School book

Exams

3rd Sec.

Algebra & Solid

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The First test

First : Answer one of the following questions

First question : Choose the correct answer :

1] If $^{n}C_{n-3} = 20$, then n = ...

a] 3 b] 4 c] 5 d] 6

The Solution

$\frac{\lfloor n}{\lfloor n-3 \ \lfloor 3} = 20$ $\therefore \frac{n(n-1)(n-2)\lfloor n-3}{\lfloor n-3 \ \lfloor 3} = 20$

$\therefore n(n-1)(n-2) = 20 \times \lfloor 3 = 6 \times 5 \times 4$ $\therefore n = 6$

2] $i + i^2 + i^3 + \cdots \ldots + i^{100} = \ldots$

a] 0 b] 1 c] 2 d] 100

The Solution

$(i - 1 - i + 1) + (i - 1 - i + 1) + \ldots$ to 25 — terms $= 0 + 0 + \ldots$ to 25 terms $= 0$

Another Solution

It is a geometry sequence a = i , r = i and $L = i^{100} = (i^4)^{25} = 1$

$S_n = \frac{L\,r - a}{r-1} = \frac{1 \times i - i}{i-1} = \frac{0}{i-1} = zero$

3] If A (7 , −1 , 8) , B (11 , 2 , − 4) , then $\overline{AB}$ = ...cm

a] 10 b] 11 c] 12 d] 13

The Solution

Let A $(x_1, y_1, z_1) = (7, -1, 8)$ & B $(x_2, y_2, z_2) = (11, 2, -4)$

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(11-7)^2 + (2+1)^2 + (-4-8)^2} = 13$ cm.

4] If $x^2 + y^2 + z^2 + 4x - 6y + 8z + 4 = 0$ is an equation of a sphere , whose diameter length = ...cm

a] 5 b] 10 c] 15 d] 20

The Solution

$\left(-\frac{1}{2}\ coefficient\ of\ x , -\frac{1}{2}\ coefficient\ of\ y , -\frac{1}{2}\ coefficient\ of\ z\right)$

= The center of the sphere = (L , k , n) = (−2 , 3 , −4) & d = 4

$\therefore r = \sqrt{L^2 + k^2 + n^2 - d} = \sqrt{(-2)^2 + +(3)^2 + (-4)^2 - 4} = 5$ cm $\therefore$ **The diameter = 10 cm.**

Another Solution

$(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 8z + 16) + 4 - (4 + 9 + 16) = 0$

$\therefore (x + 2)^2 + (y - 3)^2 + (z + 4)^2 = 25$ $\therefore r = 5$ **cm.** $\therefore$ **The diameter = 10 cm.**

5] If $L_1 : \frac{x-3}{1} = \frac{y+2}{-2} = \frac{z+1}{-4}$ is parallel to $L_2 : \frac{x+5}{-2} = \frac{y}{k+1} = \frac{z-1}{8}$, then k = ...

a] 3 b] 4 c] 5 d] 6

The Solution

$\because L_1 // L_2$ $\therefore \frac{1}{-2} = \frac{-2}{k+1} = \frac{-4}{8}$ $\therefore k + 1 = 4$ $\therefore k = 3$

6] If θ is the measure of the angle included between the two vectors $\vec{A} = (-2 , -6 , 1)$, $\vec{B} = (2 , 6 , -1)$, then θ = ...

a] 30^0 b] 60^0 c] 120^0 d] 180^0

The Solution

$\cos\theta = \frac{\vec{A} \bullet \vec{B}}{\|\vec{A}\| \, \|\vec{B}\|} = \frac{(-2 , -6 , 1) \bullet (2 , 6 , -1)}{\sqrt{4+36+1} \ \sqrt{4+36+1}} = \frac{-4-36-1}{\sqrt{41} \times \sqrt{41}} = \frac{-41}{41} = -1$ $\therefore m(\angle\theta) = 180°$

Second question : Complete each of the following :

1] The coefficient of x^5 in the expansion of $(3 - 2x)^7$ equals ....

The Solution

Let the term contains x^5 in the expansion is T_{r+1}

$\because T_{r+1} = {}^7C_r \times (-2x)^r \ (3)^{7-r} = {}^7C_r \times (-2)^r \times (3)^{7-x} \times x^r$ $\therefore r = 5$

$\therefore$ The term which contains x^5 in the expansion is T_6

$\therefore$ The coefficient of $T_6 = {}^7C_5 \times (-2)^5 \times (3)^2 = -32 \times 9 = -6048$

2] The solution set of $\begin{vmatrix} x & 1 & 2 \\ 0 & x & 3 \\ 0 & 0 & x \end{vmatrix} - 8 = 0$ in R is ....

The Solution

$\because$ The determinant in the triangular form $\therefore x^3 - 8 = 0$ $\therefore x^3 = 8$ $\therefore x = 2$ $\therefore$ **S.S. = { 2 }**

3] If $\vec{A} = 2\hat{i} + 3\hat{j} + m\hat{k}$, $\vec{B} = -6\hat{i} - 4\hat{j} + 4\hat{k}$ and $\vec{A} \perp \vec{B}$, then m = ...

The Solution

$\because \vec{A} \perp \vec{B}$ $\quad \therefore \vec{A} \cdot \vec{B} = 0$ $\quad \therefore (2, 3, m) \cdot (-6, -4, 4) = 0$

$\therefore -12 - 12 + 4m = 0$ $\quad \therefore 4m = 24$ $\quad \therefore m = 6$

4] If $\vec{A} = (3, 0, 4)$, $\vec{B} = \hat{i} - 2\hat{j} + 3\hat{k}$, then $\vec{A} \times \vec{B} =$ ....

The Solution

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 4 \\ 1 & -2 & 3 \end{vmatrix} = \hat{i}(0+8) - \hat{j}(9-4) + \hat{k}(-6-0) = 8\hat{i} - 5\hat{j} - 6\hat{k}$$

5] The equation of the sphere whose center is $(2, -3, 1)$ and its radius length equals $2\sqrt{5}$ is ....

The Solution

The equation of the sphere $\quad (x-L)^2 + (y-k)^2 + (z-n)^2 = r^2$

$\because (2, -3, 1) = (L, k, n)$ $\quad \therefore (x-2)^2 + (y+3)^2 + (z-1)^2 = 20$

6] The equation of the straight line passing through the two points A $(2, -1, 4)$, B$(-1, 0, 2)$ is ......

The Solution

$\vec{d} = \overrightarrow{AB} = \vec{B} - \vec{A} = (-1, 0, 2) - (2, -1, 4) = (-3, 1, -2)$

Its enough to write one of the following.

$\therefore$ **The equation of the straight line** $\frac{x-2}{-3} = \frac{y+1}{1} = \frac{z-4}{-2}$ **The cartesian form**

$\therefore x = 2 - 3t, \quad y = -1 - t, \quad z = 4 - 2t$ **The parametric form**

$\therefore \vec{r} = A + t\vec{d} = (2, -1, 4) + t(-3, 1, -2)$ **The vector form.**

Answer the following question :

Questions three :

3 a] In the expansion of $(2x+\frac{1}{x^2})^{15}$, **find the value of the term free of x** and then **prove that** the expansion does not contain a term includes x^5 .

The Solution

Let the term free of x is T_{r+1}

$\therefore T_{r+1} = {}^{15}C_r\left(\frac{1}{x^2}\right)^r (2x)^{15-r} = {}^{15}C_r\ (2)^{15-r}(x)^{-2r}(x)^{15-r} = {}^{15}C_r\,(2)^{15-r}(x)^{15-3r}$

Let $15-3r=0$ $\therefore r=5$ $\therefore$ The term free of x is $T_6 = {}^{15}C_5 \times (2)^{10} = 3075072$

Let the term contains x^5 is a general term $\therefore 15-3r=5$ $\therefore 10=3r$ $\therefore r=\frac{10}{3} \notin z^+$

$\therefore$ **There is no term Contains x^5 in this expansion .**

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3 b] Find all the different forms of the equation of the straight line : $\frac{x+3}{2}=\frac{2y-1}{5}=\frac{3z+2}{4}$

The Solution

Let $\frac{x+3}{2}=\frac{2y-1}{5}=\frac{3z+2}{4}=t$ $\therefore \frac{x+3}{2}=t$ $\therefore x+3=2t$ $\therefore x=-3+2t$

$\frac{2y-1}{5}=t$ $\therefore 2y-1=5t$ $\therefore 2y=1+5t$ $\therefore y=\frac{1}{2}+\frac{5}{2}t$

$\frac{3z+2}{4}=t$ $\therefore 3z+2=4t$ $\therefore 3z=-2+4t$ $\therefore z=-\frac{2}{3}+\frac{4}{3}t$

$\therefore x=-3+2t,\quad y=\frac{1}{2}+\frac{5}{2}t,\quad z=-\frac{2}{3}+\frac{4}{3}t$ **The parametric form**

$\therefore \vec{r}\left(-3,\frac{1}{2},\frac{-2}{3}\right)+t\left(2,\frac{5}{2},\frac{4}{3}\right)$ **The vector form .**

$\therefore \frac{x+3}{2}=\frac{y-\frac{1}{2}}{\frac{5}{2}}=\frac{z+\frac{2}{3}}{\frac{4}{3}}$ **The cartesian form**

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4 a] Find the multiplicative inverse of the matrix $A=\begin{pmatrix}1 & -1 & 2\\ 2 & -3 & 1\\ 1 & 5 & 21\end{pmatrix}$

The Solution

$|A|=\begin{vmatrix}1 & -1 & 2\\ 2 & -3 & 1\\ 1 & 5 & 21\end{vmatrix}=1\begin{vmatrix}-3 & 1\\ 5 & 21\end{vmatrix}+1\begin{vmatrix}2 & 1\\ 1 & 21\end{vmatrix}+2\begin{vmatrix}2 & -3\\ 1 & 5\end{vmatrix}=1(-63-5)+1(42-1)+2(10+3)=-1$

The cofactor of $A = \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} & \overline{a_{13}} \\ \overline{a_{21}} & \overline{a_{22}} & \overline{a_{23}} \\ \overline{a_{31}} & \overline{a_{32}} & \overline{a_{33}} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} -3 & 1 \\ 5 & 21 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 21 \end{vmatrix} & \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 5 & 21 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 21 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 5 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} \end{pmatrix}$

$\therefore$ The matrix of cofactor of A is $F = \begin{pmatrix} -68 & -41 & 13 \\ 31 & 19 & -6 \\ 5 & 3 & -1 \end{pmatrix}$

$\therefore Adj(A) = F^t = \begin{pmatrix} -68 & 31 & 5 \\ -41 & 19 & 3 \\ 13 & -6 & -1 \end{pmatrix}$

$A^{-1} = \frac{1}{|A|} \times F^t = \frac{1}{-1} \times \begin{pmatrix} -68 & 31 & 5 \\ -41 & 19 & 3 \\ 13 & -6 & -1 \end{pmatrix} = \begin{pmatrix} 68 & -31 & -5 \\ 41 & -19 & -3 \\ -31 & 6 & 1 \end{pmatrix}$

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4 b] Find the **two square roots** of the complex number $z = 2 - 2\sqrt{3}\,i$ in the trigonometric form.

The Solution

$\because z = 2 - 2\sqrt{3}\,i \quad \therefore x = 2 \;\&\; y = -2\sqrt{3} \quad \therefore r = \sqrt{x^2 + y^2} = \sqrt{4 + 12} = 4$

$\tan\theta = \frac{y}{x} = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$, $\because x > 0, y < 0 \quad \therefore z$ in the 4^{th} quad $\quad \therefore m(\angle\theta) = \frac{-\pi}{3}$

$z = 4\left[\text{Cos}\left(\frac{-\pi}{3}\right) + i\,\text{Sin}\left(\frac{-\pi}{3}\right)\right]$

$\sqrt{z} = \sqrt{4}\left[\text{Cos}\left(\frac{\frac{-\pi}{3} + 2\pi m}{2}\right) + i\,\text{Sin}\left(\frac{\frac{-\pi}{3} + 2\pi m}{2}\right)\right]$ where $m = 0$, $m = 1$

At m = 0 $\quad \therefore \sqrt{z} = 2\left[\text{Cos}\left(\frac{-\pi}{6}\right) + i\,\text{Sin}\left(\frac{-\pi}{6}\right)\right]$

At m = 1 $\quad \therefore \sqrt{z} = 2\left[\text{Cos}\left(\frac{5\pi}{6}\right) + i\,\text{Sin}\left(\frac{5\pi}{6}\right)\right]$

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5 a] Solve the following equations : $x + 3y + 2z = 13$, $2x - y + z = 3$, $3x + y - z = 2$ using the multiplicative inverse of the matrix .

The Solution

The matrix equation AX = B Where $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 13 \\ 3 \\ 2 \end{pmatrix}$

$|A| = \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix}$

$= 1(1 - 1) - 3(-2 - 3) + 2(2 + 3) = 0 + 15 + 10 = 25$

The cofactor of $A = \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} & \overline{a_{13}} \\ \overline{a_{21}} & \overline{a_{22}} & \overline{a_{23}} \\ \overline{a_{31}} & \overline{a_{32}} & \overline{a_{33}} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \end{pmatrix}$

$\therefore$ The matrix of cofactor of A is $F = \begin{pmatrix} 0 & 5 & 5 \\ 5 & -7 & 8 \\ 5 & 3 & -7 \end{pmatrix}$

$\therefore \therefore Adj\,(A) = F^t = \begin{pmatrix} 0 & 5 & 5 \\ 5 & -7 & 3 \\ 5 & 8 & -7 \end{pmatrix}$ $\quad \therefore A^{-1} = \frac{1}{|A|} \times F^t = \frac{1}{25}\begin{pmatrix} 0 & 5 & 5 \\ 5 & -7 & 3 \\ 5 & 8 & -7 \end{pmatrix}$,

$\therefore X = A^{-1}B \quad \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 0 & 5 & 5 \\ 5 & -7 & 3 \\ 5 & 8 & -7 \end{pmatrix}\begin{pmatrix} 13 \\ 3 \\ 2 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 25 \\ 50 \\ 75 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\therefore x = 1, y = 2, z = 3$

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5 b] Find the point of intersection of the planes

$2x + y - z = -1$, $x + y + z - 2 = 0$, $3x - y - z = 6$

The Solution

$2x + y - z = -1$ ---- (1) , $x + y + z = 2$ ----- (2) , $3x - y - z = 6$ ----- (3)

By adding (2) , (3) $\therefore 4x = 8 \quad \therefore x = 2$ by adding (1) , (2) $\therefore 3x + 2y = 1$ ----- (4)

$\because x = 2 \quad \therefore 6 + 2y = 1 \quad \therefore 2y = -5 \quad \therefore y = -\frac{5}{2}$

$\because x + y + z = 2 \quad \therefore 2 - \frac{5}{2} + z = 2 \quad \therefore z = \frac{5}{2}$

$\therefore$ The point of intersection of the planes is $\left(2, -\frac{5}{2}, \frac{5}{2}\right)$

Another method :

To get the point of intersection between 3 planes , the cramers method can be used to find the solution .

$\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix} = 2(0) - 1(-4) - 1(-4) = 8$

$\Delta_x = \begin{vmatrix} -1 & 1 & -1 \\ 2 & 1 & 1 \\ 6 & -1 & -1 \end{vmatrix} = -1(0) - 1(-8) - 1(-8) = 16$

$\Delta_y = \begin{vmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 3 & 6 & -1 \end{vmatrix} = 2(-8) + 1(-4) - 1(0) = -20$

$\Delta_z = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 2 \\ 3 & -1 & 6 \end{vmatrix} = 2(8) - 1(0) - 1(-4) = 20$

$x = \frac{\Delta_x}{\Delta} = \frac{16}{8} = 2$, $y = \frac{\Delta_y}{\Delta} = \frac{-20}{8} = \frac{-5}{2}$, $z = \frac{\Delta_z}{\Delta} = \frac{20}{8} = \frac{5}{2}$

$\therefore$ The point of intersection between the three planes is $\left(2, -\frac{5}{2}, \frac{5}{2}\right)$

The Second test

First : Answer one of the following questions

First question : Choose the correct answer :

1] If the two equations $2x + y = 1$, $4x + 2y = k$ have an infinite number of solutions , then k =……………..

a] zero b] 1 c] 2 d] 3

The Solution

$\because$ The system have infinite number of solution .

$\therefore$ *The two equations are the same equation .* $\therefore k = 2$

2] If ${}^{n+1}C_3 : {}^{n}C_4 = 2 : 3$, then n = ……….

a] 2 b] 3 c] 6 d] 11

The Solution

$\because {}^{n+1}C_3 : {}^{n}C_4 = \frac{2}{3}$ $\therefore \frac{\lfloor n+1}{\lfloor n-2 \lfloor 3} \div \frac{\lfloor n}{\lfloor n-4 \lfloor 4} = \frac{2}{3}$

$\therefore \frac{(n+1)\lfloor n}{(n-2)(n-3)\lfloor n-4 \lfloor 3} \times \frac{4 \lfloor 3 \lfloor n-4}{\lfloor n} = \frac{2}{3}$ $\therefore \frac{4(n+1)}{(n-2)(n-3)} = \frac{2}{3}$ $\therefore 2(n-2)(n-3) = 12(n+1)$

$\therefore n^2 - 5n + 6 = 6n + 6$ $\therefore n^2 - 11n = 0$ $\therefore n(n-11) = 0$ $\therefore$ ($n = 0$ refused) $\therefore n = 11$

3] If $x^2 + y^2 + z^2 + 6x - 4y + 10z - 8 = 0$ is the equation of a circle whose center is M, then $M =$ ……………

a] $(-3, 2, -5)$ b] $(4, -2, -5)$ c] $(-3, -2, -5)$ d] $(3, 2, 5)$

The Solution

$\therefore$ The Centre of the sphere is $= (L, k, n)$

$M = \left(-\frac{1}{2}\text{ coefficicent of } x, -\frac{1}{2}\text{ coefficient of } y, -\frac{1}{2}\text{ coefficicent of } z\right) = (-3, 2, -5)$

4] If $\vec{A} = (-2, 4, 6)$, $\vec{B} = (0, k, 3)$ where $k \in Z^{+}$ and $\|\overline{AB}\| = 7$, then the value of k = ____

a] 10 b] 8 c] 6 d] 4

The Solution

Let $A(x_1, y_1, z_1) = (-2, 4, 6)$ & $B(x_2, y_2, z_2) = (0, k, 3)$

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(0 - -2)^2 + (k-4)^2 + (3-6)^2} = 7$ cm.

by squaring both sides $\therefore 4 + (k-4)^2 + 9 = 49$

$\therefore (k-4)^2 = 36$ $\therefore k - 4 = 6$ $\therefore k = 10$ or $k - 4 = -6$ $\therefore k = -2$ refused because $k \in z^{+}$

5] If θ is the measure of the angle included between $\vec{A} = (2, 0, 2)$, $\vec{B} = (0, 0, 4)$, then θ = ...

a] 30^0 b] 45^0 c] 60^0 d] 90^0

The Solution

$\text{Cos}\,\theta = \frac{\vec{A} \bullet \vec{B}}{\|\vec{A}\|\|\vec{B}\|} = \frac{(2, 0, 2) \bullet (0, 0, 4)}{\sqrt{8}\sqrt{16}}$ $\therefore \text{Cos}\,\theta = \frac{8}{8\sqrt{2}} = \frac{1}{\sqrt{2}}$ $\therefore m(\angle\theta) = 45°$

6] If $L_1 : \frac{x-3}{2} = \frac{-y-1}{6} = \frac{z}{k}$ is parallel to $L_2 : \frac{x+2}{6} = \frac{y-4}{m} = \frac{z-1}{3}$, then k + m = ...

a] -17 b] -10 c] 10 d] 17

The Solution

$\because L_1 // L_2$ $\therefore \frac{2}{6} = \frac{-6}{m} = \frac{k}{3}$ $\therefore m = -18$, k = 1 $\therefore k + m = 1 - 18 = -17$

Another Solution

$\because L_1 // L_2$, $\therefore (2, -6, K) = t(6, m, 3)$ $\therefore 2 = 6t$ $\therefore t = \frac{1}{3}$, $-6 = t\,m$, $\therefore -6 = \frac{1}{3}m$

$\therefore m = -18$, $k = 3t = 3 \times \frac{1}{3} = 1$ $\therefore k + m = 1 - 18 = -17$

Another Solution

$\because L_1 // L_2$ $\therefore \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -6 & k \\ 6 & m & 3 \end{vmatrix} = \vec{0}$ $\therefore (-18 + mK)\hat{\imath} - (6 - 6k)\hat{\jmath} + (2M + 36)\hat{k} = \vec{0}$

$\therefore 6 - 6k = 0$ $\therefore K = 1$, $2m = -36$ $\therefore m = -18$, $\therefore k + M = 1 - 18 = -17$

Second question : Complete

1] $w + w^2 + \cdots \ldots\ldots\ldots\ldots + w^{100} =$ ……………………………

The Solution

$33(w + w^2 + w^3) + [(w^3)^{33} \times w] = 33 \times 0 + 1 \times w = w$

Another Solution

It is geometric sequence a = w , r = w and the last term , $L = w^{100} = (w^3)^{33} \times w = w$

$S_n = \frac{Lr - a}{r - 1}$ $\quad \therefore S_{100} = \frac{w \times w - w}{w - 1} = \frac{w(w - 1)}{w - 1} = w$

2] If a , b , c are the lengths of the sides of the triangle ABC then the value of

$\begin{vmatrix} a & b & c \\ 5 & 7 & 8 \\ \sin A & \sin B & \sin C \end{vmatrix} =$ ……………………

The Solution

$\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin c} = k$ $\quad \therefore a = k \sin A$ & $b = k \sin B$ & $c = k \sin C$

$\therefore \begin{vmatrix} k\sin A & k\sin B & k\sin C \\ 5 & 7 & 8 \\ \sin A & \sin B & \sin C \end{vmatrix} = k \begin{vmatrix} \sin A & \sin B & \sin C \\ 5 & 7 & 8 \\ \sin A & \sin B & \sin C \end{vmatrix} = 0$ because $R_1 = R_3$

3] If $\vec{A} = (-1, 4, 2)$, $\vec{B} = (2, 2, 1)$, then the component of $\vec{A}$ in the direction of $\vec{B} =$ ……………

The Solution

The component of $\vec{A}$ in the direction of $\vec{B}$ is $\frac{|\vec{A} \bullet \vec{B}|}{\|\vec{B}\|} = \frac{(-1, 4, 2) \bullet (2, 2, 1)}{\sqrt{9}} = \frac{8}{3}$

4] $x^2 + y^2 + z^2 - 4k\,x + 4y - 8z + 2k = 0$ is the equation of a sphere , the length of its radius equals $2\sqrt{5}$ the value of k = …………………

The Solution

$\because$ The center of the sphere $= \left(-\frac{1}{2}\, coefficient\ of\ x , -\frac{1}{2}\, coefficient\ of\ y , -\frac{1}{2}\, coefficient\ of\ z\right)$

$= (L , k , n) = (2k , -2 , 4)$ & $d = 2k$

$\therefore r = \sqrt{L^2 + k^2 + n^2 - d} = \sqrt{(2k)^2 + (-2)^2 + (4)^2 - 2k} = 2\sqrt{5}$ By squaring both sides

$\therefore 4k^2 + 4 + 16 - 2k = 20 \quad \therefore 4k^2 - 2k = 0 \quad \therefore 2k(2k - 1) = 0 \quad \therefore k = 0$ or $k = \frac{1}{2}$

5] If the two planes $3x - y + 2z + 3 = 0$, and $kx - 4y + z - 5 = 0$ are perpendicular , then the value of k = ...

The Solution

∵ The ⊥ vectors for the two planes are $\overrightarrow{d_1} = (3, -1, 2)$, $\overrightarrow{d_2} = (k, -4, 1)$

∵ The two planes are ⊥ ∴ $\overrightarrow{d_1} \cdot \overrightarrow{d_2} = 0$ ∴ $(3, -1, 2) \cdot (k, -4, 1) = 0$

∴ $3k + 4 + 2 = 0$ ∴ $3k = -6$ ∴ $k = -2$

6] If C $(-1, 6, -5)$ is the midpoint of $\overline{AB}$ where A $(k - 2, -1, m + 3)$, B $(2, n - 7, -2)$, then k + m − n = ...

The Solution

∵ C is the midpoint of $\overline{AB}$ ∴ $C = \frac{A+B}{2}$ ∴ $(-1, 6, -5) = \frac{(k-2, -1, m+3) + (2, n-7, -2)}{2}$

∴ $\frac{k-2+2}{2} = -1$ ∴ $k = -2$, $\frac{-1+n-7}{2} = 6$ ∴ $n - 8 = 12$ ∴ $n = 20$

$\frac{m+3-2}{2} = -5$ ∴ $m + 1 = -10$ ∴ $m = -11$ ∴ $k + m - n = -2 - 11 - 20 = -33$

Third question :

3a] Find the coefficient of x^5 in the expansion of $(1 - x + x^2)(1 + x)^{11}$

The Solution

$(1 - x + x^2)(1 + x)^{11} = (1 - x + x^2)[1 + {}^{11}C_1 x + {}^{11}C_2 x^2 + {}^{11}C_3 x^3 + {}^{11}C_4 x^4 + {}^{11}C_5 x^5 + ... + x^{11}]$

The terms includes x^5 are $1 \times {}^{11}C_5 x^5$, $-x \times {}^{11}C_4 x^4$, $x^2 \times {}^{11}C_3 x^3$

The coefficient of $x^5 = {}^{11}C_5 - {}^{11}C_4 + {}^{11}C_3 = 462 - 330 + 165 = 297$

3b] Prove that the straight line $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z}{3}$ intersects the plane $3x + 2y + z - 8 = 0$ at a point and **find the measure of the inclination angle of the line with the plane**.

The Solution

∵ $(2, -1, 3)$ is the direction vector of the straight line .

∵ $(3, 2, 1)$ is the normal direction vector of the plane .

$(2, -1, 3) \cdot (3, 2, 1) = 7 \neq 0$ ∴ The straight line is not parallel to the plane .

∴ The straight line line intersects the plane .

Let the angle between the straight line and the perpendicular to the plane = θ

$\cos\theta = \frac{(3,\ 2,\ 1)\bullet(2,\ -1,\ 3)}{\sqrt{14}\sqrt{14}} = \frac{7}{14} = \frac{1}{2}$ $\quad \therefore m(\angle\theta) = 60°$

$\therefore$ **The angle of inclination of the line with the plane = 90° − 60° = 30°**

Another solution for the first part.

Let $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z}{3} = t$ $\therefore x = 1 + 2t,\ y = -3 - t,\ z = 3t$

by substitution in the equation of the plane $\therefore 3(1 + 2t) + 2(-3 - t) + 3t - 8 = 0$

$\therefore 3 + 6t - 6 - 2t + 3t - 8 = 0 \qquad \therefore 7k = 11 \qquad \therefore k = \frac{11}{7}$

$\therefore x = 1 + 2 \times \frac{11}{7} = \frac{29}{7}\ , \quad y = -3 - \frac{11}{7} = -\frac{32}{7},\ z = 3 \times \frac{11}{7} = \frac{33}{7}$

$\therefore$ The point of intersection between the line and the plane is $\left(\frac{29}{7},\ -\frac{32}{7},\ \frac{33}{7}\right)$

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4 a] Calculate the rank of the matrix $A = \begin{pmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 3 & -5 & 2 \end{pmatrix}$ hence prove that the equations $2x - y - 3z = 0$, $x + 2y + z = 1$, $3x - 5y + 2z = 13$ have a unique solution and find this solution using the multiplicative inverse of the matrix.

The Solution

$|A| = \begin{vmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 3 & -5 & 2 \end{vmatrix} = 2(4+5) + 1(2-3) - 3(-5-6) = 50 \neq 0$ $\therefore Rk(A) = order\ of\ |A| = 3$

$\because$ The number of unknown = 3 $\qquad \therefore$ The equations non homogeneous

$\therefore$ The equations has unique solution the matrix equation AX = B

Where $\quad A = \begin{pmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 3 & -5 & 2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ 1 \\ 13 \end{pmatrix}$

The cofactor of $A = \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} & \overline{a_{13}} \\ \overline{a_{21}} & \overline{a_{22}} & \overline{a_{23}} \\ \overline{a_{31}} & \overline{a_{32}} & \overline{a_{33}} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ -5 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} \\ -\begin{vmatrix} -1 & -3 \\ -5 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 3 & -5 \end{vmatrix} \\ \begin{vmatrix} -1 & -3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \end{pmatrix}$

$\therefore$ The matrix of cofactor of A is $F = \begin{pmatrix} 9 & 1 & -11 \\ 17 & 13 & 7 \\ 5 & -5 & 5 \end{pmatrix}$

$\therefore Adj(A) = F^t = \begin{pmatrix} 9 & 17 & 5 \\ 1 & 13 & -5 \\ -11 & 7 & 5 \end{pmatrix}$ $\qquad \therefore A^{-1} = \frac{1}{|A|} \times F^t = \frac{1}{50}\begin{pmatrix} 9 & 17 & 5 \\ 1 & 13 & -5 \\ -11 & 7 & 5 \end{pmatrix}$

$\therefore X = A^{-1}B \qquad \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 9 & 17 & 5 \\ 1 & 13 & -5 \\ -11 & 7 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 13 \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 100 \\ -50 \\ 50 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\therefore x = 2 , y = -1 , z = 1 \qquad \therefore S.S. = \{ (2 , -1 , 1) \}$

---

4 b] Find the exponential form of the complex number $z = \frac{2+6i}{3-i}$, then find $z^{-1}, \bar{z}$ و $\sqrt{z}$ in the trigonometric form .

The Solution

$z = \frac{2+6i}{3-i} \times \frac{3+i}{3+i} = \frac{6+20i-6}{9+1} = 2i = 2\left[\text{Cos}\frac{\pi}{2} + i\,\text{Sin}\frac{\pi}{2}\right] = 2e^{\frac{\pi}{2}i} \quad \because 1 = \text{Cos}\,0 + i\,\text{Sin}\,0$

$\therefore z^{-1} = \frac{1}{z} = \frac{1}{2}\left[\frac{\text{Cos}\,0 + i\,\text{Sin}\,0}{\text{Cos}\frac{\pi}{2} + i\,\text{Sin}\frac{\pi}{2}}\right] = \frac{1}{2}\left[\text{Cos}\left(-\frac{\pi}{2}\right) + i\,\text{Sin}\left(-\frac{\pi}{2}\right)\right]$

$\overline{Z} = 2\left[\text{Cos}\frac{\pi}{2} - i\,\text{Sin}\frac{\pi}{2}\right] = 2\left[\text{Cos}\left(0-\frac{\pi}{2}\right) + i\,\text{Sin}\left(0-\frac{\pi}{2}\right)\right] = 2\left[\text{Cos}\left(-\frac{\pi}{2}\right) + i\,\text{Sin}\left(-\frac{\pi}{2}\right)\right]$

$\sqrt{z} = \sqrt{2}\left[\text{Cos}\frac{\pi}{2} + i\,\text{Sin}\frac{\pi}{2}\right]^{\frac{1}{2}} = \sqrt{2}\left[\text{Cos}\left(\frac{\frac{\pi}{2}+2m\pi}{2}\right) + i\,\text{Sin}\left(\frac{\frac{\pi}{2}+2m\pi}{2}\right)\right]$ where $m = 0 , m = -1$

at $m = 0$ $\quad \therefore \sqrt{z} = \sqrt{2}\left[\text{Cos}\frac{\pi}{4} + i\,\text{Sin}\frac{\pi}{4}\right]$

at $m = 1$ $\quad \therefore \sqrt{z} = \sqrt{2}\left[\text{Cos}\left(\frac{-3\pi}{4}\right) + i\,\text{Sin}\left(\frac{-3\pi}{4}\right)\right]$

5a] Prove that one of the values of the expression $\sqrt{i} - \sqrt{-i} = \sqrt{2}$

The Solution

Let $z_1 = i = \text{Cos}\frac{\pi}{2} + i\,\text{Sin}\frac{\pi}{2} \quad \therefore \sqrt{z_1} = \sqrt{i} = \text{Cos}\left(\frac{\frac{\pi}{2}+2m\pi}{2}\right) + i\,\text{Sin}\left(\frac{\frac{\pi}{2}+2m\pi}{2}\right)$ where $m = 0 , -1$

At $m = 0$ $\quad \therefore \sqrt{z_1} = \sqrt{i} = \text{Cos}\frac{\pi}{4} + i\,\text{Sin}\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i$

At $m = -1 \quad \therefore \sqrt{z_1} = \sqrt{i} = \text{Cos}\left(\frac{-3\pi}{4}\right) + i\,\text{Sin}\left(\frac{-3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i$

Let $z_2 = -i = \text{Cos}\left(-\frac{\pi}{2}\right) + i\,\text{Sin}\left(-\frac{\pi}{2}\right)$

$\therefore \sqrt{z_2} = \sqrt{-i} = \text{Cos}\left(\frac{-\frac{\pi}{2}+2m\pi}{2}\right) + i\,\text{Sin}\left(\frac{-\frac{\pi}{2}+2m\pi}{2}\right)$ where $m = 0 , 1$

at $m = 0 \quad \therefore \sqrt{z_2} = \sqrt{-i} = \text{Cos}\left(-\frac{\pi}{4}\right) + i\,\text{Sin}\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i$

at $m = 1$ $\quad \therefore \sqrt{z_2} = \sqrt{-i} = \text{Cos}\left(\frac{3\pi}{4}\right) + i\,\text{Sin}\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i$

$\therefore \sqrt{i} - \sqrt{-i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i = \frac{1}{\sqrt{2}} \times 2 = \sqrt{2}$

Another Solution

By squaring the required value $(\sqrt{i} - \sqrt{-i})^2 = (\sqrt{i})^2 - 2 \times \sqrt{i} \times \sqrt{-i} + (\sqrt{-i})^2$

$= i - 2 \times \sqrt{i} \times i \times \sqrt{i} - i = -2 \times \sqrt{i} \times i \times \sqrt{i} = -2 \times i \times i = 2$ Its square roots $= \pm\sqrt{2}$

one of the values of the expression $\sqrt{i} - \sqrt{-i} = \sqrt{2}$

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5 b] If $(x-2)^2 + (y+4)^2 + (z-2)^2 = 1$, $(x+4)^2 + (y-4)^2 + (z-2)^2 = 4$ are the equations of two spheres, find the distance between the centers of the two spheres and show that the two spheres do not intersect.

The Solution

The Centre of the first sphere $= M_1 = (2, -4, 2)$ $\therefore r_1 = 1$

The Centre of the second sphere $= M_2 = (-4, 4, 2)$ $\therefore r_2 = 2$

$\therefore M_1M_2 = \sqrt{(-4-2)^2 + (4+4)^2 + (2-2)^2} = \sqrt{100} = 10$ $\because r_1 + r_2 = 3$

$\because M_1M_2 > r_1 + r_2$ $\therefore$ The two spheres do not intersect $\therefore$ The two spheres are distant.

The third test

First : Answer one of the following questions

First question : Choose the correct answer :

1] The sum of the coefficients of the expansion of $(1+x)^5$ equals

a] zero b] 5 c] 32 d]

The Solution

The sum of the coefficients of the expansion $(1+1)^5 = (2)^5 = 32$

Another Solution

${}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 1 + 5 + 10 + 10 + 5 + 1 = 32$

2] If x is a complex number, then the number of solutions of the equation $\begin{vmatrix} x^3+1 & x-1 \\ x+1 & x^3-1 \end{vmatrix} = 0$

a] 6 b] 5 c] 4 d] 3

The Solution

$\Delta = (x^3+1)(x^3-1) - (x+1)(x-1) = (x^6-1) - (x^2-1)$

$= x^6 - x^2 = x^2(x^4-1) = x^2(x^2-1)(x^2+1) = x^2(x-1)(x+1)(x^2+1) = 0$

$\because$ x is a complex number $\therefore$ The number of solution of $x^2+1=0$ is 2

& the number of solution of $x^2(x+1)(x-1)=0$ is 3 $\therefore$ The number of solution is 5 .

3] If (x,y,z) *is the midpoint of* $\overline{AB}$ *where* $A(-4\ ,\ 0,\ 5)$, $B\ (-2,\ 4,\ -13)$

, *then* $x+y+z=$ ....

a] -5 b] -6 c] 3 d] 4

The Solution

$\because\ (x,y,z) = \frac{A+B}{2}$ $\therefore (x,y,z) = \left(\frac{-4-2}{2}, \frac{0+4}{2}, \frac{5-13}{2}\right) = (-3,2,-4)$

$\therefore x+y+z = -3+2-4=-5$

4] If A $(-4,-2,3)$, B $(1,2,k)$ and the length of $\overline{AB}=\sqrt{77}$, then one of the values of k is ....

a] 2 b] 4 c] 6 d] 9

The Solution

$\because$ $AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ $\because AB=\sqrt{77}$ $\therefore (AB)^2=77$

$\therefore (1+4)^2+(2+2)^2+(k-3)^2=77$ $\therefore 25+16+(k-3)^2=77$ $\therefore (k-3)^2=36$

$\therefore k-3=6$ $\therefore k=9$ (one of values of k) $\therefore k-3=-6$ $\therefore k=-3$

5] If $\vec{A}=(-1,3,4)$, $\vec{B}=(0,-2,5)$, then $\|\overrightarrow{AB}\|=$ ⋯ … …

a] $2\sqrt{3}$ b] $3\sqrt{3}$ c] $4\sqrt{3}$ d] $5\sqrt{3}$

The Solution

Let $\vec{A}=(-1,3,4)=(x_1,y_1,z_1)$ & $\vec{B}=(0,-2,5)=(x_2,y_2,z_2)$

$\because$ $AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$AB=\sqrt{(0+1)^2+(-2-3)^2+(5-4)^2}=\sqrt{27}=3\sqrt{3}$ *unit length*

6] The length of the perpendicular drawn from point A(3 , 0 , −5) on the plane $2x + \sqrt{5}\,y + 4z - 6 = 0$ equals ....

a] 4 b] 5 c] 6 d] 7

The Solution

The length of the perpendicular $= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|2 \times 3 + \sqrt{5} \times 0 + 4 \times -5 - 6|}{\sqrt{4 + 5 + 16}} = \frac{20}{5} = 4$ *unit length*

Question 2 : Complete each of the following:

1] If $z = \text{Sin } 60^0 - i \text{ Cos } 60^0$, then the principle amplitude of z = ...

The Solution

$z = \text{Sin } 60 - i \text{ Cos } 60 = \frac{\sqrt{3}}{2} - \frac{1}{2}i$ $\therefore x = \frac{\sqrt{3}}{2} , y = -\frac{1}{2}$ $\therefore r = \sqrt{x^2 + y^2} = 1$

$\tan\theta = \frac{y}{x} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$, $\therefore m(\angle\theta) = -30 = -\frac{\pi}{6}$ $\therefore z = \text{Cos}\left(-\frac{\pi}{6}\right) + i\,\text{Sin}\left(-\frac{\pi}{6}\right)$

$\therefore$ **The principle amplitude** $= -\frac{\pi}{6} = -30^0$

2] The rank of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{pmatrix}$ equals = ...

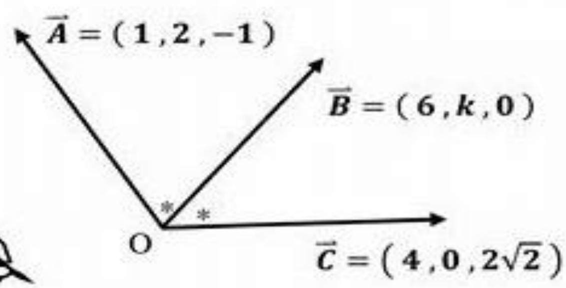
The Solution

$\because |A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{vmatrix} = 10 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$ $\therefore R(A) < 3$

$\because$ All the small determinant = 0 For example $\begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix} = 0$ $\therefore R(A) < 2$ $\therefore R(A) = 1$

3] In the opposite figure

, the value of k = ...



The Solution

Let the angle between $\vec{A}$ & $\vec{B}$ is θ and between $\vec{B}$ & $\vec{C}$ is β $\because m(\angle\theta) = m(\angle\beta)$

$\therefore \text{Cos}\,\theta = \text{Cos}\,\beta$ $\therefore \frac{\vec{A} \bullet \vec{B}}{\|\vec{A}\|\|\vec{B}\|} = \frac{\vec{B} \bullet \vec{C}}{\|\vec{B}\|\|\vec{C}\|}$ $\therefore \frac{6 + 2k}{\sqrt{6} \times \sqrt{36 + k^2}} = \frac{24}{\sqrt{36 + k^2} \times 2\sqrt{6}}$ $\therefore 6 + 2k = 12$ $\therefore 2k = 6$ $\therefore k = 3$

4] The radius length of the sphere $x^2 + y^2 + z^2 + 4x - 6y + 8z + 4 = 0$ equals ....

The Solution

The Centre of the sphere = (L , k , n)

$\left(-\frac{1}{2}\ coefficient\ of\ x, -\frac{1}{2}\ coefficient\ of\ y, -\frac{1}{2}\ coefficient\ of\ z\right) = (-2, 3, -4)$

$\therefore\ r = \sqrt{x^2 + y^2 + z^2 - d} = \sqrt{4 + 9 + 16 - 4} = 5$ unit length

---

5] If the straight line $\frac{x+3}{2} = \frac{y+1}{-6} = \frac{z-2}{k}$ is parallel to the straight line $\frac{x+2}{4} = \frac{y-5}{m} = \frac{z-1}{3}$

then $k + m =$ ....

The Solution

$\because$ The two straight lines are parallel $\therefore \frac{2}{4} = \frac{-6}{m} = \frac{k}{3}$ $\therefore 2m = -24$ $\therefore m = -12$

& $4k = 6$ $\therefore k = \frac{3}{2}$ $\therefore k + m = \frac{3}{2} - 12 = -10.5$

6] If the straight line $\frac{x+2}{6} = \frac{y-1}{m} = \frac{z-1}{3}$ is perpendicular to the straight line $\frac{x-9}{-2} = \frac{y+8}{1}$, $z = 3$

, then m = ...

The Solution

The directed vectors of the two straight lines are (6 , m , 3) , (−2 , 1 , 0)

$\because$ The two lines are perpendicular $\therefore (6, m, 3) \cdot (-2, 1, 0) = 0$ $\therefore -12 + m = 0$ $\therefore m = 12$

Answer the following question :

3a] If $(m + x)^n = 3a + 6a\,x + 5a\,x^2 + \ldots$ where $n \in Z^+$

, find the value of each of m and a .

The Solution

The coefficient of $T_1 = 3a$, The coefficient of $T_2 = 6a$, The coefficient of $T_3 = 5a$

$\frac{\text{The coefficient of } T_2}{\text{The coeficient of } T_1} = \frac{6a}{3a} = 2$ $\therefore \frac{n-1+1}{1} \times \frac{1}{m} = 2$ $\therefore \frac{n}{m} = 2$ $\therefore n = 2m$

$\frac{\text{The coefficient of } T_3}{\text{The coeficient of } T_2} = \frac{5a}{6a} = \frac{5}{6}$ $\therefore \frac{n-2+1}{2} \times \frac{1}{m} = \frac{5}{6}$ $\therefore \frac{n-1}{2m} = \frac{5}{6}$ $\therefore 6n - 6 = 10m$ $\because n = 2m$

$\therefore 6(2m) - 6 = 10m$ $\therefore 12m - 6 = 10m$ $\therefore 12m - 10m = 6$ $\therefore 2m = 6$ $\therefore m = 3$ $\therefore n = 6$

$\because T_1 = {}^nC_0\,(x)^0\,m^n = 3a$ $\therefore {}^6C_0\ m^n = 3a$ $\therefore 3^6 = 3a$ $\therefore a = 3^5 = 243$

3b] Prove that the following system of equations has a solution except the non zero solution and write the general form of these solutions.

$2x - y + 3z = 0$, $4x + 5y - z = 0$, $2x + 3y - z = 0$

The Solution

$A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 5 & -1 \\ 2 & 3 & -1 \end{pmatrix}$ $\because |A| = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 5 & -1 \\ 2 & 3 & -1 \end{vmatrix} = 2\begin{vmatrix} 5 & -1 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 4 & -1 \\ 2 & -1 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix}$

$= 2(-5+3) + 1(-4+2) + 3(12-10) = 2 \times -2 + 1 \times -2 + 3 \times 2 = -4 - 2 + 6 = 0$

$\therefore Rk(A) < 3$ Less than the number of variables.

$\therefore$ **The equations have infinite number of solution except the zero solution.**

The equations have a solution other than the zero solution

By subtraction (3) From (2) $\therefore 2x + 2y = 0$ $\therefore x = -y$ ....(4)

By subtraction (3) From (1) $\therefore -4y + 4z = 0$ $\therefore y = z$ ....(5)

Let $x = L$ $\therefore y = -L$, $z = -L$ $\therefore$ **The general form of the solution (L , − L , −L)**

4a] If $|z_1| = |z_2| = 1$, and the $\arg(z_1 z_2^3) = 81^0$, $\arg\left(\frac{z_1}{z_2}\right) = 33^0$

, write in the form of $x + yi$ the number $(z_1^{15} z_2^{15})$

The Solution

Let the org of $z_1 = \theta_1$ & the org of $z_2 = \theta_2$ $\therefore \theta_1 + 3\theta_2 = 81 , \theta_1 - \theta_2 = 33$

by subtraction $\therefore 4\theta_2 = 48$ $\therefore \theta_2 = 12°$, $\theta_1 = 45°$

$\therefore z_1 = \cos 45 + i \sin 45$ $z_2 = \cos 12 + i \sin 12$

$\therefore (z_1^{15} z_2^{15}) = (z_1 z_2)^{15} = (\cos 57 + i \sin 57)^{15} = \cos(855) + i \sin(855)$

$= \cos 315 + i \sin 315 = \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

4b] Find the length of the perpendicular drawn from point A $(-2, 3, 1)$ to the line

$\frac{x+2}{2} = \frac{y-3}{4} = \frac{z-1}{4}$

The Solution

$\therefore$ The point $(-2, 3, 1) \in$ the straight line $\therefore$ **The required length of** the perpendicular **= zero**

5a] Prove that : $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$

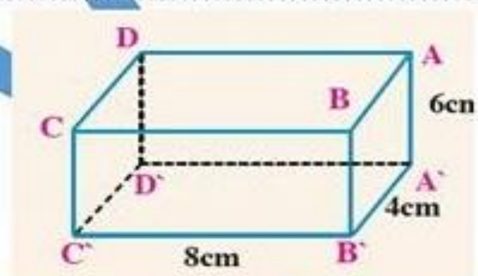
The Solution

by multiply $C_1 \times a$, $C_2 \times b$, $C_3 \times c$

$\therefore$ L.H.S. $= \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix}$ by take **abc** common factor from R_3

$\therefore$ **L.H.S.** $= \frac{abc}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$ = **R.H.S.**

5b] In the opposite figure



A B C D $A^\backslash B^\backslash C^\backslash D^\backslash$ is a cuboid , **find** $\overline{BD^\backslash} \cdot \overline{CA^\backslash}$

The Solution

Let $D^\backslash$ is the origin point (0 , 0 , 0) $\therefore$ A(0 , 8 , 6) , B(4 , 8 , 6) , C (4 , 0 , 6) , $A^\backslash$ **(0 , 8 , 0)**

$\therefore \overrightarrow{BD^\backslash} = \overrightarrow{D^\backslash} - \vec{B} = (0, 0, 0) - (4, 8, 6) = (-4, -8, -6)$

$\overrightarrow{CA^\backslash} = \overrightarrow{A^\backslash} - \vec{C} = (0, 8, 0) - (4, 0, 6) = (-4, 8, -6)$

$\therefore \overrightarrow{BD^\backslash} \cdot \overrightarrow{CA^\backslash} = (-4, -8, -6) \cdot (-4, 8, -6) = 16 - 64 + 36 = -12$

The fourth test

First : Answer one of the following questions:

First Question : Choose the correct answer :

1] If $^{10}C_{r+1} : {}^{10}C_{r-1} = 21 : 10$, then the value of r = ...

a] 3 b] 4 c] 5 d] 6

The Solution

$\frac{^{10}C_{r+1}}{^{10}C_{r-1}} = \frac{21}{10}$ $\therefore \frac{^{10}C_{r+1}}{^{10}C_{r}} \times \frac{^{10}C_{r}}{^{10}C_{r-1}} = \frac{21}{10}$ $\therefore \frac{10-r-1+1}{r+1} \times \frac{10-r+1}{r} = \frac{21}{10}$ $\therefore \frac{10-r}{r+1} \times \frac{11-r}{r} = \frac{21}{10}$

$\therefore 21\,r(r+1) = 10(11-r)(10-r)$ $\therefore 21r^2 + 21r = 1100 - 210r + 10r^2$

$\therefore 11r^2 + 231r - 1100 = 0$ $\therefore r^2 + 21r - 100 = 0$

$\therefore (r+25)(r-4) = 0$ $\therefore r = -25$ refused $\therefore r = 4$

2] If $\begin{vmatrix} Log_2 3 & 3 & 9 \\ 0 & Log_3 5 & 7 \\ 0 & 0 & Log_5 x \end{vmatrix} = 4$, then $x =$ ....

a] 16 b] 32 c] 64 d] 128

The Solution

$\because$ The determinate in the triangular form $\therefore \Delta = log_2 3 \times log_3 5 \times log_5 x = 4$

$\therefore \frac{\log 3}{\log 2} \times \frac{\log 5}{\log 3} \times \frac{\log x}{\log 5} = 4$ $\therefore \frac{\log x}{\log 2} = 4$ $\therefore Log_2 x = 4$ $\therefore x = 2^4 = 16$

3] If $\vec{A} = (1, -1, 2)$, $\vec{B} = (0, 2, -3)$, $\vec{C} = (-2, 1, 0)$, then $\|3\vec{A} - \vec{B} + \vec{C}\|$

a] $8\sqrt{3}$ b] 11 c] 12 d] $7\sqrt{2}$

The Solution

$3\vec{A} - \vec{B} + \vec{C} = 3(1, -1, 2) - (0, 2, -3) + (-2, 1, 0)$

$= (3, -3, 6) - (0, 2, -3) + (-2, 1, 0) = (1, -4, 9)$

$\therefore \|3\vec{A} - \vec{B} + \vec{C}\| = \|(1, -4, 9)\| = \sqrt{1+16+81} = \sqrt{98} = 7\sqrt{2}$ unit length

---

4] If $L_1: \frac{x+2}{-1} = \frac{y+3}{3} = \frac{z+5}{2}$ is perpendicular to $L_2: \frac{x}{2} = \frac{y-5}{k} = \frac{z-6}{m}$ then 3k + 2m =..........

a] −1 b] 0 c] 2 d] 4

The Solution

From the equation of the first straight line $\overrightarrow{d_1} = (-1, 3, 2)$

From the equations of the second straight line $\overrightarrow{d_2} = (2, k, m)$

∵ The two straight lines are perpendicular ∴ $(-1, 3, 2) \cdot (2, k, m) = 0$

∴ $-2 + 3k + 2m = 0$ ∴ $3k + 2m = 2$

5] The measure of the angle between the two straight lines

$x - 1 = \frac{y+2}{\sqrt{2}} = -z + 1$, $-x = z + 3$, $y = 4$ equals ....

a] 45^0 b] 120^0 c] 135^0 d] 150^0

The Solution

From the equation of the first straight line $\overrightarrow{d_1} = (1, \sqrt{2}, -1)$

From the equations of the second straight line $\overrightarrow{d_2} = (-1, 0, 1)$

Let the angle between the two lines is θ

$\cos\theta = \frac{|\overrightarrow{d_1} \cdot \overrightarrow{d_2}|}{\|\overrightarrow{d_1}\| \|\overrightarrow{d_2}\|} = \frac{|(1,\sqrt{2},-1) \cdot (-1,0,1)|}{\sqrt{4} \times \sqrt{2}} = \frac{|-2|}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$ ∴ $m(\angle\theta) = 45°$

6] The direction cosines of the vector $(2, -4, 4)$ are

a] $(2, -4, 4)$ b] $(1, -2, 2)$ c] $(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3})$ d] $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$

The Solution

$\|(2, -4, 4)\| = \sqrt{4 + 16 + 16} = 6$

∴ The direction cosines of the vector is $\left(\frac{2}{6}, -\frac{4}{6}, \frac{4}{6}\right) = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$

Second questions : complete:

1] $(3 + 7\omega + 3\omega^2)(3 - 7\omega^2 + 3\omega) =$ ....

The Solution

$(3 + 7\omega + 3\omega^2)(3 - 7\omega^2 + 3\omega) = [3(1 + w^2) + 7w][3(1 + w) - 7w^2]$

$= (-3w + 7w)(-3w^2 - 7w^2) = 4w \times -10w^2 = -40w^3 = -40$

2] The rank of the matrix $A = \begin{pmatrix} 2 & -6 \\ -3 & 3 \\ 4 & -12 \end{pmatrix}$ equals = ...

The Solution

$\because$ A is a matrix of order 3×2

$\therefore$ The greatest degree of the order of the determinant contained from it is 2

$\therefore \begin{vmatrix} 2 & -6 \\ -3 & 3 \end{vmatrix} = 6 - 18 = -12 \neq 0$ $\therefore R(A) = 2$

3] The Centre of the sphere $x^2 + y^2 + z^2 + 8x - 12y + 2z + 1 = 0$ *equals* ....

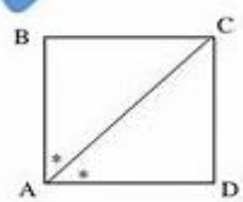
The Solution

$\because$ The Centre of the sphere = (L , k , n)

$= (-\frac{1}{2}$ coefficient of x , $-\frac{1}{2}$ coefficient of y , $-\frac{1}{2}$ coefficient of z) = $(-4 , 6 , -1)$

4] A B C D is a square of side length 10 cm , then $\overrightarrow{AB} \cdot \overrightarrow{AC} =$ ....

The Solution



$\because$ ABCD is a square of side length 10 cm.

$\because$ The diagonal of the square bisect the angle of the vertex .

$\therefore \|\overrightarrow{AB}\| = 10$ & $\|\overrightarrow{AC}\| = 10\sqrt{2}$ $\therefore m(\angle BAC) = 45°$

$\therefore \overrightarrow{AB} \cdot \overrightarrow{AC} = \|\overrightarrow{AB}\| \times \|\overrightarrow{AC}\| \times \cos 45° = 10 \times 10\sqrt{2} \times \frac{1}{\sqrt{2}} = 100\ cm^2$

5] The unit vector in the direction of $\vec{A} = (2 , 3 , 2\sqrt{3})$ *equals* ....

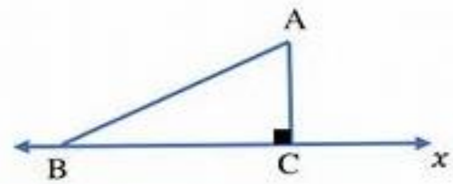
The Solution

$\because \|\vec{A}\| = \sqrt{4 + 9 + 12} = \sqrt{25} = 5$

$\therefore$ The unit vector in the direction of $\vec{A} = \vec{A}^{\bullet} = \frac{\vec{A}}{\|\vec{A}\|} = \frac{(2\ ,\ 3\ ,\ 2\sqrt{3})}{5} = \left(\frac{2}{5}\ ,\ \frac{3}{5} , \frac{2}{5}\sqrt{3}\right)$

6] The length of the perpendicular drawn from point $(-2,-3,1)$ to x −axis equals ....

The Solution

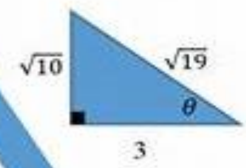


Let A $(-2,-3,1)$, B$(1,0,0)$ located in the axis

The projection of A on x −axis is C

$\therefore \overrightarrow{BA} = (-2,-3,1) - (1,0,0) = (-3,-3,1)$

$\therefore \overrightarrow{BC}$ is the projection of $\overrightarrow{BA}$ in x −axis



$\therefore \|\overrightarrow{BC}\| = \frac{|\overrightarrow{BA} \bullet \vec{B}|}{\|\vec{B}\|} = \frac{|(-3\ ,\ -3\ ,\ 1) \bullet (1\ ,\ 0\ ,\ 0)|}{\sqrt{1+0+0}} = |-3| = 3$

$\|\overrightarrow{BA}\| = \sqrt{9+9+1} = \sqrt{19}$ **unit length**

$\therefore AC = \sqrt{19-3^2} = \sqrt{10}$ **unit length**

Another Solution

Let θ is the angle between $\overrightarrow{BA}$ and x −axis

$\because \cos\theta = \frac{|\overrightarrow{BA} \bullet \vec{x}|}{\|\overrightarrow{BA}\|\ \|\vec{x}\|} = \frac{(-3\ ,\ -3\ ,\ 1) \bullet (1\ ,\ 0\ ,\ 0)}{\sqrt{9+9+1} \times \sqrt{1+0+0}} = \frac{3}{\sqrt{19}}$ $\sin\theta = \frac{\sqrt{10}}{\sqrt{19}}$

$\therefore AC = \|\overrightarrow{BA}\| \sin\theta = \sqrt{19} \times \frac{\sqrt{10}}{\sqrt{19}} = \sqrt{10}$ **unit length**

Another Solution

Let θ is the angle between $\overrightarrow{BA}$ and x −axis $\because \sin\theta = \frac{\|\overrightarrow{BA} \times \vec{x}\|}{\|\overrightarrow{BA}\|\ \|\vec{x}\|}$

$\because \overrightarrow{BA} \times \vec{x} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3 & -3 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \hat{\jmath} + 3\hat{k} = (0,1,3)$ $\therefore \|\overrightarrow{BA} \times \vec{x}\| = \|(0\ ,\ 1\ ,\ 3)\| = \sqrt{10}$ unit length

$\therefore \sin\theta = \frac{\sqrt{10}}{\sqrt{19}}$ $\therefore AC = \|\overrightarrow{BA}\| \sin\theta = \sqrt{19} \times \frac{\sqrt{10}}{\sqrt{19}} = \sqrt{10}$ **unit length**

3a] Find the greatest term in the expansion of $(3+2x)^6$ at $x = 1$

The Solution

Let T_{r+1} is the greatest term in the expansion $\therefore T_{r+1} > T_r$ $\therefore \frac{T_{r+1}}{T_r} \geq 1$ when $x = 1$

$\therefore \frac{6-r+1}{r} \times \frac{2}{3} \geq 1$ $\therefore \frac{2(7-r)}{3r} \geq 1$ $\therefore 14 - 2r \geq 3r$ $\therefore 14 - 2r \geq 3r$

$\therefore 14 \geq 5r$ $\therefore \frac{14}{5} \geq r$ $\therefore r = 2$ $\therefore$ The greatest term is T_3

$\therefore T_3 = {}^6C_2 \times (2x)^2 \times (3)^4$ at $x = 1$ $\therefore T_3 = {}^6C_2 \times (2)^2 \times (3)^4 = 4860$

3b] Find the volume of a parallelepiped in which three adjacent sides are represented by the vectors :

$\vec{A} = (1, -1, 2)$, $\vec{B} = (3, -2, 0)$, $\vec{C} = (0, 2, 4)$

The Solution

The volume of the parallelepiped $= |\vec{A} \cdot \vec{B} \times \vec{C}|$

$$\because |\vec{A} \cdot \vec{B} \times \vec{C}| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & -2 & 0 \\ 0 & 2 & 4 \end{vmatrix} = |1(-8+0) + 1(12+0) + 2(6+0)| = 16 \text{ unit volume}$$

4a] Find the roots of the equation $z^4 + 4 = 0$ in the trigonometric form .

The Solution

$z^4 = -4 = 4 [\, \text{Cos}(\pi) + i\, \text{Sin}(\pi) \,]$

$Z = \sqrt[4]{4} \, [\, \text{Cos}\left(\frac{\pi + 2m\pi}{4}\right) + i \, \text{Sin}\left(\frac{\pi + 2m\pi}{4}\right)]$ where $m = 0, 1, -1, -2$

At $m = 0$ $\therefore z = \sqrt{2} \, [\, \text{Cos}(\frac{\pi}{4}) + i \, \text{Sin}(\frac{\pi}{4})]$

At $m = 1$ $\therefore z = \sqrt{2} \, [\, \text{Cos}(\frac{3\pi}{4}) + i \, \text{Sin}(\frac{3\pi}{4})]$

At $m = -1$ $\therefore z = \sqrt{2} \, [\, \text{Cos}(\frac{-\pi}{4}) + i \, \text{Sin}(\frac{-\pi}{4})]$

At $m = -2$ $\therefore z = \sqrt{2} \, [\, \text{Cos}(\frac{-3\pi}{4}) + i \, \text{Sin}(\frac{-3\pi}{4})]$

4b] If $\vec{A}$, $\vec{B}$, $\vec{C}$ are three mutually perpendicular unit vectors Find :

a] $\|2\vec{A} - \vec{B} + 3\vec{C}\|$ b] If $\vec{A} = (\frac{1}{3}, \frac{-2}{3}, \frac{2}{3})$, $\vec{B} = (\frac{-2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}})$ Find $\vec{C}$

The Solution

a] $(\|2\vec{A} - \vec{B} + 3\vec{C}\|)^2 = (2\vec{A} - \vec{B} + 3\vec{C}) \cdot (2\vec{A} - \vec{B} + 3\vec{C})$

$= 4\|\vec{A}\|^2 - 2\vec{A} \cdot \vec{B} + 6\vec{A} \cdot \vec{C} - 2\vec{B} \cdot \vec{A} + \|\vec{B}\|^2 - \vec{B} \cdot \vec{C} + 6\vec{C} \cdot \vec{A} - 2\vec{C} \cdot \vec{B} + 4\|\vec{C}\|^2$

$\because \vec{A}, \vec{B}, \vec{C}$ are three mutually perpendicular unit vectors

$\therefore \|\vec{A}\| = \|\vec{B}\| = \|\vec{C}\| = 1$, $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} = \vec{A} \cdot \vec{C} = \vec{C} \cdot \vec{A} = \vec{B} \cdot \vec{C} = \vec{C} \cdot \vec{B} = 0$

$(\|2\vec{A} - \vec{B} + 3\vec{C}\|)^2 = 4 + 1 + 9 = 14$ $\therefore \|2\vec{A} - \vec{B} + 3\vec{C}\| = \sqrt{14}$

b] $$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{vmatrix} = \left(\frac{-2}{3\sqrt{5}}\right)\hat{\imath} - \left(\frac{5}{3\sqrt{5}}\right)\hat{\jmath} + \left(\frac{-4}{3\sqrt{5}}\right)\hat{k}$$

5a] Discuss the possibility for solving the following equations and write this solution , if exists :

$x + y = 2$, $2x + 3y = 5$

The Solution

$\because A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, $A^{*} = \left(\begin{matrix} 1 & 1 \\ 2 & 3 \end{matrix}\middle|\begin{matrix} 2 \\ 5 \end{matrix}\right)$, A^{*} is of order 2×3 $\because |A| = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1 \neq 0$

$\therefore Rk(A) = 2$ $\therefore$ The greatest order of the determinant can be constructed from A^{*} is 2 and the value of all these determinant $\neq 0$ $\therefore R(A^{*}) = 2$

$\therefore Rk(A) = Rk(A^{*}) = 2 =$ number of unknown

$\therefore$ **The group has unique solution and its matrix equation is AX = B**

where $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ $\therefore A^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \frac{1}{1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\therefore x = 1$, $y = 1$

5b] If $z = \sin\frac{\pi}{9} + i\cos\frac{\pi}{9}$, find ($\bar{z}$) in the trigonometric form and find the cubic roots of the number $(\bar{z})^9$

The Solution

$\bar{z} = \sin\frac{\pi}{9} - i\cos\frac{\pi}{9} = \cos\left(90° - \frac{\pi}{9}\right) - i\sin\left(90° - \frac{\pi}{9}\right) = \cos(70) - i\sin(70)$

$= \cos(360 - 70) + i\sin(360 - 70) = \cos(290) + i\sin(290)$

$\therefore (\bar{z})^9 = \cos(290 \times 9) + i\sin(290 \times 9) = \cos(2610) + i\sin(2610) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$\sqrt[3]{(\bar{z})^9} = \cos\left(\frac{\frac{\pi}{2} + 2m\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2m\pi}{3}\right)$ where $m = 0 , 1 , -1$

At $m = 0$ $\therefore \sqrt[3]{(\bar{z})^9} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

At $m = 1$ $\therefore \sqrt[3]{(\bar{z})^9} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

At $m = -1$ $\therefore \sqrt[3]{(\bar{z})^9} = \cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2}$

The fifth test

First : Answer one of the following questions :

First question: Choose the correct answer :

1] If $36\ {}^{2n-1}P_{n-1} = 9\ {}^{2n}P_n$, then n = ________

a] 1 b] 2 c] 3 d] 4

The Solution

$\frac{36\times \underline{|2n-1}}{\underline{|n}} = 9\times\frac{\underline{|2n}}{\underline{|n}}$ $\therefore 4\times\underline{|2n-1} = \underline{|2n} = (2n)\underline{|2n-1}$ $\therefore 2n = 4$ $\therefore n = 2$

2] If the two equation $x + y = 2$, $2x + ky = 4$ has more than one solution , then k = ...

a] -2 b] -1 c] 1 d] 2

The Solution

From 3rd prep.

$\therefore$ These two equations are the same . $\therefore k = 2$

3] If $\overrightarrow{AB} = -3\hat{\imath} + 3\hat{\jmath} + 7\hat{k}$, $\overrightarrow{BC} = \hat{\jmath} + 5\hat{k}$, then $\|\overrightarrow{AC}\|$

a] 13 b] 12 c] 10 d] 9

The Solution

$\because \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = -3\hat{\imath} + 4\hat{\jmath} + 12\hat{k}$ $\therefore \|\overrightarrow{AC}\| = \sqrt{9+16+144} = \sqrt{169} = 13$

4] *If* $\vec{A} = (-7, 3, 10)$, $\vec{B} = (-4, -1, -2)$ then the unit vector in the direction of $\overrightarrow{AB}$ = ...

a] $(\frac{3}{13}, \frac{4}{13}, \frac{12}{13})$ b] $(\frac{3}{13}, \frac{-4}{13}, \frac{-12}{13})$ c] $(\frac{-3}{13}, \frac{4}{13}, \frac{12}{13})$ d] $(\frac{-3}{13}, \frac{-4}{13}, \frac{-12}{13})$

The Solution

$\overrightarrow{AB} = (-4, -1, -2) - (-7, 3, 10) = (3, -4, -12)$ $\therefore \|\overrightarrow{AB}\| = \sqrt{9+16+144} = \sqrt{169} = 13$

$\therefore$ The unit vector in the direction of $\overrightarrow{AB} = \frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \left(\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$

5] If $\vec{A} = (1, -1, 2)$, $\vec{B} = (3, -2, 0)$, $\vec{C} = (0, 2, 4)$, then $\vec{A} \bullet \vec{B} \times \vec{C}$

a] 10 b] 12 c] 14 d] 16

The Solution

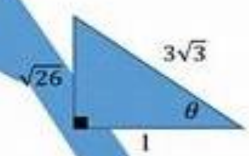
$$\vec{A} \bullet \vec{B} \times \vec{C} = \begin{vmatrix} 1 & -1 & 2 \\ 3 & -2 & 0 \\ 0 & 2 & 4 \end{vmatrix} = 1(-8+0) + 1(12+0) + 2(6+0) = 16$$

6] The length of the perpendicular drawn from point A(1 , 0 , 2) to the straight line

$\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z-3}{-2}$ equals ... .

a] $\frac{\sqrt{26}}{4}$ b] $\frac{\sqrt{26}}{5}$ c] $\frac{\sqrt{26}}{3}$ d] $\frac{\sqrt{26}}{6}$

The Solution



The direction vector of the given straight line $\vec{d} = (2, -1, -2)$

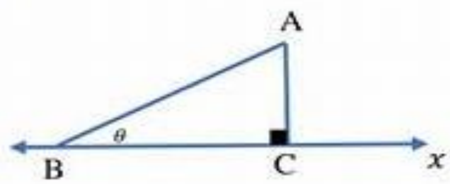
The point $B(2, -1, 3) \in$ the straight line.

Let C is the projection of the point A on the straight line

and θ is the measure of the angle between $\overline{BA}$ and the straight line.

$\therefore \overrightarrow{BA} = (1, 0, 2) - (2, -1, 3) = (-1, 1, -1)$

$\because \cos\theta = \frac{|\overrightarrow{BA} \bullet \vec{d}|}{\|\overrightarrow{BA}\| \ \|\vec{d}\|} = \frac{|(-1, 1, -1) \bullet (2, -1, -2)|}{\sqrt{1+1+1}\ \sqrt{4+1+4}} = \frac{1}{3\sqrt{3}}$



$\therefore \sin\theta = \frac{\sqrt{26}}{3\sqrt{3}}$ $\because \|\overrightarrow{BA}\| = \sqrt{1+1+1} = \sqrt{3}$

$\frac{AC}{\sin\theta} = \frac{AB}{\sin 90}$ $\therefore AC = \|\overrightarrow{BA}\| \sin\theta = \sqrt{3} \times \frac{\sqrt{26}}{3\sqrt{3}} = \frac{\sqrt{26}}{3}$ unit length.

Second Question : Complete:

1] $\left(2+\frac{3}{\omega}\right)\left(2+\frac{3}{\omega^2}\right)\left(3-\frac{2}{\omega}\right)\left(3-\frac{2}{\omega^2}\right) =$ ....

The Solution

The value $= \left(2+\frac{3\omega^3}{\omega}\right)\left(2+\frac{3\omega^3}{\omega^2}\right)\left(3-\frac{2\omega^3}{\omega}\right)\left(3-\frac{2\omega^3}{\omega^2}\right) = (2+3w^2)(2+3w)(3-2w^2)(3-2w)$

$= (4+6w+6w^2+9)(9-6w-6w^2+4) = [13+6(w+w^2)][13-6(w+w^2)]$

$= (13-6)(13+6) = 7 \times 19 = 133$

2] If the coefficients of T_6, T_{16} in the expansion of $(a+b)^n$ are equal, then n = ...

The Solution

coefficient of T_6 = coefficient of T_{16} $\therefore {}^nC_5 = {}^nC_{15}$ $\therefore n = 5 + 15 = 20$

3] Cosine the measure of the angle between the two lines :

$\frac{x}{1} = \frac{y}{-2} = \frac{z+1}{-2}$ and $\frac{x}{1} = \frac{y-2}{-2} = \frac{z}{2}$ equals ....

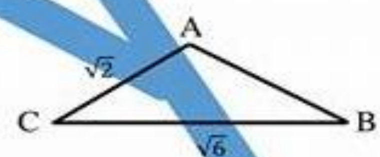
The Solution

The directed vector of the two straight lines (1 , −2 , −2) & (1 , −2 , 2)

Let the angle between the two lines is θ $\therefore \cos\theta = \frac{|(1,-2,-2)\bullet(1,-2,2)|}{\sqrt{1+4+4}\sqrt{1+4+4}} = \frac{1}{9}$

4] In the opposite figure , If $\|\overline{BC}\| = \sqrt{6}$, $\|\overline{AC}\| = \sqrt{2}$

$\overline{BA} = (-1, 0, 1)$, then $\overline{BA} \bullet \overline{BC}$ = ...



The Solution

$\|\overline{AB}\| = \sqrt{1+0+1} = \sqrt{2}$ $\because$ By using Cos law $\therefore \cos B = \frac{2+6-2}{2\sqrt{2}\times\sqrt{6}} = \frac{3}{2\sqrt{3}}$

$\therefore \overline{BA} \bullet \overline{BC} = \|\overline{BA}\|\|\overline{BC}\| \cos B = \sqrt{2} \times \sqrt{6} \times \frac{3}{2\sqrt{3}} = 3$

5] The general equation of the sphere whose Centre is (3 , 4 , −5) and touches y z plane is ....

The Solution

The sphere touch yz plane $\therefore r = |3| = 3$ unit length

$\therefore$ The equation of the sphere is $(x-3)^2 + (y-4) + (z+5)^2 = 9$

6] The vectors form of the equation of the straight line which passes through point (2 , −1 , 4) and its direction vector is $\vec{d} = (4, 7, 1)$ is ....

The Solution

The vector form of the equation of the straight line $\vec{r} = (2, -1, 4) + t(4, 7, 1)$

3a] In the expansion of $(1+x)^{18}$ according to the ascending powers of x, If the coefficients of T_{2r+4}, T_{r-2} are equal, **find the value of r.**

The Solution

The coefficient of T_{2r+4} = The coefficient of T_{r-2}

${}^{18}C_{2r+3} = {}^{18}C_{r-3}$ $\quad \therefore 2r+3 = r-3$ $\quad \therefore r = -6$ (refused)

Or $\quad 2r+3+r-3 = 18 \quad \therefore r = 6$

3b] If the length of the perpendicular drawn from point A(0, −1, 2) to the plane $\sqrt{2}\,x + y - z + k = 0$ equals 2 unit length, find the value of k.

The Solution

The length of the perpendicular $= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}} = \frac{|0\times\sqrt{2} - 1\times 1 + 2\times -1 + k|}{\sqrt{2+1+1}} = 2$

$\therefore |-3+k| = 4 \qquad \therefore -3+k = 4 \quad \therefore k = 7 \qquad Or \quad -3+k = -4 \quad \therefore k = -1$

4a] Solve the following equations $2x + y - 2z = 10$, $x + 2y + 2z = 1$, $5x + 4y + 3z = 6$ using the multiplicative inverse of the matrix.

The Solution

$A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 5 & 4 & 3 \end{pmatrix} \qquad \therefore |A| = \begin{vmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 5 & 4 & 3 \end{vmatrix} = 2(6-8) - 1(3-10) - 2(4-10) = 15 \neq 0$

$\therefore$ The group has a unique solution

The matrix equation is $AX = B$ where $A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 5 & 4 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ & $B = \begin{pmatrix} 10 \\ 1 \\ 6 \end{pmatrix}$

The cofactor of A $= \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} & \overline{a_{13}} \\ \overline{a_{21}} & \overline{a_{22}} & \overline{a_{23}} \\ \overline{a_{31}} & \overline{a_{32}} & \overline{a_{33}} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} \\ -\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 2 & -2 \\ 5 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \end{pmatrix}$

$\therefore$ The matrix of cofactor of A is $F = \begin{pmatrix} -2 & 7 & -6 \\ -11 & 16 & -3 \\ 6 & -6 & 3 \end{pmatrix}$

The cofactor matrix $\therefore Adj(A) = F^t = \begin{pmatrix} -2 & -11 & 6 \\ 7 & 16 & -6 \\ -6 & -3 & 3 \end{pmatrix}$

$\therefore A^{-1} = \frac{1}{|A|} \times F^t = \frac{1}{15}\begin{pmatrix} -2 & -11 & 6 \\ 7 & 16 & -6 \\ -6 & -3 & 3 \end{pmatrix}$

$\therefore X = A^{-1}B$, $\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{15}\begin{pmatrix} -2 & -11 & 6 \\ 7 & 16 & -6 \\ -6 & -3 & 3 \end{pmatrix}\begin{pmatrix} 10 \\ 1 \\ 6 \end{pmatrix} = \frac{1}{15}\begin{pmatrix} 5 \\ 50 \\ -45 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{10}{3} \\ -3 \end{pmatrix}$

$\therefore x = \frac{1}{3}$, $y = \frac{10}{3}$, $z = -3$ $\therefore$ S.S. $= \left\{\left(\frac{1}{3}, \frac{10}{3}, -3\right)\right\}$

4b] If $z_1 = \frac{6+4i}{1+i}$, $z_2 = \frac{26}{5-i}$ If $Z = 4(z_1 - z_2)$ find the cubic roots of z in the exponential form

The Solution

$z_1 = \frac{6+4i}{1+i} \times \frac{1-i}{1-i} = \frac{6-6i+4-4i}{2} = \frac{10-2i}{2} = 5 - i$, $z_2 = \frac{26}{5-i} \times \frac{5+i}{5+i} = \frac{26(5+i)}{26} = 5 + i$

$\therefore z = (z_1 - z_2) = 4(5 - i - 5 - i) = -8i = x + iy$ $\therefore x = 0$, $y = -8$

$\therefore r = \sqrt{x^2 + y^2} = 8$ & $\tan\theta = \frac{y}{x} = \frac{-8}{0}$ $\therefore z = 8\left[\text{Cos}\left(-\frac{\pi}{2}\right) + i\,\text{Sin}\left(-\frac{\pi}{2}\right)\right]$

$\therefore \sqrt[3]{z} = 2\left[\text{Cos}\left(\frac{-\frac{\pi}{2} + 2m\pi}{3}\right) + i\,\text{Sin}\left(\frac{-\frac{\pi}{2} + 2m\pi}{3}\right)\right]$ where $m = 0, 1, -1$

At $m = 0$ $\therefore \sqrt[3]{z} = 2\left[\text{Cos}\left(-\frac{\pi}{6}\right) + i\,\text{Sin}\left(-\frac{\pi}{6}\right)\right] = 2e^{-\frac{1}{6}\pi i}$

At $m = 1$ $\therefore \sqrt[3]{z} = 2\left[\text{Cos}\left(\frac{\pi}{2}\right) + i\,\text{Sin}\left(\frac{\pi}{2}\right)\right] = 2e^{\frac{\pi}{2}i}$

At $m = -1$ $\therefore \sqrt[3]{z} = 2\left[\text{Cos}\left(\frac{-5\pi}{6}\right) + i\,\text{Sin}\left(\frac{-5\pi}{6}\right)\right] = 2e^{\frac{-5\pi}{6}}$

---

5 a] **Without expanding** expansion the determinant **prove that :**

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = a^2 + b^2 + c^2 + 1$$

The Solution

$L.H.S. = \begin{vmatrix} a^2 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} + \begin{vmatrix} 1 & ab & ac \\ 0 & b^2+1 & bc \\ 0 & bc & c^2+1 \end{vmatrix}$

a common factor in R_1 & first C_1 and write the second determinant as the sum of two determinants

$L.H.S. = a^2\begin{vmatrix} 1 & b & c \\ b & b^2+1 & bc \\ c & bc & c^2+1 \end{vmatrix} + \begin{vmatrix} 1 & ab & ac \\ 0 & b^2 & bc \\ 0 & bc & c^2+1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & ac \\ 0 & 1 & bc \\ 0 & 0 & c^2+1 \end{vmatrix} =$

$-b\,R_1 + R_2$, $-c\,R_1 + R_3$ in the first determinant

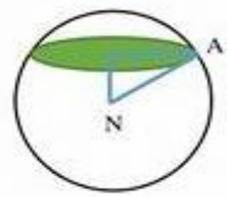
$L.H.S. = a^2\begin{vmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + b^2\begin{vmatrix} 1 & a & ac \\ 0 & 1 & c \\ 0 & c & c^2+1 \end{vmatrix} + c^2 + 1$ $R_3 - c\,R_2$ in the 2^{nd} determinant

$$\therefore L.H.S. = a^2 \begin{vmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + b^2 \begin{vmatrix} 1 & a & ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix} + c^2 + 1 = a^2 + b^2 + c^2 + 1 = R.H.S.$$

5b] If the plane $2x - y - 2z + 12 = 0$ cut the sphere $(x+3)^2 + (y+2)^2 + (z-1)^2 = 15$, find the area of the cross section (trace) .

The Solution

The Centre of the sphere is $N = (-3, -2, 1)$ & $AN = \sqrt{15}$



$\therefore MN$ = The length of the perpendicular line from N to the plane

$$= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|2\times -3 - 1\times -2 - 2\times 1 + 12|}{\sqrt{4+1+4}} = \frac{|-6+2-2+12|}{3} = \frac{6}{3} = 2 \text{ unit length}$$

r of the circle $= \sqrt{15-4} = \sqrt{11}$ unit length

$\therefore$ The area of the cross section circle $= \pi r^2 = 11\pi$ unit area

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The sixth test

First : Answer one of the following :

First question : Choose the correct answer:

1] If ${}^{n}C_3 : {}^{n-1}C_4 = 8 : 5$, then the value of n ....

a] 5 b] 7 c] 8 d] 9

The Solution

$$\frac{\lfloor n}{\lfloor 3 \ \lfloor n-3} \div \frac{\lfloor n-1}{\lfloor 4 \ \lfloor n-5} = \frac{8}{5} \qquad \therefore \frac{\lfloor n}{\lfloor 3 \ \lfloor n-3} \times \frac{\lfloor 4 \ \lfloor n-5}{\lfloor n-1} = \frac{8}{5} \qquad \therefore \frac{n \lfloor n-1 \ (4) \lfloor 3 \ \lfloor n-5}{\lfloor 3 \ (n-3)(n-4) \lfloor n-5 \ \lfloor n-1} = \frac{8}{5}$$

$$\therefore \frac{4n}{(n-3)(n-4)} = \frac{8}{5} \qquad \therefore 8[n^2 - 7n + 12] = 20n \qquad \therefore 8n^2 - 56n + 96 = 20n$$

$$\therefore 8n^2 - 76n + 96 = 0 \text{ ----- } (\div 4) \qquad \therefore 2n^2 - 19n + 24 = 0 \qquad \therefore (2n-3)(n-8) = 0$$

$2n - 3 = 0 \ \therefore 2n = 3 \ \therefore n = \frac{3}{2}$ (refused) Or $n - 8 = 0 \ \therefore n = 8$

2] The coefficient of the middle term in the expansion of $(3x - \frac{1}{6})^{10}$ *equals* ………

a] $\frac{-63}{8}$ b] $\frac{-67}{8}$ c] $\frac{63}{8}$ d] $\frac{67}{8}$

The Solution

The order of the middle term is $= \frac{10}{2} + 1 = 6$ $\quad \therefore T_6 = 10_{C_5}(3x)^5\left(-\frac{1}{6}\right)^5$

$\therefore$ **The coefficient of** $T_6 = 10_{C_5}\,(3)^5\left(-\frac{1}{6}\right)^5 = -\frac{63}{8}$

****

3] The measure of the angle included between the two planes :

$x + y - 1 = 0 \;,\; y + z - 1 = 0$ equals

a] $30°$ b] $45°$ c] $60°$ d] $75°$

The Solution

The normal vector to the first plane $\overrightarrow{n_1} = (1, 1, 0)$

The normal vector to the second plane $\overrightarrow{n_2} = (0, 1, 1)$

The measure angle between the two planes is θ where

$\cos\theta = \frac{|\overrightarrow{n_1} \bullet \overrightarrow{n_2}|}{\|\overrightarrow{n_1}\|\;\|\overrightarrow{n_2}\|} = \frac{|(1,\,1,\,0)\bullet(0,\,1,\,1)|}{\sqrt{1+1}\;\sqrt{1+1}} = \frac{|1\times0+1\times1+0\times1|}{\sqrt{2}\times\sqrt{2}} = \frac{1}{2}$ $\quad \therefore m(\angle\theta) = 60°$

****

4] If $\vec{A} = (2, 1, -2)$, $\vec{A} + \vec{B} = \vec{A} \times \vec{B}$, then $\vec{B} =$ ……

a] $(2, -1, -2)$ b] $(2, 1, -2)$ c] $(-2, -1, 2)$ d] $(-2, -1, 3)$

The Solution

Let $\vec{B} = (b_x, b_y, b_z)$ $\because \vec{A} + \vec{B} = \vec{A} \times \vec{B}$ $\therefore (2, 1, -2) + (b_x, b_y, b_z) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -2 \\ b_x & b_y & b_z \end{vmatrix}$

$\therefore (2 + b_x, 1 + b_y, -2 + b_z) = \hat{\imath}(b_z + 2b_y) - \hat{\jmath}(2b_z + 2b_x) + \hat{k}(2b_y - b_x)$

$\therefore 2 + b_x = b_z + 2b_y \qquad \therefore b_x - 2b_y - b_z = -2$ ……………………………(1)

& $1 + b_y = -2b_z - 2b_x \qquad \therefore 2b_x + b_y + 2b_z = -1$ ………………………………(2)

& $-2 + b_z = 2b_y - b_x \qquad \therefore b_x - 2b_y + b_z = 2$ ………………………………(3)

by solving using cramers rule

$$\Delta = \begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & 2 \\ 1 & -2 & 1 \end{vmatrix} = 1(1+4) + 2(2-2) - 1(-4-1) = 5 + 0 + 5 = 10$$

$$\Delta_{b_x} = \begin{vmatrix} -2 & -2 & -1 \\ -1 & 1 & 2 \\ 2 & -2 & 1 \end{vmatrix} = -2(1+4) + 2(-1-4) - 1(2-2) = -10 - 10 - 0 = -20$$

$$\Delta_{b_y} = \begin{vmatrix} 1 & -2 & -1 \\ 2 & -1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 1(-1-4) + 2(2-2) - 1(4+1) = -5 + 0 - 5 = -10$$

$$\Delta_{b_z} = \begin{vmatrix} 1 & -2 & -2 \\ 2 & 1 & -1 \\ 1 & -2 & 2 \end{vmatrix} = 1(2-2) + 2(4+1) - 2(-4-1) = 0 + 10 + 10 = 20$$

$b_x = \frac{\Delta_{b_x}}{\Delta} = \frac{-20}{10} = -2$ & $b_y = \frac{\Delta_{b_y}}{\Delta} = \frac{-10}{10} = -1$ & $b_z = \frac{\Delta_{b_z}}{\Delta} = \frac{20}{10} = 2$

$\therefore \overrightarrow{B} = (b_x, b_y, b_z) = (-2, -1, 2)$

5] If A (−2 , 0 , 3) , B (4 , 2 , −5), then $\|\overrightarrow{AB}\|$ = ... length unit

a] $\sqrt{12}$ b] $\sqrt{40}$ c] $\sqrt{44}$ d] $\sqrt{104}$

The Solution

$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (4, 2, -5) - (-2, 0, 3) = (6, 2, -8)$

$\therefore \|\overrightarrow{AB}\| = \sqrt{(6)^2 + (2)^2 + (-8)^2} = \sqrt{104}$ unit length

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6] If $\vec{A} \perp \vec{B}$, $\vec{A} \perp \vec{C}$, $\vec{B} = (2, 3, 2)$, $\vec{C} = (1, 2, 1)$ and $\|\vec{A}\| = 4\sqrt{2}$, then $\vec{A}$ = ...

a] (2 , 3 , 1) b] (−4 , 0 , 4) c] (4, 4, 0) d] (0, −4 , 4)

The Solution

Let $\overrightarrow{A} = (A_x, A_y, A_z)$ $\because \overrightarrow{A} \perp \overrightarrow{B}$ $\therefore \overrightarrow{A} \odot \overrightarrow{B} = 0$

$\therefore (A_x, A_y, A_z) \odot (2, 3, 2) = 0$ $\therefore 2A_x + 3A_y + 2A_z = 0$ ------ (1)

$\because \overrightarrow{A} \perp \overrightarrow{C}$ $\therefore \overrightarrow{A} \odot \overrightarrow{C} = 0$ $\therefore (A_x, A_y, A_z) \odot (1, 2, 1) = 0$

$\therefore A_x + 2A_y + A_z = 0$ ------ (2) $\because \|\overrightarrow{A}\| = 4\sqrt{2}$ $\therefore \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2} = 4\sqrt{2}$

By solving the three equations equation (1) − 2 × equation (2) $\therefore -A_y = 0$ $\therefore A_y = 0$

$\therefore \overrightarrow{A} = (-4, 0, 4)$

Second question : Complete:

1] $(1-\frac{1}{w})(1-\frac{1}{w^2})(1-\frac{1}{w^3})(1-\frac{1}{w^4})$ .... to 10 factors = ...

The Solution

$(1-\frac{1}{w})(1-\frac{1}{w^2})(1-\frac{1}{w^3})(1-\frac{1}{w^4})$ .... to 10 factors = $(1-\frac{1}{w}-\frac{1}{w^2}-\frac{1}{w^3})^5$

$= (1 - w^2 - w + 1)^5 = (1+1+1)^5 = (3)^5 = 243$

2] The rank of the matrix $\begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ equals ....

The Solution

$\because |A| = 1 \neq 0$ $\therefore$ The rank of the matrix = 3

3] The direction vector of the straight line $\frac{x+2}{3} = \frac{z-1}{2}$, y = 2 equals ....

The Solution

(3 , 0 , 2)

4] If the measure of the angle between the two lines $\frac{x}{a} = \frac{y}{2} = \frac{z}{1}$, $\frac{x}{2} = \frac{y}{1} = \frac{z}{-1}$ equals 60^0 , then the value of a = ...

The Solution

$\because$ **(a , 2 , 1)** is the direction vector of the first straight line .

$\because$ **(2 , 1 , -1)** is the direction vector of the second straight line .

$\cos\theta = \frac{a\times 2 + 2\times 1 + 1\times -1}{\sqrt{a^2+4+1}\times\sqrt{4+1+1}}$ $\therefore \frac{2a+1}{\sqrt{a^2+5}} = \frac{1}{2}$ by squaring both sides

$\therefore \frac{(2a+1)^2}{6(a^2+5)} = \frac{1}{4}$ $\therefore 4(4a^2+4a+1) = 6a^2+30$, $16a^2+16a+4 = 6a^2+30$

$10a^2 + 16 - 26 = 0$ ------ $(\div 2)$, $5a^2 + 8a - 13 = 0$, $(5a+13)(a-1) = 0$ $\therefore a = -\frac{13}{5}$ **or a = 1**

5] If A (1 , 0 , 0) and B (0 , 1 , 1) lie on the plane $k\,x + y + m\,z + 2 = 0$, then $k + m =$ ....

The Solution

$\because$ A $\in$ the plane $\quad \therefore k \times 1 + 0 + 0 = -2 \quad \therefore k = -2$

$\because B \in$ the plane $\therefore 0 \times k + 1 + m = -2 \quad \therefore m = -3 \quad \therefore k + m = -2 - 3 = -5$

****

6] If $\vec{A} = (1,\ 0,\ 2)$, $\vec{B} = (2, -1, -2)$, then $(\vec{A} \times \vec{B}) \bullet (\vec{B} \times \vec{A}) =$ ....

The Solution

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 2 \\ 2 & -1 & -2 \end{vmatrix} = (0+2)\hat{\imath} - (-2-4)\hat{\jmath} + (-1-0)\hat{k} = (2,\ 6, -1)$$

$\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A}) \quad \therefore \vec{B} \times \vec{A} = -2\hat{\imath} - 6\hat{\jmath} + \hat{k} = (-2, -6, 1)$

$\therefore (\vec{A} \times \vec{B}) \bullet (\vec{B} \times \vec{A}) = (2, 6, -1) \bullet (-2, -6, 1) = -4 - 36 - 1 = -41$

****

Answer the following questions :

Third question :

3−a] If the coefficients of the fourth , fifth and sixth terms in the expansion $(2x + y)^n$ according to the descending powers of x form an arithmetic sequence, **find the value of n**

The Solution

$\because$ The coefficient of T_4 , The coefficient T_5 , The coefficient of T_6 form A.S. ,

$\therefore$ The coefficient of T_5 is the arithmetic mean between The coefficient of T_4 & The coefficient of T_6

coefficient of T_4 + coefficient of $T_6 = 2$ coefficient of T_5

$\therefore \frac{coeeficient\ of\ T_4}{coefficicnent\ T_5} + \frac{coeeficient\ of\ T_6}{coefficicnent\ T_5} = 2 \quad \therefore \frac{4}{n-4 \times 1} \times \frac{2}{1} + \frac{n-5+1}{5} \times \frac{1}{2} = 2 \quad \therefore \frac{8}{n-3} + \frac{n-4}{10} = 2$

$\therefore \frac{8(10)+(n-4)(n-3)}{10(n-3)} = 2 \quad \therefore \frac{80+n^2-7n+12}{10n-30} = 2 \quad \therefore 80 + n^2 - -7n + 12 = 20n - 60$

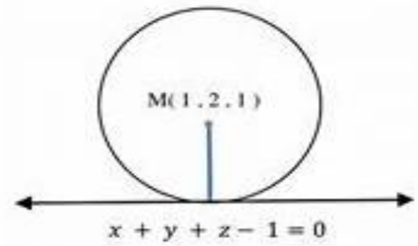
$\therefore n^2 - 27n + 80 + 12 + 60 = 0 \qquad \therefore n^2 - 27n + 152 = 0$

$\therefore (n-19)(n-8) = 0 \qquad \therefore n = 19$ or $n = 8$

****

3−b] A sphere of Centre (1 , 2 , 1) touches the plane $x + y + z = 1$, find the equation of the sphere

The Solution



r of the sphere = The length of the perpendicular drawn from its centre to the plane .

$$r = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|1\times1 + 2\times1 + 1\times-1|}{\sqrt{(1)^2+(1)^2+(1)^2}} = \sqrt{3} \text{ unit length}$$

∴ The equation of the circle is $(x-1)^2 + (y-2)^2 + (z-1)^2 = 3$

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4 − a] Discuss the possibility of solving the set of the following system equations : $4x + 3y - 5z = 6$, $3x + 2y + 4z = 12$, $5x - 2y - 7z = 1$, then find the solution set of these equations using the multiplicative inverse

The Solution

$$A = \begin{pmatrix} 4 & 3 & -5 \\ 3 & 2 & 4 \\ 5 & -2 & -7 \end{pmatrix} \quad \therefore |A| = \begin{vmatrix} 4 & 3 & -5 \\ 3 & 2 & 4 \\ 5 & -2 & -7 \end{vmatrix} = 4\begin{vmatrix} 2 & 4 \\ -2 & -7 \end{vmatrix} - 3\begin{vmatrix} 3 & 4 \\ 5 & -7 \end{vmatrix} - 5\begin{vmatrix} 3 & 2 \\ 5 & -2 \end{vmatrix}$$

$$= 4[-14+8] - 3[-21-20] \; -5[-6-10] = 4\times-6 - 3\times-41 - 5\times-16 = -24 + 123 + 80 = 179$$

$\therefore |A| \neq 0 \quad \therefore Rk(A) = 3 \quad \because$ The number of unknown = 3

∴ The equations are non homogeneous . ∴ The equations have unique solution .

The matrix equation is AX = B where $A = \begin{pmatrix} 4 & 3 & -5 \\ 3 & 2 & 4 \\ 5 & -2 & -7 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 6 \\ 12 \\ 1 \end{pmatrix}$

The cofactor of A $= \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} & \overline{a_{13}} \\ \overline{a_{21}} & \overline{a_{22}} & \overline{a_{23}} \\ \overline{a_{31}} & \overline{a_{32}} & \overline{a_{33}} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} 2 & 4 \\ -2 & -7 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 5 & -7 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 5 & -2 \end{vmatrix} \\ -\begin{vmatrix} 3 & -5 \\ -2 & -7 \end{vmatrix} & \begin{vmatrix} 4 & -5 \\ 5 & -7 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 5 & -2 \end{vmatrix} \\ \begin{vmatrix} 3 & -5 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 4 & -5 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix} \end{pmatrix}$

∴ The matrix of cofactor of A is $F = \begin{pmatrix} -6 & 41 & -16 \\ 31 & -3 & 23 \\ 22 & -31 & -1 \end{pmatrix}$

$$\text{Adj}(A) = \begin{pmatrix} -6 & 31 & 22 \\ -41 & -3 & -31 \\ 16 & 23 & -1 \end{pmatrix} \quad \therefore A^{-1} = \frac{1}{|A|}\, adj(A) = \frac{1}{179}\begin{pmatrix} -6 & 31 & 22 \\ -41 & -3 & -31 \\ -16 & 23 & -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{179}\begin{pmatrix} -6 & 31 & 22 \\ -41 & -3 & -31 \\ -16 & 27 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 12 \\ 1 \end{pmatrix} \qquad \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad \therefore S.S. = \{ (2 , 1 , 1) \}$$

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4 – b] If $Z_1 = \left(\frac{\sqrt{3}+i}{2}\right)^4$, $Z_2 = Sin\,\frac{\pi}{3} + i\,Cos\,\frac{\pi}{3}$, $i^2 = -1$, and $Z = \frac{Z_1}{Z_2}$

Find the square roots of z in the trigonometric form .

The Solution

$\frac{\sqrt{3}+i}{2} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ $\quad \therefore x = \frac{\sqrt{3}}{2}$, $y = \frac{1}{2}$ $\quad \therefore\ r = \sqrt{x^2+y^2} = 1$ & $\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{3}}$

$\therefore m(\angle\theta) = 30^0$ $\quad \therefore z_1 = \left(\text{Cos}\frac{\pi}{6} + i\,\text{Sin}\frac{\pi}{6}\right)^4 = \text{Cos}\frac{2\pi}{3} + i\,\text{Sin}\frac{2\pi}{3}$

$Z_2 = \text{Sin}\frac{\pi}{3} + i\,\text{Cos}\frac{\pi}{3} = \text{Cos}(\,90 - \frac{\pi}{3}) + i\,\text{Sin}(\,90 - \frac{\pi}{3}) = \text{Cos}\,30^0 + i\,\text{Sin}\,30^0$

$\therefore z = \frac{z_1}{z_2} = \text{Cos}(\frac{2\pi}{3} - \frac{\pi}{6}) + i\,\text{Sin}(\frac{2\pi}{3} - \frac{\pi}{6}) = \text{Cos}\ \frac{\pi}{2} + i\,\text{Sin}\ \frac{\pi}{2}$

$\therefore\ \sqrt{Z} = \text{Cos}\left(\frac{\frac{\pi}{2}+2m\pi}{2}\right) + i\,\text{Sin}\left(\frac{\frac{\pi}{2}+2m\pi}{2}\right)$ where $m = 0\ ,\ -1$

At $m = 0$ $\quad \therefore \sqrt{Z} = \text{Cos}\frac{\pi}{4} + i\,\text{Sin}\frac{\pi}{4}$

At $m = -1$ $\quad \therefore \sqrt{Z} = \text{Cos}(\frac{-3\pi}{4}) + i\,\text{Sin}(\frac{-3\pi}{4})$

5–a] Without expanding the determinant ,

prove that : $\begin{vmatrix} x & a & b \\ a & x & b \\ b & a & x \end{vmatrix} = (x + a + b)\,(x - a)\,(x - b)$

The Solution

L.H.S. = $\begin{vmatrix} x & a & b \\ a & x & b \\ b & a & x \end{vmatrix}$ $\qquad c_1 + c_2 + c_3$

$\therefore$ L.H.S. = $\begin{vmatrix} x+a+b & a & b \\ x+a+b & x & b \\ x+a+b & a & x \end{vmatrix} = (x+a+b)\begin{vmatrix} 1 & a & b \\ 1 & x & b \\ 1 & a & x \end{vmatrix}$ $\qquad r_2 - r_1$ & $r_3 - r_1$

$\therefore$ L.H.S. $= (x+a+b)\begin{vmatrix} 1 & a & b \\ 0 & x-a & 0 \\ 0 & 0 & x-b \end{vmatrix} = (x+a+b)(x-a)(x-b) =$ R.H.S.

5–b] Find the different forms of the equation of the straight line passing through (2 , 1, −3) and Parallel to the straight line $\frac{x-1}{5} = \frac{y+3}{2} = \frac{1-z}{3}$

The Solution

The direction vector = (5 , 2 , 3)

The equation of the straight line in the vector form $\vec{r} = \vec{A} + t\,\vec{d}$

$\vec{r} = (2,1,-3) + t(5,2,3)$

The parametric form $x = 2 + 5t$, $y = 1 + 2t$, $z = -3 + 3t$

The cartesian form $\frac{x-2}{5} = \frac{y-1}{2} = \frac{z+3}{3}$

The seventh test

First Answer one of the following :

First question: Choose the correct answer:

1] If $30_{C_r} = 30_{C_{r+10}}$, $n_{P_7} = 90 \times n - 2_{P_5}$, then $\lfloor n - r$

a] 0 b] 1 c] 10 d] $\lfloor 20$

The Solution

$\frac{\lfloor n}{\lfloor n-7} = \frac{90 \lfloor n-2}{\lfloor n-7}$ $\therefore n(n-1)\lfloor n-2 = 90 \lfloor n-2$ $\therefore n(n-1) = 10 \times 9$ $\therefore n = 10$

$\because 30_{C_r} = 30_{C_{r+10}}$ $\therefore r + r + 10 = 30$ $\therefore 2r = 20$ $\therefore r = 10$ $\therefore \lfloor n - r = \lfloor 10 - 10 = \lfloor 0 = 1$

2] If the equations $3x - 2y + z = 0$, $6x - 5y + 2z = 0$, $9x - 6y + kz = 0$ have solutions other than the zero solution , then k =………………….

a] 0 b] 1 c] 3 d] 4

The Solution

∴ The equations are homogeneous have solutions other than the zero solution

$\therefore |A| = 0$ $\therefore |A| = \begin{vmatrix} 3 & -2 & 1 \\ 6 & -5 & 2 \\ 9 & -6 & k \end{vmatrix} - 3(-5k + 12) + 2(6k - 18) + 1(-36 + 45) = 0$

$\therefore -15k + 36 + 12k - 36 - 36 + 45 = 0$, $\therefore 3k = 9$ $\therefore k = 3$

3] The length of the perpendicular drawn between the two planes

$3x + 12y - 4z = 9$, $3x + 12y - 4z = -17$ equals ……………….

a] 2 b] 3 c] 4 d] 5

The Solution

Take a point belongs to the first plane Let $y = 0 , z = 0$ $\therefore x = 3$ $\therefore A = (3,0,0)$

the length of the perpendicular drawn between the two planes = the length of the

⊥ line from A to the second .

plane $= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}} = \frac{|3\times3+12\times0-4\times0+17|}{\sqrt{9+144+16}} = \frac{26}{13} = 2$ unit length

Another solution

$\because$ The two lines are parallel $\therefore$ The length of the perpendicular drawn between the two planes

$= \frac{|-9-17|}{\sqrt{9+144+16}} = \frac{26}{13} = 2$ unit length

---

4] If $\overrightarrow{A} = (4, -k, 6)$, $\overrightarrow{B} = (2, 2, m)$ and $\overrightarrow{A} \parallel \overrightarrow{B}$, then $k + m = ...$

a] -3 b] -2 c] -1 d] zero

The Solution

$\because \overrightarrow{A} \parallel \overrightarrow{B}$ $\therefore \frac{4}{2} = \frac{-k}{2} = \frac{6}{m}$ $\therefore k = -4$ & $m = 3$ $\therefore k + m = -4 + 3 = -1$

5] If the straight line $x = 3y = az$ is parallel to the plane $x + 3y + 2z + 4 = 0$

, then a = ...

a] 3 b] 2 c] 1 d] -1

The Solution

$\because x = 3y = az$ (divide by 3a) $\therefore$ The equation of the straight line $\frac{x}{3a} = \frac{y}{a} = \frac{z}{3}$

$\therefore$ The directed vector of the straight line = (3a , a , 3)

The directed vector of the normal to the plane = (1 , 3 , 2)

$\because$ The straight line // plane

$\therefore$ The directed vector of the line ⊥ the directed vector of the normal to the plane

$\therefore (3a, a, 3) \cdot (1, 3, 2) = 0$ $\therefore 3a + 3a + 6 = 0$ $\therefore 6a = -6$ $\therefore a = -1$

Another solution :

$\because x = 3y = az$ (divide by 3) $\therefore \frac{x}{3a} = \frac{y}{a} = \frac{z}{3}$ $\because$ The line // The plane

$\therefore 3a \times 1 + 3 \times a + 6 = 0$ $\therefore 6a = -6$ $\therefore a = -1$

6] If $\overrightarrow{A} = (1, -2, 1)$, $\overrightarrow{B} = (-2, 1, 2)$, then the component of $\overrightarrow{A}$ in the direction of $\overrightarrow{B}$

a] $(\frac{4}{9}, \frac{-2}{9}, \frac{-4}{9})$ b] $(\frac{4}{9}, \frac{2}{9}, \frac{4}{9})$ c] $(\frac{-4}{9}, \frac{-2}{9}, \frac{-2}{9})$ d] $(\frac{4}{9}, \frac{-2}{9}, \frac{-4}{9})$

The Solution

The component of $\overrightarrow{A}$ in the directed of $\overrightarrow{B}$

$$= \frac{\overrightarrow{A} \bullet \overrightarrow{B}}{\|\overrightarrow{B}\|}\left(\frac{\overrightarrow{B}}{\|\overrightarrow{B}\|}\right) = \frac{(1,\ -2,\ 1)\bullet(-2,\ 1,\ 2)}{\sqrt{9}}\left(\frac{(-2,1,\ 2)}{\sqrt{9}}\right) = -\frac{2}{9}(-2, 1, 2) = \left(\frac{4}{9}, -\frac{2}{9}, -\frac{4}{9}\right)$$

Second question : Complete :

1] $\left(\frac{3+5\omega}{5+3\omega^2} + \frac{5+3\omega^2}{3+5\omega}\right)^8 =$ ....

The Solution

$$\left(\frac{3w^3+5w}{5+3w^2} + \frac{5w^3+3w^2}{3+5w}\right)^8 = \left[\frac{w(3w^2+5)}{5+3w^2} + \frac{w^2(5w+3)}{3+5w}\right]^8 = (w+w^2)^8 = (-1)^8 = 1$$

2] **The rank of the** of the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -1 \\ 1 & -3 & 4 \end{pmatrix}$ equals ....

The Solution

$$\therefore |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & -1 \\ 1 & -3 & 4 \end{vmatrix} = 1(4-3) - 1(4+1) + 3(-3-1) = -16 \neq 0 \quad \therefore R(A) = 3$$

3] If the plane X : $x - z + 1 = 0$, and the plane Y : $2x - 2y - z = 0$, then the measure of the angle between the two planes = ...

The Solution

The normal vector to the first plane $\overrightarrow{n_1} = (1, 0, -1)$

The normal vector to the second plane $\overrightarrow{n_2} = (2, -2, -1)$

The measure angle between the two planes is θ where

$$\cos\theta = \frac{|\overrightarrow{n_1} \bullet \overrightarrow{n_2}|}{\|\overrightarrow{n_1}\|\ \|\overrightarrow{n_2}\|} = \frac{|(1,\ 0,\ -1)\bullet(2,-2,-1)|}{\sqrt{1+0+1}\quad\sqrt{4+4+1}} = \frac{2+0+1}{\sqrt{1+0+1}\sqrt{4+4+1}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \quad \therefore m(\angle\theta) = 45^\circ$$

4] The radius length of the sphere $(x-2)^2+(y+4)^2+(z-5)^2=64$ equals ....

The Solution

$r=\sqrt{64}=8$ cm

5] If $\overrightarrow{A}=(4,-5,1)$, $\overrightarrow{B}=(2,-k,-2)$, $\overrightarrow{C}=(-4,4,m-2)$ and $\overrightarrow{AB}\parallel\overrightarrow{C}$. then $k+m=$ ....

The Solution

$\overrightarrow{AB}=(2,-k,-2)-(4,-5,1)=(-2,-k+5,-3)$ $\because \overrightarrow{AB}\parallel\overrightarrow{C}$

$\therefore \frac{-2}{-4}=\frac{-k+5}{4}=\frac{-3}{m-2}$ $\therefore -k+5=2$ $\therefore k=3$ & $m-2=-6$ $\therefore m=-4$ $\therefore k+m=-1$

6] If $\|\overrightarrow{A}\|=2$, $\|\overrightarrow{B}\|=3$, $\|\overrightarrow{C}\|=12$ and $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$ are mutually orthogonal , then $\|\overrightarrow{A}+\overrightarrow{B}+\overrightarrow{C}\|=$ ....

The Solution

$(\|\overrightarrow{A}+\overrightarrow{B}+\overrightarrow{C}\|)^2=(\overrightarrow{A}+\overrightarrow{B}+\overrightarrow{C})\cdot(\overrightarrow{A}+\overrightarrow{B}+\overrightarrow{C})$

$=(\|\overrightarrow{A}\|)^2+\overrightarrow{A}\cdot\overrightarrow{B}+\overrightarrow{A}\cdot\overrightarrow{C}+\overrightarrow{B}\cdot\overrightarrow{A}+(\|\overrightarrow{B}\|)^2+\overrightarrow{B}\cdot\overrightarrow{C}+\overrightarrow{C}\cdot\overrightarrow{A}+\overrightarrow{C}\cdot\overrightarrow{B}+(\|\overrightarrow{C}\|)^2$

$\because$ The vectors mutually orthogonal $\therefore \overrightarrow{A}\cdot\overrightarrow{B}=\overrightarrow{B}\cdot\overrightarrow{A}=\overrightarrow{A}\cdot\overrightarrow{C}=\overrightarrow{C}\cdot\overrightarrow{A}=\overrightarrow{B}\cdot\overrightarrow{C}=\overrightarrow{C}\cdot\overrightarrow{B}=0$

$\therefore \|\overrightarrow{A}\|=4$, $\|\overrightarrow{B}\|=9$, $\|\overrightarrow{C}\|=12$

$\therefore \|\overrightarrow{A}+\overrightarrow{B}+\overrightarrow{C}\|=\sqrt{(\|\overrightarrow{A}\|)^2+(\|\overrightarrow{B}\|)^2+(\|\overrightarrow{C}\|)^2}=\sqrt{4+9+144}=$ $\sqrt{157}$ unit length .

Answer the following questions :

3a] If $Z_1=(\sin\frac{\pi}{9}+i\cos\frac{\pi}{9})^5$, $Z_2=(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})^4$ *and* $z=\frac{z_1}{z_2}$ find the square roots of z in its exponential form .

The Solution

$z_1=(\sin\frac{\pi}{9}+i\cos\frac{\pi}{9})^5=[\cos(\frac{\pi}{2}-\frac{\pi}{9})+i\sin(\frac{\pi}{2}-\frac{\pi}{9})]^5=[\cos(\frac{7\pi}{18})+i\sin(\frac{7\pi}{18})]^5=$

$=\cos\left(\frac{35\pi}{18}\right)+i\sin\left(\frac{35\pi}{18}\right)=\cos\left(\frac{-\pi}{18}\right)+i\sin\left(\frac{-\pi}{18}\right)$

$z_2=\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^4=\cos 2\pi+i\sin 2\pi=\cos 0+i\sin 0$

$\therefore z = \frac{z_1}{z_2} = \frac{\text{Cos}\left(\frac{-\pi}{18}\right) + i\,\text{Sin}\left(\frac{-\pi}{18}\right)}{\text{Cos}\,0 + i\,\text{Sin}\,0} = \text{Cos}\left(\frac{-\pi}{18} - 0\right) + i\,\text{Sin}\left(\frac{-\pi}{18} - 0\right) = \text{Cos}\left(\frac{-\pi}{18}\right) + i\,\text{Sin}\left(\frac{-\pi}{18}\right)$

$\therefore \sqrt{z} = \text{Cos}\left(\frac{\frac{-\pi}{18} + 2m\pi}{2}\right) + i\,\text{Sin}\left(\frac{\frac{-\pi}{18} + 2m\pi}{2}\right)$ where $m = 0, 1$

At $m = 0$ $\therefore \sqrt{z} = \text{Cos}\left(\frac{-\pi}{36}\right) + i\,\text{Sin}\left(\frac{-\pi}{36}\right) = e^{\frac{-1}{36}\pi i}$

At $m = 1$ $\therefore \sqrt{z} = \text{Cos}\left(\frac{35\pi}{36}\right) + i\,\text{Sin}\left(\frac{35\pi}{36}\right) == e^{\frac{35}{36}\pi i}$

3b] If $\overrightarrow{A} = (2\,\text{Cos}\,\theta\,,\,Log_3 x\,,\,Sin\,\theta)$, $\overrightarrow{B} = (\text{Cos}\,\theta\,,\,Log_5 27\,,\,2\,Sin\,\theta)$ and $\overrightarrow{A} \cdot \overrightarrow{B} = 11$ find the value of x.

The Solution

$\because \overrightarrow{A} \cdot \overrightarrow{B} = 11$ $\therefore (2\,\text{Cos}\,\theta\,,\,Log_3 x\,,\,Sin\,\theta) \cdot (\text{Cos}\,\theta\,,\,Log_5 27\,,\,2Sin\,\theta) = 11$

$\therefore 2Cos^2\theta + log_3 x \times log_5 27 + 2\,Sin^2\theta = 11 = 2(Cos^2\theta + Sin^2\theta) + \frac{\text{Log}\,x}{\text{Log}\,3} \times \frac{\text{Log}\,27}{\text{Log}\,5}$

$= 2 + \frac{\text{Log}\,x}{\text{Log}\,3} \times \frac{3\,\text{Log}\,3}{\text{Log}\,5} = 11$ $\therefore \frac{3\log x}{\log 5} = 9$ $\therefore Log_5 x = 3$ $\therefore x = (5)^3 = 125$

4a] In the expansion of $(1 + x)^n$ according to the ascending power of x if $T_3 = 17$, $3\,T_2 \times T_4 = 544$, find the value for each of n and x.

The Solution

$\because T_3 = 17$ $\therefore {}^nC_2\,x^2 = 17$ ----- (1) $\therefore 3T_2 \times T_4 = 544$ ---- $\div (T_3)^2$

$\therefore 3 \times \frac{T_2}{T_3} \times \frac{T_4}{T_3} = \frac{544}{17 \times 17}$ $\therefore 3 \times \frac{2}{n-2+1} \times \frac{1}{x} \times \frac{n-3+1}{3} \times \frac{x}{1} = \frac{32}{17}$ $\therefore \frac{n-2}{n-1} = \frac{16}{17}$

$\therefore 17(n-2) = 16(n-1)$ $\therefore 17n - 34 = 16n - 16$ $\therefore 17n - 16n = 34 - 16$ $\therefore n = 18$

by substitution in (1) $\therefore {}^{18}C_2\,x^2 = 17$ $\therefore 153\,x^2 = 17$ $\therefore x^2 = \frac{1}{9}$ $\therefore x = \pm\frac{1}{3}$

4b] without expanding the determinant, **prove that :**

$$\begin{vmatrix} a+b+2 & a & b \\ 1 & 2a+b+1 & b \\ 1 & a & a+2b+1 \end{vmatrix} = 2\,(a+b+1)^3$$

The Solution

$r_1 - r_3\,,\; r_2 - r_3$ $\therefore L.H.S. = \begin{vmatrix} a+b+1 & 0 & -a-b-1 \\ 0 & a+b+1 & -a-b-1 \\ 1 & a & a+2b+1 \end{vmatrix}$

Common factor $(a+b+1)$ from r_1 , r_2

$$\therefore L.H.S. = (a+b+1)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & a & a+2b+1 \end{vmatrix} \quad C_3 + C_2 + C_1$$

$$\therefore L.H.S. = (a+b+1)^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & a & 2a+2b+2 \end{vmatrix} = (a+b+1)^2(1)(1)(2a+2b+2) = 2(a+b+1)^3 = R.H.S.$$

5a] If $A = \begin{pmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{pmatrix}$ *and* $A^t = A^{-1}$, , find the value for each of x, y, z

The Solution

$\because A^t = A^{-1}$ multiply $\times A$ *from the right side* . $\therefore A^t A = A^{-1} A = I$

$$\therefore \begin{pmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{pmatrix} \begin{pmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{pmatrix} = I \quad \therefore \begin{pmatrix} 2x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3z^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore 2x^2 = 1 \therefore x^2 = \frac{1}{2} \therefore x = \pm\frac{1}{\sqrt{2}}$ & $6y^2 = 1 \therefore y^2 = \frac{1}{6} \therefore y = \pm\frac{1}{\sqrt{6}}$ & $3z^2 = 1 \therefore z^2 = \frac{1}{3} \therefore z = \pm\frac{1}{\sqrt{3}}$

5c] Find the point of intersection of the straight line $x = y = z$ and the plane

$x + 2y + 3z = 12$

The Solution

Let $x = y = z = t$ $\therefore$ By substitution in the equation of the plane

$\therefore t + 2t + 3t = 12 \therefore 6t = 12 \therefore t = 2$ $\therefore$ The point of intersection $(2, 2, 2)$

The eighth test

First: Answer one of the following questions

First question : Complete :

1] If $\lfloor 1+Log x = 1$, then $x =$ .... or ....

The Solution

$1 + \text{Log}\, x = 1 \quad \therefore \text{Log}\, x = 0 \quad \therefore x = 1$ or $1 + \text{Log}\, x = 0 \quad \therefore \text{Log}\, x = -1 \quad \therefore x = 10^{-1} = \frac{1}{10}$

2] If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ a & b & c \end{vmatrix} = 5$, then the value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ a+5 & b+5 & c+5 \end{vmatrix} =$ ....

The Solution

Write the determinant as the sum of two determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ a & b & c \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 5 & 5 & 5 \end{vmatrix}$

take 5 common factor 3rd row of 2nd determinant

$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ a & b & c \end{vmatrix} + 5 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 5 + 5 \times 0 = 5$ because $R_1 = R_3$ in 2nd determinant

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3] The measure of the angle between the two lines $\vec{r_1} = (-2, 5, -7) + k\,(-6, 6, 8)$

$\vec{r_2} = (1, -2, 3) + k`\,(4, 12, -6)$ equals ....

The Solution

$\because \vec{d_1} = (-6, 6, 8)$ is the direction vector of the first straight line .

$\because \vec{d_2} = (4, 12, -6)$ is the direction vector of the second straight line .

Let the angle between the two straight lines is θ

$\therefore \text{Cos}\,\theta = \frac{|\vec{d_1} \bullet \vec{d_2}|}{\|\vec{d_1}\|\ \|\vec{d_2}\|} = \frac{|(-6\ ,\ 6,\ 8) \bullet (4\ ,\ 12\ ,\ -6)|}{\sqrt{36+36+64}\ \sqrt{16+144+36}} = \frac{-24+72-48}{\sqrt{136}\ \sqrt{196}} = 0 \qquad \therefore m(\angle\theta) = 90^\circ$

4] If $\|\vec{A}\| = 4$, $\|\vec{B}\| = 6$ and the measure of the angle between the two vectors $\vec{A}$, $\vec{B}$ equals 60^0 , then $(2\vec{A} + \vec{B}) \bullet (\vec{A} - \vec{B}) =$ ....

The Solution

∵ The angle between the two vectors is θ where $m(\angle \theta) = 60°$

$\cos\theta = \frac{\overrightarrow{A} \bullet \overrightarrow{B}}{\|\overrightarrow{A}\| \ \|\overrightarrow{B}\|}$ $\quad \therefore \frac{1}{2} = \frac{\overrightarrow{A} \bullet \overrightarrow{B}}{4 \times 6}$ $\quad \therefore \overrightarrow{A} \bullet \overrightarrow{B} = 12$

$\therefore (2\overrightarrow{A} + \overrightarrow{B}) \bullet (\overrightarrow{A} - \overrightarrow{B}) = 2(\|\overrightarrow{A}\|)^2 - 2\overrightarrow{A} \bullet \overrightarrow{B} + \overrightarrow{B} \bullet \overrightarrow{A} - (\|\overrightarrow{B}\|)^2$

$= 2 \times 16 - 2 \times 12 + 12 - 36 = 32 - 24 + 12 - 36 = -16$

5] The equation of the sphere whose diameter is $\overline{AB}$ where A (7, 1, −4) , B (3, −1, 2) is ....

The Solution

The center of the sphere $= \left(\frac{7+3}{2}, \frac{1-1}{2}, \frac{-4+2}{2}\right) = (L, k, n) = (5, 0, -1)$

$\because \overrightarrow{AB} = (3, -1, 2) - (7, 1, -4) = (-4, -2, 6)$

$\therefore$ *The diameter* $= \|\overrightarrow{AB}\| = \sqrt{16 + 4 + 36} = \sqrt{56} = 2\sqrt{14}$ unit length

$\therefore$ The radius of the sphere $= r = \sqrt{14}$ unit length

$\therefore$ The equation of the sphere $(x-5)^2 + y^2 + (z+1)^2 = 14$

6] If $\overrightarrow{A} = (1, 2, -4)$, $\overrightarrow{B} = (1, 1, k-1)$ and $\|\overrightarrow{A} + \overrightarrow{B}\| = 7$ unit of length , then k = ...

The Solution

$\overrightarrow{A} + \overrightarrow{B} = (1, 2, -4) + (1, 1, k-1) = (2, 3, k-5)$

$\because \|\overrightarrow{A} + \overrightarrow{B}\| = 7$ $\quad \therefore (\|\overrightarrow{A} + \overrightarrow{B}\|)^2 = 49$ $\quad \therefore 4 + 9 + (k-5)^2 = 49$ $\quad \therefore (k-5)^2 = 36$

$\therefore k - 5 = 6$ || $k - 5 = -6$

k = 11 || **k = −1**

Second question : Choose the correct answer

1] $\frac{a^2+b^2}{a+bi} = 2 + 3i$, then $a \times b =$ .... Where $a, b \in R$

a] −6 b] −5 c] 5 d] 6

The Solution

$a^2 + b^2 = (a + bi)(2 + 3i) = (2a - 3b) + (3a + 2b)i$

$\therefore a^2 + b^2 = 2a - 3b$ -----(1) $\quad 3a + 2b = 0 \quad \therefore a = \frac{-2}{3}b$ ----- (2)

by substitution $\quad \therefore \frac{4}{9}b^2 + b^2 = \frac{-4}{3}b - 3b$ $\quad$ multiply by 9

$\therefore 4b^2 + 9b^2 = -12b - 27b \quad \therefore 13b^2 + 39b = 0 \quad \therefore 13\,b(b+3) = 0$

$\therefore$ **b = 0 refused**, because $a \times b = 0$

or **b = −3** by substitution in (2) $\therefore a = 2 \qquad \therefore a \times b = -6$

2] The rank of the matrix $A = \begin{pmatrix} 0 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{pmatrix}$

a] 3 b] 2 c] 1 d] zero

The Solution

$\because |A| = \begin{vmatrix} 0 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{vmatrix} = 0$ because $C_3 = -1.5C_1 \quad \therefore R(A) < 3 \because \begin{vmatrix} 0 & -2 \\ -2 & 4 \end{vmatrix} = -4 \neq 0 \therefore R(A) = 2$

3] A B C D is a parallelogram in which $\overrightarrow{AB} = (2, 2, -1)$, $\overrightarrow{AD} = (-1, 2, -3)$, then the surface area of the parallelogram = ... cm^2

a] 6 b] $7\sqrt{2}$ c] $3\sqrt{11}$ d] $\sqrt{101}$

The Solution

$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 2 & -1 \\ -1 & 2 & -3 \end{vmatrix} = -4\hat{\imath} + 7\hat{\jmath} + 6\hat{k}$

Area of the parallelogram ABCD = $\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \sqrt{16 + 49 + 36} = \sqrt{101}\ cm^2$

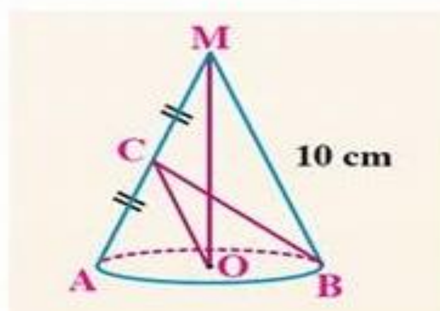
4] In the opposite figure :

a right circular cone , the perimeter of its base = 12π cm ,

C is the midpoint of $\overline{AM}$, then $\overrightarrow{BC} \cdot \overrightarrow{CO}$ = ...

a] − 43 b] − 40

c] − 37 d] − 33



The Solution

The perimeter of the cane = 12π cm. $\therefore 2\pi r = 12\pi \therefore r = BO = 6\,cm$, $\cos(\angle MBO) = \frac{6}{10} = \frac{3}{5}$

$\because MC = CA$, BO = OA $\therefore \overline{MB} \,//\, \overline{OC} \quad \therefore m(\angle COB) + m(\angle MBO) = 180°$

$\therefore \cos(\angle COB) = -\cos(\angle MBO) = -\frac{3}{5}$, In Δ MAO $\because m(\angle MOA) = 90°$

MC = CA $\therefore CO = \frac{1}{2} MA = 5\,cm$

In Δ BOC : $(BC)^2 = (BO)^2 + (OC)^2 - 2(BO)(OC)\ Cos\ (\angle BOC) = 36 + 25 - 2 \times 6 \times 5 \times -\frac{3}{5} = 99$

$\therefore BC = 3\sqrt{11}$ cm. & $\text{Cos}(\angle BCO) = \frac{25+99-36}{2\times 5 \times 3\sqrt{11}} = \frac{43}{15\sqrt{11}}$

$\therefore \overrightarrow{BC} \bullet \overrightarrow{CO} = -(\overrightarrow{CB} \bullet \overrightarrow{CO}) = -\|\overrightarrow{CB}\|\|\overrightarrow{CO}\|\ \text{Cos}(\angle BCO) = -\left(3\sqrt{11} \times 5 \times \frac{43}{15\sqrt{11}}\right) = -43$

Another Solution

In Δ BOC : $\overrightarrow{BO} + \overrightarrow{OC} = \overrightarrow{BC}$, $\therefore \overrightarrow{BC} \bullet \overrightarrow{CO} = (\overrightarrow{BO} + \overrightarrow{OC}) \bullet \overrightarrow{CO} = \overrightarrow{BO} \bullet \overrightarrow{CO} + \overrightarrow{OC} \bullet \overrightarrow{CO}$

$= \overrightarrow{BO} + \overrightarrow{CO} - \overrightarrow{OC} \bullet \overrightarrow{OC} = \|\overrightarrow{BO}\|\|\overrightarrow{CO}\|\ \text{Cos}(COB) - \|\overrightarrow{OC}\|^2 = 6 \times 5 \times -\frac{3}{5} - 25 = -43$

5] If $\vec{A} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{B} = 2\hat{\imath} - \hat{\jmath} - \hat{k}$, then $\vec{A} \times (\vec{A} - \vec{B}) = \ldots\ldots\ldots\ldots$

a] $\hat{\imath} + \hat{k}$

b] $-3\hat{\imath} + 3\hat{k}$

c] $-3\hat{\imath} - 3\hat{\jmath}$

d] $3\hat{\imath} - 2\hat{\jmath}$

The Solution

$\vec{A} - \vec{B} = -\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$ $\therefore \vec{A} \times (\vec{A} - \vec{B}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ -1 & 2 & 2 \end{vmatrix} = -3\hat{\jmath} + 3\hat{k}$

6] If $L_1 : x = 0, y = z$, $L_2 : y = 0, x = z$ are two straight lines in space , the measure of the angle between them is θ , then $\theta = \ldots\ldots\ldots\ldots$

a] 45^0

b] 60^0

c] 70^0

d] 90^0

The Solution

$\because$ $\overrightarrow{d_1} = (0, 1, 1)$ is the direction vector of the first straight line .

$\because$ $\overrightarrow{d_2} = (1, 0, 1)$ is the direction vector of the second straight line .

Let the angle between the two straight lines is θ

$\therefore \text{Cos}\,\theta = \frac{|\overrightarrow{d_1} \bullet \overrightarrow{d_2}|}{\|\overrightarrow{d_1}\|\ \|\overrightarrow{d_2}\|} = \frac{|(0,1,1) \bullet (1,0,1)|}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$ $\therefore m(\angle \theta) = 60°$

3a] Use the multiplicative inverse of a matrix to solve the following equations :

$2x - y + z = -1$, $x - z = 2$, $x + y = 3$

The Solution

$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$ $\therefore |A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 2(0+1) + 1(0+1) + 1(1-0) = 4$, $\therefore |A| \neq 0$

$\therefore R(A) = 3$ $\because$ The number of unknown = 3 $\because$ The equations are homogeneous

∴ The equations has unique solution , the equation of the matrix AX = B

where $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$

The cofactor of A $= \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} & \overline{a_{13}} \\ \overline{a_{21}} & \overline{a_{22}} & \overline{a_{23}} \\ \overline{a_{31}} & \overline{a_{32}} & \overline{a_{33}} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} \end{pmatrix}$

∴ The matrix of cofactor of A is $F = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & -3 \\ 1 & 3 & 1 \end{pmatrix}$

$\therefore Adj\,(A) = F^t = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 3 \\ 1 & -3 & 1 \end{pmatrix}$ $\therefore A^{-1} = \frac{1}{|A|} \times F^t = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 3 \\ 1 & -3 & 1 \end{pmatrix}$

$\therefore X = A^{-1}B = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 3 \\ 1 & -3 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 \\ 8 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ $\therefore x = 1 , y = 2 , z = -1$

3b] Find the point of intersection of the planes

$2x + y - z = -1$, $x + y + z = 2$, $3x - y - z = 6$

The Solution

By adding (3) & (2) $\therefore 4x = 8$ $\therefore x = 2$

By adding (1) & (2) $\therefore 3x + 2y = 1$ $\therefore 6 + 2y = 1$ $\therefore y = -2.5$

From (1) $\therefore 2(2) - 2.5 - z = -1$ $\therefore z = 2.5$

∴ The point of intersection of the planes **is (2 , −2.5 , 2.5)**

4a] If $z_1 = 1 - \sqrt{3}\,i$, $z_2 = \text{Cos}\,\theta + i\,\text{Sin}\,\theta$, $z_3 = (\text{Cos}\frac{\theta}{2} - i\,\text{Sin}\frac{\theta}{2})^2$ and $z = \frac{z_1 z_2}{z_3}$, find the modulus and the principle amplitude of z , then find the square roots of z in its trigonometric form when $\theta = \frac{\pi}{6}$

The Solution

$z_1 = 1 - \sqrt{3}i$ $\therefore x = 1 , y = -\sqrt{3}$ $\therefore r = \sqrt{(1)^2 + (-\sqrt{3})^2} = 4$

$\tan\theta = \frac{y}{x} = -\frac{\sqrt{3}}{1} = -\sqrt{3}$ where $x > 0 , y < 0$ $\therefore z_1$ in the 4th quad.

$\therefore m(\angle\theta) = \frac{-\pi}{3}$ $\therefore z_1 = 2\left[\text{Cos}\frac{-\pi}{3} + i\,\text{Sin}\frac{-\pi}{3}\right]$

$z_2 = \cos\theta + i\sin\theta$, $z_3 = \left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)^2 = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \cos\theta - i\sin\theta$

$\therefore z_3 = \cos(-\theta) + i\sin(-\theta)$

$\therefore z = \frac{z_1 z_2}{z_3} = 2\left[\cos\left(\frac{-\pi}{3} + \theta - -\theta\right) + i\sin\left(\frac{-\pi}{3} + \theta - -\theta\right)\right] = 2\left[\cos\left(\frac{-\pi}{3} + 2\theta\right) + i\sin\left(\frac{-\pi}{3} + 2\theta\right)\right]$

$\therefore |z| =$ The modulus $= 2$ the amplitude $= \frac{-\pi}{3} + 2\theta$ when $\theta = \frac{\pi}{6} = 30$

$\therefore z = 2[\cos 0 + i\sin 0]$ $\therefore \sqrt{z} = \sqrt{2}\left[\cos\left(\frac{0 + 2m\pi}{2}\right) + i\sin\left(\frac{0 + 2m\pi}{2}\right)\right]$ where $m = 0, 1$

At $m = 0$ $\therefore \sqrt{z} = \sqrt{2}[\cos 0 + i\sin 0]$, At $m = 1$ $\therefore \sqrt{z} = \sqrt{2}[\cos\pi + i\sin\pi]$

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4 b] Discuss the possibility of existence of a solution except the zero solution for the system of linear equations : $x + 3y - 2z = 0$, $x - 8y + 8z = 0$, $3x - 2y + 4z = 0$

The Solution

$A = \begin{pmatrix} 1 & 3 & -2 \\ 1 & -8 & 8 \\ 1 & -2 & 4 \end{pmatrix}$, $|A| = \begin{vmatrix} 1 & 3 & -2 \\ 1 & -8 & 8 \\ 1 & -2 & 4 \end{vmatrix} = 1(-32 + 16) - 3(4 - 24) - 2(-2 + 24) = 0$

$\therefore R(A) < 3$, $\because \begin{vmatrix} 1 & 3 \\ 1 & -8 \end{vmatrix} = -11 \neq 0$ $\therefore R(A) = 2$ $\because$ number of unknown $= 3$

$\therefore R(A) <$ number of unknown $\therefore$ The equations are homogeneous

$\therefore$ There exists a solution except the zero solution for these system of linear equations .

To find the solution put $x = L$

$4 \times$ First equation + second equation $\therefore 4x + 12y - 8z + x - 8y + 8z = 0$

$\therefore 5x + 4y = 0$ $\therefore y = \frac{-5}{4}L$ by substitution from any equation $\therefore z = \frac{-11}{8}L$

$\therefore$ The form of the equation $= \left(L, \frac{-5}{4}L, \frac{-11}{8}L\right)$

5a] In the expansion of $\left(x^2 + \frac{1}{2x}\right)^{3n}$ according to the descending powers of x :

First : Prove that the term free of x is of order $(2n + 1)$.

Second : Find the ratio between the term free of x and the middle term when $n = 4$, $x = 1$

The Solution

Let the term free of x is T_{r+1}

$\therefore T_{r+1} = {}^{3n}C_r \left(\frac{1}{2x}\right)^r (x^2)^{3n-r} = {}^{3n}C_r \times (2)^{-r} \times x^{-r} \times x^{6n-2r} = {}^{3n}C_r \times (2)^{-r} \times x^{6n-3r}$

Let $6n - 3r = 0$ $\therefore 6n = 3r$ $\therefore r = 2n$ $\therefore$ The term free of x is of order $(2n + 1)$

Second : when n = 4 ∴ number of terms = 13 ∴ the order of the middle term = 7

The order of the term free of $x = 2 \times 4 + 1 = 9$

$\therefore \frac{T_9}{T_7} = \frac{T_9}{T_8} \times \frac{T_8}{T_7} = \frac{12-8+1}{8} = \frac{\frac{1}{2}}{1} + \frac{12-7+1}{7} \times \frac{\frac{1}{2}}{1} = \frac{15}{112}$

5b] If the two spheres $(x-3)^2 + y^2 + (z-3)^2 = 16$, $(x+1)^2 + (y-4)^2 + (z-k)^2 = 25$ are tangential , **find the value of k** .

The Solution

with respect to the first sphere $M_1 = (3, 0, 3)$ and $r_1 = 4$

with respect to the second sphere , $M_2 = (-1, 4, k)$ and $r_2 = 5$

∵ The two sphere touch each other .

(i) If the two spheres touch each other externally ∴ $M_1M_2 = r_1 + r_2 = 9$ ∴ $(M_1M_2)^2 = 81$

∴ $(3+1)^2 + (0-4)^2 + (3-k)^2 = 81$ ∴ $16 + 16 + (3-k)^2 = 81$ ∴ $(3-k)^2 = 49$

∴ $3 - k = 7$ ∴ $k = -4$ or $3 - k = -7$ ∴ $k = 10$

(ii) If the two spheres touches each other internally ∴ $M_1M_2 = r_2 - r_1 = 1$

$(M_1M_2)^2 = 1$ ∴ $16 + 16 + (3-k)^2 = 1$ ∴ $(3-k)^2 = -31$ refused

The two spheres can't be touching internally .

The ninth test

First : Answer one of the following:

First question : Complete :

1] If ${}^{x+y}P_4 = 360$, $\underline{|2x+y} = 5040$, then ${}^{y}C_{2x} =$ ....

The Solution

∵ ${}^{x+y}P_4 = 360$ ∴ ${}^{x+y}P_4 = {}^{6}P_4$ ∴ $x + y = 6$ .... (1)

∵ $\underline{|2x+y} = 5050$ ∴ $\underline{|2x+y} = \underline{|7}$ ∴ $2x + y = 7$ .... (2)

by subtraction (2) – (1) ∴ $x = 1$ by substitution ∴ $y = 5$ ∴ ${}^{y}P_{2x} = {}^{5}C_2 = 10$

2] The solution set of the equation $\begin{vmatrix} a+1 & 3 & 2 \\ 0 & a-1 & 5 \\ 0 & 0 & 7 \end{vmatrix} = 21$ is ………………………

The Solution

$\because$ The determinant in the triangular form its value = 7(a + 1) (a − 1) = 21

$\therefore 7(a^2 - 1) = 21$ $\therefore a^2 - 1 = 3$ $\therefore a^2 = 4$ $\therefore a = \pm 2$ $\therefore$ S.S. = { 2 , −2 }

3] Cosine the angle between the two vectors $\vec{A} = (1, -3, 0)$, $\vec{B} = (2, 0, 1)$ equals ……………

The Solution

Let the angle between the two vectors is θ $\therefore \cos\theta = \frac{(1,-3,0) \cdot (2,0,1)}{\sqrt{1+9+1}\sqrt{4+0+1}} = \frac{2}{\sqrt{10} \times \sqrt{5}} = \frac{\sqrt{2}}{5}$

4] The radius length of the sphere : $x^2 + y^2 + z^2 + 2x - 2y - 4z - 3 = 0$ equals ……………

The Solution

The Centre of the sphere = (−1 , 1 , 2) and d = −3

$\therefore$ The radius of the sphere = $r = \sqrt{x^2 + y^2 + z^2 - d} = \sqrt{1+1+4+3} = \sqrt{9} = 3$ unit length

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5] If $\vec{A} = (\frac{-1}{2}, \frac{3}{4}, k)$ is a unit vector , then the value of k = …………… or ……………

The Solution

$\therefore \vec{A}$ is a unit vector $\therefore \|\vec{A}\| = 1$ $\therefore \frac{1}{4} + \frac{9}{16} + k^2 = 1$ $\therefore k^2 = \frac{3}{16}$ $\therefore k = \pm\frac{\sqrt{3}}{4}$

6] If $\vec{A} = (k, -3, 1)$, $\vec{B} = (2, 3, -k)$ are perpendicular , then the value of k = ……………

The Solution

$\because$ The two vectors are perpendicular $\therefore (k, -3, 1) \cdot (2, 3, -k) = 0$ $\therefore 2k - 9 - k = 0$ $\therefore k = 9$.

Second question : Complete :

1] $(1 + w)^4 + (1 + w^2)^4 + (w + w^2)^4 = \cdots$ ………………………

The Solution

The value = $(-w^2)^4 + (-w)^4 + (-1)^4 = w^8 + w^4 + 1 = w^6 \times w^2 + w^3 \times w + 1 = w^2 + w + 1 = 0$

2] The rank of the matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ equals ....

The Solution

$\because |A| = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 2(4-1)\ -1(-2-3) + 3(-1-6) = -13 \therefore |A| \neq 0 \therefore R(A) = 3$

3] If $\vec{A} = (3, -2, k)$, $\vec{B} = (1, m, 2)$ and $\vec{A} \parallel \vec{B}$, then k = ... , m = ...

The Solution

$\because$ The two vectors are parallel $\therefore \frac{3}{1} = \frac{-2}{m} = \frac{k}{2}$ $\therefore k = 6 , m = -\frac{2}{3}$

4] If the measure of the angle which $\vec{C} = (2, 4, k)$ makes with the positive direction of y− axis equals 45^0 , then k = ...

The Solution

The directed vector of the +ve y−axis is $(0, 1, 0)$

$\because$ The measure of the angle = 45 $\therefore \frac{1}{\sqrt{2}} = \frac{(2, 4, k) \cdot (0, 1, 0)}{\sqrt{4+16+k^2}\ \sqrt{0+1+0}} = \frac{4}{\sqrt{20+k^2}}$

by squaring both sides $\therefore 20 + k^2 = 32$ $\therefore k^2 = 12$ $\therefore k = \pm 2\sqrt{3}$

5] If the two planes : $x + 2y + kz = 2$, $3x - y + 2z + 4 = 0$ are perpendicular , then k = ...

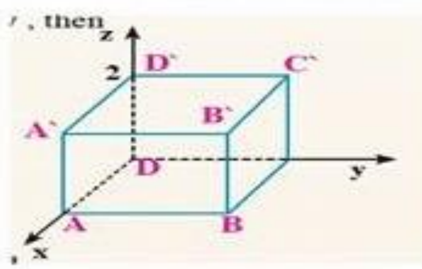
The Solution

The directed vector of the two planes are $(1, 2, k)$, $(3, -1, 2)$

$\because$ The two planes are perpendicular $\therefore (1, 2, k) \cdot (3, -1, 2) = 0 \therefore 3 - 2 + 2k = 0 \therefore k = -\frac{1}{2}$

6] In the opposite figure :

A B C D $A^\backslash$ $B^\backslash$ $C^\backslash$ $D^\backslash$ is a cube of side length unity , then $\overrightarrow{AB^\backslash} \cdot \overrightarrow{BD} =$ ....



The Solution

Let D is the origin point

$\therefore D(0,0,0)\ ,\ A(1,0,0)\ ,\ B`(1,1,1)\ ,\ D`(0,0,1)\ ,\ B(1,1,0)$

$\therefore \overrightarrow{BD} = (0,0,0) - (1,1,0) = (-1,-1,0) \quad , \quad \overrightarrow{AB`} = (1,1,1) - (1,0,0) = (0,1,1)$

$\therefore \overrightarrow{AB`} \cdot \overrightarrow{BD} = (0,1,1) \cdot (-1,-1,0) = -1$

3a] If $Z_1 = 2\left(\sin\frac{\pi}{3} + i\cos\frac{\pi}{3}\right)$, $Z_2 = \sqrt{2}\left(\sin\frac{\pi}{4} - i\cos\frac{\pi}{4}\right)$, $Z_3 = 1 + \sqrt{3}\,i$

Find the number $z = \dfrac{Z_1{}^3 \times Z_2{}^4}{Z_3{}^5}$ in its exponential form , then find the square roots of z in its trigonometric form .

The Solution

$z_1 = 2\left[\sin\frac{\pi}{3} + i\cos\frac{\pi}{3}\right] = 2\left[\cos\left(90 - \frac{\pi}{3}\right) + i\sin\left(90 - \frac{\pi}{3}\right)\right] = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$

$\therefore z_1^3 = 8\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]^3 = 8\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$

$z_2 = \sqrt{2}\left[\sin\left(\frac{\pi}{4}\right) - i\cos\left(\frac{\pi}{4}\right)\right] = \sqrt{2}\left[\cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right]$

$= \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right] = \sqrt{2}\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$

$\therefore (z_2)^4 = (\sqrt{2})^4\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]^4 = 4[\cos(-\pi) + i\sin(-\pi)]$

$\because z_3 = 1 + \sqrt{3}i \quad \therefore x = 1\ \&\ y = \sqrt{3} \quad \therefore r = \sqrt{1+3} = 2 \quad \therefore \tan\theta = \frac{y}{x} = \sqrt{3} \quad \because x > 0 , y > 0$

$\therefore z_3$ lies in the 1st quad. $\quad \therefore m(\angle\theta) = \frac{\pi}{3} \quad \therefore z_3 = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$

$\therefore (z_3)^5 = (2)^5\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]^5 = 32\left[\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right] = 32\left[\cos\left(\frac{5\pi}{3} - 2\pi\right) + i\sin\left(\frac{5\pi}{3} - 2\pi\right)\right]$

$= 32\left[\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right]$

$\therefore z = \dfrac{Z_1{}^3 \times Z_2{}^4}{Z_3{}^5} = \frac{8\times4}{32}\left[\cos\left(\frac{\pi}{2} - \pi + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{2} - \pi + \frac{\pi}{3}\right)\right] = \cos\left(-\frac{1}{6}\pi\right) + i\sin\left(-\frac{1}{6}\pi\right) = e^{-\frac{1}{6}\pi i}$

$\because z = \cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right) \qquad \therefore \sqrt{z} = \cos\left(\frac{\frac{-\pi}{6} + 2m\pi}{2}\right) + i\sin\left(\frac{\frac{-\pi}{6} + 2m\pi}{2}\right)$ where $m = 0 , 1$

At $m = 0 \quad \therefore \sqrt{z} = \cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)$ & At $m = 1 \quad \therefore \sqrt{z} = \cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)$

3b] If the plane $2ax - 3ay + 4az + 6 = 0$ passes through the midpoint of the line segment joining the centers of the two spheres $x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$, $x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$, find the value of a .

The Solution

$\because$ The Centre of the first sphere $M_1 = (-3 , 4 , 1)$

The center of the second sphere $M_2 = (5, -2, 1)$

The midpoint of $\overline{M_1M_2} = \left(\frac{-3+5}{2}, \frac{4-2}{2}, \frac{1+1}{2}\right) = (1, 1, 1) \in$ The plane

∵ **This point must satisfied the equation of the plane** $\therefore 2a \times 1 - 3a \times 1 + aa \times 1 + 6 = 0$

$\therefore 2a - 3a + 4a + 6 = 0 \quad \therefore 3a + 6 = 0 \quad \therefore a = -2$

4a] Use the multiplicative inverse of a matrix to solve the following equations :

$x - 2y + 2z = 2 \;,\; 3x + 4z = 10 \;,\; 6z - y = 5$

The Solution

$A = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 0 & 4 \\ 0 & -1 & 6 \end{pmatrix}, |A| = \begin{vmatrix} 1 & -2 & 2 \\ 3 & 0 & 4 \\ 0 & -1 & 6 \end{vmatrix} = 1(0+4) + 2(18-0) + 2(-3-0) = 34 \neq 0$

$\therefore$ The matrix equation is AX = B where $A = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 0 & 4 \\ 0 & -1 & 6 \end{pmatrix}, = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 2 \\ 10 \\ 5 \end{pmatrix}$

The cofactor of A $= \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} & \overline{a_{13}} \\ \overline{a_{21}} & \overline{a_{22}} & \overline{a_{23}} \\ \overline{a_{31}} & \overline{a_{32}} & \overline{a_{33}} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} 0 & 4 \\ -1 & 6 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 0 & 6 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} -2 & 2 \\ -1 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} \\ \begin{vmatrix} -2 & 2 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & 0 \end{vmatrix} \end{pmatrix}$

$\therefore$ The matrix of cofactor of A is $F = \begin{pmatrix} 4 & -18 & -3 \\ 10 & 6 & 1 \\ -8 & 2 & 6 \end{pmatrix}$

$\therefore Adj(A) = F^t = \begin{pmatrix} 4 & 10 & -8 \\ -18 & 6 & 2 \\ -3 & 1 & 6 \end{pmatrix} \qquad \therefore A^{-1} = \frac{1}{|A|} \times F^t = \frac{1}{34}\begin{pmatrix} 4 & 10 & -8 \\ -18 & 6 & 2 \\ -3 & 1 & 6 \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{34}\begin{pmatrix} 4 & 10 & -8 \\ -18 & 6 & 2 \\ -3 & 1 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ 10 \\ 5 \end{pmatrix} = \frac{1}{34}\begin{pmatrix} 68 \\ 34 \\ 34 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad \therefore x = 2, y = 1, z = 1$

4b] Prove that the term free of x in the expansion of $(x^2 + \frac{1}{x^3})^{5n}$ where $n \in Z^+$ equals $\frac{\lfloor 5n}{\lfloor 2n \; \lfloor 3n}$

The Solution

Let the term free of x is T_{r+1}

$\therefore T_{r+1} = {}^{5n}C_r \left(\frac{1}{x^3}\right)^r \times (x^2)^{5n-r} = {}^{5n}C_r \times x^{-3r} \times x^{10n-2r} = {}^{5n}C_r \times x^{10n-5r}$

Let $10n - 5r = 0 \quad \therefore 10n = 5r \quad \therefore r = 2n \quad \therefore$ The term free of $x = {}^{5n}C_{2n} = \frac{\lfloor 5n}{\lfloor 2n \; \lfloor 3n}$

5a] Find the value of k which makes the equations : $kx + y + z = 1$, $x + ky + z = 1$, $x + y + kz = 1$ have an infinite number of solutions .

The Solution

$A = \begin{pmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix}$ $\therefore |A| = \begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix}$, $A^* = \left(\begin{array}{ccc|c} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{array}\right)$

The equations have an infinite number of solutions when $RK(A) = RK(A^*) <$ the number of variable .

Let $|A| = 0$ $\therefore k(k^2 - 1) - 1(k - 1) + 1(1 - k) = k(k - 1)(k + 1) - (k - 1) - (k - 1)$

$= (k - 1)[k(k + 1) - 1 - 1] = (k - 1)[k^2 + k - 2] = (k - 1)(k - 1)(k + 2) = 0$

$\therefore k = 1$ or $k = -2$

where k = 1 $\therefore |A| = 0$ $\therefore 1 \le Rk(A) < 3$ & $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\because A^* = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right)$

$\because$ All the determinants of 2nd degree $= 0$ $\therefore Rk(A) = Rk(A^*) = 1$

$\because$ The equations are homogeneous when k = 1 $\therefore$ **There are infinite number of solution**

when k = −2 $\therefore |A| = 0$ $\therefore 1 \le Rk(A) < 3$ & $A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$

$\because \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = -2 - 1 = -5 \ne 0$ $\therefore R(A) = 2$

$\because A^* = \left(\begin{array}{ccc|c} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{array}\right)$ & $\begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 1(1 + 2) - 1(-2 - 1) + 1(4 - 1) = 9 \ne 0$

$\therefore Rk(A^*) = 3$ $\therefore Rk(A) \ne Rk(A^*)$

$\because$ The equations are homogeneous $\therefore$ **when k = −2 no solution at any way**

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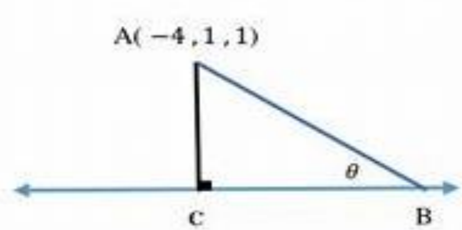
5b] Find the length of the perpendicular drawn from point $(-4, 1, 1)$ on the line $\frac{x+3}{1} = \frac{y-1}{\sqrt{5}} = \frac{z+2}{2}$.

The Solution



The directed vector of the straight line $\overrightarrow{d_1} = (1, \sqrt{5}, 2)$

, The point $B(-3, 1, -2) \in$ straight line

Let C is the projection of A in the straight line

θ is the angle between $\overrightarrow{BA}$ and the starlight line

$A = (-4, 1, 1)$ $\therefore \overrightarrow{d_2} = \overrightarrow{BA} = (-4, 1, 1) - (-3, 1, -2) = (-1, 0, 3)$

$\cos\theta = \frac{|\overline{d_2} \cdot \overline{d_1}|}{\|\overline{d_2}\|\,\|\overline{d_1}\|} = \frac{(-1,0,\ 3)\cdot(1,\sqrt{5},2)}{\sqrt{1+0+9}\ \sqrt{1+5+4}} = \frac{5}{10} = \frac{1}{2}$ $\therefore m(\angle\theta) = 60°$, $\therefore \|\overline{BA}\| = \sqrt{1+0+9} = \sqrt{10}$

$\frac{AC}{\sin 60} = \frac{AB}{\sin 90}$ $\therefore AC = AB \sin 60 = \sqrt{10} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{30}}{2}$ length unit .

The tenth test

First : Answer one of the following :

First question : Complete :

1] If $x = \frac{-1-\sqrt{3}\,i}{2}$, $i^2 = -1$, then the numerical value of $x^8 + x^4 + 5 = \cdots$ ……………………

The Solution

$x = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = w$ $\therefore x^8 + x^4 + 5 = w^8 + w^4 + 5 = w^2 + w + 5 = -1 + 5 = 4$

Or $x = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = w^2$ $\therefore x^8 + x^4 + 5 = w^{16} + w^6 + 5 = w + w^2 + 5 = -1 + 5 = 4$

2] If $\lfloor n$, $\lfloor n-2$, $n\lfloor 2-n$ are the side lengths of a triangle , then the numerical value of the perimeter of the triangle = ……………

The Solution

$\because$ The length of the first side of a triangle = $\lfloor n$ $\therefore n \geq 0$ $\therefore n \in \{0,\ 1, 2, \ldots\ldots\}$ ----- (1)

$\because$ **The length of the second side of a triangle =** $\lfloor n-2$

$\therefore n - 2 \geq 0$ $\therefore n \geq 2$ $\therefore n \in \{2, 3, 4, \ldots\}$ ----- (2)

$\because$ The length of the third side of a triangle = $n\lfloor 2-n$

$\therefore 2 - n \geq 0$ $\therefore 2 \geq n$, $\therefore n \in \{0, 1, 2\}$ ---- (3) from (1) , (2) , (3)

$\therefore$ n = 2 $\therefore$ The lengths of the sides of the triangle $\lfloor n = 2$, $\lfloor 0 = 1$ & $2\lfloor 0 = 2$

$\therefore$ **The perimeter of the triangle = 2 + 1 + 2 = 5 length unit**

3] If $\vec{A} = (-2, k, -3)$ is parallel to the straight line $\frac{x+3}{4} = \frac{y}{8} = \frac{z-1}{6}$, then k = ……………

The Solution

The directed vector of the straight line = (4 , 8 , 6)

$\therefore \overline{A}$ // the straight line $\therefore \frac{-2}{4} = \frac{k}{8} = \frac{-3}{6}$ $\therefore k = -4$

4] The measure of the angle which the vector $\vec{A} = (3, 4, \sqrt{11})$ makes with the positive direction of x −axis equals ....

The Solution

The positive unit vector in the direction of x −axis is (1 , 0 , 0) Let the required angle is θ

$\therefore \cos\theta = \frac{(3,4,\sqrt{11}) \bullet (1,0,0)}{\sqrt{9+16+11}\sqrt{1+0+0}} = \frac{3}{6} = \frac{1}{2}$ $\therefore m(\angle\theta) = 60°$

....

5] If the two planes $x - 3y + mz = 5$, and $3x + ky + 6z = 10$ are parallel , then k × m = ________

The Solution

The perpendicular directed vector for each planes are (1 , −3 , m) , (3 , k , 6)

$\because$ The two planes are parallel $\therefore \frac{1}{3} = \frac{-3}{k} = \frac{m}{6}$ $\therefore k = -9 , m = 2$ $\therefore k \times m = -18$

....

6] The distance between the two parallel planes $4x + 6y + 12z + 18 = 0$ and $4x + 6y + 12z - 10 = 0$ is ________

The Solution

Take point belongs to the first plane put $x = 0 , z = 0 \therefore y = -3$

$\therefore A(0 , -3 , 0) \in$ the first plane .

$\therefore$ The length of the perpendicular planes = the length of the ⊥ line from A to the second plane $\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}} = \frac{|4\times 0 + 6\times -3 + 12\times 0 - 10|}{\sqrt{16+36+144}} = \frac{28}{14} = 2$ **length unit**

....

Second question : Choose the correct answer :

1] $1 - 6x + \frac{6\times 5}{2\times 1}x^2 - \frac{6\times 5\times 4}{3\times 2\times 1}x^3 + \cdots \ldots\ldots\ldots\ldots + x^6 = 64$, *then* $x =$ ________

a] −1 b] 3 c] { −1 , 3 } d] 2

The Solution

L.H.S. $= (1-x)^6 = 64 = 2^6$ $\therefore 1 - x = 2$ $\therefore x = -1$ or $1 - x = -2$ $\therefore x = 3$ $\therefore$ c]

....

2] $(\frac{5-3w^2}{5w-3}+\frac{2-7w}{2w^2-7})^2$

a] 3 b] −3 c] 3 i d] −3i

The Solution

$(\frac{5w^3-3w^2}{5w-3}+\frac{2w^3-7w}{2w^2-7})^2=\left(\frac{w^2(5w-3)}{5w-3}+\frac{w(2w^2-7)}{2w^2-5}\right)^2=(w^2-w)^2=(-\sqrt{3}i)^2=-3$

3] If the two straight lines : $\frac{x+1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$, $\frac{x}{3}=\frac{y+1}{4}=\frac{z-1}{k}$ are perpendicular then k =……

a] 4 b] −4 c] $\frac{9}{2}$ d] $-\frac{9}{2}$

The Solution

∵ (2 , 3 , 4) is the direction vector of the first straight line .

∵ (3 , 4 , k) is the direction vector of the second straight line .

∵ The two straight lines are perpendicular ∴ (2 , 3 , 4) • (3 , 4 , k) = 0 ∴ $6+12+4k=0$ ∴ $k=-\frac{9}{2}$

4] The equation of the sphere whose center is $(3,-2,1)$ and its radius length equals = 5 cm is

a] $(x+3)^2+(y-2)^2+(z+1)^2=5$

b] $(x+3)^2+(y-2)^2+(z+1)^2=25$

c] $(x-3)^2+(y+2)^2+(z-1)^2=25$

d] $(x-3)^2+(y+2)^2+(z-1)^2=\sqrt{5}$

The Solution

c] $(x-3)^2+(y+2)^2+(z-1)^2=25$

5] The measure of the angle included between the two planes $x+\sqrt{2}\,y-z=5$, $x-\sqrt{2}\,y+z=1$ equals

a] 0^0 b] 45^0 c] 90^0 d] 135^0

The Solution

The normal direction vectors of the two planes are $\overrightarrow{n_1}=(1,\sqrt{2},-1)$ & $\overrightarrow{n_2}=(1,-\sqrt{2},1)$

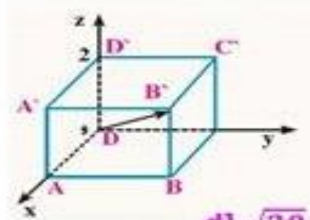
Let the angle between the two planes is θ

$\text{Cos}\,\theta=\frac{|\overrightarrow{n_1}\bullet\overrightarrow{n_2}|}{\|\overrightarrow{n_1}\|\ \|\overrightarrow{n_2}\|}=\frac{|(1,\sqrt{2},-1)\bullet(1,-\sqrt{2},1)|}{\sqrt{1+2+1}\,\sqrt{1+2+1}}=\frac{0}{4}=0$ ∴ $m(\angle\theta)=90°$

6] In the opposite figure :

A B C D A\`B\`C\`D\` is a cuboid A (4 , 0 , 0) ,

C (0, 9, 0) D\` (0, 0, 7) , then $\| \overrightarrow{AC`} \|$ = ...



a] $\sqrt{146}$ b] $\sqrt{114}$ c] 5 d] $\sqrt{20}$

The Solution

From the graph C\`( 0 , 9 , 7 ) , $\therefore \overrightarrow{AC`} = (0,9,7)-(4,0,0)=(-4,9,7)$

$\therefore \| \overrightarrow{AC`} \| = \sqrt{16+81+49} = \sqrt{146}$ **unit length**

3a] In the expansion of $(2x-3)^{15}$ according to the descending powers of x , find the values of x which makes $13\,T_3 + 10\,T_4 + T_5 = 0$.

The Solution

$\because 13\,T_3 + 10\,T_4 + T_5 = 0$ **divide by T_4** $\therefore 13 \times \frac{T_3}{T_4} + 10 + \frac{T_5}{T_4} = 0$

$\therefore 13 \times \frac{3}{15-3+1} \times \frac{-3}{2x} + 10 + \frac{15-4+1}{4} \times \frac{2x}{-3} = 13 \times \frac{3}{13} \times \frac{-3}{2x} + 10 + \frac{12}{4} \times \frac{2x}{-3} = 0$

$\therefore \frac{-9}{2x} + 10 - 2x = 0$ (multiply by $-2x$) $\therefore 4x^2 - 20x + 9 = 0$

$\therefore (2x-9)(2x-1) = 0$ $\therefore x = \frac{9}{2}$ or $x = \frac{1}{2}$

3b] Without expanding the determinant , prove that :

$$\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = 2\begin{vmatrix} 0 & z & z \\ z & 0 & x \\ y & x & 0 \end{vmatrix}$$

The Solution

$R_1 + R_3 - R_2$ $\therefore$ L.H.S. $= \begin{vmatrix} 2z & 0 & 2x \\ y & z+x & y \\ z & z & x+y \end{vmatrix}$ 2 common factor R_1

$\therefore$ L.H.S. $= 2\begin{vmatrix} z & 0 & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix}$, $R_3 - R_1$, $\therefore$ L.H.S. $= 2\begin{vmatrix} z & 0 & x \\ y & z+x & y \\ 0 & z & y \end{vmatrix}$, $R_2 - R_3$

$\therefore$ L.H.S. $= 2\begin{vmatrix} z & 0 & x \\ y & x & 0 \\ 0 & z & y \end{vmatrix}$, exchange R_1 & R_3 then R_2 & R_3 $\therefore$ **L.H.S.** $= 2\begin{vmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{vmatrix}$ = **R.H.S.**

4a] Prove that : $\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$

The Solution

$\because 1 = \sin^2\theta + \cos^2\theta = \sin^2\theta - i^2\cos^2\theta = (\sin\theta + i\cos\theta)(\sin\theta - i\cos\theta)$

$\therefore \frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta} = \frac{(\sin\theta+i\cos\theta)(\sin\theta-i\cos\theta)+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta} = \frac{(\sin\theta+i\cos\theta)[\sin\theta-i\cos\theta+1]}{1+\sin\theta-i\cos\theta}$

$= \sin\theta + i\cos\theta = \cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)$

$\therefore$ **L.H.S.** $= \left[\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n = \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$ = R.H.S.

Another solution :

Let $z = \sin\theta + i\cos\theta \qquad \therefore \bar{z} = \sin\theta - i\cos\theta$

$\frac{1+z}{1+\bar{z}} = \frac{1+z}{1+\bar{z}} \times \frac{1-\bar{z}}{1-\bar{z}} = \frac{1+z-\bar{z}-z\bar{z}}{1-(\bar{z})^2} = \frac{1+2i\cos\theta-1}{1-[\sin^2\theta-2i\sin\theta\cos\theta-\cos^2\theta]} = \frac{2i\cos\theta}{2\cos^2\theta+2i\sin\theta\cos\theta} = \frac{i}{\cos\theta+i\sin\theta}$

$= \frac{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}}{\cos\theta+i\sin\theta} = \cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)$

$\therefore$ **L.H.S.** $= \left[\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n = \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$ = R.H.S.

Another solution :

Let $\theta = \frac{\pi}{2} - 2\beta$

$L.H.S. = \left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \left(\frac{1+\sin[\frac{\pi}{2}-2\beta]+i\cos[\frac{\pi}{2}-2\beta]}{1+\sin[\frac{\pi}{2}-2\beta]-i\cos[\frac{\pi}{2}-2\beta]}\right)^n$

$L.H.S. = \left(\frac{1+\cos 2\beta+i\sin 2\beta}{1+\cos 2\beta-i\sin 2\beta}\right)^n = \left(\frac{1+2\cos^2\beta-1+2i\sin\beta\cos\beta}{1+2\cos^2\beta-1-2i\sin\beta\cos\beta}\right)^n =$

$= \left[\frac{2\cos\beta\ (\cos\beta+2i\sin\beta)}{2\cos\beta\ (\cos\beta-2i\sin\beta)}\right]^n = \left[\frac{2\cos\beta\ (\cos\beta+2i\sin\beta)}{2\cos\beta\ (\cos(-\beta)+2i\sin(-\beta))}\right]^n =$

$[\cos(2\beta) + i\sin(2\beta)]^n = \cos n(2\beta) + i\sin n(2\beta) = \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$

4b] Find the equation of the straight line passing through the point $(3, -1, 0)$ and intersects the straight line $\vec{r} = (2, 1, 1) + t\,(1, 2, -1)$ orthogonally .

The Solution

Let the two straight lines intersect at C from the given **equation**

$\therefore$ The coordinates of the point C is $(2+t, 1+2t, 1-t)$

$\because$ The required line passes through A$(3, -1, 0)$

$\therefore \overrightarrow{CA} = (3, -1, 0) - (2+t, 1+2t, 1-t) = (1-t, -2-2t, -1+t)$

Which is the directed vector of the required line.

$\because$ The directed vector of the given line is $(1, 2, -1)$ and the two lines are $\perp$

$\therefore (1-t, -2-2t, -1+t) \cdot (1, 2, -1) = 0 \qquad \therefore 1 - t - 4 - 4t + 1 - t = 0$

$\therefore 6t = -2 \quad \therefore t = -\frac{1}{3} \qquad \therefore \overrightarrow{CA} = \left(\frac{4}{3}, \frac{-4}{3}, \frac{-4}{3}\right) = (1, -1, -1)$

$\therefore$ The required equation $\vec{r} = (3, -1, 0) + t(1, -1, -1)$

5a] Use the multiplicative inverse of the matrix to solve the set of following equations:

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \ , \ \frac{1}{x} - \frac{1}{y} + \frac{2}{z} = \frac{1}{2} \ , \ \frac{2}{x} + \frac{3}{y} - \frac{4}{z} = \frac{4}{3}$ where x, y and z are not equal to zero.

The Solution

Let $\frac{1}{x} = L \ , \ \frac{1}{y} = m \ , \frac{1}{z} = n \qquad \therefore$ *The three equations becomes.*

$L + m + n = 1 \ , \ L - m + 2n = \frac{1}{2} \ , \ 2L + 3m - 4n = \frac{4}{3}$

$\therefore A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 3 & -4 \end{pmatrix} \qquad \therefore |A| = 11 \neq 0$

The cofactor of A $= \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} & \overline{a_{13}} \\ \overline{a_{21}} & \overline{a_{22}} & \overline{a_{23}} \\ \overline{a_{31}} & \overline{a_{32}} & \overline{a_{33}} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \end{pmatrix}$

$\therefore$ The matrix of cofactor of A is $F = \begin{pmatrix} -2 & 8 & 5 \\ 7 & -6 & -1 \\ 3 & -1 & -2 \end{pmatrix}$

$\therefore Adj(A) = F^t = \begin{pmatrix} -2 & 7 & 3 \\ 8 & -6 & -1 \\ 5 & -1 & -2 \end{pmatrix} \ \& \quad A^{-1} = \frac{1}{|A|} \times F^t = \frac{1}{11}\begin{pmatrix} -2 & 7 & 3 \\ 8 & -6 & -1 \\ 5 & -1 & -2 \end{pmatrix}$

$A^{-1} = \frac{1}{11}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 3 & -4 \end{pmatrix} \ , \quad X = \begin{pmatrix} L \\ m \\ n \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{4}{3} \end{pmatrix}$

$\therefore X = A^{-1}B = \begin{pmatrix} L \\ m \\ n \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -2 & 7 & 3 \\ 8 & -6 & -1 \\ 5 & -1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{4}{3} \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 5.5 \\ \frac{11}{3} \\ \frac{11}{6} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix}$

$\therefore L = \frac{1}{2} \ \& \ m = \frac{1}{3} \ , \ n = \frac{1}{6} \qquad \therefore x = 2, y = 3, z = 6$

5b] Find the **component** of $\overrightarrow{AB}$ in the direction of $\vec{m}$ where A (2 , 1 , 0) , B (3 , 1 , $\sqrt{3}$) $\vec{m} = (3, 2, 2\sqrt{3})$

The Solution

$\overrightarrow{AB} = (3, 1, \sqrt{3}) - (2, 1, 0) = (1, 0, \sqrt{3})$ & $\|\vec{m}\| = \sqrt{9+4+12} = 5$

The vector component of $\overrightarrow{AB}$ in the directed of $\vec{m}$

$= \frac{\vec{A} \bullet \vec{m}}{\|\vec{m}\|} \times \left(\frac{\vec{m}}{\|\vec{m}\|}\right) = \frac{(1, 0, \sqrt{3}) \bullet (3,\ 2,\ 2\sqrt{3})}{5} \times \frac{9}{5} \times \frac{(3,\ 2,\ 2\sqrt{3})}{5} = \left(\frac{27}{25}, \frac{18}{25}, \frac{18}{25}\sqrt{3}\right)$