes_R -$ to the map $m \otimes_R a \mapsto m \otimes_R f(a)$, and is sent under $\mathrm{Hom}_R(N, -)$ to the map which post-composes a morphism $N \to A$ with $f : A \to B$. Dually, $\mathrm{Hom}_R(-, N)$ sends a morphism $g : B \to A$ to the map which pre-composes a morphism $A \to N$ with $g : A \to B$.

Applying $\mathrm{Hom}(\mathbb{Z}/2,-)$ and $\mathrm{Hom}(-,\mathbb{Z}/2)$ to this sequence yield the following two sequences, which are no longer exact:

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$
$$0 \longleftarrow \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{\simeq} \mathbb{Z}/2 \longleftarrow 0 \tag{1.16}$$

Notice however that exactness only fails at position C, where in both cases the kernel of $\mathbb{Z}/2 \to 0$ is $\mathbb{Z}/2$ but the image of $0 \to \mathbb{Z}/2$ is 0 and thus are not equal. In fact, if we were to try any short exact sequence in R-mod, Hom would only fail to preserve the exactness of position C: see Proposition 1.1 below.

This kind of functor is called *left-exact*, which in the case of $\mathrm{Hom}_R(N,-)$ means that for any short exact sequence $0 \to A \to B \to C \to 0$, the sequence $0 \to \mathrm{Hom}_R(N,A) \to \mathrm{Hom}_R(N,B) \to \mathrm{Hom}_R(N,C)$ is exact. In other words, $\mathrm{Hom}_R(N,B) \to \mathrm{Hom}_R(N,C)$ need no longer be surjective.

Proposition 1.1. *The functors* $\mathrm{Hom}_R(N,-) : R\text{-mod} \to \mathbf{Ab}$ *and* $\mathrm{Hom}_R(-,N) : R\text{-mod}^{op} \to \mathbf{Ab}$ *are left-exact.*

Proof. Consider the following short exact sequence in R-mod:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1.17}$$

To prove that $\mathrm{Hom}_R(N,-)$ is left-exact, we need show the exactness of the following sequence in $\mathbf{Ab}$:

$$0 \longrightarrow \mathrm{Hom}_R(N,A) \xrightarrow{\tilde{f}} \mathrm{Hom}_R(N,B) \xrightarrow{\tilde{g}} \mathrm{Hom}_R(N,C) \tag{1.18}$$

Here, $\tilde{f}$ and $\tilde{g}$ are the images of f and g under $\mathrm{Hom}_R(N,-)$. In other words, $\tilde{f} = f \circ -$ and $\tilde{g} = g \circ -$ are just post-composition by f and g. To show that (1.18) is exact, we have to check exactness at positions A and B, which means checking that $\tilde{f}$ is injective and $\ker(\tilde{g}) = \mathrm{im}(\tilde{f})$.

Suppose $\tilde{f}(\alpha) = f \circ \alpha = 0$ for some $\alpha : N \to A$. Then by the injectivity of f, α must be the zero map (since $f(\alpha(n)) = 0$ for any $n \in N$) and so $\tilde{f}$ is injective. Now, $\mathrm{im}(\tilde{f}) \subset \ker(\tilde{g})$ follows immediately from the fact that $g \circ f = 0$ by the exactness of (1.17). For the reverse inclusion, suppose $\beta \in \mathrm{Hom}_R(N,B)$ is such that $\tilde{g}(\beta) = g \circ \beta = 0$. By the universal property of the kernel, the map $\beta : N \to B$ must factor through $\ker(g) = A$ and so $\beta = f \circ \alpha = \tilde{f}(\alpha)$ for some $\alpha : N \to A$. Thus $\ker(\tilde{g}) \subset \mathrm{im}(\tilde{f})$ and we are done. The case for $\mathrm{Hom}_R(-,N)$ is dual, and uses cokernels and the surjectivity of g.

□

Remark: A very similar argument proves the converse, which is that the exactness of (1.18) for every R-module N implies that (1.17) is left-exact. Indeed, since $\bar{f}$ is injective, we have that if $f \circ \alpha : N \to B$ is zero then $\alpha : N \to A$ must be zero. Since N and α are arbitrary, for each $a \in A$ we can take $N = R$ and $\alpha(1) = a$ to see that f itself must be injective. Exactness at position B follows from using this same trick.

1.2.2 Tensor is right-exact

Let $R = k[x]$ for some field k. Suppose we have a short exact sequence of R-modules as follows:

$$0 \longrightarrow k[x] \xrightarrow{\cdot x} k[x] \twoheadrightarrow k \longrightarrow 0 \tag{1.19}$$

As an abelian group, k is isomorphic to $R/(x-a)$ for any $a \in k$. However, for this sequence to be exact we need x to act as 0 on k, and thus we think of k as the quotient module $R/(x)$. Applying the functor $k \otimes_R -$ gives the following sequence:

$$0 \longrightarrow k \xrightarrow{0} k \xrightarrow{\simeq} k \longrightarrow 0 \tag{1.20}$$

This sequence is no longer exact at position A, and so we see that the functor $M \otimes_R -$ does not preserve exact sequences for a general R-module M. However, this is the only position of a short exact sequence in which Tensor fails to preserve exactness. To show that the tensor product functor is indeed *right-exact*, we will first give a proof which will be elementary at the cost of being slightly tedious. We will then prove the useful *Tensor-Hom adjunction* and subsequently give a neater proof of Proposition 1.2.

Proposition 1.2. *The functor* $M \otimes_R - : R\text{-mod} \to \mathbf{Ab}$ *is right-exact.*

Proof. Consider the following short exact sequence in R-mod:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1.21}$$

To prove that $M \otimes_R -$ is right-exact, we'll need to show the exactness of the following sequence in $\mathbf{Ab}$:

$$M \otimes_R A \xrightarrow{\tilde{f}} M \otimes_R B \xrightarrow{\tilde{g}} M \otimes_R C \longrightarrow 0 \tag{1.22}$$

The maps $\tilde{f}$ and $\tilde{g}$ are the images of f, g under the functor $M \otimes_R -$, which means that $\tilde{f} : M \otimes_R A \to M \otimes_R B$ is given by $\tilde{f}(\sum m_i \otimes a_i) = \sum m_i \otimes f(a_i)$ and similarly for g. The surjectivity of $\tilde{g}$ is straightforward, since for any $\sum m_i \otimes c_i \in M \otimes_R C$ we can just use the surjectivity of g to pick preimages $b_i \mapsto c_i$ which will give us that $g(\sum m_i \otimes b_i) = \sum m_i \otimes g(b_i) = \sum m_i \otimes c_i$. The functoriality[7] of $M \otimes_R -$ ensures that $\text{im}(\tilde{f}) \subset \ker(\tilde{g})$, and so to prove the exactness of (1.22) we are left to check that any element $\sum m_i \otimes b_i$ which is mapped to 0 by $\tilde{g}$ must be in the image of $\tilde{f}$. So, if $\sum m_i \otimes b_i \mapsto 0$, then simply by the definition of the tensor product in **0.3.1** there exist elements $c'_\alpha, c''_\alpha, c_\beta, c_\gamma, m_\alpha, m'_\beta, m''_\beta, m_\gamma, r_\gamma$ such that

$$\begin{gathered}\sum_i m_i \otimes g(b_i) + \sum_\alpha [m_\alpha \otimes (c'_\alpha + c''_\alpha) - m_\alpha \otimes c'_\alpha - m_\alpha \otimes c''_\alpha] \\ + \sum_\beta [(m'_\beta + m''_\beta) \otimes c_\beta - m'_\beta \otimes c_\beta - m''_\beta \otimes c_\beta] + \sum_\gamma [m_\gamma r_\gamma \otimes c_\gamma - m_\gamma \otimes r_\gamma c_\gamma]\end{gathered} \tag{1.23}$$

is equal to zero in the free abelian group F on the set of symbols $S = \{m \otimes c \mid m \in M \text{ and } c \in C\}$. Quotienting F by the first set of relations $m \otimes (c' + c'') = m \otimes c' + m \otimes c''$ gives the abelian group $\mathfrak{C} := \bigoplus_{m \in M} C$, where an element c in the m^{th} copy of C corresponds to the element $m \otimes c \in F$, and addition in $\mathfrak{C}$ is given by the relation we have quotiented by. Thus $\sum m_i \otimes b_i \mapsto 0$ implies that

$$\sum_i m_i \otimes g(b_i) + \sum_\beta [(m'_\beta + m''_\beta) \otimes c_\beta - m'_\beta \otimes c_\beta - m''_\beta \otimes c_\beta] + \sum_\gamma [m_\gamma r_\gamma \otimes c_\gamma - m_\gamma \otimes r_\gamma c_\gamma] = 0 \in \mathfrak{C} \tag{1.24}$$

Now, using the surjectivity of g we can pick preimages $b_\beta, b_\gamma \in B$ of $c_\beta, c_\gamma \in C$ and form the following element of $\mathfrak{B}$:

$$\sum_i m_i \otimes b_i + \sum_\beta [(m'_\beta + m''_\beta) \otimes b_\beta - m'_\beta \otimes b_\beta - m''_\beta \otimes b_\beta] + \sum_\gamma [m_\gamma r_\gamma \otimes b_\gamma - m_\gamma \otimes r_\gamma b_\gamma] \tag{1.25}$$

So, why have we gone through all this trouble to work with such unwieldy abelian groups? Becuase the exactness of (1.21) implies the exactness of the following sequence:

$$0 \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{B} \longrightarrow \mathfrak{C} \longrightarrow 0 \tag{1.26}$$

The maps here are just given by the direct sums of the maps in (1.21). Thus $y \in \mathfrak{B}$ is in the image of some element $x \in \mathfrak{A}$ since y maps to zero in $\mathfrak{C}$. Let $[x], [y]$ respectively denote the images of x, y in $M \otimes_R A$ and $M \otimes_R B$ respectively. Then since $x \mapsto y$, we have that $[x] \mapsto [y]$ and we are done. ∎

1.2.3 The Tensor-Hom adjunction

Proposition 1.3. X

Proof. X ∎

[7] For a functor $F : \mathcal{C} \to \mathcal{D}$, using the *functoriality* of F means that we're using the property $F(g \circ f) = F(g) \circ F(f)$ in the definition of a functor, where $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$ are arbitrary.

Second proof of Proposition 1.2: As noted in Proposition 1.3, we have the following natural isomorphism in **Ab**:

$$\mathrm{Hom}(X \otimes_R Y, Z) \simeq \mathrm{Hom}(Y, \mathrm{Hom}_R(X, Z)) \tag{1.27}$$

The proof of the right-exactness of $M \otimes_R -$ is then given as follows. The sequence (1.21) is exact, and so Proposition 1.1 tells us that when we apply the contravariant functor $\mathrm{Hom}(-, \mathrm{Hom}_R(M, K))$ we get the following exact sequence in **Ab** for any R-module K:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(C, \mathrm{Hom}_R(M,K)) & \longrightarrow & \mathrm{Hom}(B, \mathrm{Hom}_R(M,K)) & \longrightarrow & \mathrm{Hom}(A, \mathrm{Hom}_R(M,K)) \\ & & \big\downarrow \simeq & & \big\downarrow \simeq & & \big\downarrow \simeq \\ 0 & \longrightarrow & \mathrm{Hom}(C \otimes_R M, K) & \longrightarrow & \mathrm{Hom}(B \otimes_R M, K) & \longrightarrow & \mathrm{Hom}(A \otimes_R M, K) \end{array} \tag{1.28}$$

We can now use the remark given after Proposition 1.1 to deduce the exactness of the sequence $M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$. As a consequence of the construction and subsequent naturality of the isomorphism (1.27) as shown in Proposition 1.3, the resulting maps in the bottom row of (1.28) will indeed be the same as those in (1.22) and so we have proven the exactness of (1.22) done.

Chapter 2

Abelian Categories and Homology

Roughly speaking, a category $\mathcal{A}$ is an *abelian category* if we can add together morphisms and objects, and we have the existence of kernels and cokernels which are well-behaved. The structure of an abelian category is motivated by that of the category **Ab** of abelian groups, and more generally the category R-mod. Indeed, the notion of an abelian category extracts precisely those properties of **Ab** which will be crucial for homological algebra. These categories are *stable* under many useful operations: if $\mathcal{A}$ is abelian, then so is the opposite category $\mathcal{A}^{op}$, the category of chain complexes $\mathbf{Ch}(\mathcal{A})$, the functor category $[\mathcal{C}, \mathcal{A}]$ for any small category $\mathcal{C}$, and so on.

There exists a bridge between the abstract setting of abelian categories and the concrete setting of R-modules, given by the Freyd-Mitchell Embedding Theorem (1964), which states that any small abelian category $\mathcal{A}$ can be embedded into R-mod for some ring R, in the sense that there exists an exact and fully faithful functor $\mathcal{A} \to R$-mod such that $\mathrm{Hom}_{\mathcal{A}}(M, N) \simeq \mathrm{Hom}_R(M, N)$. See [**weibel**] Sections 1.6 and A.2 for more details.

2.1 Abelian categories

Section **2.1.1** introduces terminology from category theory, together with many illustrative examples which may be helpful for those unfamiliar with these notions. Ultimately, what we need to understand is just the definition of the abelian category in **2.1.2**, and the examples in **2.1.1** are merely given to help develop intuition for the categorical ideas introduced here.

2.1.1 Categorical framework

Objects: Let $\mathcal{C}$ be a category. An object $X \in \mathcal{C}$ is said to be an *initial object* of $\mathcal{C}$ if for every $A \in \mathcal{C}$, there exists a unique morphism $X \to A$. This means that $\mathrm{Hom}(X, A)$ has precisely one element. Dually, $Y \in \mathcal{C}$ is a *terminal object* of $\mathcal{C}$ if for every $B \in \mathcal{C}$, there exists a unique morphism $B \to Y$, so that $\mathrm{Hom}(B, Y)$ has precisely one element. By the uniqueness of the morphisms in the definitions of initial and terminal objects X and Y, it then follows that X and Y must be unique up to unique isomorphism. When X and Y coincide, we have the notion of a *zero object* $Z \in \mathcal{C}$, which is both initial and terminal in $\mathcal{C}$. See [**leinster**] Section 1.1 for the definitions of some of the categories involved in the following examples.

Examples: In the category **Set**, every object $A \in \mathbf{Set}$ admits a morphism $A \to 1$ to the set with one element, which is necessarily unique since every object must get mapped to the unique element of 1. Conversely, the empty set admits a unique map $\emptyset \to B$ to every $B \in \mathbf{Set}$, which is known as the 'empty map'. Thus we have our first example of an initial and terminal object in a category.

In the category $\mathbf{Vect}_k$, the 0-dimensional vector space $\{0\}$ admits a unique map $\{0\} \to V$ to any vector space V, given by the zero map. But the zero map is also the unique map from any vector space $W \to \{0\}$, and thus $\{0\}$ is the zero object of $\mathbf{Vect}_k$. The case for **Grp** is identical, whereas in **Ring** the initial element is $\mathbb{Z}$ and the terminal element is the one-element ring.

Morphisms: A morphism $f : X \to Y$ in a category $\mathcal{C}$ is a *monomorphism* if for any $A \in \mathcal{C}$ and pair of maps $g_1, g_2 : A \to X$, we have that $f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$. Dually, a morphism $f : X \to Y$ in $\mathcal{C}$ is an *epimorphism* if for any $B \in \mathcal{C}$ and pair of maps $h_1, h_2 : Y \to B$, we have that $h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$.

An aside on morphisms: Monomorphisms and epimorphisms are the categorical generalisation of injective and surjective maps, the latter of which are defined using elements of sets. In familiar algebraic categories like R-mod and $\mathbf{Vect}_k$ whose objects have underlying sets, injections are monomorphisms and surjections are epimorphisms.[1]However, the converse need not hold! In **Ring**, the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is clearly not surjective, but it is an epimorphism since ring homomorphisms out of $\mathbb{Q}$ are determined by their values on integers. Another example would the the inclusion $\mathbb{Q} \hookrightarrow \mathbb{R}$ in the category of Hausdorff topological spaces with morphisms given by continuous maps, since any continuous map on $\mathbb{R}$ with the usual topology is determined by its values on $\mathbb{Q}$.

Monomorphisms which are not injective are slightly more elusive. One reason why is because in familiar categories involving objects with underlying sets, the existence of a free object on the one-element set implies that monomorphisms are injective, and such free objects often exist.[2]But such non-injective monomorphisms do exist: the projection map $\pi : \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ in the category of divisible abelian groups is in fact a monomorphism. Let's examine why:

Suppose we have maps $f, g : G \to \mathbb{Q}$ in this category, such that $\pi \circ f = \pi \circ g$. Then for any $x \in G$, $f(x) - g(x) \in \ker(\pi)$ and so $f(x) - g(x) = n$ for some $n \in \mathbb{Z}$. Suppose for contradiction that $n \neq 0$. Then choose $y \in G$ such that $2ny = x$, which exists by the divisibility of G. Then $f(x) = 2nf(y)$ so $f(y) = \frac{1}{2n} f(x)$, and similarly $g(y) = \frac{1}{2n} g(x)$. Now, $f(y) - g(y) \in \ker(\pi)$ and so must equal an integer. But $f(y) - g(y) = \frac{1}{2n}(f(x) - g(x)) = \frac{1}{2}$ which gives the required contradiction. Hence $n = 0$ and so $f = g$, which means that $\pi : \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ is a monomorphism.

2.1.2 Defining the abelian category

A category $\mathcal{A}$ is a *pre-additive* category if every Hom-set has the structure of an abelian group, and for any $X, Y, Z \in \mathcal{A}$ the composition $\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is additive in both arguments.[3]$\mathcal{A}$ will further be called *additive* if it also admits a zero object and all finite direct sums (equivalently, all finite direct products). Finally, $\mathcal{A}$ is an *abelian* category if it is additive, and the following holds: every monomorphism $f : X \rightarrowtail Y$ is the kernel of the morphism $Y \to \mathrm{coker}(f)$, and every epimorphism $g : Y \twoheadrightarrow Z$ is the cokernel of the morphism $\ker(g) \to Y$.

Remark: Intuitively, kernels and cokernels are 'well-behaved' in abelian categories. For the analysts among you, an example of when this may not happen is in the additive category of separated topological vector spaces over $\mathbb{R}$: consider the continuous and 2-integrable functions over [0,1] in the sequence

$$\mathcal{C}[0,1] \hookrightarrow \mathcal{L}^2[0,1] \longrightarrow 0 \tag{2.1}$$

Then 0 is indeed the cokernel of the inclusion map because $\mathcal{C}[0,1]$ is dense in $\mathcal{L}^2[0,1]$, but the the kernel of the second map is all of $\mathcal{L}^2[0,1]$. Another example would be the category of even-dimensional vector spaces over a field: this category is additive, but there will not exist kernels for maps of odd rank.

2.1.3 Morphisms in abelian categories

Before studying homological algebra in abelian categories, we first prove some basic results regarding the behaviour of morphisms in this setting. These will be invaluable tools for the journey ahead, and studying these proofs may provide good practice for working with morphisms in abelian categories. The following results will hold in any additive category $\mathcal{A}$ (and thus in particular if $\mathcal{A}$ is abelian).

[2]If $f : X \to Y$ is injective, then $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for any $x_1, x_2 \in X$. Thus if we have maps $g_1, g_2 : A \to X$ such that $f \circ g_1 = f \circ g_2$, then for any $a \in A$ we must have $f(g_1(a)) = f(g_2(a)) \Rightarrow g_1(a) = g_2(a)$ by the injectivity of f. Thus $g_1 = g_2$ and so f is a monomorphism. The case for surjectivity is dual.

[2]To see why, let F_x be the free object on the set $\{x\}$, and let $\alpha : X \to Y$ be a monomorphism. If $\alpha(a) = \alpha(b)$ for some $a, b \in X$, then the maps $f, g : F_x \to X$ induced respectively by $x \mapsto a$ and $x \mapsto b$ satisfy $\alpha \circ f = \alpha \circ g$. Because α is a monomorphism, $f = g$ and so $a = b$, proving the injectivity of α.

[3]A bi-additive map of abelian groups is the same thing as an R-balanced module homomorphism in the case where $R = \mathbb{Z}$ (see Chapter 1 footnote 6 for details). Thus by the universal property of the tensor product, bi-additive composition is equivalent to the existence of a natural extension $\mathrm{Hom}(X, Y) \otimes_{\mathbb{Z}} \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$.

Lemma 2.1. *If ι exhibits K as the kernel of a morphism $f : A \to B$ in $\mathcal{A}$, then ι is a monomorphism. Dually, if q exhibits C as the cokernel of f then q is an epimorphism.*

Proof. X □

Lemma 2.2. *A morphism $f : A \to B$ in $\mathcal{A}$ is a monomorphism if and only if ker$(f) = 0$. Dually, f is an epimorphism if and only if coker$(f) = 0$.*

Proof. X □

Lemma 2.3. *Given morphisms $A \xrightarrow{h} B \xrightarrow{g} C \overset{f}{\rightarrowtail} D$, the morphism h is a kernel of g if and only if h is a kernel of $f \circ g$. In other words, post-composing with a monomorphism does not change kernels. Dually, pre-composing with epimorphisms does not change cokernels.*

Proof. X □

2.1.4 Exact functors

In Chapter 1, we introduced Tensor and Hom as functors R-mod $\to$ **Ab**, and saw that they are right- and left-exact functors respectively. We now give the general definition of an *exact* functor in an arbitrary abelian category. In the following, $\mathcal{A}$ and $\mathcal{B}$ will be abelian categories.[4]

A functor $F : \mathcal{A} \to \mathcal{B}$ is called *additive* if $\mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(B))$ is a group homomorphism for any $A, B \in \mathcal{A}$. F is further called *exact*, *left-exact* and *right-exact* respectively if for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, the following sequences in $\mathcal{B}$ are respectively exact:[5]

$$\begin{aligned} 0 \to F(A) \to &F(B) \to F(C) \to 0 \\ 0 \to F(A) \to &F(B) \to F(C) \\ F(A) \to &F(B) \to F(C) \to 0 \end{aligned} \tag{2.2}$$

In other words, exact functors $\mathcal{A} \to \mathcal{B}$ send short exact sequences in $\mathcal{A}$ to short exact sequences in $\mathcal{B}$. As a consequence of this definition, we can prove that exact functors send long exact sequences to long exact sequences:[6]

Lemma 2.4. X

Proof. X □

By the exactness of (2.3) and (2.4), we see that left-exact functors preserve kernels and thus monomorphisms, and right-exact functors preserve cokernels and thus epimorphisms. We prove in Section 3.3 a powerful result from category theory which links adjoint functors, exact functors and (co)limits (the last of which (co)kernels are a special case). The theorem states that if we have an adjunction $F \dashv G$ of functors defined on abelian categories, then F preserves all colimits and G preserves all limits. As a consequence, F is right-exact and G is left-exact since (co)kernels are just instances of (co)limits. Indeed, the Tensor and Hom functors form an adjoint pair $- \otimes_R X \dashv \mathrm{Hom}_S(X, -)$ for an (R, S)-bimodule X, as we saw at the end of Chapter 1. This is one reason why notion the of an exact functor occurs so often in modern mathematics: they arise naturally from adjoint functors, and according to Saunders MacLane, '*adjoint functors arise everywhere*' (see [**leinster**] Chapter 2).

[4]We only need the notion of an additive category (see Section 2.1.2) to define an additive functor, but as we are interested in the exactness of additive functors we will work with abelian categories here.

[5]We can more succinctly write to condition of a functor F to be exact as follows: for any exact sequence $X \to Y \to Z$ in $\mathcal{A}$, the sequence $F(X) \to F(Y) \to F(Z)$ is exact. So we no longer assume that the map into Y is a monomorphism and the map out of Y is an epimorphism, but now we only need to check exactness at one object.

[6]A *long exact sequence* is simply a $\mathbb{Z}$-indexed exact sequence, and so a short exact sequence is the special case of this where at most three of the objects involved are non-zero. Most long exact sequences arise as a case of Theorem 2.1.

2.2 Homology

The abstract framework of abelian categories provides the right setting for defining homology in its most general form, but it does not reflect its concrete origins. Indeed, category theory and homological algebra were born from efforts in algebraic topology in the first half of the 20th century. Samuel Eilenberg and Saunders Mac Lane founded category theory in the 1940s after seeing a need to formalise and understand the idea of a natural transformation; this required defining functors, which in turn motivated the definition of a category.

Until this same time, homology theory remained part of topology: following work by Poincaré and others at the end of the 19th century on 'homology numbers', Emmy Noether shifted the focus to 'homology groups', which arise from a topological space as illustrated in **2.2.1** below. But it was the period of the 1940s and 1950s which saw these topologically-motivated techniques of homology theory applied to study the homology and cohomology of algebraic systems: Tor and Ext (which we study in Chapter 5), the (co)homology of Lie algebras and associative algebras, and the cohomology of sheaves among them. Then the revolutionary work [**cartan-eilenberg**] of Henri Cartan and Samuel Eilenberg united all the previously disparate homology theories with the language of resolutions and derived functors (which we'll see in Chapter 4) and crystallised the field completely. And it was in the search for a general setting of derived functors that the notion of an abelian category was born. See [**james**] Chapter 28 for more information.

2.2.1 Motivation

Topology:

Algebra:

2.2.2 Defining homology

Let $(C_\bullet, d_\bullet)$ be a chain complex of R-modules, denoted in this subsection by (C, d) for simplicity. Motivated by its topological origins, we define the n-th homology group of C as follows:

$$H_n(C) := \frac{Z_n}{B_n} = \frac{\ker(d_n : C_n \to C_{n-1})}{\operatorname{im}(d_{n+1} : C_{n+1} \to C_n)} \tag{2.3}$$

Thus homology measures the deviation of a chain complex from exactness: C is exact if and only if $H_n(C) = 0$ for all $n \in \mathbb{Z}$. And just as in **2.2.1**, the R-module $Z_n \subset C_n$ denotes the *cycles* of C_n and the R-module $B_n \subset C_n$ denotes the *boundaries* of C_n. The chain complex condition $d \circ d = 0$ ensures that $B_n \subset Z_n$ and thus all homology groups are well-defined.

If we have chain complexes (C, d^C) and (D, d^D) then a *chain map* $f : C \to D$ is a collection of maps $f_n : C_n \to D_n$ such that $f_{n-1} \circ d_n^C = d_n^D \circ f_n$ for all $n \in \mathbb{Z}$. In other words, the following diagram commutes:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}^C} & C_n & \xrightarrow{d_n^C} & C_{n-1} & \xrightarrow{d_{n-1}^C} & C_{n-2} & \longrightarrow & \dots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \downarrow f_{n-2} & & \\ \dots & \longrightarrow & D_{n+1} & \xrightarrow[d_{n+1}^D]{} & D_n & \xrightarrow[d_n^D]{} & D_{n-1} & \xrightarrow[d_{n-1}^D]{} & D_{n-2} & \longrightarrow & \dots \end{array} \tag{2.4}$$

The identity $f_{n-1} \circ d_n^C = d_n^D \circ f_n$ implies that f induces maps $\bar{f}_n := H_n(f) : H_n(C) \to H_n(D)$. Indeed, working in R-mod first suppose that $c \in Z_n$ so that $d_n^C(c) = 0$, then $f_{n-1} \circ d_n^C(c) = d_n^D \circ f_n(c) = 0$. And if $c \in B_n$, so that $c = d_{n+1}^C(c')$ for some $c' \in C_{n+1}$, then $f_{n-1} \circ d_n^C(c) = 0$. Thus f_n maps cycles to cycles and boundaries to boundaries for each $n \in \mathbb{Z}$ and so the induced maps $\bar{f}_n$ are well-defined. We say that f is a *quasi-isomorphism* if the induced maps $\bar{f}_n : H_n(C) \to H_n(D)$ are isomorphisms for all $n \in \mathbb{Z}$.

How shall we formulate homology in categorical terms? Since we don't necessarily have quotients in an arbitrary abelian category $\mathcal{A}$, we'll need to find a way of constructing homology groups in this setting such that when $\mathcal{A} = R$-mod we recover the quotient module (2.3). First recall that in R-mod, the cokernel of a map $f : X \to Y$ is the quotient of Y by the image $f(X)$. Secondly, we note that the chain complex condition ensures that the map $d_{n+1} : C_{n+1} \to C_n$

factors through $\ker(d_n)$, and so we can think of d_{n+1} as a map $C_{n+1} \to \ker(d_n)$. So if (C, d) is a chain complex in $\mathcal{A}$, we can generalise our definition of homology as follows:[7]

$$H_n(C) \coloneqq \operatorname{coker}(C_{n+1} \xrightarrow{d_{n+1}} \ker(d_n)) \tag{2.5}$$

In this way, we can view homology in each degree $n \in \mathbb{Z}$ as a functor $H_n : \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}$, where $\mathbf{Ch}(\mathcal{A})$ is the category whose objects are chain complexes in $\mathcal{A}$ and whose morpshisms are chain maps. The functoriality of homology is an immediate consequence of the argument above which shows that chain maps induce maps on the level of homology: for chain maps $f : C \to D$ and $g : D \to E$, the induced maps satisfy $H_n(g \circ f) = H_n(g) \circ H_n(f)$, and also $H_n(\mathrm{id}_C) = id_{H_n(C)}$. We will return to the category of chain complexes in Chapter 4.

2.3 The long exact sequence in homology

The upcoming Theorem 2.1 will be one of the main results in the course, the proof of which constructs a long exact sequence from certain short exact sequences. In fact, stringing together short exact sequences as such to obtain a long exact sequence is the way in which many long exact sequences arise in homological algebra. And it is the long exact sequence which gives homological algebra its computational power, since this construction often enables us to compute homology groups simply by studying the maps which these groups receive and admit. We'll see this in theorem being used in an essential way during Chapter 4 and 5.

First, we need a definition: we say that $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a *short exact sequence of chain complexes* in an abelian category $\mathcal{A}$ if $(A_\bullet, d^A_\bullet), (B_\bullet, d^B_\bullet)$ and $(C_\bullet, d^C_\bullet)$ are chain complexes of objects in $\mathcal{A}$ and there exist maps $f_\bullet : A_\bullet \to B_\bullet$ and $g_\bullet : B_\bullet \to C_\bullet$ such that all rows form short exact sequences which make the following diagram commute:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^A_{n+1}} & & \downarrow{\scriptstyle d^B_{n+1}} & & \downarrow{\scriptstyle d^C_{n+1}} & & \\
0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{2.6}$$

We now end this chapter with an excerpt from *The Lord of the Rings and Modules*:

Across the great bridge of Khazad-Dûm lay the passage through the mountain, and there now lay neither goblin nor orc between the Fellowship and their escape from Moria. But the dwarves of old had delved too greedily, and too deeply, and had awoken ancient powers of evil that would spell their demise. This was known only to Gandalf, for when the goblins suddenly fled from the caverns he alone did not rejoice, but instead tightened his grip on his staff and whispered spells of protection.

The last of the hobbits had scrambled to safety across the bridge as a great menace filled the air. And so it came to be that Durin's Bane, creation of Morgoth and survivor of the War of Wrath, came to face Gandalf the Grey. Far along the bridge, Gandalf stood alone above the yawning chasm. He turned to face the Balrog, and the hobbits looked on with terror as the Balrog uttered thus:

[7] We *can* formulate in categorical terms the image of the map $d_{n+1} : C_{n+1} \to \ker(d_n)$ as the kernel of its cokernel map (since in R-mod the cokernel of a map $f : X \to Y$ quotients Y by $f(X)$ and thus has kernel $f(X)$). But in the situation of homology this means we take the kernel of (2.5) and then take its cokernel to compute homology, which is unecessarily complicated when (2.5) itself recovers the definition of homology in R-mod.

Theorem 2.1. *Let $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ be a short exact sequence of chain complexes of objects in an arbitrary abelian category $\mathcal{A}$. Then there exist natural homomorphisms $\partial_n : H_{n+1}(C) \to H_n(A)$ for all $n \in \mathbb{Z}$ which together with the maps $f_\bullet, g_\bullet$ from (2.3) induce the following long exact sequence in homology:*

$$\ldots \longrightarrow H_{n+1}(C) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{\tilde{f}_n} H_n(B) \xrightarrow{\tilde{g}_n} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \longrightarrow \ldots \tag{2.7}$$

But Gandalf was Maiar, Istari to the Elves, and no corruption of Melkor's would beget his abandonment of the Fellowship. As the Balrog advanced along the bridge, with the might of Eru Iluvatar within him he smote the stone beneath him with his ragged staff and cried:

*'SEE ... [***hatcher***] ... PAGES 116-117!'*

The great bridge of Khazad-Dûm broke under the weight of the Gandalf's response. As the Balrog fell, Gandalf added: 'Hatcher's proof is for abelian groups, but the generalisation to R-mod is straightforward: just show R-linearity of the connecting homomorphism'.

But the Balrog's great fiery whip snaked out of the darkness and caught Gandalf by the ankles, and Gandalf was pulled over the bridge, his fingertips desperately searching for purchase on the stone. Gandalf looked down and bellowed: 'Apply the Freyd-Mitchell Embedding theorem to get the result for an arbitrary abelian category!'

'Very good', said the falling Balrog, and the great beast kindly uncoiled the whip from Gandalf's leg.

Gandalf gave silent thanks to Eru Iluvatar. And then, smiling now, he looked up and said: 'Help me up will you, Frodo? One old man can't do everything'.

Chapter 3

R-mod reprise

We now have an extremely powerful result at our disposal, given by Theorem 2.1. We will use the full weight of this theorem in Chapter 4 to study derived functors and the resulting special cases of Tor and Ext, which are absolutely central to homological algebra. But in order to do this we will need to work with *resolutions* of particular objects (which we'll study in the following chapters), and so this chapter will be devoted to studying these objects in the setting of R-mod, in which they can be formulated in quite concrete terms. We'll see in particular that if R is a principal ideal domain, then modules which are *projective, injective* and *flat* become equivalent to free, divisible and torsion-free modules respectively.[1]

3.1 Projective modules

Suppose we are given a free R-module $F = \bigoplus_{i \in \mathcal{I}} R$ together with maps $f : F \to N$ and $g : M \twoheadrightarrow N$ for some R-modules M and N. Then we can lift the map $h : F \to N$ to a map $F \to M$ such that $f = g \circ h$. Why can we do this? Because g is surjective, for each basis element e_i of F we can pick an element $a_i \in M$ such that $g(a_i) = f(e_i)$. Then the map h is just $h(e_i) \coloneqq a_i$, and this defines h on F since maps out of a free module are determined by their action on a basis.

We can't always perform such a lift if we aren't working with a free module. For example, the identity map $\mathrm{id} : \mathbb{Z}/2 \to \mathbb{Z}/2$ and projection map $\pi : \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$ in **Ab** do not admit a lift $h : \mathbb{Z}/2 \to \mathbb{Z}/4$ such that $\mathrm{id} = \pi \circ h$ since $h(1)$ must equal an element of order 2 and thus $h(1) = 2$, but then $\pi(h(1)) = \pi(2) = 0$.

This notion of lifting such a map will be helpful to us and so motivates us to define an object by this property. We say that an R-module P is *projective* if for any map $P \to B$ and surjection $A \twoheadrightarrow B$ there exists a map $P \to A$ such that the following diagram commutes:[2]

$$\begin{array}{ccc} & \overset{\exists}{\nearrow} & A \\ & & \downarrow\!\!\downarrow \\ P & \longrightarrow & B \end{array} \tag{3.1}$$

Lemma 3.1. *The following two statements are equivalent to the definition of projectivity for an R-module P:*

1. *P is a direct summand of a free module. In other words, there exists a free module F and a module Q such that $F = P \oplus Q$.*

2. *The functor* $\mathrm{Hom}_R(P, -) : R\text{-mod} \to \mathbf{Ab}$ *is exact.*

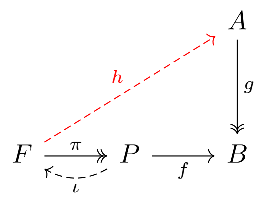
Proof. We first prove the equivalence of (1). Suppose P is projective, and pick[3]a free R-module $F \twoheadrightarrow P$ with kernel K. This gives us a short exact sequence $0 \to K \to F \to P \to 0$. Because P is projective, if we take $A = F$ and $B = P$

[1]An R-module M is *torsion-free* if for any $r \in R$ and $m \in M$, the equality $rm = 0_M$ implies that either $r = 0_R$ or $m = 0_M$. And R-module N is *divisible* if for any $n \in N$ and $r \in R$ there exists $n' \in N$ such that $n = rn'$, which intuitively means that any element of N can be divided by elements of the ring R.

[2]The map $P \to A$ need not be unique, so this isn't a universal property.

in (1.19) with $P \to P$ being the identity map, we get a section $P \to F$. Hence by Lemma 1.1 this sequence splits, which means that $F = P \oplus K$.

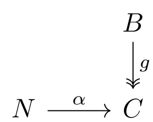
Conversely, suppose $F = P \oplus Q$ for a free module F and a module Q, and suppose we have maps $f : P \to B$ and $g : A \twoheadrightarrow B$ for some modules A and B. Consider the following diagram:



(3.2)

The map $\pi : F \to P$ is just the projection onto the first factor in the direct sum $F = P \oplus Q$. F is free, so in particular F is projective (see the argument given before (1.19)), which gives the existence of a map $h : F \to A$ making diagram (1.20) commute. Since P is a direct summand of F, we have the existence of the inclusion map $\iota : P \hookrightarrow F$ such that $\pi \circ \iota : P \hookrightarrow F \twoheadrightarrow P$ is the identity map. Thus $f = (f \circ \pi) \circ \iota = (g \circ h) \circ \iota$ by the projectivity of F, and so $h \circ \iota : P \to A$ makes the diagram starting from P commute and hence shows that P is projective.

We now prove the equivalence of (2). We know from Proposition 1.1 that $\mathrm{Hom}_R(N, -)$ is certainly left-exact, and thus will be *exact* precisely when $\mathrm{Hom}_R(N, -)$ preserves the surjection $g : B \twoheadrightarrow C$ in a short exact sequence $0 \to A \to B \to C \to 0$. In other words, we need the map $g \circ - : \mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, C)$ to be surjective. For any map $\alpha : N \to C$, consider the following diagram:



(3.3)

Thus N will be projective if and only if the above diagram admits a map $\beta : N \to B$ such that $\alpha = g \circ \beta$, which is precisely the requirement for the map $g \circ - : \mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, C)$ to be surjective. □

Warning: We prove in Theorem A.2 that any submodule of a free module over a principal ideal domain R is free. As a consequence, any projective R-module is free. This is because Lemma 3.1 gives us that projective modules are direct summands and thus submodules of free modules, and so projective modules must then also be free. But this need not hold for modules over an arbitrary ring!

One simple example is the case $R = M_2(k)$ for some field k. We can decompose the free module $R = P \oplus Q$, where $P = \{(\begin{smallmatrix} a & 0 \\ c & 0 \end{smallmatrix}) \mid a, c \in k\}$ and $Q = \{(\begin{smallmatrix} 0 & b \\ 0 & d \end{smallmatrix}) \mid b, d \in k\}$ are each isomorphic to k^2. Then neither of P and Q are free, because $\dim_k P = \dim_k Q = 2$ but any free R-module must have dimension equal to a multiple of 4 because $\dim_k R = 4$. More generally, if R is a product (equivalently, direct sum) of rings $R = A \times B$ then $A \times 0$ and $0 \times B$ will be projective but not free.

3.2 Flat modules

Proposition 1.2 shows the right-exactness of the tensor product functor. An R-module M is called *flat* if the functor $M \otimes_R -$ is exact.

Lemma 3.2. *Projective modules are flat.*

Proof. The first step of this proof is to show that free modules are flat. This is immediate because if $M \simeq R^{\mathcal{I}}$ (where $R^{\mathcal{I}} := \bigoplus_{i \in \mathcal{I}} R$) then for any R-module A, we have a natural isomorphism $A \otimes_R M \simeq A^{\mathcal{I}}$ by Proposition 0.1. Thus for any short exact sequence $0 \to A \to B \to C \to 0$, the sequence $0 \to A^{\mathcal{I}} \to B^{\mathcal{I}} \to C^{\mathcal{I}} \to 0$ is also be exact, proving that free modules are flat.

[3]We can always do this, since we can take F to be the free R-module on the generators of P, and for a generator $p \in P$ we just map the p-basis element to p. Of course, P need not have a nice set of generators, and indeed we could just take F to be the free R-module on the set of all elements of P.

Suppose now that P is a projective R-module. Then by Lemma 3.1, there exists R-modules Q and $F = P \oplus Q$ where F is free. Using Proposition 0.1 again, for any R-module A we have a natural isomorphism $A \otimes_R (P \oplus Q) \simeq (A \oplus P) \otimes_R (A \oplus Q)$. Thus the direct sum $F = P \oplus Q$ is flat if and only if each direct summand is flat, and because F is flat by the previous argument we have that P must be flat too. □

Just as we saw in section **1.3.1** that there exist projective modules which are not free, so too do there exist flat modules which are not projective. Such modules however are quite rare, and the only standard example of a flat module which is not projective is the abelian group $\mathbb{Q}$. We look at proofs of the flatness of $\mathbb{Q}$: one involving localisation, and then (after a digression on limits and colimits in R-mod) one using colimits. In the following theorem, S is a *multiplicatively closed* subset of R if $1 \in S$ and S is closed under multiplication.[4]

Theorem 3.1. *Let R be an integral domain. Then for any multiplicatively closed subset $S \subset R$, the R-module $S^{-1}R$ is flat. Thus in particular, the field of fractions $Q \coloneqq \mathrm{Frac}(R)$ is flat.*

Proof. The proof proceeds in two steps. The first is to show that the localisation functor S^{-1} is exact, where S is a multiplicatively closed subset of R. The second step is show that the functor $S^{-1}R \otimes_R -$ is naturally isomorphic to the localisation functor, which by the exactness proved in the first step then implies that $S^{-1}R$ is indeed flat. For this proof we follow [**atiyah-macdonald**] Chapter 3.

Let's begin. Our first step is to show that

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \quad \text{exact at } M \quad \Rightarrow \quad S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M'' \quad \text{exact at } S^{-1}M \tag{3.4}$$

We first observe[5]that $S^{-1}g \circ S^{-1}f = S^{-1}(g \circ f) = S^{-1}(0) = 0$ and so $\mathrm{im}(S^{-1}f) \subset \ker(S^{-1}g)$. For the converse, suppose that $\frac{m}{s} \in \ker(S^{-1}g)$. This means that $\frac{g(m)}{s} = 0 \in S^{-1}M''$ and so by the definition of localisation there exists $t \in S$ such that $tg(m) = 0 \in M''$. But g is an R-module homomorphism and so $tg(m) = g(tm)$ which means that $tm \in \ker(g)$. By exactness at M we deduce that $tm \in \mathrm{im}(f)$ and so $tm = f(m')$ for some $m' \in M'$. Thus in $S^{-1}M$ we have that $\frac{m}{s} = \frac{f(m')}{st} = (S^{-1}f)(\frac{m'}{st}) \in \mathrm{im}(S^{-1}f)$ and so have proven exactness at $S^{-1}M$.

For the second part, we wish to prove that for an R-module M, the $S^{-1}R$-modules $S^{-1}M$ and $S^{-1}R \otimes_R M$ are naturally isomorphic. To do this we establish the canonical isomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ given by $\frac{r}{s} \otimes m \mapsto \frac{rm}{s}$ for all $r \in R, s \in S, m \in M$ as follows.

The map $S^{-1}R \times M \to S^{-1}M$ given by $(\frac{r}{s}, m) \mapsto \frac{rm}{s}$ is R-balanced and so by the universal property of the tensor product induces the R-module homomorphism f defined above. The surjectivity of f is clear: an element of $S^{-1}M$ is given by $\frac{m}{s}$ and so has a preimage under f given by $\frac{1}{s} \otimes m$. For injectivity, consider $\sum \frac{r_i}{s_i} \otimes m_i$ be an arbitrary element of $S^{-1}R \otimes_R M$. Let $s = \prod_i s_i \in S$ and $t_i = \prod_{j \neq i} s_i \in S$. Then

$$\sum \frac{r_i}{s_i} \otimes m_i = \sum \frac{r_i t_i}{s_i} \otimes m_i = \sum \frac{1}{s} \otimes r_i t_i m_i = \frac{1}{s} \otimes \sum r_i t_i m_i \tag{3.5}$$

Thus every element of $S^{-1}R \otimes_R M$ is of the form $\frac{1}{s} \otimes m$ for some $s \in S$ and $m \in M$. If $f(\frac{1}{s} \otimes m) = 0$ then $\frac{m}{s} = 0$, and so by the definition of localisation there exists $t \in S$ such that $tm = 0$. But this means that $\frac{1}{s} \otimes m = \frac{t}{st} \otimes m = \frac{1}{st} \otimes tm = 0$ and so f is injective and we are done.

□

Corollary 3.1. *$\mathbb{Q}$ is a flat $\mathbb{Z}$-module which is not projective.*

Proof. We have that $\mathbb{Q}$ is flat by Theorem 3.1 because $\mathbb{Q} = \mathrm{Frac}(\mathbb{Z})$ (in other words, $\mathbb{Q} = S^{-1}\mathbb{Z}$ where $S = \mathbb{Z} \setminus \{0\}$). Suppose for contradiction that $\mathbb{Q}$ is projective. Then by Lemma 3.1, $\mathbb{Q}$ is a direct summand of a free module $F \simeq \mathbb{Z}^{\mathcal{I}}$ and so the inclusion map gives an injective homomorphism $\iota : \mathbb{Q} \hookrightarrow F$. But the only homomorphism $\mathbb{Q} \to F$ is the zero map, which is not injective. To see this, suppose we have a homomorphism $f : \mathbb{Q} \to F$. Then

[4] For those unfamiliar with *localisation*, see [**atiyah-macdonald**] Chapter 3 for an excellent introduction. Localisation in commutative algebra is a method to formally introduce denominators to a given ring or module, which in the case of $S = R \setminus \{0\}$ produces the field of fractions of R. Localisation has become a fundamental tool in algebraic geometry (which in its modern form has is built from commutative algebra), since it allows the study of the local behaviour of geometric objects such as varieties and schemes.

[5] For R-modules A and B, an R-module homomorphism $\phi : A \to B$ induces an $S^{-1}R$-module homomorphism $S^{-1}\phi : S^{-1}A \to S^{-1}B$ given by $\phi(\frac{a}{s}) = \frac{\phi(a)}{s}$ and with this definition we see that S^{-1} is functorial (which means that $S^{-1}(\psi \circ \phi) = S^{-1}\psi \circ S^{-1}\phi$ for composable R-module homomorphisms ϕ and ψ).

$$f(1) = 2^n f\left(\frac{1}{2^n}\right) \in 2^n F \implies f(1) \in \bigcap_{n\geq 0} 2^n F = 0.$$

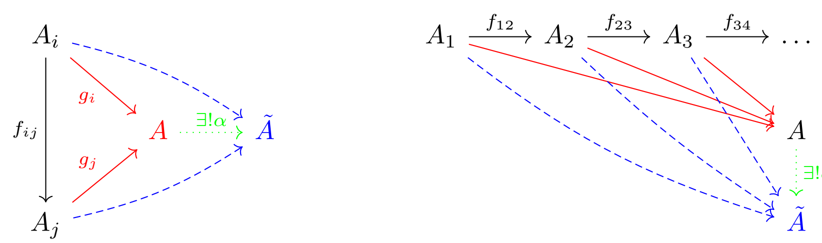
Thus $f(1) = 0$ and so $f(n) = 0$ for any $n \in \mathbb{Z}$. Suppose now $f\left(\frac{a}{b}\right) = x$ for some non-zero $x \in F$, where $a, b \in \mathbb{Z}\setminus\{0\}$ are arbitrary. Then $bx = b \cdot f\left(\frac{a}{b}\right) = f(a) = 0$ since f is zero on $\mathbb{Z}$, which means that x is a non-zero torsion element of F. But this is impossible because $F \simeq \mathbb{Z}^{\mathcal{I}}$ is torsion-free. This gives the required contradiction. □

3.3 Limits and colimits in R-mod

Limits and colimits are notions from category theory which unify many familiar constructions in mathematics. In this section we will study *inverse* and *direct* limits which in R-mod can be formulated concretely. These are particular kinds of limits and colimits respectively. We will do this in order to help us prove statements about flat and injective modules. Those wishing to learn more about limits and colimits can refer to [**leinster**] Chapter 5.

3.3.1 Directed systems

We begin by constructing direct limits, from which inverse limits will follow by duality. We say that a poset $\mathcal{I}$ is *directed* if it is equipped with a reflexive and transitive binary relation (call it $\leqslant$) such that any pair of objects in $\mathcal{I}$ have an upper bound in $\mathcal{I}$.[6]Now let $\{A_i\}_{i\in\mathcal{I}}$ be a family of R-modules with morphisms $f_{ij} : A_i \to A_j$, such that $f_{ii} = \mathrm{id}_{A_i}$ and $f_{ik} = f_{jk} \circ f_{ij}$ for all $i, j, k \in \mathcal{I}$. We then say that the collection $\langle A_i, f_{ij}\rangle$ forms a *directed system* over $\mathcal{I}$. We now define the *colimit* of this directed system (also called the *direct limit*) to be the universal object $A \in R$-mod such that there exist morphisms $g_i : A_i \to A$ for each $i \in \mathcal{I}$ with the property that $g_i = g_j \circ f_{ij}$ for all i, j in $\mathcal{I}$. In other words, for any other object $\tilde{A}$ satisfying these conditions, there exists a unique morphism $\alpha : A \to \tilde{A}$ such that the first diagram below commutes. When $\mathcal{I} = \mathbb{N}$, which will often be the case for us, this universal property takes the simpler form of the second commutative diagram below:



(3.6)

Inverse limits are simply the dual case of direct limits, and thus are obtained by simply reversing all arrows in (3.6). In detail, this means that in our directed system $\langle A_i, f_{ij}\rangle$ the morphisms are now $f_{ij} : A_j \to A_i$ with composition rule $f_{ik} = f_{ij} \circ f_{jk}$, the red arrows become morphisms $g_i : A \to A_i$ satisfying $g_i = f_{ij} \circ g_j$, and the unique morphism arising from universality is now a map $\alpha : \tilde{A} \to A$. The reason why in R-mod (and other algebraic categories like **Ab**) we are interested in limits of these directed systems in particular is because we can deduce their algebraic structure as follows:

Lemma 3.3. *Let X and Y respectively be the inverse and direct limit of the directed system $\langle A_i, f_{ij}\rangle$ over $\mathcal{I}$ defined above. Then X and Y are precisely the following two R-modules:*

$$X = \left\{ a \in \prod_{i\in\mathcal{I}} A_i \mid a_i = f_{ij}(a_j) \text{ for all } i \leqslant j \text{ in } \mathcal{I} \right\}, \qquad Y = \bigsqcup_{i\in\mathcal{I}} A_i / \sim \tag{3.7}$$

where if $x_i \in A_i$ and $x_j \in A_j$, then $x_i \sim x_j$ if and only if there exists some $k \in \mathcal{I}$ such that $i, j \leqslant k$ and $f_{ik}(x_i) = f_{jk}(x_j)$. In other words, we identify $x_i \sim x_j$ if they become equal at some stage $k \in \mathcal{I}$.

Proof. X □

[6]For example, any totally ordered set (like $\mathbb{Z}$ and $\mathbb{R}$) are directed sets. For a topological space X, the set of open neighbourhoods around a point $x \in X$ also form a directed set when ordered by inclusion, which unlike the previous example is not totally ordered set (since we needn't have $U \subset V$ or $V \subset U$ for arbitrary open neighbourhoods U, V of x).

Corollary 3.2. *Suppose* $\{A_i\}_{i \in \mathcal{I}}$ *is a collection of* R*-modules which are partially ordered by inclusion. Then their colimit is given by* $Y = \bigcup_{i \in \mathcal{I}} A_i$.

Proof. X □

3.3.2 Examples of inverse and direct limits

We have seen from Corollary 3.2 that direct limits (in other words, colimits of directed systems) are well-behaved under inclusions. We will use this result to express some familiar abelian groups as colimits of such directed systems.

Example: What is the colimit over $\mathbb{N}$ of the following directed system of free abelian groups?

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \xrightarrow{\cdot 4} \ldots \tag{3.8}$$

These morphisms *are* injections, but note that the maps are not inclusions (for example, the first map sends $1 \mapsto 2 \neq 1$). To use Corollary 3.2 we'll need to view each copy of $\mathbb{Z}$ as a submodule of the next in the sequence, and so we rewrite this system as follows:

$$\begin{array}{ccccccc} \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \xrightarrow{\cdot 3} & \mathbb{Z} & \xrightarrow{\cdot 4} & \ldots \\ \downarrow{\scriptstyle \cdot 1} & & \downarrow{\scriptstyle \cdot \frac{1}{2}} & & \downarrow{\scriptstyle \cdot \frac{1}{6}} & & \\ \mathbb{Z} & \hookrightarrow & \frac{1}{2}\mathbb{Z} & \hookrightarrow & \frac{1}{6}\mathbb{Z} & \hookrightarrow & \ldots \end{array} \tag{3.9}$$

Then because each vertical map is an isomorphism which is compatible with the horizontal maps (in other words, the vertical maps make the squares commute), we can use the universal property of the colimit to deduce that both rows will have the same colimit. Thus by Corollary 3.2 we have:

$$\operatorname{colim}_{\mathbb{N}}(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \xrightarrow{\cdot 4} \ldots) = \operatorname{colim}_{\mathbb{N}}(\mathbb{Z} \hookrightarrow \tfrac{1}{2}\mathbb{Z} \hookrightarrow \tfrac{1}{6}\mathbb{Z} \hookrightarrow \ldots) = \bigcup_{n \in \mathbb{N}} \tfrac{1}{n!}\mathbb{Z} = \mathbb{Q} \tag{3.10}$$

We can use this same idea to deduce the following:

$$\operatorname{colim}_{\mathbb{N}}(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \ldots) = \operatorname{colim}_{\mathbb{N}}(\mathbb{Z} \hookrightarrow \tfrac{1}{2}\mathbb{Z} \hookrightarrow \tfrac{1}{4}\mathbb{Z} \hookrightarrow \ldots) = \bigcup_{n \in \mathbb{N}} \tfrac{1}{2^n}\mathbb{Z} = \mathbb{Z}[\tfrac{1}{2}] \tag{3.11}$$

Example: Let us now see how to compute $X = \lim_{\mathbb{N}}(\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \ldots)$. From Lemma 3.3, an element of the directed system of R-modules $\lim_{\mathbb{N}}(M_1 \leftarrow M_2 \leftarrow \ldots)$ is an $\mathbb{N}$-indexed family of elements $m_i \in M_i$ such that $m_k \mapsto m_{k+1}$ for all $k \in \mathbb{N}$. So we are looking for an element $a = (a_1, a_2, \ldots) \in \prod_{\mathbb{N}} \mathbb{Z}$ such that $a_k = 2a_{k+1}$ for each $k \in \mathbb{N}$. But then $a_1 = 2^{k-1}a_k$ for all $k \in \mathbb{N}$ and the only such possibility is $a_i = 0$ for all $i \in \mathbb{N}$, and thus $X = 0$.

Example: Our last example will be to compute $X = \lim_{\mathbb{N}}(A \twoheadleftarrow A^2 \twoheadleftarrow A^3 \twoheadleftarrow \ldots)$ and $Y = \operatorname{colim}_{\mathbb{N}}(A \hookrightarrow A^2 \hookrightarrow A^3 \hookrightarrow \ldots)$ for an arbitrary $A \in R$-mod, where the maps in these systems are the canonical projection and inclusions maps. So in the first case the maps $f_{k,k+1}$ are $(a_1, \ldots, a_k, a_{k+1}) \mapsto (a_1, \ldots, a_k)$ and in the second case they are $(a_1, \ldots, a_k) \mapsto (a_1, \ldots, a_k, 0)$. Using Lemma 3.3, we immediately have that $X = \prod_{\mathbb{N}} A$. And since we are working with inclusions in the second system, Corollary 3.2 gives us that $Y = \bigcup_{n \in \mathbb{N}} A^n$. But then an element of $a \in Y$ must lie in A^N for some $N \in \mathbb{N}$ and thus a is of the form $a = (a_1, \ldots, a_N, 0, 0, \ldots)$ and so we see that in fact $Y = \bigoplus_{\mathbb{N}} A$.

3.3.3 Direct limits and flatness

We conclude Section 3.3 with the following theorem, which gives an equivalent characterisation of flatness when R is a principal ideal domain. Since $\mathbb{Q}$ is torsion-free, this provides a second approach to proving that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. To prove this theorem, we will use three different results, two of whose proofs have been deferred to the appendix for succinctness. The third of these results is as follows:

Proposition 3.1. *Direct limits in R-mod are exact functors. In other words, if we have $\mathcal{I}$-indexed directed systems $\langle K_i, f_{ij}^K \rangle, \langle M_i, f_{ij}^M \rangle$ and $\langle N_i, f_{ij}^N \rangle$ and an $\mathcal{I}$-indexed sequence of chain complexes $K_\bullet \to M_\bullet \to N_\bullet$ which is exact at M_i for all $i \in \mathcal{I}$, then $\mathrm{colim} K_\bullet \to \mathrm{colim} M_\bullet \to \mathrm{colim} N_\bullet$ is exact at $\mathrm{colim}(M_\bullet)$.*

Proof. X □

Theorem 3.2. *Let A be a module over a principal ideal domain R. Then A is a flat R-module if and only if A is torsion free.*

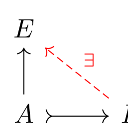
Proof. Suppose that A is a flat R-module, which by the definition of flatness means that $- \otimes_R A$ is an exact functor. Consider the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$ with the usual maps, where $n \in \mathbb{Z}$. This maps to the short exact sequence $0 \to A \to A \to A/n \to 0$ where the first map (which by the construction of the isomorphism in Proposition 0.3 is still given by multiplication by n) remains injective by the exactness of $- \otimes_R A$. Thus for non-zero $a \in A$, we see that $na = 0$ implies $n = 0$ and so A is torsion-free.

Conversely, suppose that A is a torsion-free R-module. Let $\{H_i\}_{i \in \mathcal{I}}$ be the collection of finitely-generated submodules of A for some indexing set $\mathcal{I}$. Then $\langle H_i, f_{ij} \rangle$ form a directed system over $\mathcal{I}$ where the maps f_{ij} are given by inclusions $H_i \hookrightarrow H_j$. So by Corollary 3.2 we see that $\mathrm{colim}_{i \in \mathcal{I}} H_i = \bigcup_{i \in \mathcal{I}} H_i = A$. Thus $- \otimes_R A \simeq - \otimes_R \mathrm{colim} H_i \simeq \mathrm{colim}(- \otimes_R H_i)$, where the latter isomorphism follows from Theorem A.1. Because A is torsion-free, each H_i must be torsion-free and so by Theorem A.3 we see that each H_i must in fact be free. Since free modules are projective which in turn are flat (using Lemma 3.1 and 3.2 respectively), we see that H_i is in fact flat for all $i \in \mathcal{I}$. This means that every $- \otimes_R H_i$ is exact. Thus by Proposition 3.1 since directed limits in R-mod preserve exactness, we see that $- \otimes_R A$ must itself be exact and so A is a flat R-module. □

3.4 Injective modules I: Baer's Criterion

Suppose we are given a morphism $f : \mathbb{Z} \to \mathbb{Q}$ and a monomorphism $g : \mathbb{Z} \rightarrowtail \mathbb{Q}$. Can we always find a morphism $h : \mathbb{Z} \to \mathbb{Q}$ such that $f = h \circ g$? Well, $f(1) = a$ for some $a \in \mathbb{Q}$ and the map $g : \mathbb{Z} \to \mathbb{Z}$ must be multiplication by n for some $n \in \mathbb{Z}$. So we can just define h to be the map given by $1 \mapsto \frac{a}{n} \in \mathbb{Q}$ and then h will satisfy $f = h \circ g$.

In fact, we could replace $\mathbb{Z}$ by *any* abelian groups A and B and we would always be able to lift[7] a map $f : A \to \mathbb{Q}$ along a monomorphism $g : A \rightarrowtail B$ to a map $h : B \to \mathbb{Q}$ which restricts to f on $A \simeq g(A) \subset B$. This is because $\mathbb{Q}$ is *injective* as a $\mathbb{Z}$-module (which will be a consequence of Corollary 3.3). Injective modules are the categorical dual to projective modules, and are thus defined in a similar fashion as follows. We say that an R-module E is *injective* if for any R-modules A and B and any map $A \to E$ and monomorphism $A \rightarrowtail B$, there exists a (not necessarily unique) map $B \to E$ such that the following diagram commutes:



(3.12)

Comparing (3.6) to (3.1) shows us that projective modules in R-mod are the same as injective modules in (R-mod)^{op}, which as an immediate consequence of Lemma 3.1 gives us that an R-module E is injective if and only if the functor $\mathrm{Hom}_R(-, E) : R\text{-mod}^{op} \to \mathbf{Ab}$ is exact. And by the same argument in the proof of Lemma 3.1 which shows that a short exact sequence $0 \to K \to F \to P \to 0$ splits whenever P is projective, so too does any short exact sequence $0 \to E \to X \to C \to 0$ split whenever E is injective.

The notion of injective modules were invented by Reinhold Baer in 1940, long before projective modules were first introduced in [**cartan-eilenberg**] in 1956. The following theorem is known as *Baer's Criterion*, and it provides an equivalent condition for an R-module to be injective which is often much easier to work with than the definition given above, from which a number of nice corollaries will follow.

Theorem 3.3. *An R-module E is injective if and only if for every ideal $J \triangleleft R$, every map $J \to E$ can be extended to a map $R \to E$.*

[6] $A = \bigcup_{i \in \mathcal{I}} H_i$ simply because A is the union of all its finitely-generated submodules: $A = \bigcup_{a \in A} a \subset \bigcup_{a \in A} Ra \subset \bigcup_{i \in \mathcal{I}} H_i \subset A$.

[7] The terms *lifting* and *extending* have the same meaning in this context. For maps $f : X \to Z$ and $g : X \to Y$, a map $h : Y \to Z$ such that $f = h \circ g$ is said to be a *lift* or *extension* of f along g.

Proof. The forwards implication is straightforward: take $A \rightarrowtail B$ in (3.6) to be the inclusion map $J \hookrightarrow R$, and then by the injectivity of E we have the required lift $R \to E$. For the converse, suppose we are given two R-modules A and B together with a morphism $\alpha : A \to E$ and a monomorphism $i : A \to B$. Since monomorphisms in R-mod are equivalent to injections (see **2.1.1**), we may view A as a submodule of B via the isomorphism $A \simeq i(A) \subset B$. We need to find a map $B \to E$ which is an extension of α.

To do this, we look at the poset[8] $\mathcal{E} := \{\alpha' : A' \to E \mid A \subset A' \subset E \text{ and } \alpha'|_A = \alpha\}$ of all R-module homomorphisms which extend $\alpha : A \to E$. The partial order on $\mathcal{E}$ is given as follows: $\alpha' \leqslant \alpha''$ precisely when α'' extends α'. By Zorn's lemma[9], there exists a maximal extension $\alpha' : A' \to E$. To finish, we need to show that $A' = B$.

Suppose for contradictions there exists $b \in B \setminus A'$. Then the set $J = \{r \in R \mid rb \in A'\}$ is an ideal of R. By assumption, the map $J \to A' \to E$ given by the composition $\alpha' \circ b$ extends to a map $f : R \to E$. Define $A'' := A + Rb \subset B$ and $\alpha'' : A'' \to E$ by $\alpha''(a + rb) = \alpha'(a) + f(r)$ for $a \in A'$ and $r \in R$. Then α'' is well-defined (because $\alpha'(rb) = f(r)$ for $rb \in A' \cap Rb$) and also extends α', contradicting the maximality of α'. Thus $A' = B$ and so we have found our extension $\alpha' : B \to E$. □

Example: Let $A = \mathbb{Z}/n$ for some $n \in \mathbb{Z}$. Then A is certainly not an injective $\mathbb{Z}$-module for $n \geqslant 2$: taking $J = n\mathbb{Z}$ we see that the map $J \hookrightarrow \mathbb{Z}$ given by $n \mapsto 1$ cannot lift to a non-zero map $\mathbb{Z} \to A$. We can however use Baer's Criterion to prove that A is injective *as a module over itself.* This is a consequence of the following result:

Corollary 3.3. *Let R be a principal ideal domain, and $a \in R \setminus \{0\}$. Then $E := R/Ra$ is injective as module over itself.*

Proof. From elementary ring theory, the ideals of R/Ra are in bijection with the ideals of R containing a, and so because R is a principal ideal domain the ideals of R/Ra correspond precisely divisors of a. So to invoke Baer's Criterion, we need to show that for any $b \in R$ such that $b|a$, an arbitrary E-module homomorphism $f : dE \to E$ lifts to an E-module homomorphism $f : E \to E$. Now, a homomorphism $f : bE \to E$ is defined by where it sends the generator b of bE. Using that $b|a$ we can write $\frac{a}{b}b = 0 \in E$ and so $f(\frac{a}{b}b) = \frac{a}{b}f(b) = 0 \in E$ which implies that $b|f(b)$[10]. Thus we can extend f to a map $E \to E$ defined by $1_E \mapsto \frac{f(b)}{b}$ and by Theorem 3.2 we are done. □

Corollary 3.4. *Let R be an integral domain. If an R-module E is both torsion-free and divisible, then E is injective. In particular, $Q :=$*Frac*(R) is an injective R-module.*

Proof. Let $J \triangleleft R$ and suppose we have a map $f : J \to E$. By Baer's Criterion, if we can extend f to a map $R \to E$ then we will have shown that E is injective. If $J = \{0\}$ then f trivially extends to the zero map $R \to E$, so we may assume that $J \neq \{0\}$ and pick a non-zero element $x \in J$. Because E is divisible, $f(x) = xe$ for some $e \in E$. Thus for any $a \in J$ we have that $xf(a) = f(ax) = af(x) = axe$ and so $x(f(a) - ae) = 0$ which because E is torsion-free gives us that $f(a) = ae$ for all $a \in J$. Thus we may extend f to a map $R \to E$ by send $r \mapsto re$ for all $r \in R$. □

3.5 Injective modules II: Working over $\mathbb{Z}$

Why has our study of projective modules in this chapter has been relatively short in comparison with the more extensive treatment given to injective modules? As noted in Section 3.1, over principal ideal domains (among other kinds of rings) all projective modules are free. As a result, projective modules will be particularly simple to exhibit in many of the cases in which we'll be working. However, the same is rarely true for injective modules. Since we will need to be able construct and work with both projective and injective modules in Chapters 4 and 5, we study injective modules in detail in Sections 3.4 and 3.5 as preparation. Indeed, from our efforts in this section, we will be able to give a classification of injective modules over $\mathbb{Z}$.

Proposition 3.2. *Let A be a module over a principal ideal domain R. Then A is injective if and only if A is divisible.*

[8] A set X is a *partially ordered set* (which we shorten to *poset*) if it is equipped with binary relation $\leqslant$ which satisfies reflexivity (a $\leqslant$ a), antisymmetry ($a \leqslant b$ and $b \leqslant a$ implies $a = b$), and transitivity ($a \leqslant b$ and $b \leqslant c$ implies $a \leqslant c$), where $a, b, c \in X$ are arbitrary.

[9] Zorn's lemma is equivalent to the Axiom of Choice, and one formulation of the lemma is as follows. If a poset X has the property that every totally ordered subset of X has an upper bound in X, then X must have at least one maximal element. This has fundamental applications to all parts of mathematics: for example, it gives us the existence of maximal ideals of a ring.

[10] If $\frac{a}{b}f(b) = 0 \in E$ then $\frac{a}{b}f(b) = ra$ for some $r \in R$. Since R is an integral domain we can cancel a to see that $f(b) = rb$ and thus $b|f(b)$.

Proof. Suppose A is an injective R-module, and consider non-zero elements $a \in A$ and $x \in R$. We need to show that there exists $b \in A$ such that $a = xb$. Let us first observe that the principal ideal $I \coloneqq rR$ and R are isomorphic as R modules via the map $\phi : R \to I$ given by $r \mapsto rx$ because R has no zero divisors. Now, we pre-compose the map $R \to A$ given by $r \mapsto ra$ with ϕ^{-1} to obtain a map of R-modules $f : I \to A$ such that $f(xr) = ra$ for all $r \in R$. By Theorem 3.2 we can use the injectivity of A to lift f and obtain an R-module homomorphism $g : R \to A$ such that $g|_I = f$. Let $b = g(1)$, then $xb = xg(1) = g(x) = f(x) = a$ and we are done.

Conversely, suppose that A is a divisible R-module. Suppose we have an ideal $I \triangleleft R$ and an R-module homomorphism $f : I \to A$. Since R is a principal ideal domain, $I = Rx$ for some $x \in R$. And since A is divisible, we can write $f(x) = xa$ for some $a \in A$. Then the map $g : R \to A$ defined by $g(1) = a$ is the required lift of f and thus by Theorem 3.2 A is injective. □

Example: If V is a k-vector space, then by Proposition 3.2 it must be injective over k. Furthermore, both Corollary 3.2 and Proposition 3.2 immediately give us that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module because $\mathbb{Q}$ is divisible. Now, quotients of a divisible R-module are themselves divisible[11], and so by Proposition 3.2 $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.[12]

To prepare ourselves for Theorem 3.3 which will give us our classification of injective modules over $\mathbb{Z}$, we will first need to see why the *Prüfer groups* $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \coloneqq \mathrm{colim}_{\mathbb{N}}(\mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^2 \hookrightarrow \mathbb{Z}/p^3 \hookrightarrow \ldots)$ are divisible for any prime $p \in \mathbb{N}$. The Prüfer groups are countable abelian groups which, as we'll see in Theorem 3.3, along with $\mathbb{Q}$ form the building blocks of all divisible abelian groups.[13]

Lemma 3.4. *$\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ is an injective $\mathbb{Z}$-module for all primes $p \in \mathbb{N}$.*

Proof. First observe that an element $x \in \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ can be uniquely written as a sum $x = \frac{a_1}{p} + \frac{a_2}{p^2} + \ldots + \frac{a_m}{p^m}$ for some $m \in \mathbb{N}$, where each $a_i \in \{0, 1, \ldots, p-1\}$. Taking these terms over the common denominator p^m we see that the element x is of the form $x = \frac{b}{p^m}$ for some $b \in \mathbb{Z}$. Thus we see that $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ is p-divisible since $y = \frac{x}{p} \in \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ satisfies $y = px$.

We now prove that $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ is q-divisible for any prime $q \neq p$. Because q and p^m are coprime we must have that $\langle q, p^n \rangle = \mathbb{Z}$, where $\langle q, p^m \rangle$ is the subgroup of $\mathbb{Z}$ generated by p^m and q. So by Bézout's lemma we can find $k, l \in \mathbb{Z}$ such that $b = kq + lp^m$. We may assume $p^m > q$ since if necessary we can just rewrite x as $\frac{bp^{m'}}{p^{m+m'}}$ for any $k \in \mathbb{N}$, and use $bp^{m'}$ instead of b. Hence $x = q\bar{k}$ and so $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ is q-divisible for all primes $q \neq p$. Thus $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ is divisble by all primes and so is a divisible abelian group, which by Proposition 3.2 imples that it is an injective $\mathbb{Z}$-module. □

Lemma 3.5. *Let $\{E_i\}_{i \in \mathcal{I}}$ be a collection of R-modules. Then E_i is injective for all $i \in \mathcal{I}$ if and only if $\prod_{i \in \mathcal{I}} E_i$ is injective. Furthermore, if R is noetherian[14]then E_i is injective for all $i \in \mathcal{I}$ if and only if $\bigoplus_{i \in \mathcal{I}} E_i$ is injective.*

Proof. If $\{E_i\}_{i \in \mathcal{I}}$ is a collection of R-modules, then it is straightforward to see that they are each injective if and only if their direct produce is injective. This follows from Proposition 0.1 and (as mention in Section 3.4) the exactness of $\mathrm{Hom}_R(-, E_i)$ being equivalent to the injectivity of E_i:

[11] If A is divisible then for any $a \in A$ and $r \in R$ we can find $b \in A$ such that $a = rb$. But then in any quotient module $\bar{A}$ of A, for $\bar{a} \in \bar{A}$ we have that $\bar{a} = r\bar{b}$ and so $\bar{A}$ is divisible.

[12] In light of Theorem 3.3, we can expect to be able to decompose $\mathbb{Q}/\mathbb{Z}$ as a direct sum of Prüfer groups (the decomposition won't involve $\mathbb{Q}$ since every element of $\mathbb{Q}/\mathbb{Z}$ is torsion). As an exercise, can you explicitly construct an isomorphism $\alpha : \bigoplus_{p \in \mathbb{N}} \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$? Hint: by the universal property of the direct sum, the inclusions $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$ give rise to this group homomorphism α, and you just need to show that α is both injective and surjective.

[13] We can also think of the Prüfer group $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ as the group of all p^n-th roots of unity on the unit circle with the group operation given by multiplication, for all $n \in \mathbb{N}$. Let's look at the example $p = 2$. We start off with $\mathbb{Z}/2 = \{\pm 1\}$, include this into $\mathbb{Z}/4 = \{\pm 1, \pm i\}$, include this into $\mathbb{Z}/8 = \{\pm 1, \pm i, \pm \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i\}$, and so on. Then using the formulation of colimits in [SECTION 3.3] we can also view $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ as $\bigcup_{n \in \mathbb{N}} \mathbb{Z}/p^n$ and so the Prüfer groups are often denoted as $\mathbb{Z}/p^\infty$.

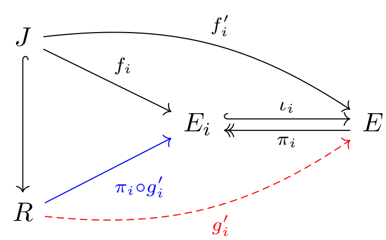
[14] A ring R is *noetherian* if any ascending chain of ideals of R stabilises. This means that for any sequence $I_1 \subset I_2 \subset I_3 \subset \ldots$ of ideals of R, there exists some $N \in \mathbb{N}$ such that $I_k = I_N$ for all $k \geqslant N$. Equivalently, any ideal of R is finitely-generated. Two important examples of noetherian rings are principal ideal domains and finitely-generated R-algebras, the latter of which is due to a result known as *Hilbert's Basis Theorem*. Noetherian rings are named after Emmy Noether, who made fundamental contributions to abstract algebra and theoretical physics. Among her far-reaching influence in modern mathematics, she developed the theory of ideals in commutative rings, and subsequently united the representation theory of groups with the theory of rings and modules.

$$\prod_{i\in\mathcal{I}} E_i \text{ injective} \iff \operatorname{Hom}_R(-, \prod_{i\in\mathcal{I}} E_i) \simeq \prod_{i\in\mathcal{I}} \operatorname{Hom}(-, E_i) \text{ exact} \iff E_i \text{ injective for all } i \in \mathcal{I} \tag{3.13}$$

We now prove the stronger equivalence, under the assumption that R is noetherian. Suppose we are given an ideal $I \lhd R$ together with a map $h : I \to E_1 \oplus E_2$. Then $h = f_1 \oplus f_2$, where $f_i \coloneqq \pi_i \circ h$ is the post-composition of h with the i-projection map. Because E_1 and E_2 are injective, the maps $f_i : I \to E_i$ extend to maps $g_i : R \to E_i$ for $i \in \{1, 2\}$. Thus we can extend $h : I \to E_1 \oplus E_2$ to a map $g = g_1 \oplus g_2 : R \to E_1 \oplus E_2$, and so by induction we have that finite direct sums of injective modules are injective.

Let $E \coloneqq \bigoplus_{i\in\mathcal{I}} E_i$. Suppose that $J \lhd R$ and that we have a map $\alpha : J \to E$. Because R is noetherian, J is finitely-generated and so $\alpha(J) \subset \bigoplus_{i\in\mathcal{I}'} E_i$ for some finite subset $\mathcal{I}' \subset \mathcal{I}$. Thus by the previous paragraph we can extend α to a homomorphism $R \to \bigoplus_{i\in\mathcal{I}'} E_i$ whose composition with the canonical inclusion $\bigoplus_{i\in\mathcal{I}'} E_i \hookrightarrow E$ gives the desired result.

Conversely, suppose that $E \coloneqq \bigoplus_{i\in\mathcal{I}} E_i$ is an injective R-module. For any ideal $J \lhd R$ and map $f_i : J \to E_i$ for some $i \in \mathcal{I}$, we can post-compose f_i with the inclusion $\iota_i : E_i \to E$ to obtain a map $f_i' : J \to E$. By the injectivity of E, we can extend f_i' to a map $g_i' : R \to E$, which when post-composed with $\pi_i : E \to E_i$ gives us a map $g_i : R \to E_i$ such that $f_i = g_i|_{E_i}$ and so E_i is injective for all $i \in \mathcal{I}$. This construction is illustrated in the following commutative diagram:



(3.14)

□

Theorem 3.4. *An abelian group is injective if and only if it is a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ for some primes $p \in \mathbb{N}$.*

Proof. The reverse direction is immediate by Lemma 3.3 and 3.4, together with the example following Proposition 3.2 explaining why $\mathbb{Q}$ is an injective abelian group. To prove the forward direction, we let E be an injective abelian group. Consider the collection of all submodules of E which decompose as a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. This forms a poset under inclusions which respect the direct sum decomposition, and so by Zorn's lemma this poset admits a maximal element M. If $M = E$ then we are done by Lemma 3.4, so suppose for contradiction that $M \neq E$.

The short exact sequence $0 \to M \to E \to E/M \to 0$ splits: this is because M is injective (since it is a direct sum of injective abelian groups), and so the monomorphism $M \hookrightarrow E$ lifts along the identity map $M \to M$ to give a retraction $E \to M$. This means that $E = M \oplus E/M$ and so to conclude that E is injective we just need to find a submodule of $N \coloneqq E/M$ which is isomorphic to $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, because this will contradict the maximality of M and thus give us that $M = E$.[15]

We first note that N is a direct summand of the injective module E and so by Lemma 3.4 N itself must be injective. Now pick a non-zero element $x \in N$. Let C_0 be the cyclic subgroup of N which is generated by x, and let $C \subset C_0$ be a subgroup of C_0 isomorphic to either $\mathbb{Z}/p$ or $\mathbb{Z}$. Let $D = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ in the first case, and $D = \mathbb{Q}$ in the latter case. Then we have inclusion maps $f : C \hookrightarrow N$ and $g : C \hookrightarrow D$, so by the injectivity of N we have a map $h : D \to N$ such that $f = h \circ g$. We want to show that $h : D \to N$ is injective, because this will give us a copy of either $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ or $\mathbb{Q}$ inside N, which by the above argument will conclude the proof.

To this end, suppose that $h(d) = h(d') \in N$ for some $d, d' \in D$. Now, because D is either $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ we can find $a, b \in \mathbb{Z}$ such that $ad, bd' \in C$. Let $N = ab$, then also $Nd, Nd' \in C$. But $h|_C$ is injective and so $h(d) = h(d') \Rightarrow h(Nd) = Nh(d) = Nh(d') = h(Nd') \Rightarrow Nd = Nd'$, and because D is divisible we have that $d = d'$. Thus $h : D \to N$ is injective and so we have found a submodule of N which is isomorphic to D which is either $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ and so we are done. □

[15] If we find such a submodule $L \subset N$ such that L is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, then by the injectivity of L the short exact sequence $0 \to L \to N \to N/L \to 0$ splits. This means that $N = L \oplus N/L$ and so $E = M \oplus L \oplus N/L$ which contradicts the maximality of M, since L is either $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$.

Remark: If R is a principal ideal domain then we are in a good setting to use Baer's Criterion since we understand the ideals of R well, but for commutative noetherian rings whose ideals are harder to understand (for example, quotients of finitely-generated algebras over a field) we may find Baer's Criterion difficult to use since it requires access to all ideals of R. Theorem 3.3 is in fact a special case of a much more general result which provides an alternative way to find injective R-modules, which we state here for completeness but will not prove in this course: if R is a commutative noetherian ring, then associated to every prime ideal $P \triangleleft R$ there is a unique injective R-module I_P, and any injective module decomposes as a direct sum of these distinguished injective modules. So in the context of $\mathbb{Z}$, the injective module $\mathbb{Q}$ corresponds to the zero ideal, and the Prüfer groups correspond to the ideals $p\mathbb{Z} \triangleleft \mathbb{Z}$.

Chapter 4

Resolutions and Derived Functors

In mathematics, there are many instances of objects which are simple to understand but may not necessarily have desirable properties when considering some mathematical construction. This is when we may wish to use *resolutions*, which in essence is a tool to replace these objects with something which may be more complicated, but has the properties we desire and is in some sense equivalent to our original object. For example, we may study a CW-complex structure on a topological space X, which - though it may make our situation more complicated, because we have to keep track of cells and attaching maps - gives us a space which is homotopy equivalent to X and has many nice properties, such as enabling the efficient computation of the homology groups of X.

In homological algebra and modern algebraic geometry, we are interested also in studying *functors* using such approximations. This leads us to *derived functors*, which in some sense repair the failure of a particular functor to be exact, and in algebraic geometry gives rise to the central notion of sheaf cohomology. The functors of interest for us will be Tor and Ext, which respectively are the approximations of the Tensor and Hom functors, and we will meet these in Chapter 5.

The resolutions which are relevant to this setting are those involving objects with the properties of being free, flat, projective or injective. We have seen that in R-mod they can be expressed in conrecte forms, but we can also formulate the latter two of these constructions in the general setting of an abelian category, which (as mentioned in Section 2.2) is the appropriate setting for derived functors. In this chapter, $\mathcal{A}$ will denote an arbitrary abelian category.

4.0.1 Projective and injective objects

Since we are no longer working with modules over a ring R in the setting of an arbitrary abelian category, we have no obvious generalisation of the notion of free and flat modules: the former is defined in terms of a direct sum of copies of R, and the latter in terms of the tensor product functor which in turn is defined only on categories of modules. However, the definitions of projective and injective modules in terms of their lifting properties given in Chapter 3 can be used to formulate their generalisation to *projective* and *injective objects* in an arbitrary abelian category $\mathcal{A}$. Indeed, we can use the same definition as those given by (3.1) and (3.6). That is, for any objects $A, B, X, Y \in \mathcal{A}$ there exist lifts given as follows:

(4.1)

It is clear from these two diagrams that $P \in \mathcal{A}$ is projective if and only if $P \in \mathcal{A}^{op}$ is injective and vice versa. Thus projective and injective objects are dual and so any statements proven for one will also hold for the other in the opposite category[1]. Furthermore, studying the proof of Lemma 3.1 (2) we see that the arguments are given only in terms of morphisms, and so the equivalence of the projectivity of $P \in \mathcal{A}$ and the exactness of $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathbf{Ab}$ holds in $\mathcal{A}$. Dually, $E \in \mathcal{A}$ is an injective object if and only if $\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{op} \to \mathbf{Ab}$ is exact.

[1] When working in R-mod, we know by the Freyd-Mitchell Embedding theorem that $(R\text{-mod})^{op}$ can be thought of as a module category over some ring S, but there need not be any nice relationship between R and S. Thus we need to understand *both* projective and injective objects separately when working in module categories, instead of relying on their duality for one to provide insight to the other, and this

4.1 Resolutions

Let $A \in \mathcal{A}$. We say that a chain complex $(P_\bullet, d_\bullet) = (\dots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0)$ of objects $P_i \in \mathcal{A}$ is a *left resolution* of A if the following chain complex is exact:

$$\dots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0 \tag{4.2}$$

We usually write this as $P_\bullet \to A$, and we call the map $\epsilon : P_0 \twoheadrightarrow A$ the *augmentation* map. We say further that $P_\bullet \to A$ is a *projective resolution* if each $P_i \in \mathcal{A}$ is projective. Thus $P_\bullet \to A$ is a projective resolution if and only if $P_\bullet$ has zero homology in all degrees except 0, for which $H_0(P_\bullet) = A$. This is precisely the same thing as stating that the following commutative diagram is a quasi-isomorphism of chain complexes:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \epsilon} & & \downarrow \\ \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 \end{array} \tag{4.3}$$

We'll write $A[n]$ when we wish to express an object $A \in \mathcal{A}$ as a chain complex concentrated in degree n, so in this language (4.3) can be written as the quasi-isomorphism $\epsilon : P_\bullet \to A[0]$. We write this quasi-isomorphism also using the augmentation map ϵ since it is given by $\epsilon : P_0 \to A$ in degree 0 and the zero map elsewhere.

Dually, $(I^\bullet, d^\bullet)$ is an *right resolution* of $A \in \mathcal{A}$ if it is a left resolution of $A \in \mathcal{A}^{op}$, in which case we have a quasi-isomorphism $\epsilon : A[0] \to I^\bullet$ as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^0 & \xrightarrow{d^0} & I^1 & \xrightarrow{d^1} & I^2 & \longrightarrow & \dots \\ \uparrow & & \uparrow{\scriptstyle \epsilon} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \end{array} \tag{4.4}$$

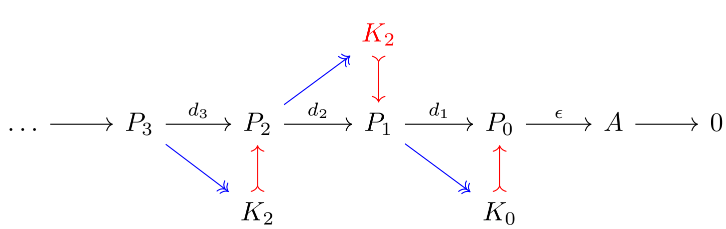
It is further an *injective resolution* if each $I_i \in \mathcal{A}$ is injective. Note that the augmentation map $\epsilon : A \rightarrowtail I^0$ is now a monomorphism $A \to I^0$, as this is the dual to the epimorphism $\epsilon : P_0 \twoheadrightarrow A$.

4.1.1 Existence of resolutions

We say that $\mathcal{A}$ has *enough projectives* if every object $A \in \mathcal{A}$ admits an epimorphism $P \twoheadrightarrow A$ where $P \in \mathcal{A}$ is projective. Dually, $\mathcal{A}$ has *enough injectives* if every $A \in \mathcal{A}$ admits a monomorphism $A \rightarrowtail I$ where $I \in \mathcal{A}$ is injective.

Proposition 4.1. *If $\mathcal{A}$ has enough projectives, then every $A \in \mathcal{A}$ admits a projective resolution. Dually, if $\mathcal{A}$ has enough injectives, then every $A \in \mathcal{A}$ admits an injective resolution.*

Proof. We construct a projective resolution by *splicing* as illustrated in (4.5) below, following [**weibel**] Lemma 2.2.5. Let $\mathcal{P}$ be the property that $\mathcal{A}$ has enough projectives. To begin, using $\mathcal{P}$ there exist a projective object and epimorphism $\epsilon : P_0 \twoheadrightarrow A$. Define $K_0 \coloneqq \ker(\epsilon)$ and then using $\mathcal{P}$ we have a projective object P_1 epimorphism $P_1 \twoheadrightarrow K_0$. Composing this with the canonical monomorphism $K_0 \rightarrowtail P_0$ gives us a map $d_1 : P_1 \to P_0$. Again, $\mathcal{P}$ gives us the existence of $P_2 \twoheadrightarrow K_1 \coloneqq \ker(d_1)$ which when post-composed with $K_1 \rightarrowtail P_1$ gives us a map $d_2 : P_2 \to P_1$. Continuing inductively gives us a chain complex of projectives, which by construction is exact. The argument for injective resolutions follows by duality.



(4.5)

□

is why Chapter 3 devotes much detail to studying both individually. But in the abstract setting of an arbitrary abelian category, in which we don't have a concrete formulation of what these objects look like, it is their duality that we will rely upon to make our proofs efficient.

To show that R-mod has enough projectives is straightforward. Indeed, if $M \in R$-mod then define P to be the free module indexed by the elements of M (or any other generating set of M), which since free modules are projective gives us a projective object P which surjects onto M. Moreover, we can ensure that a projective resolution $P_\bullet \to M$ as constructed in (4.5) is in a fact *free resolution*, by taking P_{i+1} to be the free R-module indexed by the elements of K_i for $i \geqslant 0$. Thus when looking to construct a projective resolution in R-mod we can simply use the construction given in Proposition 4.1 to obtain a resolution $P_\bullet \to M$ where each $P_i = R^{J_i}$ for some indexing set J_i.

There is no similarly easy way to construct injective resolutions in R-mod for a general ring R, and this is reflected in the comparatively more involved proof of the following theorem. But first, we need to introduce and prove two results about *pushouts* in R-mod. Given morphisms $\phi : A \to B$ and $\psi : A \to C$ in R-mod, we define their pushout to be the universal object X which makes the first diagram below commute. The universality of X means that for any other $\tilde{X}$ which makes such a square commute, there exists a unique map $\alpha : X \to \tilde{X}$ which makes the second diagram below commute:[2]

$$\begin{array}{ccc} B & \longrightarrow & X \\ {\scriptstyle\phi}\uparrow & & \uparrow \\ A & \underset{\psi}{\longrightarrow} & C \end{array} \qquad\qquad \begin{array}{cccc} & & & \tilde{X} \\ & & \overset{\alpha}{\nearrow} & \\ B & \longrightarrow & X & \\ {\scriptstyle\phi}\uparrow & & \uparrow & \\ A & \underset{\psi}{\longrightarrow} & C & \end{array} \tag{4.6}$$

Lemma 4.1. *In R-mod the categorical pushout of a diagram $B \overset{\phi}{\leftarrow} A \overset{\psi}{\rightarrow} C$ can be expressed as the R-module which is the quotient of $B \oplus C$ by the submodule $\{(f(a), -g(a)) \mid a \in A\}$.*

Proof. X ☐

Lemma 4.2. *Let X be the pushout of the diagram $B \overset{\phi}{\leftarrow} A \overset{\psi}{\rightarrow} C$. If ψ is injective then so is the map $\tilde{\psi} : B \to X$, and if ϕ is injective then so is the map $\tilde{\phi} : C \to X$, where $\tilde{\phi}$ and $\tilde{\psi}$ are the canonical pushout maps.*

Proof. Suppose $b \in B$ is such that $\tilde{\psi}(b) = 0 \in X$. Lemma 4.1 gives us that X is the quotient of $B \oplus C$ by the submodule $\{((\phi(a), -\psi(a)) \mid a \in A\}$. By the definition of the canonical pushout map $\tilde{\psi}$, this means that $\tilde{\psi}(b) = [(b, 0)] = 0 \in X$, and so $(b, 0) = (f(a), -g(a))$ for some $a \in A$. Thus $\psi(a) = 0$, which by the injectivity of ψ imples that $a = 0$. Hence $(b, 0) = (0, 0)$ and so $b = 0$, which proves the injectivity of $\tilde{\psi}$. By symmetry we see that the injectivity of ϕ implies that of $\tilde{\phi}$. ☐

Theorem 4.1. *R-mod has enough injectives.*

Proof. We begin with the case in which R is noetherian. Given an R-module M, we wish to find an injective module E such that there exists an injection $M \hookrightarrow E$. To this end, let $S_0 := \{(I, f) : I \lhd R \text{ and } f \in \mathrm{Hom}_R(I, M)\}$. Consider the following pushout diagram where $\mathfrak{P}(M)$ denotes the pushout:

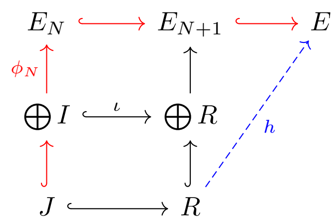
$$\begin{array}{ccc} M & \longrightarrow & \mathfrak{P}(M) \\ {\scriptstyle\phi_0}\uparrow & & \uparrow \\ \bigoplus I & \overset{\iota}{\hookrightarrow} & \bigoplus R \end{array} \tag{4.7}$$

Here, $\phi_0 := \bigoplus f$ and the direct sums are indexed over all elements $(I, f) \in S$. Define $E_0 := M$ and $E_{n+1} := \mathfrak{P}(E_n)$, and let $E := \mathrm{colim}_{n\in\mathbb{N}}(E_0 \to E_1 \to \ldots)$. By Lemma 4.2, all the maps $E_k \to E_{k+1}$ are injections and so by Corollary 3.2 we see that $E = \bigcup_{k\in\mathbb{N}} E_k$ and so we have an injection $M \hookrightarrow E$. It remains to show that E is indeed an injective R-module, and we will do this using Baer's Criterion.

We now invoke the noetherian property of R: for any ideal $J \lhd R$ and map $g : J \to E$, the image of g must factor through some finite stage of the colimit because J is finitely-generated. In other words, $g : J \to E$ is in fact a map $g' : J \to E_N$ composed with the canonical inclusion $E_N \hookrightarrow E$, because we can choose $N \in \mathbb{N}$ large enough to ensure that E_N includes the images of all generators of J. We will now prove the existence of a map $h : R \to E$ such that

[2] The categorical dual of the pushout is the *pullback*, obtained from (4.6) by simply reversing all arrows. Lemma 4.1 exhibits the R-module structure of the pushout, we can similarly formulate the pullback in R-mod as the submodule $\{(b, c) \in B \oplus C \mid \phi(b) = \psi(c)\}$.

$h|_J = g$ which by Baer's Criterion will prove the injectivity of E. Consider the following commutative diagram arising from the N-th pushout:



(4.8)

Similarly to (4.6), the map $\phi_N = \bigoplus f$ and the direct sums in (4.7) are indexed over all pairs $(I, f) \in S_N$, where for $k \in \mathbb{N}$ we define S_k to be $\{(I, f) : I \lhd R \text{ and } f \in \mathrm{Hom}_R(I, E_k)\}$. The inclusion maps $J \hookrightarrow \bigoplus I$ and $R \hookrightarrow \bigoplus R$ in (4.7) are those which include into the (J, g')-summand. Thus by the definition of ϕ_N, the map $J \hookrightarrow \bigoplus I \to E_N$ above given by the pre-composition of ϕ_N with the (J, g')-summand inclusion map is precisely our map $g' : J \to E_N$. Post-composing g' with the inclusion $E_N \hookrightarrow E_{N+1}$ yields a map $g'' : J \to E_{N+1}$, and by the commutativity of (4.7) the map $R \hookrightarrow \bigoplus R \to E_{N+1}$ gives a lift of g''. Post-composing both this map and g'' with the inclusion $E_{N+1} \hookrightarrow E$ gives us a lift $h : R \to E$ of our map $g : J \to E$, which by Baer's Criterion proves the injectivity of E. The maps g and h are denoted in (4.7) by the red maps and the blue dashed map respectively.

We now turn to the case when R is no longer noetherian. We have used this property precisely once in our proof so far, from which we obtained that $g : J \to E$ factors through some *finite* stage of the colimit defining E because $J \lhd R$ is finitely-generated. In other words, we used the fact that the colimit contains infinitely many terms, which by the finiteness property on J requires g to factor through the N-th pushout diagram (4.7) for some $N \in \mathbb{N}$, which gives us the required lift h of g. To adapt our proof to the case in which an arbitrary ideal J need not be finitely generated, we turn to *ordinal* and *cardinal* numbers[3]to ensure that g factors at some stage through a pushout diagram as in (4.7).

Let λ be an ordinal number whose cardinality is greater than that of R. Now define $E_0 := M$, and for each ordinal $\alpha < \lambda$ define inductively $E_\alpha := \mathfrak{P}(E_\beta)$ when $\alpha = \beta + 1$ for some ordinal β and $E_\alpha := \mathrm{colim}_{\beta<\alpha} E_\beta$ when α is a limit ordinal. Then $E := M_\lambda$ is an injective module which contains M as a submodule by the same argument as above, using now that $g : J \to E$ factors through the pushout diagram above at some ordinal-indexed stage $\gamma < \lambda$.

□

4.1.2 Constructing resolutions in R-mod

4.1.3 Preparing for derived functors

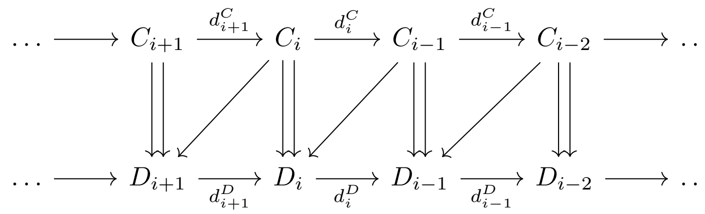
As mentioned at the start of this chapter, the idea behind replacing an an object $A \in \mathcal{A}$ with a resolution $X_\bullet$ of A is that, while $X_\bullet$ may be an infinite chain complex and thus more unwieldy than A, it may have suitably nice properties for our purposes (for example, each X_i may be projective) and it is quasi-isomorphic to A. In other words, $X_\bullet$ it is equivalent to $A[0]$ on the level of homology. However, if we want any two projective or injective resolutions of A to be unique up to some equivalence (which will be essential for our construction of derived functors in Section 4.2), then quasi-isomorphism is not a strong enough condition and so we turn to the notion of *chain homotopy.*[4]

Recalling the definition of a chain map from Section 2.2, we will say that two chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are *chain-homotopic* if there exists a chain map $h_\bullet : C_\bullet \to D_{\bullet+1}$ such that $d^D_{i+1} \circ h_i + h_{i-1} \circ d^C_i = f_i - g_i$. It is straightforward to show that the notion of chain homotopy is indeed an equivalence relation. The chain homotopy given by h is denoted by the diagonal maps in (4.8), and the algebraic relation between f and g is shown diagrammatically in (4.9) below:

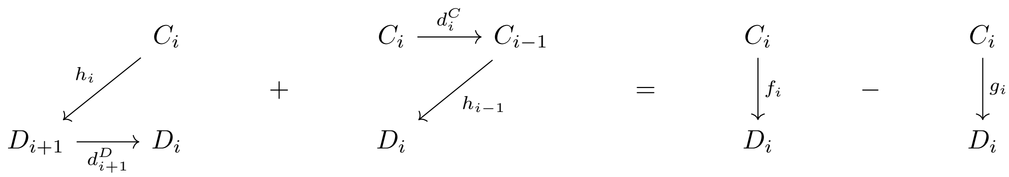
[3]A set X can be *well-ordered* if there exists a total order on X such that any subset of X has a least element. For example, with the usual relation $\leqslant$ we know that $\mathbb{N}$ is well-ordered, but $\mathbb{Z}$ is not since the subset of negative integers has no least element. Equivalent to the axiom of choice and Zorn's lemma is the *well-ordering theorem*, which states that every set can be well-ordered.

Intuitively speaking, *ordinal* numbers are a generalisation of the natural numbers to the setting of different types of infinity, and can be thought of as arbitrarily large infinities whose well-order extends that of $\mathbb{N}$. To each ordinal number α we can associate a *cardinal* number, namely its cardinality $|\alpha|$. As a consequence of the well-ordering theorem, on any set we can perform *transfinite induction*. This uses the total order on ordinal numbers to generalise induction over $\mathbb{N}$, which allows us to exhibit inductive arguments over infinities of any cardinality.

[4]As you may have guessed by its name, the notion of chain homotopy does indeed have topological origins, as explained in the subsection *Homotopy Invariance* in [**hatcher**] Section 2.1.



(4.9)



(4.10)

For the rest of this chapter we assume that our arbitrary abelian category $\mathcal{A}$ has enough projectives and injectives, so that by Proposition 4.1 for any object in $\mathcal{A}$ there exist projective and injective resolutions.

Proposition 4.2. *Let M and N be objects of $\mathcal{A}$ and $P_\bullet \to M$ a chain complex such that each P_i is projective. Then for any morphism $\alpha : M \to N$ and left resolution $Q_\bullet \to N$, there exists a chain map $f_\bullet : P_\bullet \to Q_\bullet$ unique up to homotopy equivalence which lifts α, in the sense that the following diagram commutes:*

4.2 Derived functors

Recall from Section 2.2 the definition of the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes of objects in $\mathcal{A}$. We will now form the *derived category*: for the purposes of this course we won't go into detail with the derived category, but will just use it to to define derived functors. To construct this category, we first pass from $\mathbf{Ch}(\mathcal{A})$ to the *homotopy category* $K(\mathcal{A})$ by identifying chain maps which are chain homotopic. In other words, the objects of $K(\mathcal{A})$ are precisely those of $\mathbf{Ch}(\mathcal{A})$, and its morphisms are $\mathrm{Hom}_{K(\mathcal{A})}(X, Y) = \mathrm{Hom}_{\mathbf{Ch}(\mathcal{A})}(X, Y)/\sim$, where $f \sim g$ if and only if there exists a chain homotopy between f and g. To produce the derived category $D(\mathcal{A})$ we now formally invert all quasi-isomorphisms in $K(\mathcal{A})$. While $\mathbf{Ch}(\mathcal{A})$ is an abelian category whenever $\mathcal{A}$ is, the homotopy category and derived category need no longer be abelian categories.

The idea of the derived category is to treat chain complexes and their resolutions on an equal footing, which means that we treat quasi-isomorphisms as isomorphisms. This provides the right setting for derived functors, since we will be defining them in terms of homology and so we will want to identify those complexes which are quasi-isomorphic as these will be isomorphic on the level of homology. We can think of the construction of $D(\mathcal{A})$ as analogous to forming the field of fractions of an integral domain, in which we formally invert elements and then impose an equivalence relation: in the case of $D(\mathcal{A})$ we invert quasi-isomorphisms. See [**weibel**] Chapter 10 for more details.

4.2.1 Derivations of the inverse limit functor

Chapter 5

Tor and Ext

5.1 Motivation

5.2 Defining Tor and Ext

5.3 Total complexes

5.4 Computing Tor and Ext

Chapter 6

Understanding Ext

6.1 Extensions and the Baer sum

6.2 The Ext ring

Appendix A

Results in abelian categories

Theorem A.1. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be functors between abelian categories $\mathcal{A}$ and $\mathcal{B}$ such that $F \dashv G$. Then F preserves colimits of $\mathcal{A}$ and G preserves limits of $\mathcal{B}$. This means that for diagrams $X : \mathcal{J}_1 \to \mathcal{C}$ and $Y : \mathcal{J}_2 \to \mathcal{B}$ whose colimits and limits exist in $\mathcal{A}$ and $\mathcal{B}$ respectively, we have the following natural isomorphisms:*

$$F(\mathrm{colim}_j X_j) \simeq \mathrm{colim}_j F(X_j), \qquad G(\lim_j Y_j) \simeq \lim_j G(Y_j) \tag{A.1}$$

Proof. X $\square$

Theorem A.2. *Let R be a principal ideal domain. Then any submodule of a free R-module will itself be free.*

Proof. X $\square$

Theorem A.3. *Let A be a finitely-generated module over a principal ideal domain R. Then there exists a unique decreasing sequence of proper ideals $(d_1) \supset (d_2) \supset \ldots \supset (d_n)$ such that $A \simeq \bigoplus_{i=1}^{n} R/(d_i)$.*

Proof. X $\square$