Norman Schofield · Gonzalo Caballero · Daniel Kselman *Editors*

Advances in Political Economy

Institutions, Modelling and Empirical Analysis

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unemployment continues to rise. At some point, public spending cuts may seem an inappropriate, unjust and harsh response to a problem that is increasingly viewed as intractable to short-term solutions.

Finally, the fact that valence politics variables do much to drive the composite vote intention model indicates that attitudes toward the spending cuts will not be the sole drivers of party support in the next general election. Rather than respond directly and reflexively to the conditions around them, British voters place economic hardships and policy in broader context with images of party leaders, partisan attachments and more global assessments of party performance. Differing attitudes about the harsh austerity measures are exerting substantial effects on party support, but these attitudes have not negated the force of valence politics considerations. Rather, reactions to the evolving state of the economy coupled with mutable partisan attachments and the more general evaluations of party and leader performance that voters are making can be expected to animate the model in predictable ways in the years ahead. Performance politics remains important for understanding electoral choice in Britain and other mature democracies as the present era of economic hardship and austerity policies unfolds.

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Modeling Elections with Varying Party Bundles: Applications to the 2004 Canadian Election

Kevin McAlister, Jee Seon Jeon, and Norman Schofield

1 Introduction

Early work in formal political theory focused on the relationship between constituencies and parties in two-party systems. It generally showed that in these cases, parties had strong incentive to converge to the electoral median (Hotelling 1929; Downs 1957; Riker and Ordeshook 1973). These models assumed a one-dimensional policy space and non-stochastic policy choice, meaning that voters would certainly vote for a party. These models showed that there exists a Condorcet point at the electoral median. However, when extended into spaces with more than one dimension, these two-party pure-strategy Nash equilibria generally do not exist. While attempts were made to reconcile this difference, the conditions necessary to assure that there is a pure-strategy Nash equilibrium at the electoral median were strong and unrealistic with regards to actual electoral systems (Caplin and Nalebuff 1991).

Instead of pure-strategy Nash equilibria (PNE) there often exist mixed strategy Nash equilibria, which lie in the subset of the policy space called the uncovered set (Kramer 1978). Many times, this uncovered set includes the electoral mean, thus giving some credence to the median voter theorem in multiple dimensions (Poole and Rosenthal 1984; Adams and Merrill 1999; Merrill and Grofman 1999; Adams

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2001). However, this seems at odds with the chaos theorems which apply to multidimensional policy spaces.

The contrast between the instability theorems and the stability theorems suggest that a model in which the individual vote is not deterministic is most appropriate (Schofield et al. 1998; Quinn et al. 1999). This kind of stochastic model states that the voter has a vector of probabilities corresponding to the choices available in the election. This insinuates that if the voter went to the polls for the same election multiple times, he might not make the same vote every time. This model is in line with multiple theories of voter behavior and still yields the desirable property of showing that rational parties will converge to the electoral mean given the simple spatial framework.

Using this framework, Schofield (2007) shows that convergence to the mean need not occur given that valence asymmetries are accounted for. In this context, valence is taken to mean any sorts of quality that a candidates has that is independent of his location within a policy space. In general, valence is linked to the revealed ability of a party to govern in the past or the predicted ability of a party to govern well in the future. In recent years, models with a valence measure have been developed and utilized in studies of this sort. Schofield extends upon these models and demonstrates a necessary and sufficient condition for convergence to the mean, meaning that the joint electoral mean is a local pure-strategy Nash equilibrium (LNE) in the stochastic model with valence.

Valence can generally be divided into two types of valence: aggregate valence (or character valence) and individual valence (or sociodemographic valence). Both types of valence are exogenous to the position that a party takes in an election, meaning that these valence measures rely on some other underlying characteristic. Aggregate valence is a measure of valence which is common to all members in an electorate, and can be interpreted as the average perceived governing ability of a party for all members of an electorate (Penn 2003). Individual valence is a bit more specific, where this kind of valence depends upon the characteristics of a voter. This kind of valence differs from individual to individual. For example, in United States elections, African-American voters are very much more likely to vote for the Democratic candidate than they are to vote for the Republican candidate. Thus, it can be said that the Democratic candidate is of higher valence among African-American voters than the Republican candidate is. Both kinds of valence can be important in determining the outcomes of elections and are necessary to consider when building models of this sort.

Recent empirical work on the stochastic vote model has relied upon the assumption of Type-I extreme value distributed errors (Dow and Endersby 2004). These errors, commonly associated with microeconometric models, are typical of models that deal with individual choice, where individual utility is determined by the valence terms and the individual's distance from the party in the policy space. This distance is weighted by β, a constant that is determined by the average weight that individuals give to their respective distances from the parties. The workhorse of individual choice models is the multinomial logit distribution, which is an extension of the dichotomous response logit distribution. This distribution assumes that the probability that an individual votes for a party follows the Type-I extreme value distribution, thus matching the assumed distribution of the stochastic voting model. This creates a natural empirical partner for the stochastic vote model.

Using this statistical framework and the assumption that individual choice follows this distribution, Schofield (2007) introduced the idea of the convergence coefficient, c, which is a measure of attraction to the electoral mean in an electoral system. This coefficient is unitless, thus it can be compared across models. Low values of this value indicate strong attraction to the electoral mean, meaning that the electoral mean is a local pure-strategy Nash equilibrium (Patty 2005, 2007). High values indicate the opposite. He also lays out a necessary and a sufficient condition for convergence to the electoral mean with regards to the convergence coefficient:

1. When the dimension of the policy space is 2, then the sufficient condition for convergence to the electoral mean is $c < 1$.
2. The necessary condition for convergence is if $c < w$, where w is the number of dimensions of the policy space of interest.

When the necessary condition fails, at least one party will adopt a position away from the electoral mean in equilibrium, meaning that a LNE does not exist at the electoral mean. As a LNE must exist for the point to be a pure strategy equilibrium, this implies non-existence of a PNE at the center. Given the definition of the convergence coefficient, the general conclusion is that the smaller β is, the smaller the valence differences are among candidates, and the lower the variance of the electoral distribution is, the more likely there is to be a LNE at the electoral center.

However, this only answers the question where the local Nash equilibria are in the simplest case of having one electoral mean that parties are responding to. This problem can quickly become more complicated. Imagine a country with five parties and two different regions. Four of the parties run in both regions, and are thus attempting to appeal to voters in both regions. However, one of these parties only runs in one of the regions and is only trying to appeal to the voters of this region. Thus, it would be unreasonable for it to position itself with regards to the electoral mean for the entire electorate. Rather, it wants to maximize its vote share within in the region in which it runs. Parties can choose to run in select regions for a variety of reasons. They may run for historical reasons or responsive reasons or even choose not to run in regions where they know they will not do well at all. As parties have limited resources, sometimes this kind of decision must be made.

In order to assess convergence to the electoral mean in this case, one must take into account the electoral centers that parties are responding to. In the above example, convergence to the electoral mean would mean that the first four parties converge to the overall electoral mean, or the mean of all voters in the electorate, while the fifth party would converge to the electoral mean of those individuals in its respective region. Thus, the convergence coefficient would no longer be appropriate, as it is proven only when the position for all parties is equal to zero on all dimensions. Similarly, when there are parties which run in different combinations of regions, the typical multinomial logit model is no longer appropriate because the underlying assumption of "independence of irrelevant alternatives" (IIA) is no longer met (Train 2003). Given that there are problems with estimation of parameters from the currently utilized empirical methodology and problems with the underlying theoretical mechanism that drives the reasoning behind the convergence coefficient, we are left without the useful information gained about party tendencies in the stochastic model. Under the current framework, researchers can only analyze convergence, valence, and spatial adherence within specific regions. However, in this paper we propose a method for handling more structurally complex electorates.

In this chapter, we introduce methods for analyzing the stochastic vote model in electorates where individuals do not all vote for the same party bundle. First, this chapter will demonstrate that the convergence coefficient first defined by Schofield can be adjusted to handle any vector of party positions. We will determine the first and second order conditions necessary to show that a vector of policy positions is a local Nash equilibrium (LNE). From this, we will show that the convergence coefficient for a more complex electorate can be derived in a similar manner to that used previously. We will also show the necessary and sufficient conditions for convergence. Secondly, we will introduce a method that can be used to estimate the parameters necessary to find equilibria in the model. This empirical model, an extension of the mixed logit model, will utilize the same Type-I extreme value distribution assumptions used previously, but will not rely upon the IIA assumption necessary to use the basic multinomial logit model. This varying choice set logit (VCL: see Yamamoto 2011) will allow for aggregate estimation of parameters to occur while also allowing regional parameters to be estimated. This method of estimation along with the notions of convergence that will allow analysis of the stochastic voting model in more complex situations.

Finally, to illustrate these methods, we will analyze the Canadian elections in 2004. Canada has a regional party which only runs in one region of the country, however, in 2004, the regional party gained seats in the Parliament. As this election is an ideal testing point for these new methods, they can tell us whether or not these new methods give logical results. From this analysis, some insight can be gained as to the way in which parties can organize themselves to maximize the number of votes received.

2 The Formal Stochastic Model

The data in the spatial model is distributed $x_i \in X$ where $i \in N$ represents a member of the electorates's ideal point and n is the number of members in the sample. We assume that X is an open convex subset of Euclidian space, $\mathbb{R}^w$, where w is finite and corresponds to the number of dimensions selected to represent the policy space.

Each of the parties, $j \in P$, where $P = \{1, \dots, j, \dots, p\}$ chooses a policy, $z_j \in X$, to declare to the electorate prior to the election. Let $\mathbf{z} = (z_1, z_2, \dots, z_p)$ be the vector of party positions. Given $\mathbf{z}$, each voter i is described by a vector:

$$u_i(x_i, \mathbf{z}) = \big(u_{i1}(x_i, z_1), u_{i2}(x_i, z_2), \ldots, u_{ip}(x_i, z_p)\big)$$

$$\text{where } u_{ij}(x_i, z_j) = u^*_{ij}(x_i, z_j) + \epsilon_{ij}$$

$$\text{and } u^*_{ij}(x_i, z_j) = \lambda_j - \beta\|z_j - x_i\|^2 + \alpha_{ij}$$

Here, $u^*_{ij}(x_i, z_j)$ is the observable utility for i, associated with party j. λ_j is an exogenous valence term for agent j which is common throughout all members of a population (i.e. party quality).[1] β is a positive constant and $\|.\|$ is the Euclidian distance between individual i and party j.[2] α_{ij} is an exogenous sociodemographic valence term, meaning that this term can be viewed as the average assessment of a party's governing ability to the members of a specific group.[3] The error term, ϵ_{ij} is assumed to be commonly distributed among individuals. In particular, we assume that the cumulative distribution of the errors follows a Type-I extreme value distribution. This is not only the norm in individual choices, it also allows the theoretical model to match the corresponding empirical model, making the transition between the two easier.

Given the stochastic assumption of the model, the probability that i votes for j given z, $\rho_{ij}(z)$ is equal to:

$$\rho_{ij}(\mathbf{z}) = Pr\big[u_{ij}(x_i, z_j) > u_{il}(x_i, z_l),\ \forall l \neq j\big]$$

In turn, we assume that the expected vote share for agent j given $\mathbf{z}$, is $V_j(\mathbf{z})$ where:

$$V_j(\mathbf{z}) = \frac{1}{n}\sum_{\forall i \in N} \rho_{ij}(\mathbf{z})$$

We assume in this model that agent j chooses z_j to maximize $V_j(\mathbf{z})$ given the positions of the other parties. We seek equilibria of the model where each of the parties attempts to maximize vote share.

For the purposes of this paper, when we talk about an equilibria, we refer to a local Nash equilibria (LNE). This definition of equilibrium relies on maximizing the expected vote share gained by a party given the positions of the other parties. A vector of positions, $\mathbf{z}^*$, is said the be a LNE if $\forall j$, z^*_j is a critical point of the

[1] This can be conceptualized as an average assessment of the parties quality to govern among all members of the electorate, regardless of sociodemographic identity.

[2] To match up with the empirical applications later in the paper, the utility individual i gains from having party j in office is compared to a base party, $j = 1$. As is normal, we assume this party has a utility of zero and the other utilities are compared to this party. Thus, the utility gained by i by voting for j can also be seen as $u^*_{ij}(x_i, z_j) = \lambda_j - \beta(\sum_{m=1}^{w}((x_{jm} - x_{im})^2 - (x_1 m - x_{im})^2)) + \alpha_{ij}$ where the summation is of the Euclidian distances for each dimension of the policy space. This places our model in line with the latent utility models that are commonly used in microeconometric theory and bridges the gap between our theoretical model and the corresponding empirical model.

[3] In this paper, we assume that this term is common among all members of a specific sociodemographic group. However, we can set up these terms to represent individuals with individual level random effects.

vote function and the Hessian matrix of second derivatives is non-positive, meaning that the eigenvalues are all non positive. More simply put, a vector, $\mathbf{z}^*$, is a LNE if each party locates itself at a local maximum in its respective vote function. This means, that given the opportunity to make moves in the policy space and relocate its platform, no vote-maximizing party would choose to move. We assume that parties can estimate how their vote shares would change if they marginally move their policy position. The local Nash equilibrium is that vector z of party positions so that no party may shift position by a small amount to increase its vote share. More formally a LNE is a vector $\mathbf{z} = (z_1, \dots, z_j, \dots, z_p)$ such that each $V_j(\mathbf{z})$ is weakly locally maximized at the position z_j. To avoid problems with zero eigenvalues we also define a strict local Nash equilibrium (SLNE) to be a vector that strictly locally maximizes $V_j(\mathbf{z})$. We typically denote an LNE by $\mathbf{z}(K)$ where K refers to the model we consider. Using the estimated MNL coefficients we simulate these models and then relate any vector of party positions, $\mathbf{z}$, to a vector of vote share functions $V(\mathbf{z}) = (V_1(\mathbf{z}), \dots, V_p(\mathbf{z}))$, predicted by the particular model with p parties.

Given that we have defined the errors as cumulatively coming from a Type-I extreme value distribution, the probability $\rho_{ij}(z)$ has a multinomial logit specification and can be estimated. For each voter i and party j the probability that i votes for j given z is given by:

$$\begin{aligned}
\rho_{ij}(\mathbf{z}) &= \frac{\exp(u^*_{ij}(x_i, z_j))}{\sum_{k=1}^{p} \exp(u^*_{il}(x_i, z_k))} \\
&= \left[1 + \sum_{k \neq j}^{p} \exp(f_k) \right]^{-1} \\
\text{where } f_k &= \sum_{k=1}^{p} \bigl(u^*_{il}(x_i, z_k) \bigr) - \bigl(u^*_{ij}(x_i, z_j) \bigr).
\end{aligned}$$

$$\begin{aligned}
\text{Thus} \quad \frac{d\rho_j(\mathbf{z})}{dz_j} &= 2\beta(z_j - x_i) \left[1 \times \left[1 + \sum_{k \neq j}^{p} \exp(f_k) \right] \right]^{-2} \left[\sum_{k \neq j}^{p} \exp(f_k) \right] \\
&= 2\beta(z_j - x_i) \times \bigl[\rho_{ij}(\mathbf{z})\bigr]\bigl[1 - \rho_{ij}(\mathbf{z})\bigr]
\end{aligned}$$

in region k, with population, N_k, of size n_k the first order condition becomes

$$\frac{dV_{jk}(\mathbf{z}_k)}{dz_j}\Big|_{z_{-j}=z} = \frac{1}{n_k} 2\beta_k \sum\nolimits_{i \in N_k} \rho_{ijk}(1 - \rho_{ijk})(z_j - x_i) = 0, \tag{1}$$

$$\text{so } z_j = \sum\nolimits_{i \in N_k} w_{ij} x_i, \tag{2}$$

$$\text{where } w_{ij} = \frac{\rho_{ijk}(1 - \rho_{ijk})}{\sum_{k=1} \rho_{ijk}(1 - \rho_{ijk})}. \tag{3}$$

In order to show that points are LNE, we need to show that given $\mathbf{z}$, all agents are located at a critical point of their respective vote functions, $V_j(\mathbf{z})$. Thus, we need to show that the first derivative of the vote function, given $\mathbf{z}$, is equal to zero. Then we need to show the Hessian matrices at these points and compute their eigenvalues.

In this paper, we make two key departures from previous papers that have used this stochastic vote model. First, and certainly the most important departure, we intend to assess convergence in a model where the position vector of interest does not have all of the parties at the joint aggregate electoral origin. As explained before, in cases where there are regional parties that do not run in all parts of an electorate, there is no incentive for these agents to locate at the overall electoral mean. Rather, in line with other median voter results, these parties have incentives to locate at their respective electoral means, meaning that they position themselves on the ideal point of the average voter that actually has the choice to vote for that party. Thus, should we find that parties in an electoral system converge to the electoral mean in equilibrium, we should find that parties that run in all regions of an electorate converge to the joint electoral mean and regional parties converge to their respective regional electoral means. Previous papers have adjusted the scale of the policy space such that the electoral mean corresponds to the origin of the policy space and this allowed for some convenient cancelation to occur in proofs. For the purposes of this paper, though, we cannot make those cancelations and, thus, we are assessing convergence for a general vector of party positions rather than a zero vector. Second, we assume a second kind of valence, an individual valence, that was not previously included in utility equation. We intend to assess convergence to the mean given these individual valence measures as well, showing proofs including these variables.

The first derivative of $V_j(z)$ with respect to one dimension of the policy space is:

$$\frac{dV_j(\mathbf{z})}{dz_j} = \frac{2\beta}{n}\sum_{i=1}^{n}(z_j - x_i)\rho_{ij}(1-\rho_{ij})$$

Of course, a LNE has to be at a critical point, so all the set of possible LNE can be obtained by setting this equation to 0. Note that this derivative is somewhat different than that from earlier works as we do not assume that ρ_{ij} equals ρ_j (being independent of i). This is due to the fact that we do not assume that all parties are located at the electoral mean.

This result is important in a couple of ways. First, we see that the first derivative does not rely on λ_j or α_{ij} in any way aside from the calculation of the probability, ρ_{ij}, that an individual i votes for party j. This is an encouraging result because any resulting measures that assess convergence (i.e. the convergence coefficient) will not depend on the individual level valences. Previously, Schofield (2007) only showed that the convergence coefficient could be calculated when we assume a common valence for agent j across all members of an electorate. This finding allows us to expand the convergence coefficient notion to include these individual level valences as long as they are exogenous of a voter's ideal point. Second, after doing some simple algebra, it is easy to see that when a party locates at its respective electoral mean, the equation always equals zero, meaning that it is always at a critical point. This is also a good result, because it gives further support to the idea that the electoral mean is always a possible LNE.

To test if a critical point is a local maximum in the vote function, thus a LNE, we need a second order condition. The Hessian matrix of second derivatives is a $w \times w$ matrix defined as follows:

- Let $v_t = (x_{1t}, x_{2t}, \ldots, x_{nt})$ be the vector of the tth coordinates of the positions of the n voters and let. Let $z_j = (z_{1j}, z_{2j}, \ldots, z_{tj})$ and $\langle v_t - z_{tj}, v_s - z_{sj}\rangle$ be the scalar product, with $\Delta_0 = [\langle v_t - 0, v_s - 0\rangle]$ the electoral covariance matrix about the origin.Then diagonal entries of the Hessian for candidate j have the following form:

$$\frac{1}{n}\sum_{i=1}^{n} 2\beta(\rho_{ij})(1-\rho_{ij})\big(2\beta(x_{it}-z_{tj})^2(1-2\rho_{ij})-1\big)$$

- The off diagonal elements have the following form:

$$\frac{1}{n}\sum_{i=1}^{n} 4\beta^2(x_{is}-z_{sj})(x_{is}-z_{tj})\rho_{ij}(1-\rho_{ij})(1-2\rho_{ij})$$

- where $s \neq t$, and $s = 1, \ldots, w$, and $t = 1, \ldots, w$.

Given this matrix, if all w eigenvalues of the Hessian are negative given $\mathbf{z}$, then we can say that the position of interest is a LNE.

Unlike previous models of this sort, there is no characteristic matrix that the Hessian can be reduced to in order to assess whether or not a point is a local Nash equilibria. Thus, for the proper second order test, the eigenvalues of the Hessian must be found. However, as in earlier works, a reduced equation can be used to find a convergence coefficient, a unitless measure of how quickly the second derivative is changing at a given point. This convergence coefficient can be viewed substantively as a measure of how much a rational, vote-optimizing party is attracted to a certain position. As the coefficient becomes large, the party is repelled from the position.

We know that the trace of the Hessian is equal to the sum of the eigenvalues associated with the matrix. In order to be a local maximum, and thus a LNE, the eigenvalues have to all be negative. Thus, the trace of the Hessian must be negative as well in order for the point to be a local maximum. Given the equation for the main diagonal elements, we can see that it relies on β, $\rho_i j$, and the squared distance between the individual's ideal point on one dimension and the party's position on the same dimension. As β and $\rho_i j$ are necessarily positive, the only way in which the second derivative can be negative is if $2\beta(x_i - z_i)^2(1-2\rho_{ij})$ is greater than 1. Thus, this is the value of interest when trying to assess whether or not a point is a local maximum. This value can be viewed as the measure of how fast the probability that voter i votes for party j changes as the party makes small moves. We reason that the mean of $2\beta(x_i - z_i)^2(1 - 2\rho_{ij})$ over all voters is an equivalent concept to the convergence coefficient that does not rely on parties being positioned at the electoral origin. However, this is only for one dimension, so the full definition of the convergence coefficient is:

$$c(\mathbf{z}) = \frac{1}{n}\sum_{i=1}^{w}\sum_{i=1}^{n} 2\beta(x_{it} - z_{tj})^2(1 - 2\rho_{ij})$$

In words, the convergence coefficient is equal to the sum of mean values of

$$2\beta(x_i - z_i)^2(1 - 2\rho_{ij})$$

over all individuals in the electorate for each dimension of the policy space. This notion is supported by the fact that when all parties do locate at the electoral origin, this definition of the convergence coefficient is equivalent to the definition provided in Schofield (2007).

Given this definition of the convergence coefficient, we can derive necessary and sufficient conditions for convergence to a given vector of party positions. Given a vector of party positions, a sufficient condition for the vector being a local Nash equilibrium is that $c(\mathbf{z}) < 1$. If $c(\mathbf{z})$ is less than 1, then we can guarantee that the second derivatives with respect to each dimension are less than 0. This eliminates the possibility that the party is located at a saddle point. A *necessary* condition for convergence to the vector of interest is that $c(\mathbf{z}) < w$. However, for the position to be a LNE, each second derivative has to be negative. Thus, each constituent part of $c(\mathbf{z})$ must be less than 1.

It is important to note that a convergence coefficient can be calculated for each party in the electoral system. Previously, given that all of the parties have been attempting to optimize over the same population, an assumption could be made that the highest convergence coefficient would belong to the party which had the lowest exogenous valence. However, with the slight restructuring of the model to include individual level valences and parties which run in singular regions, as ρ_j can no longer be reduced down to a difference of valences, we can no longer make the assumption that the lowest valence party will be the first to move away from the mean should that be equilibrium behavior. In fact, given that there are multiple definitions of valence in the equation and multiple values of these valences for each region, a notion of lowest valence party becomes very difficult to define. Thus, the convergence coefficient should be calculated for each party to ensure a complete analysis of convergence behavior. Then the party with the highest convergence coefficient represents the electoral behavior of the system. Thus, for an electoral system, the convergence coefficient is:

$$c(\mathbf{z}) = \arg_{p} c_p(z)$$

In summary, the method for assessing whether or not a vector of party positions is a LNE is as follows: