9

Transmission-Line Analysis

9.1 Introduction

Transmission-line analysis is an important topic in many electrical engineering areas that range from power line systems to telecommunications, including computer networks.

The simplest examples of a transmission line are the two-wire line and the coaxial cable. In both cases we are dealing with a system made of two conductors separated by a dielectric medium where waves propagate from one end to the other guided by the geometrical configuration of the line conductors.

Since typical transmission-line systems have longitudinal dimensions of the order of the working wavelength λ (or above), the standard approximations of quasi-stationary regimes no longer apply – lumped-parameters circuit approaches simply cannot be used. Some examples follow:

- Transmission power lines: length 1000 km; frequency 60 Hz; wavelength 5000 km.
- Telephone lines: length 100 km; frequency 3 kHz; wavelength 100 km.
- Transmitter–antenna line links: length 100 m; frequency 10 MHz; wavelength 30 m.
- Computer boards: length 20 cm; frequency 1 GHz; wavelength 30 cm.

Nonetheless, we will assume in our analysis that the transversal system dimensions are negligibly small compared to λ, that is transversal wave phenomena are absent.

Besides the simple cases of the two-wire line and the coaxial cable, transmission-line analysis is also of crucial importance in the study of multiconductor transmission-line structures; however, such a topic will only be tackled superficially due to the introductory nature of this textbook. Likewise, the topic of non-uniform lines – those where the transverse profile of the line changes along the line length – will only be briefly referred to.

Before we get into the mathematical details of the equations that govern transmission-line phenomena, let us introduce the subject from a qualitative point of view.

Consider a transmission line of length l connecting a generator to a load (Figure 9.1), where, for simplification purposes, the line conductors are assumed to be perfect conductors.

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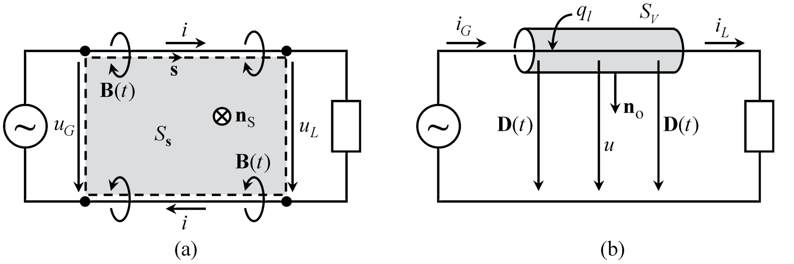

Figure 9.1 (a) Generator and load voltages are not the same, due to magnetic field distributed effects along the line. (b) Generator and load currents are not the same, due to electric field distributed effects along the line

You may recall that in the introductory matter to Part III we emphasized that the equalities $u_G(t) = u_L(t)$ and $i_G(t) = i_L(t)$ would not be valid for rapid time-varying field phenomena. Let us see why this happens.

As shown in Figure 9.1(a), conductor currents give rise to a distributed time-varying magnetic field along the line. By application of the induction law (Chapter 5) you find

$$-u_G(t) + u_L(t) = -\frac{d}{dt}\psi_l(t) = -\int_{S_s} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}_S \, dS \tag{9.1}$$

where $\psi_l(t)$ denotes the flux linked with the rectangular path of longitudinal length l. Therefore you can see that the presence of distributed inductive effects is ultimately responsible for the inequality $u_G(t) \neq u_L(t)$.

Also, as shown in Figure 9.1(b), conductor voltages give rise to a distributed time-varying electric field along the line (displacement currents between conductors). By application of the charge continuity equation in integral form (Chapter 6) you find

$$-i_G(t) + i_L(t) = -\frac{d}{dt}q_l(t) = -\int_{S_V} \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{n}_o \, dS \tag{9.2}$$

where $q_l(t)$ denotes the total electric charge distributed along the top conductor of length l. Therefore you can see that the presence of distributed capacitive effects is ultimately responsible for the inequality $i_G(t) \neq i_L(t)$.

For slow time-varying regimes, where both the inductive and capacitive distributed effects are negligibly small, we have $u_G(t) = u_L(t)$ and $i_G(t) = i_L(t)$. For rapid time-varying regimes, the voltages and currents at the generator and load terminals do not coincide; nevertheless, a simple identity between their time integrals still exists.

Assume that the driving voltage is a natural time-limited function; that is, assume the line is inactive for $t = \pm\infty$. Try a time integration of (9.1) and (9.2) from $t = -\infty$ to $t = \infty$.

When you do that, you will find, from (9.1),

$$-\int_{-\infty}^{\infty} u_G(t)\, dt + \int_{-\infty}^{\infty} u_L(t)\, dt = \underbrace{\psi_l(-\infty)}_{0} - \underbrace{\psi_l(\infty)}_{0}$$

Hence,

$$\int_{-\infty}^{\infty} u_G(t)\ dt = \int_{-\infty}^{\infty} u_L(t)\ dt \tag{9.3}$$

Likewise, from (9.2),

$$-\int_{-\infty}^{\infty} i_G(t)\ dt + \int_{-\infty}^{\infty} i_L(t)\ dt = \underbrace{q_l(-\infty)}_{0} - \underbrace{q_l(\infty)}_{0}$$

Hence,

$$\int_{-\infty}^{\infty} i_G(t)\ dt = \int_{-\infty}^{\infty} i_L(t)\ dt \tag{9.4}$$

The conclusions in (9.3) and (9.4) are not just theoretical curiosities; they can be very helpful for checking how correct your results are when solving a given problem, whether you do it by hand or employ dedicated software tools.

The fact that voltages and currents at the accessible ends of the line are different, $u_G(t) \neq u_L(t)$, $i_G(t) \neq i_L(t)$, is a clear indication that line voltages and line currents will undergo a continuous evolution along the line's longitudinal coordinate, $u = u(z, t)$, $i = i(z, t)$. In the next section we will obtain the laws that govern such an evolution.

9.2 Time-Domain Transmission-Line Equations for Lossless Lines

Consider a two-conductor line immersed in a non-magnetic linear homogeneous dielectric medium, whose conductors run parallel to the longitudinal z axis.

If you assume that the current-carrying conductors are perfect, the electric field component E_z will be absent. This means that outside the conductors, the field lines of $\mathbf{E}$ and $\mathbf{H}$ exist in planes transversal to the z axis; in this case we say that we are dealing with transverse electromagnetic waves (TEM),

$$\mathbf{E} = E_x(z, t)\vec{e}_x + E_y(z, t)\vec{e}_y, \quad \mathbf{H} = H_x(z, t)\vec{e}_x + H_y(z, t)\vec{e}_y$$

In these circumstances we can unambiguously define line voltages and line currents as

$$u(z, t) = \int_{\overrightarrow{ba}} \mathbf{E}(z, t) \cdot d\mathbf{s}$$

$$i(z, t) = \int_{S_1} \mathbf{J}_1(z, t) \cdot \vec{e}_z\ dS = \int_{S_2} \mathbf{J}_2(z, t) \cdot (-\vec{e}_z)\ dS = \oint_{\mathbf{s}} \mathbf{H}(z, t) \cdot d\mathbf{s}$$

where, as shown in Figure 9.2, the arbitrary open path $\overrightarrow{ba}$, the arbitrary closed path $\mathbf{s}$ and the conductor cross-sections S_1 and S_2 all belong to the same transversal plane $z =$ constant.

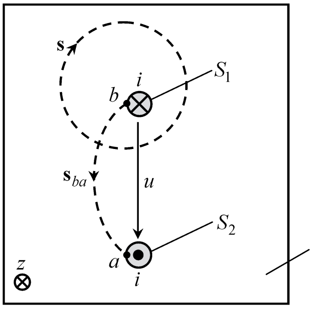

Figure 9.2 Line voltages and line currents are unambiguously defined in the transverse plane of the transmission line. **E**-field integration along an arbitrary open path from b to a yields the same result for the voltage u. Similarly, **H**-field integration along an arbitrary closed path encircling the upper conductor yields the same result for the current i

However, as you move from a z plane to another z plane both u and i will change. Our next goal is to determine the equations that govern those changes.

Consider the application of the induction law to a very small line section of length $\Delta z \to 0$ (Figure 9.3).

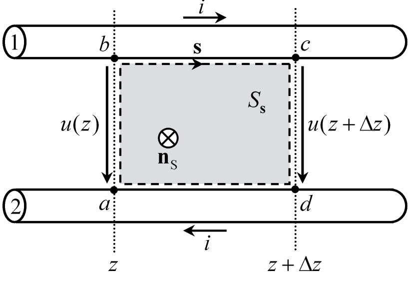

Figure 9.3 Application of the induction law to a very small line section of length $\Delta z \to 0$

The rectangular circulation path $\mathbf{s} = \overrightarrow{abcda}$ includes four contributions: one on the plane z, another on the plane $z + \Delta z$ and two others along the conductors' surfaces (where $E_z = 0$). The magnetic flux $\psi_{\Delta z}$ linked to the circulation path originated by the line currents can be evaluated by taking into account the per-unit-length external inductance L_e of the line

$$\begin{cases} \oint_{\mathbf{s}} \mathbf{E} \cdot d\mathbf{s} = \int_{\overrightarrow{ab}} \mathbf{E} \cdot d\mathbf{s} + \int_{\overrightarrow{bc}} \mathbf{E} \cdot d\mathbf{s} + \int_{\overrightarrow{cd}} \mathbf{E} \cdot d\mathbf{s} + \int_{\overrightarrow{da}} \mathbf{E} \cdot d\mathbf{s} = -u(z, t) + 0 + u(z + \Delta z, t) + 0 \\ -\dfrac{d}{dt} \int_{S_s} \mathbf{B} \cdot \mathbf{n}_S \, dS = -\dfrac{d}{dt} \psi_{\Delta z} = -L_e \Delta z \dfrac{\partial}{\partial t} i(z, t) \end{cases}$$

Equating the results above, dividing by Δz and taking the limit as $\Delta z \to 0$ we obtain

$$\lim_{\Delta z \to 0} \frac{u(z + \Delta z, t) - u(z, t)}{\Delta z} = \frac{\partial}{\partial z} u(z, t) = -L_e \frac{\partial}{\partial t} i(z, t)$$

Next, consider the application of the integral form of the charge continuity equation to a very small line section of length $\Delta z \to 0$; conductor 1 is enclosed by the integration surface S_V (Figure 9.4).

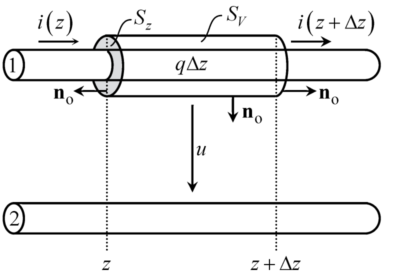

Figure 9.4 Application of the integral form of the charge continuity equation to a very small line section of length $\Delta z \to 0$

The current intensities through S_V are $-i(z, t)$ and $i(z+\Delta z, t)$, on the left and right cross-sections, respectively. The electric charge $q\Delta z$ inside S_V (originating the electric field between line conductors) can be evaluated by taking into account the per-unit-length capacitance C of the line

$$\begin{cases} \int\limits_{S_V} \mathbf{J} \cdot \mathbf{n}_o \, dS = \int\limits_{S_z} \mathbf{J} \cdot (-\vec{e}_z) \, dS + \int\limits_{S_{z+\Delta z}} \mathbf{J} \cdot \vec{e}_z \, dS = -i(z, t) + i(z+\Delta z, t) \\ -\int\limits_V \rho dV = -q\Delta z = -C\Delta z \; u(z, t) \end{cases}$$

Equating the results above, dividing by Δz and taking the limit as $\Delta z \to 0$ we obtain

$$\lim_{\Delta z \to 0} \frac{i(z+\Delta z, t) - i(z, t)}{\Delta z} = \frac{\partial}{\partial z} i(z, t) = -C \frac{\partial}{\partial t} u(z, t)$$

Summarizing, the time-domain transmission-line equations for lossless lines are

$$\frac{\partial}{\partial z} u(z, t) = -L_e \frac{\partial}{\partial t} i(z, t) \tag{9.5}$$

$$\frac{\partial}{\partial z} i(z, t) = -C \frac{\partial}{\partial t} u(z, t) \tag{9.6}$$

The line voltage rate of change in space is proportional to the line current rate of change in time; the link between those two rates is established via the per-unit-length inductance of the line. The line current rate of change in space is proportional to the line voltage rate of change in time; the link between those two rates is established via the per-unit-length capacitance of the line.

9.2.1 Wave Parameters, Propagation Velocity, Characteristic Wave Resistance

If you take the z derivative of (9.5) and then use (9.6) or, the other way around, if you take the z derivative of (9.6) and then use (9.5), you will obtain the already familiar one-dimensional wave equation, which applies indistinctly to line voltages and line currents

$$\frac{\partial^2}{\partial z^2}\left\{\begin{matrix} u(z,t) \\ i(z,t) \end{matrix}\right\} - L_e C \frac{\partial^2}{\partial t^2}\left\{\begin{matrix} u(z,t) \\ i(z,t) \end{matrix}\right\} = 0 \tag{9.7}$$

where the product $L_e C$ is the inverse of the squared wave propagation velocity, that is

$$v^2 = \frac{1}{L_e C} \tag{9.8}$$

from which two solutions can be found, $v = \pm 1/\sqrt{L_e C}$. While one solution describes a propagating wave along the positive z direction (the so-called incident wave), the other solution describes a propagating wave along the negative z direction (the so-called reflected wave). This means that, in general, the line voltage and line current solutions are a superposition of two counter-propagating waves, with velocity $v = 1/\sqrt{L_e C}$, which we can write in the form

$$u(z,t) = u_i(z - vt) + u_r(z + vt) \tag{9.9a}$$

$$i(z,t) = i_i(z - vt) + i_r(z + vt) = \frac{u_i(z - vt)}{(R_w)_i} + \frac{u_r(z + vt)}{(R_w)_r} \tag{9.9b}$$

where subscripts i and r are remainders for the incident and reflected wave solutions, respectively.

From (9.7), you can see that voltage and current waves are similar except for a scale factor with the physical dimensions of a resistance. For the incident wave, the scaling factor $(R_w)_i = u_i/i_i$ is the so-called characteristic resistance of the incident wave. The same thing applies, identically, to the reflected wave, with $(R_w)_r = u_r/i_r$.

Let us now determine the characteristic resistances concerning both waves.

By making $\xi = z - vt$, $\zeta = z + vt$, and taking into account that

$$\frac{\partial u}{\partial z} = \frac{\partial(u_i + u_r)}{\partial z} = \overbrace{\left(\frac{du_i}{d\xi}\right)}^{u_i'}\frac{\partial \xi}{\partial z} + \overbrace{\left(\frac{du_r}{d\zeta}\right)}^{u_r'}\frac{\partial \zeta}{\partial z} = u_i' + u_r'$$

$$\frac{\partial i}{\partial t} = \frac{\partial(i_i + i_r)}{\partial t} = \frac{di_i}{d\xi}\frac{\partial \xi}{\partial t} + \frac{di_r}{d\zeta}\frac{\partial \zeta}{\partial z} = \frac{u_i'}{(R_w)_i}(-v) + \frac{u_r'}{(R_w)_r}(v)$$

substituting (9.9) into (9.5) we find

$$u_i' + u_r' = \frac{L_e v}{(R_w)_i}u_i' + \frac{-L_e v}{(R_w)_r}u_r'$$

from which you see that the characteristic wave resistances are

$$(R_w)_i = -(R_w)_r = R_w = L_e v = \frac{1}{Cv} = \sqrt{\frac{L_e}{C}} \tag{9.10}$$

In the same way that two counter-propagating waves have reciprocal velocities ($\pm v$), the same thing happens with their characteristic resistances ($\pm R_w$).

According to this conclusion, we rewrite the transmission-line wave equation solutions in the final form

$$u(z, t) = u_i(z - vt) + u_r(z + vt) \tag{9.11a}$$

$$i(z, t) = i_i(z - vt) + i_r(z + vt) = \frac{u_i(z - vt)}{R_w} + \frac{u_r(z + vt)}{-R_w} \tag{9.11b}$$

In Chapter 8 we saw that the velocity of electromagnetic waves in a lossless homogeneous medium was determined solely by the intrinsic properties of the medium, that is by the parameters μ and ε. For non-magnetic media we have found

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0\varepsilon_r}} = \frac{c}{\sqrt{\varepsilon_r}} \tag{9.12}$$

We can check that this result still applies to waves guided by lossless homogeneous transmission lines. In fact, if you make use of the per-unit-length capacitance and inductance parameters established in Chapters 2 and 4, concerning the coaxial cable and the two-thin-wire line configuration, you will obtain v as given in (9.12) – see Table 9.1.

Table 9.1 Transmission-line parameters

	Coaxial cable	Two-thin-wire line
Per-unit-length capacitance C (F/m)	$\dfrac{2\pi\varepsilon}{\ln(r_2/r_1)}$	$\dfrac{\pi\varepsilon}{\ln(2d/r)}$
Per-unit-length inductance L_e (H/m)	$\dfrac{\mu_0}{2\pi}\ln(r_2/r_1)$	$\dfrac{\mu_0}{\pi}\ln(2d/r)$
Propagation velocity $v = \dfrac{1}{\sqrt{L_e C}}$ (m/s)	$\dfrac{1}{\sqrt{\mu_0\varepsilon}}$	$\dfrac{1}{\sqrt{\mu_0\varepsilon}}$
Characteristic resistance $R_w = \sqrt{\dfrac{L_e}{C}}$ (Ω)	$\sqrt{\dfrac{\mu_0}{\varepsilon}}\dfrac{\ln(r_2/r_1)}{2\pi}$	$\sqrt{\dfrac{\mu_0}{\varepsilon}}\dfrac{\ln(2d/r)}{\pi}$

However, the transmission-line characteristic wave resistance R_w is not to be confused with the free-space characteristic resistance we spoke about in Chapter 8; while the latter is a relationship between electric and magnetic field intensities, the former relates the voltage and the current of a given wave. Transmission-line characteristic wave resistances depend not only on the dielectric medium properties, but also on the transverse profile of the line itself – see Table 9.1.

When reading Table 9.1, you should bear in mind that, for the coaxial cable configuration, r_1 and r_2 denote the radius of the inner conductor and the inner radius of the outer conductor, respectively. For the two-thin-wire line, r denotes the radius of both conductors, and $2d$ denotes the axial separation between them.

9.2.2 Pulse Propagation, Pulse Reflection

Consider a transmission line of length l, and assume that at the input end of the line ($z = 0$) a triangular pulse voltage of short duration ΔT is provided by an ideal generator (Figure 9.5(a)).

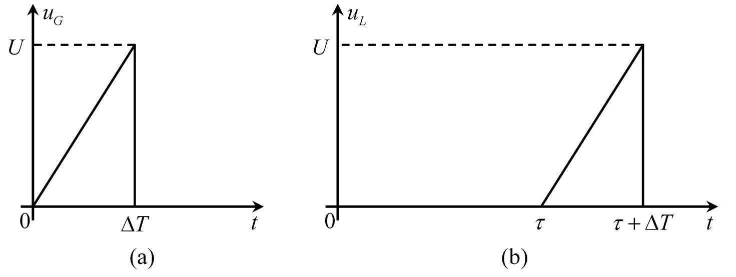

Figure 9.5 (a) Generator voltage shape. (a) Load voltage shape for the particular case of a matched line; the delay time τ is defined as $\tau = l/v$

The time delay between the emission of the pulse and its arrival at the load terminal is $\tau = l/v$ (with $\Delta T < \tau$). If you take a photo of the line voltage and line current along the line length at an instant $t < \tau$ you will only observe the propagating incident wave (Figure 9.6).

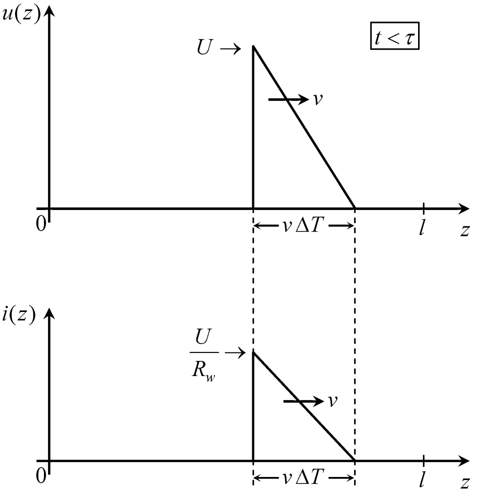

Figure 9.6 Snapshot, at instant t ($t < \tau$), of the line voltage and line current (incident wave) observed along z

A parenthetical remark: at this point some of you may be thinking that there is a mistake in the line drawing shown in Figure 9.6, for the shape of the triangular voltage pulse is the reverse of the triangular pulse shape in Figure 9.5(a). Well, you are thinking wrong; there is no mistake in Figure 9.6. The thing is that Figure 9.5(a) and Figure 9.6 are different representations of the pulse; while the first is a representation in time, the second is a representation in space. From Figure 9.5(a) you can see that the wavefront ($t = 0$) starts with zero voltage and only later in time ($t = \Delta T$) is the peak value U reached; as a consequence, when the wave travels in space, the pulse tail is the peak value.

As the incident wave reaches $z = l$ part of its energy is absorbed by the load and another part is reflected back towards the generator (reflected wave).

You may be wondering whether the reflection process can be prevented in some way. The answer is yes. If the load is properly chosen it can happen that the incident pulse is totally absorbed by the load; in such a case we say that the line is perfectly matched. In this case the voltage at the load terminals is simply a delayed image of the generator voltage (Figure 9.5(b)).

Let us see how you should choose the load R_L in order to get a matched line.

If you enforce the goal condition $u_r = 0$ in (9.11), and particularize the result to $z = l$, you will find

$$u(l, t) = u_i(l - vt); \quad i(l, t) = \frac{u_i(l - vt)}{R_w}$$

Noting that $R_L = u(l, t)/i(l, t)$, you readily conclude

$$\text{Matched line} : R_L = R_w \tag{9.13}$$

Whenever a lossless line is terminated by a resistor whose resistance is equal to the characteristic resistance of the line, reflected waves will be absent. This result provides you with a physical interpretation for the concept of characteristic wave resistance.

Take for instance a $50\,\Omega$ coaxial cable. Such a specification has nothing to do with the resistance of the cable conductors; the $50\,\Omega$ information concerns the cable's characteristic resistance, and tells you that if you want to avoid reflected waves inside the cable then you must terminate it with a $50\,\Omega$ resistor.

Next, let us examine the reflection processes that take place in a transmission line when the line is a mismatched one. For exemplification purposes, we keep considering that at $z = 0$ the voltage pulse is the one shown in Figure 9.5(a), but now we assume that the load terminals are left open, that is $i(l, t) = 0$.

As before in Figure 9.6, if you take a photo of the line voltage and line current along the line length at an instant $t < \tau$ you will observe only the propagating incident wave.

Since the load is an open circuit, it cannot absorb energy and a reflected wave is produced such that, at $z = l$,

$$i = 0 = i_i + i_r \quad \rightarrow \quad i_r = -i_i \text{ and } u_r = -R_w i_r = +R_w i_i = u_i \tag{9.14}$$

Now, if you take a photo of the line voltage and line current along the line length at an instant $\tau < t < 2\tau$ you will observe only the propagating reflected wave where the current pulse has changed its polarity – see Figure 9.7.

When the wave reaches the generator terminal, the generator voltage is zero (remember that the pulse duration ΔT is smaller than τ); put another way, the incoming wave sees a

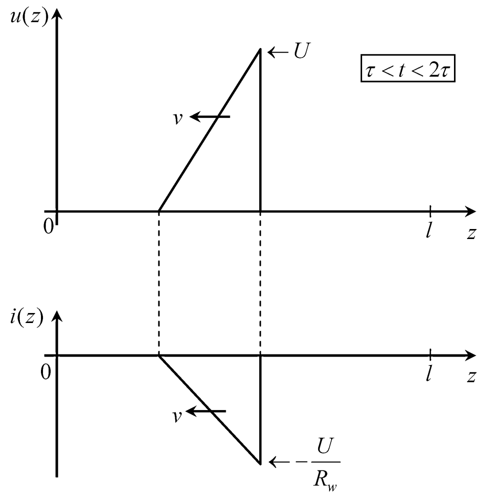

Figure 9.7 Snapshot, at instant t ($\tau < t < 2\tau$), of the line voltage and line current (reflected wave) observed along z

short circuit at $z = 0$. A new reflection phenomenon takes place and a new incident wave is originated, such that

$$u = 0 = u_i + u_r \quad \rightarrow \quad u_i = -u_r \text{ and } i_i = u_i/R_w \tag{9.15}$$

Again, if you take a photo of the line voltage and line current along the line length at an instant $2\tau < t < 3\tau$ you will observe only the propagating incident wave, where the voltage pulse has changed its polarity – see Figure 9.8.

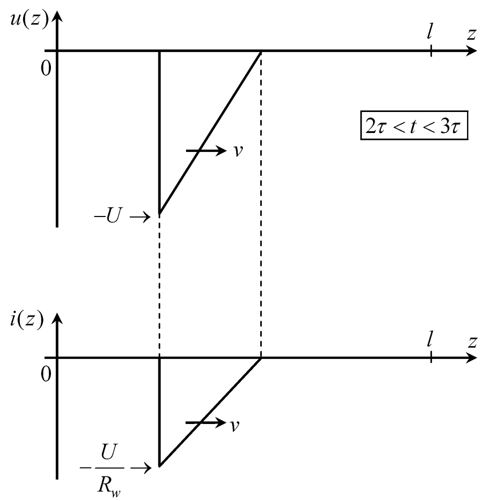

Figure 9.8 Snapshot, at instant t ($2\tau < t < 3\tau$), of the line voltage and line current (incident wave) observed along z

Since the line is lossless and its terminals do not include any dissipative components, reflection phenomena at both ends will occur in succession, and indefinitely.

Before ending this section, an important remark is in order. You have seen in the preceding analysis that current pulse waves travel along the line at a velocity $v = 1/\sqrt{\mu_0 \varepsilon}$, which typically is of the order of 10^8 m/s. Does this mean that the conductors' free electrons are moving at that same speed?

Absolutely not! Electron speeds inside good conductors are of the order of 1 m/s (Chapter 3). So, where is the catch?

Have you ever witnessed the so-called Mexican wave that fans usually perform in football stadiums? They get up and sit down sector after sector in a synchronized way producing a revolving wave around the stadium. Football fans are not running at the wave velocity, indeed they barely move. Well, the same thing happens with the conductor charges. They barely move. When the guided electromagnetic wave travels along the line, the free charges are pulled from their rest position to the conductor surface. After the wave has passed, the free charges return to their rest position.

9.3 Application Example (Parallel-Plate Transmission Line)

Consider a transmission line consisting of two parallel-plate perfect conductors of length l and width w. The space between the plates is filled with a non-magnetic dielectric material of very small thickness δ and permittivity $\varepsilon = \varepsilon_r \varepsilon_0$.

As shown in Figure 9.9, the line is fed at $z = 0$ by a voltage source described by its internal resistance R_G plus an ideal generator. The latter produces a single rectangular voltage pulse of amplitude U and duration ΔT. At the opposite end, $z = l$, the line is terminated by a resistor R_L.

Data: $l = 20\,\text{cm}$, $w = 25.13\,\text{mm}$, $\delta = 1\,\text{mm}$, $\varepsilon_r = 2.25$, $U = 4\,\text{V}$, $\Delta T = 1\,\text{ns}$, $R_L = 30\,\Omega$.

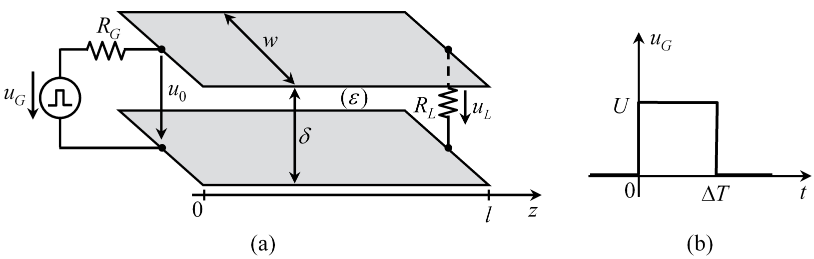

Figure 9.9 Parallel-plate transmission line driven by a pulse generator. (a) Perspective view of the transmission line. (b) Generator voltage plotted against time

Questions

Q_1 Determine the propagation velocity of the TEM waves inside the dielectric medium, v.

Q_2 Determine the per-unit-length capacitance and inductance of the line, C and L_e.

Q_3 Determine the characteristic wave resistance of the line, R_w.

Q_4 Assume that $R_G = R_w$.
Obtain the time shape of the voltages $u_0(t)$ and $u_L(t)$.
Check that

$$\int_{-\infty}^{\infty} u_0(t)\ dt = \int_{-\infty}^{\infty} u_L(t)\ dt$$

Evaluate the energy brought into play by the generator as well as the total energy dissipated in the system.

Q_5 Repeat the calculations in Q_4, but considering that $R_G = 0$.

Solutions

Q_1 $v = 1/\sqrt{\mu_0\varepsilon} = c/\sqrt{\varepsilon_r} = 2 \times 10^8\,\text{m/s}$.

Q_2 The electrostatic capacitance of the configuration of length l is $C_l = \varepsilon wl/\delta$. The per-unit-length capacitance of the line is $C = C_l/l = \varepsilon w/\delta = 500\,\text{pF/m}$.

$$v^2 = \frac{1}{L_e C} \to L_e = \frac{1}{v^2 C} = \frac{\mu_0 \delta}{w} = 50\,\text{nH/m}$$

Q_3

$$R_w = \sqrt{\frac{L_e}{C}} = \sqrt{\frac{\mu_0}{\varepsilon}} \times \frac{\delta}{w} = 10\,\Omega$$

Q_4 We start by noting that the time delay coincides with the pulse duration

$$\tau = \frac{l}{v} = 1\text{ns} = \Delta T$$

When the process begins, only the incident wave is present, $u = u_i$, $i = i_i = u_i/R_\text{w}$.

At $z = 0$ we have

$$u_0 = u_G - R_G i_0 \text{ or } u_i = u_G - R_G \frac{u_i}{R_w}$$

Since $R_G = R_w$, we find $u_0 = u_i = u_G/2 = 2\,\text{V}$ and $i_0 = i_i = u_G/(2R_w) = 200\,\text{mA}$.

At $t = \tau$, when the incident wave reaches the load, reflection occurs. The reflected wave is determined using the boundary condition at $z = l$, that is

$$u_L = u_i + u_r,\ i_L = \frac{u_i - u_r}{R_w},\ u_L = R_L i_L \tag{9.16}$$

from which we find

$$u_r = \Gamma u_i = \underbrace{\left(\frac{R_L - R_w}{R_L + R_w}\right)}_{\Gamma} u_i, \quad \Gamma = 0.5 \tag{9.17}$$

where the dimensionless factor Γ is the so-called load reflection coefficient. The reflected wave voltage is a positive pulse with half the magnitude of the incident one, $u_r = 1\,\text{V}$. The reflected wave current is a negative pulse, $i_r = -u_r/R_w = -100\,\text{mA}$.

From (9.16) and (9.17), we obtain the load voltage,

$$u_L = (1+\Gamma)\ u_i = (1+\Gamma)\frac{u_G}{2} = 3\ \text{V}$$

The reflected wave travels along the line towards the generator ($\tau < t < 2\tau$) and when it gets there it finds a matched load and the process ceases (note that the generator voltage is zero for $t > \tau$). The voltage at $z = 0$ is simply determined by the arriving wave u_r, that is $u_0 = 1\,\text{V}$, the corresponding current is $i_0 = -100\,\text{mA}$.

Figure 9.10 summarizes the preceding results, showing the time shape of both $u_0(t)$ and $u_L(t)$:

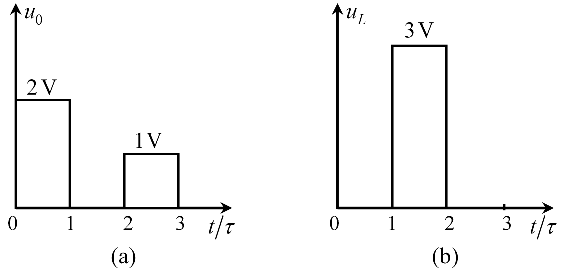

Figure 9.10 Voltages observed at the input and output ends of the line when $R_G = R_w$. (a) Input voltage $u_0(t)$. (b) Output voltage $u_L(t)$

$$\int_{-\infty}^{\infty} u_0(t)\ dt = \int_0^{\tau} u_0(t)\ dt + \int_{2\tau}^{3\tau} u_0(t)\ dt = 3\ \text{nWb};\ \int_{-\infty}^{\infty} u_L(t)\ dt = \int_{\tau}^{2\tau} u_L(t)\ dt = 3\ \text{nWb}$$

$$W_{R_G} = \int_{-\infty}^{\infty} R_G i_0^2\ dt = \int_0^{\tau} R_G i_0^2\ dt + \int_{2\tau}^{3\tau} R_G i_0^2\ dt = 0.5\ \text{nJ}$$

$$W_{R_L} = \int_{-\infty}^{\infty} \frac{u_L^2}{R_L}\ dt = \int_{\tau}^{2\tau} \frac{u_L^2}{R_L}\ dt = 0.3\ \text{nJ}$$

$$W_G = \int_{-\infty}^{\infty} u_G i_0\ dt = \int_0^{\tau} u_G i_0\ dt = 0.8\ \text{nJ}$$

Note that $W_G = W_{R_G} + W_{R_L}$.

Q_5 With $R_G = 0$ we have $u_0(t) = u_G(t)$.

The initially launched incident wave is $u_i = u_G = U = 4\,\text{V}$, $i_i = u_i/R_w = 400\,\text{mA}$.

When the incident wave reaches the load at $t = \tau$, reflection occurs dictated by the same equations as in (9.16) and (9.17), from which we obtain $u_r = \Gamma u_i = \Gamma U = 2\,\text{V}$ and $u_L = (1+\Gamma)U = 6\,\text{V}$.

The reflected wave travels along the line towards the generator ($\tau < t < 2\tau$) and when it gets there it finds a short-circuit condition, $u = 0$, hence

$$u = 0 = u_i + u_r \quad \rightarrow \quad u_i = -u_r$$

A new reflection occurs, and a new incident voltage wave consisting of a negative pulse is produced, $u_i = -u_r = -\Gamma U = -2\,\mathrm{V}$.

For $t = 3\tau$, the negative pulse arrives at the load, where it is reflected back. The new reflected voltage wave is obtained from (9.17), $u_r = \Gamma u_i = -\Gamma^2 U = -1\,\mathrm{V}$. The load voltage is determined from $u_L = (1+\Gamma)u_i = -(1+\Gamma)\Gamma U = -3\,\mathrm{V}$.

The reflected wave travels along the line towards the generator ($3\tau < t < 4\tau$) and when it gets there it again finds a short-circuit condition; consequently, a new incident wave is produced such that $u_i = -u_r = \Gamma^2 U = 1\,\mathrm{V}$.

This wave reaches the load at $t = 5\tau$, and produces a new reflection, the load voltage being given by $u_L = (1+\Gamma)u_i = (1+\Gamma)\Gamma^2 U = 1.5\,\mathrm{V}$.

The process of successive reflections at both ends of the line lasts indefinitely with u_L tending to zero as time passes.

For clarity, we summarize next the results obtained for the load voltage:

For $\tau < t < 2\tau$: $u_L = (1+\Gamma)U = 6\,\mathrm{V}$.
For $3\tau < t < 4\tau$: $u_L = -\Gamma(1+\Gamma)U = -3\,\mathrm{V}$.
For $5\tau < t < 6\tau$: $u_L = \Gamma^2(1+\Gamma)U = 1.5\,\mathrm{V}$.

In general: For $(2n-1)\tau < t < 2n\tau$: $u_L = (-\Gamma)^{n-1}(1+\Gamma)U$, with $n \geq 1$.

Figure 9.11 shows the time shape of both $u_0(t)$ and $u_L(t)$:

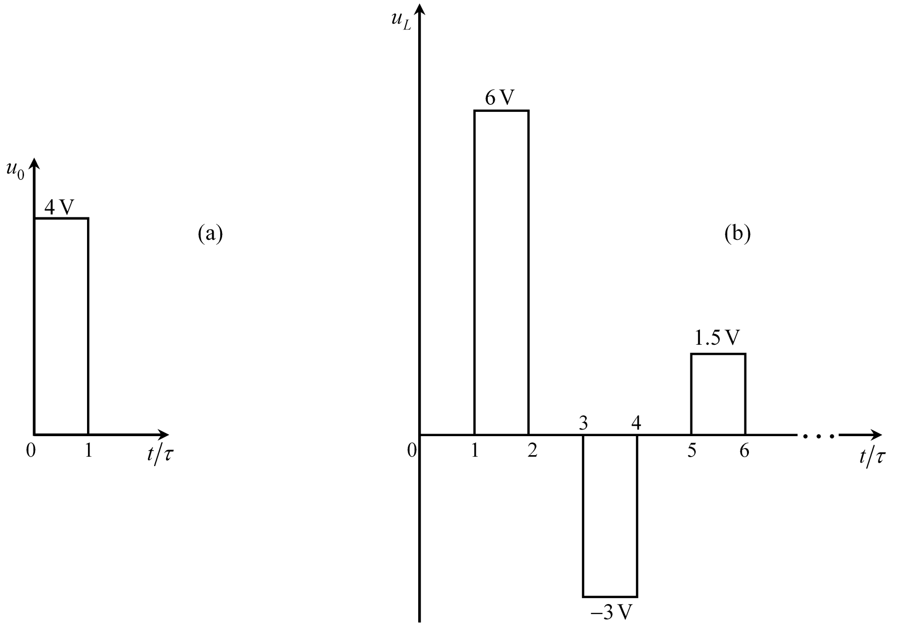

Figure 9.11 Voltages observed at the input and output ends of the line when $R_G = 0$. (a) Input voltage $u_0(t)$. (b) Output voltage $u_L(t)$

$$\int_{-\infty}^{\infty} u_0(t)\ dt = \int_0^{\tau} U\ dt = U\tau = 4\,\text{nWb}$$

$$\int_{-\infty}^{\infty} u_L(t)\ dt = \sum_{n=1}^{\infty} \int_{(2n-1)\tau}^{2n\tau} u_L\,dt = (1+\Gamma)U\tau \times \sum_{n=1}^{\infty} (-\Gamma)^{n-1}$$

Taking into account that the geometric series $\sum_{n=1}^{\infty} (-\Gamma)^{n-1}$ converges to $1/(1+\Gamma)$, we conclude that $\int_{-\infty}^{\infty} u_L(t)dt = U\tau$, as expected:

$$W_G = \int_{-\infty}^{\infty} u_G i_0\,dt = \int_0^{\tau} u_G i_0\,dt = \frac{U^2}{R_w}\tau = 1.6\,\text{nJ}$$

$$W_{R_L} = \int_{-\infty}^{\infty} \frac{u_L^2}{R_L}\,dt = \frac{1}{R_L}\sum_{n=1}^{\infty} \int_{(2n-1)\tau}^{2n\tau} u_L^2\,dt = \frac{(1+\Gamma)^2 U^2}{R_L}\tau \times \sum_{n=1}^{\infty} \Gamma^{2(n-1)}$$

Taking into account that the geometric series $\sum_{n=1}^{\infty} \Gamma^{2(n-1)}$ converges to $1/(1-\Gamma^2)$, and noting, from (9.17), that

$$R_L = R_w \frac{1+\Gamma}{1-\Gamma}$$

we conclude

$$W_{R_L} = \frac{(1+\Gamma)^2 U^2}{(1-\Gamma^2)R_L}\tau = \frac{U^2}{R_w}\tau = W_G$$

9.4 Frequency-Domain Transmission-Line Equations for Lossy Lines

Actual transmission-line structures are made of imperfect conductors and imperfect dielectric media, which are a cause of undesirable losses. Regrettably, the perturbations introduced by loss mechanisms cannot be easily taken into account directly in the time-domain transmission-line equations. To help you understand the difficulty, it may suffice to recall what you learnt in Section 8.7 and Problem 8.8.5 about the skin-effect phenomenon in imperfect conductors. In Section 8.7 we established, in the frequency domain, an equation for the per-unit-length impedance of a circular cylindrical conductor, $\overline{Z}(\omega) = R(\omega) + j\omega L_i(\omega)$. Nonetheless, in Problem 8.8.5, we showed that such an equation does not translate into the time domain as $u = Ri + L_i\ di/dt$ (that would only be possible if R and L_i were frequency-independent parameters, which, clearly, is not the case).

Therefore, our approach to the analysis of lossy lines is conducted entirely in the frequency domain, where line voltages and line currents are sinusoidal functions described by the corresponding phasors $\overline{U}(z)$ and $\overline{I}(z)$, that is

$$u(z,t) = \Re\left(\overline{U}(z)\ e^{j\omega t}\right),\ \ i(z,t) = \Re\left(\overline{I}(z)\ e^{j\omega t}\right)$$

A word of advice: since the assumption of perfect conductors has been dropped, a longitudinal component for the electric field does now exist. The electromagnetic field structure is no longer of the TEM type, and some ambiguity arises in the definition of line voltages. However, for good conductors, the longitudinal component of the **E** field is, in general, much smaller than the transverse component. This allows us to keep using the same formalism we established for lossless lines. But, in any case, you must realize that this approach is in fact an approximation (quasi-TEM approach).

9.4.1 Per-Unit-Length Longitudinal Impedance, Per-Unit-Length Transverse Admittance

Let us return to the (lossless) transmission-line equation in (9.5), which was obtained considering the application of the induction law: $\partial u/\partial z = -L_e \partial i/\partial t$. The corresponding frequency-domain equation is

$$\frac{d}{dz}\overline{U}(z) = -j\omega L_e \overline{I}(z) \tag{9.18}$$

In order to take into account the skin-effect phenomena in the two imperfect conductors of the line, two additional contributions should be added to the right-hand side of this equation. Those contributions are associated with the longitudinal voltage drops along line conductors 1 and 2,

$$\int_{\overrightarrow{bc}} \mathbf{E} \cdot d\mathbf{s} \text{ and } \int_{\overrightarrow{da}} \mathbf{E} \cdot d\mathbf{s}$$

which were null in the case of perfect conductors (see again Figure 9.3).

Therefore, if we add the per-unit-length conductor impedances $\overline{Z}_1(\omega) = R_1(\omega) + j\omega L_1(\omega)$ and $\overline{Z}_2(\omega) = R_2(\omega) + j\omega L_2(\omega)$ as perturbations to the main term $j\omega L_e$ in (9.18), we obtain the generalized equation

$$\frac{d}{dz}\overline{U}(z) = -\{\underbrace{(R_1(\omega) + R_2(\omega))}_{R} + j\omega \underbrace{(L_1(\omega) + L_2(\omega) + L_e)}_{L}\}\overline{I}(z) \tag{9.19}$$

This result can be written compactly if we introduce, as is usual, the so-called longitudinal impedance per unit length of the line $\overline{Z}_l$ (units: Ω/m)

$$\overline{Z}_l = R(\omega) + j\omega L(\omega) = Z_l \, e^{j(\pi/2 - \delta_Z)} \tag{9.20}$$

where δ_Z is the longitudinal loss angle.

Now, let us return to the (lossless) transmission-line equation in (9.6), which was obtained considering the application of the charge continuity equation: $\partial i/\partial z = -C \partial u/\partial t$. The corresponding frequency-domain equation is

$$\frac{d}{dz}\overline{I}(z) = -j\omega C \, \overline{U}(z) \tag{9.21}$$

Dielectric losses can occur because of two distinct mechanisms: one is the presence of transverse conduction currents across the imperfect insulation medium ($\sigma_{\text{dielectric}} \neq 0$), the other is the periodic polarization of the dielectric material – a phenomenon that can be accounted for by replacing the permittivity ε by a complex permittivity $\overline{\varepsilon} = \varepsilon'(\omega) - j\varepsilon''(\omega)$; see Section 8.4.

The first contribution can be included in the right-hand side of (9.21) by adding a perturbation term $G_\sigma \overline{U}$. The second contribution can be incorporated into (9.21) by replacing C by a complex per-unit-length capacitance $\overline{C} = C' - jC''$, where, typically, $C'' \ll C'$ and $C' \simeq C$. On doing this we obtain the generalized equation

$$\frac{d}{dz}\overline{I}(z) = -\{\underbrace{(G_\sigma + \omega C''\omega)}_{G} + j\omega C\}\overline{U}(z) \tag{9.22}$$

This result can be written compactly if we introduce, as is usual, the so-called transverse admittance per unit length of the line $\overline{Y}_t$ (units: S/m)

$$\overline{Y}_t = G(\omega) + j\omega C = Y_t\ e^{j(\pi/2 - \delta_Y)} \tag{9.23}$$

where δ_Y is the transversal loss angle.

In short, the frequency-domain transmission-line equations for the steady-state harmonic analysis of lossy lines read as

$$\frac{d}{dz}\overline{U}(z) = -\overline{Z}_l(\omega)\overline{I}(z) \tag{9.24}$$

$$\frac{d}{dz}\overline{I}(z) = -\overline{Y}_t(\omega)\overline{U}(z) \tag{9.25}$$

9.4.2 Propagation Constant, Phase Velocity, Characteristic Wave Impedance

Let us seek the solutions to the pair of equations in (9.24) and (9.25).

Taking the z derivative of (9.24) and substituting $-\overline{Y}_t(\omega)\overline{U}(z)$ for $d\overline{I}/dz$ yields

$$\frac{d^2\overline{U}(z)}{dz^2} - \overline{Z}_l\overline{Y}_t\overline{U}(z) = 0 \tag{9.26}$$

This is a homogeneous linear differential equation with constant coefficients of order 2.

Note that we have already dealt with equations of this type in the analysis of transient regimes (Chapter 7); the only difference is that, now, the equation is on z (not on t). Since this a familiar subject we can skip some details, and at once write

$$\text{Characteristic equation}: s^2 - \overline{Z}_l\overline{Y}_t = 0$$

$$\text{Characteristic roots}: s_{1,2} = \pm\overline{\gamma}, \quad \overline{\gamma} = \sqrt{\overline{Z}_l\overline{Y}_t} \tag{9.27}$$

$$\text{Line voltage solution}: \overline{U}(z) = \overline{U}_i e^{-\overline{\gamma}z} + \overline{U}_r e^{+\overline{\gamma}z} \tag{9.28}$$

In (9.27) the complex constant $\overline{\gamma}$ is designated a propagation constant

$$\overline{\gamma}(\omega) = \sqrt{\overline{Z}_l\overline{Y}_t} = \sqrt{Z_l Y_t}\ e^{j(\pi - \delta_Z - \delta_Y)/2} = \alpha + j\beta \tag{9.29}$$

Since both the loss angles δ_Z and δ_Y are confined to the range $[0, \pi/2]$, the propagation constant must necessarily belong to the first quadrant of the complex plane and, consequently, α and β are positive numbers; α is the so-called attenuation constant (units: Np/m), and β is the so-called phase constant (units: rad/m).

As shown in (9.28), the voltage solution is made up of two contributions, the first associated to the incident wave and the second to the reflected wave. The constants $\overline{U}_i$ and $\overline{U}_r$ are to be determined upon imposition of the pertinent boundary conditions at the ends of the line.

The solution for the line current $\overline{I}(z)$ is determined from (9.24),

$$\overline{I}(z) = -\frac{1}{\overline{Z}_l}\frac{d\overline{U}(z)}{dz}$$

yielding

$$\text{Line current solution:} \quad \overline{I}(z) = \frac{\overline{U}_i e^{-\overline{\gamma}z} - \overline{U}_r e^{+\overline{\gamma}z}}{\overline{Z}_w} \tag{9.30}$$

where $\overline{Z}_w$ is the characteristic wave impedance of the line (units: Ω) – a concept that, in the frequency domain, can be viewed as the generalization of the characteristic wave resistance introduced in Section 9.2.1 for lossless line analysis,

$$\overline{Z}_w(\omega) = \sqrt{\frac{\overline{Z}_l}{\overline{Y}_t}} = Z_w\ e^{j\theta_w},\ Z_w = \sqrt{\frac{Z_l}{Y_t}},\ \theta_w = \frac{\delta_Y - \delta_Z}{2} \tag{9.31}$$

Since both the loss angles δ_Z and δ_Y are confined to the range $[0, \pi/2]$, the characteristic wave impedance must belong to the first or eighth octant of the complex plane, that is $\theta_w \in [-\pi/4, \pi/4]$ and $\Re\left(\overline{Z}_w\right) > 0$.

As in our prior discussion on lossless lines, here too we can state that in order to avoid the presence of the reflected wave, $\overline{U}_r = 0$ (matched line case), all you have to do is to terminate the line by a load impedance equal to the characteristic wave impedance, $\overline{Z}_L = \overline{Z}_w$. Nevertheless, you should bear in mind that $\overline{Z}_w$ is, now, a frequency-dependent parameter and, therefore, if the working frequency changes you will most likely fail with perfect matching (unless you properly readjust the load impedance value).

To better grasp the significance of the wave parameters $\overline{\gamma}$ and $\overline{Z}_w$ introduced in this section, we are going to obtain space–time descriptions of $u(z, t)$ and $i(z, t)$. To simplify matters we assume that the working frequency and the line termination are such that the line is perfectly matched, that is

$$\overline{U}(z) = \overline{U}_i\ e^{-(\alpha+j\beta)z},\ \text{with}\ \overline{U}_i = U\ e^{j\phi_u}$$

$$\overline{I}(z) = \frac{\overline{U}_i\ e^{-(\alpha+j\beta)z}}{Z_w\ e^{j\theta_w}}$$

Taking into account that $u(z, t) = \Re\left(\overline{U}(z)\ e^{j\omega t}\right)$ and $i(z, t) = \Re\left(\overline{I}(z)\ e^{j\omega t}\right)$, we get

$$u(z, t) = U\ e^{-\alpha z}\cos(\omega t - \beta z + \phi_u) \tag{9.32a}$$

$$i(z, t) = I\ e^{-\alpha z}\cos(\omega t - \beta z + \phi_i) \tag{9.32b}$$

where $I = U/Z_w$ and $\phi_i = \phi_u - \theta_w$.

Since the preceding expressions are very similar, let us pay attention to only one of them, for example the line voltage $u(z, t)$.

The phase under the cosine function depends on z and t, $\phi(z, t) = \omega t - \beta z + \phi_u$. At which speed v should you run along the line in order to always observe the same phase value?

By making $z = z_0 + vt$ you find the condition

$$\omega t - \beta v t - \beta z_0 + \phi_u = \text{time-invariant constant}$$

from which you readily get

$$v = \frac{\omega}{\beta} \tag{9.33}$$

The velocity defined in (9.33) is the so-called phase velocity. It coincides with the propagation velocity defined in (9.8) and (9.12) in the case of lossless lines.

Next, consider that, at a given instant t_0, you take a photo of the voltage distribution along the line. You will observe an oscillating function on z with decaying amplitude – see Figure 9.12.

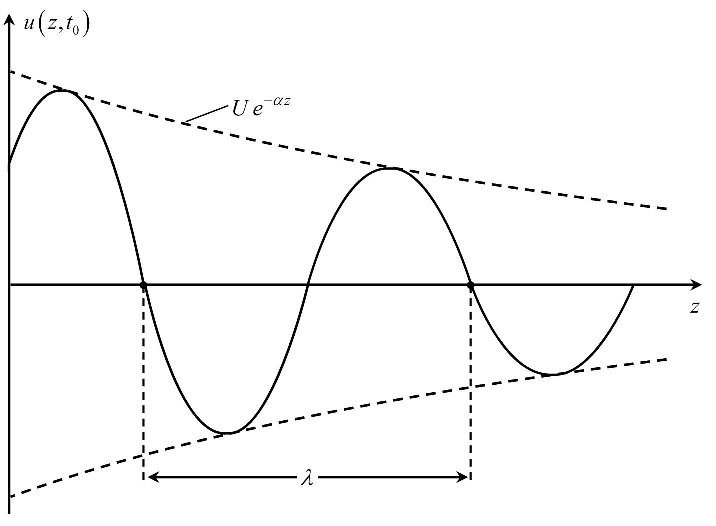

Figure 9.12 Snapshot, taken at $t = t_0$, of the line voltage distribution along the axial coordinate z, for the case of a lossy line submitted to a sinusoidal steady-state regime

The rate of decay is determined by the attenuation constant α. On the other hand, the periodicity of the oscillating function is determined by the phase constant β. In fact, the space period λ (wavelength) is such that the product $\beta\lambda$ must equal 2π, that is

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi v}{\omega} = \frac{v}{f}$$

which, manifestly, is no surprise.

To end this section, imagine that you place an oscilloscope at $z = z_0$ to observe the time evolution of $u(z_0, t)$. The image you will obtain on the screen is shown in Figure 9.13; it is a pure sinusoidal function with a time period $T = \omega/(2\pi) = 1/f$, which displays no decay at all. Pay attention to the fact that, according to (9.32), the attenuation constant affects the function evolution on z, but not on t. However, if you place another oscilloscope at $z_1 > z_0$ you will observe another sinusoidal time function with smaller magnitude (see again Figure 9.13).

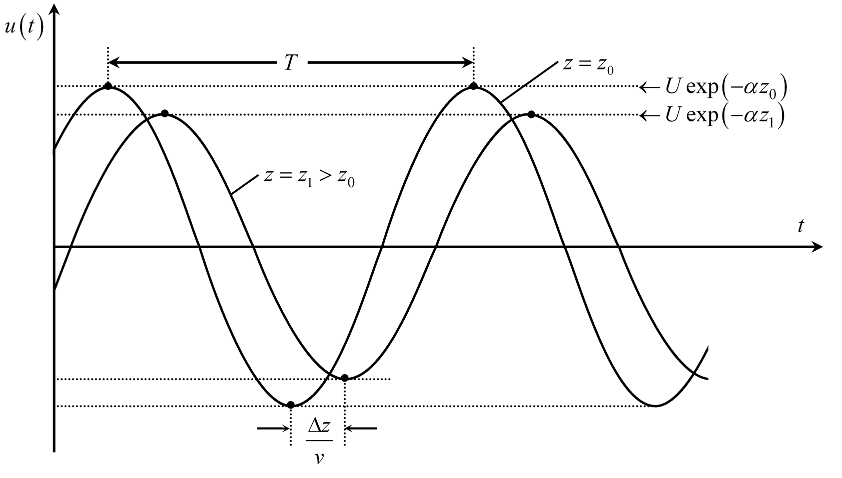

Figure 9.13 Oscilloscope readings of the time evolution of the line voltage at two different places $z = z_0$ and $z = z_1 = z_0 + \Delta z$

9.4.3 Transfer Matrix, Non-Uniform Line Analysis

In many applications a detailed knowledge of the voltage and current distributions along the line is of little concern. Quite often the line is simply treated as a black box, a two-port network, where only the voltages and currents at the accessible ends really matter – see Figure 9.14.

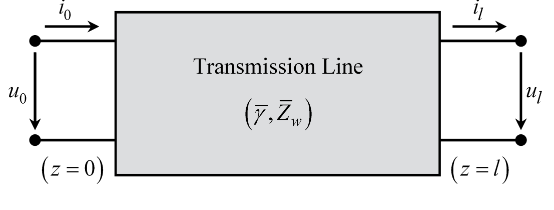

Figure 9.14 A two-port, black box representation of a transmission line of length l

Consider a transmission-line section of length l where $\overline{U}_0$ and $\overline{I}_0$ are the input quantities, and $\overline{U}_l$ and $\overline{I}_l$ are the output quantities. Our next goal is to establish a matrix relationship among those quantities, that is

$$\begin{bmatrix} \overline{U}_0 \\ \overline{I}_0 \end{bmatrix} = [T] \begin{bmatrix} \overline{U}_l \\ \overline{I}_l \end{bmatrix} \tag{9.34}$$

where $[T]$ is the transfer matrix (also termed the chain matrix).

Making use of (9.28) and (9.30) we get

$$z = l: \begin{cases} \overline{U}_l = \overline{U}_i e^{-\overline{\gamma}l} + \overline{U}_r e^{+\overline{\gamma}l} \\ \overline{Z}_w \overline{I}_l = \overline{U}_i e^{-\overline{\gamma}l} - \overline{U}_r e^{+\overline{\gamma}l} \end{cases} \rightarrow \begin{cases} \overline{U}_i = \frac{1}{2}(\overline{U}_l + \overline{Z}_w \overline{I}) e^{+\overline{\gamma}l} \\ \overline{U}_r = \frac{1}{2}(\overline{U}_l + \overline{Z}_w \overline{I}) e^{-\overline{\gamma}l} \end{cases}$$

$$z = 0: \begin{cases} \overline{U}_0 = \overline{U}_i + \overline{U}_r = \cosh(\overline{\gamma}l)\overline{U}_l + \sinh(\overline{\gamma}l)\overline{Z}_w \overline{I}_l \\ \overline{Z}_w \overline{I}_0 = \overline{U}_i - \overline{U}_r = \sinh(\overline{\gamma}l)\overline{U}_l + \cosh(\overline{\gamma}l)\overline{Z}_w \overline{I}_l \end{cases}$$

from which the transfer matrix is determined

$$[T] = \begin{bmatrix} \cosh(\overline{\gamma}l) & \sinh(\overline{\gamma}l)\overline{Z}_w \\ \overline{Z}_w^{-1} \sinh(\overline{\gamma}l) & \cosh(\overline{\gamma}l) \end{bmatrix} \tag{9.35}$$

The transfer matrix is a precious tool for handling non-uniform line analysis. Imagine that you have a transmission-line structure whose transverse profile changes along z (see the example in Figure 9.15). You may, conceptually, break down the structure into a chain of N small line sections, of appropriate lengths, so that each and every section can be considered a uniform line.

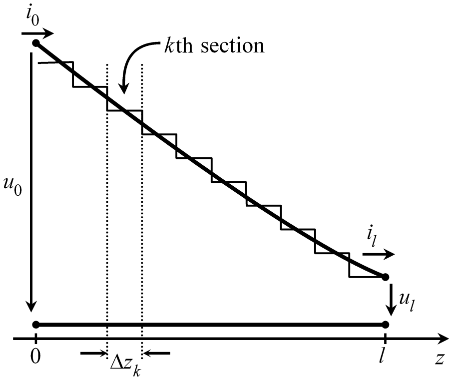

Figure 9.15 Segmentation technique ordinarily used for the analysis of non-uniform lines

Once you have evaluated the transfer matrices $[T_1], \ldots, [T_k], \ldots, [T_N]$ pertaining to the individual line sections, you can finally determine the global transfer matrix $[T]$ of the non-uniform structure through

$$\begin{bmatrix} \overline{U}_0 \\ \overline{I}_0 \end{bmatrix} = \underbrace{[T_1]\ [T_2]\ [T_3] \ldots [T_N]}_{[T]} \begin{bmatrix} \overline{U}_l \\ \overline{I}_l \end{bmatrix}$$

The accuracy of this method obviously depends on the number of line sections you have utilized. The thinner the segmentation, the better the results.

9.5 Frequency-Domain Transmission-Line Equations for Lossless Lines

In a lossy line, the attenuation and phase constants increase with the frequency, but while $\alpha \propto \sqrt{f}$, we have $\beta \propto f$. This means that the loss angles

$$\delta_Z + \delta_Y = 2\arctan(\alpha/\beta)$$

ordinarily tend to zero as the frequency increases. Therefore, for very high-frequency regimes, it makes sense to consider the approximations $\overline{Z}_l \to j\omega L_e$ and $\overline{Y}_t \to j\omega C$, which in turn imply

$$\overline{\gamma}(\omega) = \sqrt{\overline{Z}_l \overline{Y}_t} = j\omega\sqrt{L_e C} = 0 + j\beta$$

$$\overline{Z}_w = \sqrt{\frac{\overline{Z}_l}{\overline{Y}_t}} = \sqrt{\frac{L_e}{C}} = R_w + j0$$

This means that we are back again to the framework of lossless-line analysis. Consequently, the transmission-line solutions established in (9.28) and (9.30) greatly simplify:

$$\overline{U}(z) = \overline{U}_i e^{-j\beta z} + \overline{U}_r e^{+j\beta z} \tag{9.36a}$$

$$\overline{I}(z) = \frac{\overline{U}_i e^{-j\beta z} - \overline{U}_r e^{+j\beta z}}{R_w} \tag{9.36b}$$

where $\overline{U}_i$ and $\overline{U}_r$ denote, respectively, the complex amplitudes of the incident and reflected voltage waves at the line sending end, $z = 0$.

9.5.1 Terminated Line, Load Reflection Coefficient, Line Input Impedance

Consider a transmission line of length l driven by a sinusoidal voltage generator at one end and terminated at the opposite end by a passive load impedance $\overline{Z}_L$.

For the analysis of terminated lines it is frequent, and convenient, to introduce a longitudinal y axis with origin at the load, antiparallel to z – see Figure 9.16.

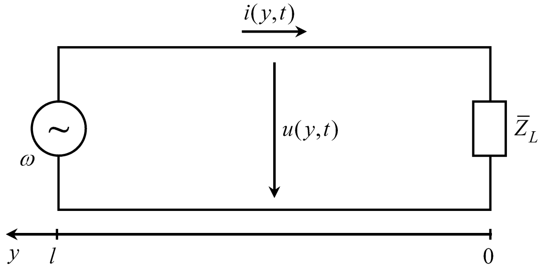

Figure 9.16 Terminated transmission line showing the longitudinal y axis starting at the load terminals

Taking into account the change of variable, $y = l - z$, the solution in (9.36) reads as

$$\overline{U}(y) = \overline{U}_i' \; e^{+j\beta y} + \overline{U}_r' \; e^{-j\beta y} \tag{9.37a}$$

$$\overline{I}(y) = \frac{\overline{U}_i' \; e^{+j\beta y} - \overline{U}_r' \; e^{-j\beta y}}{R_w} \tag{9.37b}$$

where $\overline{U}_i' = \overline{U}_i \; e^{-j\beta l}$ and $\overline{U}_r' = \overline{U}_r \; e^{+j\beta l}$ respectively denote the incident wave voltage and reflected wave voltage phasors, measured at the load terminals. A complex dimensionless number, the load reflection coefficient $\overline{\Gamma}$, is defined as the ratio of those wave phasors, that is

$$\overline{\Gamma} = \frac{\overline{U}_r'}{\overline{U}_i'} \tag{9.38}$$

Taking into account the definition of $\overline{\Gamma}$, we can rewrite (9.37) as

$$\overline{U}(y) = \overline{U}_i' \; e^{+j\beta y} \times (\overbrace{1 + \overline{\Gamma} \; e^{-j2\beta y}}^{\overline{V}_U}) \tag{9.39a}$$

$$\overline{I}(y) = \frac{\overline{U}_i' \; e^{+j\beta y}}{R_w} \times (\underbrace{1 - \overline{\Gamma} \; e^{-j2\beta y}}_{\overline{V}_I}) \tag{9.39b}$$

On the right-hand side of (9.39) the leading factor is to be interpreted as the incident wave contribution; the factors on the right, $\overline{V}_U$ and $\overline{V}_I$, are complex vectors where the influence of the terminating load is included via $\overline{\Gamma}$. Those two auxiliary vectors are such that

$$\sphericalangle \overline{V}_U, \overline{V}_I = \sphericalangle \; \overline{U}, \overline{I} = \varphi(y); \qquad \frac{\overline{V}_U + \overline{V}_I}{2} = 1 \tag{9.40}$$

If the equations in (9.39) are particularized to the load terminals, $y = 0$, we get

$$\overline{U}_L = \overline{U}_i'(1 + \overline{\Gamma}); \;\; \overline{I}_L = \frac{\overline{U}_i'}{R_w}(1 - \overline{\Gamma})$$

Dividing the two equations above, we find

$$\overline{Z}_L = R_w \frac{1 + \overline{\Gamma}}{1 - \overline{\Gamma}} \tag{9.41}$$

Solving (9.41) for $\overline{\Gamma}$, we obtain

$$\overline{\Gamma} = \Gamma \; e^{j\theta} = \frac{\overline{Z}_L - R_w}{\overline{Z}_L + R_w} \tag{9.42}$$

For passive loads (consisting of R, L, C arrangements), where $\Re\left(\overline{Z}_L\right) \geq 0$, we have $\Gamma \leq 1$ and $0 \leq \theta \leq 2\pi$. The physical reason behind the fact that Γ cannot exceed unity is that the energy reflected by the load cannot exceed the energy carried by the incident wave.

Total reflection ($\Gamma = 1$) can only occur in the case of non-dissipative loads, that is

$$\text{Short circuit}: \overline{Z}_L = 0 \rightarrow \overline{\Gamma} = 1\ e^{j\pi}$$

$$\text{Open circuit}: \overline{Z}_L = \infty \rightarrow \overline{\Gamma} = 1$$

$$\text{Reactive load}: \overline{Z}_L = jX_L \rightarrow \overline{\Gamma} = 1\ e^{j[\pi - 2\arctan(X_L/R_w)]}$$

Total absorption ($\Gamma = 0$) happens when the line is perfectly matched, $\overline{Z}_L = R_w$.

The line input impedance $\overline{Z}_{in}$, defined as the ratio $\overline{U}/\overline{I}$ at $y = l$, depends on the frequency, on the line length and on the line load. If the equations in (9.39) are particularized to the input end, we get

$$\overline{U}_{in} = \overline{U}_i' e^{j\beta l}(1 + \overline{\Gamma}\ e^{-j2\beta l}) \text{ and } \overline{I}_{in} = \left[\overline{U}_i'\ e^{j\beta l}(1 - \overline{\Gamma}\ e^{-j2\beta l})\right]/R_w$$

Hence

$$\overline{Z}_{in} = \frac{\overline{U}_{in}}{\overline{I}_{in}} = R_w \frac{1 + \overline{\Gamma}\ e^{-j2\beta l}}{1 - \overline{\Gamma}\ e^{-j2\beta l}} \tag{9.43}$$

9.5.2 Matched Line, Open Line, Short-Circuited Line

We now particularize the results from Section 9.5.1 to three particularly important cases.

9.5.2.1 Matched Line ($\overline{Z}_L = R_w,\ \overline{\Gamma} = 0$)

In the frequency domain:

$$\begin{cases} \overline{U}(y) = \overline{U}_i' e^{j\beta y} \\ \overline{I}(y) = \overline{U}_i' e^{j\beta y}/R_w \end{cases} \tag{9.44}$$

where $\overline{U}_i' = U'\ e^{j\phi_u'}$.

The input impedance is

$$\overline{Z}_{in} = \frac{\overline{U}_{y=l}}{\overline{I}_{y=l}} = R_w$$

In the time domain:

$$\begin{cases} u(y, t) = U' \cos(\omega t + \beta y + \phi_u') \\ i(y, t) = \dfrac{U'}{R_w} \cos(\omega t + \beta y + \phi_u') \end{cases} \tag{9.45}$$

In a matched line, the input impedance $\overline{Z}_{in}$ is equal to the characteristic wave resistance; it depends neither on the frequency nor on the line length. Since the reflected wave is absent, the propagation phenomena along the line are entirely described by the incident wave component. In this case we say we are dealing with a purely traveling wave. If you take two photos of $u(y)$, at $t = t_1$ and $t = t_2 > t_1$, you will observe two sinusoidal functions of the same magnitude but shifted by a distance $\Delta y = v(t_2 - t_1)$ – see Figure 9.17.

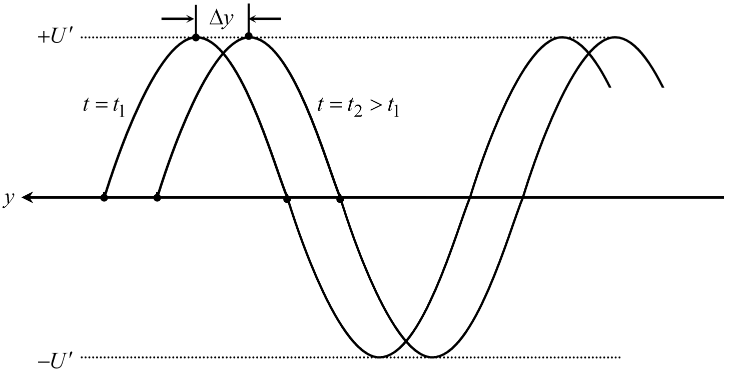

Figure 9.17 Snapshots, taken at $t = t_1$ and $t = t_2 > t_1$, of the line voltage distribution along the axial coordinate y, for the case of a matched lossless line (traveling wave). $\Delta y = v(t_2 - t_1)$

9.5.2.2 Open Line ($\overline{Z}_L = \infty,\ \overline{\Gamma} = 1$)

In the frequency domain:

$$\begin{cases} \overline{U}(y) = \overline{U}_i' \left(e^{+j\beta y} + e^{-j\beta y}\right) \\ \overline{I}(y) = \dfrac{\overline{U}_i'}{R_w} \left(e^{+j\beta y} - e^{-j\beta y}\right) \end{cases} \tag{9.46a}$$

or, which is the same thing,

$$\begin{cases} \overline{U}(y) = 2\overline{U}_i' \cos(\beta y) \\ \overline{I}(y) = \dfrac{2j\overline{U}_i'}{R_w} \sin(\beta y) \end{cases} \tag{9.46b}$$

where $\overline{U}_i' = U' e^{j\phi_u'}$.

The input impedance is

$$\overline{Z}_{in} = \frac{\overline{U}_{y=l}}{\overline{I}_{y=l}} = \frac{-jR_w}{\tan(\beta l)} \tag{9.47}$$

In the time domain:

$$\begin{cases} u(y, t) = 2U' \cos(\omega t + \phi_u') \cos(\beta y) \\ i(y, t) = 2\dfrac{U'}{R_w} \cos(\omega t + \phi_u' + \pi/2) \sin(\beta y) \end{cases} \tag{9.48}$$

In an open-line situation the input impedance is purely reactive, its sign depending on the value of the product $\beta l = 2\pi l/\lambda$. For instance, if $l = \lambda/4$ you will find $\beta l = \pi/2$ and the input impedance is zero; the generator at the sending end of an open line faces a short-circuit situation at its own terminals! On the contrary, if $l = \lambda/2$ the generator will face an open-circuit situation. See Figure 9.18.

For those of you who are studying transmission lines for the first time, the preceding results must be shocking and unexpected. However, you must realize that transmission-line

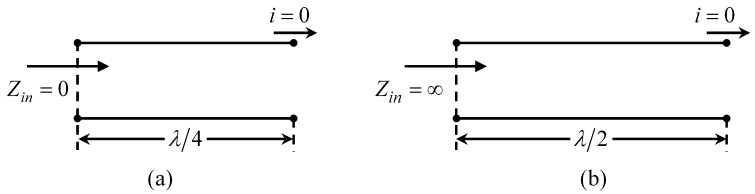

Figure 9.18 Lossless transmission line terminated by an open circuit. (a) The input impedance is zero when $l = \lambda/4$. (b) The input impedance is infinite when $l = \lambda/2$

behavior is completely inexplicable if you insist on using the ordinary tools and concepts from circuit analysis (which pertain to slow time-varying field phenomena, and are not valid here).

As you can see from (9.46a), the line voltage and line current solutions are the result of two counter-propagating waves, both of the same magnitude. These two waves interact with each other giving rise to constructive or destructive interference at certain critical points along the y axis. If you compare the time-domain solutions in (9.48) and (9.45) you will see that in the latter the space and time variables are linearly combined under the cosine function (which is typical of traveling waves); however, in (9.48) the space and time variables appear separated under different trigonometric functions. When two counter-propagating traveling waves of the same magnitude add together they give rise to a purely stationary wave, that is a wave that pulses but does not move. In fact, if you take a series of photos of $i(y)$ at different time instants you will observe the result shown in Figure 9.19.

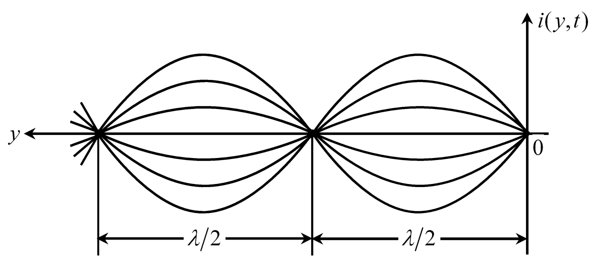

Figure 9.19 Successive snapshots, taken at different time instants, of the line current distribution along the axial coordinate y, for the case of an open lossless line (stationary wave)

In addition you may note, from (9.48), that at certain instants of time the line current is zero everywhere along the line length, the same thing happening to the line voltage, but at other instants:

$$\begin{cases} i(y,t) = 0; \ \text{for } \omega t = (k-1)\dfrac{\pi}{2} - \phi_u' \\ u(y,t) = 0; \ \text{for } \omega t = k\dfrac{\pi}{2} - \phi_u' \end{cases}$$

with k an odd integer.

9.5.2.3 Short-Circuited Line ($\overline{Z}_L = 0,\ \overline{\Gamma} = -1$)

This case can be considered the dual of the preceding situation, where the roles of u and i are interchanged.

In the frequency domain:

$$\begin{cases} \overline{U}(y) = \overline{U}_i'\left(e^{+j\beta y} - e^{-j\beta y}\right) \\ \overline{I}(y) = \dfrac{\overline{U}_i'}{R_w}\left(e^{+j\beta y} + e^{-j\beta y}\right) \end{cases} \tag{9.49a}$$

or, which is the same thing,

$$\begin{cases} \overline{U}(z) = 2j\overline{U}_i' \sin(\beta y) \\ \overline{I}(y) = \dfrac{2\overline{U}_i'}{R_w}\cos(\beta y) \end{cases} \tag{9.49b}$$

where $\overline{U}_i' = U'\ e^{j\phi_u'}$.

The input impedance is

$$\overline{Z}_{in} = \frac{\overline{U}_{y=l}}{\overline{I}_{y=l}} = jR_w \tan(\beta l) \tag{9.50}$$

In the time domain:

$$\begin{cases} u(y,t) = 2U' \cos(\omega t + \phi_u' + \pi/2)\sin(\beta y) \\ i(y,t) = 2\dfrac{U'}{R_w}\cos(\omega t + \phi_u')\cos(\beta y) \end{cases} \tag{9.51}$$

In a short-circuited line the input impedance is purely reactive, its sign depending on the value of the product $\beta l = 2\pi l/\lambda$. For instance, if $l = \lambda/4$ you will find $\beta l = \pi/2$ and the input impedance goes to ∞; the generator at the sending end of a short-circuited line faces an open-circuit situation at its own terminals! On the contrary, if $l = \lambda/2$ the generator will face a short-circuit situation. See Figure 9.20.

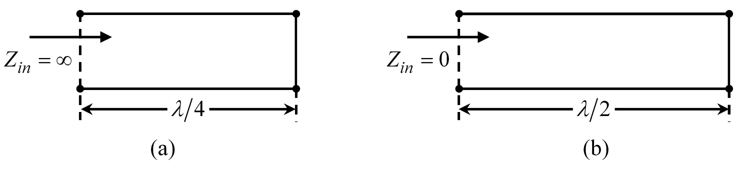

Figure 9.20 Lossless transmission line terminated by a short circuit. (a) The input impedance is infinite when $l = \lambda/4$. (b) The input impedance is zero when $l = \lambda/2$

Again, as in the case of an open line, here too the line voltage and line current are described by purely stationary waves. Apart from a vertical scale factor, the graph for $i(y)$ in Figure 9.19 could be used here to reproduce the evolution $u(y)$ at several instants of time.

9.5.3 Standing Wave Pattern, Standing Wave Ratio, Active Power

In the case of a matched line, if you record the rms values of the line voltage and line current distributions along the length of the line you will obtain constant readings. In fact, from (9.44), you have

$$\begin{cases} U_{rms}(y) = \dfrac{\left|\overline{U}(y)\right|}{\sqrt{2}} = \dfrac{U_i}{\sqrt{2}} = (U_i)_{rms} = \text{constant} \\ I_{rms}(y) = \dfrac{\left|\overline{I}(y)\right|}{\sqrt{2}} = \dfrac{(U_i)_{rms}}{R_w} = \text{constant} \end{cases}$$

However, when the line is mismatched, the superposition of the incident and reflected waves gives rise to more or less pronounced fluctuations in both $U_{rms}(y)$ and $I_{rms}(y)$. The larger the magnitude of the reflection coefficient Γ, the more important the fluctuation. In fact, the complex vectors $\overline{V}_U$ and $\overline{V}_I$ introduced in (9.39), which vary with y, have their minimum and maximum magnitudes given by $1-\Gamma$ and $1+\Gamma$, the overall fluctuation being 2Γ. In addition, note that when $|\overline{V}_U|$ is at a maximum, $|\overline{V}_I|$ is at a minimum, and vice versa – see Figure 9.21.

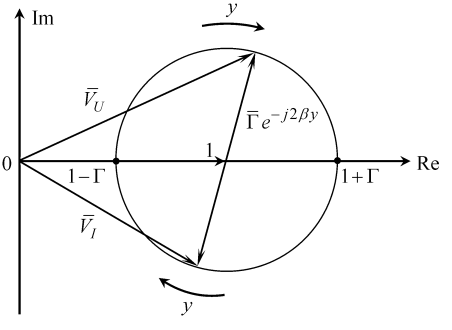

Figure 9.21 Complex plane vector diagram used for analyzing the evolution of $\overline{V}_U$ and $\overline{V}_I$ as one moves away from the load terminals (increasing y)

The evolution of $U_{rms}(y)$ and $I_{rms}(y)$ can be retrieved from the geometrical vector diagram in Figure 9.21 where, as you move away from the load towards the generator terminals, the tips of $\overline{V}_U(y)$ and $\overline{V}_I(y)$ move clockwise, synchronously, along a circumference of radius Γ.

An analytical alternative for obtaining the evolution of $U_{rms}(y)$ and $I_{rms}(y)$ is also available. Go back to the general equations in (9.39), take their absolute value and divide by $\sqrt{2}$. This enables you to write U_{rms} and I_{rms} as follows:

$$U_{rms}(y) = (U_i)_{rms}\left|1+\Gamma\ e^{j(\theta-2\beta y)}\right| = (U_i)_{rms}\sqrt{1+\Gamma^2+2\Gamma\cos(2\beta y-\theta)} \tag{9.52a}$$

$$I_{rms}(y) = \frac{(U_i)_{rms}}{R_w}\left|1-\Gamma\ e^{j(\theta-2\beta y)}\right| = \frac{(U_i)_{rms}}{R_w}\sqrt{1+\Gamma^2-2\Gamma\cos(2\beta y-\theta)} \tag{9.52b}$$

where $\theta = \sphericalangle\overline{\Gamma}$.

Typical standing wave patterns representative of $U_{rms}(y)$ and $I_{rms}(y)$ are depicted in Figure 9.22.

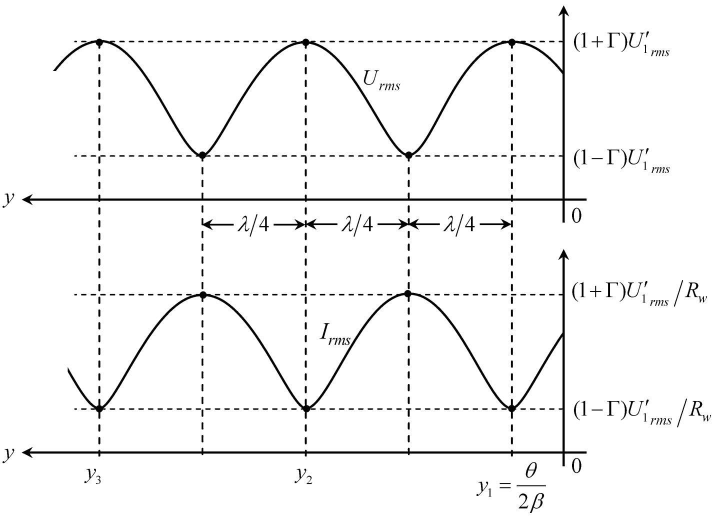

Figure 9.22 Standing wave patterns representative of $U_{rms}(y)$ and $I_{rms}(y)$

The main features of the standing wave patterns are summarized next:

- The patterns are periodic, with period $\lambda/2$.
- Consecutive maxima and minima are separated by a distance equal to $\lambda/4$.
- A point of maximum voltage is a point of minimum current, and vice versa.
- Voltage maxima and minima are given by

$$(U_{rms})_{\max} = (U_i)_{rms} \times (1+\Gamma); \quad (U_{rms})_{\min} = (U_i)_{rms} \times (1-\Gamma) \tag{9.53}$$

- Current maxima and minima are given by

$$(I_{rms})_{\max} = \frac{(U_i)_{rms}}{R_w} \times (1+\Gamma); \quad (I_{rms})_{\min} = \frac{(U_i)_{rms}}{R_w} \times (1-\Gamma) \tag{9.54}$$

- The abscissas y_k where the line voltage is at a maximum are given by

$$y_k = \frac{\theta}{2\beta} + k\frac{\lambda}{2}, \text{ with } k = 0, 1, 2, \ldots \tag{9.55}$$

- From (9.53) and (9.54) you have

$$\frac{(U_{rms})_{\max}}{(I_{rms})_{\max}} = \frac{(U_{rms})_{\min}}{(I_{rms})_{\min}} = R_w \tag{9.56}$$

- The voltage and current standing wave patterns are essentially identical. They just differ in two details: a translation of $\lambda/4$ in the horizontal axis, and a scale factor of R_w in the vertical axis.

In order to quantify the magnitude of the fluctuations observed in the standing wave patterns, one usually defines the so-called standing wave ratio, SWR, a dimensionless parameter that can vary from 1 to ∞:

$$\mathrm{SWR} = \frac{(U_{rms})_{\max}}{(U_{rms})_{\min}} = \frac{(I_{rms})_{\max}}{(I_{rms})_{\min}} = \frac{1+\Gamma}{1-\Gamma} \tag{9.57}$$

For matched loads, where $\Gamma = 0$, you have an SWR of 1 (fluctuations do not exist). For loads where total reflection occurs, $\Gamma = 1$, you have an SWR of ∞ (fluctuations are as large as possible, and $(U_{rms})_{\min} = (I_{rms})_{\min} = 0$).

Note especially that, in (9.57), the absolute value of $\overline{\Gamma}$ is used, not the complex $\overline{\Gamma}$ itself.

The active power delivered by a lossless line is constant along y because, by definition, loss mechanisms along the line are absent. You may remember that the active power was defined in Chapter 7 through $P = (p(t))_{av} = U_{rms}I_{rms}\cos\varphi$. However, in a mismatched line, the quantities U_{rms}, I_{rms} and φ – all of them – vary with y. So, the obvious question is how can you evaluate P in as simple a manner as possible?

The key for this question can be found in (9.40).

Whenever $\overline{V}_U(y)$ and $\overline{V}_I(y)$ happen to be parallel vectors you have $\varphi = 0$. But, from Figure 9.21, you will see that such a situation corresponds exactly to extremal points in the standing wave patterns. Therefore, the answer you are looking for is

$$P = (U_{rms})_{\max}\,(I_{rms})_{\min} = (U_{rms})_{\min}\,(I_{rms})_{\max} = (1-\Gamma^2)\frac{(U_i)^2_{rms}}{R_w}$$

9.5.4 *The Low-Frequency Limit Case, Short Lines*

The transmission-line results developed in the foregoing sections should agree with the familiar results from circuit analysis when low frequencies are considered. To confirm that this statement is true, let us examine two illustrative situations.

For an open-line situation ($\overline{Z}_L = \infty$) we saw in (9.47) that the input impedance observed at the generator terminals was given by $\overline{Z}_{in} = -jR_w/\tan(\beta l)$. For low-frequency regimes we have $\beta l \ll 1$, and consequently $\tan(\beta l) \simeq \beta l$. Therefore, taking into account the definitions of R_w and β, we find

$$\overline{Z}_{in} = \frac{-jR_w}{\tan(\beta l)} \simeq \frac{-jR_w}{\beta l} = -j\frac{\sqrt{L_e/C}}{\omega\sqrt{L_eC}l} = \frac{1}{j\omega(Cl)}$$

from which you can see that the input impedance of an insulated two-conductor system of length l immersed in a dielectric medium is nothing but the impedance of a capacitor whose capacitance is Cl, as expected from a circuit theory approach.

Likewise, for a short-circuited line ($\overline{Z}_L = 0$) we saw in (9.50) that the input impedance observed at the generator terminals was given by $\overline{Z}_{in} = jR_w\tan(\beta l)$. For low-frequency regimes we have $\beta l \ll 1$, and $\tan(\beta l) \simeq \beta l$. Therefore, taking into account the definitions of R_w and β, we find

$$\overline{Z}_{in} = jR_w\tan(\beta l) \simeq jR_w\,\beta l = j\sqrt{\frac{L_e}{C}}\,\omega\sqrt{L_eC}l = j\omega(L_el)$$

from which you can see that the input impedance of a two-conductor system of length l short-circuited at one end is nothing but the impedance of a rectangular single-loop inductor whose inductance is $L_e l$, as expected from a circuit theory approach.

9.6 Application Example (Line-Matching Techniques)

The problem of mismatched loads can be handled by using several techniques. Here we address two commonly used solutions, the quarter-wave transformer and the single matching stub.

Consider an aerial lossless transmission line characterized by $R_w = 150\ \Omega$. The line, of length $l = 3\,\text{m}$, is terminated by a resistive load $R_L = 50\,\Omega$. The line is driven by an ideal generator whose voltage is $u_G(t) = \sqrt{2}\,(U_G)_{rms}\cos(\omega t)$, where $(U_G)_{rms} = 200\,\text{V}$, $f = 100\,\text{MHz}$.

Questions

Q$_1$ Determine the following parameters: v, λ, $\overline{\Gamma}$ and SWR. Determine the complex amplitude $\overline{U}_i'$ of the incident wave voltage at the load terminals. Obtain the load voltage $\overline{U}_L$. Draw the standing wave pattern of the rms voltage along the line. Find the active power delivered to the load.

Q$_2$ Consider that the aerial line of length l is made of two distinct sections (Figure 9.23), the first of length $l - d$ with $R_w = 150\,\Omega$, and the second of length $d = \lambda/4$ with characteristic resistance R_{w_0} (this second section is called a quarter-wave transformer).

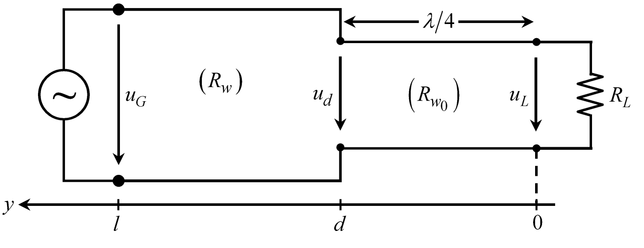

Figure 9.23 Line-matching technique using a quarter-wave line transformer

The purpose of the quarter-wave transformer is to ensure that at the transverse plane $y = d$ the line impedance is equal to R_w, therefore ensuring that the first line section is perfectly matched.

Regarding the second line section, determine d, R_{w_0}, $\overline{\Gamma}_0$ and SWR$_0$. Obtain $\overline{U}_d$ and $\overline{U}_L$. Draw the standing wave pattern of the rms voltage along the line. Find the active power delivered to the load.

Q$_3$ As shown in Figure 9.24, the aerial line of length l includes a stub positioned at $y = d$. The stub consists of a short-circuited line section of adjustable length h (with the same characteristics of the original line). Parameters d and h are the unknowns of the problem. The minimal distance d is chosen so that the line admittance to the right of $y = d$ is of the type $\overline{Y}_{y=d^+} = R_w^{-1} + jS$. The minimal distance h is chosen so that the input admittance of the auxiliary stub is $\overline{Y}_{st} = -jS$. The parallel association of these two admittances results in an equivalent impedance $\overline{Z}_{y=d} = (\overline{Y}_{y=d^+} + \overline{Y}_{st})^{-1}$ equal to R_w, thus ensuring that the line section fed by the generator is perfectly matched.

Determine d, S and h. Obtain $\overline{U}_d$ and $\overline{U}_L$. Draw the standing wave pattern of the rms voltage along the line. Find the active power delivered to the load.

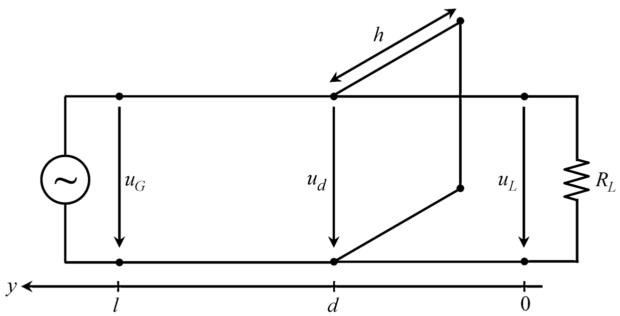

Figure 9.24 Line-matching technique using a parallel-connected stub

Solutions

Q$_1$ $v = 1/\sqrt{\mu_0 \varepsilon_0} = 3 \times 10^8\,\text{m/s}$; $\lambda = v/f = 3\,\text{m}$;

$$\overline{\Gamma} = \frac{R_L - R_w}{R_L + R_w} = 0.5\, e^{j\pi}; \quad \text{SWR} = \frac{1+\Gamma}{1-\Gamma} = 3$$

From $\overline{U}_G = \overline{U}_i' e^{j\beta l}(1 + \overline{\Gamma} e^{-j2\beta l})$, noting that $\beta l = 2\pi$, we get

$$\overline{U}_i' = \frac{\overline{U}_G}{1 + \overline{\Gamma}} = \sqrt{2}\; 400\,\text{V}$$

From $\overline{U}_L = \overline{U}_i'(1 + \overline{\Gamma})$, we get $\overline{U}_L = \overline{U}_G$ (note that the line is one wavelength long).
From $(U_{rms})_{\max} = (U_i')_{rms}\,(1 + \Gamma)$, we get $(U_{rms})_{\max} = 600\,\text{V}$.
From $(U_{rms})_{\min} = (U_i')_{rms}\,(1 - \Gamma)$, we get $(U_{rms})_{\min} = 200\,\text{V}$.
From $P = (1 - \Gamma^2)\,(U_i')^2_{rms}/R_w$, we get $P = 800\,\text{W}$.
See the sketch of the voltage standing wave pattern in Figure 9.25(a).

Q$_2$ $d = \lambda/4 = 75\,\text{cm}$. Taking into account that for a quarter-wave line $\beta d = \pi/2$, we can write, from (9.39),

$$\overline{U}_d = j\overline{U}_i'(1 - \overline{\Gamma}_0) \quad \text{and} \quad \overline{I}_d = j\overline{U}_i'(1 + \overline{\Gamma}_0)/R_{w_0}$$

Dividing both equations and enforcing $\overline{U}_d/\overline{I}_d = R_w$, we obtain

$$R_w = R_{w_0} \frac{1 - \overline{\Gamma}_0}{1 + \overline{\Gamma}_0}$$

Making use of

$$\overline{\Gamma}_0 = \frac{R_L - R_{w_0}}{R_L + R_{w_0}}$$

we finally conclude

$$R_w = R_{w_0} \frac{R_{w_0}}{R_L} \quad \rightarrow \quad R_{w_0} = \sqrt{R_L R_w} = 86.6\ \Omega$$

from which we evaluate

$$\overline{\Gamma}_0 = 0.268\ e^{j\pi} \text{ and } \mathrm{SWR}_0 = \frac{1+\Gamma_0}{1-\Gamma_0} = 1.73$$

Since the first line section is matched, the voltage $\overline{U}_d$ is equal to $\overline{U}_G$ apart from a phase delay $\beta(l-d) = 3\pi/2$, that is

$$\overline{U}_d = \sqrt{2}\,200\,e^{-j3\pi/2} = j\sqrt{2}\,200\,\mathrm{V}.$$

Knowledge of $\overline{U}_d$ permits the determination of

$$\overline{U}_i' = \frac{\overline{U}_d}{j(1-\overline{\Gamma}_0)} = \sqrt{2}\ 157.73\ \mathrm{V}$$

which finally allows for the evaluation of the load voltage,

$$\overline{U}_L = \overline{U}_i'(1+\overline{\Gamma}_0) = \sqrt{2}\,115.47\,\mathrm{V}.$$

You may note that

$$\frac{(U_d)_{rms}}{(U_L)_{rms}} = \mathrm{SWR}_0$$

The active power can be determined via

$$P = \frac{(U_G)^2_{rms}}{R_w} \text{ or } P = \frac{(U_L)^2_{rms}}{R_L}$$

yielding $P = 266.7\,\mathrm{W}$.

See the sketch of the voltage standing wave pattern in Figure 9.25(b).

Q_3 The input admittance of the end section of length d is obtained from (9.43) as

$$\overline{Y}_{y=d^+} = R_w^{-1} \frac{1-\overline{\Gamma}\,e^{-j2\beta d}}{1+\overline{\Gamma}\,e^{-j2\beta d}}$$

Taking into account, from Q_1, that $\overline{\Gamma} = -0.5$, we can write

$$R_w^{-1} + jS = \overline{Y}_{y=d^+} = R_w^{-1} \frac{2+e^{-jx}}{2-e^{-jx}}, \text{ where } x = 2\beta d = \frac{4\pi d}{\lambda}$$

After some algebraic manipulation we find

$$1 + jSR_w = \frac{3 - j4\sin x}{5 - 4\cos x} \rightarrow \begin{cases} 1 = \dfrac{3}{5 - 4\cos x} \\ SR_w = \dfrac{-4\sin x}{5 - 4\cos x} \end{cases} \rightarrow \begin{cases} x = \pi/3 \\ SR_w = -1.155 \end{cases}$$

In conclusion, $d = \lambda/12 = 25\,\text{cm}$ and $S = -7.7\,\text{mS}$.

The input admittance of the stub is determined from (9.50),

$$\overline{Y}_{st} = -jS = \frac{-j}{R_w \tan(\beta h)}$$

from which we get $\tan(\beta h) = (SR_w)^{-1} = -0.866$, yielding $h = 116\,\text{cm}$ (in your calculations do not forget that the angle βh is expressed in radians, not degrees!).

Since the first line section is matched, the voltage $\overline{U}_d$ is equal to $\overline{U}_G$ apart from a phase delay $\beta(l - d) = 11\pi/6$, that is $\overline{U}_d = \sqrt{2}\,200\,e^{+j\pi/6}\,\text{V}$.

For the end line section to the right of the stub insertion we utilize

$$\overline{U}(y) = \overline{U}'_i e^{+j\beta y}\left(1 + \overline{\Gamma} e^{-j2\beta y}\right)$$

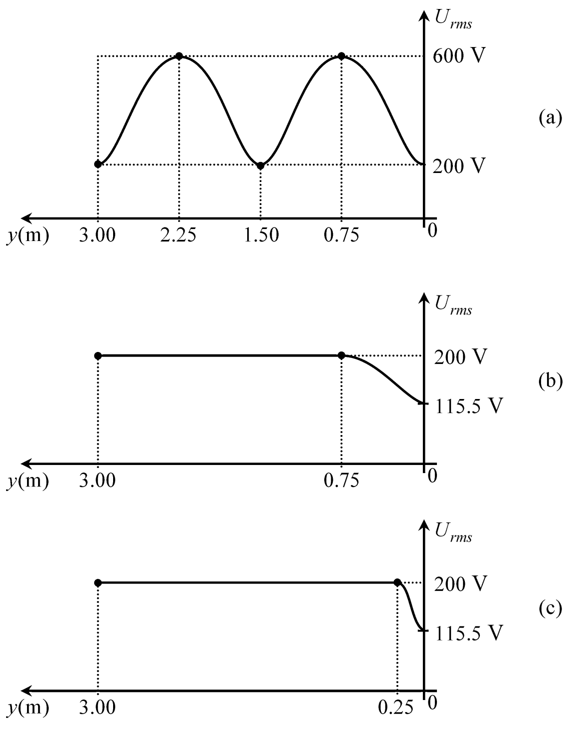

Figure 9.25 Standing wave voltage patterns. (a) Without the use of mitigation matching techniques. (b) Using a quarter-wave line transformer. (c) Using a parallel-connected stub

with the boundary condition $\overline{U}_{y=d} = \sqrt{2}\,200\,e^{+j\pi/6}\,\mathrm{V}$, from which we find

$$\overline{U}_i' = \frac{\sqrt{2}\,200\,e^{j\pi/6}}{e^{j\pi/6}\,(1-0.5\;e^{-j\pi/3})} = \sqrt{2}\;230.95\;e^{-j\pi/6}\;\mathrm{V}$$

Finally, at the load terminals, we obtain $\overline{U}_L = \overline{U}_i'\left(1+\overline{\Gamma}\right) = \sqrt{2}\,115.47\;e^{-j\pi/6}\,\mathrm{V}$.

The active power can be determined via

$$P = \frac{(U_G)^2_{rms}}{R_w} \text{ or } P = \frac{(U_L)^2_{rms}}{R_L}$$

yielding $P = 266.7\,\mathrm{W}$.

See the sketch of the voltage standing wave pattern in Figure 9.25(c).

9.7 Multiconductor Transmission Lines

In the foregoing sections we paid attention to the analysis of two-conductor transmission lines; however, many transmission-line structures include more than just two conductors. As a subject of the utmost interest to electrical engineers, multiconductor transmission lines can be found almost everywhere, with applications in overhead power lines, railway systems, printed circuit boards, flatpack or ribbon cables for electronic systems interconnection, as well as several microwave structures, to name just a few.

In this section we address the topic of multiconductor transmission lines (MTLs) at an introductory level.

Consider a uniform transmission-line system made of $N+1$ conductors parallel to the longitudinal z axis, where conductor (0) is taken as the reference conductor – see Figure 9.26.

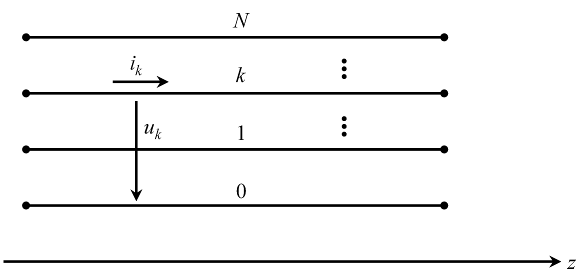

Figure 9.26 Multiconductor transmission-line structure with $N+1$ coupled conductors

For $N > 1$, the generalization of the frequency-domain two-conductor line equations in (9.24) and (9.25) take the matrix form

$$\frac{d}{dz}\left[\overline{U}(z)\right] = -\left[\overline{Z}_l\right]\left[\overline{I}(z)\right];\quad \frac{d}{dz}\left[\overline{I}(z)\right] = -\left[\overline{Y}_t\right]\left[\overline{U}(z)\right] \tag{9.58}$$

where $\left[\overline{U}(z)\right]$ and $\left[\overline{I}(z)\right]$ are complex column vectors, of size N, gathering the complex amplitudes of the line voltages and line currents

$$\left[\overline{U}(z)\right] = \begin{bmatrix} \overline{U}_1(z) \\ \overline{U}_2(z) \\ \vdots \\ \overline{U}_N(z) \end{bmatrix}; \quad \left[\overline{I}(z)\right] = \begin{bmatrix} \overline{I}_1(z) \\ \overline{I}_2(z) \\ \vdots \\ \overline{I}_N(z) \end{bmatrix}$$

and $\left[\overline{Z}_l\right]$ and $\left[\overline{Y}_t\right]$ respectively denote the longitudinal impedance and transverse admittance matrices per unit length of the line.

The coupled differential equations in (9.58) can be decoupled by introducing an appropriate change of variables

$$\left[\overline{U}\right] = [T]\left[\overline{U}_m\right], \quad \left[\overline{I}\right] = [W]\left[\overline{I}_m\right] \tag{9.59}$$

where $\left[\overline{U}_m\right]$ and $\left[\overline{I}_m\right]$ are arrays for the so-called modal voltages and modal currents, respectively. Substituting (9.59) into (9.58), we find

$$\frac{d}{dz}\left[\overline{U}_m(z)\right] = -\underbrace{\left([T]^{-1}\left[\overline{Z}_l\right][W]\right)}_{\left[\overline{Z}_m\right]}\left[\overline{I}_m(z)\right] \tag{9.60a}$$

$$\frac{d}{dz}\left[\overline{I}_m(z)\right] = -\underbrace{\left([W]^{-1}\left[\overline{Y}_t\right][T]\right)}_{\left[\overline{Y}_m\right]}\left[\overline{U}_m(z)\right] \tag{9.60b}$$

where $\left[\overline{Z}_m\right]$ and $\left[\overline{Y}_m\right]$ are the per-unit-length longitudinal impedance and transverse admittance matrices in modal coordinates, respectively.

In order to ensure that the equations in (9.60) define a decoupled set, both $\left[\overline{Z}_m\right]$ and $\left[\overline{\mathrm{Y}}_m\right]$ must be diagonal matrices, that is

$$\frac{d}{dz}\begin{bmatrix} \overline{U}_{1_m} \\ \vdots \\ \overline{U}_{k_m} \\ \vdots \\ \overline{U}_{N_m} \end{bmatrix} = -\begin{bmatrix} \overline{Z}_{1_m} & & & & \\ & \ddots & & & \\ & & \overline{Z}_{k_m} & & \\ & & & \ddots & \\ & & & & \overline{Z}_{N_m} \end{bmatrix}\begin{bmatrix} \overline{I}_{1_m} \\ \vdots \\ \overline{I}_{k_m} \\ \vdots \\ \overline{I}_{N_m} \end{bmatrix} \tag{9.61a}$$

$$\frac{d}{dz}\begin{bmatrix} \overline{I}_{1_m} \\ \vdots \\ \overline{I}_{k_m} \\ \vdots \\ \overline{I}_{N_m} \end{bmatrix} = -\begin{bmatrix} \overline{Y}_{1_m} & & & & \\ & \ddots & & & \\ & & \overline{Y}_{k_m} & & \\ & & & \ddots & \\ & & & & \overline{Y}_{N_m} \end{bmatrix}\begin{bmatrix} \overline{U}_{1_m} \\ \vdots \\ \overline{U}_{k_m} \\ \vdots \\ \overline{U}_{N_m} \end{bmatrix} \tag{9.61b}$$

or, which is the same,

$$\text{For } k=1 \text{ to } N: \begin{cases} \dfrac{d}{dz}\overline{U}_{k_m} = -\overline{Z}_{k_m}\ \overline{I}_{k_m} \\ \dfrac{d}{dz}\overline{I}_{k_m} = -\overline{Y}_{k_m}\ \overline{U}_{k_m} \end{cases} \tag{9.61c}$$

The equations in (9.61c) are the familiar two-conductor transmission-line equations, whose solution we have already established in (9.28) and (9.30)

$$\overline{U}_{k_m}(z) = \left(\overline{U}_i\right)_k e^{-\overline{\gamma}_k z} + \left(\overline{U}_r\right)_k e^{+\overline{\gamma}_k z}; \quad \overline{I}_{k_m}(z) = \frac{\left(\overline{U}_i\right)_k e^{-\overline{\gamma}_k z} - \left(\overline{U}_r\right)_k e^{+\overline{\gamma}_k z}}{\overline{Z}_{w_k}}$$

where $\overline{\gamma}_k = \sqrt{\overline{Z}_{k_m}\overline{Y}_{k_m}}$ and $\overline{Z}_{w_k} = \sqrt{\overline{Z}_{k_m}/\overline{Y}_{k_m}}$ respectively denote the propagation constant, and the characteristic wave impedance for the kth mode.

Once the modal voltages and currents $\left[\overline{U}_m\right]$ and $\left[\overline{I}_m\right]$ have been found, we reuse (9.59) to determine the natural voltages and currents in the multiconductor line.

So far, the procedure described above is straightforward. However, we still have to deal with the problem of the determination of the transformation matrices $[T]$ and $[W]$ – a major point in this formalism.

Taking into account the definition of the diagonal matrices $\left[\overline{Z}_m\right]$ and $\left[\overline{Y}_m\right]$ in (9.60), and bearing in mind that diagonal matrices always commute, we find

$$\left[\overline{Z}_m\right]\left[\overline{Y}_m\right] = \left[\overline{\gamma}^2\right] = [T]^{-1}\left[\overline{Z}_l\overline{Y}_t\right][T] \tag{9.62a}$$

$$\left[\overline{Y}_m\right]\left[\overline{Z}_m\right] = \left[\overline{\gamma}^2\right] = [W]^{-1}\left[\overline{Y}_t\overline{Z}_l\right][W] \tag{9.62b}$$

where $\left[\overline{\gamma}^2\right]$ is a diagonal matrix that gathers the squared propagation constants of the N independent propagation modes.

From (9.62a) you can see that the transformation matrix $[T]$ is determined by solving an eigenvalue/eigenvector problem concerning the $\left[\overline{Z}_l\overline{Y}_t\right]$ matrix product; that is,

$$\left[\overline{Z}_l\overline{Y}_t\right]\ [T] = [T]\ \left[\overline{\gamma}^2\right] \quad \text{or} \quad \left(\left[\overline{Z}_l\ \overline{Y}_t\right] - \overline{\gamma}_k^2[1]\right)\ [t_k] = 0 \tag{9.63}$$

where $[t_k]$, the column k of $[T]$, represents the kth eigenvector of $\left[\overline{Z}_l\ \overline{Y}_t\right]$ associated to the eigenvalue $\overline{\gamma}_k^2$. Matrix $[1]$ is the identity matrix.

The numerical task involved in solving (9.63) is a heavy one, particularly if N is large. Fortunately, standard software packages exist for finding the eigenvalues and eigenvectors of general complex matrices. Note that the procedure in (9.63) allows you not only you to find $[T]$ but also to obtain the modal propagation constants.

For the determination of the transformation matrix $[W]$ we may adopt diverse strategies. For example, we may start by arbitrarily defining $\left[\overline{Z}_m\right]$ as a non-singular diagonal matrix. Then we determine $\left[\overline{Y}_m\right]$ by using $\left[\overline{Y}_m\right] = \left[\overline{Z}_m\right]^{-1}\left[\overline{\gamma}^2\right]$. Finally, from (9.60), we obtain, indifferently,

$$[W] = \begin{cases} \left[\overline{Z}_l\right]^{-1}[T]\left[\overline{Z}_m\right] \\ \left[\overline{Y}_t\right][T]\left[\overline{Y}_m\right]^{-1} \end{cases} \tag{9.64}$$

Alternatively, let us take the transpose of the matrix equation in (9.62b):

$$\left[\overline{Z}_m\right]\left[\overline{Y}_m\right] = \left[\overline{\gamma}^2\right] = [W]^T\left[\overline{Z}_l\overline{Y}_t\right][W]^{-1T} \tag{9.65}$$

where account is taken of the symmetry of $\left[\overline{Z}_l\right]$ and $\left[\overline{Y}_t\right]$. The comparison between (9.65) and (9.62a) indicates one possible choice:

$$[W]^T = [T]^{-1} \tag{9.66}$$

Final remarks are as follows:

- For lossy lines, extraordinary cases exist where the problem posed in (9.63) of transforming $\left[\overline{Z}_l\,\overline{Y}_t\right]$ into diagonal form cannot be solved at all. However, for typical MTL configurations, those abnormal situations are extremely rare.
- The solution of (9.63) does not uniquely define $[T]$. If you find one solution for $[T]$ and multiply it on the right by an arbitrary non-singular diagonal matrix, the resulting new matrix will also be a solution of (9.63).
- The degrees of freedom contained in the construction of $[T]$ and $[W]$ imply some sort of arbitrariness in the definition of $\left[\overline{Z}_m\right]$, $\left[\overline{Y}_m\right]$ and $\left[\overline{Z}_{w_m}\right]$; this remark should put you on your guard as far as the physical significance of those matrices is concerned.
- For lossy lines, the transformation $[T]$ is in general a complex frequency-dependent matrix.
- For lossless lines, where $\left[\overline{Z}_l\right] = j\omega[L]$ and $\left[\overline{Y}_t\right] = j\omega[C]$, a real frequency-independent transformation matrix $[T]$ can always be found.
- For lossless homogeneous lines, where the dielectric medium around the conductors is characterized by ε and μ, the eigenvalues of the product matrix $\left[\overline{Z}_l\,\overline{Y}_t\right]$ are all equal, that is $\overline{\gamma}_k = j\omega\sqrt{\mu\varepsilon}$, for $k = 1$ to N.

9.8 Application Example (Even and Odd Modes)

Consider a lossless inhomogeneous MTL structure consisting of two identical dielectric-coated cylindrical conductors in close contact seated on a reference conducting plane. The structure is operating at $\omega = 1$ Grad/s. The ratio of the radii r/r_0 is 0.8. The insulation coating material is characterized by $\varepsilon = 4\varepsilon_0$, $\mu = \mu_0$. As shown in Figure 9.27, the system under analysis displays a vertical plane of symmetry; the corresponding per-unit-length capacitance and inductance matrices have been evaluated as

$$[C] = \begin{bmatrix} 225.62 & -69.80 \\ -69.80 & 225.62 \end{bmatrix} \text{pF/m}, \quad [L] = \begin{bmatrix} 126.51 & 35.26 \\ 35.26 & 126.51 \end{bmatrix} \text{nH/m}$$

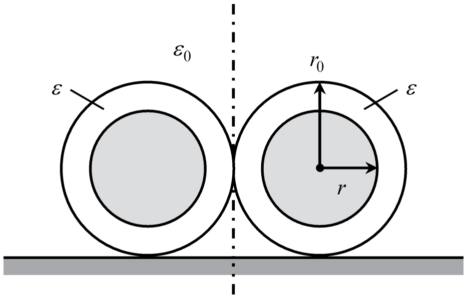

Figure 9.27 Cross-sectional view of an MTL structure consisting of two identical dielectric-coated cylindrical conductors in close contact seated on a reference ground plane

Questions

Q_1 Determine the matrix products $[\overline{Z}_l \overline{Y}_t]$ and $[\overline{Y}_t \overline{Z}_l]$.

Q_2 Determine the eigenvectors (transformation matrices) and eigenvalues of both matrices. Evaluate the phase velocities of the two propagation modes.

Q_3 Determine the modal matrices $[\overline{Z}_m]$ and $[\overline{Y}_m]$, and then evaluate the characteristic wave impedances of the two propagation modes.

Q_4 Assume that the receiving end of the structure is terminated in a matched load. Determine the solution for the complex amplitudes of the line voltages and currents along the line.

Q_5 Determine the characteristic impedance matrix describing the matched load.

Solutions

Q_1

$$[\overline{Z}_l \overline{Y}_t] = [\overline{Y}_t \overline{Z}_l] = (j\omega)^2 [L][C] = (j\omega)^2 [C][L] = \begin{bmatrix} -26.08 & 0.875 \\ 0.875 & -26.08 \end{bmatrix} \text{ m}^{-2}$$

Q_2

$$[T] = [W] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The eigenvector

$$[t_1] = [w_1] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

defines the excitation of mode 1, the so-called even mode (or common mode), where $\overline{U}_1 = \overline{U}_2$ and $\overline{I}_1 = \overline{I}_2$.

The eigenvector

$$[t_2] = [w_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

defines the excitation of mode 2, the so-called odd mode (or differential mode), where $\overline{U}_1 = -\overline{U}_2$ and $\overline{I}_1 = -\overline{I}_2$.

Regarding the eigenvalues, we find

$$\left[\overline{\gamma}^2\right] = \begin{bmatrix} -25.21 & 0 \\ 0 & -26.96 \end{bmatrix} \text{ m}^{-2}$$

From $\overline{\gamma}_k = j\beta_k$ we obtain $\beta_1 = 5.021\,\text{rad/m}$ and $\beta_2 = 5.192\,\text{rad/m}$.
From $\beta_k = \omega/v_k$ we obtain $v_1 = 1.992 \times 10^8\,\text{m/s}$ and $v_2 = 1.926 \times 10^8\,\text{m/s}$.

Note that the normalization factor $1/\sqrt{2}$ which appears in both $[T]$ and $[W]$ is arbitrary; it has been chosen with the single purpose of getting unit norm eigenvectors.

Q_3

$$\left[\overline{Z}_m\right] = j\omega\,[T]^{-1}\,[L]\,[W] = \begin{bmatrix} 161.8 & 0 \\ 0 & 91.25 \end{bmatrix} \Omega/\text{m}$$

$$\left[\overline{Y}_m\right] = j\omega\,[W]^{-1}\,[C]\,[T] = \begin{bmatrix} 155.8 & 0 \\ 0 & 295.4 \end{bmatrix} \text{ mS/m}$$

From $\overline{Z}_{w_k} = \sqrt{\overline{Z}_{k_m}/\overline{Y}_{k_m}}$ we obtain $\overline{Z}_{w_1} = R_{w_1} = 32.22\,\Omega$ and $\overline{Z}_{w_2} = R_{w_2} = 17.58\,\Omega$.

Q_4 Voltage solution:

$$\left[\overline{U}\right] = [T]\begin{bmatrix} \left(\overline{U}_i\right)_1 e^{-j\beta_1 z} \\ \left(\overline{U}_i\right)_2 e^{-j\beta_2 z} \end{bmatrix} \rightarrow \begin{cases} \overline{U}_1(z) = \frac{1}{\sqrt{2}}\left(\left(\overline{U}_i\right)_1 e^{-j\beta_1 z} + \left(\overline{U}_i\right)_2 e^{-j\beta_2 z}\right) \\ \overline{U}_2(z) = \frac{1}{\sqrt{2}}\left(\left(\overline{U}_i\right)_1 e^{-j\beta_1 z} - \left(\overline{U}_i\right)_2 e^{-j\beta_2 z}\right) \end{cases}$$

Current solution:

$$\left[\overline{I}\right] = [W]\begin{bmatrix} \dfrac{\left(\overline{U}_i\right)_1}{R_{w_1}} e^{-j\beta_1 z} \\ \dfrac{\left(\overline{U}_i\right)_2}{R_{w_2}} e^{-j\beta_2 z} \end{bmatrix} \rightarrow \begin{cases} \overline{I}_1(z) = \dfrac{1}{\sqrt{2}}\left(\dfrac{\left(\overline{U}_i\right)_1}{R_{w_1}} e^{-j\beta_1 z} + \dfrac{\left(\overline{U}_i\right)_2}{R_{w_2}} e^{-j\beta_2 z}\right) \\ \overline{I}_2(z) = \dfrac{1}{\sqrt{2}}\left(\dfrac{\left(\overline{U}_i\right)_1}{R_{w_1}} e^{-j\beta_1 z} - \dfrac{\left(\overline{U}_i\right)_2}{R_{w_2}} e^{-j\beta_2 z}\right) \end{cases}$$

Q_5 When a transmission-line structure is terminated in a matched load (as we have assumed in Q_4), the relationship between $\left[\overline{U}(z)\right]$ and $\left[\overline{I}(z)\right]$ is independent of z. Therefore, we can use the results derived in Q_4, particularized to $z = 0$, to get

$$\begin{cases} \sqrt{2}\,\overline{U}_1(0) = \left(\overline{U}_i\right)_1 + \left(\overline{U}_i\right)_2 \\ \sqrt{2}\,\overline{U}_2(0) = \left(\overline{U}_i\right)_1 - \left(\overline{U}_i\right)_2 \end{cases} \quad \text{and} \quad \begin{cases} \sqrt{2}\,\overline{I}_1(0) = \left(\overline{U}_i\right)_1/R_{w_1} + \left(\overline{U}_i\right)_2/R_{w_2} \\ \sqrt{2}\,\overline{I}_2(0) = \left(\overline{U}_i\right)_1/R_{w_1} - \left(\overline{U}_i\right)_2/R_{w_2} \end{cases}$$

Eliminating $\left(\overline{U}_i\right)_1$ and $\left(\overline{U}_i\right)_2$, we find

$$\begin{bmatrix}\overline{U}_1\\ \overline{U}_2\end{bmatrix} = \left[\overline{Z}_w\right]\begin{bmatrix}\overline{I}_1\\ \overline{I}_2\end{bmatrix}$$

where

$$\left[\overline{Z}_w\right] = \frac{1}{2}\begin{bmatrix}(R_{w_1}+R_{w_2}) & (R_{w_1}-R_{w_2})\\ (R_{w_1}-R_{w_2}) & (R_{w_1}+R_{w_2})\end{bmatrix} = \begin{bmatrix}24.90 & 7.32\\ 7.32 & 24.90\end{bmatrix}\,\Omega$$

You can check that this impedance matrix corresponds to the line termination shown in Figure 9.28.

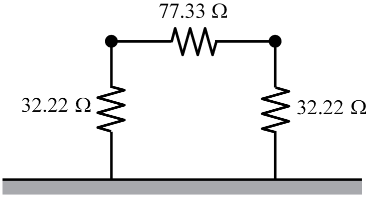

Figure 9.28 Matched termination for the MTL structure depicted in Figure 9.27

9.9 Proposed Homework Problems

Problem 9.9.1

Consider a lossless transmission line, 300 m long, characterized by $v = 3\times10^8$ m/s and $R_w = 60\,\Omega$. As shown in Figure 9.29, the line is excited at $z = 0$ by a step voltage $U_0 = 120$ V. A resistor R terminates the line at $z = l$.

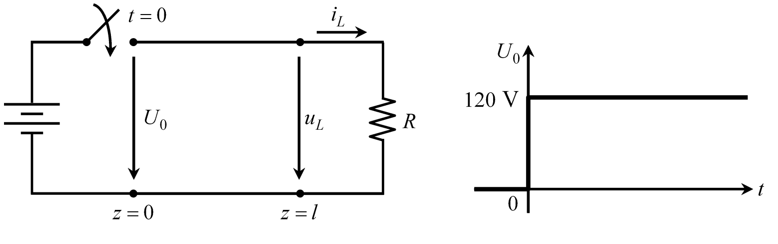

Figure 9.29 A step voltage is applied to the input end of a transmission line loaded by a resistor

Q_1 Determine the per-unit-length L and C parameters of the line. Determine the time delay τ between the two ends of the line.

Q_2 Assuming that $R = 0$, find the evolution of $i_L(t)$.

Q_3 Assuming that $R = \infty$, find the evolution of $u_L(t)$.

Q_4 Assuming that $R = 20\,\Omega$, find the evolution of $u_L(t)$.

Answers

Q_1

$$L = \frac{R_w}{v} = 200.0\,\text{nH/m}; \quad C = \frac{1}{vR_w} = 55.55\,\text{pF/m}; \quad \tau = l/v = 1\,\mu\text{s}$$

Q_2

$$i_L(t) = \begin{cases} 0 & ;\ \text{for } 0 < t < \tau \\ 4\text{ A} & ;\ \text{for } \tau < t < 3\tau \\ 8\text{ A} & ;\ \text{for } 3\tau < t < 5\tau \\ \vdots & \qquad \vdots \\ 2k\text{ A} & ;\ \text{for } (k-1)\tau < t < (k+1)\tau \end{cases}$$

(with k an even number). See Figure 9.30(a).

Q_3

$$u_L(t) = \begin{cases} 0 & ;\ \text{for } 0 < t < \tau \\ 240\text{ V} & ;\ \text{for } \tau < t < 3\tau \\ 0 & ;\ \text{for } 3\tau < t < 5\tau \\ 240\text{ V} & ;\ \text{for } 5\tau < t < 7\tau \\ 0 & ;\ \text{for } 7\tau < t < 9\tau \\ \vdots & \qquad \vdots \end{cases}$$

See Figure 9.30(b).

Q_4

$$u_L(t) = \begin{cases} 0 & ;\ \text{for } 0 < t < \tau \\ 60\text{ V} & ;\ \text{for } \tau < t < 3\tau \\ 90\text{ V} & ;\ \text{for } 3\tau < t < 5\tau \\ 105\text{ V} & ;\ \text{for } 5\tau < t < 7\tau \\ \vdots & \qquad \vdots \\ 120\text{ V} & \qquad t = \infty \end{cases}$$

See Figure 9.30(c).

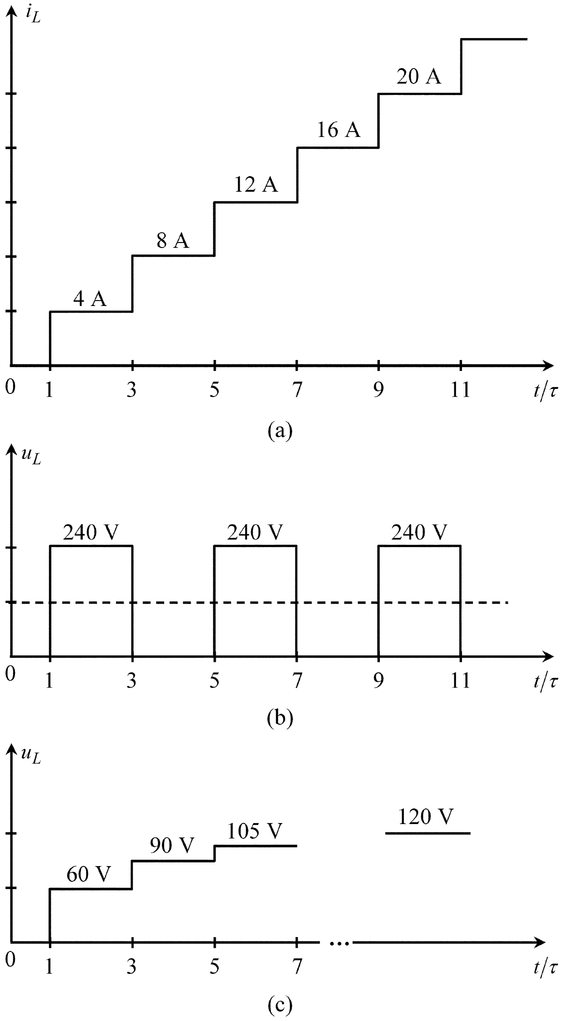

Figure 9.30 Load analysis. (a) Plot of the load current against time for the case $R = 0$. (b) Plot of the load voltage against time for the case $R = \infty$. (b) Plot of the load voltage against time for the case $R = 20\,\Omega$

Problem 9.9.2

Consider a coaxial cable of length $l = 1\,\text{m}$, whose per-unit-length parameters are known, $L_e = 0.25\,\mu\text{H/m}$ and $C = 100\,\text{pF/m}$. The operating frequency is $f = 100\,\text{MHz}$. The dielectric medium is assumed to be lossless, but the line conductors not. Assume that the perturbation arising from skin effect phenomena on the line conductors is such that the corresponding per-unit-length impedance correction is $\overline{Z}_1 + \overline{Z}_2 = \overline{Z}_{cond} = 10\,e^{j40^\circ}\,\Omega/\text{m}$.

Q_1 Determine the per-unit-length longitudinal impedance and transverse admittance of the line, $\overline{Z}_l$ and $\overline{Y}_t$.

Q_2 Determine the phase velocity v and the attenuation constant α.

Q_3 Determine the load impedance $\overline{Z}_L$ necessary for perfect matching. For this case determine the relationship between the active powers at the generator and load terminals; that is, find $\eta = P_L/P_G$.

Answers

Q_1 $\overline{Z}_l = j\omega L_e + \overline{Z}_{cond} = 163.6\, e^{j87.32^\circ}\ \Omega/\text{m}$; $\overline{Y}_t = j\omega C = j62.83\,\text{mS/m}$.

Q_2 $\overline{\gamma} = \alpha + j\beta = \sqrt{\overline{Z}_l \overline{Y}_t} = 3.206\, e^{j88.66^\circ} = 0.075 + j3.205\,\text{m}^{-1}$.

$\alpha = 75\,\text{mNp/m}$; $v = \omega/\beta = 1.96 \times 10^8\,\text{m/s}$.

Q_3

$$\overline{Z}_L = \overline{Z}_w = Z_w\, e^{j\theta_w} = \sqrt{\frac{\overline{Z}_l}{\overline{Y}_t}} = 51.03\, e^{-j1.34^\circ} = 51 - j1.2\,\Omega$$

$$\begin{cases} \overline{U}(z) = \overline{U}_G\, e^{-(\alpha+j\beta)z} \\ \overline{I}(z) = \dfrac{\overline{U}_G}{Z_w\, e^{j\theta_w}}\, e^{-(\alpha+j\beta)z} \end{cases} \rightarrow P(z) = U_{rms}(z)\ I_{rms}(z)\ \cos(\theta_w) = \frac{(U_G)^2_{rms}}{Z_w}\ \cos(\theta_\omega)\, e^{-2\alpha z}$$

$$\eta = \frac{P_L}{P_G} = \frac{P_{z=l}}{P_{z=0}} = e^{-2\alpha l} = 0.86$$

Problem 9.9.3

A transfer matrix representation of a two-conductor lossy line of length l was obtained in (9.35) in Section 9.4.3:

$$\begin{bmatrix} \overline{U}_0 \\ \overline{I}_0 \end{bmatrix} = \underbrace{\begin{bmatrix} \cosh(\overline{\gamma} l) & \sinh(\overline{\gamma} l)\, \overline{Z}_w \\ \overline{Z}_w^{-1} \sinh(\overline{\gamma} l) & \cosh(\overline{\gamma} l) \end{bmatrix}}_{[T]} \begin{bmatrix} \overline{U}_l \\ \overline{I}_l \end{bmatrix}$$

Consider the equivalent ‘T’ circuit shown in Figure 9.31, whose component blocks are two identical impedances $\overline{Z}_x$ and one admittance $\overline{Y}_x$.

Q_1 Determine the transfer matrix of the equivalent circuit.

Q_2 By comparing the result obtained above with the transfer matrix of the line section, determine both $\overline{Z}_x$ and $\overline{Y}_x$.

Q_3 Assume that the transmission line is operated at a low-frequency regime; that is, assume that $l \ll \lambda$. Find the corresponding approximations for $\overline{Z}_x$ and $\overline{Y}_x$, and redraw the equivalent circuit.

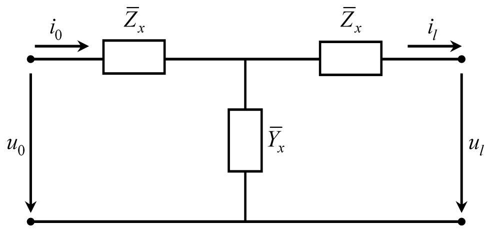

Figure 9.31 An equivalent model, with the shape of a 'T' circuit, aimed at representing a lossy line section of length l

Answers

Q_1

$$[T] = \begin{bmatrix} 1+\overline{Z}_x\overline{Y}_x & \overline{Z}_x(2+\overline{Z}_x\overline{Y}_x) \\ \overline{Y}_x & 1+\overline{Z}_x\overline{Y}_x \end{bmatrix}$$

Q_2

$$\overline{Y}_x = \overline{Z}_w^{-1}\sinh(\overline{\gamma}l); \quad \overline{Z}_x = \overline{Z}_w \tanh\frac{\overline{\gamma}l}{2}$$

Q_3 For $|\overline{\gamma}l| \ll 1$ we have

$$\sinh(\overline{\gamma}l) \simeq \overline{\gamma}l = \sqrt{\overline{Z}_l\overline{Y}_t}\ l \quad \text{and} \quad \tanh\frac{\overline{\gamma}l}{2} \simeq \frac{\overline{\gamma}l}{2} = \sqrt{\overline{Z}_l\overline{Y}_t}\ \frac{l}{2}$$

$$\overline{Y}_x = \overline{Z}_w^{-1}\sinh(\overline{\gamma}l) \simeq \sqrt{\frac{\overline{Y}_t}{\overline{Z}_l}} \times \sqrt{\overline{Z}_l\overline{Y}_t}\ l = \overline{Y}_t l = Gl + j\omega(Cl)$$

$$\overline{Z}_x = \overline{Z}_w \tanh\frac{\overline{\gamma}l}{2} \simeq \sqrt{\frac{\overline{Z}_l}{\overline{Y}_t}} \times \sqrt{\overline{Z}_l\overline{Y}_t}\frac{l}{2} = \frac{\overline{Z}_l l}{2} = \frac{Rl}{2} + j\omega\left(\frac{Ll}{2}\right)$$

The corresponding simplified equivalent circuit is shown in Figure 9.32.

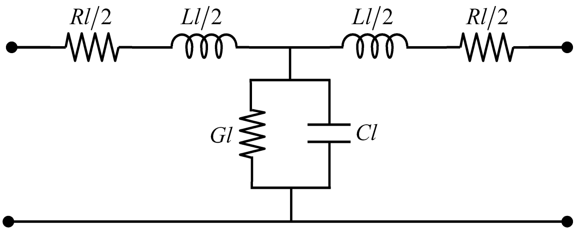

Figure 9.32 Identification of the constituent components of the 'T' circuit in Figure 9.31, for the special case of low-frequency regimes

Problem 9.9.4

A power distribution line of length $l = 100\,\text{km}$ is fed by an AC generator whose voltage is given by $u_G(t) = \sqrt{2}\ (U_G)_{rms} \cos \omega t$, where $(U_G)_{rms} = 20\,\text{kV}$, and $f = 50\,\text{Hz}$.

The per-unit-length parameters of the line are known: $R = 0.15\,\Omega/\text{km}$, $L = 2\,\text{mH/km}$, $G = 0$, $C = 6\,\text{nF/km}$.

Q_1 Determine the propagation constant $\overline{\gamma}$ and the wavelength λ. Check that $l \ll \lambda$.

Q_2 Taking into account the results obtained in Problem 9.9.3, show that $\left|\overline{Y}_x^{-1}\right| \gg \left|\overline{Z}_x\right|$.

Q_3 Assume that the receiving end of the line is short-circuited. Determine $\overline{I}_G$.

Q_4 Assume that the receiving end is left open. Determine $\overline{I}_G$.

Answers

Q_1 $\overline{\gamma} = \alpha + j\beta = \sqrt{(R + j\omega L)(j\omega C)} = (0.129 + j1.096) \times 10^{-6}\,\text{m}^{-1}$.

$\lambda = 2\pi/\beta = 5733\,\text{km} \gg l$.

Q_2 $\overline{Y}_x^{-1} = -j5.3\,\text{k}\Omega$; $\quad \overline{Z}_x = 7.5 + j31.4 = 32.3\, e^{j76.57^\circ}\,\Omega$.

$\left|\overline{Y}_x^{-1}\right| \gg \left|\overline{Z}_x\right|$.

Q_3 By making use of the equivalent 'T' circuit in Figure 9.32 you get

$$\overline{I}_G \approx \frac{\overline{U}_G}{2\overline{Z}_x} = \frac{\overline{U}_G}{(R + j\omega L)\, l} = \sqrt{2}\ 310\, e^{-j76.57^\circ}\ \text{A}$$

Q_4 By making use of the equivalent 'T' circuit in Figure 9.32 you get

$$\overline{I}_G \approx \overline{Y}_x \overline{U}_G = j\omega C l \overline{U}_G = \sqrt{2}\ 3.77\, e^{j90^\circ}\ \text{A}$$

Problem 9.9.5

Consider a homogeneous lossless line, 50 cm long, whose dielectric medium is characterized by $\mu = \mu_0$ and $\varepsilon = 2.25\varepsilon_0$. An ideal generator is positioned at $y = l$, the corresponding voltage being given by $u_G(t) = \sqrt{2}\ (U_G)_{rms} \cos \omega t$, where $(U_G)_{rms} = 60\,\text{V}$, and $f = 0.1\,\text{GHz}$.

The characteristic wave resistance of the line is $R_w = 60\,\Omega$. As shown in Figure 9.33, the line is terminated by a load impedance $\overline{Z}_L$.

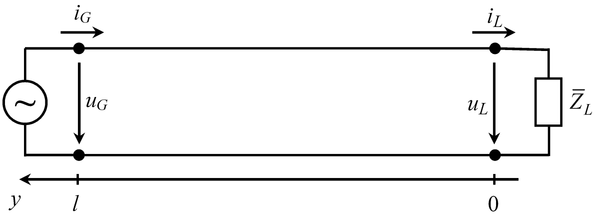

Figure 9.33 A homogeneous uniform lossless line terminated by a load impedance $\overline{Z}_L$

Q_1 Determine the propagation velocity v as well as the associated wavelength λ. Evaluate the per-unit-length L and C parameters.

Q_2 The load reflection coefficient is $\overline{\Gamma} = -1/3$. Determine the load impedance.

Q_3 Find the complex amplitude of the incident wave voltage at the load terminals, and determine the representative phasors for the voltage and current at $y = 0$ and $y = l$.

Q_4 Draw the standing wave patterns of $U_{rms}(y)$ and $I_{rms}(y)$ and, based on them, find SWR and the active power P transmitted along the line.

Q_5 Assume that the dielectric permittivity quadruples its original value. How would the parameters L, C, R_w, v and λ change? How would the new standing wave patterns look? Does the active power change?

Answers

Q_1 $v = 1/\sqrt{\mu_0 \varepsilon} = 2 \times 10^8$ m/s; $\lambda = v/f = 2$ m (note that $l = \lambda/4$).

$$L = \frac{R_w}{v} = 300\,\text{nH/m};\ \ C = \frac{1}{vR_w} = 83.33\,\text{pF/m}$$

Q_2

$$\overline{Z}_L = R_w \frac{1+\overline{\Gamma}}{1-\overline{\Gamma}} = R_L = 30\,\Omega$$

Q_3

$$\begin{cases} \overline{U}_G = \overline{U}_i' \, e^{j\beta l}(1 + \overline{\Gamma}\, e^{-j2\beta l}) \\ \overline{I}_G = \dfrac{\overline{U}_i'}{R_w}\, e^{j\beta l}(1 - \overline{\Gamma}\, e^{-j2\beta l}) \end{cases} \quad \text{where } \overline{\Gamma} = -\frac{1}{3}, \quad \beta l = \frac{2\pi l}{\lambda} = \frac{\pi}{2}, \quad \overline{U}_G = \sqrt{2}\ 60\ \text{V}$$

$$\overline{U}_i' = -j\tfrac{3}{4}\overline{U}_G = -j\sqrt{2}\ 45\ \text{V}, \quad \overline{I}_G = \sqrt{2}\ 0.5\,\text{A}.$$

$$\begin{cases} \overline{U}_L = \overline{U}_i' \ (1 + \overline{\Gamma}) = -j\sqrt{2}\ 30\ \text{V} \\ \overline{I}_L = \dfrac{\overline{U}_i'}{R_w}\ (1 - \overline{\Gamma}) = -j\sqrt{2}\ 1\,\text{A} \end{cases}$$

Q_4 The standing wave patterns are depicted in Figure 9.34.

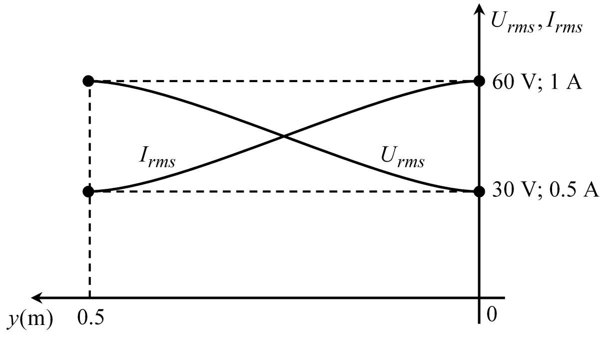

Figure 9.34 Voltage and current standing wave patterns (note that $l = \lambda/4$)

The SWR is 2; $P = (U_G)_{rms}(I_G)_{rms} = (U_L)_{rms}(I_L)_{rms} = 30\,\text{W}$.

Q_5 L remains unaltered. C quadruples.

R_w, v and λ reduce to half of their original values (note that the new $R_w = 30\,\Omega = R_L$). Now the line is perfectly matched. The new standing wave patterns are horizontal lines, $U_{rms}(y) = 60\,\text{V}$ and $I_{rms}(y) = 2\,\text{A}$.

The active power quadruples, $P = 120\,\text{W}$.

Problem 9.9.6

Consider a lossless transmission line of length l subjected to a time-harmonic regime. The line is left open at both ends. The generator that drives the line is positioned at an arbitrary location $y = y_G$ – see Figure 9.35.

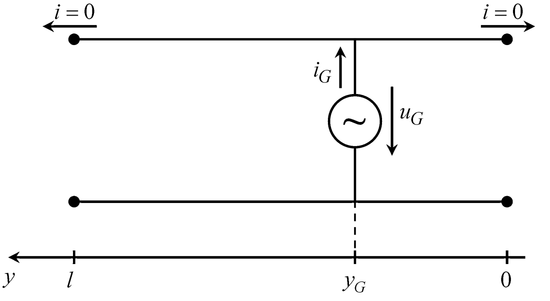

Figure 9.35 A two-conductor transmission line, left open at both ends, is excited by an ideal voltage generator placed at $y = y_G$

Q_1 Determine the complex amplitude of the generator current $\overline{I}_G$ as a function of its position.

Q_2 Consider the particular situation where the line length is an integer multiple of one half-wavelength, $l = n\lambda/2$. Find $\overline{I}_G$.

Answers

Q_1 Input admittance to the right of the generator:

$$\left(\overline{Y}_{in}\right)_R = j\frac{\tan(\beta y_G)}{R_w}$$

Input admittance to the left of the generator:

$$\left(\overline{Y}_{in}\right)_L = j\frac{\tan\left(\beta(l-y_G)\right)}{R_w}$$

$$\overline{I}_G = \overline{U}_G\left(\left(\overline{Y}_{in}\right)_R + \left(\overline{Y}_{in}\right)_L\right) = \frac{j\overline{U}_G}{R_w}\left(\tan(\beta y_G) + \tan(\beta(l-y_G))\right)$$

Q_2 From trigonometry: $\tan(\beta y) + \tan(\beta(l-y)) = \tan(\beta l) \times (1 - \tan(\beta y) \times \tan(\beta(l-y)))$. Therefore

$$\overline{I}_G = \frac{j\overline{U}_G}{R_w}\tan(\beta l) \times (1 - \tan(\beta y_G) \times \tan(\beta(l-y_G)))$$

For $l = n\lambda/2$ you have $\beta l = n\pi$, $\tan(\beta l) = 0$, $\overline{I}_G = 0$.

Problem 9.9.7

Consider a non-uniform transmission-line structure consisting of a chain connection of two uniform lossless coaxial cable sections, as shown in Figure 9.36. The first cable section, of length $l_1 = 10\,\text{m}$, is characterized by $R_{w_1} = 75\,\Omega$ and $v_1 = 2 \times 10^8\,\text{m/s}$. The second cable section, of length $l_2 = 7.5\,\text{m}$, is characterized by $R_{w_2} = 50\,\Omega$ and $v_2 = 1 \times 10^8\,\text{m/s}$. The working frequency is $f = 10\,\text{MHz}$. The load is an antenna that radiates 400 W and whose input impedance is $R_L = 100\,\Omega$.

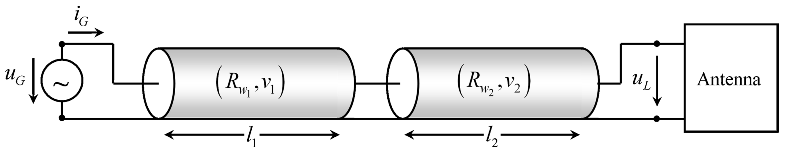

Figure 9.36 An antenna fed by the chain connection of two distinct coaxial cables

Q_1 Determine the transfer matrices describing both cable sections.

Q_2 Determine the transfer matrix of the global non-uniform transmission-line structure.

Q_3 Assuming that $\overline{U}_L = U_L\, e^{j0}$, determine $\overline{U}_G$ and $\overline{I}_G$.

Answers

Q_1

$$\lambda_1 = v_1/f = 20\,\text{m};\quad \beta_1 = 2\pi/\lambda_1 = \pi/10\,\text{rad/m};\quad \beta_1 l_1 = \pi.$$
$$\lambda_2 = v_2/f = 10\,\text{m};\quad \beta_2 = 2\pi/\lambda_2 = \pi/5\,\text{rad/m};\quad \beta_2 l_2 = 3\pi/2.$$

$$[T_1] = \begin{bmatrix} \cos(\beta_1 l_1) & jR_{w_1}\sin(\beta_1 l_1) \\ jR_{w_1}^{-1}\sin(\beta_1 l_1) & \cos(\beta_1 l_1) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} \cos(\beta_2 l_2) & jR_{w_2}\sin(\beta_2 l_2) \\ jR_{w_2}^{-1}\sin(\beta_2 l_2) & \cos(\beta_2 l_2) \end{bmatrix} = \begin{bmatrix} 0 & -j50\ \Omega \\ -j0.02\ \text{S} & 0 \end{bmatrix}$$

Q_2

$$[T] = [T_1][T_2] = \begin{bmatrix} 0 & j50\,\Omega \\ j0.02\,\text{S} & 0 \end{bmatrix}$$

Q_3

$$\overline{U}_L = \sqrt{2}\ 200\,\text{V};\quad \overline{I}_L = \sqrt{2}\ 2\,\text{A}.$$

$$\begin{bmatrix} \overline{U}_G \\ \overline{I}_G \end{bmatrix} = [T]\begin{bmatrix} \overline{U}_L \\ \overline{I}_L \end{bmatrix} = j\sqrt{2}\begin{bmatrix} 100\,\text{V} \\ 4\,\text{A} \end{bmatrix}$$

Problem 9.9.8

Consider the connections shown in Figure 9.37, where a 100 MHz voltage generator simultaneously drives two lossless coaxial cables of length $l = 75\,\text{cm}$. The cables are geometrically identical, but they differ in their dielectric media properties, $\varepsilon_1 = 4\varepsilon_0$ and $\varepsilon_2 = 9\varepsilon_0$, $\mu_1 = \mu_2 = \mu_0$. The generator voltage is $u_G(t) = \sqrt{2}\,(U_G)_{rms}\cos(\omega t)$, with $(U_G)_{rms} = 48\,\text{V}$.

Q_1 For each cable, determine the propagation velocities v_1 and v_2 as well as the corresponding values of the wavelength λ_1 and λ_2.

Q_2 The per-unit-length inductance common to both cables is $L = 400\,\text{nH/m}$. Find the per-unit-length capacitances C_1 and C_2 of the cables, as well as the corresponding characteristic wave resistances R_{w_1} and R_{w_2}.

Q_3 Cable 1 is matched to its load. Cable 2 is short-circuited at $y = 0$. Determine R. Determine the load reflection coefficients $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ for both cables.
Find the complex amplitudes of all of the voltages and currents marked in Figure 9.37.

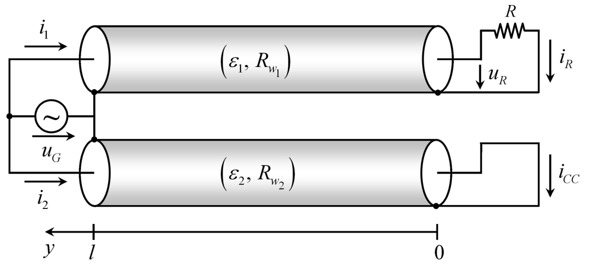

Figure 9.37 A voltage generator drives two lossless coaxial cables of length l. While one of the cables is matched, the other is short-circuited

Answers

Q$_1$ $v_k = c/\sqrt{\varepsilon_r} \to v_1 = 1.5 \times 10^8\,\text{m/s};\ v_2 = 1.0 \times 10^8\,\text{m/s}$.
$\lambda_1 = 1.5\,\text{m};\ \lambda_2 = 1.0\,\text{m}$.

Q$_2$

$$C_k = \frac{1}{v_k L} \to C_1 = 111\,\text{pF/m};\ C_2 = 250\,\text{pF/m}.$$

$$R_{w_k} = \sqrt{\frac{L}{C_k}} \to R_{w_1} = 60\,\Omega; \quad R_{w_2} = 40\,\Omega$$

Q$_3$ $R = R_{w_1} = 60\,\Omega$. $\overline{\Gamma}_1 = 0;\ \overline{\Gamma}_2 = -1$.
$\overline{U}_G = \sqrt{2}\ 48\ e^{j0}\ \text{V};\ \overline{U}_R = \sqrt{2}\ 48\ e^{j\pi}\ \text{V}.\ \overline{I}_1 = \sqrt{2}\ 0.8\ e^{j0}\ \text{A};\ \overline{I}_R = \sqrt{2}\ 0.8\ e^{j\pi}\ \text{A}.$
$\overline{I}_2 = 0;\ \overline{I}_{cc} = \sqrt{2}\ 1.2\ e^{j\pi/2}\ \text{A}.$

Problem 9.9.9

Consider an aerial ($\mu = \mu_0$, $\varepsilon = \varepsilon_0$) lossless transmission line, of length $l = 1\,\text{m}$, subjected to a time-harmonic regime. The generator voltage is $u_G(t) = \sqrt{2}\,(U_G)_{rms}\cos(\omega t)$, with $(U_G)_{rms} = 100\,\text{V}$. The resistors included in the load circuit are characterized by $R = 1\,\text{k}\Omega$. As shown in Figure 9.38, two standing wave patterns were recorded, one with the switch S open, the other with the switch closed.

Q$_1$ Why can you say that the line is matched when S is closed? Determine R_w and v.

Q$_2$ Consider the situation with S open. By analyzing the standing wave pattern, determine the wavelength λ, the operating frequency f and the phase constant β. Evaluate $\overline{\Gamma}$ and SWR, and obtain $(U_{rms})_{\min}$.
Find the complex amplitudes $\overline{U}_G, \overline{I}_G, \overline{U}_L$ and $\overline{I}_L$.
Determine the standing wave pattern concerning the evolution of $I_{rms}(y)$.

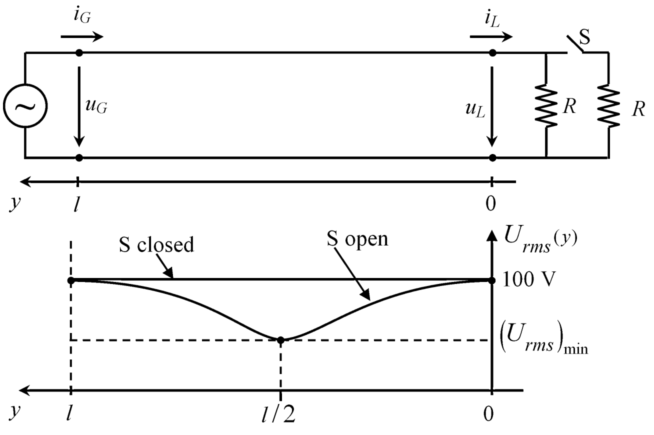

Figure 9.38 A transmission line is terminated by a load configuration that depends on the status of the switch. The corresponding voltage standing wave patterns are shown for S closed and open

Answers

Q_1 With S closed ($R_L = R/2 = 500\,\Omega$) the line is matched because $U_{rms}(y) = \text{constant}$. $R_w = R_L = 500\,\Omega$; $\quad v = 1/\sqrt{\mu_0\varepsilon_0} = 3\times 10^8\,\text{m/s}$.

Q_2 From $\lambda/4 = l/2$, you get $\lambda = 2\,\text{m}$. $f = v/\lambda = 150\,\text{MHz}$. $\beta = 2\pi/\lambda = \pi\,\text{rad/m}$.

$$\overline{\Gamma} = \Gamma = \frac{R - R_w}{R + R_w} = \frac{1}{3};\ \text{SWR} = \frac{1+\Gamma}{1-\Gamma} = 2;\ (U_{rms})_{\min} = \frac{(U_{rms})_{\max}}{\text{SWR}} = 50\,\text{V}$$

$$\overline{U}_G = \sqrt{2}\ 100\,\text{V};\ \overline{U}_L = -\sqrt{2}\ 100\,\text{V};\ \overline{I}_G = \sqrt{2}\ 100\,\text{mA};\ \overline{I}_L = -\sqrt{2}\,100\,\text{mA}.$$

The standing wave pattern concerning the evolution of $I_{rms}(y)$ is characterized by two minima $(I_{rms})_{\min} = 100\,\text{mA}$ at the line ends, $y = 0$ and $y = l$, and by a maximum $(I_{rms})_{\max} = 200\,\text{mA}$ occurring at $y = l/2$.