Appendix C

Approximate Riemann Solvers

C.1. The HLL and HLLC solvers

C.1.1. *The HLL solver*

C.1.1.1. *Principle*

The HLL solver is named after the initials of Harten, Lax and Van Leer [HAR 83b]. The HLL solver was developed for 2×2 systems of hyperbolic conservation laws. In most applications of fluid mechanics, the two equations solved are the continuity and the momentum equations.

The HLL solver is based on the *a priori* assumption that the solution is made of two discontinuities moving at speeds λ^- and λ^+ that separate the left and right states of the Riemann problem from an intermediate region of constant state (Figure C.1). The discontinuities do not need to be physically permissible (i.e. the entropy principle does not need to be verified). Assuming that estimates can be provided for the speeds λ^- and λ^+, the Rankin-Hugoniot conditions can be written across the two waves, thus providing two relationships between the intermediate region of constant state and the states of the Riemann problem.

Writing the jump relationship across the wave λ^- gives:

$$F_L - F_* = (U_L - U_*)\lambda^- \qquad [C.1]$$

Wave Propagation in Fluids: Models and Numerical Techniques, Second Edition Vincent Guinot
© 2010 ISTE Ltd. Published 2010 by ISTE Ltd.

where U_L and U_R are the values of U and F in the left state of the Riemann problem respectively, and U* and F* are the values of U and F in the intermediate region respectively.

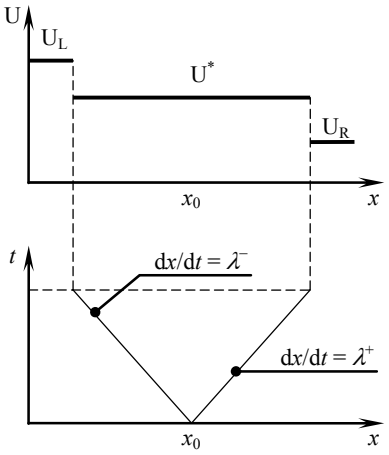

Figure C.1. *Principle of the HLL solver. Definition sketch in the physical space (top) and in the phase space (bottom)*

Writing the jump relationship across the wave λ^+ leads to:

$$F^* - F_R = (U^* - U_R)\lambda^+ \qquad [C.2]$$

where U_R and F_R are the values of U and F in the right state of the Riemann problem. Solving equations [C.1–2] for U* and F* yields:

$$\left.\begin{aligned} U^* &= \frac{\lambda^+ U_R - \lambda^- U_L}{\lambda^+ - \lambda^-} + \frac{F_L - F_R}{\lambda^+ - \lambda^-} \\ F^* &= \frac{\lambda^+ F_L - \lambda^- F_R}{\lambda^+ - \lambda^-} - \frac{\lambda^- \lambda^+}{\lambda^+ - \lambda^-}(U_L - U_R) \end{aligned}\right\} \qquad [C.3]$$

F* as given by the second equation [C.3] is the approximate value of the flux in the intermediate region of constant state. For the sake of comparison with other approximate solvers (see sections C.2 and C.3), the second equation [C.3] can be rewritten as:

$$F^* = \frac{F_L + F_R}{2} + \left[\frac{1}{2}\frac{\lambda^- + \lambda^+}{\lambda^+ - \lambda^-}\tilde{A} + \frac{\lambda^- \lambda^+}{\lambda^+ - \lambda^-} I\right](U_L - U_R) \qquad [C.4]$$

where I and Ã are respectively the identity matrix and Roe's matrix, that is, the matrix such that $F_L - F_R = \widetilde{A}(U_L - U_R)$ (see section 6.8 for details). Equation [C.4] indicates that the HLL flux can be seen as the sum of a centered estimate of the flux (that is unconditionally unstable when a first-order explicit method is used, see section 6.5.2) and a diffusive flux that contributes to stabilize the numerical solution. The diffusive flux is often called an artificial viscosity term.

The flux F_0 at the location of the initial discontinuity is equal to F^* if $\lambda^- < 0 < \lambda^+$, that is, for a subsonic/subcritical flow configuration. For supersonic/supercritical situations, F_0 cannot be taken equal to F^*, unless the estimate of the wave speeds λ^- and λ^+ is adapted as described in section C.1.1.2.

C.1.1.2. *Wave speed estimates*

Davis [DAV 88] proposed the following formula for the estimates of λ^- and λ^+:

$$\left.\begin{aligned} \lambda^- &= \min\left[\lambda^{(1)}(U_L), \lambda^{(1)}(U_R), 0\right] \\ \lambda^+ &= \max\left[\lambda^{(2)}(U_L), \lambda^{(2)}(U_R), 0\right] \end{aligned}\right\} \qquad [C.5]$$

with $\lambda^{(1)} < \lambda^{(2)}$. Formula [C.4] allows subcritical conditions as well as supercritical conditions to be accounted for in a single formula. Assume indeed a supercritical flow to the left, $\lambda^{(1)} < \lambda^{(2)} < 0$. Equations [C.5] become:

$$\left.\begin{aligned} \lambda^- &= \min\left[\lambda^{(1)}(U_L), \lambda^{(1)}(U_R)\right] \\ \lambda^+ &= 0 \end{aligned}\right\} \qquad [C.6]$$

and equation [C.3] leads to:

$$F_0 = F^* = F_R \qquad [C.7]$$

This is not the correct expression for the flux F^* in the intermediate region of constant state. However, this is the correct formula for the value F_0 of F at the location of the initial discontinuity.

Einfeldt [EIN 88] shows that the following estimates lead to a better resolution of shock waves:

$$\left.\begin{aligned} \lambda^- &= \min\left[\lambda^{(1)}(U_L), \widetilde{\lambda}^{(1)}(U_R, U_R), 0\right] \\ \lambda^+ &= \max\left[\widetilde{\lambda}^{(2)}(U_R, U_R), \lambda^{(2)}(U_R), 0\right] \end{aligned}\right\} \qquad [C.8]$$

where the wave speeds $\tilde{\lambda}^{(p)}(U_R, U_R)$ are the eigenvalues of Roe's matrix $\tilde{A}$ obtained from the states U_L and U_R (see section 6.8 for details). An alternative option proposed in [EIN 88] consists of using the property, proved in [LAX 57] for convex conservation laws, that the propagation speed for a shock is approximated by the average of the wave speeds on both sides of the shocks:

$$\left.\begin{aligned} \lambda^- &= \min\left[\frac{\lambda^{(1)}(U_L)+\lambda^{(1)}(U_R)}{2}, 0\right] \\ \lambda^+ &= \max\left[\frac{\lambda^{(2)}(U_L)+\lambda^{(2)}(U_R)}{2}, 0\right] \end{aligned}\right\} \qquad [C.9]$$

C.1.2. *The HLLC solver*

The HLLC solver is an extension of the HLL solver to 3×3 hyperbolic systems of conservation laws where the middle wave is a contact discontinuity. Examples of such systems are the Euler equations of gas dynamics seen in section 2.6, or the one-dimensional restriction of the two-dimensional shallow water equations seen in section 5.4. Combining the linear advection equation seen in section 1.2 to hydrodynamic equations such as the water hammer equations (section 2.4) or the Saint Venant equations (section 2.5).

Note that the HLL solver may also be applied to 3×3 systems because equation [C.3] does not use any particular assumption as to the number of components of U and F. However, the contact discontinuity in 3×3 systems moves at the speed u, while the speeds of the other two waves are $u - c$ and $u + c$. The contact discontinuity is located in the intermediate region of constant state. Equation [C.3] that leads us to overestimate the speed of the contact discontinuity induces strong numerical diffusion, thus causing unacceptable smoothing of the contact discontinuity in the numerical solution. The more accurate HLLC solver [TOR 94] was developed to better account for the propagation speed of the contact discontinuity.

The HLLC solver is applied to the one-dimensional restriction of the two-dimensional shallow water equations seen in section 5.4. Remember that the one-dimensional restriction of two-dimensional equations in the direction normal to the interfaces between the computational cells is the key tool to the finite volume solution of two-dimensional hyperbolic conservation laws (see section 7.1.2). For

the sake of simplicity, the source term is assumed to be zero in what follows. Equation [2.2] is to be solved, with the following definitions for U, F and S:

$$\mathrm{U} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}, \qquad \mathrm{F} = \begin{bmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{bmatrix}, \qquad \mathrm{S} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [C.10]$$

where g is the gravitational acceleration, h is the water depth, u and v are the flow velocities in the x- and y-directions respectively. The first two components of the vectors account for the continuity and x-momentum equations, that yield the waves $u - c$ and $u + c$. The third component of the vectors U and F accounts for the conservation of the y-momentum. It is easy to check that the following characteristic equation holds for the third wave:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 0 \quad \text{for} \frac{\mathrm{d}x}{\mathrm{d}t} = u \qquad [C.11]$$

The y-velocity v is a Riemann invariant along the characteristic of speed u. The x-velocity u being independent from v, the propagation speed of the invariant v is not a function of it and the wave is a contact discontinuity.

The wave pattern of the solution is the following (see Figure C.2). The central wave is a contact discontinuity moving at a speed u. The left and right waves, the speeds of which are $u - c$ and $u + c$ respectively, may be rarefaction waves or shock waves depending on the left and right states of the Riemann problem.

The intermediate region of constant state is divided into two parts. The subregion to the left of the contact discontinuity is denoted by the superscript *,1, while the subregion on the right-hand side of the contact discontinuity is denoted by the superscript *,2. The first two components F_1 and F_2 of the flux are the same in both subregions. They are computed using equation [C.3] as follows:

$$\left.\begin{aligned} F_1^{*,1} = F_1^{*,2} &= \frac{\lambda^+ \mathrm{F}_{1,\mathrm{L}} - \lambda^- \mathrm{F}_{1,\mathrm{R}}}{\lambda^+ - \lambda^-} - \frac{\lambda^- \lambda^+}{\lambda^+ - \lambda^-}(h_\mathrm{L} - h_\mathrm{R}) \\ F_2^{*,1} = F_2^{*,2} &= \frac{\lambda^+ \mathrm{F}_{2,\mathrm{L}} - \lambda^- \mathrm{F}_{2,\mathrm{R}}}{\lambda^+ - \lambda^-} - \frac{\lambda^- \lambda^+}{\lambda^+ - \lambda^-}\left[(hu)_\mathrm{L} - (hu)_\mathrm{R}\right] \end{aligned}\right\} \qquad [C.12]$$

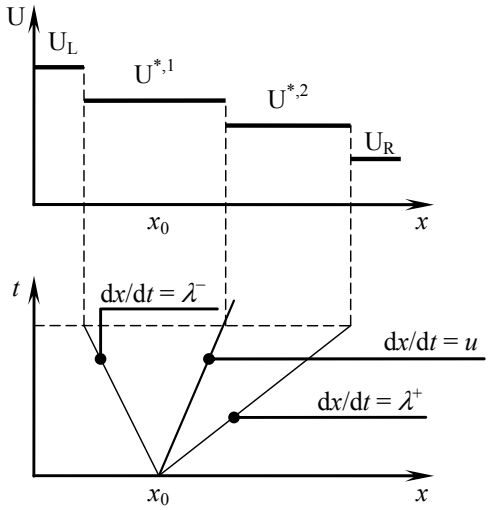

Figure C.2. *Principle of the HLLC solver in the physical space (top) and in the phase space (bottom)*

where $F_1 = hu$ and $F_2 = hu^2 + gh^2/2$. The values of v on the left- and right-hand sides of the contact discontinuity are v_L and v_R respectively. The expression of v in the two subregions is:

$$\left.\begin{aligned} v^{*,1} &= v_L \\ v^{*,2} &= v_R \end{aligned}\right\} \qquad [C.13]$$

which leads to the following expression for the third component of the flux vector:

$$\left.\begin{aligned} F_3^{*,1} &= F_1^{*,1} v_L \\ F_3^{*,2} &= F_1^{*,2} v_R \end{aligned}\right\} \qquad [C.14]$$

The flux F_3 at the interface is equal to $F_3^{*,1}$ if $F_1^{*,1}$ is positive. It is equal to $F_3^{*,2}$ if $F_1^{*,1}$ is negative.

C.2. Roe's solver

C.2.1. *Principle*

Roe's solver [ROE 81] is based on the non-conservation form [2.5] of the governing equations:

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = S'$$

where the matrix A is defined so as to enforce conservation, as explained in section 6.8.2. The coefficients of Roe's matrix are constant coefficients, functions of the left and right states of the Riemann problem. Therefore the solution of the Riemann problem with the non-conservation form [2.5] of the equations is made of m contact discontinuities separating intermediate regions of constant state (Figure C.3).

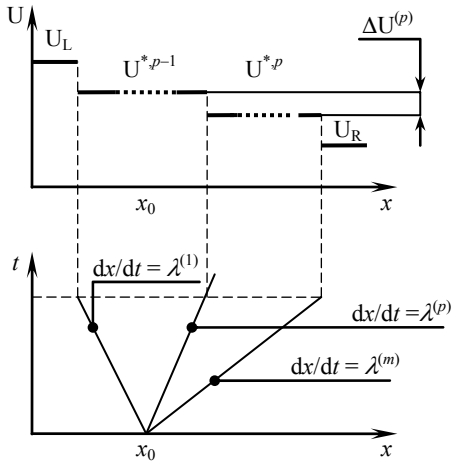

Figure C.3. *Principle of Roe's solver. Definition sketch in the physical space (top) and in the phase space (bottom)*

The eigenvalues and eigenvectors of A being known, the difference between the left and right states can be written in the base of eigenvectors of A as the sum of elementary jumps across the various waves:

$$U_R - U_L = \sum_{p=1}^{m} \Delta U^{(p)} \qquad [C.15]$$

where the quantity $\Delta U^{(p)}$ is the variation in U across the pth wave. The wave strengths $\alpha^{(p)}$ are introduced as:

$$\Delta U^{(p)} = \alpha^{(p)} K^{(p)} \quad [C.16]$$

where $K^{(p)}$ is the pth eigenvector of A. The pth wave moves at the (known) speed $\lambda^{(p)}$, which allows the jump in U to be related to the jump in F across the wave via the Rankin-Hugoniot condition:

$$\Delta F^{(p)} = \lambda^{(p)} \Delta U^{(p)} \quad [C.17]$$

Substituting equation [C.16] into equation [C.17] yields:

$$\Delta F^{(p)} = \lambda^{(p)} \alpha^{(p)} K^{(p)} \quad [C.18]$$

Applying equation [C.18] across the waves with negative speeds leads to the following relationship between the flux F_L and the flux F_0 at the location x_0 of the initial discontinuity:

$$F_0 - F_L = \sum_{\lambda^{(p)}<0} \Delta F^{(p)} = \sum_{\lambda^{(p)}<0} \lambda^{(p)} \alpha^{(p)} K^{(p)} \quad [C.19]$$

Applying equation [C.18] across the waves with positive speeds leads to the following relationship between F_R and F_0:

$$F_R - F_0 = \sum_{\lambda^{(p)}>0} \Delta F^{(p)} = \sum_{\lambda^{(p)}>0} \lambda^{(p)} \alpha^{(p)} K^{(p)} \quad [C.20]$$

Subtracting equation [C.19] from equation [C.20] yields:

$$F_0 = \frac{F_L + F_R}{2} + \frac{1}{2} \sum_{\lambda^{(p)}<0} \lambda^{(p)} \alpha^{(p)} K^{(p)} - \frac{1}{2} \sum_{\lambda^{(p)}>0} \lambda^{(p)} \alpha^{(p)} K^{(p)} \quad [C.21]$$

Equation [C.21] can be rewritten in condensed form as:

$$F_0 = \frac{F_L + F_R}{2} - \frac{1}{2} \sum_{p=1}^{m} \left| \lambda^{(p)} \right| \alpha^{(p)} K^{(p)} \quad [C.22]$$

The original algorithm for Roe's solver is the following:

1) Compute Roe's matrix, its eigenvalues and its eigenvectors.

2) Compute the wave strengths $\alpha^{(p)}$ as the coordinates of ΔU in the base of eigenvectors of A.

3) Apply equation [C.19].

C.2.2. *Algorithmic simplification*

The algorithm provided in section C.2.1 requires the computation of the eigenvectors of the matrix $\tilde{A}$ and the subsequent computation of the wave strengths. This can be avoided by noticing that equations [C.15–16] can be rewritten as:

$$\Delta U = K\alpha \qquad [C.23]$$

where K and a are respectively the matrix of eigenvectors of $\tilde{A}$ and the vector formed by the wave strengths $\alpha^{(p)}$. Conversely, equation [C.22] can be written as:

$$F_0 = \frac{F_L + F_R}{2} - \frac{1}{2} K|\Lambda|\alpha \qquad [C.24]$$

where $|\Lambda|$ is the diagonal matrix formed by the absolute values of the eigenvalues of $\tilde{A}$. Equation [C.24] is rewritten as:

$$F_0 = \frac{F_L + F_R}{2} - \frac{1}{2} K|\Lambda|K^{-1}K\alpha = \frac{F_L + F_R}{2} - \frac{1}{2} K|\Lambda|K^{-1}\,\Delta U \qquad [C.25]$$

Introducing the matrix $|\tilde{A}| = K\,|\Lambda|\,K^{-1}$, equation [C.25] becomes:

$$F_0 = \frac{F_L + F_R}{2} + \frac{1}{2}\left|\tilde{A}\right|(U_L - U_R) \qquad [C.26]$$

This flux formula is to be compared with equation [C.4] for the HLL solver. As the HLL solver, Roe's flux may be seen as the sum of a centered flux and a diffusive flux (artificial viscosity) that contributes to stabilize the solution.

As mentioned in [EIN 88], Roe's diffusion flux is smaller than the diffusion flux in the HLL solver. This yields problems at sonic/critical points because the numerical diffusion is not strong enough at such points. The treatment of such points is dealt with in the next section.

C.2.3. *Entropy violation and fixes*

Roe's solver is well-known to yield local violations of the entropy condition in the neighborhood of sonic/critical points. This results in the appearance of "rarefaction shocks" that violate the entropy condition. Various methods have been introduced to eliminate the problem.

1) An entropy fix described in [TOR 97] consists of checking the presence of a sonic/critical point between the left (or right) state of the Riemann problem and the intermediate region of constant state. If a sonic/critical point is detected between the left state U_L and the intermediate region of constant state U_*, the solver is applied to the Riemann problem (U_L, U_*). The new value of the flux is used as F_0. Conversely, if a sonic point is detected between U_* and U_R, the solver is applied to the problem (U_*, U_R). The new value of the flux computed from this Riemann problem is used as U_0.

2) Another option consists of modifying slightly the calculation of the matrix $|\tilde{A}|$. Indeed the entropy violation at sonic/critical points is due to the fact that one of the eigenvalues $\tilde{\lambda}^{(p)}$ of Roe's matrix is close to zero. Consequently, the stabilizing, diffusive flux is not large enough, which triggers a local increase in the gradient of U. A simple solution consists of imposing that the absolute values of the eigenvalues of $|\Lambda|$ should not be smaller than a given threshold ε:

$$|\Lambda| = \begin{bmatrix} \ddots & & 0 \\ & \max\left(\left|\tilde{\lambda}^{(p)}\right|, \varepsilon\right) & \\ 0 & & \ddots \end{bmatrix} \qquad [C.27]$$

C.2.4. *Application example: the shallow water equations*

Roe's solver is applied to the one-dimensional shallow water equations. These equations can be written in the form [2.2], with U, F, S and K defined as:

$$U = \begin{bmatrix} h \\ hu \end{bmatrix},\ F = \begin{bmatrix} hu \\ hu^2 + gh^2/2 \end{bmatrix},\ S = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ K = \begin{bmatrix} 1 & 1 \\ u-c & u+c \end{bmatrix} \qquad [C.28]$$

The matrix $|\Lambda|$ is given by:

$$|\Lambda| = \begin{bmatrix} |u-c| & 0 \\ 0 & |u+c| \end{bmatrix} \qquad [C.29]$$

which leads to the following expression for $\left|\tilde{\mathrm{A}}\right|$:

$$\mathrm{K}\left|\Lambda\right|\mathrm{K}^{-1} = \frac{1}{2}\begin{bmatrix} (u+c)\left|\mathrm{Fr}-1\right| + (c-u)\left|\mathrm{Fr}+1\right| & \left|\mathrm{Fr}+1\right| - \left|\mathrm{Fr}-1\right| \\ (c^2-u^2)\left|\mathrm{Fr}+1\right| & (c-u)\left|\mathrm{Fr}-1\right| \\ -(c^2-u^2)\left|\mathrm{Fr}-1\right| & +(u+c)\left|\mathrm{Fr}+1\right| \end{bmatrix} \qquad [C.30]$$

If the entropy fix 2) described in section C.2.3 is to be used, $|\mathrm{Fr} \pm 1|$ must be replaced with $\max(|\mathrm{Fr} \pm 1|, \varepsilon)$ in equation [C.30].

C.3. The Lax-Friedrichs solver

The Lax-Friedrichs solver [LAX 54] is used in a number of recently proposed Discontinuous Galerkin (DG) techniques (see Chapter 8 and more specifically section 8.4). It combines solution robustness with the simplicity of the formulation. The flux F_0 at the location x_0 of the initial discontinuity is given by:

$$\mathrm{F}_0 = \frac{\mathrm{F_L} + \mathrm{F_R}}{2} + \frac{1}{2}\frac{\Delta x}{\Delta t}(\mathrm{U_L} - \mathrm{U_R}) \qquad [C.31]$$

This formula is equal to the sum of a centered flux (that leads to unconditionally unstable solutions when used with first-order explicit schemes) and a diffusive flux, or artificial viscosity term. The diffusion coefficient $\Delta x/(2\,\Delta t)$ is the maximum possible value that preserves solution stability.

In [COC 98], a so-called "local" Lax-Friedrichs method, also known as the Rusanov flux, is used:

$$\left.\begin{aligned} \mathrm{F}_0 &= \frac{\mathrm{F_L} + \mathrm{F_R}}{2} + \frac{\lambda_{\max}}{2}(\mathrm{U_L} - \mathrm{U_R}) \\ \lambda_{\max} &= \max_p\left(\left|\lambda_{\mathrm{L}}^{(p)}\right|, \left|\lambda_{\mathrm{R}}^{(p)}\right| \right) \end{aligned}\right\} \qquad [C.32]$$

C.4. Approximate-state solvers

C.4.1. *Principle*

Approximate-state solvers use a property of hyperbolic systems of conservation laws so that the Riemann invariants (or generalized Riemann invariants) provide an approximation of the Rankin-Hugoniot conditions across shocks [LAX 57]. This means that the approximate solution should be a satisfactory approximation of the exact solution of the Riemann problem even though the nature of the waves present in the Riemann solution has been wrongly guessed.

This property is used by approximate-state solvers to make an *a priori* guess on the nature of the waves in the solution, which influences directly the nature of the system of equations to be solved. The assumption on the nature of the waves is not revised, even if the calculation of the variables in the intermediate zone of constant state shows that the initial guess was wrong. Although formally invalid, the method is shown to give good results when applied to a number of hyperbolic systems. There are two main families of approximate-state Riemann solvers:

– shock wave-based solvers, that use the assumption that the solution is made of shocks across which the Rankin-Hugoniot conditions are applied;

– rarefaction wave-based solvers, where the solution is assumed to be made of rarefaction waves across which the Riemann invariants or generalized Riemann invariants are applied.

C.4.2. *Shock-based solvers*

The earliest type of approximate state Riemann solvers are two-shock solvers applied to the Euler equations [COL 82]. Assuming that the left and right states are separated from the intermediate region of constant state by two shocks, applying the Rankin-Hugoniot relationships across the shocks gives:

$$\left.\begin{aligned} \mathrm{F_L} - \mathrm{F^*} &= (\mathrm{U_L} - \mathrm{U^*})c_{s,\mathrm{L}} \\ \mathrm{F_R} - \mathrm{F^*} &= (\mathrm{U_R} - \mathrm{U^*})c_{s,\mathrm{R}} \end{aligned}\right\} \qquad [\mathrm{C.33}]$$

where $c_{s,\mathrm{L}}$ and $c_{s,\mathrm{R}}$ are respectively the propagation speeds of the leftward and rightward shocks. In contrast with the HLL/HLLC solver or Roe's solver, these propagation speeds are not known a priori. For a 2×2 system of equations, system [C.33] leads to 4 equations, with the two components of U* and the shock speeds $c_{s,\mathrm{L}}$ and $c_{s,\mathrm{R}}$ as unknowns.

System [C.33] is nonlinear. Its solution usually requires an iterative procedure. Dukowicz [DUK 85] introduced a simplification for a number of hyperbolic systems of conservation laws (including the Euler equations) whereby the jump relationships are approximated with second-degree polynomials, for which an analytical solution is available.

C.4.3. *Rarefaction wave-based solvers*

C.4.3.1. *Principle*

Two rarefaction wave-based Riemann solvers have been applied successfully to various types of hyperbolic systems of conservation laws such as the water hammer equations and 2×2 simplified systems describing two-phase flows in pipes [GUI 00, GUI 01a, GUI 01b], as well as the two-dimensional shallow water equations [GUI 03a, LHO 07] and the two-dimensional shallow water equations with porosity [FIN 10] for the large scale simulation of urban floods. These solvers use the assumption that the solution is made of rarefaction waves. Different options have been explored in the literature:

1) The Riemann invariants applied across the waves are written as functions of the conserved variable U. As shown in section 2.2.1, using the Riemann invariants allows the solution U to be determined uniquely in the intermediate region(s) of constant state. Knowing the solution U at all points allows the flux F_0 at the abscissa x_0 of the initial discontinuity to be computed. This approach can be used for pipe transients in pipes with uniform geometry and in the computation of the shallow water equations on flat bottoms [GUI 00, GUI 01a, GUI 01b, GUI 03a]. The main drawback of this method is that balancing of the source terms in the momentum equations is not straightforward.

2) A second option consists of expressing the Riemann invariants as functions of the flux F. The source terms are included in the characteristic form of the equations, which allows their influence to be taken into account in the determination of the flux F_0. The broad lines of the approach is given hereafter. The method is presented briefly hereafter. More details on the practical application of the solver to the water hammer, shallow water equations and shallow water equations with porosity can be found in [LHO 07, FIN 10].

C.4.3.2. *Computation of the intermediate state*

The conservation form [2.2] is recalled:

$$\frac{\partial \mathrm{U}}{\partial t} + \frac{\partial \mathrm{F}}{\partial x} = \mathrm{S}$$

The purpose is to transform equation [2.2] into an equation in characteristic form where the Riemann invariants are expressed as functions of the flux F. This is done by multiplying equation [2.2] with the Jacobian matrix $\mathrm{A} = \partial \mathrm{F} / \partial \mathrm{U}$:

$$\mathrm{A}\frac{\partial \mathrm{U}}{\partial t} + \mathrm{A}\frac{\partial \mathrm{F}}{\partial x} = \mathrm{AS} \qquad [\mathrm{C.34}]$$

Noting that A dU = dF, equation [C.34] is rewritten as:

$$\frac{\partial \mathrm{F}}{\partial t} + \mathrm{A}\frac{\partial \mathrm{F}}{\partial x} = \mathrm{AS} \qquad [\mathrm{C.35}]$$

The Riemann invariants appear by left-multiplying equation [C.35] by the inverse of the matrix K (see section 2.1.3):

$$\frac{\partial \mathrm{W}}{\partial t} + \Lambda\frac{\partial \mathrm{W}}{\partial x} = \mathrm{S}' \qquad [\mathrm{C.36}]$$

where Λ is the diagonal matrix formed by the eigenvalues of A, and W and S' are defined as:

$$\left.\begin{aligned} \mathrm{dW} &= \mathrm{K}^{-1}\,\mathrm{dF} \\ S' &= \mathrm{K}^{-1}\mathrm{AS} \end{aligned}\right\} \qquad [\mathrm{C.37}]$$

Equation [C.36] can be rewritten as:

$$\frac{\mathrm{d}W_p}{\mathrm{d}t} = S'_p \qquad \text{for}\ \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda^{(p)}, \qquad p = 1,\ldots,m \qquad [\mathrm{C.38}]$$

Solving the $m \times m$ system [C.38] for the components F_p yields directly the expression of F at the abscissa x_0 of the initial discontinuity.

C.4.3.3. *Wave pattern update and flux calculation*

Knowing the flow variables in the intermediate region of constant state allows the wave pattern to be updated [LHO 07]. This is done in two steps:

(1) The speeds λ^- and λ^+ of the waves are computed for the left, intermediate and right state.

(2) The wave pattern is identified as follows:

(2.1) if $\lambda_{\mathrm{L}}^{-} < 0 < \lambda_{*}^{-}$, the point x_0 is a critical/sonic point in a rarefaction wave. A critical point formula is used for the flux;

(2.2) if $\lambda_{\mathrm{L}}^{-} \le \lambda_{*}^{-} \le 0 \le \lambda_{*}^{+}$, the point x_0 is located within the intermediate region of constant state computed as in section C.4.3.2;

(2.3) if $\lambda_{*}^{+} < 0 < \lambda_{\mathrm{R}}^{+}$, the point x_0 is a critical/sonic point in a rarefaction wave. A critical point formula is used for the flux;

(2.4) if $\lambda_{\mathrm{L}}^{-} > \lambda_{*}^{-}$, the left-hand wave is a shock. The shock speed c_s is computed using the jump relationships between the left and intermediate states. $\mathrm{F}_0 = \mathrm{F}_{\mathrm{L}}$ if $c_s > 0$, $\mathrm{F}_0 = \mathrm{F}_{*}$ otherwise;

(2.5) if $\lambda_{*}^{+} > \lambda_{\mathrm{R}}^{+}$, the right-hand wave is a shock. The shock speed c_s is computed using the jump relationships between the intermediate and right states. $\mathrm{F}_0 = \mathrm{F}_{\mathrm{R}}$ if $c_s < 0$, $\mathrm{F}_0 = \mathrm{F}_{*}$ otherwise.

Practical applications to the water hammer and shallow water equations indicate that the flux formulae given by such solvers bear similarities with those given by the HLL solver, with enhanced balancing properties (see Chapter 9 for balancing issues of schemes in the presence of source terms).