Appendix A

Linear Algebra

A.1. Definitions

A vector v is an ordered set of m numbers $v_1, \ldots, v_m$, called the components of the vector. The components of the vector are arranged in a single column. The following notation is used:

$$\mathrm{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_i & \cdots & v_m \end{bmatrix}^T = [v_i] \qquad \text{[A.1]}$$

where v_i is the ith component of v and T is the transposition operator.

An $m \times n$ matrix A is formed by a set of numbers $a_{i,j}$ arranged in m rows and n columns. i and j are respectively the indices for the row and the column of the matrix. The following notation is used:

$$\mathrm{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & & \vdots & & \vdots \\ a_{i,1} & \cdots & a_{i,j} & \cdots & a_{i,n} \\ \vdots & & \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,j} & \cdots & a_{m,n} \end{bmatrix} = [a_{i,j}] \qquad \text{[A.2]}$$

An $m \times n$ matrix may be viewed as a set of n vectors of size m arranged in a single line. The matrix A in equation [A.3] may be defined as:

$$\mathrm{A} = \left[\mathrm{a}^{(1)} \cdots \mathrm{a}^{(j)} \cdots \mathrm{a}^{(n)}\right] \qquad [A.3]$$

where the vectors $\mathrm{a}^{(j)}, j = 1, \ldots, n$, are defined as:

$$\mathrm{a}^{(j)} = [a_{1,j} \cdots a_{i,j} \cdots a_{m,j}]^T \qquad [A.4]$$

Note that a vector is a single-columned matrix.

Also note the following, particular cases:

– A square matrix has the same number of rows and columns, $m = n$.

– A symmetric matrix is a matrix that is left invariant by transposition (a symmetric matrix is necessarily a square matrix):

$$a_{i,j} = a_{j,i} \qquad \forall \begin{cases} i = 1, \ldots, m \\ j = 1, \ldots, m \end{cases} \qquad [A.5]$$

– The identity matrix is a symmetric matrix, the elements of which are all zero, except the diagonal terms that are equal to one:

$$\left.\begin{aligned} &\mathrm{I} = [\delta_{i,j}] \\ &\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \end{aligned}\right\} \qquad [A.6]$$

where $\delta_{i,j}$ is known as Kronecker's operator.

A.2. Operations on matrices and vectors

A.2.1. *Addition*

Let $\mathrm{A} = [a_{i,j}]$ and $\mathrm{B} = [b_{i,j}]$ be two $m \times n$ matrices. Adding A and B yields the matrix $\mathrm{C} = [c_{i,j}]$ defined as follows:

$$\left.\begin{aligned} &\mathrm{C} = \mathrm{A} + \mathrm{B} \\ &c_{i,j} = a_{i,j} + b_{i,j} \qquad \forall (i = 1, \ldots, m;\ j = 1, \ldots, n) \end{aligned}\right\} \qquad [A.7]$$

The sum of two vectors is defined exactly in the same way:

$$\left.\begin{aligned} \mathrm{w} &= \mathrm{u} + \mathrm{v} \\ w_i &= u_i + v_i \qquad \forall\ i = 1,\dots,m \end{aligned}\right\} \qquad [A.8]$$

Note that matrices or vectors may be added only if they have the same size.

A.2.2. *Multiplication by a scalar*

Let $\mathrm{A} = [a_{i,j}]$ and β be a matrix and a scalar respectively. Multiplying A by β yields the matrix $\mathrm{B} = [b_{i,j}]$ defined as:

$$\left.\begin{aligned} \mathrm{B} &= \beta \mathrm{A} \\ b_{i,j} &= \beta a_{i,j} \qquad \forall (i = 1,\dots,m;\ j = 1,\dots,n) \end{aligned}\right\} \qquad [A.9]$$

The product of a scalar and a vector is defined in the same way:

$$\left.\begin{aligned} \mathrm{v} &= \beta \mathrm{u} \\ v_i &= \beta u_i \qquad \forall\ i = 1,\dots,m \end{aligned}\right\} \qquad [A.10]$$

A.2.3. *Matrix product*

Let $\mathrm{A} = [a_{i,j}]$ be an $m \times l$ matrix and $\mathrm{B} = [b_{i,j}]$ be an $l \times n$ matrix. The product of A and B is an $m \times n$ matrix C defined as:

$$\left.\begin{aligned} \mathrm{C} &= \mathrm{AB} \\ c_{i,j} &= \sum_{k=1}^{l} a_{i,k} b_{k,j} \qquad \forall (i = 1,\dots,m;\ j = 1,\dots,n) \end{aligned}\right\} \qquad [A.11]$$

A vector being nothing else than a matrix with only one column, the product between the matrix A and the vector u is defined as:

$$\left.\begin{aligned} \mathrm{v} &= \mathrm{Au} \\ v_i &= \sum_{k=1}^{n} a_{i,k} u_k \qquad \forall\ i = 1,\dots,m \end{aligned}\right\} \qquad [A.12]$$

NOTE.– In contrast with scalar multiplication, the matrix product is not commutative. The product AB is not equal to the product BA in the general case.

A.2.4. *Determinant of a matrix*

Let A be a square matrix of size $m \times m$. The determinant of A, denoted by Det(A), or |A|, is defined using the following recurrence relationship:

$$\begin{aligned} \left|\mathrm{A}\right| &= \sum_{i=1}^{m} (-1)^{i+q} a_{i,q} \left|\mathrm{A}_{i,q}\right| \qquad \forall q = 1,\dots,m \\ &= \sum_{j=1}^{m} (-1)^{j+p} a_{p,j} \left|\mathrm{A}_{p,j}\right| \qquad \forall p = 1,\dots,m \end{aligned} \quad \text{[A.13]}$$

where the matrix $\mathrm{A}_{i,q}$ is the $(m-1)\times(m-1)$ square matrix obtained from A by removing the row q and the column i. The final result is the same, regardless of the row q and the column i chosen in the sum [A.13].

The determinant verifies the following properties:

$$\left.\begin{aligned} \left|\mathrm{AB}\right| &= \left|\mathrm{BA}\right| = \left|\mathrm{A}\right|\left|\mathrm{B}\right| \\ \left|\mathrm{A}^T\right| &= \left|\mathrm{A}\right| \\ \left|\mathrm{I}\right| &= 1 \end{aligned}\right\} \quad \text{[A.14]}$$

A.2.5. *Inverse of a matrix*

Let A be an $m \times m$ square matrix. The inverse A^{-1} of A is an $m \times m$ matrix defined as:

$$\mathrm{A}^{-1}\mathrm{A} = \mathrm{A}\mathrm{A}^{-1} = \mathrm{I} \quad \text{[A.15]}$$

The first relationship [A.14] indicates that a matrix has an inverse only if its determinant is non-zero. The third relationship implies that the determinant of the inverse of A is the inverse of the determinant of A.

A.3. Differential operations using matrices and vectors

A.3.1. *Differentiation*

Let $A = [a_{i,j}]$ be an $m \times n$ matrix. A is differentiated with respect to a given parameter or variable t by differentiating all its components individually:

$$\frac{\partial A}{\partial t} = \left[\frac{\partial a_{i,j}}{\partial t}\right] \quad [A.16]$$

This definition also applies to the particular case of a vector that can be seen as a single-columned matrix:

$$\frac{\partial u}{\partial t} = \left[\frac{\partial u_i}{\partial t}\right] \quad [A.17]$$

A.3.2. *Jacobian matrix*

Let $u = [u_i]$ be a vector of size m and $v = [v_i]$ be a vector of size n. The Jacobian matrix A of u with respect to v is an $m \times n$ matrix defined as:

$$\left.\begin{aligned} A &= \frac{\partial u}{\partial v} \\ a_{i,j} &= \frac{\partial u_i}{\partial v_j} \qquad \forall (i = 1,\ldots,m; j = 1,\ldots,n) \end{aligned}\right\} \quad [A.18]$$

A.4. Eigenvalues, eigenvectors

A.4.1. *Definitions*

The scalar λ is an eigenvalue of the matrix A if there is a non-zero vector v, called and eigenvector, such that:

$$Av = \lambda v \quad [A.19]$$

The characteristic polynomial of A is defined as:

$$P(\lambda) = \left|A - \lambda I\right| \quad [A.20]$$

The eigenvalues of A are the roots of the characteristic polynomial:

$$P(\lambda) = 0 \qquad [A.21]$$

The eigenvector v associated with a given eigenvalue λ is obtained by substituting the (known) value of λ into equation [A.19]. A linear algebraic system is obtained. Since at least one of the components of u is non-zero, it can be set to any arbitrary value, e.g. one, that serves as a basis in the computation of the remaining components of v.

A.4.2. *Example*

Consider the matrix A obtained for the Saint Venant equations (see section 2.5.3.1):

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{bmatrix}$$

The eigenvalues of A verify equations [A.20–21]:

$$\begin{vmatrix} -\lambda & 1 \\ c^2 - u^2 & 2u - \lambda \end{vmatrix} = 0 \qquad [A.22]$$

which leads to:

$$-(2u - \lambda)\lambda - (c^2 - u^2) = 0 \qquad [A.23]$$

Equation [A.23] can be rewritten as:

$$(\lambda - u)^2 = c^2 \qquad [A.24]$$

which leads to the following two solutions:

$$\left.\begin{aligned} \lambda^{(1)} &= u - c \\ \lambda^{(2)} &= u + c \end{aligned}\right\} \qquad [A.25]$$

The eigenvector $K^{(1)}$ associated with the first eigenvalue $\lambda^{(1)}$ verifies:

$$\begin{bmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{bmatrix} \begin{bmatrix} K_1^{(1)} \\ K_2^{(1)} \end{bmatrix} = (u-c) \begin{bmatrix} K_1^{(1)} \\ K_2^{(1)} \end{bmatrix} \quad [A.26]$$

that is:

$$\left. \begin{aligned} K_2^{(1)} &= (u-c)K_1^{(1)} \\ (c^2 - u^2)K_1^{(1)} + 2uK_2^{(1)} &= (u-c)K_2^{(1)} \end{aligned} \right\} \quad [A.27]$$

These two conditions can easily be checked to be equivalent. The first eigenvector is therefore:

$$K^{(1)} = \begin{bmatrix} K_1^{(1)} \\ (u-c)K_1^{(1)} \end{bmatrix} \quad [A.28]$$

The vector $K^{(1)}$ verifies equation [A.19] for any non-zero value of $K_1^{(1)}$. Using the obvious choice $K_1^{(1)} = 1$ leads to:

$$K^{(1)} = \begin{bmatrix} 1 \\ u-c \end{bmatrix} \quad [A.29]$$

It is easy to check that the second eigenvector is given by:

$$K^{(2)} = \begin{bmatrix} 1 \\ u+c \end{bmatrix} \quad [A.30]$$