APPENDICES

Appendix 1

Notion of Probability

Whether it is experiments carried out in reverberation chambers or theoretical simulations of these tools, the data such as the electric field or the power collected in the room behave like random variables. The developments related in Chapter 3 frequently use the main results of the theory of probabilities. Some notions about this theory will be recalled in this appendix [BAS 67].

A1.1. The random variable concept

A random variable is represented by a number, whose quite unpredictable behavior can be attached to a *probability distribution*. In the context of the book, the random variables will follow a continuous description. These variables, often denoted by a small letter, will belong to the set of real numbers. However, their domain can be bounded and restricted in the positive real numbers or extended to all real numbers, as illustrated below:

$$x \in \left[X_{\text{mini}} \quad X_{\text{maxi}}\right] \quad \text{or} \quad x \in \left[0 \quad +\infty\right[\quad \text{or} \quad x \in \left]-\infty \quad +\infty\right[\qquad \text{[A1.1]}$$

A1.2. Probability concept from intuition

Let us consider an experiment, whose aim is to take a variable x staggered on N discrete values, which are defined by the following conventions:

$$x \in (X_1, X_2 \ldots X_i \ldots X_N) \qquad \text{[A1.2]}$$

During t_N experiments, the X_i value appears t_i times. An estimate of the P_i probability, of instance X_i, will thus be given by the t_i/t_N ratio, i.e.:

$$\hat{\mathrm{P}}_i = \frac{t_i}{t_N} \quad \text{where} \quad \hat{\mathrm{P}}_i = \mathrm{Prob}\left[x = X_i\right] \qquad \text{[A1.3]}$$

Intuitively, we realize that by indefinitely increasing the t_N number of experiments, the estimate of P_i must converge on the rigorous P_i probability.

Knowing that the sum of the t_i tests corresponds to all t_N number of experiments, the sum of the N probabilities is necessarily equal to the unit, hence:

$$\lim_{t_N \to \infty}\left[\hat{\mathrm{P}}_i\right] = \mathrm{P}_i \qquad \text{[A1.4]}$$

$$\sum_{i=1}^{N} t_i = t_N \quad \to \quad \sum_{i=1}^{N} \hat{\mathrm{P}}_i = 1 \qquad \text{[A1.5]}$$

This expression is called the *normalized condition* of the probability.

A1.3. Probability density function (pdf)

In case of random variables with continuous spread, the definition of the probability attached to the x variable will be transposed to the differential element $d\mathrm{p}$. Regarding the probability for a quantity α belonging to the $[x\ x+dx]$ interval, i.e.:

$$d\mathrm{p} = \mathrm{Prob}\left[x < \alpha < x + dx\right] \qquad \text{[A1.6]}$$

This relation and the properties of the differential calculation result in the function $\mathrm{p}(x)$, called the *probability density function* noted with the abbreviation pdf:

$$d\mathrm{p} = \mathrm{p}(x)\,dx \qquad \text{[A1.7]}$$

The normalized condition [A1.5] can be extended to the pdf, by solving the integral below:

$$\int_{-\infty}^{+\infty} \mathrm{p}(x)\,dx = 1 \qquad \text{[A1.8]}$$

If the domain to which the x variable belongs is bounded, the calculation of the integral will be bounded as well.

A1.4. Computation of moments

From the probability concept introduced in section A1.2, we find that an estimate of the mean value of the variable x, taking the notation $< x >$, can be expressed as follows:

$$< x >= \frac{\sum_{i=1}^{N} X_i\, t_i}{t_N} = \sum_{i=1}^{N} X_i \frac{t_i}{t_N} \qquad [A1.9]$$

For an infinite number of experiments, the estimate of the mean value tends towards the rigorous mean value established by expression [A1.10], where P_i represents the rigorous probability:

$$\overline{x} = E[x] = \sum_{i=1}^{N} X_i P_i \qquad [A1.10]$$

This calculation is then called the *expected value* or moment computation of the x variable.

A1.4.1. *Computation of the moment of the x random variable*

Calculation of the moment formulated in equation [A1.10] can be extended to the random continuous variables, by forming integral [A1.11]:

$$m_{x_1} = E[x] = \int_{-\infty}^{+\infty} x\, p(x)\, dx \qquad [A1.11]$$

The moment of the x variable merged with its rigorous mean value is denoted here by the m_x notation. The index 1 recalls that it concerns the x variable calculation.

A1.4.2. *Computation of the moment of the x squared random variable*

Calculation of the moment can be extended to the x^2 variable, the so-called x squared random variable.

$$m_{x_2} = \overline{x^2} = \mathrm{E}\left[x^2\right] = \int_{-\infty}^{+\infty} x^2 \, \mathrm{p}(x)\, dx \qquad \text{[A1.12]}$$

The result of this integral is then strictly similar to the mean value of the square of x, sometimes denoted by the x^2 notation topped by a stroke.

A1.5. Centered and normalized variables

For the convenience of some calculations, we carry out the transformation of the random variables, then presented under centered or normalized variables answering to the following definitions.

A1.5.1. *Centered variables*

The centered variable is designated here by the x_0 notation. It represents the random function of the x variable at both side of the average amplitude. x_0 takes the definition:

$$x_0 = x - m_{x_1} \quad \rightarrow \quad \mathrm{E}\left[x_0\right] = 0 \qquad \text{[A1.13]}$$

Calculation of the moment of x_0 necessarily gives zero.

A1.5.2. *Normalized variables*

In order to avoid the physical dimension of the random variables, we use normalized variables given by the ratio of x over the mean value of x given by the computation of the moment of x as found in equation [A1.11]. The normalized variable is designed by the notation x with a bottom index r.

$$x_r = \frac{x}{m_{x_1}} \qquad \text{[A1.14]}$$

A1.6. Computation of the variance and standard deviation

The variance and the standard deviation of an x variable enable us to put a figure on the amplitude of the random behavior of x around its mean value.

A1.6.1. *Definition of the variance*

The variance is defined by the calculation of the moment of the square amplitude of the x_0 centered variable brought back to x. The variance takes the notation of the square of the σ_x symbol, i.e.:

$$\sigma_x^2 = \mathrm{E}\left[x_0^2\right] = \mathrm{E}\left[(x - m_{x_1})^2\right] \quad \text{[A1.15]}$$

From the development of this expression, we can show that the variance is the difference between the moment of the square of x and the squared moment of x:

$$\sigma_x^2 = m_{x_2} - \left(m_{x_1}\right)^2 = \mathrm{E}\left[x^2\right] - \left(\mathrm{E}[x]\right)^2 \quad \text{[A1.16]}$$

A1.6.2. *Definition of the standard deviation*

The standard deviation corresponds to the σ_x notation. It represents the square root of the variance:

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{E\left[(x - m_{x_1})^2\right]} \quad \text{[A1.17]}$$

Compared to the variance, the standard deviation leads to a result expressed with a similar physical dimension to the one of the random variable x.

Let us specify that the normalized variable of a centered variable is the ratio of this variable with its standard deviation.

A1.7. Probability distributions

Under some physical and mathematical considerations, we manage to develop probability distributions, to which we can attach functions either expressed in terms of X_i discrete random variables or x continuous random variables. In the second case, they are the pdf, which are defined above with the p(x) notation.

Only the knowledge of the probability distribution enables us to calculate the moments, as well as the variance and the standard deviation. As an example, we mention two distributions, which are frequently used in this book: the *uniform probability distribution* and the *normal distribution*, also called the Gaussian distribution.

A1.7.1. *Uniform probability distribution*

Let us consider a random variable x, whose behavior is uniformly distributed in the [0 2π] interval. Under this assumption, the pdf attached to x takes a p_0 invariant value, i.e.:

$$x \in [0 \quad 2\pi] \quad \to \quad \mathrm{p}(x) = \mathrm{p}_0 \qquad \text{[A1.18]}$$

Knowing that, for the considered interval, the p(x) function must meet the normalized condition [A1.8], we immediately find the value of p_0:

$$\int_0^{2\pi} \mathrm{p}(x)\, dx = 1 \quad \to \quad \mathrm{p}_0 = \frac{1}{2\pi} \qquad \text{[A1.19]}$$

The generation of random numbers using the uniform distribution law is frequently used in order to produce other random numbers, which are developed according to the *Monte Carlo trials*.

A1.7.2. *Normal probability distribution*

The normal distribution aims at random variables, whose amplitude distribution around the mean value is carried out with the ideal random distribution. We will see in Chapter 3, that the conditions of maximum entropy and minimum energy of the x variable enable us to find the analytical form of the p(x) pdf.

If this is a centered normalized variable x_r which is defined according to the left side of equation [A1.20], p(x) takes the expression:

$$x_r = \frac{x - m_{x_1}}{\sigma_x} \quad \rightarrow \quad \mathrm{p}(x_r) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_r^2} \qquad [A1.20]$$

Section A1.12 includes the summary of the main probability distributions found in this book.

A1.8. The cumulative distribution function (cdf)

The cumulative distribution function, designated with the cdf abbreviation, is denoted by the $\mathrm{F}(X_0)$ function and represents the probability of locating the x random variable below an arbitrary X_0 value. Coming back to section A1.3 shows that the distribution function is the integral of the $\mathrm{p}(x)$ pdf, which is established between the lower bound of the variation domain of x and X_0. If the x variable belongs to the positive real numbers, $\mathrm{F}(X_0)$ takes the expression:

$$x \in \mathrm{R}^+ \quad \rightarrow \quad \mathrm{F}(X_0) = \mathrm{Prob}\left[x \leq X_0\right] = \int_0^{X_0} \mathrm{p}(x)\, dx \qquad [A1.21]$$

We will see later on that the cdf is efficient for the construction of histograms established on experimental data.

A1.9. The ergodism notion

We will carry out the intuitive definition of the ergodic property, which is then applied to the calculation of the autocorrelation function.

A1.9.1. *Intuitive definition of the ergodic property*

Let us consider a $x(t)$ function depending on a non-random variable t, which can be the time or a spatial location brought back to any coordinate system. If the variations of $x(t)$ as a function of t behave in such an unpredictable way, this is a *random function*. During a process of data collection, $x(t)$ can be evaluated at periodic intervals T_e, called sampling periods.

A series of N samples of $x(t)$ then takes the form of a series in which appears the Dirac function $\delta(t)$. This means that $\delta(t)$ vanishes at any point, except for $t = 0$, i.e.:

$$\tilde{x}(t) = \sum_{k=0}^{N-1} x_k \, \delta(t - kT_e) \qquad \text{[A1.22]}$$

The ~ notation shows that it is a sampled function. If we indefinitely increase the size N of the statistical sample thus constituted, the x_k terms found in [A1.22] form a series of random variables, to which we can attach a pdf p(x). If we manage to determine p(x), calculation of the moment of x will give the rigorous mean value of x denoted by the x notation topped with a stroke:

$$\overline{x} = m_{x_1} = \mathrm{E}[x] = \int_{\mathsf{D}} x \, \mathrm{p}(x) \, dx \qquad \text{[A1.23]}$$

Calculation of the integral is extended within the D domain, covered by the x variable.

An alternative way to this previous integral calculation consists of the computation of the mean value of $x(t)$ versus the t variable. If we restrict the domain to the positive real numbers, the x_{mv}average value thus obtained, takes the expression:

$$x_{\mathrm{mv}} = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t) \, dt \qquad \text{[A1.24]}$$

To say that the random variable x satisfies the *ergodic principle*, means that [A1.23] and [A1.24] tend to similar mean values, i.e.:

$$\text{Ergodism} \quad \Rightarrow \quad \overline{x} = x_{\mathrm{mv}} \qquad \text{[A1.25]}$$

The insertion of the sampled function [A1.22] into integral [A1.24], results in the estimate of the mean value of x, i.e.:

$$< x >= \frac{1}{N} \sum_{k=0}^{N-1} x_k \qquad \text{[A1.26]}$$

A1.9.2. ***Use of ergodism to the calculation of the autocorrelation function***

Let us allocate to the random function $x(t)$, two values, x_1 and x_2. One comes from a determination, when we allocate the t_1 value on t, and the other for t_2. The following step of the calculation will be eased by the use of the shift τ mentioned below:

$$x_1 = x(t_1) \quad x_2 = x(t_2) \quad \text{with} \quad t_2 = t_1 + \tau \qquad \text{[A1.27]}$$

When t is the time variable, a negative shift τ means that x_2 corresponds to a value of $x(t)$ occurring before x_1. Conversely, it is an advanced value when τ is positive.

If the statistical properties of x_1 and x_2 remain invariant regardless of t (what we will call in the next section stationary state), x_1 and x_2 are two random variables, to which we can attach a *joint probability density function* which is designated by the $p_{12}(x_1,x_2)$ notation. We can say that the joint pdf feature in an analytical way, the degree of dependence of the random behavior of the x_1 and x_2 variables. If they are independent variables, $p_{12}(x_1,x_2)$ is reduced to the product of the own pdf of x_1 or x_2:

$$\text{Independent variables} \quad \rightarrow \quad p_{12}(x_1, x_2) = p(x_1)\,p(x_2) \qquad \text{[A1.28]}$$

Given the stationary state assumptions of the variables, the pdf appearing in equation [A1.28] must be independent from the t variable.

The autocorrelation function taking the symbol $C_{xx}(\tau)$ is defined by the moment computation of the product x_1x_2:

$$C_{xx}(\tau) = \mathrm{E}\left[x_1x_2\right] = \iint_{\mathrm{D}} x_1x_2\, p_{12}(x_1, x_2)\, dx_1dx_2 \qquad \text{[A1.29]}$$

The value and the shape of the autocorrelation function are directly related to the degree of dependence of the random behavior of the variables x_1, x_2. This means that any usual random variable produces a decreasing $C_{xx}(\tau)$ function, when the shift τ increases.

Except for some specific conditions the computation of the joint pdf cannot be processed analytically. To determine the autocorrelation, it is then more convenient to use the ergodism principle, i.e. adopting the integral [A1.30] brought back to the t variable, i.e.:

$$C_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)\, x(t+\tau)\, dt \qquad \text{[A1.30]}$$

This expression assumes that the domain of the variable t of the random function $x(t)$ is contained in the domain of positive real numbers.

From this relation, we find that the value of the autocorrelation function in $\tau = 0$ is the mean of the x squared variable. Conversely, for the infinitely large shifts $|\tau|$, the x_1 and x_2 variables become necessarily independent. In this case, the combination of [A1.28] and [A1.29] shows that $C_{xx}(\tau)$ is the square of the average value of x.

Inserting a finite size sampled function in equation [A1.30], we obtain the estimate of $C_{xx}(\tau)$ given by the series mentioned below. We then talk about the estimate of $C_{xx}(\tau)$, which is designated by the suitable convention, i.e.:

$$\hat{C}_{xx}(rT_e) = \frac{1}{N} \sum_{-N/2}^{(N-1)/2} x_k\, x_{k+r} \qquad \text{[A1.31]}$$

The estimate of the autocorrelation function is used in Chapter 4 and 8, in order to characterize the efficiency of a mode stirrer.

A1.10. Features of the random stationary variables

The stationary state of a variable mentioned in the previous section assumes that the statistical features of the variable remains unchanged at anytime and anywhere. Let us take a look at the example of a x variable, which is assumed to follow a normal probability distribution introduced in [A1.20]. If it is proven that disjoined samples of the normalized form of this variable remains in agreement with the same normal distribution, we can conclude from this experiment that x_r responds to a stationary state. However, if the standard deviation of x significantly varies from one sample to another, x loses the stationary state.

We will see in Chapters 3 and 4 that only a fine statistical analysis of the behavior of the variables enables us to decide the stationary state criterion. The characterization and calibration of the reverberation chambers are mostly based on the stationary state feature of the data collected during the measurements.

A1.11. The characteristic function

Although the characteristic function is not explicitly used in this book, we can find it in the definition of some probability distributions. The characteristic function is also involved in the central limit theorem.

Let us consider a random real and continuous variable x, which is linked to the p(x) pdf. We introduce the real variable t, in order to form the complex variable z, which is defined as follows:

$$z = e^{jxt} \qquad [A1.32]$$

The characteristic function usually designated by the $\varphi(t)$ notation, corresponds to the moment computation of the complex random z variable, i.e.:

$$\varphi(t) = \mathrm{E}[z] = \int_{-\infty}^{+\infty} z\, \mathrm{p}(x)\, dx \qquad [A1.33]$$

The form of this expression shows that the characteristic function is the Fourier integral of the p(x) pdf:

$$\varphi(t) = \int_{-\infty}^{+\infty} \mathrm{p}(x)\, e^{jxt}\, dx \qquad [A1.34]$$

Consequently, if we manage to determine the characteristic function attached to a random variable x, the calculation of the inverse Fourier integral leads to p(x):

$$\mathrm{p}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(t)\, e^{-jxt}\, dt \qquad [A1.35]$$

According to the demonstration of the central limit theorem as proposed in the book by Papoulis [PAP 91], let us consider a continuous random variable x_S made up of the sum of N variables x. Each of these N variables obey the same p(x) pdf; the new x_S variable is then expressed:

$$x_s = \sum_{i=1}^{N} x_i \qquad \text{[A1.36]}$$

Assuming that the x_i variables are all independent, the $p_s(x_s)$ pdf attached to x_s will be made up of the product:

$$p_s(x_s) = \prod_{i=1}^{N} p(x_i) = [p(x)]^N \qquad \text{[A1.37]}$$

The characteristic $\varphi_s(t)$ function attached to the x_s variable can thus be written:

$$\varphi_s(t) = \int_{-\infty}^{+\infty} p_s(x_s)\, e^{j x_s t} dx_s \qquad \text{[A1.38]}$$

After insertion of equations [A1.36] and [A1.37] within the integral and after the calculation of the inverse Fourier integral of $\varphi_s(t)$, we reach the expression made up of N convolution products of the p(x) functions, hence:

$$p_s(x_s) = p(x_1) * p(x_2) \ldots * p(x_i) \ldots * p(x_N) \qquad \text{[A1.39]}$$

It is then shown when the size N of the sample forming the x_s variable indefinitely increases, the resulting $p_s(x_s)$ pdf tends to the normal distribution [PAP 91]:

$$N \to \infty \quad \Rightarrow \quad p_s(x_s) \to \frac{1}{\sigma_{x_s} \sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x_s - m_{x_s 1})}{\sigma_{x_s}^2}} \qquad \text{[A1.40]}$$

We find in this formula the moment and the variance of x_s, which are presented with the suitable notation conventions. We easily deduce from the properties formulated previously in this appendix, the following expressions:

$$m_{x_s 1} = E[x_s] = N\, m_{x1} \quad \text{where} \quad m_{x1} = E[x] \qquad \text{[A1.41]}$$

$$\sigma_{x_s}^2 = \mathrm{E}\left[(x_s)^2\right] - \left(\mathrm{E}\left[x_s\right]\right)^2 = N\sigma_x^2 \qquad \text{[A1.42]}$$

Thus, the sum of N independent random variables following the same probability distribution, is scattered at both sides of its mean value with an ideal random fashion expressed in terms of the normal distribution.

A1.12. Summary of the main probability distributions

Readers can only find in this summary the probability distributions found in the book.

A1.12.1. *Uniform distribution*

Let x be a continuous variable. Uniform distribution means that the p(x) probability density function is invariant, whatever the value of x, which belongs to its variation domain.

The uniform probability distribution can thus correspond to relation [A1.43] below:

$$\mathrm{p}(x) = \mathrm{p}_0 \qquad \text{[A1.43]}$$

A1.12.2. *Normal distribution*

A continuous x variable follows a normal distribution or a Gaussian distribution, when the p(x) pdf appropriates the following general form:

$$\mathrm{p}(x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-m_{x1})^2}{\sigma_x^2}} \qquad \text{[A1.44]}$$

The variation domain of x is assumed to be infinitely extended. We can find in this relation the moment m_{x1}, as well as the σ_x standard deviation.

If it is a normalized and centered variable x_r, this expression takes the simplified form:

$$\mathrm{p}(x_r) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_r)^2}{2}} \qquad [A1.45]$$

A1.12.3. *Chi-squared distribution*

Let us consider a χ^2 variable made up of the sum of N terms, themselves made up of the square amplitude of the normalized and centered x_i variables, according to all the normal and stationary distributions:

$$\chi^2 = \alpha = \sum_{i=1}^{N} x_i^2 \qquad [A1.46]$$

It can be shown that any value α of the χ^2 variable is characterized by a pdf, which is expressed in the following form:

$$\mathrm{p}(N,\alpha) = \frac{1}{2^{\frac{N}{2}} \Gamma(N/2)} \alpha^{\frac{N}{2}-1} e^{-\frac{\alpha}{2}} \qquad [A1.47]$$

In this formula, the integer N parameter indicates the degree of freedom of the p(N,α) function, which is called a chi-squared distribution. In formula [A1.47], Γ(N/2) requires the calculation of an Euler integral taking the general form:

$$\Gamma(N/2) = \int_0^{+\infty} v^{\frac{N}{2}-1} e^{-v} dv \qquad [A1.48]$$

A1.12.4. *Weibull distribution*

The Weibull distribution aims at the continuous random positive x variable and two parameters, k and λ, which belong to the positive real numbers. The Weibull distribution takes the general expression:

$$\mathrm{p}(x,k,\lambda) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \quad \text{with} \quad x,k,\lambda \in \mathbb{R}^+ \qquad [A1.49]$$

In this formula, k is a shape parameter, and λ is a scale parameter.

A1.12.5. *Exponential distribution*

The exponential distribution aims at random positive variables x. The exponential distribution tends to the Weibull distribution, when we allocate the value 1 to the shape parameter k, i.e.:

$$\mathrm{p}(x,\lambda) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \qquad [A1.50]$$

In this case, the λ scale parameter is reduced to the moment of the x variable:

$$\lambda = m_{x1} = \mathrm{E}\left[x\right] \qquad [A1.51]$$

The exponential distribution is also a specific case of the chi-square distribution.

A1.12.6. *Rayleigh distribution*

The Rayleigh distribution aims at random continuous variables x. This is the Weibull distribution with shape parameter $k = 2$:

$$\mathrm{p}(x,\lambda) = 2\frac{x}{\lambda^2} e^{-\frac{x^2}{\lambda^2}} \qquad [A1.52]$$

The λ scale parameter is related to the variance by the equation:

$$\lambda^2 = 2\sigma_x^2 \qquad [A1.53]$$

As shown in Chapter 3, the Rayleigh distribution is also a particular case of the chi-square distribution with two degrees of freedom.

With the prospect of comparing random data collected during experiments with known probabilities distributions, the contribution of the two independent parameters found in the Weilbull distribution, simplifies the adjustment tests.

A1.13. Tables of numerical values of the normal distribution integrals

Knowing that the pdf expressing the normal distributions [A1.44] and [A1.45] includes the Gaussian function, the computation of the cdf can only be carried out by using a numerical calculation. It is to this purpose that the data of Table A1.1 correspond to the calculation of the integral of the normal distribution [A1.54] when the reduced x_r variable belongs to the $[\lambda \quad +\infty[$ domain for lower λ bounds ranging between 0 and 4.5.

A1.13.1. *Calculation of the integral*

$$I = \frac{2}{\sqrt{2\pi}} \int_{\lambda}^{+\infty} e^{-\frac{(x_r)^2}{2}} dx_r \qquad \text{[A1.54]}$$

λ	0	0.2	0.4	0.6	0.8	1	1.2	1.4
100 I	1	84.148	68.916	54.851	42.371	31.731	23.014	16.151

λ	1.6	1.8	2	2.2	2.4	2.6	2.8	3
100 I	10.960	7.186	4.550	2.781	1.640	0.932	0.511	0.270

λ	3.2	3.4	3.6	3.8	4	4.5
100 I	0.137	0.067	0.032	0.014	0.006	0.0006

Table A1.1. *Numerical values taken from the book by Bass [BAS 67]*

Conversely, in some statistical problems using the central limit theorem, the integral equation [A1.55] needs to be solved.

A1.13.2. *Solution to the integral equation*

$$I = \frac{2}{\sqrt{2\pi}} \int_{\lambda}^{+\infty} e^{-\frac{(x_r)^2}{2}} dx_r \qquad \text{[A1.55]}$$

Table A1.2 has the λ solutions corresponding to the values of the I integral ranging between 10^{-4} and 1.

100 I	100	95	90	85	80	75	70	65
λ	0.0000	0.0627	0.1257	0.1891	0.2533	0.3186	0.3853	0.4538

100 I	60	55	50	45	40	35	30	25
λ	0.5244	0.5978	0.6745	0.7554	0.8416	0.9346	1.0364	1.1503

Table A1.2. *Numerical values taken from the book by Bass [BAS 67]*

A1.14. Bibliography

[BAS 67] BASS J., *Eléments de calcul des probabilités, théorique et appliqué*, Masson, Paris, 1967.

[PAP 91] PAPOULIS A., *Probability, Random Variables, and Stochastic Process*, McGraw-Hill, New York, 1991.