Inverse Trigonometric Functions

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In This Presentation…

- We will give a definition
- Discuss some of the inverse trig functions
- Learn how to use it
- Do example problems

Definition

- In Calculus, a function is called a one-to-one function if it never takes on the same value twice; that is f(x1)~= f(x2) whenever x1~=x2.
- Following that, if f is a one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by

    f^(-1)y=x

  => f(x)=y

A Note with an Example

- Domain of f^{-1}= Range of f
- Range of f^{-1}= Domain of f
- For example, the inverse function of $f(x) = x^3$ is

  $f^{-1}(x)=x^{1/3}$ because if $y=x^3$, then

  $f^{-1}(y)=f^{-1}(x^3)=(x^3)^{1/3}=x$

Caution Rule: the -1 in f^{-1} is not an exponent.

Thus $f^{-1}(x)$ does not mean $1/f(x)$

Cancellation Equations and Finding the Inverse Function:

- $f^{-1}(f(x))=x$ for every x in A
- $f(f^{-1}(x))=x$ for every x in B

- To find the Inverse Function
- Step 1: Write $y=f(x)$
- Step 2: Solve this equation for x in terms of y (if possible).
- Step 3: To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y=f^{-1}(x)$.

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Example:

- Find the inverse function of $f(x) = x^3+2$

So, $y= x^3+2$

Solving the equation for x:

$$x^3=y-2$$

$$x=(y-2)^{1/3}$$

Finally interchanging x and y:

$$y=(x-2)^{1/3}$$

Therefore the inverse function is

$$f^{-1}(x)=(x-2)^{1/3}$$

Inverse Trigonometric Functions:

- The domains of the trigonometric functions are restricted so that they become one-to-one and their inverse can be determined.

- Since the definition of an inverse function says that

$$f^{-1}(x)=y$$

$$=> f(y)=x$$

We have the inverse sine function,

$$\sin^{-1}x=y$$

$$=> \sin y=x \qquad \text{and} \qquad {}^{-\pi}/_{2} <= y <= {}^{\pi}/_{2}$$

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Example and cancellation equations:

- Evaluate $\sin^{-1}(^1/_2)$
- We have

$$\sin^{-1}(^1/_2) = ^{\pi}/_6$$

because $\sin(^{\pi}/_6) = ½$

and $^{\pi}/_6$ lies between $^{-\pi}/_2$ and $^{\pi}/_2$

- Cancellation Eq:

$\sin^{-1}(\sin x) = x$ for $^{-\pi}/_2 <= x <= ^{\pi}/_2$

$\sin(\sin^{-1} x) = x$ for $-1 <= x <= -1$

More Inverse Functions:

- Inverse Cosine function:

$\cos^{-1}x=y$

$\Rightarrow \cos y=x$ and $0 <= y <= \pi$

The Cancellation Equations:

$\cos^{-1}(\cos x)= x$ for $0<=x<=\pi$

$\cos(\cos^{-1} x)= x$ for $-1 <= x <= -1$

* Inverse Tangent Function:

$\tan^{-1}x=y$

$\Rightarrow \tan y=x$ and $^{-\pi}/_{2} < y < {}^{\pi}/_{2}$

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More Inverse Functions

Example:

Simplify cos $(\tan^{-1}x)$

* Simplify cos $(\tan^{-1}x)$
* Let $y=\tan^{-1}x$

Then $\tan y=x$ and $-\pi/2 < y < \pi/2$

Since tan y is known, it is easier to find sec y first:

$\sec^2 y=1+\tan^2 y= 1+x^2$

$\sec y=(1+x^2)^{1/2}$

Thus $\cos(\tan^{-1}x)=\cos y= \frac{1}{secy} = \frac{1}{\sqrt{1+x^2}}$

More on inverse

* Inverse Cotangent Function:

$\cot^{-1}x=y$

=> $\cot y=x$ and $0< y < \pi$

- Inverse Cosecant Function:

$\text{cosecant}^{-1}x=y$

=> cosecant $y=x$ and $y \in (0, {}^{\pi}/_{2}] \cup (\pi, 3\pi/2)$

- Inverse Secant Function:

$\text{Secant}^{-1}x=y$

=> Secant $y=x$ and $y \in (0, {}^{\pi}/_{2}] \cup (\pi, {}^{3\pi}/_{2})$

Inverse Tangent

- $\lim_{x\to\infty} tan^{-1}x = \frac{\pi}{2}$

- $\lim_{x\to\infty} tan^{-1}x = -\frac{\pi}{2}$

- Limits of arctan can be used to derive the formula for the derivative (often an useful tool to understand and remember the derivative formulas)

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Derivatives of Inverse Trig Functions

- $\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}(sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}(tan^{-1}x) = -\frac{1}{1+x^2}$

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Examples

- Differentiate (a) $y=\frac{1}{\sin^{-1}x}$ and (b) $f(x)=x \arctan\sqrt{x}$
- Solution:

(a) $\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x)$

$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1-x^2}}$

(b) $f'(x) = x\frac{1}{1+(\sqrt{x})^2}(\frac{1}{2}x^{-\frac{1}{2}}) + \arctan\sqrt{x}$

$= \frac{\sqrt{x}}{2(1+x)} + \arctan\sqrt{x}$

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Example

- Prove the identity $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$
- Prove:

$$f(x) = \tan^{-1}x + \cot^{-1}x$$

Then,

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0 \text{ for all values of x.}$$

Therefore f(x) =C, a constant.

To determine the value of C, we put x=1. Then

$$C = f(1) = \tan^{-1}1 + \cot^{-1}1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Thus $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

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Useful Integration Formulas

- $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1}x + C$ (1)

- $\int \frac{1}{x^2+1}\, dx = \tan^{-1}x + C$ (2)

- $\int \frac{1}{x2+a2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$ (3)

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Example

- Example:

Find $\int \frac{x}{x^4+9}\, dx$

Solution:

We substitute u= x^2 because then du= 2x dx

and we can use (3) with a=3:

$$\int \frac{x}{x^4+9}\, dx = \frac{1}{2} \int \frac{du}{u^2+9} = \frac{1}{2} * \frac{1}{3} \tan^{-1}\left(\frac{u}{3}\right) + C$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{x^2}{3}\right) + C$$

Summary

- This outlines the basic procedure for solving and computing inverse trig functions

- Remember a triangle can also be drawn to help with the visualization process and to find the easiest relationship between the trig identities. It almost always helps in double checking the work.

References

- Calculus – Stewart 6th Edition
  - Section 7.1 “Inverse Trigonometric Functions”
  - Section 7.6 “Trigonometric Substitution”
  - Appendixes A1, D “Trigonometry”

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Thank you!

Enjoy those trig functions…!