Course 311: Hilary Term 2006
Part VI: Introduction to Affine Schemes

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Contents

6 Introduction to Affine Schemes

6.1 Rings and Modules of Fractions

Let R be a unital commutative ring. A subset S of R is said to be a *multiplicative subset* if $1 \in S$ and $ab \in S$ for all $a \in S$ and $b \in S$.

Let M be a module over a unital commutative ring R, and let S be a multiplicative subset of R. We define a relation $\sim$ on $M \times S$, where elements (m, s) and (m', s') of $M \times S$ satisfy the relation $(m, s) \sim (m', s')$ if and only if $us'm = usm'$ for some element u of S.

The relation $\sim$ on $M \times S$ is clearly reflexive and symmetric. It is also transitive. Indeed if (m, s), (m', s') and (m'', s'') are elements of $M \times S$, and if $(m, s) \sim (m', s')$ and $(m', s') \sim (m'', s'')$, then there exist elements u and v of S such that $us'm = usm'$ and $vs''m' = vs'm''$. But then $uvs' \in S$, and

$$\begin{aligned}(uvs')(s''m) &= (vs'')(us'm) = (vs'')(usm') = (us)(vs''m') \\ &= (us)(vs'm'') = (uvs')(sm''),\end{aligned}$$

and hence $(m, s) \sim (m'', s'')$. This shows that the relation $\sim$ on $M \times S$ is transitive. We conclude that the relation $\sim$ is an equivalence relation on $M \times S$. Let $S^{-1}M$ denote the set of equivalence classes arising from this relation. Given any element (m, s) of $M \times S$, we denote by m/s the element of $S^{-1}M$ that represents the equivalence class of (m, s). Note that ordered pairs (m, s) and (m'/s') in $M \times S$ satisfy the equation $m/s = m'/s'$ if and only if there exists some element u of S such that $us'm = usm'$.

We claim that there is a well-defined operation of addition on the set $S^{-1}M$, defined such that

$$m_1/s_1 + m_2/s_2 = (s_2m_1 + s_1m_2)/(s_1s_2).$$

To verify this, let $m_1, m_1', m_2, m_2' \in M$ and $s_1, s_1', s_2, s_2' \in S$ satisfy the equations $m_1/s_1 = m_1'/s_1'$ and $m_2/s_2 = m_2'/s_2'$ Then there exist elements u_1 and u_2 of S such that such that $u_1s_1'm_1 = u_1s_1m_1'$ and $u_2s_2'm_2 = u_2s_2m_2'$. Then

$$\begin{aligned}(u_1u_2)(s_1's_2')(s_2m_1 + s_1m_2) &= (u_2s_2s_2')(u_1s_1'm_1) + (u_1s_1s_1')(u_2s_2'm_2) \\ &= (u_2s_2s_2')(u_1s_1m_1') + (u_1s_1s_1')(u_2s_2m_2') \\ &= (u_1u_2)(s_1s_2)(s_2'm_1' + s_1'm_2')\end{aligned}$$

and therefore

$$(s_2m_1 + s_1m_2)/(s_1s_2) = (s_2'm_1' + s_1'm_2')/(s_1's_2').$$

This shows that the operation of addition on $S^{-1}M$ is well-defined. Moreover, given elements $m, m_1, m_2, m_3 \in M$ and $s, s_1, s_2, s_3 \in S$, it is easy to verify that

$$\begin{aligned} m_1/s_1 + m_2/s_2 &= m_2/s_2 + m_1/s_1, \\ (m_1/s_1 + m_2/s_2) + m_3/s_3 &= (s_2 s_3 m_1 + s_1 s_3 m_2 + s_1 s_2 m_3)/(s_1 s_2 s_3) \\ &= m_1/s_1 + (m_2/s_2 + m_3/s_3), \\ 0/1 + m/s &= m/s, \\ m/s + (-m)/s &= 0/s^2 = 0/1. \end{aligned}$$

It follows that $S^{-1}M$ is an Abelian group with respect to the operation of addition. The zero element of this group is the element $0/1$.

Any unital commutative ring R may be considered as a module over itself. Thus, given any non-empty multiplicative subset S of R, we can form an Abelian group $S^{-1}R$. Each ordered pair (r, s) in $R \times S$ determines an element r/s of $S^{-1}R$. Moreover the elements r/s and r'/s' determined by ordered pairs (r, s) and (r', s') satisfy the equation $r/s = r'/s'$ if and only if $us'r = usr'$ for some element u of S. The operation of addition on the Abelian group $S^{-1}R$ is defined by the equation

$$r_1/s_1 + r_2/s_2 = (s_2 r_1 + s_1 r_2)/(s_1 s_2)$$

for all $r_1, r_2 \in R$ and $s_1, s_2 \in S$.

Let R be a unital commutative ring, let S be a non-empty multiplicative subset of R, and let M be a module over the ring R. If r and r' are elements of R, m and m' are elements of M, and s, s', t and t' are elements of S, and if $r/s = r'/s'$ and $m/t = m'/t'$, then there exist elements u and v of S such that $us'r = usr'$ and $vt'm = vtm'$. Then

$$(uv)(s't')(rm) = (uv)(st)(mm'),$$

and hence $(rm)/(st) = (r'm')(s't')$. It follows that there is a well-defined operation that multiplies elements r/s of $S^{-1}R$ by elements m/t of $S^{-1}R$, where $(r/s)(m/t) = (rm)/(st)$.

In particular we can define a multiplication operation on the Abelian group $S^{-1}R$ by defining

$$(r_1/s_1)(r_2/s_2) = (r_1 r_2)/s_1 s_2$$

for all elements r_1/s_1 and r_2/s_2 of $S^{-1}R$. This multiplication operation is commutative and associative, and, for any element s of S, the element s/s

of $S^{-1}R$ is a multiplicative identity element for $S^{-1}R$. Also

$$\begin{aligned} (r/s)(r_1/s_1 + r_2/s_2) &= (r/s)((s_2r_1 + s_1r_2)/s_1s_2) \\ &= (s_2rr_1 + s_1rr_2)(ss_1s_2) \\ &= (ss_2rr_1 + ss_1rr_2)(s^2s_1s_2) \\ &= (rr_1)/(ss_1) + (rr_2)/(ss_2) \\ &= (r/s)(r_1/s_1) + (r/s)(r_2/s_2) \end{aligned}$$

for all $r, r_1, r_2 \in R$ and $s, s_1, s_2 \in S$, and therefore the operations of addition and multiplication on $S^{-1}R$ satisfy the distributive law. Therefore the operations of addition and multiplication on $S^{-1}R$ therefore give $S^{-1}R$ the structure of a unital commutative ring. Moreover it is a straightforward exercise to verify that if M is a module over the ring R, then the Abelian group $S^{-1}M$ is a module over the ring $S^{-1}R$, where $(r/s)(m/t) = (rm)/(st)$ for all $r \in R$, $m \in M$ and $s, t \in S$.

Example Let R be the ring $\mathbb{Z}$ of integers, and let S be the set $\mathbb{Z}^*$ of non-zero integers. Then $S^{-1}R$ represents the field $\mathbb{Q}$ of rational numbers. Also $\mathbb{Z}^{*-1}\mathbb{Z}^n \cong \mathbb{Q}^n$. Indeed the function from $\mathbb{Z}^{*-1}\mathbb{Z}^n$ to $\mathbb{Q}^n$ that sends $(m_1, \ldots, m_n)/s$ to $(s^{-1}m_1, \ldots, s^{-1}m_n)$ for all integers $m_1, \ldots, m_n$ and for all non-zero integers s is well-defined, and is an isomorphism.

Any Abelian group may be regarded as a module over the ring $\mathbb{Z}$ of integers. If A is a finite Abelian group then $\mathbb{Z}^{*-1}A = \{0\}$. For there exists an non-zero integer n such that $na = 0$ for all $a \in A$. (We can take n to be the order $|A|$ of the group.) Then $a/s = (na)/ns = 0/ns = 0/1$ for all $a \in A$ and $s \in \mathbb{Z}^*$.

Example Let S be the set $\{1, 2, 4, 8, \ldots\}$ of non-negative powers of the integer 2. Then $S^{-1}\mathbb{Z}$ is the ring of dyadic rational numbers. (A *dyadic rational number* is a rational number of the form $m/2^n$ for some integers m and n.)

Let R be an integral domain. Then the set R^* of non-zero elements of R is a multiplicative subset. Let $Q(R) = R^{*-1}R$. Then $Q(R)$ is a field. Its elements may be represented in the form r/s, where $r, s \in R$ and $s \neq 0$. Let r, r', s and s' be elements of R, where s and s' are non-zero. Then $r/s = r'/s'$ if and only if $s'r = sr'$. For it follows from the definition of $R^{*-1}R$ that if $r/s = r'/s'$ then there exists some non-zero element u of R such that $u(s'r - sr') = 0$. But then $s'r - sr' = 0$, since the product of non-zero elements of an integral domain is always non-zero. The field $Q(R)$ is referred to as the *field of fractions* of the integral domain R. There is a homomorphism $i: R \to Q(R)$ from R to $Q(R)$, where $i(r) = r/1$ for all $r \in R$.

If r_1 and r_2 are elements of the integral domain R and if $r_1/1 = r_2/1$ then $r_1 = r_2$. It follows that the homomorphism $i\colon R \to Q(R)$ is injective, and gives a natural embedding of the integral domain R in its field of fractions, enabling one to view R as a subring of the field $Q(R)$.

The field of fractions of the ring $\mathbb{Z}$ of integers is the field $\mathbb{Q}$ of rational numbers.

6.2 The Spectrum of a Unital Commutative Ring

Let R be a unital commutative ring. A *prime ideal* of R is a proper ideal $\mathfrak{p}$ of R with the property that, given any two elements r_1 and r_2 of R for which $r_1 r_2 \in \mathfrak{p}$, either $r_1 \in \mathfrak{p}$ or $r_2 \in \mathfrak{p}$.

Let $\operatorname{Spec} R$ denote the set of prime ideals of the ring R. For each ideal $\mathfrak{a}$ of R, let $V(\mathfrak{a})$ denote the subset of $\operatorname{Spec} R$ consisting of all prime ideals $\mathfrak{p}$ for which $\mathfrak{a} \subset \mathfrak{p}$. We claim that there is a well-defined topology on $\operatorname{Spec} R$ whose closed sets are the sets that are of the form $V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of R.

Given any collection $\{\mathfrak{a}_\lambda : \lambda \in \Lambda\}$ of ideals of R, we can form their sum $\Sigma_{\lambda\in\Lambda}\mathfrak{a}_\lambda$, which is the ideal consisting of all elements of R that can be expressed as a finite sum of the form $x_1 + x_2 + \cdots + x_r$ where each summand x_i is an element of some ideal $\mathfrak{a}_{\lambda_i}$ belonging to the collection.

Also given any two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R, we can form their product $\mathfrak{a}\mathfrak{b}$. This ideal $\mathfrak{a}\mathfrak{b}$ is the ideal of R consisting of all elements of R that can be expressed as a finite sum of the form $x_1y_1 + x_2y_2 + \cdots + x_ry_k$ with $x_i \in \mathfrak{a}$ and $y_i \in \mathfrak{b}$ for $i = 1, 2, \ldots, r$.

Proposition 6.1 *Let R be a unital commutative ring, let $\operatorname{Spec} R$ be the spectrum of R, and for each ideal $\mathfrak{a}$ of R let $V(\mathfrak{a})$ denote the subset of $\operatorname{Spec} R$ defined by*

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{a} \subset \mathfrak{p}\}.$$

Then these subsets of $\operatorname{Spec} R$ have the following properties:

(i) $V(\{0\}) = \operatorname{Spec} R$ *and* $V(R) = \emptyset$*;*

(ii) $\bigcap_{\lambda\in\Lambda} V(\mathfrak{a}_\lambda) = V\left(\sum_{\lambda\in\Lambda}\mathfrak{a}_\lambda\right)$ *for every collection* $\{\mathfrak{a}_\lambda : \lambda \in \Lambda\}$ *of ideals of R;*

(iii) $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$ *for all ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R.*

Thus there is a well-defined topology on $\operatorname{Spec} R$ whose closed sets are the sets that are of the form $V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of R.

Proof The zero element of R belongs to every ideal, and therefore $V(\{0\}) = \operatorname{Spec} R$. Also $V(R) = \emptyset$ since every prime ideal is by definition a proper ideal of R. This proves (i).

Let $\mathfrak{p}$ be a prime ideal of R. Then

$$\begin{aligned} \mathfrak{p} \in \bigcap_{\lambda\in\Lambda} V(\mathfrak{a}_\lambda) &\iff \mathfrak{p} \in V(\mathfrak{a}_\lambda) \text{ for all } \lambda \in \Lambda \\ &\iff \mathfrak{a}_\lambda \subset \mathfrak{p} \text{ for all } \lambda \in \Lambda \\ &\iff \sum_{\lambda\in\Lambda} \mathfrak{a}_\lambda \subset \mathfrak{p} \\ &\iff \mathfrak{p} \in V\left(\sum_{\lambda\in\Lambda} \mathfrak{a}_\lambda\right) \end{aligned}$$

It follows that

$$\bigcap_{\lambda\in\Lambda} V(\mathfrak{a}_\lambda) = V\left(\sum_{\lambda\in\Lambda} \mathfrak{a}_\lambda\right)$$

for any collection $\{\mathfrak{a}_\lambda : \lambda \in \Lambda\}$ of ideals in R. This proves (ii).

Now let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R, and let $\mathfrak{a}\mathfrak{b}$ denote the ideal consisting of all elements of R that can be expressed as a finite sum of the form $x_1y_1 + x_2y_2 + \cdots + x_ry_k$ with $x_i \in \mathfrak{a}$ and $y_i \in \mathfrak{b}$ for $i = 1, 2, \ldots, r$.

If $\mathfrak{p}$ is a prime ideal of R, and if $\mathfrak{p} \notin V(\mathfrak{a})$ and $\mathfrak{p} \notin V(\mathfrak{b})$ then the sets $\mathfrak{a} \setminus \mathfrak{p}$ and $\mathfrak{b} \setminus \mathfrak{p}$ are non-empty. Let $x \in \mathfrak{a} \setminus \mathfrak{p}$ and $y \in \mathfrak{b} \setminus \mathfrak{p}$. Then $xy \in \mathfrak{a}\mathfrak{b} \setminus \mathfrak{p}$, and therefore $\mathfrak{p} \notin V(\mathfrak{a}\mathfrak{b})$. It follows from this that $V(\mathfrak{a}\mathfrak{b}) \subset V(\mathfrak{a}) \cap V(\mathfrak{b})$. But also $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{a}$ and $\mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{b}$, and therefore $V(\mathfrak{a} \cap \mathfrak{b}) \subset V(\mathfrak{a}\mathfrak{b})$, $V(\mathfrak{a}) \subset V(\mathfrak{a} \cap \mathfrak{b})$ and $V(\mathfrak{b}) \cap V(\mathfrak{a} \cap \mathfrak{b})$. Thus

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) \subset V(\mathfrak{a} \cap \mathfrak{b}) \subset V(\mathfrak{a}\mathfrak{b}) \subset V(\mathfrak{a}) \cap V(\mathfrak{b}).$$

and therefore

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}).$$

This proves (iii). ∎

We shall regard the spectrum $\operatorname{Spec} R$ of any unital commutative ring as a topological space whose closed sets are the subsets of $\operatorname{Spec} R$ that are of the form $V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of R.

Let R_1 and R_2 be unital commutative rings, and let 1_{R_1} and 1_{R_2} denote the multiplicative identity elements of R_1 and R_2. A function $\varphi R_1 \to R_2$ from R_1 to R_2 is said to be a *unital homomorphism* if $\varphi(x + y) = \varphi(x) + \varphi(y)$, $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(1_{R_1}) = \varphi(1_{R_2})$ for all $x, y \in R_1$.

Lemma 6.2 *Let $\varphi\colon R_1 \to R_2$ be a unital homomorphism between unital commutative rings R_1 and R_2. Then $\varphi\colon R_1 \to R_2$ induces a continuous map $\varphi^*\colon \operatorname{Spec} R_2 \to \operatorname{Spec} R_1$ from $\operatorname{Spec} R_2$ to $\operatorname{Spec} R_1$, where $\varphi^*(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ for every prime ideal $\mathfrak{p}$ of R_2.*

Proof Let $\mathfrak{p}$ be a prime ideal of R_2. Now $1_{R_2} \notin \mathfrak{p}$, because $\mathfrak{p}$ is a proper ideal of R_2, and any ideal of R_2 that contains the identity element 1_{R_2} must be the whole of R_2. But then $1_{R_1} \notin \varphi^{-1}(R_2)$, since $\varphi(1_{R_1}) = 1_{R_2}$. It follows that $\varphi^{-1}(\mathfrak{p})$ is a proper ideal of R_1.

Let x and y be elements of R. Suppose that $xy \in \varphi^{-1}(\mathfrak{p})$. Then $\varphi(x)\varphi(y) = \varphi(xy)$ and therefore $\varphi(x)\varphi(y) \in \mathfrak{p}$. But $\mathfrak{p}$ is a prime ideal of R_2, and therefore either $\varphi(x) \in \mathfrak{p}$ or $\varphi(y) \in \mathfrak{p}$. Thus either $x \in \varphi^{-1}(\mathfrak{p})$ or $y \in \varphi^{-1}(\mathfrak{p})$. This shows that $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of R_1. We conclude that there is a well-defined function $\varphi^*\colon \operatorname{Spec} R_2 \to \operatorname{Spec} R_1$ such that $\varphi^*(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of R_2.

Let $\mathfrak{a}$ be an ideal of R_1, and let $\mathfrak{b}$ be the ideal of R_2 generated by $\varphi(\mathfrak{a})$. (This ideal $\mathfrak{b}$ is the intersection of all ideals $\mathfrak{c}$ of R_2 for which $\varphi(\mathfrak{a}) \subset \mathfrak{c}$.) Then

$$\begin{aligned}
\varphi^{*-1}(V(\mathfrak{a})) &= \{\mathfrak{p} \in \operatorname{Spec} R_2 : \varphi^{-1}(\mathfrak{p}) \in V(\mathfrak{a})\} \\
&= \{\mathfrak{p} \in \operatorname{Spec} R_2 : \mathfrak{a} \subset \varphi^{-1}(\mathfrak{p})\} \\
&= \{\mathfrak{p} \in \operatorname{Spec} R_2 : \varphi(\mathfrak{a}) \subset \mathfrak{p}\} \\
&= \{\mathfrak{p} \in \operatorname{Spec} R_2 : \mathfrak{b} \subset \mathfrak{p}\} = V(\mathfrak{b}).
\end{aligned}$$

Thus the preimage (under φ^*) of every closed set in $\operatorname{Spec} R_1$ is a closed set in $\operatorname{Spec} R_2$. It follows from this that the function $\varphi^*\colon \operatorname{Spec} R_2 \to \operatorname{Spec} R_1$ is continuous, as required. ∎

6.3 The Spectrum of a Quotient Ring

Let $\mathfrak{a}$ be a proper ideal of a unital commutative ring R. Then the quotient ring $R/\mathfrak{a}$ is a unital commutative ring. The *quotient homomorphism* $\pi_{\mathfrak{a}}\colon R \to R/\mathfrak{a}$ is the surjective homomorphism that sends each element x of R to $\mathfrak{a}+x$.

Proposition 6.3 *Let R be a unital commutative ring, let $\mathfrak{a}$ be a proper ideal of R, and let $\pi_{\mathfrak{a}}\colon R \to R/\mathfrak{a}$ be the corresponding quotient homomorphism onto the quotient ring $R/\mathfrak{a}$. Then the induced map $\pi_{\mathfrak{a}}^*\colon \operatorname{Spec} R/\mathfrak{a} \to \operatorname{Spec} R$ maps $\operatorname{Spec} R/\mathfrak{a}$ homeomorphically onto the closed set $V(\mathfrak{a})$.*

Proof Let $\mathfrak{q}$ be a prime ideal of $R/\mathfrak{a}$. Then $\mathfrak{a} \subset \pi_{\mathfrak{a}}^{-1}(\mathfrak{q})$ and therefore $\pi_{\mathfrak{a}}^*(\mathfrak{q}) \subset V(\mathfrak{a})$. We conclude that $\pi_{\mathfrak{a}}^*(\operatorname{Spec} R/\mathfrak{a}) \subset V(\mathfrak{a})$.

Let $\mathfrak{p}$ be a prime ideal of R belonging to $V(\mathfrak{a})$, and let $\mathfrak{q} = \pi_{\mathfrak{a}}(\mathfrak{p})$. Now $\mathfrak{a} \subset \mathfrak{p}$, and therefore $\pi_{\mathfrak{a}}^{-1}(\mathfrak{q}) = \mathfrak{a} + \mathfrak{p} = \mathfrak{p}$. It follows from that that $\mathfrak{q}$ must be a proper ideal of $R/\mathfrak{a}$. Let x and y be elements of R with the property that $(\mathfrak{a}+x)(\mathfrak{a}+y) \in \mathfrak{q}$. Then $\pi_{\mathfrak{a}}(xy) \in \mathfrak{q}$, and therefore $xy \in \mathfrak{p}$. But then either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$, and thus either $\mathfrak{a}+x \in \mathfrak{q}$ or $\mathfrak{a}+y \in \mathfrak{q}$. This shows that $\mathfrak{q}$ is a prime ideal of $R/\mathfrak{a}$. Moreover $\mathfrak{p} = \pi_{\mathfrak{a}}^*(\mathfrak{q})$. We conclude that $\pi_{\mathfrak{a}}^*(\operatorname{Spec} R/\mathfrak{a}) = V(\mathfrak{a})$.

If $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are prime ideals of $R/\mathfrak{a}$, and if $\pi_{\mathfrak{a}}^*(\mathfrak{q}_1) = \pi_{\mathfrak{a}}^*(\mathfrak{q}_2)$ then $\pi_{\mathfrak{a}}^{-1}(\mathfrak{q}_1) = \pi_{\mathfrak{a}}^{-1}(\mathfrak{q}_2)$, and therefore $\mathfrak{q}_1 = \pi_{\mathfrak{a}}(\pi_{\mathfrak{a}}^*(\mathfrak{q}_1)) = \pi_{\mathfrak{a}}(\pi_{\mathfrak{a}}^*(\mathfrak{q}_2)) = \mathfrak{q}_2$. It follows that the map $\pi_{\mathfrak{a}}^*\colon \operatorname{Spec} R/\mathfrak{a} \to \operatorname{Spec} R$ is injective. We have now shown that $\pi_{\mathfrak{a}}^*\colon \operatorname{Spec} R/\mathfrak{a} \to \operatorname{Spec} R$ maps the spectrum $\operatorname{Spec} R/\mathfrak{a}$ bijectively onto the closed subset $V(\mathfrak{a})$ of the spectrum $\operatorname{Spec} R$ of R.

Let $\mathfrak{b}$ is an ideal of $R/\mathfrak{a}$, and let $\mathfrak{q}$ be a prime ideal of $R/\mathfrak{a}$. Then $\pi_{\mathfrak{a}}(\pi_{\mathfrak{a}}^{-1}(\mathfrak{b})) = \mathfrak{b}$ and $\pi_{\mathfrak{a}}(\pi_{\mathfrak{a}}^{-1}(\mathfrak{q})) = \mathfrak{q}$. It follows that $\pi_{\mathfrak{a}}^{-1}(\mathfrak{b}) \subset \pi_{\mathfrak{a}}^{-1}(\mathfrak{q})$ if and only if $\mathfrak{b} \subset \mathfrak{q}$. But then

$$\begin{aligned} V(\pi_{\mathfrak{a}}^{-1}(\mathfrak{b})) \cap V(\mathfrak{a}) &= \{\mathfrak{p} \in V(\mathfrak{a}) : \pi_{\mathfrak{a}}^{-1}(\mathfrak{b}) \subset \mathfrak{p}\} \\ &= \pi_{\mathfrak{a}}^*\{\mathfrak{q} \in \operatorname{Spec} R/\mathfrak{a} : \pi_{\mathfrak{a}}^{-1}(\mathfrak{b}) \subset \pi_{\mathfrak{a}}^{-1}(\mathfrak{q})\} \\ &= \pi_{\mathfrak{a}}^*\{\mathfrak{q} \in \operatorname{Spec} R/\mathfrak{a} : \mathfrak{b} \subset \mathfrak{q}\} = \pi_{\mathfrak{a}}^*(V(\mathfrak{b})) \end{aligned}$$

Thus the continuous function $\pi_{\mathfrak{a}}^*\colon \operatorname{Spec} R/\mathfrak{a} \to \operatorname{Spec} R$ maps closed subsets of $\operatorname{Spec} R/\mathfrak{a}$ onto closed subsets of $V(\mathfrak{a})$. But any continuous bijection between topological spaces that maps closed sets onto closed sets is a homeomorphism. (Indeed one can readily verify that the inverse of the bijection is continuous.) We conclude therefore that the function $\pi_{\mathfrak{a}}^*$ maps $\operatorname{Spec} R/\mathfrak{a}$ homeomorphically onto $V(\mathfrak{a})$, as required. ∎

An element r of a ring R is said to be *nilpotent* if $r^n = 0$ for some positive integer n. The *nilradical* of a commutative ring is the set of all nilpotent elements of the ring. Note that if r and s are elements of a commutative ring and if $r^m = 0$ and $s^n = 0$ then $(r+m)^{m+n} = 0$. Also $(-r)^m = 0$, and $(tr)^m = 0$ for all $t \in R$. It follows that the nilradical of a commutative ring is an ideal of that ring. This ideal is by definition the radical of the zero ideal.

Corollary 6.4 *Let R be a unital commutative ring, and let N be the nilradical of R. Then the quotient homomorphism $\nu\colon R \to R/N$ induces a homeomorphism $\nu^*\colon \operatorname{Spec} R/N \to \operatorname{Spec} R$ between the spectra of R/N and R.*

Proof Let r be an element of the ring R, and let $\mathfrak{p}$ be a prime ideal of R. If $r^n \in \mathfrak{p}$ for some positive integer n then $r \in \mathfrak{p}$ (since a product of elements of R belongs to a prime ideal if and only if one of the factors belongs to that prime ideal). It follows that a nilpotent element of R belongs to every prime ideal of R, and that $V(N) = \operatorname{Spec} R$, where N is the nilradical of R. But, for any ideal $\mathfrak{a}$ of R, the quotient homomorphism from R to $\mathfrak{a}$ induces a homeomorphism between $\operatorname{Spec} R/\mathfrak{a}$ and $V(\mathfrak{a})$. It follows that the quotient homomorphism $\nu\colon R \to R/N$ induces a homeomorphism between $\operatorname{Spec} R$ and $\operatorname{Spec} R/N$.

6.4 The Spectrum of a Ring of Fractions

Let R be a unital commutative ring, and let S be a non-empty multiplicative subset of R. Then there is a well-defined natural homomorphism $\iota_S\colon R \to S^{-1}R$ from R to $S^{-1}R$, where $\iota_S(r) = r/1$ for all $r \in R$.

Proposition 6.5 *Let R be a unital commutative ring, let S be a multiplicative subset of R, and let $\iota_S\colon R \to S^{-1}R$ be the natural homomorphism, where $\iota_S(r) = rs/s$ for all $r \in R$ and $s \in S$. Then the induced map $\iota_S^*\colon \operatorname{Spec} S^{-1}R \to \operatorname{Spec} R$ maps $\operatorname{Spec} S^{-1}R$ homeomorphically onto the subspace*

$$\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \cap S = \emptyset\}$$

of $\operatorname{Spec} R$.

Proof Let $\mathfrak{q}$ be a prime ideal of $S^{-1}R$, and let r and s be elements of R and S respectively for which $r/s \in \mathfrak{q}$. Then $\iota_S(r) \in \mathfrak{q}$, since $\iota_S(r) = s(r/s)$, and therefore $r \in \iota_S^*(\mathfrak{q})$. Thus

$$\mathfrak{q} = \{r/s \in S^{-1}R : r \in \iota_S^*(\mathfrak{q}) \text{ and } s \in S\}.$$

It follows from this that the function $\iota_S^*\colon \operatorname{Spec} S^{-1}R \to \operatorname{Spec} R$ is injective.

If $\mathfrak{q}$ is a prime ideal of $S^{-1}R$, then and if s is an element of S then $s/s \notin \mathfrak{q}$, because s/s is the identity element of $S^{-1}R$, and no prime ideal of a unital commutative ring contains the identity element. It follows that $s \notin \iota_S^*(\mathfrak{q})$ for all $\mathfrak{q} \in \operatorname{Spec} S^{-1}R$. Thus $\iota_S^*(\operatorname{Spec} S^{-1}R) \subset X$, where $X = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \cap S = \emptyset\}$.

Now let $\mathfrak{p}$ be an element of X, and let $\mathfrak{q}$ be the ideal of $S^{-1}R$ generated by $\iota_S(\mathfrak{p})$. Then

$$\mathfrak{q} = \{x/s \in S^{-1}R : x \in \mathfrak{p} \text{ and } s \in S\}.$$

Let $x \in \mathfrak{p}$ and $s, t \in S$. Then $s \notin \mathfrak{p}$ and $t \notin \mathfrak{p}$, and therefore $st \notin \mathfrak{p}$. But $xt \in \mathfrak{p}$. Therefore $xt \neq st$ for all $x \in \mathfrak{p}$ and $s, t \in S$. It follows from the definition of the ring of fractions $S^{-1}R$ that $x/s \neq 1$ for all $x \in \mathfrak{p}$ and $s \in S$, and therefore $\mathfrak{q}$ is a proper ideal of $S^{-1}R$.

Let x_1/s_1 and x_2/s_2 be elements of $S^{-1}R$, where $x_1, x_2 \in R$ and $s_1, s_2 \in S$. Suppose that $(x_1/s_1)(x_2/s_2) \in \mathfrak{q}$. Then $(x_1x_2)/(s_1s_2) = y/s$ for some $y \in \mathfrak{p}$ and $s \in S$. But then $tsx_1x_2 = ts_1s_2y$, and therefore $tsx_1x_2 \in \mathfrak{p}$, for some $t \in S$. But $ts \in S$ and $\mathfrak{p} \in X$, and therefore $ts \notin \mathfrak{p}$. It follows that $x_1x_2 \in \mathfrak{p}$, since $\mathfrak{p}$ is a prime ideal of R. But then either $x_1 \in \mathfrak{p}$, in which case $x_1/s_1 \in \mathfrak{q}$, or else $x_2 \in \mathfrak{p}$, in which case $x_2/s_2 \in \mathfrak{q}$. We have thus shown that if a product of elements of $S^{-1}R$ belongs to the proper ideal $\mathfrak{q}$, then at least one of the factors must belong to $\mathfrak{q}$. We conclude that $\mathfrak{q}$ is a prime ideal of $S^{-1}R$. Moreover if x is an element of R and if $\iota_S(x) \in \mathfrak{q}$ then $sx/s = y/t$ for some $y \in \mathfrak{p}$ and $s, t \in S$, and therefore $stux = suy$ for some $u \in S$. But $tuy \in \mathfrak{p}$ and $stu \notin \mathfrak{p}$. It follows that $x \in \mathfrak{p}$. We conclude that $\mathfrak{p} = \iota_S^*(\mathfrak{q})$ for any prime ideal $\mathfrak{p}$ of R satisfying $\mathfrak{p} \cap S = \emptyset$, where $\mathfrak{q}$ is the prime ideal of $S^{-1}R$ generated by $\iota_S(\mathfrak{p})$. Thus $\iota_S^*(\operatorname{Spec} S^{-1}R) = X$, and the continuous function ι_S^* maps $\operatorname{Spec} S^{-1}R$ bijectively onto X.

Let $\mathfrak{b}$ be an ideal of $S^{-1}R$, and let $\mathfrak{q}$ be a prime ideal of $S^{-1}R$. Suppose that $\iota_S^{-1}(\mathfrak{b}) \subset \iota_S^{-1}(\mathfrak{q})$. Let x/s be an element of $\mathfrak{b}$, where $x \in R$ and $s \in S$. Then $\iota_S(x) \in \mathfrak{b}$, and therefore $x \in \iota_S^{-1}(\mathfrak{b})$. But then $x \in \iota_S^{-1}(\mathfrak{q})$, and therefore $x/s \in \mathfrak{q}$. We conclude that $\iota_S^{-1}(\mathfrak{b}) \subset \iota_S^{-1}(\mathfrak{q})$ if and only if $\mathfrak{b} \subset \mathfrak{q}$. This shows that $\iota_S * (V(\mathfrak{b})) = X \cap V(\iota_S^{-1}(\mathfrak{b})$, where $V(\mathfrak{b}) = \{\mathfrak{q} \in \operatorname{Spec} S^{-1}R : \mathfrak{b} \subset \mathfrak{q}\}$ and $V(\iota_S^{-1}(\mathfrak{b})) = \{\mathfrak{p} \in \operatorname{Spec} R : \iota_S^{-1}(\mathfrak{b}) \subset \mathfrak{p}\}$. Thus the continuous map ι_S^* maps closed subsets of $\operatorname{Spec} S^{-1}R$ onto closed subsets of X, and therefore maps $\operatorname{Spec} S^{-1}R$ homeomorphically onto X, as required. ∎

Let R be a unital commutative ring. Each element f of R determines an open subset $D(f)$ of $\operatorname{Spec} R$, where

$$D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}.$$

This open set is the complement of the closed set consisting of all prime ideals of R that contain the ideal (f) generated by the element f of R. Note that $D(f) \cap D(g) = D(fg)$ for all elements f and g of R. Indeed let $\mathfrak{p}$ be any prime ideal of R. Then $fg \notin \mathfrak{p}$ if and only if $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$. Thus $\mathfrak{p} \in D(fg)$ if and only if $\mathfrak{p} \in D(f)$ and $\mathfrak{p} \in D(g)$. In particular $D(f^n) = D(f)$ for all natural numbers n.

Let $\mathfrak{a}$ be an ideal of the ring R. Then

$$\begin{aligned} \operatorname{Spec} R \setminus V(\mathfrak{a}) &= \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{a} \not\subset \mathfrak{p}\} \\ &= \bigcup_{f \in \mathfrak{a}} \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\} \\ &= \bigcup_{f \in \mathfrak{a}} D(f) \end{aligned}$$

It follows that the collection of subsets of $\operatorname{Spec} R$ that are of the form $D(f)$ for some $f \in R$ is a basis for the topology of $\operatorname{Spec} R$, since each open subset of $\operatorname{Spec} R$ is a union of open sets of this form.

Given any element f of R, let

$$S_f = \{1, f, f^2, f^3, \ldots\} = \{f^n : n \in \mathbb{Z} \text{ and } n \geq 0\}.$$

Then S_f is a non-empty multiplicative subset of R. Let R_f denote the corresponding ring of fractions defined by defined by $R_f = S_f^{-1}R$. An element of

R_f can be represented in the form r/f^m where $r \in R$ and m is a non-negative integer. Moreover two such elements r/f^m and r'/f^n of R_f satisfy the equation $r/f^m = r'/f^n$ if and only if $f^{n+l}r = f^{m+l}r'$ for some non-negative integer l.

The following result is an immediate corollary of Proposition 6.5.

Corollary 6.6 *Let R be a unital commutative ring, let f be an element of R, and let $R_f = S_f^{-1}R$, where S_f is the multiplicative subset of R consisting of all elements of R that are of the form f^n for some non-negative integer n. Let $\iota_f\colon R \to R_f$ be the homomorphism with $\iota_f(r) = r/1$ for all $r \in R$. Then the induced map $\iota_f^*\colon \operatorname{Spec} R_f \to \operatorname{Spec} R$ maps $\operatorname{Spec} R_f$ homeomorphically onto the open set $D(f)$, where $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$.*

6.5 Intersections of Prime Ideals

Theorem 6.7 *Let $\mathfrak{a}$ be an ideal of a unital commutative ring R, let $\sqrt{\mathfrak{a}}$ be the radical of the ideal $\mathfrak{a}$, consisting of those elements r of R with the property that $r^n \in \mathfrak{a}$ for some natural number n. Then $\sqrt{\mathfrak{a}}$ is the intersection of all prime ideals $\mathfrak{p}$ of R satisfying $\mathfrak{a} \subset \mathfrak{p}$.*

Proof Let f be an element of the ring R with the property that $f^n \notin \mathfrak{a}$ for all natural numbers n, let $R_f = S_f^{-1}R$, where S_f denotes the set consisting of the powers f^n of f for all non-negative integers n, let $\iota_f\colon R \to R_f$ denote the homomorphism sending $r \in R$ to $r/1$, and let $\mathfrak{b}$ denote the ideal of R_f generated by $\iota_f(\mathfrak{a})$. Then the elements of $\mathfrak{b}$ can be represented as fractions of the form x/f^m where $x \in \mathfrak{a}$ and m is some non-negative integer. We claim that $\mathfrak{b}$ is a proper ideal of R_f. If it were the case that $\mathfrak{b} = R_f$, there would exist some element x of $\mathfrak{a}$, and some non-negative integer m such that $x/f^m = 1/1$ in R_f. But then there would exist some non-negative integer k such that $f^k x = f^{k+m}$. But then $f^{k+m} \in \mathfrak{a}$, because $f^k x \in \mathfrak{a}$. But the element f has been chosen such that $f^n \notin \mathfrak{a}$ for all positive integers n. It follows that $\mathfrak{b} \neq R_f$, and therefore $\mathfrak{b}$ is a proper ideal of R_f. But then there exists a maximal ideal $\mathfrak{m}$ of R_f such that $\mathfrak{b} \subset \mathfrak{m}$ (Theorem 3.31). Moreover $\mathfrak{m}$ is a prime ideal of R_f (Lemma 3.35), and therefore $\mathfrak{m} \in \operatorname{Spec} R_f$. Let $\mathfrak{p} = \iota_f^*(\mathfrak{m})$. Then $\mathfrak{p} \in D(f)$, and therefore $f \notin \mathfrak{p}$. Also $\mathfrak{a} \subset \mathfrak{p}$. It follows from this that $\bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} \subset \sqrt{\mathfrak{a}}$. But if $r \in \sqrt{\mathfrak{a}}$, and if $\mathfrak{p}$ is a prime ideal of R with $\mathfrak{a} \subset \mathfrak{p}$ then $r^n \in \mathfrak{p}$ for some positive integer n, and then $r \in \mathfrak{p}$ (since $\mathfrak{p}$ is a prime ideal). It follows that $\sqrt{\mathfrak{a}} \subset \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$, and therefore $\bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} = \sqrt{\mathfrak{a}}$, as required. ∎

Corollary 6.8 *Let R be a unital commutative ring, let $\mathfrak{a}$ be an ideal of R, let $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{a} \subset \mathfrak{p}\}$ and, for each element f of R, let $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. Then*

$$\sqrt{\mathfrak{a}} = \{f \in R : D(f) \cap V(\mathfrak{a}) = \emptyset\}.$$

Proof It follows from Theorem 6.7 that $\sqrt{\mathfrak{a}}$ is the intersection of all prime ideals $\mathfrak{p}$ of R satisfying $\mathfrak{a} \subset \mathfrak{p}$. Thus an element f of R belongs to $\sqrt{\mathfrak{a}}$ if and only if $f \in \mathfrak{p}$ for all $\mathfrak{p} \in V(\mathfrak{a})$, and thus if and only if $D(f) \cap V(\mathfrak{a}) = \emptyset$, as required. ∎

Corollary 6.9 *Let R be a unital commutative ring, and for each element f of R, let $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. Then $D(f) = 0$ if and only if f is nilpotent.*

Proof The radical of the zero ideal $\{0\}$ is the nilradical N of R, that consists of all nilpotent elements of R. Moreover $V(N) = \operatorname{Spec} R$. It follows from Corollary 6.8 that $N = \{f \in R : D(f) = \emptyset\}$, as required. ∎

We have seen that there is a topological space naturally associated to any unital commutative ring R. This topological space is the *spectrum* of the ring, and is denoted by $\operatorname{Spec} R$. The elements of the spectrum are the prime ideals of the ring. Each ideal $\mathfrak{a}$ of R determines a closed subset $V(\mathfrak{a})$ of the spectrum $\operatorname{Spec}(R)$, where $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{a} \subset \mathfrak{p}\}$. Moreover if F is a closed subset of $\operatorname{Spec} R$ then $F = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of R. This closed set $V(\mathfrak{a})$ is homeomorphic to the spectrum $\operatorname{Spec} R/\mathfrak{a}$ of the quotient ring $R/\mathfrak{a}$ (Proposition 6.3).

Each element f of R determines an open subset $D(f)$ of $\operatorname{Spec} R$, where $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. These open subsets form a basis for the topology of $\operatorname{Spec} R$ (i.e., every open subset of $\operatorname{Spec} R$ is a union of open sets each of which is of the form $D(f)$ for some $f \in R$). Moreover, given any $f \in R$, the open set D_f is homeomorphic to $\operatorname{Spec} R_f$, where R_f is the ring of fractions $S_f^{-1}R$ determined by the multiplicative subset $\{1, f, f^2, f^3, \ldots\}$ of non-negative powers of f (Corollary 6.6).

6.6 Topological Properties of the Spectrum

Theorem 6.10 *The spectrum* $\operatorname{Spec} R$ *of any unital commutative ring R is a compact topological space.*

Proof Let $\{U_\lambda : \lambda \in \Lambda\}$ be any open cover of $\operatorname{Spec} R$. Then there exists a collection $\{\mathfrak{a}_\lambda : \lambda \in \Lambda\}$ of ideals of R such that

$$R \setminus U_\lambda = V(\mathfrak{a}_\lambda) = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{a}_\lambda \subset \mathfrak{p}\}$$

for each open set U_λ in the given collection. Let the ideal $\mathfrak{a}$ be the sum $\sum_{\lambda\in\Lambda}\mathfrak{a}_\lambda$ of all the ideals $\mathfrak{a}_\lambda$ in this collection. Then

$$V(\mathfrak{a}) = \bigcap\nolimits_{\lambda\in\Lambda} V(\mathfrak{a}_\lambda) = \bigcap\nolimits_{\lambda\in\Lambda}(\operatorname{Spec} R \setminus U_\lambda) = \operatorname{Spec} R \setminus \bigcup\nolimits_{\lambda\in\Lambda} U_\lambda = \emptyset.$$

Thus there is no prime ideal $\mathfrak{p}$ of R with $\mathfrak{a} \subset \mathfrak{p}$. But any proper ideal of R is contained in some maximal ideal (Theorem 3.31), and moreover every maximal ideal is a prime ideal (Lemma 3.35). It follows that there is no maximal ideal of R that contains the ideal $\mathfrak{a}$, and therefore this ideal cannot be a proper ideal of R. We conclude that $\mathfrak{a} = R$, and therefore every element of the ring R may be expressed as a finite sum where each of the summands belongs to one of the ideals $\mathfrak{a}_\lambda$. In particular there exist elements $x_1, x_2, \ldots, x_k$ of R and ideals $\mathfrak{a}_{\lambda_1}, \mathfrak{a}_{\lambda_2}, \ldots, \mathfrak{a}_{\lambda_k}$ in the collection $\{\mathfrak{a}_\lambda : \lambda \in \Lambda\}$, such that $x_i \in \mathfrak{a}_{\lambda_i}$ for $i = 1, 2, \ldots, k$ and $x_1 + x_2 + \cdots + x_k = 1$. But then $\sum_{i=1}^k \mathfrak{a}_{\lambda_i} = R$, and therefore

$$\operatorname{Spec} R \setminus \bigcup_{i=1}^k U_{\lambda_i} = \bigcap_{i=1}^k V(\mathfrak{a}_\lambda) = V\left(\sum_{i=1}^k \mathfrak{a}_k\right) = V(R) = \emptyset.$$

and therefore $\{U_{\lambda_i} : i = 1, 2, \ldots, k\}$ is an open cover of $\operatorname{Spec} R$. Thus every open cover of $\operatorname{Spec} R$ has a finite subcover. We conclude that $\operatorname{Spec} R$ is a compact topological space, as required. ∎

Corollary 6.11 *Let $\mathfrak{a}$ be an ideal of a unital commutative ring R. Then the closed subset $V(\mathfrak{a})$ of* $\operatorname{Spec} R$ *is compact.*

Proof This follows immediately from the result that a closed subset of a compact topological space is compact.

Corollary 6.12 *Let R be a unital commutative ring, let f be an element of R, and let $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{a}\}$. Then $D(f)$ is a compact subset of* $\operatorname{Spec} R$.

Proof Let $S_f = \{1, f, f^2, \ldots\}$, and let $R_f = S_f^{-1}R$. Then the open set $D(f)$ is homeomorphic to the spectrum of the ring R_r (Corollary 6.6). But the spectrum of any ring is a compact topological space (Theorem 6.10). Therefore $D(f)$ is compact, as required. ∎

A unital commutative ring is *Noetherian* if every ideal of the ring is finitely-generated.

Corollary 6.13 *Let R be a Noetherian ring. Then every subset of the spectrum* $\operatorname{Spec} R$ *of R is compact.*

Proof Let U be an open subset of $\operatorname{Spec} R$. Then $U = \operatorname{Spec} R \setminus V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of R (where $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{a} \subset \mathfrak{p}\}$. The ideal $\mathfrak{a}$ is finitely-generated, since R is Noetherian. Therefore there exists a finite set $f_1, f_2, \ldots, f_k$ of elements of $\mathfrak{a}$ that generates $\mathfrak{a}$. Then $V(\mathfrak{a})$ is the intersection of the closed sets $V(f_i)$ for $i = 1, 2, \ldots, k$, where

$$V(f_i) = \{\mathfrak{p} \in \operatorname{Spec} R : f_i \in \mathfrak{p}\} = \operatorname{Spec} R \setminus D(f_i)$$

and therefore $U = D(f_1) \cup D(f_2) \cup \ldots D(f_k)$. But each open set $D(f_i)$ is compact (Corollary 6.12). Therefore U, being a finite union of compact sets, is compact. Thus every open subset of $\operatorname{Spec} R$ is compact. It follows immediately from this that every subset of $\operatorname{Spec} R$ is compact, for, given any collection of open sets that covers some subset A of $\operatorname{Spec} R$, the union U of all those open sets is open, and is therefore compact, and is covered by the open sets in the given collection. But then there exists some finite collection of open sets belonging to the original collection which covers U, and therefore covers A. ∎

A topological space is said to be irreducible if the intersection of any two non-empty open sets is non-empty. Every irreducible topological space is connected.

Theorem 6.14 *Let R be a unital commutative ring, let $\mathfrak{a}$ be an ideal of R, and let $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{a} \subset \mathfrak{p}\}$. Then $V(\mathfrak{a})$ is an irreducible topological space if and only if the radical $\sqrt{\mathfrak{a}}$ of $\mathfrak{a}$ is a prime ideal of R.*

Proof Suppose that $V(\mathfrak{a})$ is an irreducible topological space. Let r_1 and r_2 be elements of $R \setminus \sqrt{\mathfrak{a}}$. Then $D(r_1) \cap V(\mathfrak{a})$ and $D(r_2) \cap V(\mathfrak{a})$ are non-empty, since

$$\sqrt{\mathfrak{a}} = \{f \in R : D(f) \cap V(\mathfrak{a}) = \emptyset\}$$

(Corollary 6.8). Now $D(r_1 r_2) = D(r_1) \cap D(r_2)$, and therefore $D(r_1 r_2) \cap V(\mathfrak{a})$ is the intersection of the non-empty open subsets $D(r_1) \cap V(\mathfrak{a})$ and $D(r_2) \cap V(\mathfrak{a})$ of $V(\mathfrak{a})$. It follows from the irreducibility of $V(\mathfrak{a})$ that $D(r_1 r_2) \cap V(\mathfrak{a})$ is itself non-empty, and therefore $r_1 r_2 \in R \setminus \sqrt{\mathfrak{a}}$. Thus if $V(\mathfrak{a})$ is an irreducible topological space then the complement $R \setminus \sqrt{\mathfrak{a}}$ of $\sqrt{\mathfrak{a}}$ is a multiplicative subset of R, and therefore $\sqrt{\mathfrak{a}}$ is a prime ideal of R.

Conversely suppose that $\sqrt{\mathfrak{a}}$ is a prime ideal of R. Let U_1 and U_2 be non-empty subsets of $V(\mathfrak{a})$. Any open subset of $V(\mathfrak{a})$ is a union of subsets of $V(\mathfrak{a})$ each of which is of the form $D(r) \cap V(\mathfrak{a})$ for some $r \in R$. Therefore there exist elements r_1 and r_2 of R such that $D(r_1) \cap V(\mathfrak{a})$ and $D(r_2) \cap V(\mathfrak{a})$ are non-empty, $D(r_1) \cap V(\mathfrak{a}) \subset U_1$, and $D(r_2) \cap V(\mathfrak{a}) \subset U_2$. Then $r_1 \notin \sqrt{\mathfrak{a}}$

and $r_2 \notin \sqrt{\mathfrak{a}}$. But then $r_1 r_2 \notin \sqrt{\mathfrak{a}}$, because the complement of a prime ideal is a multiplicative subset of R It follows that $D(r_1 r_2) \cap V(\mathfrak{a})$ is non-empty. But $D(r_1 r_2) \cap V(\mathfrak{a})$ is the intersection of $D(r_1) \cap V(\mathfrak{a})$ and $D(r_2) \cap V(\mathfrak{a})$. Therefore $D(r_1 r_2) \cap V(\mathfrak{a}) \subset U_1 \cap U_2$, and thus $U_1 \cap U_2$ is non-empty. We have thus shown that if $\sqrt{\mathfrak{a}}$ is a prime ideal of R then the intersection of any two non-empty open subsets of $V(\mathfrak{a})$ is non-empty. It follows that if $\sqrt{\mathfrak{a}}$ is a prime ideal then $V(\mathfrak{a})$ is irreducible, as required. ∎

Corollary 6.15 *The spectrum of an integral domain is an irreducible topological space.*

Proof If R is an integral domain then its nilradical is the zero ideal, and moreover the zero ideal is a prime ideal. Moreover $V(\{0\}) = \operatorname{Spec} R$. It therefore follows from Theorem 6.14 that $\operatorname{Spec} R$ is an irreducible topological space.

Corollary 6.16 *Let R be a unital commutative ring, and let N be the nilradical of R. Suppose that the spectrum $\operatorname{Spec} R$ of R is an irreducible topological space. Then R/N is an integral domain.*

Proof If $\operatorname{Spec} R$ is irreducible then N is a prime ideal of R, and therefore R/N is an integral domain (Lemma 3.34). ∎

6.7 Localization and the Structure Sheaf of a Unital Commutative Ring

Let $\mathfrak{p}$ be a prime ideal of a unital commutative ring R, and let $S_\mathfrak{p} = R \setminus \mathfrak{p}$. Then S is a non-empty multiplicative subset of R. We denote by $R_\mathfrak{p}$ the ring of fractions defined by $R_\mathfrak{p} = S_\mathfrak{p}^{-1} R$. Each element of $R_\mathfrak{p}$ may be represented as a fraction, which we will denote by $(r/s)_\mathfrak{p}$, whose numerator r is an element of R and whose denominator s is an element of $R \setminus \mathfrak{p}$. Let r and r' be elements of R, and let s and s' be elements of $R \setminus \mathfrak{p}$. Then $(r/s)_\mathfrak{p} = (r'/s')_\mathfrak{p}$ if and only if $us'r = usr'$ for some element u of $R \setminus \mathfrak{p}$. This ring $R_\mathfrak{p}$ is referred to as the *localization* of the ring R at the prime ideal $\mathfrak{p}$.

Similarly, given any module M over the unital commutative ring R, we define the *localization* $M_\mathfrak{p}$ of M at a prime ideal $\mathfrak{p}$ of R to be the module $S_\mathfrak{p}^{-1} M$ whose elements are represented as fractions $(m/s)_\mathfrak{p}$, where $m \in M$ and $s \in R \setminus \mathfrak{p}$. Two fractions $(m/s)_\mathfrak{p}$ and $(m'/s')_\mathfrak{p}$ represent the same element of $M_\mathfrak{p}$ if and only if there exists some element u of $R \setminus \mathfrak{p}$ such that $us'm = usm'$.

Let $\mathcal{O}_R$ denote the disjoint union of the rings $R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of R, and let $\pi_R \colon \mathcal{O}_R \to \operatorname{Spec} R$ denote the function that sends elements of the local ring $R_\mathfrak{p}$ to the corresponding prime ideal $\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec} R$.

Also, given any R-module M, let $\tilde{M}$ denote the disjoint union of the modules $M_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of R, and let $\pi_M: \tilde{M} \to \operatorname{Spec} R$ denote the function that sends elements of the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ to the corresponding prime ideal $\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec} R$.

We shall prove that there are natural topologies defined on the sets $\mathcal{O}_R$ and $\tilde{M}$ with respect to which the surjective functions $\pi_R: \mathcal{O}_R \to \operatorname{Spec} R$ and $\pi_M: \tilde{M} \to \operatorname{Spec} R$ are local homeomorphisms.

We recall some results from general topology. Let X be a set, and let $\mathcal{B}$ a collection of subsets of X. This collection $\mathcal{B}$ is said to be a *basis* for a topology on X if there is a well-defined topology on X such that the open subsets of X are those subsets of X that are unions of subsets belonging to the collection $\mathcal{B}$. In order that a collection $\mathcal{B}$ of subsets of X be a basis for a topology on X, it is necessary and sufficient that the following two conditions be satisfied:

(i) given any $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$;

(ii) given $B_1, B_2 \in \mathcal{B}$ for which $B_1 \cap B_2$ is non-empty, and given $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

If we define the open sets in X to be those sets that are unions of sets belonging to the collection $\mathcal{B}$ then condition (i) ensures that X is an open subset of itself. The empty set is regarded as the union of an empty collection of sets belonging to $\mathcal{B}$, and is therefore an open set. Any union of open sets is clearly an open set, and condition (ii) ensures that the intersection of two subsets belonging to $\mathcal{B}$ is a union of subsets belonging to $\mathcal{B}$, and therefore ensures that the intersection of any two open sets is an open set. It follows from this that any finite intersection of open sets is an open set. Thus any collection $\mathcal{B}$ of subsets of X satisfying conditions (i) and (ii) does indeed give rise in this way to a well-defined topology on X.

Let X and Y be topological spaces, and let $f: X \to Y$ be a continuous function from X to Y. The function f is said to be a *local homeomorphism* if, given any point x of X, there exists an open set U in X such that $x \in U$, $f(U)$ is an open set in Y, and the restriction $f|U$ of f to the open set U defines a homeomorphism from U to $f(U)$.

Proposition 6.17 *Let R be a unital commutative ring, let M be a module over the ring R, let $\tilde{M}$ be the disjoint union of the $R_{\mathfrak{p}}$-modules $M_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of R, and let $\pi_M: \tilde{M} \to \operatorname{Spec} R$ be the surjective function that sends all elements of $M_{\mathfrak{p}}$ to the prime ideal $\mathfrak{p}$. For each ordered pair (m, s) in $M \times R$, let*

$$\tilde{D}(m, s) = \{(m/s)_{\mathfrak{p}} \in \tilde{M} : \mathfrak{p} \in \operatorname{Spec} R \text{ and } s \notin \mathfrak{p}\}.$$

Then the collection of sets $\tilde{D}(m,s)$ is a basis for a topology on $\tilde{M}$, with respect to which the surjective function $\pi_M\colon \tilde{M} \to \operatorname{Spec} R$ is a local homeomorphism.

Proof Note that set $\tilde{D}(m,s)$ is the empty set if $s^n = 0$ for some positive integer n, for in those cases $s \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

Let $\mathcal{B}$ denote the collection of subsets of $\tilde{M}$ that are of the form $\tilde{D}(m,s)$ for some $(m,s) \in M \times R$. Now, given any element ρ of $\tilde{M}$, there exist a prime ideal $\mathfrak{p}$, and elements m and s of M and $R \setminus \mathfrak{p}$ respectively, such that $\rho = (m/s)_\mathfrak{p}$. But then $\rho \in \tilde{D}(m,s)$. It follows that every element of $\tilde{M}$ belongs at least one set in the collection $\mathcal{B}$.

Let (m,s) and (m',s') be ordered pairs in $M \times R$, and let $\rho \in \tilde{D}(m,s) \cap \tilde{D}(m',s')$. Then $\rho = (m/s)_\mathfrak{p} = (m'/s')_\mathfrak{p}$, and therefore there exists some element t of $R \setminus \mathfrak{p}$ such that $ts'm = tsm'$. But then

$$(m/s)_\mathfrak{q} = (ts'm/tss')_\mathfrak{q} = (tsm'/tss')_\mathfrak{q} = (m'/s')_\mathfrak{q}$$

for all prime ideals $\mathfrak{q}$ for which $tss' \notin \mathfrak{q}$. It follows that $\rho \in \tilde{D}(ts'm, tss')$ and $\tilde{D}(ts'm, tss') \subset \tilde{D}(m,s) \cap \tilde{D}(m',s')$. We conclude from this that $\{\tilde{D}(m,s) : (m,s) \in M \times R\}$ is a basis for a topology on $\tilde{M}$. Henceforth we regard $\tilde{M}$ as a topological space, with the topology defined by this basis of open sets.

We now show that the function $\pi_M\colon \tilde{M} \to \operatorname{Spec} R$ is continuous. Let $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$ for each element f of R. Now $(m/s)_\mathfrak{p} = (fm/fs)_\mathfrak{p}$ for all $(m,s) \in M \times R$, and for all prime ideals $\mathfrak{p}$ satisfying $f \notin \mathfrak{p}$ and $s \notin \mathfrak{p}$. It follows that $\tilde{D}(m,s) \cap \pi_M^{-1}(D(f)) = \tilde{D}(fm, fs)$ for all $(m,s) \in M \times R$. We conclude that $\pi_M^{-1}(D(f))$ is the union of the open sets $\tilde{D}(fm, fs)$ for all $(m,s) \in M \times R$, and is therefore itself an open set. But every open set in $\operatorname{Spec} R$ is a union of open sets that are of the form $D(f)$ for some $f \in R$. It follows that the preimage of every open subset of $\operatorname{Spec} R$ is an open subset of $\tilde{M}$. Thus $\pi_M\colon \tilde{M} \to \operatorname{Spec} R$ is continuous.

Now $\pi_M(\tilde{D}(m,s)) = D(s)$ for all $(m,s) \in M \times R$. But every open subset of $\tilde{M}$ is a union of sets of that are each of the form $\tilde{D}(m,s)$ for some $(m,s) \in M \times R$. It follows that $\pi_M(U)$ is an open subset of $\operatorname{Spec} R$ for every open subset U of $\tilde{M}$.

Finally we note that the the function $\pi_M \tilde{M} \to \operatorname{Spec} R$ maps the open subset $\tilde{D}(m,s)$ of $\tilde{M}$ injectively and continuously onto the open set $D(s)$ for all $(m,s) \in M \times R$. But it also maps open sets to open sets. Therefore it maps $\tilde{D}(m,s)$ homeomorphically onto $D(s)$. We conclude that the function $\pi_M \tilde{M} \to \operatorname{Spec} R$ is a local homeomorphism, as required. ∎

Corollary 6.18 *Let R be a unital commutative ring, let $\mathcal{O}_R$ be the disjoint union of the rings $R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of R, where $R_\mathfrak{p} = (R \setminus \mathfrak{p})^{-1}R$, and let $\pi_R\colon \mathcal{O}_R \to \operatorname{Spec} R$ be the surjective function that sends elements of* *the ring $R_\mathfrak{p}$ to $\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of R. For each pair (r,s) of elements of R, let*

$$\tilde{D}(r,s) = \{(r/s)_\mathfrak{p} \in \mathcal{O}_R : \mathfrak{p} \in \operatorname{Spec} R \text{ and } s \notin \mathfrak{p}\}.$$

Then the collection of sets $\tilde{D}(r,s)$ is a basis for a topology on $\mathcal{O}_R$, with respect to which the surjective function $\pi_R: \mathcal{O}_R \to \operatorname{Spec} R$ is a local homeomorphism.

Proof The result is an immediate corollary of Proposition 6.17 on taking the R-module M to be R itself (where the unital commutative ring R acts on itself by left multiplication). ∎

We shall henceforth regard $\mathcal{O}_R$ and $\tilde{M}$ as topological spaces, with the topologies defined by Proposition 6.17 and Corollary 6.18.

We shall refer to $\pi_R: \mathcal{O}_R \to \operatorname{Spec} R$ as the *structure sheaf* of the unital commutative ring. The preimage $\pi_R^{-1}(\mathfrak{p})$ of any element $\mathfrak{p}$ of the spectrum of R is referred to as the *stalk* of the sheaf over the prime ideal $\mathfrak{p}$. This stalk is the localization $R_\mathfrak{p}$ of the ring R at the prime ideal $\mathfrak{p}$.

Definition Let X be a topological space. A *sheaf of rings* over X consists of a topological space $\mathcal{O}$ and a continuous surjective map $\pi: \mathcal{O} \to X$ which satisfies the following conditions:

(i) the surjective map $\pi: \mathcal{O} \to X$ is a local homeomorphism;

(ii) at each point x of X, the stalk $\pi^{-1}(\{x\})$ of the sheaf over x is a ring;

(iii) the algebraic operations of addition and multiplication on the stalks $\pi^{-1}(\{x\})$ of the sheaf determine continuous functions from $\mathcal{Z}$ to $\mathcal{O}$, where

$$\mathcal{Z} = \{(r_1, r_2) \in \mathcal{O} \times \mathcal{O} : \pi(r_1) = \pi(r_2)\}.$$

Proposition 6.19 *Let R be a unital commutative ring. Then the structure sheaf $\pi_R: \mathcal{O}_R \to \operatorname{Spec} R$ of R is a sheaf of rings over the spectrum $\operatorname{Spec} R$ of R.*

Proof Let $\mathcal{Z} = \{(\rho_1, \rho_2) \in \mathcal{O}_R \times \mathcal{O}_R : \pi_R(\rho_1) = \pi_R(\rho_2)\}$, and let $\Sigma: \mathcal{Z} \to \mathcal{O}_R$ and and $\Pi: \mathcal{Z} \to \mathcal{O}_R$ be the functions defined such that $\Sigma(\rho_1, \rho_2) = \rho_1 + \rho_2$ and $\Pi(\rho_1, \rho_2) = \rho_1 \rho_2$ for all for all $(\rho_1, \rho_2) \in \mathcal{Z}$. We must prove that these functions Σ and Π are continuous.

Let

$$\begin{aligned} D(s) &= \{\mathfrak{p} \in \operatorname{Spec} R : s \notin \mathfrak{p}\}, \\ \tilde{D}(r,s) &= \{(r/s)_\mathfrak{p} : \mathfrak{p} \in \operatorname{Spec} R \text{ and } s \notin \mathfrak{p}\} \end{aligned}$$

for all $r, s \in R$. Now, given an element (ρ_1, ρ_2) of $\mathcal{Z}$, there exist elements x, y, f and g of R, and a prime ideal $\mathfrak{p}$ of R such that $\mathfrak{p} \in D(f) \cap D(g)$, $\rho_1 = (x/f)_\mathfrak{p}$ and $\rho_2 = (y/g)_\mathfrak{p}$. But $D(f) \cap D(g) = D(fg)$, $(x/f)_\mathfrak{p} = (gx/fg)_\mathfrak{p}$, $(y/g)_\mathfrak{p} = (fy/fg)_\mathfrak{p}$, $(x/f)_\mathfrak{p} + (y/g)_\mathfrak{p} = (gx + fy/fg)_\mathfrak{p}$, and $(x/f)_\mathfrak{p}(y/g)_\mathfrak{p} = (xy/fg)_\mathfrak{p}$. Let $\lambda: D(fg) \to \mathcal{O}_R$ and $\mu: D(fg) \to \mathcal{O}_R$ be the sections of $\pi_R: \mathcal{O}_R \to \operatorname{Spec} R$ defined such that $\lambda(\mathfrak{q}) = (gx + fy/fg)_\mathfrak{q}$ and $\mu(\mathfrak{q}) = (xy/fg)_\mathfrak{q}$ for all $\mathfrak{q} \in D(fg)$. Then the functions λ and μ are continuous, being the inverses of the homeomorphisms from $\tilde{D}(gx + fy, fg)$ and $\tilde{D}(xy, fg)$ to $D(fg)$ obtained on restricting the function $\pi_R: \mathcal{O}_R \to \operatorname{Spec} R$ to $\tilde{D}(gx + fy, fg)$ and $\tilde{D}(xy, fg)$ respectively. Moreover $\Sigma(\rho_1', \rho_2') = \lambda(\pi_R(\rho_1))$ and $\Pi(\rho_1', \rho_2') = \mu(\pi_R(\rho_1))$ for all $(\rho_1', \rho_2') \in \mathcal{Z} \cap (\tilde{D}(gx, fg) \times \tilde{D}(fy, fg))$. It follows from this that the restrictions of the addition and multiplication functions Σ and Π to the open neighbourhood $\mathcal{Z} \cap (\tilde{D}(gx, fg) \times \tilde{D}(fy, fg))$ of (ρ_1, ρ_2) in $\mathcal{Z}$ are continuous. This proves the continuity of the functions $\Sigma: \mathcal{Z} \to \mathcal{O}_R$ and $\Pi: \mathcal{Z} \to \mathcal{O}_R$ at any point (ρ_1, ρ_2) of $\mathcal{Z}$, as required. ∎

Definition Let X be a topological space, and let $\pi_0: \mathcal{O} \to X$ be a sheaf of rings over X. A *sheaf of $\mathcal{O}$-modules* over X consists of a topological space $\mathcal{M}$ and a continuous surjective map $\pi: \mathcal{M} \to X$ satisfying the following conditions:

(i) the surjective map $\pi: \mathcal{M} \to X$ is a local homeomorphism;

(ii) at each point x of X, the stalk $\pi^{-1}(\{x\})$ of the sheaf $\mathcal{M}$ over x is a module over the corresponding stalk $\pi_0^{-1}(\{x\})$ of the sheaf $\mathcal{O}$.

(iii) the algebraic operations of addition and multiplication on the stalks of the sheaves determine continuous functions $\Sigma: \mathcal{X} \to \mathcal{M}$ and $\Pi: \mathcal{Y} \to \mathcal{M}$,where

$$\begin{aligned} \mathcal{X} &= \{(m_1, m_2) \in \mathcal{M} \times \mathcal{M} : \pi(m_1) = \pi(m_2)\}. \\ \mathcal{Y} &= \{(r, m) \in \mathcal{O} \times \mathcal{M} : \pi_0(r) = \pi(m)\}. \end{aligned}$$

Proposition 6.20 *Let R be a unital commutative ring, and let M be a module over the ring R. Then M determines a sheaf $\pi_M: \tilde{M} \to \operatorname{Spec} R$ of $\mathcal{O}_R$-modules over the spectrum $\operatorname{Spec} R$ of R, where $\pi_M(\mathfrak{p}) = M_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof The proof of Proposition 6.19 may easily be adapted to prove the continuity of the relevant functions determined by the algebraic operations on the stalks of the sheaves. ∎

6.8 Affine Schemes

Let R be a unital commutative ring. The *affine scheme* associated with R consists of the space $\operatorname{Spec} R$ of prime ideals of R, with the Zariski topology, together with the structural sheaf $\pi_R\colon \mathcal{O}_R \to \operatorname{Spec} R$ of the ring R. The affine scheme associated with a unital commutative ring R is usually denoted by $\operatorname{Spec} R$.

Such an affine scheme is an example of a *ringed space*. A ringed space $(X, \mathcal{O})$ consists of a topological space together with a sheaf $\mathcal{O}$ of rings over that space X.

Affine schemes are examples of *schemes*. A *scheme* is a ringed space $(X, \mathcal{O})$ with the property that, given any point of the space, there exists an open set U, containing that point, such that U, together with the restriction $\mathcal{O}|U$ of the structure sheaf $\mathcal{O}$ to that subset, are isomorphic to the affine scheme associated to some unital commutative ring. Such an open set is referred to as an *affine open set*. (Here the unital commutative ring associated to an affine open set depends on the open set: different affine open sets will in general have different rings associated with them.)

6.9 Continuous Sections of the Structure Sheaf

Let $\pi_R\colon \mathcal{O}_R \to \operatorname{Spec} R$ be the structure sheaf of a unital commutative ring R. A *section* $\sigma\colon D \to \mathcal{O}_R$ of this sheaf, defined over a subset D of $\operatorname{Spec} R$, is a function with the property that $\pi_R(\sigma(\mathfrak{p})) = \mathfrak{p}$ for all $\mathfrak{p} \in D$.

Theorem 6.21 *Let $\pi_R\colon \mathcal{O}_R \to \operatorname{Spec} R$ be the structure sheaf of a unital commutative ring R, and let x and f be elements of R. Suppose that $(x/f)_\mathfrak{p} = 0$ for all elements $\mathfrak{p}$ of $D(f)$. Then there exists some non-negative integer n such that $f^n x = 0$.*

Proof It follows from the definition of the ring $R_\mathfrak{p}$, where $R_\mathfrak{p} = (R \setminus \mathfrak{p})^{-1}R$, that there exists an element $h_\mathfrak{p}$ of $R \setminus \mathfrak{p}$ for each element $\mathfrak{p}$ in $D(f)$ such that $h_\mathfrak{p} x = 0$. Let $\mathfrak{a}$ be the ideal of R generated by $\{h_\mathfrak{p} : \mathfrak{p} \in D(f)\}$, and let $V(\mathfrak{a}) = \{\mathfrak{q} \in \operatorname{Spec} R : \mathfrak{a} \subset \mathfrak{q}\}$. If $\mathfrak{p} \in D(f)$ then $h_\mathfrak{p} \notin \mathfrak{p}$ and therefore $\mathfrak{p} \notin V(\mathfrak{a})$. Thus $D(f) \cap V(\mathfrak{a}) = \emptyset$. It now follows from Corollary 6.8 that $f \in \sqrt{\mathfrak{a}}$, and therefore there exists some natural number n such that $f^n \in \mathfrak{a}$. But then there exist $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k \in D(f)$ and $u_1, u_2, \ldots, u_k \in R$ such that $f^n = \sum_{i=1}^{k} u_i h_{\mathfrak{p}_i}$. But then $f^n x = \sum_{i=1}^{k} u_i h_{\mathfrak{p}_i} x = 0$, as required. ∎

Corollary 6.22 *Let $\pi_R\colon \mathcal{O}_R \to \operatorname{Spec} R$ be the structure sheaf of a unital commutative ring R, and let x be an element of R. Suppose that $(x/1)_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Then $x = 0$.*

Corollary 6.23 *Let $\pi_R\colon \mathcal{O}_R \to \operatorname{Spec} R$ be the structure sheaf of a unital commutative ring R, and let x, y, f and g be elements of R. Suppose that $(x/f)_\mathfrak{p} = (y/g)_\mathfrak{p}$ for all elements $\mathfrak{p}$ of $D(fg)$. Then there exists some non-negative integer n such that $f^n g^{n+1} x = g^n f^{n+1} y$.*

Proof The elements x, y, f and g have the property that $(gx - fy/fg)_\mathfrak{p} = 0$ for all $\mathfrak{p} \in D(fg)$. It follows from Theorem 6.21 that there exists some natural number n such that $(fg)^n(gx - fy) = 0$. The result follows. ∎

Theorem 6.24 *Let $\pi_R\colon \mathcal{O}_R \to \operatorname{Spec} R$ be the structure sheaf of a unital commutative ring R, let $\sigma\colon \operatorname{Spec} R \to \mathcal{O}_R$ be a continuous section of the structure sheaf. Then there exists an element r of R such that $\sigma(\mathfrak{p}) = (r/1)_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof Let

$$\begin{aligned} D(f) &= \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}, \\ \tilde{D}(x, f) &= \{(x/f)_\mathfrak{p} \in \mathcal{O}_R : \mathfrak{p} \in \operatorname{Spec} R \text{ and } f \notin \mathfrak{p}\} \end{aligned}$$

for all $x, f \in R$. The collection of open sets $\tilde{D}(x, f)$ is a basis for the topology of $\mathcal{O}_R$. (Proposition 6.18). Let $\mathfrak{p}$ be a prime ideal of R, and let x and f be elements of R such that $\sigma(\mathfrak{p}) = (x/f)_\mathfrak{p}$. Then $\sigma(\mathfrak{p}) \in \tilde{D}(x, f)$. It follows from the continuity of $\sigma\colon \operatorname{Spec} R \to \mathcal{O}_R$ that $\sigma^{-1}(\tilde{D}(x, f))$ is an open neighbourhood of $\mathfrak{p}$. Now the collection $\{D(g) : g \in R\}$ of open sets is a basis for the topology of $\operatorname{Spec} R$. It follows that there exists some element g of R such that $\mathfrak{p} \in D(g)$ and $D(g) \subset D(f)$. Let $x_\mathfrak{p} = gx$ and $f_\mathfrak{p} = gf$. Then $\sigma(\mathfrak{q}) = (x/f)_\mathfrak{q} = (x_\mathfrak{p}/f_\mathfrak{p})_\mathfrak{q}$ for all $\mathfrak{q} \in D(f_\mathfrak{p})$. The collection $\{D(f_\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} R\}$ of open sets is an open cover of $\operatorname{Spec} R$. But $\operatorname{Spec} R$ is compact. It follows that there is a finite set $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$ of prime ideals of R such that

$$\operatorname{Spec} R = D(f_{\mathfrak{p}_1}) \cup D(f_{\mathfrak{p}_2}) \cup \cdots \cup D(f_{\mathfrak{p}_K}).$$

Let $x_i = x_{\mathfrak{p}_i}$ and $f_i = f_{\mathfrak{p}_i}$ for $i = 1, 2, \ldots, k$. Then $\sigma(\mathfrak{p}) = (x_i/f_i)_\mathfrak{p}$ for all $\mathfrak{p} \in D(f_i)$. It follows that if i and j are distinct integers between 1 and k then $(x_i/f_i)_\mathfrak{p} = (x_j/f_j)_\mathfrak{p}$ for all $\mathfrak{p} \in D(f_i f_j)$. It then follows from Corollary 6.23 that there exists some non-negative integer n such that $f_i^n f_j^{n+1} x_i = f_j^n f_i^{n+1} x_j$. Moreover we can choose the value of n large enough to ensure that these identities hold simultaneously for all distinct pairs of integers i and j between 1 and k. Now the union of the open sets $D(f_i)$ is the whole of the spectrum $\operatorname{Spec} R$ of R. Moreover $D(f_i) = D(f_i^{n+1})$ for $i = 1, 2, \ldots, k$. It follows that there is no prime ideal of R that contains all of the elements $f_1^{n+1}, f_2^{n+1}, \ldots, f_k^{n+1}$, and therefore the ideal of R generated by these elements is the whole of R. It follows that there exist elements $u_1, u_2, \ldots, u_k$ of R such that

$$u_1 f_1^{n+1} + u_2 f_2^{n+1} + \cdots + u_k f_k^{n+1} = 1.$$

Let

$$r = u_1 f_1^n x_1 + u_2 f_2^n x_2 + \cdots + u_k f_k^n x_k.$$

Then

$$f_i^{n+1} r = \sum_{j=1}^k u_j f_i^{n+1} f_j^n x_j = \sum_{j=1}^k u_j f_j^{n+1} f_i^n x_i = f_i^n x_i,$$

for $i = 1, 2, \ldots, k$. Thus if $\mathfrak{p}$ is a prime ideal of R and and if $\mathfrak{p} \in D(f_i)$ then $(r/1)_\mathfrak{p} = (x_i/f_i)_\mathfrak{p} = \sigma(\mathfrak{p})$. But every point of $\operatorname{Spec} R$ belongs to at least one of the open sets $D(f_i)$. Therefore $(r/1)_\mathfrak{p} = \sigma(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$, as required. ∎

It follows easily from the definition of a sheaf that the the sum and product of continuous sections of a sheaf of rings are themselves continuous sections of that sheaf. The set of continuous sections of a sheaf of ring is therefore itself a ring.

Corollary 6.25 *Let R be a unital commutative ring. Then R is isomorphic to the ring of continuous sections of the structure sheaf $\pi_R: \mathcal{O}_R \to \operatorname{Spec} R$.*

Proof For each $r \in R$, let $\sigma_r: \operatorname{Spec} R \to \mathcal{O}_R$ be defined by $\sigma_r(\mathfrak{p}) = (r/1)_\mathfrak{p}$. Then the function mapping an element r of the ring R to the section σ_r of the structure sheaf is a homomorphism of rings. It follows from Corollary 6.22 that this homomorphism is injective. It follows from Theorem 6.24 that the homomorphism is surjective. Therefore this homomorphism is an isomorphism of rings

Let R be a ring, let f be an element of R, let $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$, and let $R_f = S_f^{-1} R$, where $S_f = \{1, f, f^2, \ldots\}$. Then the function $\iota_f: R \to R_f$ that sends r to $r/1$ for all $r \in R$ induces a homeomorphism $\iota_f^*: \operatorname{Spec} R_f \to D(f)$ from $\operatorname{Spec} R_f$ to $D(f)$ (Corollary 6.6). Let $\pi_R: \mathcal{O}_R \to \operatorname{Spec} R$ and $\pi_{R_f}: \mathcal{O}_{R_f} \to \operatorname{Spec} R_f$ be the structure sheaves of R and R_f respectively. One can readily verify that if $\mathfrak{q}$ is a prime ideal of R_f, and if $\mathfrak{p} = \iota_f^*(\mathfrak{q})$, then the ring $(R_f)_\mathfrak{q}$ is isomorphic to $R_\mathfrak{p}$, where $R_\mathfrak{p} = (R \setminus \mathfrak{p})^{-1} R$ and where $(R_f)_\mathfrak{q} = (R_f \setminus \mathfrak{q})^{-1} R_f$. Indeed the function that sends $((r/f^m)/(s/f^n))_\mathfrak{q}$ to $(rf^n/sf^m)_\mathfrak{p}$ for all $r \in R$ and $s \in R \setminus \mathfrak{p}$ is an isomorphism from $(R_f)_\mathfrak{q}$ to $R_\mathfrak{p}$. It follows that the stalks of the structure sheaves of R_f and R over $\mathfrak{q}$ and $\mathfrak{p}$ respectively are isomorphic. Moreover one can readily check that these isomorphisms combine to give a homeomorphism mapping $\mathcal{O}_{R_f}$ onto the open subset $\pi_R^{-1}(D(f))$ of $\mathcal{O}_R$. The homeomorphism maps the stalk of the structure sheaf of R_f over $\mathfrak{q}$ isomorphically onto the stalk of the structure sheaf of R over $\iota_f^*(\mathfrak{q})$. It follows from this that ring of continuous sections of the structure sheaf of R defined over $D(f)$ is isomorphic to the ring of continuous sections of the structure sheaf of R_f. The following result therefore follows directly from Corollary 6.25.

Corollary 6.26 *Let R be a unital commutative ring, let f be an element of R, and let $R_f = S_f^{-1}R$, where $S_f^{-1} = \{1, f, f^2, \ldots\}$. Then R_f is isomorphic to the ring of continuous sections $\sigma: D(f) \to \mathcal{O}_R$ of the structure sheaf $\pi_R: \mathcal{O}_R \to \operatorname{Spec} R$ defined over the open set $D(f)$ of* $\operatorname{Spec} R$.

6.10 The Structural Sheaf of an Integral Domain

The theory of the structure sheaf of a commutative ring simplifies somewhat when the ring is an integral domain. This is a consequence of the fact that that the various rings of fractions determined by multiplicative subsets of the integral domain can all be embedded in the field of fractions of that integral domain.

Let R be an integral domain. The set $R \setminus \{0\}$ of non-zero elements of R is then a multiplicative set, and we can therefore form a corresponding ring of fractions K, where $K = (R \setminus \{0\})^{-1}R$. Moreover K is a field. It is referred to as the *field of fractions* of the integral domain R. Any element of K may be represented by a quotient of the form r/s, where r and s are elements of R and $s \neq 0$. Let r, r', s and s' be elements of R, where $s \neq 0$ and $s' \neq 0$. Then $r/s = r'/s'$ if only if $s'r = sr'$.

Now let S be a multiplicative subset of the integral domain, all of whose elements are non-zero. An element of the corresponding ring of fractions $S^{-1}R$ is represented as a fraction r/s, where $r \in R$ and $s \in S$. Let r and r' be elements of R, and let s and s' be elements of S. Then $r/s = r'/s'$ in the ring $S^{-1}R$ if and only if there exists some element t of S such that $ts'r = tsr'$, and thus if and only if there exists some element t of S such that $t(s'r - sr') = 0$. But the product non-zero elements of an integral domain is always non-zero, and the elements of the multiplicative subset S are all non-zero. It follows that $r/s = r'/s'$ in the ring $S^{-1}R$ if and only if $s'r = sr'$, and thus if and only if $r/s = r'/s'$ in K, where K is the field of fractions of R. It follows that $S^{-1}R$ may be regarded as a subring of K.

In particular $R_{\mathfrak{p}}$ may be regarded as a subring of K for each prime ideal $\mathfrak{p}$ of R. Similarly, if $S_f = \{1, f, f^2, \ldots\}$ for some non-zero element f of R, and if $R_f = S_f^{-1}R$ then R_f may be regarded as a subring of K.

Let $\pi_R: \mathcal{O}_R \to \operatorname{Spec} R$ be the structure sheaf of the integral domain R, and let K be the field of fractions of R. Then each stalk of the structure sheaf may be regarded as a subring of K. There is therefore a There is a injective function $\epsilon: \mathcal{O}_R \to (\operatorname{Spec} R) \times K$ which sends $(r/s)_{\mathfrak{p}}$ to $(\mathfrak{p}, r/s)$ for all $(r/s)_{\mathfrak{p}} \in R_{\mathfrak{p}}$. Now the topology on $\operatorname{Spec} R$ and the discrete topology on K together determine a product topology on $(\operatorname{Spec} R) \times K$. One can show that the injective function $\epsilon: \mathcal{O}_R \to (\operatorname{Spec} R) \times K$ maps $\mathcal{O}_R$ homeomorphically onto its range, which is a subset of $(\operatorname{Spec} R) \times K$. The continuous sections of the structure sheaf then correspond to constant functions from $\operatorname{Spec} R$ to K.

In the case where the unital commutive ring under consideration is an integral domain, Theorem 6.24 then corresponds to the following theorem.

Theorem 6.27 *Let R be an integral domain and, for each prime ideal $\mathfrak{p}$ of R, let $R_{\mathfrak{p}} = (R \setminus \mathfrak{p})^{-1}R$. Then $\bigcap_{\mathfrak{p} \in \operatorname{Spec} R} R_{\mathfrak{p}} = R$.*

6.11 Rings of Congruence Classes

In order to get some idea of how the theory of affine schemes applies to rings that are not integral domains, it is worthwhile to consider the case of the ring $\mathbb{Z}/m\mathbb{Z}$ where m is a composite number.

Thus let m be a composite number satisfying $m > 1$, and let $R = \mathbb{Z}/m\mathbb{Z}$. We express m as a product of the form $m = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where $p_1, \ldots, p_k$ are prime numbers and $k > 1$. Then each of the prime numbers p_i determines a corresponding prime ideal $p_i R$ of R, for $i = 1, 2, \ldots, k$, where $p_i = p\mathbb{Z}/m\mathbb{Z}$. The spectrum of $\operatorname{Spec} R$ in then represented by the set $\{p_1, p_2, \ldots, p_k\}$ consisting of the prime divisors of R, and is a finite set. Moreover the topology on this finite set is the discrete topology.

We now show that $R_{p_i R} \cong \mathbb{Z}/m_i\mathbb{Z}$ for $i = 1, 2, \ldots, k$ where $m_i = p_i^{n_i}$. Now each element of R is of the form $[x]_m$ for some integer x, where $[x]_m$ denotes the congruence class of m. Let $\mathfrak{p} = p_i R$ for some prime divisor p_i of R, and let $\iota_{\mathfrak{p}}: R \to R_{\mathfrak{p}}$ be the homomorphism that maps an element $[x]_m$ of R to $([x]_m/[1]_m)_{\mathfrak{p}}$. Now

$$R \setminus \mathfrak{p} = \{[t]_m : t \in \mathbb{Z} \text{ and } t \text{ is coprime to } p\}.$$

It follows that $\iota_{\mathfrak{p}}([x]_m) = 0$ if and only if $[tx]_m = 0$ for some integer t coprime to p, and thus if and only if m divides tx for some integer t coprime to p.

But there exists an integer t for which tx is divisible by m if and only if m is itself divisible by m_i. Thus an element $[x]_m$ of R satisfies $\iota_{\mathfrak{p}}([x]_m) = 0$ if and only if x is divisible by m_i. Moreover if s is any integer coprime to p_i then, given any integer x, there exists some integer y such that $x \equiv sy \bmod m_i$. But then there exists some integer t coprime to t_i such that $tx \equiv tsy \bmod m$. This ensures that $([x]_m/[s]_m)_{\mathfrak{p}} = ([y]_m/[1]_m)_{\mathfrak{p}} = \iota_{\mathfrak{p}}([y]_m)$. We conclude that $\iota_{\mathfrak{p}}\colon R \to R_{\mathfrak{p}}$ is surjective. It follows that $R_{\mathfrak{p}} \cong R/\ker \iota_{\mathfrak{p}} \cong \mathbb{Z}/m_i\mathbb{Z}$.

The result of Theorem 6.24 corresponds to the fact that

$$R \cong R_{p_1R} \times R_{p_2R} \cdots \times R_{p_kR},$$

and thus to the fact that

$$\mathbb{Z}/m\mathbb{Z} \cong (\mathbb{Z}/m_1\mathbb{Z}) \times (\mathbb{Z}/m_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/m_k\mathbb{Z}),$$

where $m_i = p_i^{n_i}$ for $i = 1, 2, \ldots, k$. This result is a form of the Chinese Remainder Theorem.