Chapter 3

Weak Solutions and their Properties

3.1. Appearance of discontinuous solutions

3.1.1. *Governing mechanisms*

As shown in section 1.4.3 with the example of the inviscid Burgers equation, initially continuous solutions may evolve into discontinuous solutions. This section focuses on the mechanisms that lead to the formation of discontinuities. Discontinuous solutions are an inevitable consequence of the nonlinearity of a hyperbolic conservation law, as shown hereafter.

Consider a scalar hyperbolic conservation law expressed in conservation form as:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \qquad [3.1]$$

This is the general form [1.1], where the source term is assumed to be zero. For the sake of simplicity, F is assumed to be a function of U only. As shown in Chapter 1, equation [3.1] can be written in the non-conservation form [1.28], recalled here:

$$\frac{\partial U}{\partial t} + \lambda \frac{\partial U}{\partial x} = 0$$

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where $\lambda = \mathrm{d}F/\mathrm{d}U$. As shown in Chapter 1, equation [1.28] can be rewritten in the characteristic form [1.30], recalled here:

$$U = \text{Const} \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda$$

U is constant along the characteristic lines $\mathrm{d}x/\mathrm{d}t = \lambda$. If F is a nonlinear function of U, the wave speed λ depends on the value of U. U being constant along a given characteristic, λ is also a constant along the characteristic and the characteristics are straight lines in the phase space (Figure 3.1).

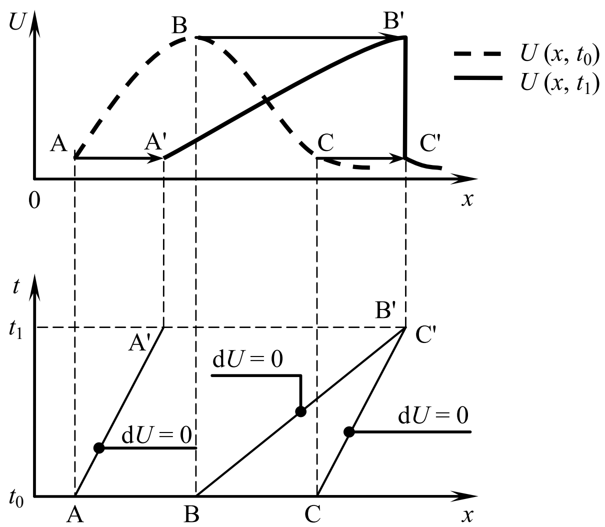

Figure 3.1. *Continuous solution evolving into a discontinuous solution in the case of a convex flux function. Sketch in the physical space (top) and in the phase space (bottom)*

Consider the case where the initial profile (ABC) at time t_0 is not monotonic. The maximum of U is reached at the point B.

If the flux function is convex, the characteristic issued from B moves faster than those issued from A and C because λ is an increasing function of U. The characteristic issued from B 'catches up' the characteristic issued from C at a time $t_1 > t_0$. At $t = t_1$, the points A, B and C move to A', B' and C'. U being constant along the characteristics, the values of U at A, B and C are identical to those at A', B' and C' respectively. Since B' and C' have the same abscissa, the profile of U is necessarily discontinuous because U simultaneously takes the value U_B and U_C at the same point.

If the flux function is concave, the characteristic issued from B is slower than those issued from A and C. Reasoning as in the paragraph above leads to the conclusion that a discontinuity appears at the point A' = B'.

A general formula can be derived for the time at which a discontinuity appears for the first time. This is achieved by deriving an expression for the space derivative $\partial U / \partial x$ from equation [1.28] and finding the date at which $\partial U / \partial x$ becomes infinite. Differentiating equation [1.28] with respect to x leads to:

$$\frac{\partial^2 U}{\partial x \partial t} + \frac{\partial}{\partial x}\left(\lambda \frac{\partial U}{\partial x}\right) = 0 \qquad [3.2]$$

Expanding the space derivative and swapping the time and space differentials leads to:

$$\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial x}\right) + \lambda \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = -\frac{\partial \lambda}{\partial x}\frac{\partial U}{\partial x} \qquad [3.3]$$

As shown in Chapter 1, equation [3.3] can be expressed in characteristic form as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial x}\right) = -\frac{\partial \lambda}{\partial x}\frac{\partial U}{\partial x} \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda \qquad [3.4]$$

Equation [3.4] is a first-order Ordinary Differential Equation (ODE) in $\partial U / \partial x$. Noting that $\partial \lambda / \partial x = \partial \lambda / \partial U \; \partial U / \partial x$, equation [3.4] is rewritten as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial x}\right) = -\frac{\partial \lambda}{\partial U}\left(\frac{\partial U}{\partial x}\right)^2 \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda \qquad [3.5]$$

Since $\partial \lambda / \partial U$ is a function of U only, U is constant along a characteristic line and $\partial \lambda / \partial U$ is also a constant along a characteristic line. ODE [3.5] has the following analytical solution:

$$\frac{\partial U}{\partial x}(t) = \left[\left(\frac{\partial U}{\partial x}(t_0)\right)^{-1} + (t - t_0)\frac{\partial \lambda}{\partial U}\right]^{-1} \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda \qquad [3.6]$$

Equation [3.6] describes the variations of $\partial U / \partial x$ as seen by an observer moving at a speed λ. $\partial U / \partial x$ becomes infinite if the quantity between the brackets in equation [3.6] becomes zero, which occurs at a time $t = t_1$ such that:

$$t_1 = t_0 - \left[\frac{\partial U}{\partial x}(t_0)\frac{\mathrm{d}\lambda}{\mathrm{d}U}\right]^{-1} = t_0 - \left[\frac{\partial \lambda}{\partial x}(t_0)\right]^{-1} \qquad [3.7]$$

The time t_d at which the first discontinuity appears in the profile is given by the minimum of all the times t_1 associated with all the possible values of x:

$$t_d = \min_x \left\{ t_0 - \left[\frac{\partial \lambda}{\partial x}(x,t_0)\right]^{-1} \right\} = t_0 - \left\{ \min_x \left[\frac{\partial \lambda}{\partial x}(x,t_0)\right]\right\}^{-1} \qquad [3.8]$$

Note that the profile can become discontinuous only if t_d is larger than t_0, that is, if there exists at least one value of x for which the following condition is satisfied:

$$\frac{\partial \lambda}{\partial x}(x,t_0) \le 0 \qquad [3.9]$$

3.1.2. *Local invalidity of the characteristic formulation – graphical approach*

The conservation form and the characteristic formulation presented in Chapters 1 and 2 for scalar hyperbolic laws and hyperbolic systems of conservation laws were derived under the assumption that the derivatives of the variables and the fluxes are defined at all points of time and space. This assumption is not valid at discontinuities. The characteristic approach cannot be used in its classical form across discontinuities and a specific treatment must be applied. Such a treatment is detailed in section 3.4.

In the case of scalar hyperbolic conservation laws, the method of characteristics may be applied even to discontinuous solutions, provided that the method is modified using the so-called "equal area rule". The equal area rule combines the properties of invariance and conservation with the necessary condition of solution uniqueness. It consists of the following two steps.

The first step consists of applying the original method of characteristics to the initial profile. The solution becomes discontinuous at the time t_1 when the characteristics issued from B and C intersect at B' = C'. Applying the method of characteristics at a time $t_2 > t_1$ leads to a multi-valued solution profile (A"B"C") as sketched in Figure 3.2 because the point B' passes the point C'. Such a profile is not physically permissible in that U may take two or three different values at the abscissa x lying within the interval $[x_{B"}, x_{C"}]$.

The second step consists of correcting the profile [A"B"C"] so as to restore the uniqueness of the solution. The correction is made as follows:

– the corrected profile is discontinuous because the discontinuity appeared at $t_1 < t_2$;

– the correction should guarantee conservation. Consequently, the area under the corrected profile should be the same as the area under the profile before the correction.

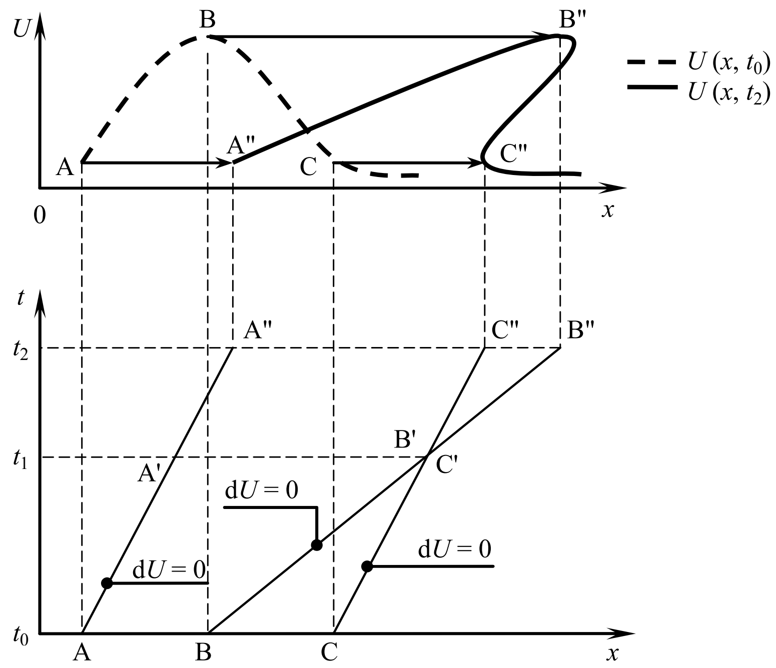

Figure 3.2. *Using the method of characteristics beyond the time at which a discontinuity appears. Sketch in the physical space (top) and in the phase space (bottom)*

The application of the correction to the profile [A"B"C"] in Figure 3.2 is illustrated by Figure 3.3. The gray-shaded areas on both sides of the discontinuity D are strictly equal, hence the term “equal area rule”. The correction is also reflected in the phase space (Figure 3.3, bottom), where the initial characteristics [B'C'] and [B'C"] are replaced with a bold line that represents the trajectory of the discontinuity.

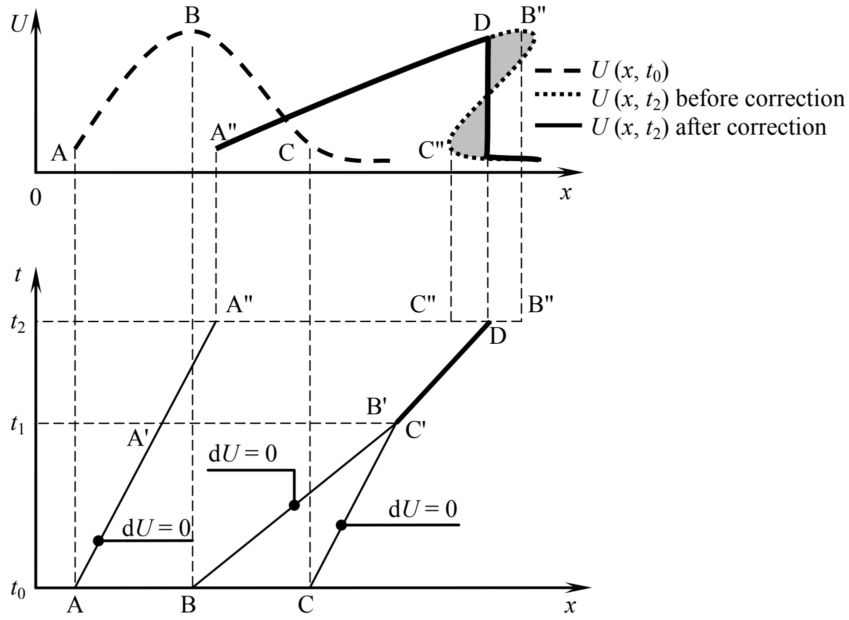

Figure 3.3. *Application of the equal area rule to a multiple-valued solution. Representation in the physical space (top) and in the phase space (bottom)*

3.1.3. *Practical examples of discontinuous flows*

3.1.3.1. *Free surface flow: the breaking of a wave*

The dependence of the wave speed on the flow variable is the main reason for the breaking of sea or ocean waves traveling to the shore. Figure 3.4 illustrates the behavior of a sea wave along a line drawn in the direction perpendicular to the shore. This direction may be seen as the longitudinal axis of a canal of infinite width, the bottom of which rises in the direction of positive x. Consider a wave traveling to the shore. The initial profile [ABC] of the free surface is continuous. The water depth h_B of the crest of the wave is larger than the depth h_C of the front. For a wave traveling over a mild beach slope, the depth h_A of the tail of the wave is larger than h_C and smaller than h_B. The average flow velocity u is small compared to the speed c of the waves in still water, with the consequence that the wave speed $u + c$ can be approximated reasonably with the speed $c = (gh)^{1/2}$. The wave speed being smaller in the regions where the flow is shallower, the characteristic lines $dx/dt = u + c$ are convex in the phase space (Figure 3.4, bottom).

The characteristic that passes at C is slower than the characteristic that passes at B, where the depth is larger. Consequently, the front side [BC] becomes steeper as the wave travels to the shore. Conversely, the characteristic issued from A is slower than that issued from B and the rear side [AB] of the wave becomes milder as time goes. After a certain time the point B' catches up the point C', the free surface becomes vertical and the wave breaks.

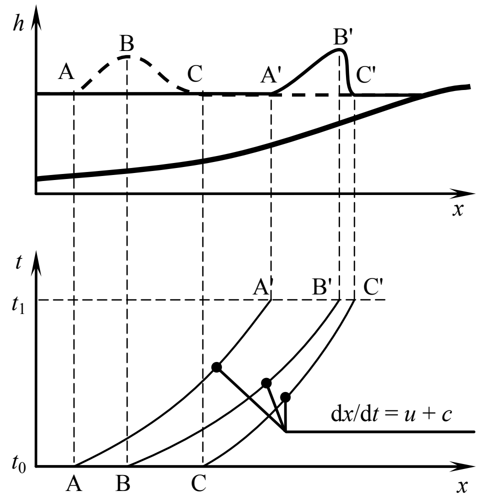

Figure 3.4. *Breaking of a wave traveling to the shoreline. Sketch in the physical space (top) and in the phase space (bottom)*

NOTE. – Although the profile [A"B"C"] in Figure 3.2 is very similar to that of a breaking wave, the resemblance is purely coincidental. The profile [A"B"C"] is derived from a purely mathematical construction that does not account for the phenomena that govern the breaking of a wave. The breaking of the wave is a two-dimensional process in the vertical plane, while the construction in Figure 3.2 involves only one dimension of space.

3.1.3.2. *Aerodynamics: supersonic flight*

By definition, the speed of an airplane (or any other flying object) in supersonic flight is larger than the speed of sound. The gas molecules immediately in front of the airplane cannot move away fast enough and are “pushed” ahead and aside. The local accumulation of the gas molecules induces a rise in the pressure and in the density. The molecules travel to the zones of lower pressure and the thickness of the high pressure zone stabilizes to an equilibrium thickness (typically, millimeters to centimeters), adopting the shape of a V as does the wake of a ship. The gas ahead of this V-shaped zone is undisturbed. The transition between the two zones is very thin (typically, millimeters). The thickness of the transition zone is negligible compared to the dimensions of the flying object and the pressure appears as discontinuous (Figure 3.5).

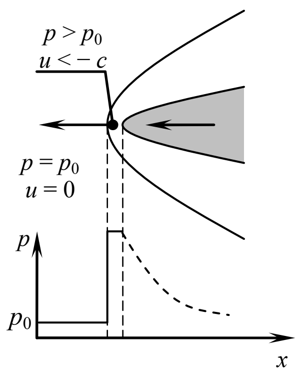

Figure 3.5. *Shock wave created by a supersonic flying object. Geometry of the wave in side view (top), pressure profile (bottom)*

In practice, thermal diffusion and turbulence phenomena induce a widening of the transition zone. The pressure profile is not strictly discontinuous but may appear so at the metric scale. The pressure decreases gradually to the rear side of the plane and the initial pressure is recovered after a sufficiently long distance.

The well-known "supersonic bang" that can be heard when an airplane flies above the speed of sound is nothing but the consequence of the sudden pressure rise across the transition zone.

3.2. Classification of waves

3.2.1. *Shock wave*

A shock wave is characterized by a discontinuity in both the conserved variable and wave speed. It obeys the following criteria (Figure 3.6):

– Criterion (S1). The solution is discontinuous across the shock. The values U_L and U_R on the left- and right-hand sides of the shock are different.

– Criterion (S2). There is at least one wave (the pth wave), the speed of which is discontinuous across the shock, such that the wave speed on the left-hand side of the shock is larger than on the right-hand side and such that the propagation speed of the discontinuity lies between these two wave speeds. The discontinuity is said to be a shock for the *p*th wave, or a *p*–shock.

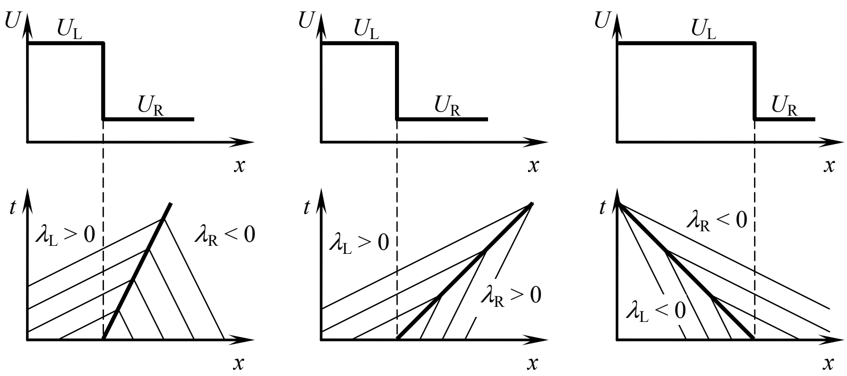

Figure 3.6. *Definition sketch of a shock in the physical space (top) and in the phase space (bottom). Sketch for a scalar variable*

When the law is scalar, there is only one wave. The criteria (S1–2) can be summarized as follows:

$$\left.\begin{array}{l} U_L \neq U_R \\ \lambda_L > c_s > \lambda_R \end{array}\right\} \qquad [3.10]$$

where c_S is the speed of the shock and the subscripts L and R denote the values taken by U and λ on the left- and right-hand sides of the discontinuity respectively. When a hyperbolic system of conservation laws is dealt with, the wave speeds are numbered in ascending order. The discontinuity is a shock for the *p*th wave, or a *p*-shock, if the following conditions are satisfied:

$$\left.\begin{array}{r} U_L \neq U_R \\ \lambda_L^{(p-1)} < c_s < \lambda_L^{(p)} \\ \lambda_R^{(p)} < c_s < \lambda_R^{(p+1)} \end{array}\right\} \qquad [3.11]$$

Note that the last two conditions [3.11] actually imply the inequality $\lambda_L^{(p)} > \lambda_R^{(p)}$. They also guarantee that the discontinuity is a shock neither for the wave $p-1$ nor for the wave $p+1$.

3.2.2. *Rarefaction wave*

A *p*–rarefaction wave (that is, a rarefaction for the *p*th wave) satisfies the following criteria (Figure 3.7):

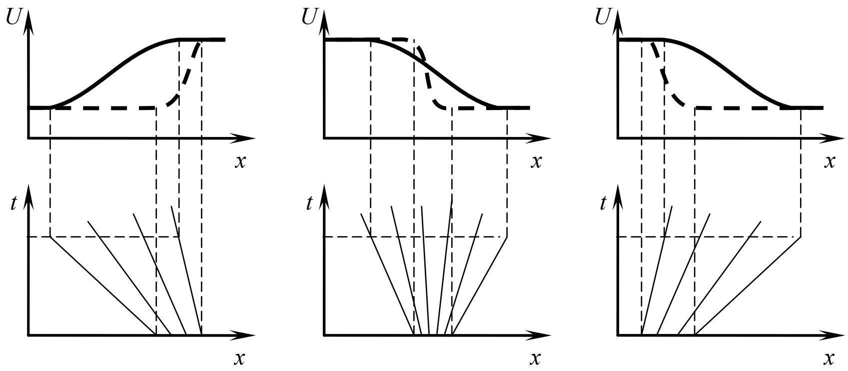

Figure 3.7. *A rarefaction wave in the physical space (top) and in the phase space (bottom). Definition sketch for a scalar variable. Initial profile (dashed line), final profile (solid line)*

– Criterion (R1). The variable U and the wave speeds vary continuously across the wave.

– Criterion (R2). The wave speed $\lambda^{(p)}$ increases from left to right across the wave

$$\frac{\partial \lambda^{(p)}}{\partial x} > 0 \qquad [3.12]$$

Rarefaction waves cause front smearing and profile smoothing.

3.2.3. *Contact discontinuity*

The *p*th wave is a contact discontinuity if it satisfies the following criteria:

– Criterion (C1). The variable is discontinuous across the wave;

– Criterion (C2). The wave speed $\lambda^{(p)}$ is continuous across the wave;

$$\left.\begin{array}{l} U_L \neq U_R \\ \lambda_L^{(p)} = \lambda_R^{(p)} \end{array}\right\} \qquad [3.13]$$

Figure 3.8 illustrates the behavior of a contact discontinuity in the physical space and in the phase space.

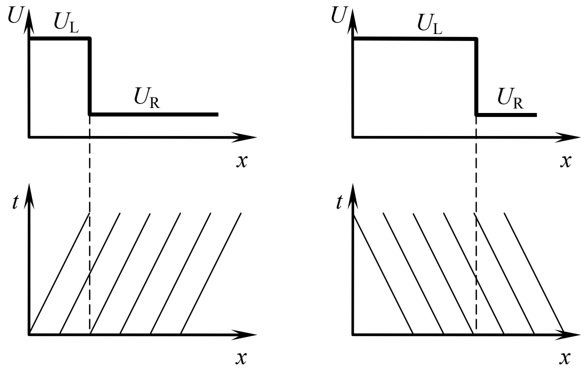

Figure 3.8. *Definition sketch for a contact discontinuity in the physical space (top) and in the phase space (bottom)*

3.2.4. *Mixed/compound wave*

Mixed waves, also known as compound waves, appear in very specific cases, such as non-convex flux functions. In such cases, a profile composed of a rarefaction wave and a shock may appear under certain combinations of initial and boundary conditions (see Chapter 4 for an example). A left-compound wave obeys the following criteria:

– Criterion (ML1). The celery $\lambda^{(p)}$ increases from left to right up to the abscissa x_s of the shock.

– Criterion (ML2). The conserved variable is discontinuous at $x = x_s$.

– Criterion (ML3). The wave speed on the right-hand side of the discontinuity is smaller than the wave speed on the left-hand side.

The criteria above can be summarized as:

$$\left.\begin{array}{ll} \mathrm{U_L} \neq \mathrm{U_R} & \\ \dfrac{\partial \lambda^{(p)}}{\partial x} > 0 & \text{for } x < x_s \\ \lambda_{\mathrm{L}}^{(p)} > \lambda_{\mathrm{R}}^{(p)} & \end{array}\right\} \qquad [3.14]$$

Conversely, a right-compound wave obeys the following definitions:

– Criterion (MR1). The wave speed decreases from right to left down to the abscissa x_s of the shock.

– Criterion (MR2). The conserved variable is discontinuous across the shock.

– Criterion (MR3). The wave speed on the left-hand side of the discontinuity is larger than the wave speed on the right-hand side.

The criteria above can be summarized as:

$$\left.\begin{array}{ll} U_L \neq U_R & \\ \dfrac{\partial \lambda^{(p)}}{\partial x} > 0 & \text{for } x > x_s \\ \lambda_L^{(p)} > \lambda_R^{(p)} & \end{array}\right\} \qquad [3.15]$$

3.3. Simple waves

3.3.1. *Definition and properties*

Consider an $m \times m$ hyperbolic system of conservation laws. By definition, its m wave speeds are all different. The pth wave is a simple wave if the conserved variable U is constant along the characteristics $\mathrm{d}x/\mathrm{d}t = \lambda^{(p)}$. By definition, the characteristic curve for a simple wave is a straight line in the phase space (see Figure 3.9) because the wave speed, that is a function of U, is also constant.

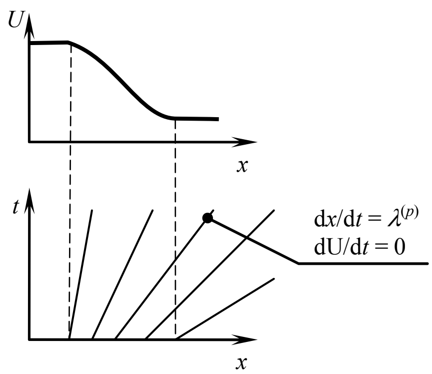

Figure 3.9. *Definition sketch for a simple wave in the physical space (top) and in the phase space (bottom)*

3.3.2. *Generalized Riemann invariants*

Generalized Riemann invariants are differential relationships that apply across simple waves. In contrast with Riemann invariants that may be used across any type of wave, generalized Riemann invariants can be used only across simple waves.

Assume that the *p*th wave is a simple wave (Figure 3.10). Then the characteristics $dx/dt = \lambda^{(p)}$ are straight lines in the phase space. Consider two such characteristics close to each other in the phase space. The leftmost characteristic is denoted by (A), the rightmost characteristic is denoted by B. In the general case, (A) and (B) are not parallel because the values of U along (A) and (B) are not identical. Since the system is hyperbolic, the wave speeds of all the remaining characteristics are different from $\lambda^{(p)}$. Consequently, the characteristics (A) and (B) can be connected using any of the remaining $m - 1$ characteristics.

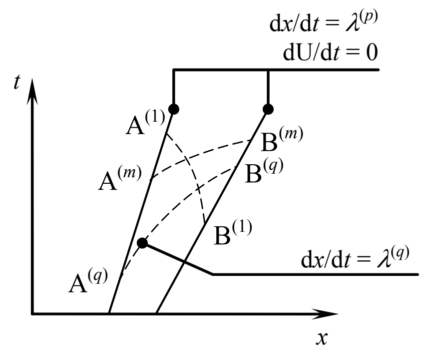

Figure 3.10. *Two neighbor characteristics in the simple wave p. Definition sketch in the phase space*

The variation dU in U between the characteristics (A) and (B) is given by (see equation [2.27]:

$$\mathrm{dU} = \mathrm{K\ dW} = \begin{bmatrix} K_1^{(1)} & \cdots & K_j^{(1)} & \cdots & K_1^{(m)} \\ \vdots & & & & \vdots \\ K_i^{(1)} & \cdots & K_j^{(1)} & \cdots & K_i^{(m)} \\ \vdots & & & & \vdots \\ K_m^{(1)} & \cdots & K_j^{(1)} & \cdots & K_m^{(m)} \end{bmatrix} \begin{bmatrix} \mathrm{d}W_1 \\ \vdots \\ \mathrm{d}W_i \\ \vdots \\ \mathrm{d}W_m \end{bmatrix} \qquad [3.16]$$

By definition, all the Riemann invariants W_q, $q \neq p$, are constant between (A) and (B):

$$\mathrm{d}W_q = 0 \;\forall\; q \neq p \tag{3.17}$$

Consequently, the only non-constant invariant between (A) and (B) is the pth Riemann invariant. Substituting equation [3.17] into equation [3.16] leads to:

$$\mathrm{d}U_k = K_k^{(p)}\mathrm{d}W_p \tag{3.18}$$

In other words, the vector dU is collinear to the pth eigenvector $\mathrm{K}^{(p)}$ across the pth wave.

$$\frac{\mathrm{d}U_1}{K_1^{(p)}} = \frac{\mathrm{d}U_2}{K_2^{(p)}} = \cdots = \frac{\mathrm{d}U_m}{K_m^{(p)}} \qquad \text{across}\ \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda^{(p)} \tag{3.19}$$

The relationships in equation [3.19] are called generalized Riemann invariants. They form a system of $m - 1$ equations that may be used across the pth wave to characterize the properties of the solution. They may be applied to specific problems such as the Riemann problem (see Chapter 4), the solution of which is made of simple waves.

Note however that the generalized Riemann invariants are meaningful only if the solution is continuous across the wave. The generalized Riemann invariants cannot be applied across shock waves. Jump relationships that are detailed in section 3.4.3 should be used instead.

3.4. Weak solutions and their properties

3.4.1. *Definitions*

Equation [1.1] can be rewritten as:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} - S = 0 \tag{3.20}$$

The weak form of equation [3.20] over a domain $[x_1, x_2]\times[t_1, t_2]$ is obtained by multiplying equation [3.20] by a function $w(x, t)$, also known as a weighting function, and by integrating the equation over the domain:

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left(\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} - S\right)w(x,t)\,\mathrm{d}x\;\mathrm{d}t = 0 \tag{3.21}$$

A solution of the weak form [3.21] is called a weak solution of [3.20]. The weak form of a vector equation in the form [2.2] is defined exactly in the same way as that of a scalar equation.

In the particular case where $w(x, t)$ is a constant, equation [3.21] can be simplified into:

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left(\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}-S\right)\mathrm{d}x\ \mathrm{d}t=0 \qquad [3.22]$$

Integrating $\partial U/\partial t$ with respect to time and $\partial F/\partial x$ with respect to x leads to:

$$\int_{x_1}^{x_2} U(x,t_2)\mathrm{d}x-\int_{x_1}^{x_2} U(x,t_1)\mathrm{d}x+\int_{t_1}^{t_2} F(x_2,t)\mathrm{d}t-\int_{t_1}^{t_2} F(x_1,t)\mathrm{d}t = \int_{t_1}^{t_2}\int_{x_1}^{x_2} S(x,t)\mathrm{d}x\ \mathrm{d}t \qquad [3.23]$$

Note that equation [3.23] is strictly equivalent to the balance [1.11–12] over the control volume $[t_0, t_0+\delta t]\times[x_0, x_0+\delta x]$ if $x_1 = x_0$, $x_2 = x_0 + \delta x$, $t_1 = t_0$, $t_2 = t_0 + \delta t$.

3.4.2. *Non-equivalence between the formulations*

Although closely connected together, the forms [3.20] and [3.23] are not strictly equivalent. They differ by two important points:

– As shown in section 1.1.2, equation [3.20] is derived from equation [3.23] by assuming that the size of the integration domain tends to zero. This allows the derivatives $\partial U/\partial t$ and $\partial F/\partial x$ to be introduced. This implies that U (and therefore F) is continuous and differentiable with respect to time and space. The form [3.20] does not account for discontinuous solutions such as shocks and compound waves.

– The assumption of continuous and differentiable solutions is not needed in the form [3.23] because the integrals in equation [3.23] can be calculated even if the solution is discontinuous in time and/or space.

In other words, a "strong solution" of equation [3.20] (that is, a solution that verifies [3.20] for all x and t) is a particular case of a weak solution, while the reciprocal is not true. The "strong form" [3.20] and the weak form [3.23] are equivalent as long as the solution is continuous in time and space. If the solution is discontinuous, equations [3.20] and [3.23] cease to be equivalent. This is of primary importance in the solution of hyperbolic PDEs with discontinuous solutions (see section 3.4.4).

3.4.3. *Jump relationships*

As shown in section 3.1.2, the characteristic form of the equation is based on the implicit assumption that the solution is continuous. It cannot be applied across discontinuities. An alternative technique is needed for the treatment of discontinuous solutions. Jump relationships, also known as "Rankin-Hugoniot relationships", are derived from a balance over a control volume that contains the discontinuity (Figure 3.11). Equation [3.23] is applied to the control volume in the limit of an infinitesimal volume width and time interval.

Consider first that the conservation law is scalar. Denoting by c_s the speed of the discontinuity, the variation between the times t_1 and t_2 in the total amount of U contained in the control volume is given by:

$$\int_{x_1}^{x_2} U(x,t_2)\mathrm{d}x - \int_{x_1}^{x_2} U(x,t_1)\mathrm{d}x = (t_2 - t_1)(U_1 - U_2)c_s \qquad [3.24]$$

where U_1 and U_2 are respectively the values of U on the left- and right-hand side of the discontinuity.

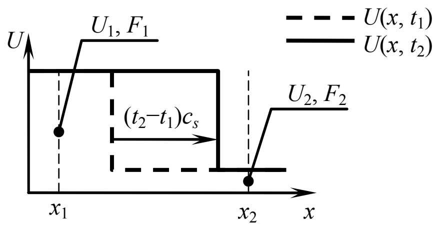

Figure 3.11. *Definition sketch for the Rankin-Hugoniot relationships*

The amount of U that crosses the boundaries of the control volume between t_1 and t_2 is given by:

$$\int_{t_1}^{t_2} F(x_2,t)\mathrm{d}t - \int_{t_1}^{t_2} F(x_1,t)\mathrm{d}t = (F_1 - F_2)(t_2 - t_1) \qquad [3.25]$$

where F_1 and F_2 denote $F(U_1)$ and $F(U_2)$ respectively. The integral of the source term is:

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} S(x,t)\mathrm{d}x\,\mathrm{d}t = (t_2 - t_1)(x_2 - x_1)\overline{S} \qquad [3.26]$$

where $\overline{S}$ is the average of S over the space-time domain $[x_1, x_2]\times[t_1, t_2]$. Substituting equations [3.24–26] into equation [3.23] and dividing by $(t_2 - t_1)$ yields:

$$(U_1 - U_2)c_s + F_2 - F_1 + (x_2 - x_1)\overline{S} = 0 \qquad [3.27]$$

When the width of the control volume tends to zero, the quantity $(x_2 - x_1)\overline{S}$ tends to zero and equation [3.27] becomes:

$$(U_1 - U_2)c_s = F_1 - F_2 \qquad [3.28]$$

Equation [3.28] is generalized to hyperbolic systems of conservation laws by noticing that it is applicable to each of the components of the vectors U and F individually. The vector form of equation [3.28] is therefore:

$$(\mathrm{U}_1 - \mathrm{U}_2)c_s = \mathrm{F}_1 - \mathrm{F}_2 \qquad [3.29]$$

where F_1 and F_2 denote $\mathrm{F}(\mathrm{U}_1)$ and $\mathrm{F}(\mathrm{U}_2)$ respectively. Equations [3.28–29] may be used to determine the speed of the discontinuity. Note that a stationary shock (i.e. a shock that does not move) satisfies the following conditions:

$$\left.\begin{array}{ll} F_1 = F_2 & \text{(scalar law)} \\ \mathrm{F}_1 = \mathrm{F}_2 & \text{(hyperbolic system)} \end{array}\right\} \qquad [3.30]$$

Also note that when the amplitude of the shock tends to zero, the shock speed c_s tends to the wave speed λ. Indeed, equation [3.28] leads to the following equivalence:

$$c_s = \frac{F_2 - F_1}{U_2 - U_1} \underset{U_2 \to U_1}{\approx} \frac{\partial F}{\partial U} = \lambda \qquad [3.31]$$

3.4.4. *Non-uniqueness of weak solutions*

3.4.4.1. *Example 1. The inviscid Burgers equation*

The weak and strong forms of hyperbolic equations not being equivalent, a nonlinear PDE may have several weak solutions, each of which is mathematically permissible. Only physical considerations allow the “correct” solution to be identified from the many possible ones. This is illustrated by the inviscid Burgers equation.

The inviscid Burgers equation derived in section 1.4.2 can be written in non-conservation form as in equation [1.66], recalled here:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

A possible conservation form of this equation is equation [1.69], recalled here:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

In this form the conserved variable is $U = u$ and the flux function is $F = U^2/2$. Applying the jump relationship [3.28] leads to the following formula for c_S:

$$c_s = \frac{1}{2}\frac{U_1^2 - U_1^2}{U_1 - U_2} = \frac{U_1 + U_2}{2} = \frac{u_1 + u_2}{2} \qquad [3.32]$$

However, equation [3.32] stems directly from the choice of the conserved variable. Other choices could be made, such as:

$$u = V^2 \qquad [3.33]$$

Substituting definition [3.33] into the non-conservation form [1.66] leads to:

$$\frac{\partial}{\partial t}(V^2) + V^2\frac{\partial}{\partial x}(V^2) = 0 \qquad [3.34]$$

Simplifying yields the following non-conservation form in V:

$$\frac{\partial V}{\partial t} + V^2\frac{\partial V}{\partial x} = 0 \qquad [3.35]$$

Equation [3.35] is equivalent to equation [1.66] because the speed λ of the wave in V is equal to V^2, that is to u, exactly as in the original equation. A continuous solution of equation [3.35] behaves exactly as a continuous solution of equation [1.66]. Differences arise when discontinuous solutions are considered. The conservation form of equation [3.35] is:

$$\frac{\partial V}{\partial t}+\frac{\partial}{\partial x}\left(\frac{V^3}{3}\right)=0 \qquad [3.36]$$

where the conserved variable is V and the flux function is $F = V^3/3$. Using the relationship [3.28] leads to the following expression for the shock speed:

$$c_s=\frac{F_1-F_2}{V_1-V_2}=\frac{V_1^2+V_1V_2+V_2^2}{3}=\frac{u_1+(u_1u_2)^{1/2}+u_2}{3} \qquad [3.37]$$

When u_1 and u_2 tend identically to a fixed value u, equations [3.32] and [3.37] tend to the same wave speed $\lambda = u$. When the solution is discontinuous however, equations [3.32] and [3.37] give different results.

There are two reasons for this:

– Equations [1.66] and [3.32] are transformed into equations [1.69] and [3.37] respectively under the assumption that the derivatives of u are defined everywhere. This is not true when the solution is discontinuous.

– The conserved variable is not the same in equation [1.69] as in equation [3.37]. Equation [1.69] is based on the implicit assumption that u is the conserved variable. Equation [3.37] is based on the implicit assumption that $V = u^{1/2}$ is the conserved variable.

The example of equations [3.32] and [3.37] shows that the solutions of equation [1.66] are not unique. Only a proper choice of the conserved variable allows the uniqueness of the solution to be ensured. It is the modeler's responsibility to define the conserved variable on the basis of physical considerations. This can be done only based on the analysis of the physical process involved, including in the case where the solutions become discontinuous.

3.4.4.2. *Example 2. The hydraulic jump*

Hydraulics specialists usually derive the steady-state, open channel flow equations using the concept of energy, also known as the hydraulic head. Such equations, however, may be obtained directly from the momentum equations, with the advantage that they remain valid even when the flow becomes discontinuous, which is not the case with the equation of energy. The equations for steady-state, open channel flow in a rectangular prismatic channel can be written as:

$$\left.\begin{aligned} \frac{\partial Q}{\partial x} &= 0 \\ \frac{\partial}{\partial x}\left(\frac{Q^2}{A}+\frac{P}{\rho}\right) &= (S_0 - S_f)gA \end{aligned}\right\} \qquad [3.38]$$

Substituting the continuity equation into the momentum equation leads to:

$$Q^2 \frac{\partial}{\partial x}\left(\frac{1}{A}\right)+\frac{\partial}{\partial x}\left(\frac{P}{\rho}\right) = (S_0 - S_f)gA \qquad [3.39]$$

By definition, $\partial(P/\rho)/\partial x = c^2 \partial A/\partial x$ and $\partial A/\partial x = b\,\partial h/\partial x$. Equation [3.39] can be rewritten as:

$$(c^2 - u^2)b\frac{\partial h}{\partial x} = (S_0 - S_f)gA \qquad [3.40]$$

Noting that $c^2 = gA/b$ and using the Froude number $\mathrm{Fr} = u/c$, equation [3.40] becomes:

$$\frac{\partial h}{\partial x} = \frac{S_0 - S_f}{1-\mathrm{Fr}^2} \qquad [3.41]$$

A more classical approach, used in most textbooks, consists of defining the hydraulic head H as the ratio of the total energy of the fluid to the product ρg:

$$H = \zeta + \frac{u^2}{2g} = h + \frac{u^2}{2g} + z_b \qquad [3.42]$$

and stating that the head loss is due to the work carried out by the friction forces:

$$\frac{\partial H}{\partial x} = -S_f \qquad [3.43]$$

Substituting equation [3.42] into equation [3.43] leads to the following expression:

$$\frac{\partial h}{\partial x} + \frac{1}{2g}\frac{\partial}{\partial x}(u^2) - S_0 = -S_f \qquad [3.44]$$

Noting that $u = Q/A$ and that Q is constant, equation [3.44] can be rewritten as:

$$\frac{\partial h}{\partial x} + \frac{Q^2}{2g}\frac{\partial}{\partial x}\left(\frac{1}{A^2}\right) = S_0 - S_f \qquad [3.45]$$

Using the differential dA = b dh again, equation [3.45] leads to:

$$\frac{\partial h}{\partial x} - \frac{Q^2 b}{2gA^3}\frac{\partial h}{\partial x} = S_0 - S_f \qquad [3.46]$$

Noting that $u = Q/A$ and introducing the Froude number, equation [3.46] is easily shown to be equivalent to equation [3.41]. Consequently, stating the conservation of momentum is equivalent to stating the conservation of energy in the continuous case.

Consider now a hydraulic jump in a rectangular channel of width b (Figure 3.12).

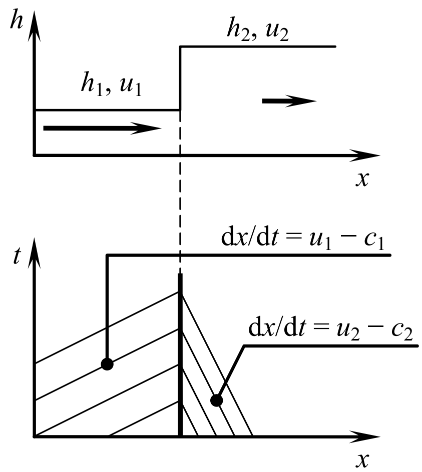

Figure 3.12. *Stationary hydraulic jump. Definition sketch in the physical space (top) and in the phase space for the characteristic u – c (bottom)*

A hydraulic jump is a stationary shock, the flow upstream of which is supercritical and subcritical downstream. Note that the depth h_1 upstream of the jump is necessarily smaller than the depth h_2 downstream of the jump because the jump is a shock for the characteristic $\mathrm{d}x/\mathrm{d}t = u - c$. For a stationary jump, equation [3.30] leads to:

$$\left.\begin{array}{c} Q_1 = Q_2 = Q \\ \dfrac{Q^2}{bh_1} + bg\dfrac{h_1^2}{2} = \dfrac{Q^2}{bh_2} + bg\dfrac{h_2^2}{2} \end{array}\right\} \qquad [3.47]$$

These two equations state the conservation of mass and momentum across the shock. Note that this is not equivalent to stating the conservation of energy. If this was the case, the head would be identical on both sides of the jump and the following relationship would hold:

$$h_1 + \frac{Q^2}{2gb^2}\frac{1}{h_1^2} = h_2 + \frac{Q^2}{2gb^2}\frac{1}{h_2^2} \qquad [3.48]$$

Equation [3.48] is not equivalent to the second relation [3.47]. Using equation [3.47], the head loss ΔH across the shock can be shown to be:

$$\Delta H = H_1 - H_2 = \frac{(h_2 - h_1)^3}{4h_1h_2} \qquad [3.49]$$

ΔH is always positive because $h_1 < h_2$ by definition. The head loss expresses the fact that part of the mechanical energy of the fluid is dissipated across the jump. The dissipation takes the form of a heat transfer to the fluid.

As in the example of the inviscid Burgers equation, the principle of conservation of momentum and energy are equivalent as long as the flow variables remain continuous. As soon as the solution becomes discontinuous, the conservation of momentum and the conservation of mechanical energy cease to be equivalent.

3.4.5. *The entropy condition*

The entropy condition allows mathematically permissible solutions that are not satisfactory from a physical point of view to be eliminated. It is also used to ensure the uniqueness of solutions of initial value problems such as the Riemann problem (see Chapter 4) [LIU 75, LIU 76]. The entropy condition is based on the following consideration. The jump relationship [3.28] allows for the existence of "rarefaction shocks", that is, solutions that satisfy the criterion (S1) in section 3.2.1 and do not satisfy the criterion (S2). Such "rarefaction shocks" would verify the following conditions:

$$\left.\begin{array}{ll} U_{\mathrm{L}} \neq U_{\mathrm{R}} & \\ \lambda_{\mathrm{L}}^{(p)} < \lambda_{\mathrm{R}}^{(p)} & \forall p = 1,\ldots,m \end{array}\right\} \qquad [3.50]$$

As illustrated by Figure 3.13, the characteristics converge to a shock in the phase space (Figure 3.13a), while they diverge from a "rarefaction shock" (Figure 3.13b).

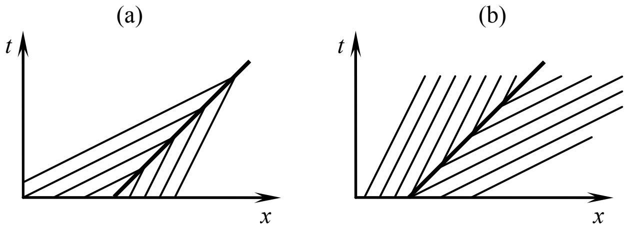

Figure 3.13. *Two mathematically permissible, discontinuous solutions: shock (a), "rarefaction shock" (b)*

The entropy principle states that "rarefaction shocks" are not physically permissible, because a discontinuous solution with wave speeds on the left-hand side of the discontinuity smaller than those on the right-hand side is not permissible. The term "entropy principle" was introduced by Courant and Friedrichs [COU 48] in their study of the Euler equations (see section 2.6). The entropy may be seen as an analog for aerodynamics of the mechanical energy, or hydraulic head, used in open channel hydraulics (the hydraulic head may be used as an entropy function for the Saint Venant or shallow water equations). As shown in section 3.4.4.2, the hydraulic head is not conserved across a shock. In a similar fashion, entropy always increases when a shock is passed in the direction of the flow.

The entropy principle may be justified as follows. A discontinuous solution may be viewed as the limit case of a continuous profile, where both sides of the discontinuity are connected to each other within a very short distance ε (see Figure 3.14). If the wave speed on the left-hand side of the discontinuity is larger than the wave speed on the right-hand side of the discontinuity, the profile becomes steeper and a discontinuity appears (Figure 3.14a). Conversely, if the wave speed on the left-hand side of the discontinuity is smaller than the wave speed on the right-hand side, a rarefaction wave appears and the profile becomes smoother (Figure 3.14b).

A shock may be seen as a self-stabilizing wave pattern in that any local smoothing of the profile (due for example to the presence of an additional source term, etc.) is automatically eliminated because the wave speed on the left-hand side of the shock is larger than on the right-hand side and the solution remains discontinuous. In contrast, a "rarefaction shock" is not a self-stabilizing wave pattern because if the profile becomes locally smooth for some reason, the difference between the wave speed on both sides of the discontinuity leads to a smoothing of the profile, thus destroying the discontinuous character of the solution.

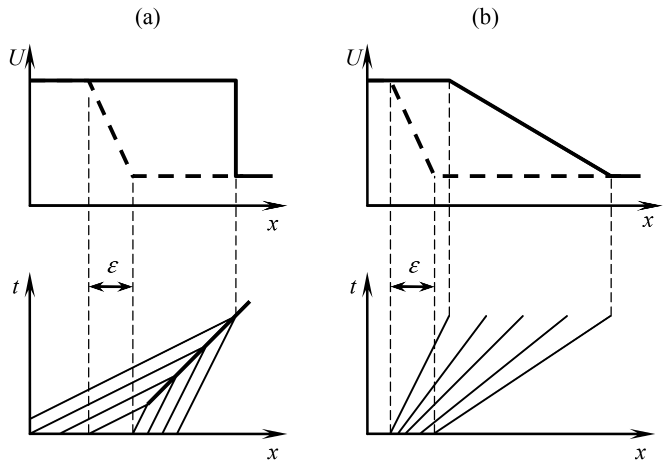

Figure 3.14. *Discontinuous profile seen as a limit of a continuous profile. Initial profile (dashed line), final profile (solid line) for a physically permissible shock (left) and for a physically non-permissible shock (right)*

3.4.6. *Irreversibility*

A salient feature of weak solutions is that their behavior is not reversible in time. In fact, two different initial conditions may lead to the same discontinuous solution, as shown in the example hereafter.

Consider the conservation form [1.69] of the inviscid Burgers equation. The initial condition is given by (see Figure 3.15a):

$$u(x,0) = \begin{cases} u_1 & \text{for } x \le x_1 \\ u_1 \dfrac{x_2 - x}{x_2 - x_1} + u_2 \dfrac{x - x_1}{x_2 - x_1} & \text{for } x_1 \le x \le x_2 \\ u_2 & \text{for } x \ge x_2 \end{cases} \qquad [3.51]$$

where $x_1 < x_2$ and $u_1 > u_2 > 0$. The initial condition takes the form of a ramp that connects the constant states u_1 and u_2 linearly between the abscissas x_1 and x_2. Since $u_1 > u_2$ the left-hand part of the profile travels faster than the right-hand part.

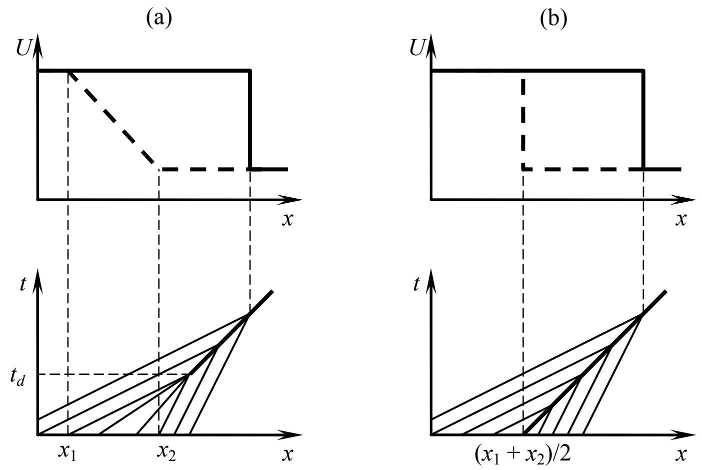

Figure 3.15. *Two different initial conditions (dashed lines) leading to identical solutions at t > t_d (solid lines)*

The profile becomes steeper and becomes discontinuous at a time t_d:

$$t_d = \frac{x_2 - x_1}{u_1 - u_2} \tag{3.52}$$

At $t = t_d$ the discontinuity is located at $x = x_d$:

$$x_d = x_1 + u_1 t_d = x_2 + u_2 t_d = \frac{u_1 x_2 - u_2 x_1}{u_1 - u_2} \tag{3.53}$$

At further times the discontinuity propagates at a speed given by the average between u_1 and u_2 (see equation [3.31]). Note however that another initial condition may be defined (Figure 3.15b):

$$u(x,0) = \begin{cases} u_1 & \text{for } x < \dfrac{x_1 + x_2}{2} \\ u_2 & \text{for } x > \dfrac{x_1 + x_2}{2} \end{cases} \tag{3.54}$$

It is easy to check that the initial conditions [3.51] and [3.54] yield exactly the same solution for $t > t_d$.

In other words, different initial conditions may lead to the same final discontinuous solution. Consequently, a discontinuous solution may not be used as a starting point to "travel backwards in time" and calculate the solution at earlier times. The irreversible behavior of the solutions stems directly from the nonlinear character of the equations that makes discontinuous solutions possible.

3.4.7. *Approximations for the jump relationships*

This section gives two approximations for the jump relationships. Such approximations have been used by a number of authors in the development of numerical methods for the calculation of discontinuous solutions. Any reader interested in the details of the proof may find it useful to refer to [COU 48] and [LAX 57].

Theorem 1. The speed of a shock for the *p*th wave in a convex conservation law can be approximated with the arithmetic mean of the wave speeds $\lambda^{(p)}$ on both sides of the shock. The approximation is second-order with respect to the variation ΔU across the shock.

Theorem 2. The variation in the *p*th generalized Riemann invariant across a *p*-shock is of third order with respect to the variation ΔU across the shock.

Theorem 1 is best illustrated by the application to the inviscid Burgers equation (see equations [3.31] and [3.37]). Equation [3.31] is applicable if u is defined as the conserved variable. Equation [3.37] applies if $u^{1/2}$ is defined as the conserved variable. It is easy to check that equation [3.37] is a second-order approximation of equation [3.31].

Theorem 2 is useful when discontinuous solutions are to be calculated. In fact, its direct implication is that the Riemann invariants provide reasonably accurate approximations of the Rankin-Hugoniot relationships. Such a property has been used to derive approximate solvers for the Riemann problem covered in Chapter 4. The Riemann problem serves as a basis for a number of numerical techniques for the solution of hyperbolic systems of conservation laws with discontinuous solutions.

3.5. Summary

3.5.1. *What you should remember*

Three main types of wave may be distinguished: shock waves (see section 3.2.1), rarefaction waves (see section 3.2.2) and contact discontinuities (see section 3.2.3). Compound waves may appear when the flux function is non-convex. A compound wave is formed by the conjunction of a shock and a rarefaction wave.

A simple wave is a wave along the characteristics of which the conserved variable is a constant. In an $m \times m$ hyperbolic system, the generalized Riemann invariants provide $m - 1$ differential relationships across simple waves.

When the flux function is nonlinear, discontinuous solutions may arise from initially continuous profiles. This is because the dependence of the wave speed on the value of the conserved variable induces a deformation in the solution profile.

A discontinuous solution of a hyperbolic conservation law is called a weak solution because it is the solution of the weak form [3.32] of the original equation [3.1]. Both formulations are non-equivalent. The "strong form" is a particular case of the weak form under the assumption of continuous and differentiable solutions.

Weak solutions may be discontinuous. They are not unique. The "correct" weak solution of a conservation law must be chosen on the basis of physical considerations, in the light of the physical processes involved that allow the conserved variable to be identified.

The behavior of weak solutions is irreversible in time. Several initial conditions may lead to the same discontinuous solution. Consequently, inverse modeling (that is, retrieving the initial condition from the solution at a later time) cannot be carried out in a straightforward manner in the presence of weak solutions.

The characteristic form of the equations, that is based on the assumption of continuous and differentiable variables, is not applicable across discontinuities. The equal area rule allows weak solutions to be calculated using the method of characteristics in the scalar case. In the general case, the jump relationships [3.28–29], also called the Rankin-Hugoniot relationships, must be used.

The main two types of discontinuity in a solution are shocks and contact discontinuities. The wave speed on the left-hand side of the shock is always larger than on the right-hand side, while they are identical in the case of a contact discontinuity.

The entropy principle states that "rarefaction shocks", the wave speed on the left-hand side of which would be smaller than on the right-hand side, are not physically permissible, even though they satisfy the jump relationship, thus being mathematically permissible.

The Riemann invariants may be viewed as an approximation of the jump relationships across shocks of small amplitude.

3.5.2. *Application exercises*

3.5.2.1. *Exercise 3.1: the kinematic wave equation*

Consider the rectangular channel used in Exercise 2.5 (see section 2.7.2.5). The flow is assumed to obey Strickler's friction law [1.81]. Steady state is assumed.

1) The initial water depth is uniformly equal to $h_0 = 1$ m. Assuming that the wide channel approximation is applicable, compute the initial discharge into the channel under the assumption of a uniform, steady flow ($S_0 = S_f$). Provide the expression of the wave speed for the kinematic wave. Carry out the numerical application for the parameters in Table 2.1.

2) A perturbation $\Delta h = 0.5$ m appears instantaneously at the upstream end of the channel. Show that a shock wave appears. Provide the expression of the propagation speed of the shock wave. Carry out the numerical application for the parameters in Table 2.1.

Indications and searching tips for the solution of this exercise can be found at the following URL: http://vincentguinot.free.fr/waves/exercises.htm.

3.5.2.2. *Exercise 3.2: the kinematic wave equation*

Consider the channel of Exercise 3.1, with the same geometry and initial conditions. The water depth at the upstream end of the channel is now assumed to increase linearly from 1 m to 1.25 m between $t = 0$ and $t = 100$ s, and to decrease linearly from 1.25 m to 1 m between $t = 100$ s and $t = 200$ s.

1) Assuming that the kinematic wave approximation is applicable, provide the expression of the time t_d at which the solution becomes discontinuous. Compute t_d and the location of the shock at $t = t_d$ from the parameters in Table 2.1.

2) Plot the water level profile at $t = 150$ s, 300 s, 450 s and 600 s. *N.B.*: it is advised to express both h and x as functions of the time t_L at which the characteristic leaves the left-hand end of the channel.

Indications and searching tips for the solution of this exercise can be found at the following URL: http://vincentguinot.free.fr/waves/exercises.htm.

3.5.2.3. *Exercise 3.3: the Buckley-Leverett equation*

Consider an aquifer, the characteristics of which are given in Table 1.5 in Exercise 1.4 (see section 1.8.2.4). The aquifer is now assumed to be uniformly contaminated with an initial hydrocarbon saturation of 90% (i.e. the initial water saturation is assumed to be 10% everywhere). As in Exercise 1.4, the aquifer is decontaminated by injecting pure water with a Darcy velocity V at the left-hand end of the domain.

1) Show that the saturation profile at $t > 0$ is a compound wave.

2) Compute the propagation speed of the shock.

3) Compute the time at which the average contamination (i.e. the average hydrocarbon saturation) in the aquifer is 5%, 1% and 0.5%.

Indications and searching tips for the solution of this exercise can be found at the following URL: http://vincentguinot.free.fr/waves/exercises.htm.

3.5.2.4. *Exercise 3.4: the Saint Venant equations*

Consider the channel of Exercise 3.1, where the Saint Venant equations are to be applied instead of the kinematic wave approximation.

1) The initial water depth is assumed to be uniformly equal to 1 m. Compute the speeds of the waves for the hydraulic parameters given in Table 2.1. Show that the flow regime depends on the slope. Provide the expression of the slope S_c for which the flow is critical.

2) A perturbation $\Delta h = 1$ m in the water level appears instantaneously at the upstream end of the channel. This triggers a moving bore that propagates to the right. Assuming that the flow regime is subcritical, provide the expression satisfied by the variation ΔQ in the discharge. Carry out the numerical approximation for $S_0 = 10^{-3}$.

Indications and searching tips for the solution of this exercise can be found at the following URL: http://vincentguinot.free.fr/waves/exercises.htm.

3.5.2.5. *Exercise 3.5: the Euler equations*

An airplane moves at Mach 1 in immobile air. For the sake of simplicity, the coordinate system is attached to the airplane.

1) Write the continuity equation and the momentum equation under the assumption of steady state. The flow velocity is assumed to be zero on the hull of the airplane. Show that the assumption of a steady state flow necessarily induces a

multidimensional flow pattern and that the air must be “evacuated” in the lateral direction.

2) Determine the lateral flow, the pressure rise and the air density next to the hull. Carry out the numerical application for the parameters in Table 3.1.

3) Check that the entropy principle is verified across the shock.

Symbol	Meaning	Value
M_0	Far field Mach number upstream of the airplane	1
p_0	Far field pressure upstream of the airplane	10^5 Pa
γ	Polytropic constant for a perfect gas	1.4
ρ_0	Far field air density upstream of the airplane	1.2 kg/m^3

Table 3.1. *Parameters for Exercise 3.5*

Indications and searching tips for the solution of this exercise can be found at the following URL: http://vincentguinot.free.fr/waves/exercises.htm.