Chapter 7

Finite Volume Methods for Hyperbolic Systems

7.1. Principle

7.1.1. *One-dimensional conservation laws*

Finite volume methods solve the conservation form of conservation laws. Scalar hyperbolic laws are expressed in conservation form as in equation [1.1], recalled here:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S$$

As shown in section 1.1.2, equation [1.1] is derived from a balance over a control volume $[x_0, x_0 + \delta x] \times [t_0, t_0 + \delta t]$, the size of which is made infinitesimal. The balance equation [1.12] is recalled:

$$\int_{x_0}^{x_0+\delta x} [U(x, t_0 + \delta t) - U(x, t_0)] \mathrm{d}x = \int_{t_0}^{t_0+\delta t} [F(x_0, t) - F(x_0 + \delta x, t)] \, \mathrm{d}t + \int_{t_0}^{t_0+\delta t} \int_{x_0}^{x_0+\delta x} S(x, t) \, \mathrm{d}x \, \mathrm{d}t$$

As shown in section 1.1.2, equation [1.1] is a particular case of equation [1.12]. In contrast with equation [1.1], equation [1.12] is not based on the assumption that

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the solution is continuous and differentiable with respect to time and space. In finite volume methods, space is discretized into volumes, also called computational cells, over which balance equation [1.12] is solved (Figure 7.1).

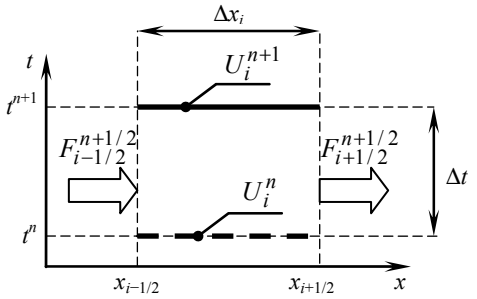

Figure 7.1. *Discretization of space and time for a one-dimensional finite volume method*

Applying balance equation [1.12] over the control volume sketched in Figure 7.1 leads to the following equation:

$$(U_i^{n+1} - U_i^n)\Delta x_i = (F_{i-1/2}^{n+1/2} - F_{i+1/2}^{n+1/2})\Delta t + \Delta t \Delta x_i S_i^{n+1/2} \qquad [7.1]$$

where Δx_i is the width of the cell i, Δt is the computational time step, U_i^n is the average value of U over the cell i at the time level n, $F_{i+1/2}^{n+1/2}$ is the average value of the flux F at the interface $i + 1/2$ between the cells i and $i + 1$ between the time levels n and $n + 1$ and $S_i^{n+1/2}$ is the average value of the source term over the cell i between the time levels n and $n + 1$. Dividing by Δx_i yields:

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x_i}(F_{i-1/2}^{n+1/2} - F_{i+1/2}^{n+1/2}) + \Delta t\ S_i^{n+1/2} \qquad [7.2]$$

The average value of U over the cell i at the next time level can be determined provided that the fluxes at the interfaces between the cells and the source term can be estimated between the time levels n and $n + 1$. How to estimate the fluxes is dealt with in the next sections. If F and S are estimated using only the known solution at the time level n, the method is said to be explicit. If the estimates of F and S are functions of the unknown solution at the time level $n + 1$, the method is said to be implicit.

Hyperbolic systems of conservation laws are written in conservation form as in equation [2.2], recalled here (see Chapter 2):

$$\frac{\partial \mathrm{U}}{\partial t} + \frac{\partial \mathrm{F}}{\partial x} = \mathrm{S}$$

Equation [2.2] is discretized by applying the scalar discretization [7.2] to each of the components of U, F and S. The vector form of equation [7.2] is obtained:

$$\mathrm{U}_i^{n+1} = \mathrm{U}_i^n + \frac{\Delta t}{\Delta x_i}(\mathrm{F}_{i-1/2}^{n+1/2} - \mathrm{F}_{i+1/2}^{n+1/2}) + \Delta t\, \mathrm{S}_i^{n+1/2} \qquad [7.3]$$

7.1.2. *Multidimensional conservation laws*

Only two-dimensional problems are considered in what follows. The extension to three-dimensional problems will not be detailed hereafter. Two-dimensional systems can be written in the conservation form [5.12], recalled here:

$$\frac{\partial \mathrm{U}}{\partial t} + \frac{\partial \mathrm{F}}{\partial x} + \frac{\partial \mathrm{G}}{\partial y} = \mathrm{S}$$

where F and G are the fluxes in the x- and y-direction respectively. Space is discretized into polygonal cells, usually triangles or quadrangles. The edges of the cells are straight lines (Figure 7.2).

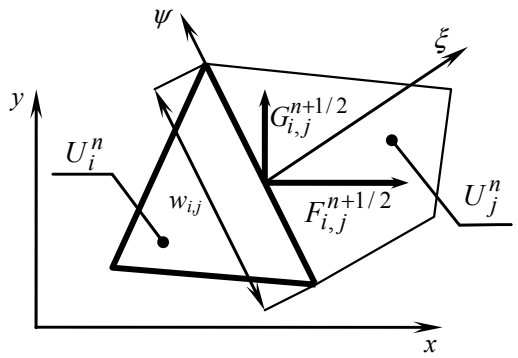

Figure 7.2. *Discretization of space for the finite volume solution of a two-dimensional hyperbolic conservation law. Notation for a scalar law*

The balance over the cell i can be written as:

$$(\mathrm{U}_i^{n+1} - \mathrm{U}_i^n)A_i + \sum_{j \in V(i)} \left[\mathrm{F}_{i,j}^{n+1/2} n_{i,j}^{(x)} + \mathrm{G}_{i,j}^{n+1/2} n_{i,j}^{(y)} \right] w_{i,j} \Delta t = \mathrm{S}_i^{n+1/2} A_i \Delta t \quad [7.4]$$

where A_i is the area of the cell, $V(i)$ is the set of the neighbor cells of the cell i, $\mathrm{F}_{i,j}^{n+1/2}$ and $\mathrm{G}_{i,j}^{n+1/2}$ are respectively the average values of the fluxes F and G at the interface (i, j) between the cells i and j between the time levels n and $n + 1$, $n_{i,j}^{(x)}$ and $n_{i,j}^{(y)}$ are the x- and y-components of the normal unit vector attached to the interface (i, j), positive from i to j, and $w_{i,j}$ is the width of the interface (i, j). Note that $n_{i,j}^{(x)}$ and $n_{i,j}^{(y)}$ are respectively the cosine and sine of the angle between the normal unit vector of the interface and the x-axis. Equation [7.4] can be rewritten as:

$$\mathrm{U}_i^{n+1} = \mathrm{U}_i^n - \frac{\Delta t}{A_i} \sum_{j \in V(i)} \left[\mathrm{F}_{i,j}^{n+1/2} n_{i,j}^{(x)} + \mathrm{G}_{i,j}^{n+1/2} n_{i,j}^{(y)} \right] w_{i,j} + \mathrm{S}_i^{n+1/2} \Delta t \quad [7.5]$$

The fluxes F and G must be estimated at each interface (i, j) in the computational domain. The apparent complexity of equation [7.5] can be reduced to some extent by noting that the quantity between brackets in equation [7.5] is nothing other than the flux in the direction normal to the interface (i, j). Consequently, the method is greatly simplified by solving the projection of the equations onto the normal unit vector. The following algorithm is used in practice:

1) For each interface (i, j), the variables U_i^n and U_j^n are expressed in the local coordinate system (ξ, ψ) attached to the interface. The directions ξ and ψ are normal and tangent to the interface respectively. Scalar variables such as the pressure, water depth, entropy, internal energy, etc. are left unchanged by the transformation, while vector variables such as the velocity or unit discharge vector are transformed using a classical rotation formula:

$$\left.\begin{aligned} u_\xi &= n_{i,j}^{(x)} u + n_{i,j}^{(y)} v \\ u_\psi &= -n_{i,j}^{(y)} u + n_{i,j}^{(x)} v \end{aligned}\right\} \quad [7.6]$$

where u_ξ and u_ψ are respectively the components of the velocity in the ξ- and ψ-direction. Given the orientation of the ξ-axis, the quantities U_i and U_j are often denoted by U_L and U_R respectively. Transformation [7.6] can be written in vector form as:

$$\left.\begin{aligned} U_L &= P^{-1}U_i^n \\ U_R &= P^{-1}U_j^n \end{aligned}\right\} \qquad [7.7]$$

where P is the conversion matrix from the global coordinate system (x, y) into the local coordinate system (ξ, ψ).

2) For each interface (i, j), compute the flux F_ξ in the direction normal to the interface using the values U_L and U_R. The various existing techniques for flux computation are detailed in the next sections and in Appendix C on Riemann solvers.

3) Convert the flux F_ξ to the global coordinate system (x, y) and apply the balance equation [7.5] that can be rewritten as:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{A_i} \sum_{j \in V(i)} PF_{\xi\, i,j}^{n+1/2} w_{i,j} + S_i^{n+1/2} \Delta t \qquad [7.8]$$

Note that a number of discretization techniques, such as the "source term upwinding technique" also use the expression of the source term in the local coordinate system attached to the interface. In this case, steps 1) to 3) must also be applied to the source term S.

7.1.3. *Application to the two-dimensional shallow water equations*

The present section deals with the application of the technique described in section 7.1.2 to the two-dimensional shallow water equations dealt with in section 5.4. The two-dimensional shallow water equations can be written in the conservation form [5.12], recalled here:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = S$$

by defining U, F, G and S as in equation [5.64], recalled hereafter:

$$\mathrm{U} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}, \qquad \mathrm{F} = \begin{bmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{bmatrix},$$

$$\mathrm{G} = \begin{bmatrix} hv \\ huv \\ hv^2 + gh^2/2 \end{bmatrix}, \qquad \mathrm{S} = \begin{bmatrix} 0 \\ (S_{0,x} - S_{f,x})gh \\ (S_{0,y} - S_{f,y})gh \end{bmatrix}$$

For the sake of clarity, the discretization of the source term is not dealt with hereafter. Steps 1) to 3) of the algorithm in section 7.1.2 are applied.

1) Expression of U in the local coordinate system. h is left unchanged, while u and v are transformed into u_ξ and u_ψ as in equation [7.6]. $\mathrm{U_L}$ and $\mathrm{U_R}$ are given by:

$$\left.\begin{aligned} \mathrm{U_L} &= \begin{bmatrix} h_i^n \\ (hu_\xi)_i^n \\ (hu_\psi)_i^n \end{bmatrix} = \begin{bmatrix} h_i^n \\ \left[n_{i,j}^{(x)} u_i^n + n_{i,j}^{(y)} v_i^n\right] h_i^n \\ \left[n_{i,j}^{(y)} u_i^n + n_{i,j}^{(x)} v_i^n\right] h_i^n \end{bmatrix} = \mathrm{P}^{-1}\mathrm{U}_i^n \\ \mathrm{U}_R &= \begin{bmatrix} h_j^n \\ (hu_\xi)_j^n \\ (hu_\psi)_j^n \end{bmatrix} = \begin{bmatrix} h_j^n \\ \left[n_{i,j}^{(x)} u_j^n + n_{i,j}^{(y)} v_j^n\right] h_j^n \\ \left[-n_{i,j}^{(y)} u_j^n + n_{i,j}^{(x)} v_j^n\right] h_j^n \end{bmatrix} = \mathrm{P}^{-1}\mathrm{U}_j^n \end{aligned}\right\} \qquad [7.9]$$

The matrix P^{-1} is derived from equation [7.9]:

$$\mathrm{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & n_{i,j}^{(x)} & n_{i,j}^{(y)} \\ 0 & -n_{i,j}^{(y)} & n_{i,j}^{(x)} \end{bmatrix} \qquad [7.10]$$

hence the expression of P:

$$\mathrm{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & n_{i,j}^{(x)} & -n_{i,j}^{(y)} \\ 0 & n_{i,j}^{(y)} & n_{i,j}^{(x)} \end{bmatrix} \qquad [7.11]$$

2) Solution of the equations in the local coordinate system. Since the interface is a straight line, the problem is locally one-dimensional and the governing equation becomes (remember that the source term is assumed to be zero):

$$\frac{\partial \mathrm{U}_\xi}{\partial t} + \frac{\partial \mathrm{F}_\xi}{\partial \xi} = 0 \qquad [7.12]$$

with:

$$\mathrm{U}_\xi = \mathrm{P}^{-1}\mathrm{U} = \begin{bmatrix} h \\ hu_\xi \\ hu_\psi \end{bmatrix}, \qquad \mathrm{F}_{\xi\, i,j}^{n+1/2} = \begin{bmatrix} h \\ hu_\xi^2 + \dfrac{gh^2}{2} \\ hu_\xi u_\psi \end{bmatrix}_{i,j}^{n+1/2} \qquad [7.13]$$

The flux at the interface is usually computed by solving a Riemann problem. Examples of exact or approximate Riemann solvers can be found in Appendix C.

3) Conversion of the flux F_ξ to the global coordinate system (x, y) and balance as in equation [7.8]:

$$\begin{bmatrix} h \\ hu \\ hv \end{bmatrix}_i^{n+1} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}_i^{n} - \sum_{j \in V(i)} \frac{w_{i,j}\Delta t}{A_i} \begin{bmatrix} 1 & 0 & 0 \\ 0 & n_{i,j}^{(x)} & -n_{i,j}^{(y)} \\ 0 & n_{i,j}^{(y)} & n_{i,j}^{(x)} \end{bmatrix} \begin{bmatrix} h \\ hu_\xi^2 + \dfrac{gh^2}{2} \\ hu_\xi u_\psi \end{bmatrix}_{i,j}^{n+1/2} \qquad [7.14]$$

7.2. Godunov's scheme

7.2.1. *Principle*

At the time of the publication of Godunov's scheme [GOD 59, GOD 99], the concept of finite volumes had not yet been formalized and Godunov presented his scheme as a finite difference scheme. Twenty years elapsed before the finite volume formalism was introduced by Van Leer in the development of the MUSCL scheme (see section 7.3).

Godunov originally saw his own scheme as a conservative generalization of the CIR scheme (see section 6.2.2). The CIR scheme is based on the characteristic form of the equations. As seen in section 6.2.2, the CIR scheme fails to preserve conservation in the presence of shocks. This is why Godunov tried to solve the conservation form of the equation by estimating the fluxes at the interface between the computational points. In Godunov's scheme, the flux $F_{\xi\,i,j}^{n+1/2}$ is computed from the solution of a Riemann problem in the local coordinate system attached to the interface (i, j). The Riemann problem is defined as:

$$\left.\begin{aligned} &\frac{\partial \mathrm{U}_\xi}{\partial t} + \frac{\partial \mathrm{F}_\xi}{\partial \xi} = 0 \\ &\mathrm{U}(\xi, t^n) = \begin{cases} \mathrm{U_L} & \text{for } \xi < 0 \\ \mathrm{U_R} & \text{for } \xi > 0 \end{cases} \end{aligned}\right\} \qquad [7.15]$$

where the interface is located at $\xi = 0$. In Godunov's scheme, the left- and right-states of the Riemann problem are taken from the average values of the variable over the cell on the left- and right-hand side of the interface respectively. The Riemann problem may be solved exactly (see Chapter 4) or approximately (see Appendix C). In the case where the Riemann problem is solved exactly, the value of U and F is known for all ξ and t. However, only the value of the flux at $\xi = 0$ is of practical interest. Approximate Riemann solvers focus on the determination of the flux at the interface. Note that the self-similarity property of the solution of the Riemann problem is of direct interest to the calculation of F because the variable and the flux are independent of time at the location of the initial discontinuity.

Also note that:

– in the one-dimensional case, the cells i and j are aligned along the x-axis, therefore $j = i + 1$ and:

$$\left.\begin{aligned} \mathrm{U_L} &= \mathrm{U}_i^n \\ \mathrm{U_R} &= \mathrm{U}_{i+1}^n \end{aligned}\right\} \qquad [7.16]$$

in this case the solution of the Riemann problem yields the flux $\mathrm{F}_{i+1/2}^{n+1/2}$;

– in multidimensional problems, $\mathrm{U_L}$ and $\mathrm{U_R}$ must be determined as in equation [7.7].

7.2.2. *Application to the scalar advection equation*

7.2.2.1. *Discretization*

This section deals with the application of Godunov's scheme to the linear advection equation [1.39], recalled here:

$$\frac{\partial}{\partial t}(AC) + \frac{\partial}{\partial x}(QC) = 0$$

where A is the cross-sectional area of the channel, C is the concentration of the dissolved substance and Q is the liquid discharge. A and Q are assumed known a priori. They satisfy the continuity equation [1.46], recalled hereafter:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

As shown in Chapter 1, combining equations [1.39] and [1.46] leads to the non-conservation form [1.48]:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = 0$$

where $u = Q/A$. The characteristic form [1.50] follows:

$$C = \text{Const} \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u$$

Applying equation [7.2] to equation [1.39] gives:

$$(AC)_i^{n+1} = (AC)_i^n + \frac{\Delta t}{\Delta x_i}\left[(QC)_{i-1/2}^{n+1/2} - (QC)_{i+1/2}^{n+1/2}\right] \qquad [7.17]$$

Equation [7.17] can be rewritten as:

$$C_i^{n+1} = \frac{A_i^n}{A_i^{n+1}} C_i^n + \frac{\Delta t}{A_i^{n+1}\Delta x_i}(Q_{i-1/2}^{n+1/2} C_{i-1/2}^{n+1/2} - Q_{i+1/2}^{n+1/2} C_{i+1/2}^{n+1/2}) \qquad [7.18]$$

where A and Q are assumed known everywhere in the computational domain at all times. C_i^{n+1} can be computed provided that the values $C_{i-1/2}^{n+1/2}$ and $C_{i+1/2}^{n+1/2}$ at the

interfaces $i - 1/2$ and $i + 1/2$ can be computed. Only the calculation of $C_{i+1/2}^{n+1/2}$ is detailed hereafter, the procedure being identical for the remaining interfaces.

7.2.2.2. *Flux calculation at internal interfaces*

The concentration $C_{i+1/2}^{n+1/2}$ at the interface $i + 1/2$ is computed from the solution of the following Riemann problem:

$$\left.\begin{aligned} &\frac{\partial}{\partial t}(AC) + \frac{\partial}{\partial t}(QC) = 0 \\ &C(x,t^n) = \begin{cases} C_{\mathrm{L}} = C_i^n & \text{for } x < x_{i+1/2} \\ C_{\mathrm{R}} = C_{i+1}^n & \text{for } x > x_{i+1/2} \end{cases} \end{aligned}\right\} \qquad [7.19]$$

The solution of problem [7.19] is described in detail in section 4.2.1. The concentration being invariant along the characteristic lines (see equation [1.50]), the concentration profile is given by:

$$C(x, t > t^n) = \begin{cases} C_{\mathrm{L}} = C_i^n & \text{for } x < x_{i+1/2} + (t - t^n)u \\ C_{\mathrm{R}} = C_{i+1}^n & \text{for } x > x_{i+1/2} + (t - t^n)u \end{cases} \qquad [7.20]$$

where u is the average velocity at the interface $i + 1/2$ between the time levels n and $n + 1$ (see section 7.2.2.4 for suggested estimates of u). $C_{i+1/2}^{n+1/2}$ is given by:

$$C_{i+1/2}^{n+1/2} = \begin{cases} C_{\mathrm{R}} = C_{i+1}^n & \text{if } u < 0 \\ C_{\mathrm{L}} = C_i^n & \text{if } u > 0 \end{cases} \qquad [7.21]$$

Note that when $u = 0$, the profile does not move. Equation [7.21] leads to indeterminacy. This however is not a problem in practice because the flux is zero when $u = 0$. The final estimate for the flux becomes:

$$Q_{i+1/2}^{n+1/2} C_{i+1/2}^{n+1/2} = \begin{cases} Q_{i+1/2}^{n+1/2} C_{i+1}^n & \text{if } Q_{i+1/2}^{n+1/2} < 0 \\ 0 & \text{if } Q_{i+1/2}^{n+1/2} = 0 \\ Q_{i+1/2}^{n+1/2} C_i^n & \text{if } Q_{i+1/2}^{n+1/2} > 0 \end{cases} \qquad [7.22]$$

Equation [7.22] can be condensed into the following expression:

$$Q_{i+1/2}^{n+1/2} C_{i+1/2}^{n+1/2} = \frac{\left|Q_{i+1/2}^{n+1/2}\right| + Q_{i+1/2}^{n+1/2}}{2} C_i^n + \frac{\left|Q_{i+1/2}^{n+1/2}\right| - Q_{i+1/2}^{n+1/2}}{2} C_{i+1}^n \qquad [7.23]$$

7.2.2.3. *Boundary conditions*

Boundary conditions are needed at every boundary where the characteristics enter the domain. If the discharge $Q_{1/2}^{n+1/2}$ at the left-hand boundary of the domain is positive (Figure 7.3a), a boundary condition must be prescribed at the interface 1/2. Conversely, if the discharge $Q_{M+1/2}^{n+1/2}$ at the right-hand boundary of the domain is negative (Figure 7.3b), a boundary condition must be prescribed at the interface $M + 1/2$. A negative discharge $Q_{1/2}^{n+1/2}$ (Figure 7.3c) and a positive discharge $Q_{M+1/2}^{n+1/2}$ (Figure 7.3d) do not require any boundary condition.

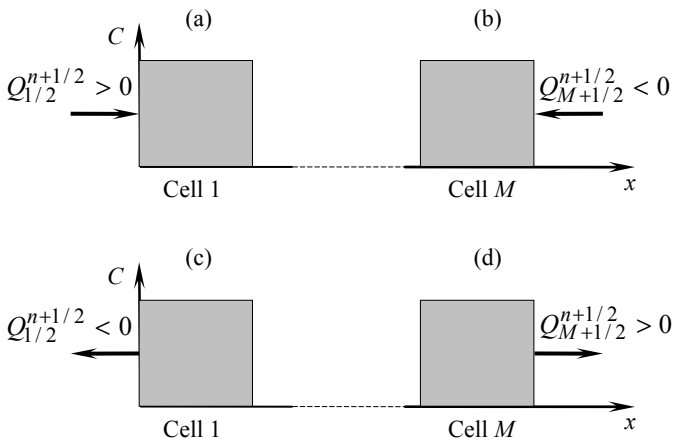

Figure 7.3. *Boundary conditions to be prescribed for the various possible flow configurations*

Two types of boundary conditions are used in practical solute transport applications:

1) Prescribed concentration C_b at the boundary. For a left-hand boundary the flux is computed as:

$$Q_{1/2}^{n+1/2} C_{1/2}^{n+1/2} = Q_{1/2}^{n+1/2} C_b \qquad [7.24]$$

2) Prescribed flux F_b at the boundary. The flux at the left-hand boundary becomes:

$$Q_{1/2}^{n+1/2} C_{1/2}^{n+1/2} = F_b \qquad [7.25]$$

Note that the fluxes at outflowing boundaries are given by:

$$\left.\begin{array}{ll} Q_{1/2}^{n+1/2} = Q_{1/2}^{n+1/2} C_1^n & \text{if } Q_{1/2}^{n+1/2} < 0 \\ Q_{M+1/2}^{n+1/2} = Q_{M+1/2}^{n+1/2} C_M^n & \text{if } Q_{M+1/2}^{n+1/2} > 0 \end{array}\right\} \qquad [7.26]$$

7.2.2.4. *Calculation of the liquid discharge at the cell interfaces*

Various options are available for the calculation of the discharge $Q_{i+1/2}^{n+1/2}$. The following approaches are proposed:

1) If A and Q are computed using a finite volume method, A is computed over the computational cells and Q is computed at the cell interfaces, which makes its use in equation [7.23] straightforward.

2) If A and Q are available from other techniques such as finite difference methods, they are not available at the interfaces between the cells but at computational points. The point value of A and Q at a given point i may be viewed as their average value over the control volume centered around i. The average discharge $Q_{i+1/2}^{n+1/2}$ over the computational time step may be obtained from an average of the point values in time and space:

$$Q_{i+1/2}^{n+1/2} = \frac{Q_i^n + Q_i^{n+1} + Q_{i+1}^n + Q_{i+1}^{n+1}}{4} \qquad [7.27]$$

3) When coupling occurs between hydrodynamics and transport, the variations in A and Q are influenced by those in C. This is the case in sediment transport models, where the suspended sediment concentration influences the erosion or deposition rate, therefore acting on the cross-sectional area. The coupling between the flow and transport processes usually requires an iterative process, whereby A and Q serve as a starting point for the calculation of C over the time step. The result of the transport calculation is used to update the hydrodynamic equation that is solved again over the time step. The result of the updated hydrodynamic equation is used to carry out a new transport calculation. This iterative process is repeated until convergence is achieved. Such a procedure being time-consuming, an explicit coupling is often

preferred. The discharge Q at the interface is computed using only the point values at the beginning of the time step:

$$Q_{i+1/2}^{n+1/2} = \frac{Q_i^n + Q_{i+1}^n}{2} \qquad [7.28]$$

7.2.2.5. *Algorithm*

The algorithm is the following:

1) Compute the liquid discharge at each interface between the computational cells. If the discharge was computed using a finite volume method, it may be used directly. If the discharge was computed using a finite difference method, equations [7.27] or [7.28] may be used.

2) Compute the flux QC at each interface using equation [7.23] at internal interfaces and equations [7.24], [7.25] or [7.26] depending on the type of the boundary condition.

3) Carry out the mass balance using equation [7.17] or [7.18].

7.2.3. *Application to the inviscid Burgers equation*

7.2.3.1. *Discretization*

The inviscid Burgers equation is derived in section 1.4. Its conservation form is given by equation [1.69], recalled hereafter:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

The equation may be written in non-conservation form as in equation [1.66]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

The characteristic form of the equation is given by equation [1.68]:

$$u = \text{Const} \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u$$

Applying the general discretization [7.2] to the conservation form [1.69] leads to:

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x_i}\left[(u_{i-1/2}^{n+1/2})^2 - (u_{i+1/2}^{n+1/2})^2\right] \qquad [7.29]$$

7.2.3.2. *Flux computation at internal interfaces*

The value $u_{i+1/2}^{n+1/2}$ at the interface $i + 1/2$ is obtained from the solution of the following Riemann problem:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \\ &u(x, t^n) = \begin{cases} u_{\mathrm{L}} = u_i^n & \text{for } x < x_{i+1/2} \\ u_{\mathrm{R}} = u_{i+1}^n & \text{for } x > x_{i+1/2} \end{cases} \end{aligned}\right\} \qquad [7.30]$$

The solution of the Riemann problem for the inviscid Burgers equation is studied in section 4.2.2. The solution may be a shock or a rarefaction wave depending on u_{L} and u_{R} (see Table 4.1). Recall that:

– if $u_{\mathrm{L}} < u_{\mathrm{R}}$, a rarefaction wave appears. If $u_{\mathrm{L}} > 0$ the wave travels to the right and u takes the value u_{L} at $x_{i+1/2}$. If $u_{\mathrm{R}} < 0$ the wave travels to the left and u takes the value u_{R} at $x_{i-1/2}$. If u_{L} and u_{R} do not have the same sign, u is zero at $x_{i+1/2}$;

– if $u_{\mathrm{L}} > u_{\mathrm{R}}$, a shock appears. The speed of the shock is the average of the speeds u_{L} and u_{R} (see equations [3.28] and [4.16]). The direction in which the shock propagates depends on the sum $u_{\mathrm{L}} + u_{\mathrm{R}}$. Note that the configuration $u_{\mathrm{L}} + u_{\mathrm{R}} = 0$ leads to indeterminacy because $u(x, t^n)$ is undefined in equation [7.30]. However, determining the solution completely is not needed in this case because the shock is stationary. Using the Rankin-Hugoniot condition [3.28] necessarily leads to a zero flux, which corresponds to $u = 0$ at the interface.

The various possible configurations are summarized in Table 7.1. The value of u at the interface may be computed using the following formula that accounts for the various possible wave configurations:

$$u_{i+1/2}^{n+1/2} = \frac{|\varepsilon_{i+1/2}| + \varepsilon_{i+1/2}}{2} u_i^n + \frac{|\varepsilon_{i+1/2}| - \varepsilon_{i+1/2}}{2} u_{i+1}^n \qquad [7.31]$$

where $\varepsilon_{i+1/2}$ is given by:

$$\varepsilon_{i+1/2} = \begin{cases} -1 & \text{if } u_i^n + u_{i+1}^n < 0 \\ 0 & \text{if } u_i^n + u_{i+1}^n = 0 \\ +1 & \text{if } u_i^n + u_{i+1}^n > 0 \end{cases} \qquad [7.32]$$

Configuration	Wave pattern	Value of u at $x_{i+1/2}$
$u_L < u_R < 0$	Rarefaction wave heading to the left	u_L
$u_L < 0$, $u_R > 0$	Rarefaction wave centered around x_0	0
$0 < u_L < u_R$	Rarefaction wave heading to the right	u_R
$u_L > u_R$, $u_L + u_R < 0$	Shock heading to the left	u_L
$u_L > u_R$, $u_L + u_R = 0$	Stationary shock	Undefined, 0 acceptable
$u_L > u_R$, $u_L + u_R < 0$	Shock heading to the right	u_R

Table 7.1. *Solution of the Riemann problem for the inviscid Burgers equation. Solution at the interface for the various possible configurations*

7.2.3.3. *Boundary conditions*

Boundary conditions must be prescribed when the characteristics enter the computational domain. No boundary conditions are needed when the characteristics leave the computational domain. Consequently, only positive velocities may be prescribed at the left-hand boundary, while only negative velocities may be prescribed at the right-hand boundary. These conditions are necessary but not sufficient in that an inflowing boundary condition may be overridden by an outgoing wave, as shown in the following example. Assume that the boundary condition $u_b > 0$ is to be prescribed at the left-hand boundary of the domain. If u_1^n is such that $u_b + u_i^n < 0$, a shock appears. The propagation speed of the shock is negative and the condition u_b cannot be prescribed at the boundary.

The following, general procedure allows the problem to be handled.

1) Define the following Riemann problem at the left-hand boundary:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \\ &u(x, t^n) = \begin{cases} u_L = u_b & \text{for } x < x_{1/2} \\ u_R = u_1^n & \text{for } x > x_{1/2} \end{cases} \end{aligned}\right\} \qquad [7.33]$$

The solution of equation [7.33] is:

$$u_{1/2}^{n+1/2} = \frac{|\varepsilon_{1/2}| + \varepsilon_{1/2}}{2} u_b + \frac{|\varepsilon_{1/2}| - \varepsilon_{1/2}}{2} u_1^n \qquad [7.34]$$

2) Define the following Riemann problem at the right-hand boundary:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \\ &u(x,t^n) = \begin{cases} u_{\mathrm{L}} = u_M^n & \text{for } x < x_{M+1/2} \\ u_{\mathrm{R}} = u_b & \text{for } x > x_{M+1/2} \end{cases} \end{aligned}\right\} \qquad [7.35]$$

The solution of problem [7.35] is:

$$u_{M+1/2}^{n+1/2} = \frac{|\varepsilon_{M+1/2}| + \varepsilon_{M+1/2}}{2} u_M^n + \frac{|\varepsilon_{M+1/2}| - \varepsilon_{M+1/2}}{2} u_b \qquad [7.36]$$

7.2.3.4. *Algorithm*

The algorithm of Godunov's scheme as applied to the inviscid Burgers equation can be summarized as follows:

1) Compute the value of u at the internal interfaces using equations [7.31–32].

2) Compute the value of u at the left- and right-hand boundary using equation [7.34] and [7.36] respectively.

3) Apply balance equation [7.29] to all the computational cells.

7.2.4. *Application to the water hammer equations*

7.2.4.1. *Discretization*

The water hammer equations studied in section 2.4 may be written in conservation form as in equation [2.2], recalled hereafter:

$$\frac{\partial \mathrm{U}}{\partial t} + \frac{\partial \mathrm{F}}{\partial x} = \mathrm{S}$$

For the sake of clarity, the cross-sectional area A and the speed of sound are assumed to be constant. The pipe is assumed to be horizontal and friction is neglected. Then, U, F and S are given by (see equation [2.68]):

$$\mathrm{U} = \begin{bmatrix} \rho A \\ \rho Q \end{bmatrix}, \quad \mathrm{F} = \begin{bmatrix} \rho Q \\ Ap \end{bmatrix}, \quad \mathrm{S} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad [7.37]$$

The equation may also be written in the non-conservation form [2.5]:

$$\frac{\partial \mathrm{U}}{\partial t} + \mathrm{A}\frac{\partial \mathrm{U}}{\partial x} = 0 \qquad [7.38]$$

where the Jacobian matrix A of F with respect to U is given by equation [2.69]:

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ c^2 & 0 \end{bmatrix}$$

The speed of sound c is constant. The characteristic form [2.82] of the equation is recalled under the assumption of a zero source term:

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(p - \rho c u) = 0 \quad \text{for} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = -c \\ &\frac{\mathrm{d}}{\mathrm{d}t}(p + \rho c u) = 0 \quad \text{for} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = c \end{aligned}\right\} \qquad [7.39]$$

Noting that c and ρ are constant, equation [7.39] is simplified into:

$$\left.\begin{aligned} &p - \rho c u = C_1 \qquad \text{for} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = -c \\ &p + \rho c u = C_2 \qquad \text{for} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = c \end{aligned}\right\} \qquad [7.40]$$

where C_1 and C_2 are constants to be determined from the initial conditions. The conservation form is discretized as in equation [7.3] under the assumption of a zero source term:

$$\mathrm{U}_i^{n+1} = \mathrm{U}_i^n + \frac{\Delta t}{\Delta x_i}(\mathrm{F}_{i-1/2}^{n+1/2} - \mathrm{F}_{i+1/2}^{n+1/2}) \qquad [7.41]$$

Equation [7.41] can be rewritten as:

$$\left.\begin{aligned}(\rho A)_i^{n+1} &= (\rho A)_i^n + \frac{\Delta t}{\Delta x_i}\left[(\rho Q)_{i-1/2}^{n+1/2} - (\rho Q)_{i+1/2}^{n+1/2}\right]\\ (\rho Q)_i^{n+1} &= (\rho Q)_i^n + \frac{\Delta t}{\Delta x_i}\left[(Ap)_{i-1/2}^{n+1/2} - (Ap)_{i+1/2}^{n+1/2}\right]\end{aligned}\right\} \quad [7.42]$$

Noting that $\mathrm{d}(Ap) = c^2\,\mathrm{d}(\rho A)$, dividing the first and second equations [7.42] by A and Q respectively leads to:

$$\left.\begin{aligned}p_i^{n+1} &= p_i^n + \frac{\rho c^2}{A}\frac{\Delta t}{\Delta x_i}(Q_{i-1/2}^{n+1/2} - Q_{i+1/2}^{n+1/2})\\ Q_i^{n+1} &= Q_i^n + \frac{A}{\rho}\frac{\Delta t}{\Delta x_i}(p_{i-1/2}^{n+1/2} - p_{i+1/2}^{n+1/2})\end{aligned}\right\} \quad [7.43]$$

7.2.4.2. *Flux calculation at internal interfaces*

The flux at the interface $i + 1/2$ is computed from the solution of the following the Riemann problem:

$$\left.\begin{aligned}&\frac{\partial \mathrm{U}}{\partial t} + \frac{\partial \mathrm{F}}{\partial x} = 0\\ &\mathrm{U}(x, t^n) = \begin{cases}\mathrm{U_L} = \mathrm{U}_i^n & \text{for } x < x_{i+1/2}\\ \mathrm{U_R} = \mathrm{U}_{i+1}^n & \text{for } x > x_{i+1/2}\end{cases}\end{aligned}\right\} \quad [7.44]$$

The solution of the Riemann problem for the water hammer equations is examined in detail in section 4.3.2. An intermediate region of constant state is separated from the left and right states by two contact discontinuities propagating in opposite directions at speeds $-c$ and $+c$. The interface $i + 1/2$ belongs to the intermediate region of constant state. The value of U and F at $x_{i+1/2}$ is determined using the first Riemann invariant [7.40] between $\mathrm{U_R}$ and the intermediate region and the second Riemann invariant [7.40] between $\mathrm{U_L}$ and the intermediate region of constant state. The following system is obtained:

$$\left.\begin{aligned}p_{i+1/2}^{n+1/2} - \rho c u_{i+1/2}^{n+1/2} &= p_{\mathrm{R}} - \rho c u_{\mathrm{R}} = p_{i+1}^n - \rho c u_{i+1}^n\\ p_{i+1/2}^{n+1/2} + \rho c u_{i+1/2}^{n+1/2} &= p_{\mathrm{L}} + \rho c u_{\mathrm{L}} = p_i^n + \rho c u_i^n\end{aligned}\right\} \quad [7.45]$$

Solving the system for p and u at the interface leads to:

$$\left.\begin{aligned} p_{i+1/2}^{n+1/2} &= \frac{p_i^n + p_{i+1}^n}{2} + \rho c \frac{u_i^n - u_{i+1}^n}{2} \\ u_{i+1/2}^{n+1/2} &= \frac{u_i^n + u_{i+1}^n}{2} + \frac{p_i^n - p_{i+1}^n}{2\rho c} \end{aligned}\right\} \quad [7.46]$$

Consequently:

$$\left.\begin{aligned} p_{i+1/2}^{n+1/2} &= \frac{1}{2}(p_i^n + p_{i+1}^n) + \frac{\rho c}{2A}(Q_i^n - Q_{i+1}^n) \\ Q_{i+1/2}^{n+1/2} &= \frac{1}{2}(Q_i^n + Q_{i+1}^n) + \frac{A}{2\rho c}(p_i^n - p_{i+1}^n) \end{aligned}\right\} \quad [7.47]$$

Substituting equation [7.47] into equation [7.43] leads to the final expression:

$$\left.\begin{aligned} p_i^{n+1} &= \frac{\mathrm{Cr}}{2}(p_{i-1}^{n+1} + p_{i+1}^{n+1}) + (1 - \mathrm{Cr})p_i^n + \mathrm{Cr}\frac{\rho c}{2A}(Q_{i-1}^{n+1} - Q_{i+1}^{n+1}) \\ Q_i^{n+1} &= \frac{\mathrm{Cr}}{2}(Q_{i-1}^{n+1} + Q_{i+1}^{n+1}) + (1 - \mathrm{Cr})Q_i^n + \mathrm{Cr}\frac{A}{2\rho c}(p_{i-1}^{n+1} - p_{i+1}^{n+1}) \end{aligned}\right\} \quad [7.48]$$

where the Courant number is defined as:

$$\mathrm{Cr} = \frac{c\Delta t}{\Delta x_i}$$

7.2.4.3. *Treatment of boundary conditions*

The following types of boundary conditions are dealt with hereafter: prescribed pressure and prescribed discharge. The calculation of the flux is detailed only for the left-hand boundary, the transposition to the right-hand boundary being straightforward.

– Prescribed pressure p_b. The pressure at the interface 1/2 is defined as:

$$p_{1/2}^{n+1/2} = p_b \quad [7.49]$$

The discharge is obtained using the first Riemann invariant between the cell 1 and the interface 1/2. The first relationship [7.40] is rewritten as:

$$p_{1/2}^{n+1/2} - \rho c u_{1/2}^{n+1/2} = p_1^n - \rho c u_1^n \quad [7.50]$$

Substituting equation [7.49] into equation [7.50] yields:

$$u_{1/2}^{n+1/2} = \frac{p_b - p_1^n}{\rho c} + u_1^n \qquad [7.51]$$

Multiplying by the cross-sectional area A gives:

$$Q_{1/2}^{n+1/2} = \frac{A}{\rho c}(p_b - p_1^n) + Q_1^n \qquad [7.52]$$

Reasoning by symmetry, the following formulae are obtained for the right-hand boundary:

$$\left.\begin{aligned} p_{M+1/2}^{n+1/2} &= p_b \\ Q_{M+1/2}^{n+1/2} &= \frac{A}{\rho c}(p_M^n - p_b) + Q_M^n \end{aligned}\right\} \qquad [7.53]$$

– Prescribed discharge Q_b at the boundary. The discharge is known directly from the boundary condition:

$$Q_{1/2}^{n+1/2} = Q_b \qquad [7.54]$$

and the pressure is obtained from the first relationship [7.40]:

$$p_{1/2}^{n+1/2} - \rho c \frac{Q_b}{A} = p_1^n - \rho c u_1^n \qquad [7.55]$$

Equation [7.55] leads to:

$$p_{1/2}^{n+1/2} = p_1^n + \frac{\rho c}{A}(Q_b - Q_1^n) \qquad [7.56]$$

Using the second relationship [7.40] at the right-hand boundary leads to:

$$p_{M+1/2}^{n+1/2} = p_M^n + \frac{\rho c}{A}(Q_M^n - Q_b) \qquad [7.57]$$

7.2.4.4. *Algorithm*

The algorithm for the solution of the water hammer equation by Godunov's scheme can be summarized as follows:

1) Compute the values of p and Q at the internal interfaces using equations [7.47].

2) Compute the values of p and Q at the boundaries of the domain using equation [7.52] or [7.54] at the left-hand boundary and using equation [7.53] or [7.57] at the right-hand boundary.

3) Apply balance equation [7.43] to all the computational cells in the domain.

7.3. Higher-order Godunov-type schemes

7.3.1. *Rationale and principle*

7.3.1.1. *Historical perspective*

Godunov's scheme [GOD 59] is first-order. First-order schemes are characterized by a strong numerical diffusion, the effect of which is to smooth out the numerical solutions, in particular in the neighborhood of shocks, contact discontinuities or sharp gradient transitions. The quality of the numerical solution can be increased only by increasing the number of computational cells, with consequences on the computational effort required. At the end of the 1970s Van Leer [VAN 77, VAN 79] introduced a new formalism for the development of higher-order conservative schemes. The purpose was twofold. Firstly, the numerical method had to be conservative, a necessary condition for the treatment of weak solutions. Secondly, the scheme had to be monotone in order to eliminate spurious oscillations in the vicinity of sharp gradients. Van Leer introduced the concepts of reconstruction and slope limiting, the principle of which is outlined in the next sections. These concepts were used by Van Leer to develop the now widely used MUSCL scheme. At the beginning of the 1980s, Colella and Woodward proposed the more complex but much more accurate PPM [COL 84]. Considerable research effort has been devoted to the development of higher-order schemes since then, leading to numerous higher-order Godunov-type schemes (see e.g. [TOR 97], [LEV 02], [GUI 03a]).

Today's formalism of Godunov-type schemes uses the following steps: (i) reconstruction, (ii) profile limiting, (iii) solution of a generalized Riemann problem, (iv) flux computation and (v) balance over the computational cells. These steps are detailed in sections 7.3.1.2 to 7.3.1.5.

7.3.1.2. *Reconstruction of the flow variable*

Higher-order Godunov-type schemes are derived from the following remark: the accuracy of the numerical solution is conditioned by that of the method used for the computation of the fluxes. The accuracy of the flux computation can be increased

only by more accurately locating the gradients in the variable than the original Godunov scheme does. The lack of accuracy of Godunov's scheme mainly stems from the fact that the Riemann problem is defined using the average value of the variable over the computational cells, that is, assuming a zero gradient in the computational cells. Van Leer proposed that the accuracy of the scheme should be increased by reconstructing the variations of U within a given cell using the average values of U over the neighboring cells.

The values U_i^n of U over the cells being known at the time level n, a reconstructed profile, denoted by $\widetilde{\mathrm{U}}_i^n(x)$, is defined over each cell (Figure 7.4). The reconstruction must satisfy conservation, that is, its average value over the cell i should be equal to U_i^n:

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \widetilde{\mathrm{U}}_i^n(x)dx = \Delta x_i \mathrm{U}_i^n \qquad [7.58]$$

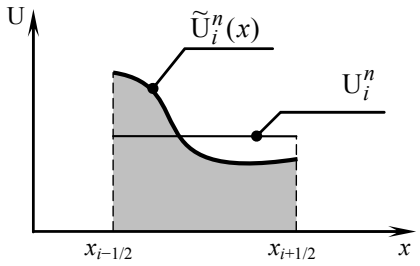

Figure 7.4. *Reconstruction of the flow variable within the computational cell i. The area below the reconstructed profile (gray-shaded region) is equal to the area below the horizontal line that indicates the average value*

7.3.1.3. *Slope limiting*

The reconstructed profile $\widetilde{\mathrm{U}}_i^n(x)$ must be corrected prior to the computation of the fluxes at the interfaces between the computational cells. As shown in [VAN 77] and [COL 84] the following necessary conditions should be fulfilled for the TVD character of the scheme to be guaranteed:

– The reconstructed profile $\widetilde{\mathrm{U}}_i^n(x)$ must be monotone over the cell i.

– The value $\widetilde{\mathrm{U}}_i^n(x_{i-1/2})$ at the interface i - 1/2 must lie between U_{i-1}^n and U_i^n.

– The value $\widetilde{\mathrm{U}}_i^n(x_{i+1/2})$ at the interface $i + 1/2$ must lie between U_i^n and U_{i+1}^n.

These conditions are summarized as follows:

$$\left.\begin{aligned}&\min(\mathrm{U}_{i-1}^n, \mathrm{U}_i^n) \le \widetilde{\mathrm{U}}_i^n(x_{i-1/2}) \le \max(\mathrm{U}_{i-1}^n, \mathrm{U}_i^n)\\&\min(\mathrm{U}_i^n, \mathrm{U}_{i+1}^n) \le \widetilde{\mathrm{U}}_i^n(x_{i+1/2}) \le \max(\mathrm{U}_i^n, \mathrm{U}_{i+1}^n)\end{aligned}\right\} \qquad [7.59]$$

In the notation [7.59], the min, max and ≤ operators are applied to each of the components of U separately.

The reconstructed profile is corrected if necessary. Applying the monotony conditions amounts to minimizing, or limiting, the average gradient of the variable over the computational cell, hence the term "slope limiting" often used to refer to the correction. The effect of slope limiting on the reconstructed profile is illustrated in Figure 7.5. Note that the corrected profile should also satisfy the conservation constraint [7.58].

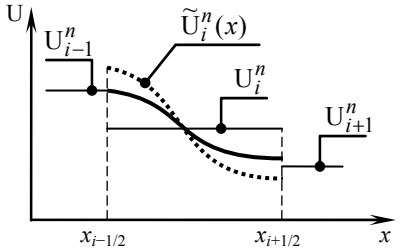

Figure 7.5. *Effect of slope limiting on a profile to meet the monotony conditions [7.59]. Reconstructed profile before the correction (dashed line), after the correction (solid line)*

7.3.1.4. *Solution of the Riemann problem at the interfaces between the cells*

The limited profiles are used to define generalized Riemann problems at the interfaces between the computational cells (Figure 7.6). Such problems take the form:

$$\left.\begin{aligned}&\frac{\partial \mathrm{U}}{\partial t} + \frac{\partial \mathrm{F}}{\partial x} = 0\\&\mathrm{U}(x, t^n) = \begin{cases}\widetilde{\mathrm{U}}_i^n(x) & \text{for } x < x_{i+1/2}\\ \widetilde{\mathrm{U}}_{i+1}^n(x) & \text{for } x > x_{i+1/2}\end{cases}\end{aligned}\right\} \qquad [7.60]$$

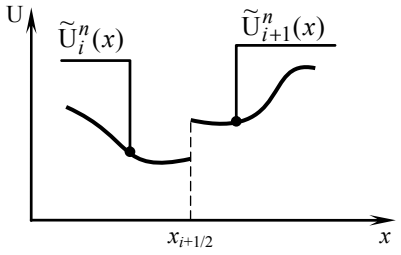

Figure 7.6. *Generalized Riemann problem at the interface i+1/2*

Such problems can in general not be solved analytically. They must be converted into equivalent Riemann problems [GUI 03a] that can be solved exactly or approximately, or they can be solved using so-called predictor-corrector methods [TOR 97]. Appendix C gives an overview of a number of approximate Riemann solvers available in the literature.

7.3.1.5. *Flux computation and balance*

Solving the Riemann problem at the interfaces between the computational cells allows the fluxes to be computed. The fluxes are used in balance equation [7.3], recalled hereafter:

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x_i}(F_{i-1/2}^{n+1/2} - F_{i+1/2}^{n+1/2}) + \Delta t\, S_i^{n+1/2}$$

7.3.2. *Example: the MUSCL scheme*

7.3.2.1. *Reconstruction*

The Monotonic Upwind Scheme for Conservation Laws (MUSCL) was developed by Van Leer [VAN 77]. Several options for the reconstruction of the conserved variable were proposed in the original publication. Only the most commonly used approach is presented here. The procedure is detailed for a scalar variable. It is generalized to vector variables by applying the reconstruction and limiting steps to each of the components of the vector variable.

The MUSCL scheme uses a linear reconstruction in the form (see Figure 7.7):

$$\tilde{U}_i^n = U_i^n + (x - x_i)a_i^n \qquad [7.61]$$

where a_i^n is the slope of the profile over the cell i and $x_i = (x_{i-1/2} + x_{i+1/2})/2$ is the abscissa of the centre of the cell i. It is easy to check that equation [7.61] satisfies the conservation property [7.58] regardless of the value of a_i^n. The slope is computed as the average slope between the cells $i - 1$ and $i + 1$ (Figure 7.7):

$$a_i^n = \frac{U_{i+1}^n - U_{i-1}^n}{x_{i+1} - x_{i-1}} = 2\frac{U_{i+1}^n - U_{i-1}^n}{\Delta x_{i-1} + 2\Delta x_{i-1} + \Delta x_{i+1}} \qquad [7.62]$$

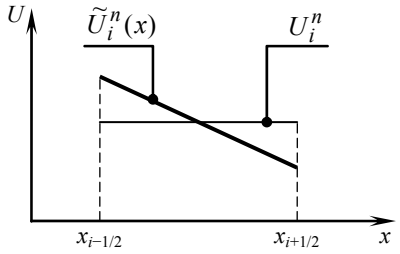

Figure 7.7. *The MUSCL reconstruction*

7.3.2.2. *Slope limiting*

The slope must be limited if at least one of the following situations occurs:

1) The cell i is a local extremum. This is true if:

$$(U_i^n - U_{i-1}^n)(U_{i+1}^n - U_i^n) \leq 0 \qquad [7.63]$$

In such a case the slope a_i^n is set to zero. Tests 2) and 3) hereafter do not need to be carried out.

2) The first condition [7.59] is not satisfied because the value at the interface $i - 1/2$ does not lie between the average values in the cells $i - 1$ and i. The slope a_i^n is adjusted to the largest possible value that allows the first condition [7.59] to be satisfied:

$$a_i^n = 2\frac{U_i^n - U_{i-1}^n}{\Delta x_i} \qquad [7.64]$$

3) The second condition [7.59] is not satisfied because the value at the interface $i + 1/2$ does not lie between the average values in the cells i and $i + 1$. The slope a_i^n

is adjusted to the largest possible value that allows the second condition [7.59] to be satisfied:

$$a_i^n = 2\frac{U_{i+1}^n - U_i^n}{\Delta x_i} \qquad [7.65]$$

7.3.2.3. *Solution of the generalized Riemann problem*

Several approaches are available for the solution of the generalized Riemann problem. The most widely used approach, referred to as the MUSCL-Hancock approach [TOR 97], is presented hereafter. Another approach, proposed by Savic and Holly [SAV 93], uses the average value of the Riemann invariants over the domain of dependence. A third approach, referred to as the EigenVector-based Reconstruction (EVR), allows for faster computations while leading to more stable solutions than the MUSCL-Hancock scheme [GUI 03a, SOA 07]. This latter approach is detailed in section 7.4.

In the MUSCL-Hancock approach, a first approximation is obtained for the fluxes by solving the following Riemann problem:

$$\left.\begin{aligned} &\frac{\partial \mathrm{U}}{\partial t} + \frac{\partial \mathrm{F}}{\partial x} = 0 \\ &\mathrm{U}(x, t^n) = \begin{cases} \widetilde{\mathrm{U}}_i^n(x_{i+1/2}) & \text{for } x < x_{i+1/2} \\ \widetilde{\mathrm{U}}_{i+1}^n(x_{i+1/2}) & \text{for } x > x_{i+1/2} \end{cases} \end{aligned}\right\} \qquad [7.66]$$

Solving the Riemann problem [7.66] yields fluxes that are used to compute the solution using equation [7.3] over half a time step:

$$\left.\begin{aligned} \mathrm{U}_{i+1/2,\mathrm{L}}^{n+1/2} &= \widetilde{\mathrm{U}}_i^n(x_{i+1/2}) \\ &\quad + \frac{\Delta t}{2\Delta x_i}\left[\mathrm{F}(\widetilde{\mathrm{U}}_i^n(x_{i-1/2})) - \mathrm{F}(\widetilde{\mathrm{U}}_i^n(x_{i+1/2}))\right] \\ \mathrm{U}_{i+1/2,\mathrm{R}}^{n+1/2} &= \widetilde{\mathrm{U}}_{i+1}^n(x_{i+1/2}) \\ &\quad + \frac{\Delta t}{2\Delta x_{i+1}}\left[\mathrm{F}(\widetilde{\mathrm{U}}_{i+1}^n(x_{i+1/2})) - \mathrm{F}(\widetilde{\mathrm{U}}_{i+1}^n(x_{i+3/2}))\right] \end{aligned}\right\} \qquad [7.67]$$

The values $U_{i+1/2,\mathrm{L}}^{n+1/2}$ and $U_{i+1/2,\mathrm{R}}^{n+1/2}$ are taken as the left and right states of a Riemann problem, the solution of which is used as a final estimate for the fluxes in equation [7.3].

7.4. EVR approach

7.4.1. *Principle of the approach*

The EVR method allows the generalized Riemann problem to be solved by converting it into an Equivalent Riemann Problem (ERP). The method was first introduced for the simulation of two-phase flows in pipes [GUI 01b]. It was then generalized to hyperbolic systems of conservation laws [GUI 03a] but no name was given to the method. The name EVR appeared when an application of the method to two-dimensional free surface flow simulations with dry beds was published [SOA 07].

The advantage of the EVR is a less computationally demanding algorithm than the MUSCL-Hancock method, with a similar precision. The method is also more robust than the classical MUSCL-Hancock approach in the presence of dry beds, with no oscillations or instabilities near wetting and drying fronts [SOA 07]. The method is presented in a one-dimensional context hereafter. However, its generalization to multiple dimensions does not introduce any particular difficulty.

Assume that the vector variable U has been reconstructed in each cell of the computational domain at time step n. The reconstructed profile in cell i at time level n is denoted by $\widetilde{U}_i^n(x)$. At interface $i+1/2$ between the cells i and $i+1$, the generalized Riemann problem is given by equation [7.60], recalled here:

$$\left.\begin{aligned} &\frac{\partial \mathrm{U}}{\partial t}+\frac{\partial \mathrm{F}}{\partial x}=0 \\ &\mathrm{U}(x,t^n)=\begin{cases}\widetilde{\mathrm{U}}_i^n(x) & \text{for } x<x_{i+1/2}\\ \widetilde{\mathrm{U}}_{i+1}^n(x) & \text{for } x>x_{i+1/2}\end{cases}\end{aligned}\right\}$$

The purpose is to determine the left and right states for an ERP that yields the same average value of the flux F at interface $i+1/2$ over the time step Δt. In other words, $\mathrm{U_L}$ and $\mathrm{U_R}$ are sought such that the Riemann problem:

$$\left.\begin{aligned} &\frac{\partial \mathrm{U}}{\partial t}+\frac{\partial \mathrm{F}}{\partial x}=0 \\ &\mathrm{U}(x,t^n)=\begin{cases}\mathrm{U_L} & \text{for } x<x_{i+1/2}\\ \mathrm{U_R} & \text{for } x>x_{i+1/2}\end{cases}\end{aligned}\right\} \qquad [7.68]$$

leads to the same flux $\mathrm{F}_{i+1/2}^{n+1/2}$ as the original Riemann problem [7.60].

The average flux $\mathrm{F}_{i+1/2}^{n+1/2}$ between time levels n and $n+1$ is estimated from a linearization of the flux function F(U):

$$\mathrm{F}_{i+1/2}^{n+1/2} \approx \mathrm{F}(\mathrm{U}_{i+1/2}^{n+1/2}) \qquad [7.69]$$

where $\mathrm{U}_{i+1/2}^{n+1/2}$ is an estimate of the average of U at interface $i+1/2$ between time levels n and $n+1$. The two Riemann problems [7.60] and [7.67] are equivalent if they lead to the same average $\mathrm{U}_{i+1/2}^{n+1/2}$. This allows necessary conditions to be written for the left and right states of the equivalent Riemann problem. This is done by writing the hyperbolic system in non-conservation form, equation [7.38]:

$$\frac{\partial \mathrm{U}}{\partial t} + A\frac{\partial \mathrm{U}}{\partial x} = 0$$

The underlying idea of the EVR approach is that the reconstructed profiles in the cells i and $i+1$ can be expressed in the base K of eigenvectors $\mathrm{K}^{(p)}$ of the Jacobian matrix A:

$$\left.\begin{aligned} \widetilde{\mathrm{U}}_i^n(x) &= \sum_{p=1}^{m} \mathrm{K}_i^{n(p)} \widetilde{\alpha}_i^{(p)}(x) \\ \widetilde{\mathrm{U}}_{i+1}^n(x) &= \sum_{p=1}^{m} \mathrm{K}_{i+1}^{n\ (p)} \widetilde{\alpha}_{i+1}^{(p)}(x) \end{aligned}\right\} \qquad [7.70]$$

where coefficients $\alpha_i^{(p)}$ and $\alpha_{i+1}^{(p)}$ are called the wave strengths by analogy with Roe's approximate solver (see Appendix C). In reconstruction [7.70], the two eigenvectors $\mathrm{K}_i^{n(p)}$ and $\mathrm{K}_{i+1}^{n\ (p)}$ are constant over each cell, while the wave strengths are functions of x. The solution U(x, t) is sought in the form:

$$\mathrm{U}(x,t) = \begin{cases} \displaystyle\sum_{p=1}^{m} \mathrm{K}_i^{n(p)} \alpha(x,t) & \text{for } x < x_{i+1/2} \\ \displaystyle\sum_{p=1}^{m} \mathrm{K}_{i+1}^{n\ (p)} \alpha(x,t) & \text{for } x > x_{i+1/2} \end{cases} \qquad [7.71]$$

Substituting equation [7.71] into equation [7.38] leads to:

$$\left.\begin{aligned} \sum_{p=1}^{m} \mathrm{K}_i^{n(p)} \frac{\partial \alpha^{(p)}}{\partial t} + \mathrm{A}_i^n \sum_{p=1}^{m} \mathrm{K}_i^{n(p)} \frac{\partial \alpha^{(p)}}{\partial x} &= 0 \\ \sum_{p=1}^{m} \mathrm{K}_{i+1}^{n\ (p)} \frac{\partial \alpha^{(p)}}{\partial t} + \mathrm{A}_{i+1}^n \sum_{p=1}^{m} \mathrm{K}_{i+1}^{n\ (p)} \frac{\partial \alpha^{(p)}}{\partial x} &= 0 \end{aligned}\right\} \qquad [7.72]$$

left-multiplying the first and second equations [7.72] by the inverse of matrices K_i^n and K_{i+1}^n respectively, we obtain:

$$\left.\begin{aligned} &\frac{\partial \alpha^{(p)}}{\partial t} + \lambda_i^{n(p)} \frac{\partial \alpha^{(p)}}{\partial x} = 0 \\ &\frac{\partial \alpha^{(p)}}{\partial t} + \lambda_{i+1}^{n\ (p)} \frac{\partial \alpha^{(p)}}{\partial x} = 0 \end{aligned}\right\} \qquad [7.73]$$

that is:

$$\left.\begin{aligned} &\frac{\mathrm{d}\alpha^{(p)}}{\mathrm{d}t} = 0 \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_i^{n(p)}, x < x_{i+1/2} \\ &\frac{\mathrm{d}\alpha^{(p)}}{\mathrm{d}t} = 0 \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_{i+1}^{n\ (p)}, x > x_{i+1/2} \end{aligned}\right\} \qquad [7.74]$$

Consequently, the wave strength $\alpha^{(p)}$ is a constant along the pth characteristic.

Equations [7.74] are used to express U at the interface as a combination of the wave strengths:

$$\begin{aligned} \mathrm{U}_{i+1/2}^{n+1/2} &= \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathrm{U}(x_{i+1/2}, t)\, \mathrm{d}t \\ &= \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \sum_{p=1}^{m} \mathrm{K}^{(p)} \alpha^{(p)}(x_{i+1/2}, t)\, \mathrm{d}t \\ &= \sum_{p=1}^{m} \mathrm{K}^{(p)} \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \alpha^{(p)}(x_{i+1/2}, t)\, \mathrm{d}t \end{aligned} \qquad [7.75]$$

The sum is broken down into:

$$\begin{aligned} \mathrm{U}_{i+1/2}^{n+1/2} = &\sum_{p, \lambda^{(p)} < 0}^{m} \mathrm{K}^{(p)} \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \alpha^{(p)}(x_{i+1/2}, t)\, \mathrm{d}t \\ &+ \sum_{p, \lambda^{(p)} > 0}^{m} \mathrm{K}^{(p)} \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \alpha^{(p)}(x_{i+1/2}, t)\, \mathrm{d}t \end{aligned} \qquad [7.76]$$

The waves with positive wave speeds originate from cell i, while those with negative wave speeds come from cell $i + 1$. Equation [7.76] becomes:

$$\begin{aligned} \mathrm{U}_{i+1/2}^{n+1/2} = & \sum_{p,\lambda^{(p)}<0}^{m} \mathrm{K}_{i+1}^{n\ (p)} \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \alpha^{(p)}(x_{i+1/2}, t)\, \mathrm{d}t \\ & + \sum_{p,\lambda^{(p)}>0}^{m} \mathrm{K}_{i}^{n(p)} \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \alpha^{(p)}(x_{i+1/2}, t)\, \mathrm{d}t \end{aligned} \qquad [7.77]$$

The average value of the wave strength $\alpha^{(p)}$ at interface $i + 1/2$ between t^n and t^{n+1} is equal to its average value over the domain of dependence of the pth wave:

$$\frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \alpha^{(p)}\, \mathrm{d}t = \begin{cases} \dfrac{1}{\lambda_i^{n(p)} \Delta t} \displaystyle\int_{x_{i+1/2} - \lambda_i^n \Delta t}^{x_{i+1/2}} \alpha^{(p)}(x, t^n)\, \mathrm{d}x & \text{if } \lambda_i^{n(p)} > 0 \\ \dfrac{1}{\lambda_{i+1}^{n\ (p)} \Delta t} \displaystyle\int_{x_{i+1/2} - \lambda_{i+1}^n \Delta t}^{x_{i+1/2}} \alpha^{(p)}(x, t^n)\, \mathrm{d}x & \text{if } \lambda_{i+1}^{n\ (p)} < 0 \end{cases} \qquad [7.78]$$

Substituting equation [7.78] into equation [7.77], we have:

$$\begin{aligned} \mathrm{U}_{i+1/2}^{n+1/2} = & \sum_{p,\lambda^{(p)}<0}^{m} \mathrm{K}_{i+1}^{n\ (p)} \frac{1}{\lambda_{i+1}^{n\ (p)} \Delta t} \int_{x_{i+1/2} - \lambda_{i+1}^n \Delta t}^{x_{i+1/2}} \tilde{\alpha}_{i+1}^{n\ (p)}(x)\, \mathrm{d}x \\ & + \sum_{p,\lambda^{(p)}>0}^{m} \mathrm{K}_{i}^{n(p)} \frac{1}{\lambda_{i}^{n(p)} \Delta t} \int_{x_{i+1/2} - \lambda_{i}^n \Delta t}^{x_{i+1/2}} \tilde{\alpha}_{i}^{n(p)}(x)\, \mathrm{d}x \end{aligned} \qquad [7.79]$$

Moreover, the left and right states $\mathrm{U_L}$ and $\mathrm{U_R}$ of the equivalent Riemann problem are sought in the form:

$$\left.\begin{aligned} \mathrm{U_L} &= \sum_{p=1}^{m} \mathrm{K}_i^{n(p)} \alpha_{\mathrm{L}}^{(p)} \\ \mathrm{U_R} &= \sum_{p=1}^{m} \mathrm{K}_{i+1}^{n\ (p)} \alpha_{\mathrm{R}}^{(p)} \end{aligned}\right\} \qquad [7.80]$$

A necessary condition for equations [7.68] and [7.60] to yield the same average value at interface $i + 1/2$ over the time step is:

$$\left.\begin{aligned} \alpha_{\mathrm{L}}^{(p)} &= \frac{1}{\lambda_i^{n\,(p)}\Delta t}\int_{x_{i+1/2}-\lambda_i^n\Delta t}^{x_{i+1/2}} \widetilde{\alpha}_i^{n\,(p)}(x)\,\mathrm{d}x && \text{if } \lambda_i^{n\,(p)} \geq 0 \\ \alpha_{\mathrm{R}}^{(p)} &= \frac{1}{\lambda_{i+1}^{n\ (p)}\Delta t}\int_{x_{i+1/2}-\lambda_{i+1}^n\Delta t}^{x_{i+1/2}} \widetilde{\alpha}_{i+1}^{n\ (p)}(x)\,\mathrm{d}x && \text{if } \lambda_{i+1}^{n\ (p)} \leq 0 \end{aligned}\right\} \qquad [7.81]$$

In cell i, the wave strengths with negative wave speeds may be chosen arbitrarily. Conversely, in cell $i + 1$, the wave strengths with positive wave speeds may be chosen arbitrarily. In practice, the average value of the wave strength over the computational cell is shown to give satisfactory results [GUI 01b, GUI 03a, SOA 07]. The final formulae of the wave strengths are:

$$\left.\begin{aligned} \alpha_{\mathrm{L}}^{(p)} &= \begin{cases} \dfrac{1}{\lambda_i^{n\,(p)}\Delta t}\displaystyle\int_{x_{i+1/2}-\lambda_i^n\Delta t}^{x_{i+1/2}} \widetilde{\alpha}_i^{n\,(p)}(x)\,\mathrm{d}x & \text{if } \lambda_i^{n\,(p)} \geq 0 \\ \alpha_i^n & \text{if } \lambda_i^{n\,(p)} < 0 \end{cases} \\ \alpha_{\mathrm{R}}^{(p)} &= \begin{cases} \dfrac{1}{\lambda_{i+1}^{n\ (p)}\Delta t}\displaystyle\int_{x_{i+1/2}-\lambda_{i+1}^n\Delta t}^{x_{i+1/2}} \widetilde{\alpha}_{i+1}^{n\ (p)}(x)\,\mathrm{d}x & \text{if } \lambda_{i+1}^{n\ (p)} \leq 0 \\ \alpha_{i+1}^n & \text{if } \lambda_{i+1}^{n\ (p)} > 0 \end{cases} \end{aligned}\right\} \qquad [7.82]$$

7.4.2. *Application to the one-dimensional shallow water equations*

The one-dimensional shallow water equations arise as a particular case of the Saint Venant equations [2.2], [2.118] under the assumption of a wide, prismatic rectangular channel. In such a case, the definition of U, F and S is modified into:

$$\mathrm{U} = \begin{bmatrix} h \\ q \end{bmatrix}, \qquad \mathrm{F} = \begin{bmatrix} q \\ q^2/h + gh^2/2 \end{bmatrix}, \qquad S = \begin{bmatrix} 0 \\ (S_0 - S_f)gh \end{bmatrix} \qquad [7.83]$$

These equations can also be seen as the one-dimensional restriction of equations [5.64]. The Jacobian matrix A is given by equation [2.119]:

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{bmatrix}$$

and the eigenvectors are given by equation [2.125]:

$$\mathrm{K}^{(1)} = \begin{bmatrix} 1 \\ u-c \end{bmatrix}, \quad \mathrm{K}^{(2)} = \begin{bmatrix} 1 \\ u+c \end{bmatrix}$$

The wave strengths are defined as:

$$\mathrm{U} = \alpha^{(1)}\mathrm{K}^{(1)} + \alpha^{(2)}\mathrm{K}^{(2)} = \mathrm{K}\alpha \qquad [7.84]$$

where α is the vector formed by the wave strengths. Substituting equations [7.83] and [2.125] into equation [7.84] leads to:

$$\left.\begin{aligned} \alpha^{(1)} + \alpha^{(2)} &= h \\ (u-c)\alpha^{(1)} + (u+c)\alpha^{(2)} &= q \end{aligned}\right\} \qquad [7.85]$$

Solving the system for $\alpha^{(1)}$ and $\alpha^{(2)}$ gives:

$$\alpha^{(1)} = \alpha(2) = \frac{h}{2} \qquad [7.86]$$

Consequently, reconstructing h suffices to reconstruct U completely. Reconstructing the unit discharge q is not necessary. Assume now that h has been reconstructed using the MUSCL approach. The reconstructed profile can be expressed as:

$$\tilde{h}_i^n(x) = h_i^n + (x - x_i)a_i^n \qquad [7.87]$$

where x_i is the abscissa of the center of cell i. The left state of the equivalent Riemann problem at interface $i + 1/2$ is:

$$\begin{aligned} \begin{bmatrix} h_\mathrm{L} \\ q_\mathrm{L} \end{bmatrix} &= \frac{h^{(1)}}{2}\begin{bmatrix} 1 \\ u_i^n - c_i^n \end{bmatrix} + \frac{h^{(2)}}{2}\begin{bmatrix} 1 \\ u_i^n + c_i^n \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} h^{(1)} + h^{(2)} \\ \dfrac{h^{(1)} + h^{(2)}}{2}u_i^n + \dfrac{h^{(2)} - h^{(1)}}{2}c_i^n \end{bmatrix} \end{aligned} \qquad [7.88]$$

where $h^{(1)}$ and $h^{(2)}$ are respectively the average values of h over the dependence domains of the waves $\lambda^{(1)} = u - c$ and $\lambda^{(2)} = u + c$. They are given by equation [7.82] as:

$$\left.\begin{aligned} h^{(1)} &= \begin{cases} \dfrac{1}{(u_i^n - c_i^n)\Delta t} \displaystyle\int_{x_{i+1/2}-(u_i^n-c_i^n)\Delta t}^{x_{i+1/2}} \widetilde{h}_i^n(x)\,\mathrm{d}x & \text{if } u_i^n - c_i^n \geq 0 \\ h_i^n & \text{if } u_i^n - c_i^n < 0 \end{cases} \\ h^{(2)} &= \begin{cases} \dfrac{1}{(u_i^n + c_i^n)\Delta t} \displaystyle\int_{x_{i+1/2}-(u_i^n+c_i^n)\Delta t}^{x_{i+1/2}} \widetilde{h}_i^n(x)\,\mathrm{d}x & \text{if } u_i^n + c_i^n \geq 0 \\ h_i^n & \text{if } u_i^n + c_i^n < 0 \end{cases} \end{aligned}\right\} \quad [7.89]$$

Substituting equation [7.87] into equation [7.89] and noting that $x_{i+1/2} = x_i + \Delta x/2$, we have:

$$\left.\begin{aligned} h^{(1)} &= \begin{cases} h_i^n + \dfrac{a_i^n}{2}\left[\Delta x_i - (u_i^n - c_i^n)\Delta t\right] & \text{if } u_i^n - c_i^n \geq 0 \\ h_i^n & \text{if } u_i^n - c_i^n < 0 \end{cases} \\ h^{(2)} &= \begin{cases} h_i^n + \dfrac{a_i^n}{2}\left[\Delta x_i - (u_i^n + c_i^n)\Delta t\right] & \text{if } u_i^n + c_i^n \geq 0 \\ h_i^n & \text{if } u_i^n + c_i^n < 0 \end{cases} \end{aligned}\right\} \quad [7.90]$$

Applying a similar reasoning to cell $i + 1$, the right state of the equivalent Riemann problem is found to be:

$$\begin{bmatrix} h_{\mathrm{R}} \\ q_{\mathrm{R}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} h^{(1)} + h^{(2)} \\ \dfrac{h^{(1)} + h^{(2)}}{2} u_{i+1}^n + \dfrac{h^{(2)} - h^{(1)}}{2} c_{i+1}^n \end{bmatrix} \quad [7.91]$$

where the wave strengths are given by:

$$\left.\begin{aligned} h^{(1)} &= \begin{cases} h_{i+1}^n - \dfrac{a_{i+1}^n}{2}\left[\Delta x_i + (u_{i+1}^n - c_{i+1}^n)\Delta t\right] & \text{if } u_{i+1}^n - c_{i+1}^n \leq 0 \\ h_{i+1}^n & \text{if } u_{i+1}^n - c_{i+1}^n > 0 \end{cases} \\ h^{(2)} &= \begin{cases} h_{i+1}^n - \dfrac{a_{i+1}^n}{2}\left[\Delta x_i + (u_{i+1}^n + c_{i+1}^n)\Delta t\right] & \text{if } u_{i+1}^n + c_{i+1}^n \leq 0 \\ h_{i+1}^n & \text{if } u_{i+1}^n + c_{i+1}^n > 0 \end{cases} \end{aligned}\right\} \quad [7.92]$$

As a conclusion, the EVR approach requires only one variable reconstruction (the water depth h), against two reconstructions for the MUSCL-Hancock approach. Moreover, the EVR approach uses a first-order, explicit time marching approach, against two steps for the second-order in time MUSCL-Hancock.

7.5. Summary

7.5.1. *What you should remember*

Finite volume schemes solve the conservation form of conservation laws. The discretized quantity is the average value of the conserved variable over the computational cells.

The change in the conserved variable from one time level to the next is computed using a balance equation over the computational cells. This requires that the fluxes be estimated at the interfaces between the cells.

Godunov-type schemes use Riemann problems to estimate the fluxes at the interfaces between the cells. The Riemann problem may be solved exactly or approximately.

Higher-order Godunov-type schemes use a reconstruction procedure to estimate the variations of the conserved variable in the computational cells. The variable in a given cell is reconstructed using the average values in the neighboring cells. This allows the gradients to be located more accurately, thus leading to more accurate estimates of the fluxes.

The second-order MUSCL scheme presented in section 7.3 uses a linear reconstruction of the conserved variable. The slope of the profile within the cell i is computed as the average slope between the cells $i-1$ and $i+1$. Spurious oscillations are eliminated from the solution by limiting the slope of the reconstructed profile, as indicated in section 7.3.2.2.

The reconstructed variables lead to generalized Riemann problems at the cell interfaces. Such generalized Riemann problems can be solved using the two-step MUSCL-Hancock approach (valid only for a linear reconstruction) presented in section 7.3.2.3 or using the EVR approach presented in section 7.4, with a particular application to the Saint Venant equations in section 7.4.2. When applied to the shallow water equations, the EVR approach has the advantage that only the water depth needs to be reconstructed and the Riemann problem needs to be solved only once (compared to two reconstructed variables and two solutions of the Riemann problem for the MUSCL-Hancock approach). The EVR approach allows the

computational effort to be reduced substantially, with an increased robustness of the numerical solution in the presence of dry beds.

7.5.2. *Application exercises*

Apply Godunov's scheme to the water hammer equations, the Saint Venant equations and the Euler equations. Use simplifying assumptions on the geometry (e.g. horizontal pipe/channel, negligible friction, etc.) so that the source term is assumed to be zero. Solve the Riemann problems dealt with in Chapter 4 and comment on the accuracy of the numerical solution.

Indications and searching tips for the solution of this exercise can be found at the following URL: http://vincentguinot.free.fr/waves/exercises.htm.