ملخص لمادة

الفيزياء الرياضية 2

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2-2-2016

الفيزياء الرياضية 2

① [Series solution of differention equation] حلول المعادلات التفاضلية باستخدام المتسلسلات

مفكوك متسلسلة تيلر

$f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \dots$

$\vec{A} = A\hat{i} + A\hat{j} + A\hat{k}$ هذه المتجهة موجودة بالفضاء

Taylor seris expansion

$$f(x) = \sum \frac{f^n(x)}{n!}\Big|_{x=x_0} (x-x_0)^n$$

let the function $y(x)$ expandable as the follwing series

$y(x) = \sum_{n=0}^{\infty} a_n x^n$ this expansion canbe used to slove differention equation involving $y(x)$

(Ex1) use elementary method to slove the follwing differentinal equation

$y' = y$

$\frac{dy}{dx} = y \rightarrow \frac{dy}{y} = dx \rightarrow \int \frac{1}{y} dy = \int dx \rightarrow \ln y + c = x$

$\ln y = x - c \rightarrow y = e^{x-c} = e^x e^{-c} = A e^x$, $\boxed{y = Ae^x}$ نجد القيمة A من الشرط الحدية $y(x=0) = 1$

(Ex2) use the method of series to slove the differentinal equation

$y' = y$

① let $y = \sum_{n=0}^{\infty} a_n x^n \rightarrow$ [convergent seris]

② $y' = \frac{d}{dx}(y) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) \rightarrow \sum_{n=0} a_n \frac{d}{dx}(x^n) \rightarrow \sum_{n=0} a_n (n)(x^{n-1})$

$(a_0 (0) x^{n-1}) = 0$ $\sum_{n=1}^{\infty} a_n x^{n-1} (n)$

الحد الأول = صفر لأنه ثابت بعد اشتقاقه

③ supstritute ① and ② out differention equation نعوض بمعادلة أعلاه

الطريقة هذه غير جيدة لأن نجعل كلا متسلسلتين تبدأ نفس العدد

$\sum_{n=1} n a_n x^{n-1} = \sum_{n=0} a_n x^n$

$\sum_{n=1} n a_n x^{n-1} - \sum_{n=0} a_n x^n = 0 \rightarrow \sum_{n=1}^{\infty} n a_n x^{n-1} - \left[a_0 x^0 + \sum_{n=1}^{\infty} a_n x^n\right] = 0$

$-a_0 x^0 + \sum_{n=1}^{\infty} (n a_n x^{n-1} - a_n x^n) = 0$

تعويض $y' = y$

او الحل يجعل الأس متشابهة

③ $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$

$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$ let $[m = n-1, n = m+1]$ نجعل الأس متشابه

$\sum_{m=0}^{\infty} (m+1) a_{(m+1)} x^m - \sum_{n=0}^{\infty} a_n x^n = 0$

$$\sum_{n=0}^{\infty}\left[(n+1)\,a_{n+1}\,x^n - a_n\,x^n\right] = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)\,a_{n+1} - a_n\right]x^n = 0$$

$$\sum_{1}(n+1)\,a_{(n+1)} - a_n = 0 \qquad a_{(n+1)} = \frac{a_n}{n+1}$$

$$a_{(n+1)} = \frac{a_n}{n+1}$$

$$a_1 = \frac{a_0}{1}, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{6}, \quad a_4 = \frac{a_3}{4} = \frac{a_0}{24}$$

$$\boxed{a_n = \frac{a_0}{n!}} \qquad y = \sum_{n=0}^{\infty} a_n x^n = \sum \frac{a_0\,x^n}{n!}$$

[4-2-2016]

[Ex p(564)] use seris solution method to slove equations

$$x^2y'' - 3xy' + 3y = 0$$

1- $y = \sum_{n=0}^{\infty} a_n x^n$

2- $y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1} n\,a_n\,x^{n-1}$

3- $y'' = \sum_{n=0}^{\infty} n(n-1)\,a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$

$$x^2\left[\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right] - 3x\left[\sum_{n=1}^{\infty} n\,a_n\,x^{n-1}\right] + 3\left[\sum_{n=0}^{\infty} a_n x^n\right] = 0$$

ادخال x على [illegible]

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^n - 3\sum_{n=1}^{\infty} n x^n a_n + 3\sum_{n=0}^{\infty} a_n x^n = 0$$

سوف نبدأ من 2 [illegible]

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^n - 3\left[(a_1 x^1) + \sum_{n=2}^{\infty} n\,x^n a_n\right] + 3\left[a_0 x^0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n\right] = 0$$

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^n - \cancel{3a_1 x} - 3\sum_{n=2}^{\infty} n x^n a_n + 3a_0 x^0 + \cancel{3a_1 x} + 3\sum_{n=2}^{\infty} a_n x^n = 0$$

$$3a_0 + \sum_{n=2}^{\infty}\left[n(n-1)\,a_n x^n - 3n\,a_n x^n + 3a_n x^n\right]$$

$$3a_0 + \sum_{n=2}^{\infty} x^n a_n\left[n(n-1) - 3n + 3\right]$$

$$3a_0 + \sum_{n=2}^{\infty} x^n a_n\left[n^2 - 4n + 3\right] = 0x^0 + 0x^1 + 0x^2 + \cdots$$

$$a_0 = 0 \qquad n^2 - 4n + 3 = 0 \qquad y = \sum_{n=0}^{\infty} a_n x^n$$

القيم مسموحة بها $n = (1, 3) \longrightarrow (a_1, a_3)$ $\qquad y = \left[a_1 x_1 + a_3 x^3\right]$

التي تجعل معادلة صحيحة

* Homework * p(564) section 1

1- $xy' = xy + y$

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$xy' = xy + y$

$x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) = x\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^n$

$x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^n\right) - \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$ $\quad m-1=n,\ m=1$

$\sum_{n=1}^{\infty} n a_n x^n - \sum_{m=1}^{\infty} a_{(m-1)} x^m - \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} a_{(n-1)} x^n - a_0 - \sum_{n=1}^{\infty} a_n x^n = 0$

$-a_0 + \sum_{n=1}^{\infty} (n a_n - a_{(n-1)} - a_n) x^n = 0$

$a_0 = 0 \qquad n a_n - a_{(n-1)} - a_n = 0 \qquad a_n(n-1) = a_{(n-1)}$

$a_0 = a_1(0) = 0$

$a_1 = a_2(1) \quad \boxed{a_2 = a_1}$

$a_2 = 2a_3$

$a_3 = \frac{a_2}{2} = \boxed{\frac{a_1}{2} = a_3}$

$a_3 = a_4(3) \Rightarrow \frac{a_3}{3} = a_4$

$a_4 = \frac{a_1}{6}$

$a_4 = 4a_5 \Rightarrow a_5 = \frac{a_4}{4}$

$a_5 = \frac{a_1}{6 \cdot 4}$

$a_1 x + \frac{a_1}{2!}x^2 + \frac{a_1}{3!}x^3 + \frac{a_1}{4!}x^4 + \cdots = a_1 \sum_{n=1}^{\infty} \frac{x^n}{n!} \Rightarrow y = a_1 e^x$

2- $y' = 3x^2 y$

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y' = 3x^2 y$

$\sum_{n=1}^{\infty} n a_n x^{n-1} - 3x^2 \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=1}^{\infty} n a_n x^{n-1} - 3\sum_{n=0}^{\infty} a_n x^{n+2} = 0 \xrightarrow{m-3=n} \sum_{n=1}^{\infty} n a_n x^{n-1} - 3\sum_{m=3}^{\infty} a_{(m-3)} x^{m-1} = 0$

$\sum_{n=1}^{\infty} n a_n x^{n-1} - 3\sum_{n=3}^{\infty} a_{(n-3)} x^{n-1} = 0 \rightarrow a_1 + 2a_2 x + \sum_{n=3}^{\infty} x^{n-1}\left[n a_n - 3a_{(n-3)}\right]$

$a_1 = 0 \qquad a_2 = 0$

$n a_n - 3a_{(n-3)} = 0 \qquad a_{(n-3)} = \frac{n a_n}{3}$

$a_0 = \frac{3a_3}{3} \quad \boxed{a_0 = a_3}$

$\boxed{a_1 = 0}$

$a_1 = \frac{4a_4}{3} \qquad a_1 = \frac{4}{3}a_4$

$a_2 = \frac{5a_5}{3} \Rightarrow a_2 = \frac{5}{3}a_5$

$\boxed{a_2 = 0}$

$a_3 = \frac{6a_6}{3} \qquad a_3 = 2a_6 \quad \Rightarrow a_6 = \frac{a_0}{2}$

$a_6 = \frac{9a_9}{3} \Rightarrow a_6 = 3a_9 \Rightarrow a_9 = \frac{a_0}{2 \cdot 3} = \frac{a_0}{6}$

$a_0 + \frac{a_0 x^3}{2!} + \frac{a_0 x^6}{3!} +$

$a_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} \rightarrow y = a_0 e^{x^3}$

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[Home Work]

3- $xy' = y$

$y = \sum_{n=0}^{\infty} a_n x^n$ $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$x \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n \longrightarrow x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0 \longrightarrow \sum_{n=1}^{\infty} n a_n x^n - a_0 - \sum_{n=1}^{\infty} a_n x^n = 0$

$-a_0 + \sum_{n=1}^{\infty} x^n a_n (n-1)$ $n-1 = 0$, $n = 1 \longrightarrow a_1$

$a_0 = 0$ $y = a_1 x$

4) $y'' = -4y$

$y = \sum_{n=0}^{\infty} a_n x^n$ $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ $y'' = \sum_{n=2} n(n-1) a_n x^{n-2}$ $m+2 = n$

$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -4 \sum_{n=0}^{\infty} a_n x^n \Longrightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$

$m+2 = n$

$\sum_{m=0}^{\infty} (m+2)(m+1) a_{(m+2)} x^m + 4 \sum_{n=0}^{\infty} a_n x^n = 0 \Longrightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} + 4 a_n\right) x^n = 0$ $\frac{(n+2)(n+1) a_{n+2}}{(n+2)(n+1)} = \frac{-4 a_n}{(n+1)(n+2)}$ $\boxed{a_{(n+2)} = \frac{-4 a_n}{(n+1)(n+2)}}$

(0) $a_2 = \frac{-4 a_0}{2} \Rightarrow a_2 = -2 a_0$ | (2) $a_4 = \frac{-4 a_2}{3 \times 4}$ $a_4 = -\frac{1}{3} a_2 \Rightarrow +\frac{2}{3} a_0$

(1) $a_3 = \frac{-4 a_1}{2 \cdot 3} \Rightarrow a_3 = \frac{-2}{3} a_1$ | (3) $a_5 = \frac{-4 a_3}{4 \cdot 5}$ $a_5 = \frac{-a_3}{5} \Rightarrow \frac{2}{15} a_1$

$y = -2a_0 - \frac{2}{3} a_1 x + \frac{2}{3} a_0 x^2 + \frac{2}{15} a_1 x^3$

5) $y' = y$

$y = \sum_{n=0}^{\infty} a_n x^n$

10) $a_{m+2} = \frac{-a_m [-4m - 2] + 4 a_{m-2}}{(m+2)(m+1)}$

8)

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6) $y'' - 2y' + y = 0$

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$k+2 = n$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{(k+2)} x^k - 2\sum_{m=0}^{\infty} (m+1) a_{(m+1)} x^m + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} x^n - 2\sum_{n=0}^{\infty} (n+1) a_{(n+1)} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)\, a_{(n+2)} - 2(n+1)\, a_{(n+1)} + a_n\right) x^n = 0$$

$$\frac{(n+2)(n+1)}{(n+2)(n+1)} a_{(n+2)} = \frac{2(n+1)\, a_{(n+1)} - a_n}{(n+2)(n+1)} \longrightarrow a_{(n+2)} =$$

(0) $a_2 = a_1 - \frac{a_0}{2}$

$a_3 = \frac{\cancel{2}(2)\, a_2}{\cancel{2}\cdot 3} - \frac{a_1}{2\cdot 3}$

(1) $a_3 = \frac{a_2}{3} - \frac{a_1}{6}$

$\frac{2a_1}{6} - \frac{a_0}{6} - \frac{a_1}{6} = \boxed{a_3 = \frac{a_1}{6} - \frac{a_0}{6}}$

(2) $a_4 = \frac{\cancel{2}\cdot \cancel{3}\, a_3}{\cancel{3}\cdot 4\cdot 2} - \frac{a_2}{3\cdot 4}$

$a_4 = \frac{a_3}{2} - \frac{a_2}{12}$

$a_4 = \frac{a_1}{12} - \frac{a_0}{12\times 6} - \frac{a_1}{12} + \frac{a_0}{24}$

$\boxed{a_{n+2} = \frac{2(n+1)\, a_{(n+1)}}{(n+2)(n+1)} - \frac{a_n}{(n+2)(n+1)}}$

$a_1 - \frac{a_0}{2} + \left(\frac{a_1}{6} - \frac{a_0}{6}\right) x^3 + \frac{a_0}{72} x^4 + \ldots$

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8) $(x^2 + 2x)\, y'' - 2(x+1)\, y' + 2y = 0$

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$(x^2 + 2x)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2(x+1)\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=2}^{\infty} n(n-1) a_n x^n + 2\sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} - 2\sum_{n=1}^{\infty} n a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^{n-1}$
$+ 2\sum_{n=0}^{\infty} a_n x^n = 0 \qquad m+1 = n \qquad m+1 = n$

$\sum_{n=2} n(n-1) a_n x^n + 2\sum_{m=1}^{\infty} (m+1) m\, a_{(m+1)} x^m - 2\sum_{n=1}^{\infty} n a_n x^n - 2\sum_{m=0}^{\infty} (m+1) a_{(m+1)} x^m + 2\sum_{n=0}^{\infty} a_n x^n = 0$

$\sum n(n-1) a_n x^n + 4a_2 x + 2\sum_{n=2} (n+1) n\, a_{(n+1)} x^n - 2a_1 x - 2\sum_{n=2} n a_n x^n - 2a_1 - 4a_2 x$
$2\sum_{n=2} (n+1) a_{n+1} x^n + 2a_0 + 2a_1 x + 2\sum_{n=2} a_n x^n = 0$

$\cancel{4a_2} - 2a_1 - \cancel{4a_2} + 2a_1 = 0 \rightarrow$

$x^0 (-2a_1 + 2a_0) = \qquad \boxed{a_1 = a_0}$

$n(n-1)\, a_n + 2(n+1) n\, a_{n+1} - 2n a_n - 2(n+1)\, a_{n+1} + 2a_n \big] x^n = 0$

$(n^2 - n - 2n + 2)\, a_n + (2n^2 + 2n - 2n - 2)\, a_{n+1} = 0$

$(n^2 - 3n + 2)\, a_n + (2n^2 - 2)\, a_{n+1}$

$$\boxed{a_{n+1} = \frac{n^2 - 3n + 2}{2n^2 - 2}\, a_n}$$

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9) $(x^2 + 1)\, y'' - 2xy' + 2y = 0$

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$(x^2+1)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$

$m+2 = n$

$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=2} n(n-1) a_n x^n + 2a_2 + 6a_3 x + \sum_{n=2} (n+1)(n+2) a_{n+2} x^n - 2a_1 x - 2\sum n a_n x^n + 2a_0$
$2a_1 x + 2\sum_{n=0} a_n x^n = 0$

$2a_2 + 2a_0 = 0 \qquad \boxed{a_2 = -a_0}$

$6a_3 - \cancel{2a_1} + \cancel{2a_1} = 0 \qquad \boxed{a_3 = 0}$

$$\sum_{n=2} \left[n(n-1) a_n + (n+2)(n+1) a_{n+2} - 2n a_n + 2a_n \right] x^n = 0$$

$[n^2 - n - 2n + 2]\, a_n = -(n+2)(n+1)\, a_{n+2}$

$\frac{(n^2 - 3n + 2)\, a_n}{-(n+1)(n+2)} = \frac{-(n+2)(n+1)\, a_{n+2}}{-(n+2)(n+1)}$

$$\boxed{a_{n+2} = \frac{(n-1)(n+2)}{(n+1)(n+2)}\, a_n}$$

$a_4 = 0$

$[y(x) = (1 - x^2)\, a_0 + a_1 x]$

$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$a_0 + a_1 x + (a_0 x^2) + 0 \ldots$

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** 7-2-2016 **

① [Legenders equations]

$$\boxed{(1-x^2)y'' - 2xy' + L(L+1)y = 0}$$ $y' = \frac{dy}{dx}$; $y'' = \frac{d^2y}{dx^2}$, $L \to$ positive integer.

* معادلة تفاضلية من درجة الثانية
تحتاج إلى شرطين لنستطيع حلها (شروط الحدية)

Ex) Which of the following belongs to the Legender equations

① $(1-x^2)y'' - 2xy' + 5y = 0$ ✗ $L(L+1)$

② $(1-x^2)y'' - 2xy' = 0$ ✓ لأنه معادلة تحتوي على $(L+1)L$

③ $(1-x^2)y'' - 2xy' + 90y = 0$ ✓ لأنه معادلة تحتوي على $L(L+1)$, $9(9+1)$

$$(1-x^2)y'' - 2xy' + L(1+L)y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n \quad ; \quad y' = \sum_{n=0}^{\infty} n a_n x^{(n-1)} = \sum_{n=1}^{\infty} n a_n x^{(n-1)}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{(n-2)} = \sum_{n=2}^{\infty} n(n-1) a_n x^{(n-2)}$$

$$(1-x^2)y'' - 2xy' + L(L+1)y = 0$$

$$(1-x^2)\left[\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right] - 2x\left[\sum_{n=1}^{\infty} n a_n x^{n-1}\right] + L(L+1)\left[\sum_{n=0}^{\infty} a_n x^n\right] = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + L(L+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2(2-1)a_2 x^0 + 3(3-1)a_3 x + \sum_{n=4}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2\sum_{n=2}^{\infty} n a_n x^n - 2a_1 x + L(L+1)a_0 + L(L+1)a_1 x + L(L+1)\sum_{n=2}^{\infty} a_n x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=4}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2\sum_{n=2}^{\infty} n a_n x^n - 2a_1 x + L(L+1)a_0 + L(L+1)a_1 x + L(L+1)\sum_{n=2}^{\infty} a_n x^n = 0$$

$$-2a_2 + L(L+1)a_0 = 0 \qquad \boxed{a_2 = -\frac{L(L+1)}{2} a_0}$$

$$6a_3 x - 2a_1 x + L(L+1)a_1 x = 0 \qquad \boxed{a_3 = \frac{2 - L(L+1)}{6} a_1}$$

$$2a_2 + 6a_3 x + \sum_{n=4}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2\sum_{n=2}^{\infty} n a_n x^n - 2a_1 x + L(L+1)a_0 + L(L+1)a_1 x + L(L+1)\sum_{n=2}^{\infty} a_n x^n = 0$$

$n = m+2$, $n = 2$

$$\sum_{m=2}^{\infty} (m+2)(m+1) a_{(m+2)} x^m$$

$$+6a_3 x - 2a_1 x + L(L+1)a_0 + L(L+1)a_1 x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{(n+2)} - n(n-1)a_n - 2na_n + L(L+1)a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} + (-n^2 - n + 2n + L(L+1))a_n = 0$$

$$(n+2)(n+1)a_{n+2} = (n^2 + n) - L(L+1)a_n = 0 \longrightarrow a_{n+2} = \frac{-n(n+1) - L(L+1)}{(n+2)(n+1)} a_n$$

$$a_{n+2} = \frac{-n^2 + n + L^2 + L}{(n+2)(n+1)} a_n \longrightarrow \boxed{a_{n+2} = \frac{-(L-n)(L+n+1)}{(n+2)(n+1)} a_n}$$

6

* [Legendres equation] **

$(1-x^2)y'' - 2xy' + l(l+1)y = 0 \qquad y(x) = \sum_{n=0}^{\infty} a_n x^n$ — $\left[\begin{matrix} a_0 \\ a_1 \end{matrix}\right.$ Boundary condition

$a_2 = \frac{-l(l+1)}{2} a_0 \;, \quad a_3 = \frac{2-l(l+1)}{6} a_1$

$$\boxed{a_{n+2} = \frac{n(n-1) - l(l+1)}{(n+2)(n+1)} a_n} \qquad \boxed{a_{n+2} = \frac{-(l-n)(l+n+1)}{(n+2)(n+1)}}$$

---

* The solution of the Legendres equation is

$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n+1} x^{n+1} + a_{n+2} x^{n+2}$

$a_0 + a_1 x + \left[\frac{-l(l+1)}{2} a_0\right] x^2 + \left[\frac{2-l(l+1)}{6} a_1 x^2\right] + \cdots + \left[\frac{-(l-(n-1))(l+(n-1)+1)}{((n-1)+2)((n-1)+1)} a_{n-1}\right] x^{n+1} + (\cdots) x^{n+2}$

$$\boxed{a_{n+2} = \frac{-(l-n)(l+n+1)}{(n+2)(n+1)} a_n}$$

* Forms of the series we notice the Legendres solution can divided into two point
The even part [depends on a_0] , The odd part [depends on a_1]

* Special cases for Legendres equation

a) $l=0 \qquad y = \underbrace{a_0}_{\text{from B.C}} + \underbrace{a_1}_{\text{B.C}} x + a_2 x^2 + a_3 x^3 + \cdots$

$a_2 = \frac{(l-0)(l+0+1)}{2} a_0 = \left[\frac{1 \cdot 1}{2}\right] a_0 = 0 \;, \quad a_4 = \frac{(l-2)(l+3)}{(2+2)(2+1)} a_2 = \left[\frac{(-2)(3)}{12}\right] = \frac{\cdot}{12} =$

$a_3 = \frac{-(l-1)(l+2)}{6} a_1 = \frac{1}{2} a_1$

$a_5 = \left[\frac{-(l-3)(l+3+1)}{(3+2)(3+1)} a_3\right] = \frac{-(-3)(4)}{5 \cdot 4} a_3 = \frac{12}{20} a_3 = \frac{6}{10} a_3 \rightarrow \frac{6}{10} \cdot \frac{1}{3} a_1 = \frac{1}{5} a_1$

$(a_2, a_4, a_6, a_8) \to$ الحدود الزوجية = صفر ، الحدود الفردية لها قيم

$y(x) = \underbrace{a_0}_{\text{الحدود الزوجية}} + \underbrace{\sum_{n=1}^{\infty} a_n x^n}_{\text{divergent series}}$ الحدود الفردية $\quad l=0$

الحدود الفردية $\left[a_1 + \frac{1}{3} a_1 + \frac{1}{5} a_1 + \frac{1}{7} a_1 \ldots\right] \quad \sum_{n=1}^{\infty} a_n x^n$ divergent series

سوف يتم استغناء عن هذا الجزء — لا نستطيع التحكم بمجموعها

$l=1$

$y(x) = a_1 x + \sum_{\substack{n=0 \\ \text{even}}}^{\infty} a_n x^n$ divergent series, use ratio test $\qquad y(x) = a_1 x$

$l=2$

$y(x) = a_0 + a_2 x^2 + \sum_{n=1}^{\infty} a_n x^n$ divergent series

$y(x) = a_0 + a_2 x^2$

$l=3$

$a_1 x + a_3 x^3 + \sum_{\substack{n=0 \\ \text{even}}}^{\infty} a_n x^n$ divergent series

The solution of Legender equation a polynomails called Legendreis polynomials denoted by $\boxed{P_l(x)}$

$P_0(x) = a_0$ $\quad$ $P_2(x) = a_0' + a_2 x^2$

$P_1(x) = a_1 x$ $\quad$ $P_3(x) = a_1' x + a_3 x^3$

With the condition $P_l(1) = 1$ | $P_1(1) = 1 = a_1(1) = , a_1 = 1$

So $a_0 = P_0(x) = a_0$ | $P_1(x) = x$

$P_0(1) = 1 = a_0$, $a_0 = 1$

$P_2(x) = a_0' + a_2 x^2 \longrightarrow P_2(1) = a_0' + a_2(1) = 1$ $\quad$ $a_{n+2}' = \frac{(l-n)(l+n+1)}{(n+2)(n+1)} a_n$

$a_0' + a_2 = 1$ ← ①

$a_0' + -3a_0' = 1$ $\quad$ $a_2 = \frac{-(2-0)(2+0+1)}{2} a_0'$

$-2a_0' = 1$

$\boxed{a_0' = \frac{-1}{2}}$ $\quad$ $a_2 = \frac{-6}{2} a_0'$ $\quad$ $\boxed{a_2 = -3a_0'}$

$\frac{-1}{2} + a_2 = 1 + \frac{1}{2}$

$\boxed{a_2 = \frac{3}{2}}$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = a_1' x + a_3 x^3$ $\quad$ $P_3(1) = a_1(1) + a_3(1) = 1$ $\quad$ $a_3 = \frac{-(l-1)(l+2)}{(3)(2)} a_1$

$a_1' + a_3 = 1$

$\frac{3}{3}a_1 - \frac{5}{3}a_1 = 1$ $\quad$ $a_3 = \frac{-(3-1)(3+2)}{6} a_1$

$\frac{-3}{2} \cdot -\frac{2}{3}a_1 = 1$ $\quad$ $a_3 = -\frac{2 \cdot 5}{3 \cdot 2} a_1 = -\frac{5}{3} a_1$

$\boxed{a_1' = \frac{-3}{2}}$ $\quad$ $\boxed{a_3 = -\frac{5}{3} a_1}$

$\frac{+3}{2} - \frac{3}{2} + a_3 = \frac{2}{2} + \frac{3}{2}$

$\boxed{a_3 = \frac{5}{2}}$

$P_3(x) = \frac{-3}{2} x + \frac{5}{2} x^3$

[Legendre's equation] **

[11-2-2016]

$\frac{d}{dx}\left[(1-x^2)y'\right] - 2xy' + l(l+1)y = 0$ $\quad$ $a_{n+2} = \frac{-(l-n)(l+1+n)}{(n+2)(n+1)}$

$y(x) = P_l(x) = a_0 + a_2 x^2 + a_4 x^4 + \dots + a_l x^l$ $\quad$ l (even)

$y(x) = P_l(x) = a_1 x + a_3 x^3 + a_5 x^5 + \dots + a_l x^l$ $\quad$ l (odd)

Ex) Find the solution for the following Legendre's equations

① $(1-x^2)y'' - 2xy' + 6y = 0$ $\quad$ $l(l+1) = 6$ $\quad$ $2(2+1) = 6$ $\quad$ $l = 2$

The solution is $P_l(x) = P_2(x) = a_0 + a_2 x^2$ $\quad$ $P_l(1) = 1$

$P_2(1) = \underbrace{a_0 + a_2 = 1}$ ---① $\quad$ $\left[P_2(x) = -\frac{1}{2} + \frac{3}{2}x^2 = \frac{1}{2}(3x^2 - 1)\right]$

$a_2 = \frac{-(2)(2+1)(3)}{2} a_0$ $\quad$ $a_0 + a_2 = 1$ $\quad$ $-2a_0 = 1$

$a_{n+2} = \frac{-(l-n)(l+n+1)}{(n+2)(n+1)} a_n$ ---② $\quad$ $\boxed{a_2 = -3a_0}$ $\quad$ $\boxed{a_0 = \frac{-1}{2}}$

$\frac{+1}{2} - \frac{1}{2} + a_2 = 1 + \frac{1}{2}$ $\quad$ $\boxed{a_2 = \frac{3}{2}}$

---

② solve $(1-x^2)y'' - 2xy' + 12y = 0$ $\quad$ $l(l+1) = 12$ $\quad$ $3(3+1) = 12$ $\quad$ $\boxed{l = 3}$

$P_l(x) = P_3(x) = a_1 x + a_3 x^3$ $\quad$ $P_3(1) = 1 = \underbrace{a_1 + a_3 = 1}$ ---① $\quad$ $a_3 = \frac{-(3-1)(3+1+1)}{(1+1)(1+2)} a_1$

$a_3 = \frac{-10}{6} = \boxed{-\frac{5}{3}a_1 = a_3}$

$a_3 = -\frac{5}{3} a_1$

$\frac{3}{3}a_1 - \frac{5}{3}a_1 = 1$ $\quad$ $-\frac{2}{3}a_1 = 1$ $\quad$ $\boxed{a_1 = \frac{-3}{2}}$ $\quad$ $a_3 = \frac{+5}{3} \times \frac{3}{2}$ $\quad$ $\boxed{a_3 = \frac{5}{2}}$

$\boxed{P_3(x) = -\frac{3}{2}x + \frac{5}{2}x^3} \Rightarrow \frac{1}{2}(5x^3 - 3x)$

* [Leibniz rule for differentional] *

Given: $\frac{d^n}{dx^n}[uy] = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{d^i}{dx^i} u\right)\left(\frac{d^{n-i}}{dx^{n-i}} v\right)$; $\binom{n}{i} = \frac{n!}{(n-i)!\,(i)!}$

function — function — any function

المشتقة نوعية لأي عدد — لو بدلنا بين u و v ينتج الاختلاف

0 → sin x
1 → cos x
2 → −sin x
3 → −cos x

* Find $\frac{d^{71}}{d^{71}x}[x \sin x]$ *

$\sum_{i=0}^{71} \binom{71}{i} \frac{d^i}{d^i x}(x) \frac{d^{71-i}}{d^{71-i}}(\sin x)$

$\binom{71}{0} \frac{d^0}{d^0 x}(x) \frac{d^{71}}{d^{71}(x)}(\sin x) + \binom{71}{1}\frac{d^1}{d^1 x}(x)\frac{d^{70}}{d^{70}(x)}(\sin x) + \binom{72}{2}\frac{d^2}{d^2 x}(x)\frac{d^{69}}{d^{69}(x)}\sin(x) + \binom{71}{3}\frac{d^3}{d^3 x}(x)\frac{d^{68}}{d^{68}x}\sin x$

$-x\cos x + 71(-\sin x) + \left(\frac{71 * 70}{2}\right)(0)(\cos x) + \frac{71 * 70 * 69}{6}(0) + \dots$

$\boxed{= -x\cos x - 71 \sin x}$

* [Home work] *

[p(567); section 2] ... $P_4(x) = a_0 + a_2 x^2 + a_4 x^4 \rightarrow \frac{3}{8} - \frac{30}{8}x^2 + \frac{35}{8}x^4$

1. Find the $P_4(x)$ $L = 4$ $P_L(1) = 1$

$a_0 + a_2 + a_4 = 1$

$a_{(n+2)} = -\frac{(L-n)(L+n+1)}{(n+1)(n+2)} a_n$

$a_2 = \frac{-(4)(5)}{2} a_0 \rightarrow [a_2 = -10 a_0]$

$a_4 = -\frac{(4-2)(4+2+1)}{(3)(4)} a_2$

$a_4 = -\frac{(2)(7)}{3 \cdot 4} a_2$

$a_4 = -\frac{7}{6} a_2$

$a_0 + -10a_0 + \frac{7}{6} \cdot 10 a_0 = 1$

$a_0\left(\frac{3}{3} - \frac{30}{3} + \frac{35}{3}\right) = 1 \rightarrow a_0\left(\frac{8}{3}\right) = 1$

$\boxed{a_0 = \frac{3}{8}}$ $\boxed{a_4 = \frac{35}{8}}$

$a_2 = -10 \cdot \frac{3}{8} \rightarrow \boxed{a_2 = -\frac{30}{8}}$ $a_4 = \frac{+7}{6} \cdot 10 \cdot \frac{3}{8} = \frac{35}{8}$

2. Show that $P_L(-1) = (-1)^L$

P_L is even function if L is even and odd if L odd

$f(-x) = f(x)$ even

$f(-x) = -f(x)$ odd

if L is even

$P_L(-x) = P_L(x)$

$P_L(-1) = P_L(x) = 1$

if L odd

$P_L(-x) = -P(x)$

$P_L(-1) = -P(1) = -1$

$\therefore P_L(-1) = (-1)^L$

[p(568); section 3]

$\left[\sum_{i=0}^{n} \binom{n}{i} \frac{d^i}{d^i x}(u) \frac{d^{n-i}}{d^{n-i}x}(w)\right] \rightarrow$ قاعدة

Find the following derivatives

2. $\frac{d^{10}}{d^{10}x}(xe^x) \longrightarrow \sum_{i=0}^{10} \binom{10}{i} \frac{d^i}{d^i x}(x) \frac{d^{10-i}}{d^{10-i}x}(e^x)$

$\binom{10}{0}(x)e^x + \binom{10}{1}(1)e^x + \binom{10}{2}(0)(e^x) \longrightarrow xe^x + 10e^x$

5. $\frac{d^{100}}{dx^{100}}\left[x^2 e^{-x}\right]$ $\sum_{i=0}^{\infty} \binom{100}{i} \frac{d^i}{d^i x}(x^2) \frac{d^{n-i}}{dx^{n-i}}(e^{-x})$

$\binom{100}{0}\frac{d^0}{d^0x}(x^2)\frac{d^{100}}{dx^{100}}(e^{-x}) + \binom{100}{1}\frac{d^1}{d^1x}(x^2)\frac{d^{99}}{dx^{99}}(e^{-x}) + \binom{100}{2}\frac{d^2}{d^2x}(x^2)\frac{d^{98}}{d^{98}}(e^{-x}) + \binom{100}{3}\frac{d^3}{d^3x}(x^3)\frac{d^{97}}{dx^{97}}(e^{-x})$

$(1) \cdot x^2(e^{-x}) + 100 \cdot 2x(-e^{-x}) + 2 \cdot (50)(99)(e^{-x}) + 0$

$= x^2e^{-x} - 200xe^{-x} + 9900e^{-x} \rightarrow e^{-x}\left[x^2 - 200x + 9900\right]$

* [Rodrigues formula] *

$$\boxed{P_L(x) = \frac{1}{2^L} \cdot \frac{1}{L!} \frac{d^L}{d^L x}(x^2-1)^L}$$

$(1-x^2)y'' - 2xy' + L(L+1)y = 0 \longrightarrow (1-x^2)P_L''(x) - 2xP_L'(x) + L(L+1)P_L(x) = 0$

[w] [P(569) : section 4] Find $P_0(x), P_1(x), P_2(x), P_3(x), P_4(x)$ from Rodrigue's Formula.

① $P_0(x) = \frac{1}{2^0}\frac{1}{0!}\frac{d^0}{d^0x}(x^2-1)^0 \rightarrow 1$ $\quad P_0(x) = 1$

② $P_1(x) = \frac{1}{2^1}\frac{1}{1!}\frac{d^1}{d^1x}(x^2-1)^1 = \frac{1}{2}(2x) = x$ $\quad P_1(x) = x$

③ $P_2(x) = \frac{1}{2^2}\frac{1}{2!}\frac{d^2}{d^2x}(x^2-1)^2 = \frac{1}{4}\cdot\frac{1}{2}(12x^2-4)$

$\frac{1}{8}\cdot 4(3x^2-1) = \frac{1}{2}(3x^2-1)$

* $\frac{d^2}{d^2x}(x^2-1)^2$
$y' = 2(x^2-1)(2x) = (4x)(x^2-1)$
$y'' = 4x(2x) + (x^2-1)(4) = 12x^2 - 4$

④ $P_3(x) = \frac{1}{2^3}\cdot\frac{1}{3!}\frac{d^3}{d^3x}(x^2-1)^3$

$\frac{1}{8}\cdot\frac{1}{6}\, 48x(x^2-1) + 48x^3 + 24x(x^2-1)$

$\frac{1}{8}\, 2x\left[2x^2-2+2x^2+x^2-1\right]$

$= \frac{1}{2}x\left[5x^2-3\right] \longrightarrow \frac{5}{2}x^3 - \frac{3}{2}x$

* $\frac{d^3}{d^3x}(x^2-1)^3$

$y' = (x^2-1)^2(6x)$
$y'' = (x^2-1)(24x^2) + 6(x^2-1)^2$
$y''' = 48x(x^2-1) + 48x^3 + 24x(x^2-1)$

$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$

⑤ $P_4(x) = \frac{1}{2^4}\frac{1}{4!}\frac{d^4}{d^4x}(x^2-1)^4$

* $\frac{d^4}{d^4x}(x^2-1)^4$

$$(1+x^2)\frac{\partial^2\Phi}{\partial^2 x} + 2x\frac{\partial\Phi}{\partial x} + h\frac{\partial^2}{\partial h^2}(h\,\Phi) = 0$$

① check this for $\Phi(x,h)$

$$\Phi = \frac{1}{\sqrt{1-2xh+h^2}} = (1-2xh+h^2)^{-1/2}$$

$$\frac{\partial\Phi}{\partial x} = \left(\frac{-1}{2}\right)(1-2xh+h^2)^{-3/2}(-2h) = h(1-2xh+h^2)^{-3/2}$$

$$\frac{\partial^2\Phi}{\partial^2 x} = h\left(\frac{-3}{2}\right)(1-2xh+h^2)^{-5/2}(-2h) = 3h^2(1-2xh+h^2)^{-5/2}$$

$$\frac{\partial\Phi}{\partial h} = h\left[\frac{-1}{2}(-2x+2h)(1-2xh+h^2)^{-3/2}\right] + (1-2xh+h^2)^{-1/2} = (hx-h^2)(1-2xh+h^2)^{-3/2} + (1-2xh+h^2)^{-1/2}$$

$$\frac{\partial^2\Phi}{\partial^2 h} = (hx+h^2)\left(\frac{-3}{2}\right)(1-2xh+h^2)^{-5/2}(-2x+2h) + (1-2xh+h^2)^{-3/2}(x-2h) + \frac{-1}{2}(1-2xh+h^2)^{-3/2}(-2x+2h)$$

$$(1-x^2)\frac{\partial^2\Phi}{\partial^2 x} - 2x\frac{\partial\Phi}{\partial x} + h\frac{\partial^2}{\partial^2 h}(h\Phi) = 0$$

$$\Phi(x,h) = \sum_{L=0}^{\infty} h^L p_L(x) \qquad h\Phi = \sum_{L=0}^{\infty} h^{L+1} p_L(x) \qquad = \sum_{L=0}^{\infty}(L+1)L\,h^{L-1}p_L(x)$$

$$(1-x^2)\sum_{L=0}^{\infty} h^L p_L''(x) - 2x\sum_{L=0}^{\infty} h^L p' + \sum_{L=0}^{\infty} L(L+1)h^L p_L = 0$$

$$\sum_{L=0}^{\infty} h^L\left[\underbrace{(1-x^2)p_L'' - 2xp_L' + L(L+1)p_L}_{\text{Legendre polynomials}}\right] = 0$$

[16-2-2016]

[Recursion Relation]

Legendr's polynamials is related to each by are lation amoving than called recursion relations

$$① \; L P_L(x) = (2L-1)x\, P_{L-1}(x) - (L-1) P_{L-2}(x)$$

$$② \; x P_L'(x) - P_{L-1}'(x) = L P_L(x)$$

$$③ \; P_L'(x) - x P_{L-1}'(x) = L P_{L-1}(x)$$

$$④ \; (1-x^2) P_L'(x) = L P_{L-1}(x) - L x P_L(x)$$

$$⑤ \; (2L+1) P_L(x) = P_{L+1}'(x) - P_{L-1}'(x)$$

$$⑥ \; (1-x^2) P_{L-1}'(x) = L x P_{L-1}(x) - L P_L(x)$$

ex. $L=2 \quad P_0 = 1 \quad P_1 = x$ find P_2 use Recursion Relation ① ?

$$L P_L(x) = (2L-1)x\, P_{L-1}(x) - (L-1) P_{L-2}$$

$$2 P_2(x) = (2(2)-1) x\, P(x) - (2-1) P_0$$

$$2 P_2(x) = 3x P_1 - P_0 \longrightarrow 2P_2(x) = 3x^2 - 1$$

$$P_2 = \frac{1}{2}\left[3x^2 - 1\right]$$

*proof ①

$$\Phi = \frac{1}{\sqrt{1-2xh+h^2}} \; ; \; \frac{\partial \Phi}{\partial h} = \left(\frac{-1}{2}\right)(1-2xh+h^2)^{-\frac{3}{2}}(-2x+2h)$$

$$\frac{\partial \Phi}{\partial h} = \frac{x-h}{(1-2xh+h^2)^{3/2}}$$

$$(1-2xh+h^2)\frac{\partial \Phi}{\partial h} = (x-h)\,\Phi \left(\frac{1}{\sqrt{1-2xh+h^2}}\right) \; \text{But } \Phi = \sum_{L=0}^{\infty} h^L P_L$$

$$\frac{\partial \Phi}{\partial h} = \sum_{L=0}^{\infty} L(h)^{L-1} P_L = \sum_{L=1}^{\infty} L h^{L-1} P_L$$

$$(1-2xh+h^2)\sum_{L=1}^{\infty} L h^{L-1} P_L = (x-h)\sum_{L=0}^{\infty} h^L P_L$$

نسحب حدين $\underbrace{\sum_{L=1}^{\infty} L h^{L-1} P_L}_{h^0 \text{ نسحب حدين}} - 2x \underbrace{\sum_{L=1}^{\infty} L h^L P_L}_{\text{نسحب حد واحد}} + \underbrace{\sum_{L=1}^{\infty} L P_L h^{L+1}}_{\text{تبقى كما هي}} = x \underbrace{\sum_{L=0}^{\infty} h^L P_L}_{\text{نسحب حدين}} - \underbrace{\sum_{L=0}^{\infty} h^{L+1} P_L}_{\text{نسحب حد واحد}}$

$$1h^0 P_1 + 2h^1 P_2 + \sum_{L=3}^{\infty} L h^{L-1} P_L - 2x h^1 P_1 - 2x\sum_{L=2}^{\infty} L h^L P_L + \sum_{L=1}^{\infty} L P_L h^{L+1} = x h^0 P_0 + x h^1 P_1 + x\sum_{L=2}^{\infty} h^L P_L - h^1 P_0 - \sum_{L=1}^{\infty} h^{L+1} P_L$$

$$h^0 P_1 = x h^0 P_0 \rightarrow \boxed{P_1 = x P_0}$$

$$2h^1 P_2 - 2x h^1 P_1 = x h^1 P_1 - h^1 P_0$$

$$h^1(2P_2 - 2xP_1) = h^1(xP_1 - P_0)$$

$$2P_2 - 3xP_1 = -P_0 \qquad 2P_2 = 3xP_1 - P_0$$

$$\boxed{P_2 = \frac{1}{2}(3xP_1 - P_0)}$$

تابع

$$\sum_{l=3}^{\infty} l h^{l-1} p_l - 2x \sum_{l=2}^{\infty} l h^{l-1} p_{l-1} + \sum_{l=1}^{\infty} l h^{l+1} p_l = x \sum_{l=2}^{\infty} h^l p_l - \sum_{l=1}^{\infty} h^{l+1} p_l$$

$m+3=l$, $m+2=l$, $m+1=l$, $m=l-2$, $m=l-1$

$$\sum_{m=0}^{\infty} (m+3) h^{m+2} p_{(m+3)} - 2x \sum_{m=0}^{\infty} (m+2) h^{m+2} p_{(m+2)} + \sum_{m=0}^{\infty} (m+1) h^{m+2} p_{(m+1)} = x \sum_{m=0}^{\infty} h^{(m+2)} p_{(m+2)} - \sum_{m=0}^{\infty} h^{(m+2)} p_{m+1}$$

$$\sum_{m=0}^{\infty} h^{m+2} \left[(m+3) p_{m+3} - 2x(m+2) p_{(m+2)} + (m+1) p_{(m+1)} = x p_{m+2} - p_{m+1}\right]$$

$$(m+3) p_{m+3} - 2x p_{m+2} m - 4x\, p_{m+2} - x p_{m+2} + p_{m+1} + m p_{m+1} + p_{m+1} = 0$$

$$(m+3) p_{m+3} - 2x m p_{m+2} - 5x p_{m+2} + 2p_{m+1} + m p_{m+1} = 0$$

$$(m+3) p_{m+3} - x(2m+5) p_{m+2} + (m+2) p_{m+1} = 0 \longrightarrow (m+3) = x(2m+5) p_{m+2} - (m+2) p_{m+1}$$

$\boxed{\begin{matrix} m+3=l \\ m=l-3 \end{matrix}}$

$$l\, p_l = x(2l-6+5) p_{l-3+2} - (l[illegible]+2) p_{l-3+1} \longrightarrow \left[l p_l = x(2l-1) p_{l-1} - (l-1) p_{l-2}\right]$$

[18-2-2016]

Legendre's equation $\left[(1-x^2)y'' - 2xy' + l(l+1)y\right]$

The solution $\left[y = \sum_{n=0}^{\infty} a_n x^n\right]$; $\left[a_{n+2} = -\frac{(l+n+1)(l-n)}{(n+2)(n+1)} a_n\right]$

For l even the odd sum is divergent [neglected]

For l odd the even sum is divergent [neglected]

The solution called Legendre's polynomial $p_l(x)$ = with $p_l(1) = 1$

* Rodrgues formula $p_l(x) = \frac{1}{2^l\, l!} \frac{d^l}{d^l x} (x^2-1)^l$

$$\Phi(x,h) = \frac{1}{\sqrt{1-2xh+h^2}} = \sum_{l=0}^{\infty} p_l(x) h^l$$

Function of Legender's polynomial

P(574)

8) $5-2x \longrightarrow C_0P_0 + C_1P_1 + \underbrace{C_2P_2 + C_3P_3}$ + ...

$5-2x = C_0P_0 + C_1P_1 = C_0(1) + C_1(x)$ ملغية لعدم التساوي في القوى

$C_0 = 5$ $C_1 = -2$ $\therefore 5-2x = 5P_0 - 2P_1$

* $6x - 3x^2 \longrightarrow \underbrace{C_0P_0 + C_1P_1 + C_2P_2} + \cancel{C_3P_3}$ ...

$C_0P_0 + C_1P_1 + C_2P_2$

$C_0(1) + C_1(x) + C_2(\frac{1}{2}(3x^2-1)) \longrightarrow C_0 + C_1x + \frac{3}{2}x^2C_2 - \frac{1}{2}C_2$

$(C_0 - \frac{1}{2}C_2) + C_1x + \frac{3}{2}C_2x^2$

$C_0 - \frac{1}{2}C_2 = 0$ $C_1 = 6$

$C_0 = \frac{1}{2}C_2$ $\frac{2}{3}\times\frac{\cancel{3}}{2}C_2 = -\cancel{3}\times\frac{2}{\cancel{3}}$

$C_0 = \frac{1}{2}\times -2$ $\boxed{C_2 = -2}$

$\boxed{C_0 = -1}$

$C_0P_0 + C_1P_1 + C_2P_2$

$= -P_0 + 6P_1 - 2P_2$

[Complete sets of orthogonal Function] المجموعة التامة من الدوال المتعامدة

Two function are said to be orthogonal in the Integral $[a,b]$ if $\int_a^b A(x)\,B(x)\,dx = 0$

Ex) $\{\sin mx \;;\; m \in N\}$ $\int_a^b \sin mx \,\sin nx\, dx = 0$ $n \neq m$

$\int_0^{2\pi} \sin x \,\sin 2x\, dx$

The set of function $\{f(x)\}$ is said to be complate on a given intravel if they spans their space

Legend're's polynemils $\{P_L(x)\}$ are ① complete ② orthogonal on the interval $[-1,1]$ That's

$\boxed{\int_{-1}^{1} P_L(x)\,P_n(x)\,dx = 0}$ $[L \neq m]$

$f(x) = \sum_{L=0}^{\infty} C_L P_L(x)$

$\int_{-1}^{1} P_L(x)\,P_L(x)\,dx = \frac{2}{2L+1}$

* the Legendre's polynoimals are normalizable using $\longrightarrow \acute{P}_L(x) = \sqrt{\frac{2(L+1)}{2}}\,P_L(x)$

Ex) Evalute the following integral

$\int_{-1}^{1} \underbrace{7(3x^2-1)}_{P_2}\;\underbrace{x}_{P_1}\,dx = 0$

Ex) Expand the function $5-2x$ using Legender polynamils

$f(x) = \sum_{L=0}^{\infty} C_L P_L(x)$ where C_L If we dot product with P $\int_{-1}^{1} f(x)\,P_m(x)\,dx$

$= \sum_{L=0}^{\infty} C_L \int_{-1}^{1} P_L(x)\,P_m(x)\,dx \Rightarrow \int_{-1}^{1} f(x)\,P_m(x)\,dx = \text{Zero} = \frac{2C_m}{2m+1}$

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section 5] #

5- Differentiate the recursion relation (5.8a) and use the recursion relation (5.8b) with l replace by $(l-1)$ to prove the recursion relation (5.8c) $\rightarrow$ $P_l'(x) - xP_{l-1}'(x) = lP_{l-1}(x)$

(5.8a)] $lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x)$; [5.8]b) : $xP_l'(x) - P_{l-1}'(x) = lP_l(x)$ $\quad l = l-1$

$lP_l'(x) = (2l-1)xP_{l-1}'(x) + (2l-1)P_{l-1}(x) - (l-1)P_{l-2}'(x)$; $xP_{(l-1)}'(x) - P_{(l-2)}'(x) = (l-1)P_{l-1}(x)$

$\boxed{P_{l-2}'(x) = xP_{l-1}'(x) - (l-1)P_{l-1}(x)}$

$lP_l'(x) = (2l-1)xP_{(l-1)}'(x) + (2l-1)P_{(l-1)}(x) - \left[(l-1)\left[xP_{(l-1)}' - (l-1)P_{(l-1)}(x)\right]\right]$

$lP_l'(x) = (2l-1)xP_{(l-1)}'(x) + (2l-1)P_{(l-1)}(x) - (l-1)xP_{(l-1)}' + (l-1)^2P_{(l-1)}(x)$

$lP_l'(x) = \underbrace{2lxP_{(l-1)}'(x)} - \cancel{xP_{(l-1)}'(x)} + 2l\cancel{P_{l-1}(x)} - \cancel{P_{l-1}(x)} - \underbrace{lxP_{(l-1)}'(x)} + \cancel{xP_{l-1}'(x)} + l^2P_{l-1}(x) - 2l\cancel{P_{l-1}(x)} + \cancel{P_{l-1}(x)}$

$lP_l(x) = lxP_{(l-1)}'(x) + l^2P_{l-1}(x) \longrightarrow \frac{l}{l}\left(P_l'(x) - xP_{l-1}'(x)\right) = \frac{l^2}{l}P_{l-1}(x)$

$\therefore P_l'(x) - xP_{l-1}'(x) = lP_{l-1}(x)$ ---(5.8c)

---

6) From (5.8b) and (5.8c) obtain (5.8d) and (5.8f)

$(P_{l-1}'(x))$ بالتعويض ; $xP_l(x)$ بالتعويض ; (5.8b) $\rightarrow xP_l'(x) - P_{l-1}'(x) = lP_l(x) \rightarrow xP_l'(x) - lP_l(x) = P_{l-1}'(x)$

5.8:d] $(1-x^2)P_l'(x) = lP_{l-1}(x) - lxP_l(x) \longrightarrow \frac{d}{dx}$ $(1-x^2)P_l''(x) + P_l'(x)(-2x) = lP_{l-1}'(x) - xlP_l'(x) - lP_l(x)$

$(1-x^2)P_l''(x) - 2xP_l'(x) = \cancel{lxP_l'(x)} - l^2P_l(x) - \cancel{xlP_l'(x)} - lP_l(x) \longrightarrow (1-x^2)P_l''(x) - 2xP_l'(x) = -(l^2+l)P_l(x)$

$(1-x^2)P_l''(x) - 2xP_l'(x) + l(l+1)P_l(x) = 0$ [Legender equation]

ضرب b بـ x

نطرح b - d ← c

---

7) write (5.8c) with l replaced by $l+1$ and use it ot eliminate the $xP_l'(x)$ terms in (5.8b). You should get (5.8e)

[5.8c] $P_l'(x) - xP_{l-1}'(x) = lP_{(l-1)}(x) \longrightarrow P_{l+1}'(x) - x\underbrace{P_l'(x)} = (l+1)P_l(x)$; $xP_l' - P_{l-1}'(x) = lP_l(x)$

$xP_l' = lP_l(x) + P_{l-1}'(x)$

$P_{l+1}'(x) - lP_l(x) - P_{l-1}'(x) = lP_l(x) + P_l(x)$

$lP_{l+1}'(x) - P_{l-1}'(x) = 2lP_l(x) + P_l(x) \longrightarrow P_{(l+1)}'(x) - P_{l-1}'(x) = (2l+1)P_l(x)$

9- $3x^2 + x - 1 \rightarrow [c_0P_0 + c_1P_1 + c_2P_2]$ c_3P_3 ; $0P_0 + P_1(1) + (2)P_2 = P_1 + 2P_2$

$c_0P_0 + c_1P_1 + c_2P_2 \longrightarrow c_0(1) + c_1(x) + c_2(\frac{1}{2}(3x^2-1))$

$c_0 + c_1x + \frac{3}{2}c_2x^2 - \frac{1}{2}c_2 \longrightarrow (c_0 - \frac{1}{2}c_2) + c_1x + \frac{3}{2}c_2x^2$

$\frac{3}{2} = \frac{3}{2}c_2 = 3x\frac{2}{3}$; $c_0 - \frac{1}{2}c_2 = -1$; $\boxed{c_1 = 1}$

$\boxed{c_2 = 2}$; $c_0 - \frac{1}{2}\times 2 = -1$

$c_0 - 1 = -1 + 1$ $\boxed{c_0 = 0}$

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$$\text{(3)}\; x^5 = \left[c_0P_0 + c_1P_1 + c_2P_2 + c_3P_3 + c_4P_4 + c_5P_5\right] c_6P_6$$

$$c_0(1) + c_1(x) + c_2\left(\tfrac{1}{2}(3x^2-1)\right) + c_3\left(\tfrac{1}{2}(5x^3-3x)\right) + c_4\left(\tfrac{1}{8}(35x^4-30x^2+3)\right) + c_5\left(\tfrac{1}{8}(63x^5-70x^3+15x)\right)$$

$$c_0 + c_1x + \frac{3}{2}x^2c_2 - \frac{1}{2}c_2 + \frac{5}{2}x^3c_3 - \frac{3}{2}xc_3 + \frac{35}{8}c_4x^4 - \frac{30}{8}x^2c_4 + \frac{3}{8}c_4 + \frac{63}{8}c_5x^5 - \frac{70}{8}x^3c_5 + \frac{15}{8}xc_5$$

$$\left(c_0 - \frac{1}{2}c_2 + \frac{3}{8}c_4\right) + \left(c_1 - \frac{3}{2}c_3 + \frac{15}{8}c_5\right)x + \left(\frac{3}{2}c_2 - \frac{30}{8}c_4\right)x^2 + \left(\frac{5}{2}c_3 - \frac{70}{8}c_5\right)x^3 + \frac{35}{8}c_4x^4 + \frac{63}{8}c_5x^5$$

$$c_0 - \frac{1}{2}c_2 + \frac{3}{8}c_4 = 0 \qquad \frac{5}{2}c_3 - \frac{70}{8}c_5 = 0 \qquad \frac{3}{2}c_2 = \frac{30}{8}c_4 \qquad \frac{35}{8}c_4 = 0 \qquad \cancel{\frac{8}{63}}\cancel{\frac{63}{8}}c_5 = 1 \times \frac{8}{63}$$

$$c_0 = 0 \qquad \frac{2}{5}c_3 - \frac{70}{8}\times\frac{8}{63} = 0 \qquad c_2 = 0 \qquad \boxed{c_4 = 0} \qquad \boxed{c_5 = \frac{8}{63}}$$

$$c_1 - \frac{3}{2}c_3 + \frac{15}{8}c_5 = 0 \qquad \frac{5}{2}\times\frac{2}{5}c_3 = \frac{70}{63}\times\frac{5}{2}$$

$$c_1 - \frac{3}{2}\times 3 + \frac{15}{8}\times\frac{8}{63} = 0 \qquad c_3 = \frac{175}{63} = 3$$

$$c_1 - 4.5 + 0.24$$

$$\boxed{c_1 = 4.26}$$

$$x^5 = \cancel{c_0P_0} + c_1P_1 + \cancel{c_2P_2} + c_3P_3 + \cancel{P_4c_4} + P_5c_5$$

$$X^3 = 4.26P_1 + 3P_3 + \frac{8}{63}P_5$$

[23 - 2 - 2016]

[The Associated Legender Equations]

$$(1-x^2)y'' - 2xy' + \left(l(l+1) - \frac{m^2}{1-x^2}\right)y = 0 \qquad \left[l\ Eq - \frac{m^2}{1-x^2}y\right] \qquad 0 < m \le l$$

The solution $\left[y = (1-x^2)^{m/2} \frac{d^m}{d^m x} P_l(x)\right]$ when $m=0 \rightarrow y = P_l(x)$

Ex) solve the following diff. eq

① $(1-x^2)y'' - 2xy' + \left[2 - \frac{1}{1-x^2}\right]y = 0 \rightarrow$ This equation the Associated Equation with $l=1, m=1$

The solution is $P_1^1(x) = (1-x^2)^{1/2} \frac{d^1}{d^1x} P_1(x)$

$= \sqrt{(1-x^2)} \frac{d}{dx}(x) = \sqrt{(1-x^2)}$

Using the Rodrigues formula: $\left[P_l(x) = \frac{1}{2^l l!} \frac{d^l}{d^l x}(x^2-1)^l\right]$

$$\boxed{P_l^m(x) = \frac{1}{2^l l!}(1-x^2)^{m/2} \frac{d^{l+m}}{d^{l+m}(x)}(x^2-1)^l}$$

Ex) Find using Rodrigues formula

$P_2^1(x) = \frac{1}{2^2\, 2!}(1-x^2)^{\frac{1}{2}} \frac{d^3}{d^3x}(x^2-1)^2$

$\frac{1}{\cancel{8}}\sqrt{(1-x^2)}\,(\overset{3}{\cancel{24}}x)$

$\frac{d}{dx}(x^2-1)^2 \rightarrow 2(x^2-1)(2x) = (4x)(x^2-1)$

$\frac{d^2}{d^2x}(4x)(x^2-1) \rightarrow (4x)(2x) + (x^2-1)4 = 8x^2 + 4x^2 - 4$

$\frac{d^3}{d^3x}(8x^2) + (4x^2-4) = 16x + 8x = 24x$

$P_2^1(x) = 3x\sqrt{(1-x^2)}$

Ex) solve the equation $(1-x^2)y'' - 2xy' + \left(6 - \frac{4}{1-x^2}\right)y = 0 \qquad l(l+1) = 6 \rightarrow l=2 \quad m=2$

$y = (1-x^2)^{\frac{2}{2}} \frac{d^2}{d^2x} P_2(x) \rightarrow (1-x^2)\frac{d^2}{d^2x}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$

$\frac{d}{dx}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) = \cancel{2}\frac{3}{\cancel{2}}x = 3x$

$\frac{d}{dx}(3x) = 3$

$y = (1-x^2)\cdot 3 \rightarrow 3(1-x^2)$

[Normalizations of the Ass. L function]

نمط أسئلة الامتحان

$\int_{-1}^{1} 3\sqrt{(1-x^2)} - (1-x^2)$

$0 = \left(\begin{matrix} m' \ne m \\ l \ne l' \end{matrix}\right) \int_{-1}^{1} P_1^1(x) - P_2^2(x) = 0$

$P_l^m(x)$

$$\int_{-1}^{1} P_l^m(x)\, P_{l'}^{m'}(x)\, dx = \int_{ll'}\int_{mm'} \left[\frac{2}{2l+1} * \frac{(m+l)!}{(l-m)!}\right]$$

$\int_{ll'} = 0 \quad l \ne l'$

$= 1 \quad l = l'$

Ex) Evalute the integral $\int_{-1}^{1} P_6^3(x)\, P_6^3(x)\, dx \rightarrow l = 6 \quad m=3$

$$= \left[\frac{2}{2\cdot 6+1} * \frac{(3+6)!}{(6-3)!}\right] = \frac{2}{13} * \frac{9!}{3!} = \frac{\cancel{2}}{13} * \frac{9!}{\cancel{2}\cdot 3\cdot 1} = \frac{9!}{39}$$

[Generlized power series [The Method of Froberous]]

Diff... Eq $\quad y = \sum_{n=0}^{\infty} a_n x^n$

In some case the solution is not polynomial for example [The solution may have negative power]

Ex) $\frac{\cos x}{x} = \frac{1}{x} - \frac{x}{2} + \frac{x^2}{3} + \ldots$

or in some case the solution may contain fraction power of x

Ex) $\sqrt{x}\sin x = x^{3/2} - \frac{x^{7/2}}{3!} + \frac{x^{11/2}}{5!} + \ldots$

$\underbrace{c_0 p_0}_{zero} + c_1 p_1 + \underbrace{c_2 p_2}_{zero} + c_3 p_3 + \underbrace{c_4 p_4}_{zero} + \ldots$

[HW] problems section 9 $\qquad \frac{3}{2}p_1 - \frac{7}{8}p_3 + \ldots$

Expand the following function in legendre series

① $f(x) = \begin{cases} -1, & -1<x<0 \\ 1, & 0<x<1 \end{cases}$ $\qquad \frac{\int_{-1}^{1} f(x)\, p_m(x)\, dx}{\frac{2}{2m+1}} = \frac{\int_{-1}^{0}(-1)\, P_m(x)\, dx}{\frac{2}{2m+1}} + \frac{\int_{0}^{1}(1)\, P_m(x)\, dx}{\frac{2}{2m+1}}$

$c_0 = \frac{-\int_{-1}^{0} P_0(x)\,dx}{2} + \frac{\int_0^1 P_0(x)\,dx}{2}$ $\qquad c_1 = \frac{\int_{-1}^{0} -P_1(x)\,dx}{\frac{2}{3}} + \frac{\int_0^1 P_1(x)\,dx}{\frac{2}{3}}$

$c_0 = \frac{-\int_{-1}^{0} 1\,dx}{2} + \frac{\int_0^1 1\,dx}{2}$ $\qquad c_1 = \frac{3}{2}\left[-\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_0^1\right]$

$c_0 = \frac{-(0+(+1))}{2} + \frac{(1-0)}{2}$ $\qquad c_1 = \frac{3}{2}\left[0+\frac{1}{2}\right] + \left(\frac{1}{2}\right)$

$c_0 = -\frac{1}{2} + \frac{1}{2} = \text{zero}$ $\qquad c_1 = \frac{3}{4} + \frac{3}{4} = \frac{6}{4} = \frac{3}{2}$

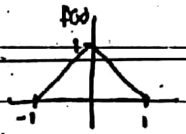
$c_3 = \frac{\int_{-1}^{0} -\frac{5}{2}x^3 + \frac{3}{2}x\,dx + \int_0^1 \frac{5}{2}x^3 - \frac{3}{2}x\,dx}{\frac{2}{7}}$

$c_3 = \frac{-\frac{5}{2}\frac{x^4}{4} + \frac{3}{2}\frac{x^2}{2}\Big|_{-1}^{0} + \frac{5}{2}\frac{x^4}{4} - \frac{3}{2}\frac{x^2}{2}\Big|_0^1}{\frac{2}{7}}$

$c_3 = \left(0 - \left(-\frac{5}{8} + \frac{3}{4}\right) + \frac{5}{8} - \frac{3}{4}\right)\frac{7}{2}$

$c_3 = \left(\frac{5}{8} - \frac{3}{4} + \frac{5}{8} - \frac{3}{4}\right)\frac{7}{2} \longrightarrow \left(\frac{10}{8} - \frac{12}{8}\right) = -\frac{2}{8}\cdot\frac{7}{2} = -\frac{7}{8}$

③



Expan of the following polynomials in a legendre series $\underbrace{c_0 p_0}_{zero} + c_1 p_1 + \underbrace{c_2 p_2}_{zero} + c_3 p_3 + \ldots$

12- $x - x^3 \longrightarrow \frac{\int_{-1}^{1} f(x)\, p_m(x)\, dx}{\frac{2}{2m+1}} \longrightarrow \frac{2}{5}p_1 + \ldots$

$\frac{\int_{-1}^{1}(x-x^3)\, P_m(x)}{\frac{2}{2m+1}} \Rightarrow c_0 = \frac{\int_{-1}^{1}(x-x^3)\cdot P_0(x)\,dx}{\frac{2}{1}} = \frac{\int_{-1}^{1}(x-x^3)}{2} = \frac{\frac{x^2}{2} - \frac{x^4}{4}\Big|_{-1}^{1}}{2} = \text{zero}$

$c_1 = \frac{\int_{-1}^{1}(x-x^3)(x)}{\frac{2}{3}} = \frac{\int_{-1}^{1} x^2 - x^4\,dx}{\frac{2}{3}} = \frac{\frac{x^3}{3} - \frac{x^5}{5}\Big|_{-1}^{1}}{\frac{2}{3}} = \frac{\left(\frac{1}{3} - \frac{1}{5}\right) - \left(-\frac{1}{3} + \frac{1}{5}\right)}{\frac{2}{3}} = \frac{\frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{5}}{\frac{2}{3}}$

$c_1 = \frac{\frac{5\times2}{5\times3} - \frac{2\times3}{5\times3}}{\frac{2}{3}} = \frac{\frac{10}{15} - \frac{6}{15}}{\frac{2}{3}} = \frac{4}{15}\cdot\frac{3}{2} = \frac{2}{5}$ $\qquad c_2 = \text{zero}$

13
$c_3 = \frac{\int_{-1}^{1}(x-x^3)\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)}{\frac{2}{7}} = \frac{7}{2}\int_{-1}^{1}\left(\frac{5}{2}x^4 - \frac{3}{2}x^2 - \frac{5}{2}x^6 + \frac{3}{2}x^4\,dx\right) \rightarrow \left(\frac{5}{2}\frac{x^5}{5} - \frac{3}{2}\frac{x^3}{3} - \frac{5}{2}\frac{x^7}{7} + \frac{3}{2}\frac{x^5}{5}\right]\Big)\frac{7}{2}$

$c_3 = \frac{7}{2}\left[\frac{x^5}{2} - \frac{x^3}{3} - \frac{5x^7}{14} + \frac{3}{10}x^5\right]_{-1}^{1} \Rightarrow \frac{7}{2}\left[\frac{1}{2} - \frac{1}{2} - \frac{5}{14} + \frac{3}{10} + \left(\frac{1}{2}\right) - \frac{1}{2} - \frac{5}{14} + \frac{3}{10}\right) \cdot \left(\frac{5}{7}\frac{10}{14} - \frac{3}{10}\right)\left(\frac{50}{70} - \frac{21}{70}\right)$

$\frac{3}{70}\cdot\frac{7}{2} =$

[HW] p.584 (section 10)

$(10.5) \Rightarrow [(1-x^2)u'' - 2xu'(m+1) + [l(l+1) - m(m+1)]u = 0$

$+[(1-x^2)u''' - 2(m+1)xu'' + [l(l+1) - (m+1)(m+2)]u'$

1- Verify equation (10.3) and (10.4)

$y = (1-x^2)^{m/2} u$

$y' = (1-x^2)^{\frac{m}{2}} u' + u\left(-2x(1-x^2)^{\frac{m}{2}-1}\right)\frac{m}{2}$

$y'' = u'(1-x^2)^{\frac{m}{2}-1}(-2x)\frac{m}{2} + (1-x^2)^{\frac{m}{2}} u'' + (1-x^2)^{\frac{m}{2}-1}(mu + u'x)$

$y'' = -2xu'm(1-x^2)^{\frac{m}{2}-1} + (1-x^2)^{\frac{m}{2}}u'' + (m-2)x^2 m u (1-x^2)^{\frac{m}{2}-2} + (u + u'x)(1-x^2)^{\frac{m}{2}-1}$

verify (10.3) $\ast \left[(1-x^2)y'' - 2xy' + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y = 0\right]$

$(1-x^2)(-2xu'm(1-x^2)^{\frac{m}{2}-1} + \dots) - 2x(u'(1-x^2)^{\frac{m}{2}} - umx(1-x^2)^{\frac{m}{2}-1}) + \left(l(l+1) - \frac{m^2}{1-x^2}\right)(1-x^2)^{m/2}u$

$-2xu'm(1-x^2)^{\frac{m}{2}} + (1-x^2)^{\frac{m}{2}+1}u'' + (m-2)x^2mu(1-x^2)^{\frac{m}{2}-1} + um(1-x^2)^{\frac{m}{2}} - u'x(1-x^2)^{\frac{m}{2}}$

$-2xu'(1-x^2)^{\frac{m}{2}} + 2x^2um(1-x^2)^{\frac{m}{2}-1} + l(l+1)(1-x^2)^{\frac{m}{2}}u - \frac{m^2}{1-x^2}(1-x^2)^{\frac{m}{2}}u$

$-2xu'(1-x^2)^{\frac{m}{2}} - mu(1-x^2)^{\frac{m}{2}} + 2x^2um(1-x)^{\frac{m}{2}-1} + \left(l(l+1) - \frac{m^2}{1-x^2}\right)(1-x)^{\frac{m}{2}}$

$(1-x^2)^{\frac{m}{2}+1}u'' + u'\left(-xm(1-x^2)^{\frac{m}{2}} - mx(1-x^2)^{\frac{m}{2}} - 2x(1-x^2)^{\frac{m}{2}}\right) + u\left(mx^2(m-2)(1-x^2)^{\frac{m}{2}-1} - m + \frac{2x^2m}{1-x^2} + l(l+1) - \frac{m^2}{1-x^2}\right) = 0$

$(1-x^2)^{\frac{m}{2}+1}u'' + u'x(1-x^2)^{\frac{m}{2}}(-2mx + 2x) - (1-x^2)^{\frac{m}{2}}u\left(\frac{(m-2)x^2m}{(1-x^2)} - m + \frac{2x^2m}{1-x^2} + l(l+1)\right) \div (1-x^2)^{\frac{m}{2}}$

$\frac{(1-x^2)^{\frac{m}{2}}}{(1-x^2)^{\frac{m}{2}}}\left[(1-x^2)u'' - (2(m+1))xu' - u\left[\frac{x^2m^2 - 2mx^2 + 2x^2m - m^2}{(1-x^2)}\right.\right.$

$(1-x^2)u'' - [2(m+1)xu' - u\left[\frac{-m^2(1-x^2)}{(1-x^2)} - m + l(l+1)\right] = 0$

$(1-x^2)u'' - [2(m+1)xu' + u[l(l+1) - m(m+1)]] = 0$ $\therefore$ the equation of (10.3)

$(1-x^2)u'' - [2(m+1)xu' + u[l(l+1) - m(m+1)]$

$\ast$ (10.4) $(1-x^2)u''' - 2(m+1)xu'' + u'[l(l+1) - m(m+1)]$

$(1-x^2)u''' + u''(-2x) - 2(m+1)xu'' + u'(2(m+1)) + u'(l(l+1)) - u'm(m+1) = 0$

$\underbrace{\quad}_{-2x - 2mx}$

$(1-x^2)u''' + u''(-2x - 2mx - 2x) + u'(2(m+1) - m(m+1) + l(l+1)) = 0$ $\therefore$ the equation of (10.4) $\ast$

$(1-x^2)u''' - u''2[(m+1) + 1]x\,u'((m+1)(m-2) + l(l+1)) = 0$

$\ast$ 2- The equation for the associated Legendre functions $\quad x = \cos\theta$

$\frac{dy}{dx} = \frac{d\theta}{dx}\frac{dy}{d\theta}$

$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]y = 0$

$x = \cos\theta \longrightarrow dx = -\sin\theta\, d\theta \to d\theta = -\frac{dx}{\sin\theta} \quad \frac{dy}{d\theta} = -\frac{dy}{dx}\sin\theta$

$\frac{1}{\sin\theta}\frac{d}{\frac{dx}{\sin\theta}}\left(\sin\theta\left(-\sin\theta\frac{dy}{dx}\right)\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)y = 0$

$\frac{\sin\theta}{\sin\theta}\frac{d}{dx}\left(\sin^2\theta\frac{dy}{dx}\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)y = 0 \longrightarrow \frac{d}{dx}\left((1-\cos^2\theta)\frac{dy}{dx}\right) + \left[l(l+1) - \frac{m^2}{1-\cos^2\theta}\right]y = 0$

$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y = 0 \longrightarrow (1-x^2)\frac{d^2y}{d^2x} + \frac{dy}{dx}(-2x) + \left[l(l+1) - \frac{m^2}{(1-x^2)}\right]y$

$(1-x^2)\frac{d^2y}{dx} - 2x\frac{dy}{dx} + \left[l(l+1) - \frac{m^2}{(1-x^2)}\right]y = 0$ The associated Legendre equation

---

4) $P_1^1(\cos\theta)$ $\quad m=1, \; l=1$

$P_l^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dm}P_l(x)$

$P_1(x) = x$

$\frac{d}{dx}P_1(x) = 1$

$P_1^1(\cos\theta) = (1-x^2)^{\frac{1}{2}}\frac{d^1}{d^1\theta}(x)$

$P_1^1(\cos\theta) = (1-\cos^2\theta)^{\frac{1}{2}} = (\sin^2\theta)^{\frac{1}{2}} = \sin\theta$

$$P_{(4)}(x) = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}$$

$$P'_4(x) = \frac{35}{8} 4x^3 - \frac{30}{4} \cdot 2x \qquad P'_4(\cos\theta) = \frac{35}{2}x^3 - \frac{15}{2}x$$

5) $P_4^1(\cos\theta)$, $m=1$, $L=4$ $\qquad P_L^m(x) = (1-x^2)^{m/2} \frac{d^m}{d^m x} P_L(x)$

$$P_4^1(\cos\theta) = (1-\cos^2\theta)^{\frac{1}{2}} \left[\frac{d^1}{d^1x} P_4(\cos\theta)\right.$$

$$P_4^1(\cos\theta) = (\sin^2\theta)^{\frac{1}{2}} \left.\left[\frac{35}{2}\cos^3\theta - \frac{15}{2}\cos\theta\right]\right]$$

$$P_4^1(\cos\theta) = \frac{1}{2}\sin\theta \left(35\cos^3\theta - 15\cos\theta\right)$$

$$P_4(x) = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}$$

$$\frac{dP(x)}{dx^4} = \frac{35}{8} 4x^3 - \frac{30}{8} 2x$$

$$\frac{d}{dx}P_4(x) = \frac{35x^3}{2} - \frac{15}{2}x$$

6) $P_3^2(\cos\theta)$, m, L $\qquad P_3^2(\cos\theta) = (1-x^2)^{\frac{2}{2}} \frac{d^2}{d^2x} P_3(x)$

$$P_3^2(\cos\theta) = (1-\cos^2\theta)^1 \left[15\cos\theta\right]$$

$$P_3^2(\cos\theta) = 15\sin^2\theta \cos\theta$$

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

$$\frac{d}{dx}P_3(x) = \frac{5}{2}3x^2 - \frac{3}{2}$$

$$\frac{d^2}{d^2x}P_3(x) = \frac{5}{2} 3 \cdot 2x = 15x$$

[problems: section 3]

6 verify problem! $\left(\frac{d^n}{d^n x}\right) d(uv)$

since $\frac{d}{dx}(uv) = D(uv) = (Du + Dv)(uv)$

where

$Du(uv) = \frac{du}{dx} v \qquad Dv(uv) = \frac{dv}{dx} u \Rightarrow \frac{d^n}{d^n x}(uv) = (Du + Dv)^n (uv)$

* using Bionomial theorem:

$$(x+y)^r = \binom{r}{0} x^r y^0 + \binom{r}{1} x^{r-1} y^1 + \binom{r}{2} x^{r-2} y^2 + \dots$$

$$x^r + r x^{r-1} y + \frac{r(r-1)}{2!} x^{r-2} y^2 + \dots$$

$$\sum_{i=0}^{r} \binom{r}{i} x^{r-i} y^i$$

* Let $Dv = x \qquad Du = y$

$$\therefore [Du + Dv]^n = \left[\frac{du}{dx} + \frac{dv}{dx}\right]^n = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{dv^{n-i}}{dx^{n-i}}\right)\left(\frac{du^i}{dx^i}\right)$$

Frobenius method

* The general method for solving diff. eq. using series solution is by using Forbenius method where instead of using the power series

$$\boxed{\sum_{n=0}^{\infty} a_n x^n \text{ we use } \sum_{n=0}^{\infty} a_n x^{n+s} \quad \text{SBC}}$$

example

$x^2y'' + 4xy' + (x^2+2)y = 0$

* Bessel's Equation :-

[1-3-2016]

$$\boxed{x^2 y'' + x y' + (x^2 - p^2) y = 0}$$

$y = \sum_{n=0}^{\infty} a_n x^n$ $\qquad$ $\underbrace{y = \sum_{n=0}^{\infty} a_n x^{n+s}}$

* The solution this called Bessel's Function wehe substatute $y = \sum_{n=0}^{\infty} a_n x^{n+s}$ the values s is $\boxed{s = \pm p}$

① The solution when $s = \pm p$ is called Bessel's function of the first kind of orther a and p by denoted $J_p(x)$

② The solution when $s = -p$ is called Bessel's function of the second kids of order p by denoted $J_{-p}(x)$

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n+p}$$

$$\boxed{\begin{array}{l} \Gamma(n+1) = n! \\ \Gamma(p+1) = p\,\Gamma(p) \end{array}}$$

any function

$$J_p(x) \quad \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n-p+1)} \quad (\ )^{-p}$$

* Write an expression for $J_0(x)$, $J_1(x)$, $J_{-1}(x)$

$$J_{(0)}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n)} \left(\frac{x}{2}\right)^{2n}$$

$$J_{(0)}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \frac{x^{2n}}{(n+1)!\, 2^{2n}}$$

$$J_{-1}(x) = \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+1)} \left(\frac{x}{2}\right)^{2n+1}$$

$$J_{-1}(x) = \frac{(-1)^n}{(n+1)!\,(n+1)!} \left(\frac{x}{2}\right)^{2n+1}$$

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)} \left(\frac{x}{2}\right)^{2n+1}$$

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!\,(n+2)!} \left(\frac{x}{2}\right)^{2n+1}$$

* Show that $J_{-p}(x) = (-1)^p J_p(x)$ for integral

$$(-1)^p \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

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$(-1)! = \mp\infty$

$$J_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+3)} \left(\frac{x}{2}\right)^{2n+2}$$

Show that $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$

$$= \frac{2}{x}\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)} \left(\frac{x}{2}\right)^{2n+1}\right] - \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+3)} \left(\frac{x}{2}\right)^{2n}\right]$$

$$= \frac{2}{x}\sum_{n=0}^{\infty} \frac{(-1)^n}{(n)!\,(n+1)!}\,\frac{x^{2n+1}}{2^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(n)!\,(n)!}\left(\frac{x}{2}\right)^{2n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n)!\,(n)!\,(n+1)}\,\frac{x^{2n}}{2^{2n}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(n)!\,n!}\,\frac{x^{2n}}{2^{2n}}\left(\frac{n+1}{n+1}\right)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2\,(n+1)}\left(\frac{x}{2}\right)^{2n} - \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{(n!)^2\,(n+1)}\left(\frac{x}{2}\right)^{2n}$$

$n!\,(n+1) = (n+1)!$

$$\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(n!)^2\,(n+1)}\,\frac{x^{2n}}{2^{2n}}\right](1-n-1) \longrightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2\,(n+1)}\,\frac{x^{2n}}{2^{2n}}(-n)$$

$$-\sum_{n=0}^{\infty} \frac{(-1)^n\,\cancel{n}}{\cancel{n}(n-1)!\,(n+1)!}\left(\frac{x}{2}\right)^{2n} \longrightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{(n-1)!\,(n+1)!}\left(\frac{x}{2}\right)^{2n}$$

$$\frac{(-1)^0}{(0-1)!\,(0+1)!}\left(\frac{x}{2}\right)^0 \longrightarrow \frac{1}{(-1)!\,(1)!} = 0$$

$m+1 = n$, $n-1 = m$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!\,(n+1)!}\left(\frac{x}{2}\right)^{2n} \longrightarrow \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+1-1)!\,(m+1+1)!}\left(\frac{x}{2}\right)^{2(m+1)}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m)!\,(m+2)!}\left(\frac{x}{2}\right)^{2m+2} \longrightarrow \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n)!\,(n+2)!}\left(\frac{x}{2}\right)^{2n+2} \longrightarrow \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\Gamma(n+1)\,\Gamma(n+3)}\left(\frac{x}{2}\right)^{2n+2}$$

[HW] section 12:

3) $J_1(x) + J_3(x) = (4/x)\,J_2(x)$

$$J_1(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)}\left(\frac{x}{2}\right)^{2n+1}$$

$$J_3(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+4)}\left(\frac{x}{2}\right)^{2n+3}$$

$$\frac{4}{x}\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+3)}\left(\frac{x}{2}\right)^{2n+2}$$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+3)}\,\frac{x^{2n+1}}{2^{2n}}\right] \longrightarrow \frac{4}{x} J_2(x)$$

$J_1(x) + J_3(x)$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)}\left(\frac{x}{2}\right)^{2n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+4)}\left(\frac{x}{2}\right)^{2n+3}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n)!\,(n+1)!}\,\frac{x^{2n+1}}{2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n)!\,(n+3)!}\,\frac{x^{2n+3}}{x^{2n+3}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n\,x^{2n+1}\,(n+3)(n+2)}{n!\,(n+3)(n+2)\,(n+1)!\,2^{2n}} + \sum_{n=0}^{\infty} \frac{(-1)^n\,x^{2n+3}}{n!\,(n+3)\,(n+2)\,(n+1)!\,2^{2n+3}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n\,x^{2n+1}}{2^{2n+1}\,n!\,(n+1)!\,(n+2)(n+3)}\left[(n+3)(n+2) + \frac{x^2}{4}\right]$$

$$\sum_{n=0}^{\infty}$$

4) $\frac{d}{dx}\left(J_0(x)\right) = -J_1(x)$ $\quad -J_1(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)} \left(\frac{x}{2}\right)^{2n+1}$

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1)} \left(\frac{x}{2}\right)^{2n} \longrightarrow \frac{d}{dx} J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,n!} \frac{2n \cdot x^{2n-1}}{2^{2n}}$$

$$\frac{(-1)^0\, x^{-1}}{(-1)!\,(0)!\,2^0} = \frac{1}{\infty} = 0 \qquad \sum_{n=0}^{\infty} \frac{(-1)^n\, 2\cancel{n}\, x^{2n-1}}{\cancel{n}(n-1)!\,(n)!\,2^{2n}} \longrightarrow \sum_{n=1}^{\infty} \frac{(-1)^n\, x^{2n-1}}{n!\,(n-1)!\,2^{2n-1}} \qquad m+1=n$$

$$\sum_{m=0}^{\infty} \frac{(-1)^{m+1}\, x^{2(m+1)-1}}{(m+1)!\,2^{2(m+1)-1}\,(m+1-1)!} \longrightarrow \sum_{n=0}^{\infty} \frac{(-1)^n(-1)\, x^{2n+1}}{(n+1)!\,2^{2n+1}\,(n)!} \longrightarrow -\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)} \left(\frac{x}{2}\right)^{2n+1}$$

$$\frac{d}{dx} J_0(x) = -J_1(x) \longrightarrow -\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)} \left(\frac{x}{2}\right)^{2n+1}$$

---

5) $\frac{d}{dx}\left[x J_1(x)\right] = x J_0(x) \longrightarrow x J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+1)} \frac{x^{2n+1}}{2^{2n}}$

$$\frac{d}{dx}\left(x \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+2)} \left(\frac{x}{2}\right)^{2n+1} \right) \longrightarrow \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+2}}{\Gamma(n+1)\,\Gamma(n+2)\,2^{2n+1}} \right)$$

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$$= \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n+2)\,x^{2n+1}}{\Gamma(n+1)\,\Gamma(n+2)\,2^{2n+1}} \longrightarrow \sum_{n=0}^{\infty} \frac{(-1)^n\, 2(n+1)\, x^{2n+1}}{(n)!\,(n+1)(n)!\,2^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(n)!\,(n)!\,2^{2n}}$$

$$\frac{d}{dx}\left[x J_1(x)\right] = x J_0(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+1)} \left(\frac{x}{2}\right)^{2n}$$

[Section 13]

1. using equation (12.9) and (13.1) which write out the first few terms of $J_0(x), J_1(x), J_{-1}(x)$, $J_2(x), J_{-2}(x)$. Show that $J_{-1}(x) = -J_1(x)$ and $J_{-2}(x) = J_2(x)$.

$$\left[J_p(x) = \frac{1}{\Gamma(1)\,\Gamma(1+p)} \left(\frac{x}{2}\right)^p + \frac{1}{\Gamma(2)\,\Gamma(2+p)} \left(\frac{x}{2}\right)^{p+2} + \frac{1}{\Gamma(3)\,\Gamma(3+p)} \left(\frac{x}{2}\right)^{3+p} + \cdots \right] \text{ equation (12.9)}$$

$$* \; J_0(x) = \frac{1}{\Gamma(1)\,\Gamma(1)} \left(\frac{x}{2}\right)^0 + \frac{1}{\Gamma(2)\,\Gamma(2)} \left(\frac{x}{2}\right)^2 + \frac{1}{\Gamma(3)\,\Gamma(3)} \left(\frac{x}{2}\right)^3 + \cdots \longrightarrow 1 + \frac{x^2}{4} + \frac{x^3}{32} + \cdots$$

$$* \; J_1(x) = \frac{1}{\Gamma(1)\,\Gamma(2)} \left(\frac{x}{2}\right)^1 + \frac{1}{\Gamma(2)\,\Gamma(3)} \left(\frac{x}{2}\right)^3 + \frac{1}{\Gamma(3)\,\Gamma(4)} \left(\frac{x}{2}\right)^4 + \cdots \longrightarrow \frac{x}{2} + \frac{x^3}{16} + \frac{x^4}{192} + \cdots$$

$$* \; J_2(x) = \frac{1}{\Gamma(1)\,\Gamma(3)} \left(\frac{x}{2}\right)^2 + \frac{1}{\Gamma(2)\,\Gamma(4)} \left(\frac{x}{2}\right)^4 + \frac{1}{\Gamma(3)\,\Gamma(5)} \left(\frac{x}{2}\right)^5 + \cdots \longrightarrow \frac{x^2}{16} + \frac{x^4}{96} + \frac{x^5}{1556} + \cdots$$

uation 1)

$$\left[J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n-p} \longrightarrow \frac{1}{\Gamma(1)\,\Gamma(-p+1)} \left(\frac{x}{2}\right)^{1-p} + \frac{1}{\Gamma(2)\,\Gamma(2-p)} \left(\frac{x}{2}\right)^{2-p} + \frac{1}{\Gamma(3)\,\Gamma(3-p)} \left(\frac{x}{2}\right)^{3-p} + \cdots \right.$$

$$J_{-1}(x) = \frac{1}{\Gamma(1)\,\Gamma(2)} \left(\frac{x}{2}\right)^1 + \frac{1}{\Gamma(2)\,\Gamma(3)} \left(\frac{x}{2}\right)^3 + \frac{1}{\Gamma(3)\,\Gamma(4)} \left(\frac{x}{2}\right)^4 + \cdots \longrightarrow \frac{x}{2} + \frac{x^3}{16} + \frac{x^4}{192} + \cdots$$

$$J_{-2}(x) = \frac{1}{\Gamma(1)\,\Gamma(3)} \left(\frac{x}{2}\right)^2 + \frac{1}{\Gamma(2)\,\Gamma(4)} \left(\frac{x}{2}\right)^4 + \frac{1}{\Gamma(3)\,\Gamma(5)} \left(\frac{x}{2}\right)^5 + \cdots \longrightarrow \frac{x^2}{16} + \frac{x^4}{96} + \frac{x^5}{1536} + \cdots$$

5-5-201[illegible]

* Section 13 *

2) Show that in general for integral n, $J_{-n}(x) = (-1)^n J_n(x)$ and $J_n(-x) = (-1)^n J_n(x)$

$$J_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(2n+1)} \left(\frac{x}{2}\right)^{3n} \qquad p=n$$

$J_{-n}(x) = (-1)^n J_n(x)$ ✗

$$J_{(-n)}(x) = \sum_{n=0} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(2n+1)} \left(\frac{x}{2}\right)^{2n+n} \longrightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(2n+1)} \left(\frac{x}{2}\right)^{3n}$$

$2n-p$, $n-p+1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(2n+1)} \left(\frac{x}{2}\right)^{3n} \neq (-1)^n \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(2n+1)} \left(\frac{x}{2}\right)^{3n}$$

$$J_n(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(2n+1)} \frac{(-x)^{3n}}{2^{3n}} \longrightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n}{\Gamma(n+1)\,\Gamma(2n+1)} \left(\frac{x}{2}\right)^{3n}$$

$$J_n(-x) = (-1)^n J_n(x) \longrightarrow \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\Gamma(n+1)\,\Gamma(2n+1)} \left(\frac{x}{2}\right)^{3n} = (-1)^n \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(2n+1)} \left(\frac{x}{2}\right)^{3n} \checkmark$$

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* [Recursion Relations] *

1. $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$
2. $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$
3. $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$
4. $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$
5. $J_p'(x) = -\frac{p}{x} J_p(x) + J_{p-1}(x) = \frac{p}{x} J_p(x) - J_{p+1}(x)$

$$\boxed{\text{* Neumann: } Y_p(x) = \frac{\cos(\pi p)\, J_p(x) - J_{-p}(x)}{\sin \pi p}}$$

* Other Kinds of Bessel Functions *

* Hankel Function or Bessel Function of Third Kind :-

$$\begin{bmatrix} H_p^{(1)}(x) = J_p(x) + i N_p(x) \\ H_p^{(2)}(x) = J_p(x) - i N_p(x) \end{bmatrix}$$

* Modified or Hyperbolic Bessel Function :-

$$\begin{bmatrix} I_p(x) = i^{-p} J_p(ix) \\ K_p(x) = \frac{\pi}{2} i^{p+1} H_p^{(1)}(ix) \end{bmatrix}$$

* Spherical Bessel Functions :-

$$\begin{bmatrix} j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{(2n+1)/2}(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right) \\ y_n(x) = \sqrt{\frac{\pi}{2x}}\, Y_{(2n+1)/2}(x) = -x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right) \\ h_n^{(1)} = j_n(x) + i y_n(x) \\ h_n^{(2)} = j_n(x) - i y_n(x) \end{bmatrix}$$

Hermite Function Hermite polynomials

The hermite diff. equation is given by $y_n'' - 2x y_n' + 2n y_n = 0$

The solution of this differential equation is given by the so called Hermite function

The solution may be obtained by way:

1. The series solution $y_x = \sum_{n=0}^{\infty} a_n x^{n+3}$
2. using operator as follows
3. Factorise the differential operation in the H. diff. eq

let $D = \frac{d}{dx}$

$(D - x)(D + x) y_n$

$= (D^2 + Dx - Dx - x^2) y_n \longrightarrow (D^2 - 1 - xD - x^2) y_n$

$\underbrace{y_n (D^2 - x^2) y_n}_{(-2n+1) y_n} + (1 - xD) y_n \longrightarrow -(2n+1) y_n + y_n - x D y_n$

$-2n y_n - x D y_n \quad (D - x)[D^2 - x^2] y_n$

$= -2(n+1)[D - x] y_n$

$(D^2 + x)(D - x) y_n$

$(D^2 - Dx + xD - x^2) y_n \longrightarrow (D^2 - x^2) y_n + (xD - 1) y_n \longrightarrow -(2n+1) - y_n + x D y_n$

$= -(2n+1) y_n + x D y_n \quad (D + x)[D^2 - x^2] y_n$

$-2m (D + x) y_n$

$(D + x) y_0 = 0$

$\frac{d}{dn} y_0 - x y_0 = 0 \longrightarrow y_0 = e^{-x^2/2}$

Ex) $y_1 = (D - x) y_0 \longrightarrow \left(\frac{d}{dx} - x\right) e^{-x^2/2}$

$= -x e^{-x^2} - x \longrightarrow y = -2x e^{-x^2/2}$

For unit polynomial

$y_0 = -e^{-x^2/2} \qquad y_1 = +2x \cdot e^{-x^2/2}$

Hermite polynomials $= e^{x^2/2} (-1)^n$ Hermit Function

$y_0 = 1 \qquad y_2 = 4x^2 - 2$

$y_1 = 2(2x) = 4x$

[HW] section 15

1- prove equation (15.2) by method similar to the one used to prove (15.1)

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$$

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = \frac{d}{dx}\left[x^{-p}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(p+1+n)} * \frac{x^{2n+p}}{2^{2n+p}}\right)\right]$$

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+p+1)} \cdot \frac{x^{2n}}{2^{2n+p}}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2n}{\Gamma(n+1)\,\Gamma(n+p+1)} \frac{x^{2n-1}}{2^{2n+p}} \rightarrow = \sum_{n=1}^{\infty} \frac{(-1)^n\, 2n}{\Gamma(n+1)\,\Gamma(n+p+1)} \frac{x^{2n-1}}{2^{2n+p}}$$

let $m+1=n$

$$\sum_{m=0}^{\infty} \frac{(-1)^{m+1}\, 2(m+1)}{\Gamma(m+2)\,\Gamma(m+p+1)} \frac{x^{2(m+1)-1}}{2^{2(m+1)+p}} = \sum_{m=0}^{\infty} \frac{(-1)^m (-1)}{\Gamma(m+2)\,\Gamma(m+p+2)} * \frac{x^{2m+1}}{2^{2m+2+p}} (2(m+1))$$

$$(-1)\sum_{m=0}^{\infty} \frac{(-1)^m\, 2\cancel{(m+1)}}{\cancel{(m+1)}(n)!\,\Gamma(m+p+2)} \frac{x^{2m+1}}{2^{2m+p+2}} \rightarrow (-1)\sum_{n=0}^{\infty} \frac{(-1)^n}{(n)!\,\Gamma(n+p+2)} \frac{x^{2n+1}}{2^{2n+p+1}} =$$

$$-1\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+p+2)} \frac{x^{2n+1}}{2^{2n+p+1}} * \frac{x^p}{x^p} \rightarrow -x^{-p}\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+p+2)} \frac{x^{2n+p+1}}{2^{2n+p+1}}$$

2- Salove equation (15.1) and (15.2) for $J_{p+1}(x)$ and $J_{p-1}(x)$. Add and subtract that two equations to get (15.3) and (15.4)

$(15.3) \rightarrow J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$ $(15.4) \rightarrow J_{p-1}(x) - J_{p+1}(x) = 2J'_p(x)$

$$\frac{2p}{x}\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n+p} \qquad 2J'_p(x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \frac{2n+p}{\Gamma(n+p+1)} \frac{x^{2n+p-1}}{2^{2n+p}}$$

$$J_{p-1}(x) + J_{p+1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+p)}\left(\frac{x}{2}\right)^{2n+p-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+p+2)}\left(\frac{x}{2}\right)^{2n+p+1} \qquad \frac{\Gamma(p+n+1)}{p+n} = \frac{(p+n)\,\Gamma(n+p)}{p+n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)}\left(\frac{x}{2}\right)^{2n+p}\left[\frac{1}{\Gamma(n+p)}\left(\frac{x}{2}\right)^{-1} + \frac{1}{\Gamma(n+p+2)}\left(\frac{x}{2}\right)^{1}\right] \rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)}\left(\frac{x}{2}\right)^{2n+p}\left[\frac{p+n\,\frac{2}{x}}{\Gamma(p+n+1)} + \frac{1\quad \frac{x}{2}}{(p+n+1)\,\Gamma(p+n+1)}\right]$$

$$J_{p-1}(x) - J_{p+1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)}\left(\frac{x}{2}\right)^{2n+p}\left[\frac{\frac{2}{x}}{\Gamma(n+p)} - \frac{\frac{x}{2}}{\Gamma(p+n+2)}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+p}}{\Gamma(n+1)}\left[\quad\right.$$

3- carry out the differentiation in equation (15.1) and (15.2) to get (15.5) $J'_p(x) = -\frac{p}{x}J_p(x) + J_{p-1}(x) = \frac{p}{x}J_p(x) - J_{p+1}(x)$

$12.2\ \boxed{x(xy')' + (x^2 - p^2)y = 0}$

$(15.1) \rightarrow \frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$

$(15.5) \rightarrow J'_p(x) = -\frac{p}{x} J_p(x) + J_{p-1}(x) = \frac{p}{x} J_p(x) - J_{p+1}(x)$

$12.6\left[\frac{a}{2n} = \frac{-a_{2n}}{2n(2n+2p)} = \frac{a_{2n-2}}{2^2 n(n+p)} \right]$

- use equations (15.1) to (15.5) to do problems 12.2 to 12.6

(Section 17) $\frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x)$

1. For problem 12.8 $J_{1/2}(x) = \sqrt{2/\pi x}\,\sin x$, use (15.2) to obtain $J_{3/2}(x)$ and $J_{5/2}(x)$. Substitute your results fro the J into (17.4) to verify the formulas stated for j_0, j_1 and j_2 in terms of $\sin x$, $\cos x$

$* \frac{d}{dx}\left[x^{-\frac{1}{2}} J_{\frac{1}{2}}(x)\right] = -x^{-\frac{1}{2}} J_{\frac{3}{2}}(x) \longrightarrow \frac{d}{dx}\left[\frac{1}{x}\sqrt{\frac{2}{\pi}}\sin x\right] = \frac{x\left(\sqrt{\frac{2}{\pi}}\cos x\right) - \sqrt{\frac{2}{x}}\sin x}{x^2}$

$* \frac{-1}{\sqrt{x}} J_{\frac{3}{2}}(x) = \left(\frac{\sqrt{\frac{2}{\pi}}\cos x}{x} - \frac{\sqrt{\frac{2}{\pi}}\sin x}{x^2}\right) \cdot -\sqrt{x} \Rightarrow J_{\frac{3}{2}}(x) = -\sqrt{\frac{2x}{\pi}}\frac{\cos x}{\sqrt{\pi}\sqrt{x}} + \sqrt{\frac{2x}{\pi}}\frac{\sin x}{x^2}$

$J_{\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\cos x + \sqrt{\frac{2}{\pi x}}\frac{\sin x}{x} \longrightarrow \sqrt{\frac{2}{\pi x}}\left[-\cos x + \frac{\sin x}{x}\right] = J_{\frac{3}{2}}$

$* \frac{d}{dx}\left[x^{-\frac{3}{2}} J_{\frac{3}{2}}(x)\right] = -x^{-3/2} J_{\frac{5}{2}}(x) \longrightarrow \frac{d}{dx}\left[-\sqrt{\frac{2}{\pi}}\frac{\cos x}{x^2} + \frac{\sin x}{x^3}\sqrt{\frac{2}{\pi}}\right] \rightarrow$

$\frac{x^2\left(+\sqrt{\frac{2}{\pi}}\sin x\right) + \sqrt{\frac{2}{\pi}}\cos x\,(2x)}{x^4} + \frac{x^3\left(\sqrt{\frac{2}{\pi}}\cos x\right) - \sin x\sqrt{\frac{2}{\pi}}(3x^2)}{x^6} \longrightarrow \frac{\sqrt{\frac{2}{\pi}}\sin x}{x^2} + 2\sqrt{\frac{2}{\pi}}\frac{\cos x}{x^3} + \frac{\sqrt{\frac{2}{\pi}}\cos x}{x^3} - \frac{3\sqrt{\frac{2}{\pi}}\sin x}{x^4}$

$J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi}}\frac{\sin x}{x^2}\cdot\frac{(-1)}{x^{-3/2}} - 3\sqrt{\frac{2}{\pi}}\frac{\cos x}{x^3\cdot x^{3/2}} + 3\sqrt{\frac{2}{\pi}}\frac{\sin x}{x^4\cdot x^{-3/2}} \rightarrow J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x - 2\sqrt{\frac{2}{\pi x}}\frac{\cos x}{x} + 2\sqrt{\frac{2}{\pi x}}\sin x$

(17.4) $j_0(x) = \sqrt{\frac{\pi}{2x}}J_{\frac{1}{2}}(x) = x^0\left(-\frac{1}{x}\frac{d}{dx}\right)^0\left(\frac{\sin x}{x}\right) \rightarrow j_0(x) = \sqrt{\frac{\pi}{2x}}J_{\frac{1}{2}}(x) = \frac{\sin x}{x} \longrightarrow$

$j_1(x) = \sqrt{\frac{\pi}{2x}}J_{\frac{3}{2}}(x) = x^1\left(-\frac{1}{x}\frac{d}{dx}\right)\left(\frac{\sin x}{x}\right) \rightarrow j_1(x) = \sqrt{\frac{\pi}{2x}}J_{\frac{3}{2}} = -\frac{d}{dx}\left(\frac{\sin x}{x}\right) \rightarrow \frac{x(\cos x) - \sin x}{x^2}$

$j_1(x) = \sqrt{\frac{\pi}{2x}}J_{\frac{3}{2}}(x) = -\frac{\cos x}{x} + \frac{\sin x}{x^2}$

$j_2(x) = \sqrt{\frac{\pi}{2x}}J_{\frac{5}{2}}(x) = x^2\left(\frac{1}{x^2}\right)\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x(\cos x) - \sin x}{x^2} \quad \frac{x^2(\cos x) + \sin x(2x)}{x^4}$

$j_2(x) = \sqrt{\frac{\pi}{2x}}J_{\frac{5}{2}}(x) = \frac{\sin x}{x} - \frac{\cos x}{x^2} - \frac{\cos x}{x^2} + \frac{2\sin x}{x^3} \rightarrow \frac{\sin x}{x} - \frac{2\cos x}{x^2} + \frac{2\sin x}{x^3} = \sqrt{\frac{\pi}{2x}}J_{\frac{5}{2}}(x)$

- $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$
- $J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[-\cos x + \frac{\sin x}{x}\right]$
- $J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\sin x - \frac{2\cos x}{x} + \frac{2\sin x}{x^2}\right]$

[HW] section 17 ...

13.3 → $N_p(x) = Y_p(x) = \frac{\cos(\pi p) J_p(x) - J_{-p}(x)}{\sin \pi p}$

From problem 13.3 $Y_{\frac{1}{2}}(x) = -\sqrt{2/\pi x}\cos x$. As in problem 2 obtain $Y_{3/2}(x)$ and $Y_{5/2}(x)$ and verify the formula (17.4) for y_0, y_1, y_2 in terms of $\sin x$ and $\cos x$.

$Y_{\frac{1}{2}}(x) = \frac{\cos(\frac{1}{2}\pi) J_{\frac{1}{2}}(x) - J_{-3/2}(x)}{\sin(\pi \frac{1}{2})} \rightarrow (\cos \frac{1}{2}\pi) \sqrt{\frac{2}{\pi x}} \cos x + \frac{\sqrt{\frac{2}{\pi x}} \sin x}{x} -$

$y_0(x) = \sqrt{\frac{\pi}{2x}} Y_{\frac{1}{2}}(x) = \cancel{-x}\left(-\frac{1}{x}\frac{d}{dx}\right)^0\left(\frac{\cos x}{x}\right) = \sqrt{\frac{\pi}{2x}} Y_{\frac{1}{2}}(x) = \frac{-\cos x}{x} \rightarrow Y_{\frac{1}{2}}(x) = -\cos x \sqrt{\frac{2}{\pi x}}$

$y_1(x) = \sqrt{\frac{\pi}{2x}} Y_{\frac{3}{2}}(x) = +x^1\left(+\frac{1}{x}\frac{d}{dx}\right)\left(\frac{\cos x}{x}\right) \rightarrow \frac{d}{dx}\left[\frac{\cos x}{x}\right] = \frac{x(-\sin x) - \cos x}{x^2} = \frac{-\sin x}{x} - \frac{\cos x}{x^2}$

$y_2(x) = \sqrt{\frac{\pi}{2x}} Y_{\frac{5}{2}}(x) = -x^2\left(\frac{+1}{x^2}\right)\frac{d^2}{dx}\left(\frac{\cos x}{x}\right) = \frac{-x(\cos x) + \sin x}{x^2} - \left[\frac{x^2(-\sin x) - 2x(\cos x)}{x^4}\right]$

$y_2(x) = \sqrt{\frac{\pi}{2x}} Y_{\frac{5}{2}}(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2} - \frac{\sin x}{x^2} - \frac{2\cos x}{x^3}$

$* \; Y_{\frac{3}{2}}(x) = -\sqrt{\frac{2}{x\pi}}\left[\frac{\sin x}{x} + \frac{\cos x}{x^2}\right] \quad * \; Y_{\frac{5}{2}}(x) = \left[\frac{\cos x}{x} - \frac{2\sin x}{x^2} - \frac{2\cos x}{x^3}\right]\sqrt{\frac{2}{x\pi}}$

---

4) using (17.3) and the result stated in problems 2 and 3 for $J_{1/2}$ and $Y_{1/2}(=N_{1/2})$ show that

$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$ and $K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$ (17.3) $I_p(x) = i^{-p} J_p(ix)$

$K_p(x) = \frac{\pi}{2} i^{p+1} H_p^{(1)}(ix)$

$H_p^{(1)}(x) = J_p(x) + i N_p(x)$ ↓ Y_p

$J_{\frac{1}{2}}(xi) = \sqrt{\frac{2}{\pi(ix)}} \sin(ix)$

$* \; I_{\frac{1}{2}}(x) = i^{-\frac{1}{2}} J_{\frac{1}{2}}(xi) \rightarrow$ $\sin(ix) = \sinh x$

$I_{\frac{1}{2}}(x) = i^{-\frac{1}{2}} \sqrt{\frac{2}{\pi(ix)}} \sinh x$

$I_{\frac{1}{2}}(x) = \cancel{i^{\frac{1}{2}}} \frac{1}{\cancel{i^{\frac{1}{2}}}} \sqrt{\frac{2}{\pi x}} \sinh x \rightarrow \therefore I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$

$* \; K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \rightarrow K_{\frac{1}{2}}(x) = \frac{\pi}{2} i^{p+1}\left[J_{\frac{1}{2}}(x) + i N_{\frac{1}{2}}(x)\right]$

$K_{\frac{1}{2}}(x) = \frac{\pi}{2} i^{\frac{1}{2}+1}\left(\sqrt{\frac{2}{\pi x}} \sin x - i\sqrt{\frac{2}{\pi x}} \cos x\right)$

$K_{\frac{1}{2}}(x) = \frac{\sqrt{\pi}\sqrt{\pi}}{\sqrt{2}\sqrt{2}} i^{\frac{1}{2}+1} \sqrt{\frac{2}{\pi x}}\left[\sin x - i\cos x\right] \rightarrow \sqrt{\frac{\pi}{2x}} i^{3/2} e^{-ix}$

$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} \, i \, e^{-x}$

5- show from (17.14) that $h_n^{(1)}(x) = -i x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{e^{ix}}{x}\right)$

$$j_n = \sqrt{\frac{\pi}{2x}}\, J_{(2n+1)/2}(x) = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$$

$$y_n(x) = \sqrt{\frac{\pi}{2x}}\, Y_{(2n+1)/2}(x) = -x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\cos x}{x}\right)$$

$$h_n^{(1)} = j_n(x) + i y_n(x)$$

$$h_n^{(1)} = x^n\left(\frac{-1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right) - i x^n\left(\frac{-1}{x}\frac{d}{dx}\right)^n\left(\frac{\cos x}{x}\right)$$

$$h_n^{(1)} = x^n\left(\frac{-d}{dx}\right)^n \left[\frac{\sin x}{x} - i\frac{\cos x}{x}\right] \times \frac{i}{i}$$

$$x^n\left(\frac{-d}{dx}\right)^n\left[i\frac{\sin x}{x} - i^2\frac{\cos x}{x}\right] \times \frac{1}{i} \longrightarrow x^n\left(\frac{-d}{dx}\right)^n \frac{1}{i}\left[i\frac{\sin x}{x} + \frac{\cos x}{x}\right]$$

$$-i x^n\left(\frac{-d}{dx}\right)^n\left(\frac{e^{ix}}{x}\right)$$

[ch 15] [partial Differential Equations]

[10-3-2016]

① Laplace's equation

$$\nabla^2 u = 0 \qquad \nabla^2 = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]$$

$$\frac{\partial^2}{\partial x^2} u(x,y,z) + \frac{\partial^2}{\partial y^2} u(x,y,z) + \frac{\partial^2}{\partial z^2} u(x,y,z) = 0$$

$\underline{u}$ could be ① Gravitational potential in free space [No mass]

② electrostatic potential in free space [No charge]

توزيع درجة الحرارة على مادة معينة بدون الاعتماد على الزمن ← ③ Steady-State temperature [time independent temperature flow] بشرط عدم وجود مصدر لحرارة

جهد سرعة أي سائل (مائع) بشرط عدم وجود تصادم أو أي شيء يسحب منه ماء ← ④ velocity potential an incompressible fluid with no vortices and no sources

② poisson's equation

$$\nabla^2 u = f(x,y,z)$$ (source function)

يمكن تعريفها في الأشياء التي لها مصدر

$$\nabla^2 \psi = \rho(x,y,z)$$

electric potinal; كثافة الشحنة

معادلة الانتشار

③ The diffusions equation

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$$

معيار على القدرة السائل على انتشار

α : diffusion aplity

صغيرة $\alpha \rightarrow$ قدرة سائل كبيرة على الانتشار

كبيرة $\alpha \rightarrow$ قدرة سائل صغيرة على الانتشار

$u \rightarrow$ non steady-state temperature flow

The concentration of a diffusing substance.

④ Wave equation

(x,y,z) مكان تعرض الموجة

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

$u \rightarrow$ The displacement from equilibrium of a vibrating string

$\psi \rightarrow$ The component at the electric or the magnetic Filed vector of electromagnetic wave

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the d Alembertion $\left[\nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]$

⑤ Helmholtz equation

$$\nabla^2 F + k^2 F = 0$$

spatial part of wave equation

⑥ Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2 \Psi + V\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

* show that the function $\sin(x-vt)$ satisfy the wave equation $u=\sin(x-vt)$

$$\nabla^2 u = \frac{\partial^2}{\partial^2 x}u + \frac{\partial^2 u}{\partial^2 y} + \frac{\partial^2}{\partial z^2}u =$$

$$\frac{\partial^2 u}{\partial^2 x} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}u\right) = \frac{\partial}{\partial x}\left(\cos(x-vt)\right) = -\sin(x-vt)$$

تعويض equation

$$= \frac{1}{v^2}\frac{\partial^2 u}{\partial t} = \frac{1}{v^2}\left[\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\sin(x-vt)\right]\right] = \frac{1}{v^2}\frac{\partial}{\partial t}\left[\cos(x-vt)(-v)\right]$$

$$= \frac{1}{v^2}\left[-\sin(x-vt)(-v)(-v)\right] = -\sin(x-vt)$$

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

* Show that any function of the form $f(x-vt)$ satisfies the wave equation

$$\nabla^2 f = \frac{\partial^2}{\partial x^2}f \longrightarrow \frac{\partial^2}{\partial x^2}f = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}f(x-vt)\right) \quad \text{let } y=(x-vt)$$

سلسلة

$$\frac{\partial}{\partial x}\left[\frac{\partial y}{\partial x}\frac{\partial f(y)}{\partial y}\right] = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial y}{\partial x}\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{1}{v^2}\frac{\partial^2}{\partial^2}f(x-vt) = \frac{1}{v^2}\frac{\partial^2}{\partial t}f(y) = \frac{1}{v^2}\frac{\partial}{\partial t}\left[\frac{\partial y}{\partial t}\frac{\partial f}{\partial y}(y)\right]$$

$$= \frac{1}{v^2}\frac{\partial y}{\partial t}\left[-v\frac{\partial f}{\partial y}(y)\right] \longrightarrow \frac{1}{v^2}\,{-v}\left[\frac{\partial^2 f}{\partial y^2}(y)(-v)\right] \longrightarrow \frac{\partial^2 f}{\partial y^2}$$

---

* Asume from electrodynamics the following equations which are valid in free space (They are called Maxwell's equations)

$$\vec\nabla\cdot\vec E = 0 \qquad \vec\nabla\cdot\vec B = 0$$

$$\boxed{\vec\nabla\times\vec E = -\frac{\partial B}{\partial t}} \qquad \vec\nabla\times\vec B = \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$\left(\epsilon_0\mu_0 = \frac{1}{c^2}\right)$

show that any component of E or B satisfies the wave equation $\Rightarrow \nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$

$$\vec\nabla\times\vec E = \frac{1}{c^2}\frac{\partial^2 \vec E}{\partial t^2} \qquad \vec\nabla\times\vec\nabla\times\vec A = \vec\nabla(\vec\nabla\cdot\vec A) - \nabla^2 A$$

$$\vec\nabla\times\vec E = -\frac{\partial}{\partial t}\vec B \qquad \frac{\partial}{\partial t}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial t}(\vec\nabla\times\vec E) = -\frac{\partial^2}{\partial t^2}\vec B$$

$$\left(\vec\nabla\times\frac{\partial\vec E}{\partial t}\right) = -\frac{\partial^2}{\partial^2 t}\vec B$$

$$(\vec\nabla\times c^2\times\nabla\times\vec B) = -\frac{\partial^2 B}{\partial t^2}$$

$$\vec\nabla\times\vec\nabla\times\vec B = \frac{-1}{c^2}\frac{\partial^2}{\partial t^2}\vec B$$

$$\vec\nabla(\vec\nabla\cdot\vec B) - \nabla^2\vec B = \frac{-1}{c^2}\frac{\partial^2 B}{\partial t^2}$$

$$-\nabla^2\vec B = \frac{-1}{c^2}\frac{\partial^2 B}{\partial t^2}$$

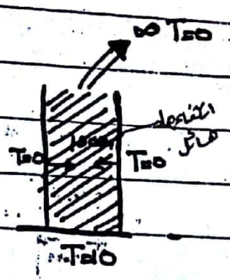
[Laplace equation]

[13-3-2016]

$\nabla^2 T = 0$

Steady is that temperater in rectangular plate معادلة لابلاس لتوزيع الحرارة على طول الصفيحة المستوية

صفيحة ممتدة لا نهائيا ← semi infinte plate located xy plane



$T(x,y)$

$$\nabla^2 T = \underbrace{\frac{\partial^2}{\partial^2 x} T + \frac{\partial^2}{\partial y^2} T}\ + \frac{\partial^2}{\partial z^2} \overset{\text{zero}}{T} = 0$$

$$\frac{\partial^2}{\partial x^2} T + \frac{\partial^2}{\partial y^2} T = 0$$

Sepration of

Let $T(x,y) = T_x(x) + T_y(y) = X(x) + Y(y)$

[Ex) is the function $(\sin(x+y))$ suprubel $\sqrt{xy}$ C
$\sin x \cos y + \sin y \cos x$]

supstitue this equation into Laplace

$$\frac{\partial^2}{\partial x^2} T + \frac{\partial^2}{\partial y^2} T = 0 \longrightarrow \frac{\partial^2}{\partial x^2} \underbrace{(X(x) y(x))}_{T} + \frac{\partial^2}{\partial y^2} \underbrace{(X(x) y(y))}_{T} = 0$$

$$Y(y) \frac{\partial^2}{\partial x^2} X(x) + X(x) \frac{\partial^2}{\partial^2 y} Y(y) = 0$$

divide by $X(x)\, Y(y) \longrightarrow \frac{\cancel{Y(y)}}{X(x)} \frac{1}{\cancel{Y(y)}} \frac{\partial^2}{\partial^2 x} X(x) + \frac{\cancel{X(x)}}{\cancel{X(x)} Y(y)} \frac{\partial^2}{\partial^2 y} Y(y) = 0$

$$\left[\frac{1}{X(x)} \frac{\partial^2}{\partial x^2} + \frac{1}{Y(y)} \frac{\partial^2}{\partial y^2} = 0 \right]$$

$f(x) + g(y) = 0 \quad f(x) = -g(x)$

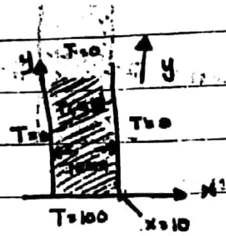
* The solution of $\left[\frac{1}{X} \frac{\partial^2}{\partial^2 x} X + \frac{1}{Y} \frac{\partial^2}{\partial^2 y} Y = 0 \right]$

* is by the assumption $\rightarrow \left(\frac{1}{X} \frac{\partial}{\partial x^2} X - \frac{1}{Y} \frac{\partial^2}{\partial y^2} Y = -K^2 \right) \longrightarrow$ wher K is $K \in R$ [constant]

* Laplace equation steady state between in a rectangular plate *

حالة مستقرة لاتتغير مع الزمن

صفيحة في بعدين درجة الحرارة لاتتغير مع زمن بل تتغير مع موقع



(Laplace) حل معادلة $\nabla^2 T = 0$ تمثل درجة الحرارة في موقع كل نقطة

$T(x,y) = X(x) + Y(y)$ باستخدام فصل المتغيرات

$\frac{\partial^2}{\partial x^2} T + \frac{\partial^2}{\partial y^2} T = 0$ معادلة تفاضلية جزئية (second order partial)

$\frac{d^2}{dx^2} X(x) = -K^2 (X(x))$ $\frac{d^2}{dy^2} Y(y) = K^2 (Y(y))$ الثابت $[X(x) - K^2$ و $(Y(y)) K^2]$ ليس متساوي بل مجموعها صفر

this second order equation

$K^2 \rightarrow$ seprac constant ثابت ناتج من تحويل معادلة تعدد الى متغيرة

$X(x) = \begin{cases} \sin Kx \\ \cos Kx \end{cases}$ $Y(y) = \begin{cases} e^{Ky} \\ e^{-Ky} \end{cases}$ → الحلول المقترحة

$T(x,y) = \begin{cases} \sin Kx \\ \cos Kx \end{cases} \begin{cases} e^{Ky} \\ e^{-Ky} \end{cases}$

عندما تكون كبيرة (y) تقترب من مالا نهاية $T=0$ e^{Ky} عندما تكبر قيمة y تقترب T هذا مستحيل لانه لا يتحقق الشرط

بينما في هذا (الوضع) يتحقق الشرط لانه كلما تزداد قيمة تقترب من الصفر وهذا يتحقق الشرط الحدي $T=0$ عند $T=0$ e^{-Ky} → الحل المعتمد

$\Rightarrow \boxed{T(x,y) = \sin Kx \, e^{-Ky}}$

$T=0$ $\sin Kx$ $\cos Kx$

$\sin 0 = 0$ يحقق شرط الحدي $\cos 0 = 1$ لا يحقق شرط الحدي

at $x=10$ $T=0$

نعوض بالمعادلة $T(10,y) = \sin(K10) \, e^{-Ky} = 0$

هذا الذي يعطي قيمة تعطي صفر — مستحيل قيمتها صفر

نحصل على زاوية التي صفر عندما تكون نقطة π Cdshee

$\sin K10 = 0$

$10K = n\pi$ $n = 0, 1, 2, \ldots$ legint value

$\frac{d^2}{dx^2} X = -K^2 (X(x))$ legient constant, legient function

نحصل عليه من استخدام شروط الحدي ← $\boxed{K = \frac{n\pi}{10}}$

$T = \underbrace{\sin \frac{n\pi}{10}}_{\text{حل 1}} \, x \, \underbrace{e^{-n\frac{\pi}{10}y}}_{\text{حل 2}}$

الحل العمومي لـ x و y لا يعطينا حل المعادلة (laplace)

بل يكمن الحل عام لمعادلة (laplace) مجموع الحدود التي نحصل عليها

عند القاعدة درجة الحرارة (T=100) القاعدة

at $y=0$ $T = 100C°$

$100 = \sin \frac{n\pi}{10} \, (x) \, e^{-\frac{n\pi}{10}(0)} \longrightarrow 100 = \sin \frac{n\pi}{10} \, x$

قيمة x متغير أخرى (مهم)

$\nabla^2 T = 0$ عند الحدود التي تحل هذه معادلة ونهائي من الحدود

* The genral solution of $\nabla^2 T = 0$ is given by:

نبدأ من (1) لأن $n=0$ عندها $=0$

$$\boxed{T = \sum_{n=1}^{\infty} b_n \sin \frac{\pi n}{10} x \; e^{-\frac{n\pi}{10} y}}$$

* fourier trick (خدعة فورييه)

$$f(x) = \sum c_n \sin n \frac{\pi}{2} x$$

$$c_n = \frac{2}{l} \int_0^l \left(f(x) \sin \frac{\pi n}{10} x \right) dx$$

تكامل $(-\cos x \leftarrow \sin x)$

نحصل على ثابت من b_n بالتحويل الفوري عند إعطائه قيمة واحدة يكون صالح لجميع القيم

bn

(عند تعويض $y=0$) $e^{-n\frac{\pi}{10}(0)}$

$$T(x,0) = 100$$

$$T = 100 \sum_{n=1}^{\infty} b_n \sin n \frac{\pi}{10} x \, (1)$$

fourier خدعة
[using fourier trick]

$$b_n = \frac{2}{10} \int_0^{10} (100) \left(\sin \frac{n\pi}{10} \right) x \, dx$$

$$b_n = 20 \int_0^{10} \sin \frac{n\pi x}{10} \, dx$$

عندما تكون n = زوجية يكون جوابها (1)
عندما يكون n = فردية يكون جوابها (-1)

(2n+1) n odd

$$b_n = 20 \left[\frac{-10}{n\pi} \cos \frac{n\pi}{10} x \Big|_0^{10} \right] \Rightarrow \frac{-200}{n\pi} \left[\cos \frac{n\pi \cdot 10}{10} - 1 \right]$$

$$y = \begin{cases} 0 & ; \ n \text{ even} \\ \frac{400}{n\pi} & ; \ n \text{ odd} \end{cases}$$

$$T = \sum_{n=0}^{\infty} \frac{400}{(2n+1)\pi} \sin \frac{(2n+1)\pi x}{10} \, e^{-\frac{(2n+1)\pi}{10} y}$$

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[HW] section 2

- Find the steady-state temperature distribution for the semi-infinite plate problem if the temperature of the bottom edge $T = f(x) = x$ / the temperature of the other sides is $0°$ and the width of the plate is 10 cm

$\nabla T^2 = 0$

$T(x,y) = X(x) + Y(y) \longrightarrow \frac{d^2}{d^2x}(x(x)) = -k^2(x) \quad \frac{d^2}{d^2y}y(y) = +k^2 y(y)$

$X(x) = \begin{cases} \sin kx \\ \cos kx \end{cases} \quad y(y) = \begin{cases} e^{-ky} \\ e^{ky} \end{cases}$

$T=0$, $T=0$, $T=0$, 10 cm, $T = f(x) = x$

$T=0 \quad y=0 \to e^{-ky}$ $\quad T=0 \quad x=0 \to \sin kx$

$x=10 \quad T=0 \quad \sin kx = 0 \to kx = n\pi \to 10k = n\pi \to \boxed{k = \frac{n\pi}{10}}$

$T(x,y) = \sin(\frac{n\pi}{10})x \; e^{-(\frac{n\pi}{10})y}$

$y=0 \quad T = f(x) = 10 = x \quad 10 = \sin(\frac{n\pi}{10})x \; e^{-(\frac{n\pi}{10})0}$ $\quad C_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{\pi n}{10}\,dx$

$10 = \sin(\frac{n\pi}{10})x$

$T = \sum_{n=1}^{\infty} b_n \sin(\frac{\pi n}{10})x \; e^{-(\frac{n\pi}{10})y} \longrightarrow T = \sum_{n=0}^{\infty} 10 \sin(\frac{\pi n}{10})x \; e^{-(\frac{n\pi}{10})y}$

$b_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin(\frac{n\pi}{10})dx \longrightarrow \frac{2}{10}\int_0^{10} x \sin(\frac{n\pi}{10})dx$

$x \quad \sin(\frac{n\pi}{10})x\,dx$

$1 \quad -\cos(\frac{n\pi}{10}) \cdot \frac{10}{n\pi}$

$-x\cos(\frac{n\pi}{10}) \cdot \frac{10}{n\pi} + \int \cos(\frac{n\pi}{10})\frac{10}{n\pi}$

$-x\cos(\frac{n\pi}{10})\frac{10}{n\pi}\Big|_0^{10} + \frac{100}{n^2\pi^2}\sin(\frac{n\pi}{10})\Big|_0^{10}$

$= \frac{-100}{n\pi}$

$b_n = \frac{2}{10}\,\frac{100}{n\pi} = \frac{20}{n\pi}$

$\begin{cases} n\text{ (odd)} & \frac{100}{n\pi} \\ n\text{ (even)} & \frac{-100}{n\pi} \end{cases}$

$T = \frac{20}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} e^{-(\frac{n\pi}{10})y} \sin(\frac{n\pi}{10})x$

2- slove the semi-infinte plate problem if bottom edge of width π is held at $T = \cos x$ and other side are $0°$

$T(x,y) = X(x) + Y(y)$

$X(x) = \begin{cases} \sin kx \\ \cos kx \end{cases} \quad Y(y) = \begin{cases} e^{-ky} \\ e^{ky} \end{cases}$

$T=0$, $T=0$, π, $T=0$, $T = f(x) = \cos x$

$T=0 \quad x=0 \quad \sin kx = 0 \qquad T=0 \quad y=0 \quad e^{-ky}$

$T=0 \quad x=\pi \quad kx = n\pi \to k\pi = n\pi \quad \boxed{k=n} \longrightarrow T(x,y) = \sin nx\, e^{-ny}$

$T = \cos x \quad y=0$

$\cos x = \sin nx \underbrace{e^{-n y(0)}}_{1} \longrightarrow \cos x = \sin nx$

$\int_0^{\pi} \cos x \; \sin nx \; dx$

$\cos x \quad \sin nx$

$-\sin x \quad -\frac{\cos nx}{n}$

$\sin x \quad \frac{1}{n}\cos nx$

$\cos x \quad \frac{1}{n^2}\sin nx$

$T = \sum_{n=0}^{\infty} b_n \sin(nx)\, e^{-ny} \longrightarrow T = \sum_{n=0} \cos x \sin nx \cdot e^{-ny}$

$-\frac{\cos x \cdot \cos nx}{n} - \int \frac{\sin x \cos nx}{n}$

$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \; \sin nx \; dx$

$\int_0^{\pi} \cos x \; \sin nx \; dx = -\frac{\cos x \cos nx}{n}\Big|_0^{\pi} - \frac{1}{n^2}\sin x \sin nx\Big|_0^{\pi} + \frac{1}{n^2}\int_0^{\pi}\cos x \sin nx\, dx$

$-\frac{\cos\pi \cos n\pi}{n} - \frac{\cos 0 \cos n0}{n}$

$b_n = \frac{2}{\pi}\cdot\frac{+2n}{1-n^2} = \frac{4}{\pi}\left(\frac{2}{1-n^2}\right)$

$\left(\frac{1}{n^2} + \frac{n^2}{n^2}\right)\int_0^{\pi}\cos x \sin nx\, dx = \frac{-2}{n}\cdot\frac{n^2}{-1+n^2} = \frac{+2n}{1-n^2}$

$T = \frac{4}{\pi}\sum_{n\text{ even}}^{\infty} \frac{n}{1-n^2} \sin(nx)\, e^{-ny}$

$\begin{cases} n\text{(even)} = \frac{2n}{n^2-1} \\ n\text{(odd)} = \text{zero} \end{cases}$

(CH 13)

Hw [sec(10.2)]

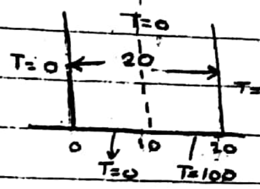
3 - Slove the semi-Infinite plate problem if bottom edge of width 20 is held at

$T = \begin{cases} f(x) \\ 100 \end{cases}$ $0<x<10$, $10<x<20$ and the otherside are at $0°$

قاعدة الشمال

$x = \begin{cases} \sin xK \\ \cos xK \end{cases}$ $y = \begin{cases} e^{-Ky} \\ e^{Ky} \end{cases}$

$T = x\,y$ $T = \begin{cases} \sin xK \\ \cos xK \end{cases} \begin{cases} e^{-Ky} \\ e^{Ky} \end{cases}$



$y=0$ $T=0 \rightarrow e^{-Ky}$ $e^{Ky} \neq 0$ X

$x=0$ $T=0 \longrightarrow \sin xK$ $\cos xK \neq 0$

$T=0$ $x=20 \longrightarrow \sin Kx = 0 \Rightarrow n\pi = Kx \longrightarrow \boxed{K = \frac{n\pi}{20}}$

$T(x,y) = \sin \frac{n\pi}{20} x \; e^{-\frac{n\pi}{20} y}$

$T = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{20}\right)x \; e^{-\frac{n\pi}{20} y}$

$T(x,0) = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{20}\right)x \; (1)$

$b_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin\left(\frac{n\pi}{\ell}\right)x$

$b_n = \frac{2}{10}\int_0^{10} (0) \sin\left(\frac{n\pi}{10}\right)x + \frac{2}{20}\int_{10}^{20} (100) \sin\left(\frac{n\pi}{20}\right)x\,dx$

$= b_n = 0 + \frac{1}{10} \cdot 100 \int_{10}^{20} \sin\left(\frac{n\pi}{20}\right)x\,dx$

$b_n = \frac{-200}{n\pi}\int_{10}^{20} \sin\left(\frac{n\pi}{20}\right)x\,dx \longrightarrow 10\left[-\cos\left(\frac{n\pi}{20}\right)x\right]_{10}^{20}$

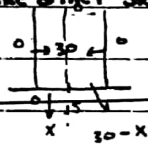
$b_n = \frac{-200}{n\pi}\left[\cos\left(\frac{n\pi\,20}{20}\right) - \cos\left(\frac{n\pi\,10}{20}\right)\right] = \frac{-200}{n\pi}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]$

$\begin{cases} n = odd \; ; \; \frac{200}{n\pi} \\ n = even \; ; \; \frac{-400}{n\pi} \end{cases}$

$T = \frac{200}{n\pi} \sum_{n=odd}^{\infty} \sin \frac{n\pi}{20} x \; e^{-\frac{n\pi}{20} y}$; $T = \frac{-400}{n\pi} \sum_{n=even}^{\infty} \sin \frac{n\pi}{20} x \; e^{-\frac{n\pi}{20} y}$

4) slove the semi-infinite plate problem if bottom edge of width 30 is held at

$T = \begin{cases} x & 0<x<15 \\ 30-x & 15<x<30 \end{cases}$ and the other sides are at $0°$



$T = \begin{cases} \sin xK \\ \cos xK \end{cases} \begin{cases} e^{-Ky} \\ e^{Ky} \end{cases}$

$T=0$ $y=0$ e^{-Ky} $x=0$ $T=0$ $\sin Kx$

$x=30$ $T=0$ $\sin Kx = 0 \longrightarrow Kx = n\pi$

$\boxed{K = \frac{n\pi}{30}}$

$T(x,y) = \sum \sin \frac{n\pi}{30} x \; e^{-\left(\frac{n\pi}{30}\right) y}$

$y=0$

$T = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi}{30} x$

$b_n = \frac{2}{\ell}\int_0^{\ell} T \sin \frac{n\pi}{30} x\,dx$

$b_n = \frac{2}{30}\left[\int_0^{15} x \sin \frac{n\pi}{30} x\,dx + \int_{15}^{30} (30-x) \sin \frac{n\pi}{30}\,dx\right]$

$b_n =$

$\int_0^{15} x \sin \frac{n\pi}{30} x\,dx$ $\quad x \quad \sin \frac{n\pi}{30} x$ $\quad 1 \quad -\cos \frac{n\pi}{30} x \left(\frac{30}{n\pi}\right)$

$\frac{-30}{n\pi} x \cos \frac{n\pi}{30} x \Big]_0^{15} + \frac{30}{n\pi}\int_0^{15} \cos \frac{n\pi}{15} x\,dx$

$\left(\frac{-15}{n\pi}(15) \cos \frac{n\pi}{15} 15 + 0\right) + \underbrace{\frac{30}{n\pi}\left(\frac{30}{n\pi}\right) \sin \frac{n\pi}{30} x \Big]_0^{15}}_{zero}$

$= \frac{-(15)^2}{n\pi} \cos n\pi$

$\int_{15}^{30} (30-x) \sin \frac{n\pi}{30} x\,dx$ $\quad 30-x \quad \sin \frac{n\pi}{30} x$ $\quad -1 \quad -\cos \frac{n\pi}{30} x \left(\frac{30}{n\pi}\right)$

$(30+x)\left(\frac{30}{n\pi}\right) \cos \frac{n\pi}{30} x \Big]_{15}^{30} - \int_{15}^{15} \left(\frac{30}{n\pi}\right) \cos \frac{n\pi}{30} x\,dx$ zero

$\underbrace{(-30+30)\left(\frac{30}{n\pi}\right) \cos n\pi}_{zero} - (-30+15)\left(\frac{30}{n\pi}\right) \cos \frac{n\pi}{2} =$

$\begin{cases} n\,even = 0 \\ n\,odd = \frac{-30}{n\pi} \end{cases}$

* The diffusion equation, The heat flow equation, Schrödinger equation *

درجة الحرارة ليست ثابتة مع زمن — The heat flow equation $\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$

توزيع درجات الحرارة على الصفيحة الخ — u represent the heat (temprature) a point (x,y,z) at time t $u(x,y,z)$

ثابت المادة التي تنتشر تعتمد على طبيعة المادة — α constant depends on the type of meterial the heat flowing through

— We assum that the solution can be written in the term

$u = \underbrace{F(x,y,z)}_{\text{يعتمد على موقع}}\ \underbrace{T(t)}_{\text{time point}}$ يعتمد على زمن

spatied point

$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \rightarrow \underbrace{T(t)}_{\text{تعامله كثابت}}\ \underbrace{\nabla^2 F}_{\text{تشتق هذه فقط}} = \frac{F}{\alpha^2}\frac{\partial T}{\partial t}$

في هذه الحالة عندما يكون الاقتران تابعاً لمتغير واحد فتكون المشتقة جزئية = مشتقة عادية: $\frac{dT}{dt} = \frac{\partial T}{\partial t}$

$\frac{\nabla^2 F}{F} = \frac{1}{\alpha^2}\frac{\frac{\partial T}{\partial t}}{T} = -K^2$

اقتران يعتمد على موقع (F) — اقتران يعتمد على الزمن (T)

في حالة معادلات التفاضلية الجزئية من الدرجة الثانية يتغير الحل بتغير إشارة K

$-K^2 \rightarrow \begin{cases}\cos Kx\\ \sin Kx\end{cases}$

$+K^2 \rightarrow \begin{cases}\cosh K\\ \sinh K\end{cases}$

* Helmholtz equation $\boxed{\nabla^2 F = -K^2 F}$

يتم اعتماد $-K^2$ لانه درجة الأطلال لا يختلف الحل باختلاف إشارة $\underline{K}$

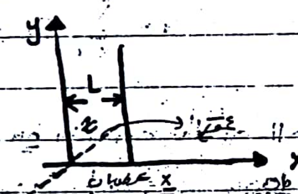
دائماً تعتمد الحل وهنا الجزء يعتمد على الزمن — $\frac{d}{dt}T = -K^2\alpha^2 T \Rightarrow T(t) = e^{-K^2\alpha^2 t} \Rightarrow$ the time dependent point of the heat flow function

انتشار الحرارة على سطح معين تتغير درجة الحرارة مع الزمن

* The Helmholtz equation * $\boxed{\nabla^2 F = -K^2 F}$ لكتابة على الشكل هذه المعادلة حيث الحرارة (تشريح)

* heat flow ... the ... of thickness L ... the temperature of ...

... تساوي عند 100 ... y



منحنى تطابق معادلة فيها اشتقاق زمني — at $t=0$ we can use the problem is steady-state and ... use Laplace $\nabla^2 u = 0 \longrightarrow \nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial T}$

الاقتران يعتمد على متغير x (x,y,z): $\frac{\partial^2 u}{\partial x^2} = 0 \longrightarrow Ax + b = u$

$u = ax + b$

$u(a \cdot (0)) + b = \boxed{b = u = 0}$ $x = 0$

$100 = ax + b$ $u(L) = 100$

$u(x) = \frac{100\,x}{L}$ $\quad a L = 100\ a \quad$ at $a = \frac{100}{L}$

[20-3-2016]

* The diffusion on heat flow equation The schrödinger equation *

$\nabla^2 = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$ $\quad u = F(x,y,z)\,T(t)$

$$\boxed{T(t) = e^{-k^2\alpha^2 t}}$$

at $t=0$ $\quad u(0)=0 \quad u(l)=100 \quad u(x)\Big|_{t=0} = \frac{100}{l}x$

The spatial part $F(x,y,z)$ satesfies the Helmheall Z equation normaly

$\nabla^2 F + k^2 F = 0$

But F depends on x $\quad \nabla^2 F + k^2 F = 0$

$\longrightarrow \frac{d^2}{dx^2}F + k^2 F = 0$

The solution is

$T(t) = \begin{cases} \sin kx \\ \cos kx \end{cases} \qquad u = \begin{cases} \sin kx \\ \cos kx \end{cases} e^{-k^2\alpha^2 t}$

$u(x=0)=0 \longrightarrow u = \sin kx\, e^{-k^2\alpha^2 t} \qquad \boxed{u(x=l)=100}$

If we assum that the temprture at $x=l$ after time peried is zero then $\sin kl = 0$ $\quad \boxed{k = \frac{n\pi}{l}}$

$\nabla^2 u = \frac{1}{\alpha^2}\frac{du}{dt} \longrightarrow \boxed{u = \sin\frac{n\pi}{l}x\, e^{-\frac{n^2\pi^2}{l^2}\alpha^2 t}}$

The general solution is the sum of all possible solution that is the sum per all values of n

$$U = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} b_n \left[\sin\left(\frac{\pi n}{l}\right)x\, e^{-\frac{n^2\pi^2}{l^2}\alpha^2 t}\right]$$

Now the rusIt of the canston b_n depends on the inital conditions at $t=0$

$u = \frac{100}{l}x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\, e^0$

$b_n = \frac{2}{l}\int_0^l \frac{100}{l}x \cdot \sin\frac{n\pi}{l}x\,dx \longrightarrow b_n = \frac{2}{l}\frac{100}{l}\int_0^l x\cdot\sin\frac{\pi n}{l}x\,dx$

$\int_0^l x\cdot\sin\frac{\pi x}{l}\,dx \longrightarrow \left(\frac{l}{n\pi}\right)x\cos\frac{n\pi}{l}x + \frac{2}{n\pi}\int_0^l \cos\frac{n\pi}{l}x\,dx \longrightarrow \underbrace{\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{l}x\Big|_0^l}_{}$ zero

$\frac{200}{l^2}\left[-\frac{l^2(-1)^n}{n\pi}\right]$

$$\boxed{b_n = \frac{200}{n\pi}(-1)^{n+1}} \quad = \sum_{n=1}\frac{200}{n\pi}(-1)^{n+1}\sin\frac{n\pi}{l}\,e^{-\frac{n^2\pi^2}{l^2}\alpha^2 t}$$

$u_0 = \sin\frac{3\pi}{l}x \qquad u_0 = \sin\frac{3\pi}{l}x + \sin\frac{5\pi}{l}x$

$b_n = \begin{cases} 0 & n\neq 3 \\ 1 & n=1 \end{cases}$

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[20-3-2016]

* Schrodanger equation

$\frac{-\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t}$ — معادلة شرودنجر

[Shor danger equation when the potential is zero] [the potential is V]

$\frac{-\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$

Time dependent shrodanger equation

the potinal could be constant, function of spatial coordinate only, function of spatial coordinat and time

$V = \frac{1}{r}\left(\frac{1}{(x,y,z)}\right)$

* let study Sch. eq when $V=0$

$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t}$

one dimension $\rightarrow \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi = i\hbar\frac{\partial\psi}{\partial t}$

if the assume the $\psi(x,t) = \varphi(x) f(t)$ supstiyce this in P_0 m equation

$\frac{-\hbar^2}{2m} f(t) \frac{d^2\varphi}{dx^2} = i\hbar\varphi\frac{df}{dt}$

divide by $\varphi \rightarrow \frac{-\hbar^2}{2m}\frac{\frac{d^2\varphi}{dx^2}}{\varphi} = i\hbar\frac{\frac{df}{dt}}{f}$

let the sepration constant be k^2 that

$\frac{\hbar^2}{2m}\frac{d^2}{dx^2} = k^2\varphi$

$i\hbar\frac{df}{dt} = k^2 f \rightarrow f e^{-k^2 t/\hbar}$, $f e^{k^2 t/\hbar}$

[22-3-2016]

* Sc'hröding equation *

The Hamiltonian operator $\hat{H}\psi = i\hbar \frac{\partial \psi}{\partial t}$

عمليات تجرى على الجسم $\hat{H} = KE + U$ طاقة حركية + potinal energy طاقة وضع

$\frac{-\hbar^2}{2m}\nabla^2 = \hat{H}$ أبسط أشكال

$\frac{-\hbar^2}{2m}\nabla^2 + \frac{1}{2}mv^2x^2 = \hat{H}$ ← النابض يتحرك بعد واحد

$\hat{H} = KE + U_g + U_e + U_s$

طاقة وضع الجاذبية

طاقة وضع نابض (صفة)

$\frac{-\hbar^2}{2m}\nabla^2 + \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} = \hat{H}$ ← الإلكترون ذرة

$\hat{H}\psi = i\hbar\frac{\partial\psi}{\partial t}$ ← أي تغير بسيط فيه سوف جواب مختلف

(free particle) لا يتعرض الجسم لأي نوع من مؤثرات مثل (photon) يتسارع طاقة

مع قوانين نيوتن

$\frac{\sum F}{m} = a = \ddot{x} = \frac{dx}{dt}$

$\sum F = m\ddot{x}$

$-kx = m\ddot{x}$

قوة إرجاع لابض

$m\ddot{x} + kx = 0$

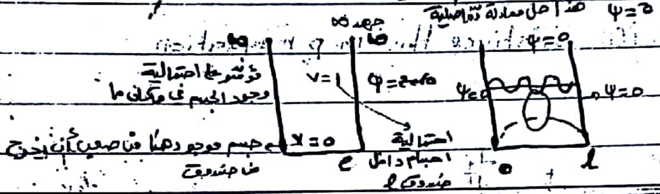
$\boxed{m\frac{d^2x}{dt} + kx = 0}$

موقع الجسم $x(t)$ → $v(t)$ → $a(t)$, KE, $u(g)$

$p = mv$

[Ex) free particle $\hat{H} = -\hbar^2\nabla^2$ all the space [ex → photons]

$$\hat{H} = \begin{cases} \frac{-\hbar^2}{2m} + 0, & 0 < x < l \\ \frac{-\hbar^2}{2m} + \infty & \text{otherwise [particle in box] infint squer well} \end{cases}$$



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* particle in a box with infinite wall

The sch. eq. is $\frac{-\hbar^2}{2m}\nabla^2\varphi = i\hbar\frac{\partial\varphi}{\partial t}$ in one dimension $\left[\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t}\right]$

* عندما تكون (Hami) تعتمد على زمن لا يمكن عمل هذه الخطوة

$\frac{-\hbar^2}{2m} T(t) \frac{d^2\psi}{dx^2} = ik\,\varphi(x)\frac{dT(t)}{dt}$

$\frac{-\hbar^2}{2m}\frac{d^2\varphi}{d^2x}\cdot\frac{1}{\psi(x)} = i \qquad \frac{dT(t)}{dt}\cdot\frac{1}{T(t)} = i$

معادلة شرودنجر لا تعتمد على زمن

let the separation constant be E $\frac{-\hbar^2}{2m}\frac{d^2\varphi}{dx^2} = E\varphi$ → [time independent shrödinger equation]

هذا لا يعتمد على زمن بل يعتمد على شكل $\hat{H}$

هذا الاشكال معروف ← $\left[\frac{dT}{dt} = E\,T(t)\right] e^{-Eit}$ ⇒ this equation is independent of the Hamitinen use as long as [The Hamitonion is time]

$T(t) = e^{-iEt}$

* [24-3-2016] *

* The wave equation *

[vibrating string] → وتر مهتز

* موجة انتقالية (transver wave) تكون عمودية على اتجاه الانتشار

* أمواج طولية (Longitudinal wave) تكون موازية على x (اتجاه حركة الجزيئات)

* we well write the wave equation is one dimantion since the string is one dimension

* the y-position of the string is changed from point to point and from time to time

y is function of (x,t) → موقع وزمن

* y is called the amplitued of the vibrating wave

* The Aamplitud y satisted the wave equation as longe as the length between two point in the string

*Q: write the $y(x,t)$ and write the new two ordinary diff. eq result from this supsititution

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2}{\partial x}\left(Y(x)\,T(t)\right) = \frac{1}{v^2}\frac{\partial^2 y}{\partial t}\left(Y(x)\,T(t)\right)$$

$$T(t)\frac{d^2}{dx^2}Y(x) = \frac{Y(x)}{v^2}\frac{d^2}{dt^2}T(t) = -k^2 \qquad \frac{d^2}{dx^2}Y(x) = -k^2\,Y(x)$$

$$\frac{d^2}{dt^2}T(t) = -k^2 v^2\, T(t)$$

$$Y''(x) = -k^2 Y(x) \qquad y(x,t) = \begin{cases}\sin kx \\ \cos kx\end{cases}\begin{cases}\sin kvt \\ \cos kvt\end{cases}$$

$$Y(x) = \begin{cases}\sin kx \\ \cos kx\end{cases}$$

$$T(t) = -k^2 v^2 T(t) \qquad T(t) = \begin{cases}\sin kvt \\ \cos kv\end{cases}$$

* Bondary and ihifraly condion

① The string is fastered at the two point $(x=0,\ x=l)$

$y(0,x) = 0 \rightarrow \cos kx$ indropped → ليس حل

$$y(x,t) = \sin kx \begin{cases}\sin kvt \\ \cos kvt\end{cases}$$

$$y = (x=l,t) = 0 \quad \sin kl \begin{cases}\cos kvt \\ \sin kvt\end{cases} = 0 \Rightarrow y(x,t) = \sin\frac{n\pi}{l}\begin{cases}\sin\frac{n\pi}{l}vt \\ \cos\frac{n\pi}{l}vt\end{cases}$$

$$\sin kl = 0 \Rightarrow \boxed{k = \frac{n\pi}{l}}$$

* The wave velocity → V

$V = \frac{\lambda}{T}$ λ → wave Length, T → Time period

f = frequency $= \frac{1}{T} = \nu$ $\lambda\nu = V$

* wave number → $\boxed{k = \frac{2\pi}{\lambda}}$ * anglear velocity → $\boxed{w = 2\pi f}$

$$w = \frac{2\pi}{T} \qquad V = \frac{w}{k} = \frac{\frac{2\pi}{T}}{\frac{2\pi}{\lambda}} \qquad \boxed{V = \frac{w}{k}}$$

[27-3-2016]

* steady - state temprtuer in a cylinder *

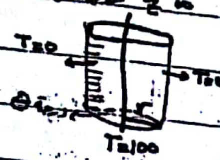
توزيع درجة الحرارة داخل الأسطوانة
(بدون مصدر)
(لا تعتمد على الزمن)

استخدام الإحداثيات الأسطوانية لأن r و θ متغيرات

عند توزيع الحرارة نستخدم (الإحداثيات الكرتيزية)

$\nabla^2 (x,y,z)$

إحداثيات cartiesuc $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$



(r,θ,z)

laplace equation $\rightarrow \nabla^2 u = 0$

$\nabla^2_{(r,\theta,z)} = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

هذا الحل غير صحيح

$$\nabla^2_{(r\theta z)} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \vec{\nabla}\cdot(\nabla_r \hat{e}_r + \nabla_\theta \hat{e}_\theta + \nabla_z \hat{e}_z)$ نسبة البعض

$\nabla^2 u = 0$

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)}_{\text{اشتقاق بالنسبة لـ } r} + \underbrace{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}_{\text{اشتقاق بالنسبة لـ } \theta} + \underbrace{\frac{\partial^2 u}{\partial z^2}}_{\text{اشتقاق بالنسبة لـ } z} = 0$$

* let as assum $u(r,\theta,z) = \underbrace{R(r)}_{\text{radiuns [radiulely part]}}\ \underbrace{\Theta(\theta)}_{\text{[angler part]}}\ Z(z) \rightarrow$ [z-part]

ثابت بالنسبة لـ r

$$\frac{\Theta(\theta)\ Z(z)}{r}\frac{\partial}{\partial r}\left(\frac{\partial R(r)}{\partial r}\right) + \frac{R(r)\ Z(z)}{r^2}\frac{\partial^2 \Theta(\theta)}{\partial \theta^2} + R(r)\ \Theta(\theta)\frac{\partial^2 Z(z)}{\partial z^2} = 0$$

فنستطيع أن إلا إذا علمنا

$$\frac{1}{rR}\left(r\frac{\partial}{\partial r}R\right) + \underbrace{\frac{1}{r^2\Theta}\frac{\partial^2\Theta}{\partial^2\theta} + \frac{1}{Z(z)}\frac{\partial^2 Z}{\partial z^2}}_{} = 0$$

علينا أخذ أحد حدود ونساويه بالثابت نشترط أن لا نحيد عليه إلى الحد الباقية

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \pm k^2 \qquad \frac{1}{Rr}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}R\right) + \frac{1}{r^2\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = \pm k^2$$

نحاول تحويل لسالب أو موجب الذي يحدده حل معادلة

e^x أفترض اشتقاقها مع زيادة

$$\frac{d^2 Z}{dz^2} = \pm k^2 Z$$

عند اختيار موجب يكون الحل e

$$Z(z) = e^{-kz} \quad +k^2 \qquad e^{+kz}$$

عندما تكون (−)

$\underbrace{\cos kz \quad \sin kz}_{\text{قيمتها غير معرفة}}$

$\lim_{n\to\infty} \sin kz$

$\lim_{n\to\infty} \cos kz$

* $Z(z) = e^{-kz}$ spration constant is $+k^2$ not $(-k^2)$

$$\frac{1}{Rr}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}R\right) + \frac{1}{r^2\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = -k^2$$

نضرب في r^2

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = k^2 r^2 \longrightarrow \left[\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - k^2 r^2\right] + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = 0$$

* then $\longrightarrow \frac{1}{\Theta}\frac{d^2}{d\theta^2}\Theta = -n^2$

فقط الأرقام الصحيحة ← (n)

اسطوانة تماثلها تتكرر لا تتحول نفس درجة حرارتها

$$\frac{r}{R}\frac{d}{d}\left(r\frac{d}{dr}R\right) - k^2 r^2 = +n^2$$

الحل الذي فيه الـ بعطي (cos, sin)

$2n\pi$

هذه اشتراطات تكرر نفسها

$$\frac{1}{\Theta}\frac{d^2}{d^2\theta}\Theta = -n^2 \longrightarrow \frac{d^2}{d\theta^2}\Theta = \Theta n^2 \Theta \qquad \Theta = \begin{cases}\sin\theta n \\ \cos\theta n\end{cases}$$

* Bessel diff. equation $\Rightarrow \left[r^2\frac{d^2}{dr^2}R + r\frac{d}{dr}R + (k^2 r^2 - n^2)R = 0\right]$

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[31-3-2016]

[14-4-2016 محاضرات]

[steady-state tempreture in cylinder]

$\nabla^2 u = 0$

$u = R(r)\,\Theta(\theta)\,Z(z)$ $\qquad \Theta(\theta)\begin{cases}\sin n\theta\\ \cos n\theta\end{cases}$

$R(r) = r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (K^2r^2 - n^2)R = 0$

$R(r) \equiv$ Bessel function $\qquad R(r) = J_n(Kr)$

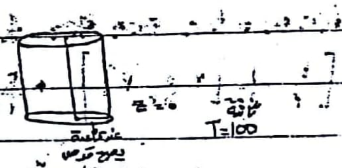
from the Bounudary condition we $u|_{r=a} = 0$ we should have $J_n(Ka) = 0$ That is Ka the Zero of Bessel function

*letus denote these Zeros by: K_m then $Ka = K_m$

$K = \frac{K_m}{a}$ (أصفار Bessel / نصف قطر اسطوانة)

*The solution of the steady-state Tempreture in cylinder is

$u(r,\theta,z) = \left\{ J\left(\frac{k_m}{a}r\right) \right\}\begin{cases}\cos n\theta\\ \sin n\theta\end{cases}\left\{ e^{-\frac{k_m}{a}z}\right.$



*at $z=0$; $u=100$ $\qquad e^{-\frac{k_m}{a}\cdot 0} \rightarrow e^{0} = 1$

$u(r,\theta) = \left\{ J_n\left(\frac{k_m}{a}r\right)\right. \quad \cos n\theta \left\{ 1 \right. = 100$

$\sin n\theta = \cos n\theta \qquad n = \text{zero}$

عندما $n=0$ نحصل على $\sin$, $\cos$ تكون ثابتة $\qquad$ غير موجودة $\underline{\sin n\theta = 100}$; $n=0$ X

الحل الوحيد الذي يحقق الشرط $T=100$ $\cos\theta n$ = constant when n = Zero = 1

*then n = Zero and only positive compination of solution is Interms of J_0 at $z=0$

the gneral solution for the cylinder is $\left[u = \sum_{m=0}^{m} c_n J_0\left(\frac{k_m}{a}r\right) e^{-\frac{k_m}{a}z}\right.$

$u = c_0\, J_0\left(\frac{k_0}{a}r\right) e^{-\frac{k_0 z}{a}}$; $u = c_1\, J_0\left(\frac{k_1}{a}r\right) e^{-\left(\frac{k_1 z}{a}\right)}$; $u = c_2\, J_0\left(\frac{k_2}{a}r\right) e^{-\frac{k_2 z}{a}} \Big]$ هذه حلول التي تمثلها Bessel function

*inorder to find the constant we note that

$u = (z=0) = 100 = \sum c_n J_0\left(\frac{k_m}{a}r\right) \cdot \underbrace{e^{-\frac{k_m}{a}r}}_{=1}$

$100 = \sum_{m=0}^{\infty} c_m J_0\left(\frac{k_m}{a}r\right)$

from the orthougully at Bessel function with respect to the weight function "r"

$c_n = \dfrac{\int 100\, J_0\left(\frac{k_m}{a}r\right) r\,dr}{\int_0^a \left[J_0\left(\frac{k_m}{a}\right)\right]^2 r\,dr} = \dfrac{\dfrac{100\,a^2\, J_1\left(\frac{k_m}{a}r\right)}{(k_m)}}{\frac{a^2}{2}\, J_1^2\left(\frac{k_m}{a}r\right)}$

(Bessel function) ايجاد قيمة c_n اشتقاقها من تعامدية

$\int_0^{\infty} \sin nx \,\sin mx\, dx = 0$

$\sin\theta_i \quad -\cos\theta_j$

$\sin^2\theta + \cos^2\theta = 1$

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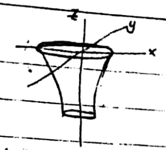
[3-4-2016]

Vibration of a circular membrane



z اتجاه الاهتزاز

$$\nabla^2 Z = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} Z$$

$$\nabla^2 Z(x,y,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} Z(x,y,t)$$

اهتزاز باتجاه z

توزيع اهتزازات معينة (x, y)

let $Z(x,y,t) = F(x,y)\,T(t)$

$$\nabla^2 F(x,y) + k^2 F(x,y) = 0 \longrightarrow \text{[Helmholtz equation]}$$

$$\ddot{T}(t) + k^2 v^2 T(t) = 0 \longrightarrow T(t) = \begin{cases} \sin kvt \\ \cos kvt \end{cases}$$

$$\nabla^2 F(x,y) + k^2 F(x,y) = 0$$

* Inorder to simplify the solution we use the polar coordinat system

$$\nabla^2_{(r,\theta)} F(r,\theta) + k^2 F(r,\theta) = 0$$

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}F(r,\theta)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}F(r,\theta)}_{\nabla^2(r,\theta)} + k^2 F(r,\theta) = 0$$

– Let $F(r,\theta) = R(r)\,\Theta(\theta)$ then:

$$\left[\frac{1}{r}\frac{d}{dr}\left[r\frac{\partial}{\partial r}R\right]\right] + \frac{R}{r^2}\frac{\partial^2}{\partial\theta^2}\Theta + k^2 R\Theta = 0$$

$$\frac{1}{R}\left(\frac{1}{r}\frac{d}{dr}\left[r\frac{\partial}{\partial r}R\right]\right) + \frac{1}{r^2\Theta}\frac{\partial^2}{\partial\theta^2}\Theta + k^2 = 0$$

$$\underbrace{\left(\frac{r^2}{R}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}R\right)\right) + k^2 r^2\right)}_{+n^2} + \underbrace{\frac{1}{\Theta}\frac{\partial^2}{\partial\theta^2}\Theta}_{-n^2} = 0$$

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + k^2 r^2 - n^2 = 0$$

$$\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = -n^2 \Rightarrow \frac{\partial^2}{\partial\theta^2}\Theta = -n^2\Theta \qquad \Theta(\theta) = \begin{cases} \sin n\theta \\ \cos n\theta \end{cases}$$

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + k^2 r^2 - n^2 = 0 \longrightarrow \text{the radial part equation Bessel's equation}$$

The solution are $J_n(kr)$ and $N_n(kr)$

$r \to 0$, $\frac{1}{0}$ = divergent

[5-4-2016]

محاضرة الاحداثيات الكروية :

احداثيات عادية ..... $v = \int \text{(dx dy dz)}$

احداثيات كروية $v = \int r^2 \sin\theta \, dr \, d\theta$

or $\hat{r}, \hat{\theta}, \hat{\phi}$

(ϕ, r, θ) اقتران بدلالة

Steady-state Temperature inasphere

$\nabla^2 u = 0$

$u \to$ is a function of the spherical coordinate system

$u \to u(r, \theta, \Phi)$

$$\nabla^2_{(r,\theta,\Phi)} = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\cdot\frac{\partial^2 u}{\partial \Phi^2} = 0$$

$\frac{d}{dr}$ ، $\frac{d}{d\theta}$ ، $\frac{d}{d\Phi}$ تفاضل جزئي

$$\underbrace{\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} u\right)\right)}_{\text{اشتقاق بالنسبة لـ } r} + \underbrace{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right)}_{\text{اشتقاق بالنسبة لـ } \theta} + \underbrace{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \Phi^2}}_{\text{اشتقاق بالنسبة لـ } \Phi} = 0$$

*let us assume that $u(r,\theta,\Phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$

then

$$\Theta(\theta)\Phi(\phi)\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} R(r)\right)\right) + \frac{R(r)\Phi(\phi)}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta(\theta)}{\partial \theta}\right) + \frac{R(r)\Theta(\theta)}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}\Phi(\phi) = 0$$

$$\frac{1}{R(r)\, r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} R(r)\right)\right) + \frac{1}{\Theta(\theta)\, r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta(\theta)}{\partial \theta}\right) + \frac{1}{\Phi(\phi)\, r^2 \sin^2\theta}\frac{\partial^2}{\partial \Phi^2}\Phi(\phi) = 0$$

نضرب بـ r^2

$$\underbrace{\frac{1}{R(r)}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} R(r)\right)\right]}_{(R(r))\text{ اقتران}} + \underbrace{\frac{r^2}{\Theta(\theta)\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta(\theta)}{\partial \theta}\right) + \frac{1}{\Phi(\phi)\sin^2\theta}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2}}_{\text{اقتران بدلالة } (\Phi, \Theta)} = 0$$

constant + constant = 0 − m

[7-4-2016]

$$\frac{1}{R r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{1}{\Theta r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\Theta\right) + \frac{1}{\Phi r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

نضرب بـ $r^2, \sin\theta^2$

$$\underbrace{\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} R\right) + \frac{\sin\theta}{\Theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\Theta\right)}_{(r,\theta) = +m^2} + \underbrace{\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}}_{\Phi = -m^2} \to \Phi(\phi) = \begin{cases} \sin\phi m \\ \cos\phi m \end{cases}$$

نقسم على $\sin^2\theta$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} R\right) + \frac{1}{\Theta \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\Theta\right) = \frac{m^2}{\sin^2\theta}$$

$$\underbrace{\left[\frac{1}{R}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} R\right)\right)\right]}_{K} + \underbrace{\left[\frac{1}{\Theta \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) - \frac{m^2}{\sin^2\theta}\right]}_{\Theta} = 0 \quad -K$$

[associated Legender Equations]

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} R\right) = K \longrightarrow \frac{1}{\Theta \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) - \frac{m^2}{\sin^2\theta} = -K$$

$$\Downarrow$$

$$\frac{1}{\Theta \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) - \frac{m^2}{\sin^2\theta} + K = 0$$

• associated Legender Equations :-

- solution $x \to$ replace by $\cos\theta$ — The solution $P_l^m(\cos\theta)$ $\qquad \frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) = l(l+1)$

$$R(r) = \begin{cases} r^l \\ r^{-l-1} \end{cases} \qquad u(r,\theta,\Phi) = r^l \, P_l^m(\cos\theta) \begin{cases} \sin m\phi \\ \cos m\phi \end{cases}$$

Stedy-state Temprature in a sphere [10-4-2016]

$\nabla^2 u(r, \theta, \Phi)$

$\Theta(\theta) = P_k^m(\cos\theta)$ $\Phi(\phi) = \begin{cases} \sin m\phi \\ \cos m\phi \end{cases}$ نختار هذا الحل لأنها اقترانات دورية

$R(r) = \begin{cases} r^{l} \\ r^{-l-1} \end{cases}$ نستثني هذا الحل لأنه يعطي عند صفر $\begin{bmatrix} e^{m\phi} \\ e^{-m\phi} \end{bmatrix}$ نهمل هذا الحل لأنه لا يكرر نفسه

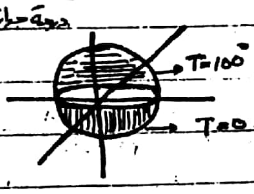
$$u = r^{l} P_l^m(\cos\theta) \begin{cases} \sin m\phi \\ \cos m\phi \end{cases}$$

درجة حرارة ثابتة وزاوية متغير

نقطة (موقع) $r = a$

محيط Φ $(x-y)$

الحل متماثل ضمن $m=0$



inorder to satisfy the given boundary condition $m=0$

$m=0$ $P_l^0(\cos\theta) = P_l(\cos\theta)$

$u = r^{l} P_l^0(\cos\theta)$

$u = r^{l} P_l(\cos\theta)$

then $u = \sum^{\infty} c_l r^{l} P_l(\cos\theta)$

$$u(r=a) = \begin{cases} 100 & 0 < \theta < \frac{\pi}{2} \quad \text{قطب شمالي كرة} \\ 0 & 0 > \theta > -\frac{\pi}{2} \quad \text{قطب جنوبي} \end{cases}$$

$\begin{cases} 0 < \cos\theta < 1 \\ -1 < \cos\theta < 0 \end{cases}$ if we let $\cos\theta = x$

$$u(r=a) = \begin{cases} 100 & 0 < x < 1 \\ 0 & -1 < x < 0 \end{cases}$$

$$u(r, x, \phi) = \sum c_l r^{l} P_l(x)$$

درجة الحرارة عند نصف القطر $u(a, x, \phi) = \begin{cases} 100 & 0 < x < 1 \\ 0 & -1 < x < 0 \end{cases} = \sum_{l=0}^{\infty} c_l a^{l} P_l(x)$

Fourier series and transforms [14-4-2016]

The Average of a function متوسط الدالة

Given the function f(x) defined on the interval (a,b) then the average of the function Avg f(x) is given by $\text{Avg } f(x) = \dfrac{\int_a^b f(x)\,dx}{b-a}$

Ex) Find the average of the following functions on the interval shown

① e^x $x \in (-5,5)$ ② $x \in (-2,2)$ ③ $\sin nx$ $x \in (0,2\pi)$ ④ $\sin^2 nx$ $x \in (0,2\pi)$

① $\text{Avg } e^x = \dfrac{\int_{-5}^{5} e^x dx}{(5-(-5))} = \dfrac{e^x\Big]_{-5}^{5}}{10} = \dfrac{e^5 - e^{-5}}{10}$

② $\text{Avg } x = \dfrac{\int_{-2}^{2} x\,dx}{(2)-(-2)} = \dfrac{\frac{x^2}{2}\Big]_{-2}^{2}}{4} = \dfrac{\frac{4}{2}-\frac{4}{2}}{4} = \dfrac{0}{4} = \text{Zero}$

③ $\text{Avg } \sin nx = \dfrac{\int_0^{2\pi} \sin nx\,dx}{(2\pi-0)} \longrightarrow \dfrac{-\cos nx}{n(2\pi)}\Big]_0^{2\pi} = \dfrac{-1}{2\pi}\left[\cos n2\pi - \cos n0\right]$ $\begin{cases} n\text{ even} = 0 \\ n\text{ odd} = 0 \end{cases} = \text{Zero}$

4) $\text{Avg } \sin^2 nx = \dfrac{\int_0^{2\pi} \sin^2 nx\,dx}{(2\pi-0)}$ $\dfrac{\int \frac{1}{2} - \frac{1}{2}\cos 2nx}{2\pi} = \dfrac{\frac{1}{2}x\Big] - \frac{1}{2}\frac{\sin 2nx}{2n}\Big]_0^{2\pi}}{2\pi}$

$\dfrac{1}{2\pi}\left[\frac{1}{2}2\pi - 0\right] - \dfrac{1}{4n}\left[\underbrace{\sin n4\pi - \sin n0}_{\text{zero}}\right] = \dfrac{1}{2\pi}\cdot\pi = \dfrac{1}{2}$

* The average value of sine equal to the average value of the function
* Show that Average value of a periodic function over it interval L is the same over any interval with the same length $\int_0^{2\pi} \cos^2 nx\,dx = \frac{1}{2} = \int_{\pi/4}^{2\pi} \cos^2 nx\,dx$

~~let~~ Let the length of the periodic the since the function is periodic

$f(x+p) = f(x)$

The average value of f(x) is $= \dfrac{\int_0^p f(x)\,dx}{p}$

Find avg f(x) of (a, a+p) then $\text{Avg } f(x) = \dfrac{\int_a^{a+p} f(x)}{p}dx$

$x = y + p \longrightarrow y = x - p$ $dx = dy$ $x = a$ $y = a - p$ $x = a+p$ $\boxed{y = a}$

$\dfrac{\int_{a-p}^{a} f(x)\,dx}{p}$ let $a = p$ $= \dfrac{\int_0^p f(x)\,dx}{p}$

أبو لحنة و ~~كيد~~ لها الحيث لتحصل على شئ ولن المعرفة

[Hw] [Section 4]

Find the average value

) $\cos^2 \frac{x}{2}$ on $(0, \frac{\pi}{2})$

$\cos^2 \frac{x}{2} = \frac{1}{2}(1 + \cos \frac{2x}{2})$

$\cos^2\frac{x}{2} = \frac{1}{2}(1 + \cos x)$

$\frac{\frac{1}{2}\int_0^{\pi/2} 1 + \cos x \, dx}{(\frac{\pi}{2} - 0)} = \frac{2}{\pi} \cdot \frac{1}{2} \int_0^{\pi/2} 1 + \cos x \, dx$

$\frac{1}{\pi}\left[x + \sin x \Big|_0^{\pi/2} \right] = \frac{1}{\pi}\left[(\frac{\pi}{2} + \sin\frac{\pi}{2}) - (0 + \sin 0) \right]$

$\frac{1}{\pi}\left[\frac{\pi}{2} + 1\right] = \frac{1}{2} + \frac{1}{\pi}$

6) $\sin x$ on $(0, \pi)$

$\frac{\int_0^\pi \sin x \, dx}{(\pi - 0)} = \frac{1}{\pi}\int_0^\pi \sin x \, dx \longrightarrow \frac{1}{\pi}(-\cos x \Big]_0^\pi)$

$-\frac{1}{\pi}[\cos \pi - \cos 0] = \frac{-1}{\pi}[-1 - 1] = \frac{-1}{\pi}(-2) = \frac{2}{\pi}$

13) using (4.3) and equation similar to (4.5) to (4.7) show that 4.3 Avg $\frac{\int_a^b f(x)\,dx}{b-a}$

$$\int_a^b \sin^2 kx \, dx = \int_a^b \cos^2 kx \, dx = \frac{1}{2}(b-a)$$

$\sin^2 kx = \frac{1}{2} - \frac{1}{2}\cos 2kx$

$\frac{1}{2}(b-a) - \frac{1}{4k}(\sin 2kb - \sin 2ka)$

$\frac{\int_a^b \frac{1}{2} - \frac{1}{2}\cos k2x \, dx}{(b-a)}$

$\frac{1}{2}x\Big|_a^b - \frac{1}{4k}\sin 2kx\Big|_a^b$

14) use the results of problem 13 to evaluate the following integrals with calculation

a) $\int_0^{4\pi/3} \sin^2(\frac{3x}{2})\,dx \longrightarrow \sin^2(\frac{3x}{2}) = \frac{1}{2}(1 - \cos 3x)$

$\frac{1}{2}\int_0^{4\pi/3} 1 - \cos 3x \, dx$

$\frac{1}{2}(x - \frac{1}{3}\sin 3x)\Big]_0^{\frac{4\pi}{3}}$

$\frac{1}{2}\Big[(\frac{4\pi}{3} - \underbrace{\frac{1}{3}\sin 3\frac{4\pi}{3}}_{\text{zero}}) - (0 - \underbrace{\frac{1}{3}\sin 3\,0}_{\text{zero}})\Big]$

$\frac{1}{2} \cdot \frac{4\pi}{3} = \frac{2\pi}{3}$

b) $\int_{-\pi/2}^{3\pi/2} \cos^2(\frac{x}{2})\,dx \qquad \cos^2(\frac{x}{2}) = \frac{1}{2}(1 + \cos 2\frac{x}{2}) = \frac{1}{2}(1 + \cos x)$

$\frac{1}{2}\int_{-\frac{\pi}{2}}^{3\pi/2} \cos x + 1 \, dx = \frac{1}{2}\left[\sin x + x\right]_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}$

$\frac{1}{2}\left[\sin\frac{3\pi}{2} + \frac{3\pi}{2} - \sin(-\frac{\pi}{2}) + \frac{\pi}{2}\right]$

$\frac{1}{2}\left[\cancel{-1} + \frac{4\pi}{2} + \cancel{1}\right] = \frac{1}{2} \cdot 2\pi = \pi$

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15) a) $\int_{-1/4}^{11/4} \cos^2 \pi x \, dx$

$\cos^2 \pi x = \frac{1}{2}(1 + \cos 2\pi x)$

$\frac{1}{2}\int_{-\frac{1}{4}}^{11/4} (1 + \cos 2\pi x)\,dx = x + \frac{\sin 2x\pi}{2\pi} \Big]_{-\frac{1}{4}}^{\frac{11}{4}}$

$\frac{1}{2}\left[\left(\frac{11}{4} + \frac{1}{2\pi}\sin 2\pi \frac{11}{4}\right) + \left(\frac{-1}{4} + \frac{1}{2\pi}\sin\left(-\frac{1}{4} \cdot 2\pi\right)\right)\right] \rightarrow \frac{1}{2}\left[\left(\frac{11}{4} + \frac{1}{2\pi}\sin \frac{11}{2}\pi\right) - \left(-\frac{1}{4} + \frac{1}{2\pi}\sin\left(-\frac{\pi}{2}\right)\right)\right]$

$\frac{1}{2}\left[\frac{11}{4} - \cancel{\frac{1}{2\pi}} + \frac{1}{4} + \cancel{\frac{1}{2\pi}}\right] = \frac{1}{2} \cdot \frac{\cancel{12}^{3}}{\cancel{4}_{1}} = \frac{3}{2}$

b) $\int_{-1}^{2} \sin^2\left(\frac{\pi x}{3}\right) dx \qquad \sin^2\left(\frac{\pi x}{3}\right) = \frac{1}{2}(1 - \cos 2x)$

$\frac{1}{2}\int_{-1}^{2} 1 - \cos \frac{2\pi x}{3}\,dx \rightarrow \frac{1}{2}\left[x - \frac{3}{2\pi}\sin \frac{2x\pi}{3}\right]_{-1}^{2} \rightarrow \frac{1}{2}\left[\left(2 - \frac{3}{2\pi}\sin \frac{4\pi}{3}\right) - \left(-1 - \frac{3}{2\pi}\sin \frac{-2\pi}{3}\right)\right] \qquad \frac{4\pi}{3} = 240$

$\frac{1}{2}\left[\left(2 - \frac{3}{2\pi}\left(\frac{-\sqrt{3}}{2}\right)\right) + \left(1 - \frac{3}{2\pi}\frac{\sqrt{3}}{2}\right)\right] = \frac{1}{2}\left[2 + \cancel{\frac{3\sqrt{3}}{4\pi}} + 1 - \cancel{\frac{3\sqrt{3}}{4\pi}}\right] = \frac{3}{2}$

Dirichlet condition

يكرر نفس الشكل في فترة محددة — الاقتران له قيمة واحدة — الفترة

Given the peroidice single valued function $f(x)$ in the intrvel $(-L, L)$ of extreman of $f(x)$ has afinite number of discontinious and infinte number exterment point with the integral $\int_{-L}^{L} |f(x)|\, dx$ bounded the $f(x)$ is expandable use fourier series expansion that is [The Fourier series converges to the function $f(x)$ and converges to the midpoint discontinouty]

غير متصل — اقتران يمكن فكه — نقاط الانقطاع

complex from of fourie series

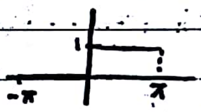
Gives the functions $f(x)$, find above Dirichlet condition then :- $f(x)$ can be expanded (complex fourier series as following)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $\quad c_n = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$

Ex) Expand the function below using complex function (fourier series)

$$f(x) \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$



$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad ; \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$c_n = \frac{1}{2\pi} \int_{0}^{\pi} (1)\, e^{-inx}\, dx \longrightarrow \frac{1}{2\pi} \left. \frac{e^{-inx}}{-in} \right]_{0}^{\pi}$$

$$= \frac{1}{2\pi} \left[\frac{e^{-in\pi}}{-in} - \frac{e^{-in0}}{-in} \right] = \frac{1}{2\pi} \left[\frac{e^{-in\pi}}{-in} - \frac{e^{0}}{-in} \right] = \frac{1}{2\pi} \left[\frac{e^{in\pi}}{-in} + \frac{1}{in} \right]$$

$(-1)^n$ even $= -1 + 1 = $ zero

$(-1)^n$ odd $\dfrac{-1-1}{-1} = \dfrac{-2}{-1} = 2\cdot\dfrac{1}{2\pi n}$

$e^{-in\pi} = \cos n\pi - i\cancel{\sin n\pi}^{\text{zero}} \qquad e^{-in\pi} = \cos n\pi$

$$\begin{cases} 0 & , n \text{ even} \\ \dfrac{1}{in\pi} & , n \text{ odd} \end{cases}$$

قيمة $n = 0$ نعوضها بالاقتران مع (تكامل)

$$c_n = \frac{1}{2\pi} \left[\frac{(-1)^n - 1}{-in} \right]$$

$$c_0 = \frac{1}{2\pi} \int_0^{\pi} e^{i0n} = \frac{1}{2\pi} \int_0^{\pi} e^{0}\, dx = \frac{1}{2\pi} [\pi - 0] = \frac{1}{2}$$

$$c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_3 e^{3ix} + c_{-3} e^{-3ix} + \dots$$

$$\frac{1}{2} + \frac{1}{2\pi} e^{ix} - \frac{1}{2\pi} e^{-ix} + \frac{1}{3i\pi} e^{3ix} - \frac{1}{3i\pi} e^{-3i\pi}$$

$$\frac{1}{2} + \frac{2}{\pi} \left[\frac{e^{ix} - e^{-ix}}{2} + \frac{1}{3} \cdot \frac{e^{3ix} - e^{-3ix}}{2i} + \dots \right] = \frac{1}{2} + \frac{2}{\pi} \left[\sin x + \frac{\sin 3x}{3} + \dots \right]$$

Interval other than $(-\pi, \pi)$

* let function $f(x)$ satisfies the Drichlect condition on the interval $(-L, L)$ then the function can be explaned use Ferient series given as folowing

$$f(x) = c_n + \sum_{n=1}^{\infty} \frac{a_0}{2} \cos \frac{n\pi}{2} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{2} x$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\,dx \qquad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\,dx \qquad c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\,dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x\,dx$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L}\,dx$$

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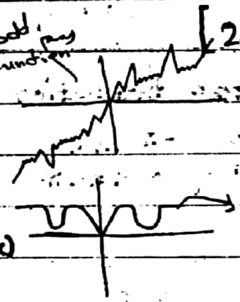
* Even and odd functions * [24-4-2016]

رسم odd function



رسم even function

- The function $f(x)$ is said to odd function if $f(-x) = -f(x)$
- The function $g(x)$ is said to even function if $g(-x) = g(x)$

*اقتران ثابت (even)

$f(x) = x + 1$

ليس odd ولا even

ليس even ولا odd

---

*[Show that any function can be written interms as sum of even and odd function ?]

[illegible]

Solution

* Given the function $M(x)$

let as $M_e(x) = \frac{M(x) + (M(-x)}{2}$ $\qquad M_o(x) = \frac{M(x) - M(x)}{2}$

*Note $M_e(x) = M_e(x)$ even

$M_o(-x) = -M_o(x)$ odd

But $\rightarrow M(x) = M_o(x) + M_e(x)$

Ex) Find the odd part of function $e^{-x^2 + \sin x}$

ـ واذا أعطيت جزء (even) مطلوب إيجاد (odd part) يكون يساوي صفر

ـ وإذا كان موجود (odd part) مطلوب إيجاد (even part) يكون = صفر

$$\leftarrow M_o(x) = \frac{M(x) - M(-x)}{2} \rightarrow \frac{e^{-x^2 + \sin x} - e^{-x^2 + (\sin(-x))}}{2} \rightarrow \frac{e^{-x^2}}{2}\left(e^{\sin x} - e^{-\sin x}\right)$$

$$\leftarrow M_e(x) = \frac{M(x) + (M(-x)}{2} \rightarrow \frac{e^{-x^2 + \sin x} + e^{-x^2 + \sin x}}{2} \rightarrow \frac{e^{-x^2}}{2}\left(e^{\sin x} + e^{-\sin x}\right)$$

Note: $\int_{-L}^{+L} f(x)\,dx = \begin{cases} 0 & \text{if } f(x) \text{ odd} \\ 2\int_0^L f(x)\,dx & f(x) \text{ even} \end{cases}$ هذا صحيح فقط بالفترة $(-L,+L)$

$$\int_{-L}^{+L} f(x)\,dx = \underbrace{\int_{L}^{0} f(x)\,dx}_{} + \int_0^L f(x)\,dx$$

let $x' = -x$ | $f(x) = f(-x)$
$x = -L$ $\boxed{x = L}$ | $\int_{-L}^{+L} f(x)\,dx = \underbrace{\int_{-L}^{0} f(x)\,dx} + \int_0^L f(x)\,dx$
$dx' = -dx$ $x = 0$, $x' = 0$

استخدم $\int_{L}^{0} f(x)\,dx$

$-\int_{L}^{0} f(-x)\,dx$

عكس حدود $\int_{0}^{L} f(-x)\,dx$

$$\int_{-L}^{L} f(x)\,dx = \int_0^L f(-x)\,dx + \int_0^L f(x)\,dx \longrightarrow \int_0^L f(x) + f(x)\,dx$$

إذا كان اقتران odd فقط بالفترة $(-L,L)$

$\int_0^L -f(x) + f(x) = \text{Zero}$

إذا كان اقتران even $\int_0^L f(x) + f(x)\,dx$

$2\int_0^L f(x)\,dx$

* Frourier series expansion for odd and even function

Given the function $f(x)$ satisfies Dricht countion the $f(x)$ is expand using Frourier seris expresion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x + \sum_{n=1} b_n \sin\frac{n\pi}{L}x$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$ $\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$

$y = |f(x)|$ even function

Note: if $f(x)$ is odd The $a_n = 0$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{\pi n}{L}x\,dx$$

Note: if $f(x)$ even $a_n = \frac{2}{L}\int_0^L \cos\frac{n\pi}{L}x\,dx$ $\quad b_n = 0$

$|x+2|$ ليست odd ولا even

$\sqrt{x}$ ← ليس odd ولا even

x^2 even, x odd

even function $[\cos x]$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$

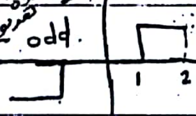
$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

the odd function

$x^2 + 2 = $ even
even, even

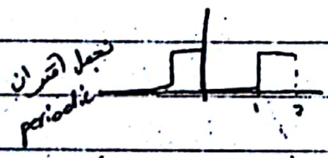
* expantion fourite series

اعادة تعريف odd



← expand part $\sin x$ odd

← expand part $\cos x$ even



odd * even = odd
$x \cdot x^2 = x^3$

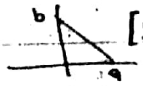
odd * odd = even
$x \cdot x = x^2$

even * even = even
$x^2 \cdot x^2 = x^4$

odd + odd = odd

even + even = even

[26-4-2016]



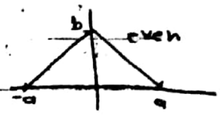
Even and odd function

Ex) Expand the following function using Fourier sine series

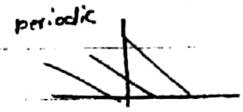
$$f(x) = \begin{cases} 1 & 0 < x < \frac{1}{2} \\ 0 & \frac{1}{2} < x < 1 \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} f(x) \sin \frac{n\pi}{L} x \, dx \qquad L = 1-(-1) = 2$$

$$2L = 2 = \boxed{L = 1}$$



$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}$$



$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx = 2 \int_0^1 f(x) \sin n\pi x \, dx$$

$$= 2 \left[\int_0^{1/2} (1) \sin n\pi x \, dx + \int_{\frac{1}{2}}^{1} (0) \sin n\pi x \, dx \right] = \frac{-2}{n\pi} \left[\cos n\pi x \right|_0^{\frac{1}{2}}$$

$$\frac{-2}{n\pi} \left(\cos \frac{n\pi}{2} - 1 \right) \qquad \frac{-2}{n\pi} \begin{cases} -1 & 1, 5, 3, 9 \ldots \\ -2 & 2, 6, 10, 14 \ldots \\ 0 & 4, 8, 12 \ldots \end{cases}$$

$$b_n = \begin{cases} \frac{2}{n\pi} & n \text{ odd} \\ \frac{4}{n\pi} & n = 2, 6, 10, 14 \ldots \\ 0 & n = 4, 8, 12 \ldots \end{cases}$$

$$\therefore f(x) = \frac{2}{\pi} \left(\sin \pi x + \frac{2 \sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \frac{2 \sin 6\pi x}{6} + \ldots \right)$$

* Fourier transforms *

$$F(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx}$$

if n becomes continous α ; then $f(x) = \int_{-\infty}^{\infty} c_\alpha e^{i\alpha x} d\alpha = \int_{-\infty}^{+\infty} y(x) e^{i\alpha x} dx$

given the function $f(x)$ satisfies centain condition then The function $f(x)$ come be transform using fouriere transforms as

$$\boxed{f(x) = \int_{-\infty}^{+\infty} g(\alpha) e^{i\alpha x} d\alpha}$$

where $g(\alpha)$ is called the fourier transformation of $f(x)$

where $g(\alpha)$ is given by the inverse fourier transforms $\boxed{g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx}$

*** Sine fourier transforms

- $f_s = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_s(\alpha) \sin\alpha x \, d\alpha$
- $g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(x) \sin\alpha x \, dx$

fourier space → فضاء الزخم → momentum space

*** cosine fourier transforms

- $f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_c(\alpha) \cos\alpha x \, d\alpha$
- $g_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos\alpha x \, dx$

[3-5-2016]

* Fourier transforms *

The complex Fourier transform

$$f(x) = \int_{-\infty}^{+\infty} g(\alpha)\, e^{ix\alpha}\, d\alpha$$

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ix\alpha}\, dx$$

Ex) Find the Fourier transforms for the function shown in the figure below

$$f(x) \begin{cases} 1 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \int_{-\infty}^{\infty} \underbrace{g(\alpha)}_{\text{Fourir transforms}}\, e^{-i\alpha x}\, d\alpha$$

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)\, e^{-i\alpha x}\, dx$$

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$g(\alpha) = \frac{1}{2\pi} \int_{-1}^{1} f(1)\, e^{-i\alpha x}\, dx = \frac{1}{2\pi} \left[\frac{e^{-i\alpha x}}{-i\alpha} \right]_{-1}^{1}$$

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$$

$$g(\alpha) = \frac{1}{2\pi} \left[\frac{e^{-i\alpha} - e^{i\alpha}}{-i\alpha} \right] = \frac{1}{\pi\alpha} \left[\frac{e^{i\alpha} - e^{-i\alpha}}{2i} \right] \qquad g(\alpha) = \frac{1}{\pi\alpha} \sin\alpha$$

* Evaluate of the following Integral

$$I = \int_{-\infty}^{\infty} \frac{\sin y}{y}\, e^{ixy}\, dy = \begin{cases} \pi & -1<x<1 \\ 0 & \text{otherwise} \end{cases}$$

f(x) نفسه

$$\text{find} \int_{-\infty}^{+\infty} \frac{\sin\alpha}{\alpha}\, d\alpha = \pi$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, e^{i2x}\, dx = \text{zero}$$

خارج الفترة

$-1 < 2x < +1$

** [The generation function of Legendre's polynomials] ** 14-2-2016

$(1-x^2)y'' - 2xy' + l(l+1)y = 0$

$P_l(x) = \sum a_n x^n \quad P(1) = 1$

$\boxed{P_l(x) = \frac{1}{2^l}\frac{1}{l!}\frac{d^l}{dx^l}(x^2-1)^l} \qquad a_{n+2} = \frac{-(l+n+1)(l-n)}{(n+1)(n+2)} a_n$

The generating function of Legendre's polynomials

$$\boxed{\Phi(x,h) = \frac{1}{\sqrt{1-2xh+h^2}}}$$

If we expand $\frac{1}{\sqrt{1-2xh+h^2}}$ using Taylor series expansion in terms of the variable "h"

if [h = 0] we found $\Phi(x,h) = P_0(x) + P_1(x)h + P_2(x)h^2 + P_3(x)h^3 + \dots$

($P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$; expansion coefficients: Legendre's polynomials)

where $P_1, P_2, P_3, P_0 \dots$ the Legendre's polynomials

$$\boxed{\Phi(x,h) = \sum_{n=0}^{\infty} \frac{\frac{\partial^n \Phi(x,h)}{\partial h^n}\Big|_{(h=0)}}{n!} h^n}$$

$n=0 \longrightarrow \frac{\partial^0 \Phi / \partial h^0}{0!}\Big|_{h=0} h^0 = \frac{1}{\sqrt{1-2xh+h^2}} = \frac{1}{1} = 1 = P_0(x)$

$n=1 \longrightarrow \frac{\partial \Phi / \partial h}{1!}\Big|_{h=0} h' = \left(\frac{-1}{2}\right)\frac{-2x+2h}{(1-2xh+h^2)^{3/2}} = x = P_1(x)$

$n=2 \longrightarrow \frac{\partial^2 \Phi / \partial^2 h}{2!}\Big|_{h=0} h^2 = \frac{1}{2!}\left[+3x^2-1\right] = \frac{1}{2}(3x^2-1) = P_2(x)$

Inorder to proved that the expansion coeffecient of the generation function $\Phi(x,h)$ is Legendre's polynomials we use the following difference equation which is satisfied by the generating function $\Phi(x,h)$

$\Phi(x,h) = P_0 + P_1 h + P_2 h^2 + \dots$

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