8

Electromagnetic Field Phenomena

8.1 Electromagnetic Waves

You most certainly have heard about waves in previous physics courses. The simplest example you may be acquainted with is the vibrating string. Assume you have a long stretched string aligned with the x axis. If you grab the free end of the string and shake it up down, you will notice that the initial movement you transferred to the string $\psi(0, t)$ will travel at constant speed v down the string, but conserving its shape – see Figure 8.1.

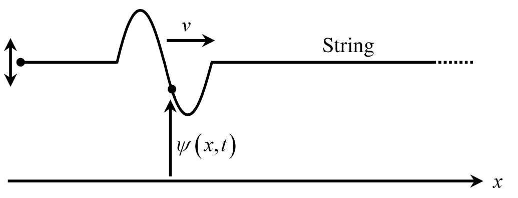

Figure 8.1 Propagation along a vibrating string

The mathematical equation describing the propagation of the oscillation (wave) along x is the well-known one-dimensional wave equation

$$\frac{\partial^2 \psi(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi(x, t)}{\partial t^2}$$

For a three-dimensional problem, where $\psi = \psi(x, y, z, t)$, the above result transforms into

$$\underbrace{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}}_{\text{lap}\,\psi} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \tag{8.1}$$

which is the scalar three-dimensional (3D) wave equation.

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After this brief introduction, let us return to Maxwell's equations in (PIV.1).

In free space ($\varepsilon = \varepsilon_0$, $\mu = \mu_0$) where currents and charges are absent ($\mathbf{J} = 0$, $\rho = 0$) Maxwell's equations assume the simplified version

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{8.2}$$

$$\operatorname{curl} \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{8.3}$$

$$\operatorname{div} \mathbf{B} = 0 \tag{8.4}$$

$$\operatorname{div} \mathbf{E} = 0 \tag{8.5}$$

By applying the curl operator on the left to (8.2), and taking into account the result in (8.3), we get

$$\operatorname{curl} \operatorname{curl} \mathbf{E} = -\frac{\partial}{\partial t} \operatorname{curl} \mathbf{B} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

From vector calculus we know that curl curl $\equiv$ grad div $-$ lap, but since div $\mathbf{E} = 0$, we find

$$\operatorname{lap} \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Likewise, had we applied the curl operator to (8.3), and taken (8.2) into account, we would get

$$\operatorname{lap} \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

In short, in free space, the electromagnetic field ($\mathbf{E}$, $\mathbf{B}$) satisfies the equation

$$\operatorname{lap} \begin{Bmatrix} \mathbf{E} \\ \mathbf{B} \end{Bmatrix} = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \begin{Bmatrix} \mathbf{E} \\ \mathbf{B} \end{Bmatrix}, \quad v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \tag{8.6}$$

Comparing this result to (8.1), you can readily see that (8.6) is nothing but the 3D wave equation of the electromagnetic field. In other words, we came to the conclusion that Maxwell's equations are compatible with wave solutions whose velocity in free space is equal to the speed of light in a vacuum, $v = 3 \times 10^8$ m/s. This fact ultimately pointed out that light itself is nothing but electromagnetic radiation; the science of optics is not an independent area of knowledge, but rather the contrary – it belongs with electromagnetism.

The first experimental proof that electromagnetic waves are real dates back to 1889 (Hertz). A remark is in order here: if the displacement current density introduced by Maxwell did not exist, (8.6) would not hold – electromagnetic waves would not exist.

For homogeneous media with $\mu \neq \mu_0$, $\varepsilon \neq \varepsilon_0$, the wave velocity in (8.6) is $v = 1/\sqrt{\mu\varepsilon}$.

Before finishing this section it may help you to have a pictorial representation of the wave phenomena. From (8.2) you can see that a time-varying $\mathbf{B}$ field may give rise to a time-varying $\mathbf{E}$ field; likewise, from (8.3), you can also see that a time-varying $\mathbf{E}$ field may give rise to a time-varying $\mathbf{B}$ field. The consequence of this is that the electromagnetic field is a self-sustainable entity

$$\cdots \to \mathbf{B}(t) \to \mathbf{E}(t) \to \mathbf{B}(t) \to \mathbf{E}(t) \to \mathbf{B}(t) \to \mathbf{E}(t) \to \cdots$$

Once it has been started at a certain region in space (for example, at an antenna) it will propagate and spread through space indefinitely and for ever, the wave process progressing at a constant speed. The sketch in Figure 8.2 gives you a glimpse of the ideas we have been talking about.

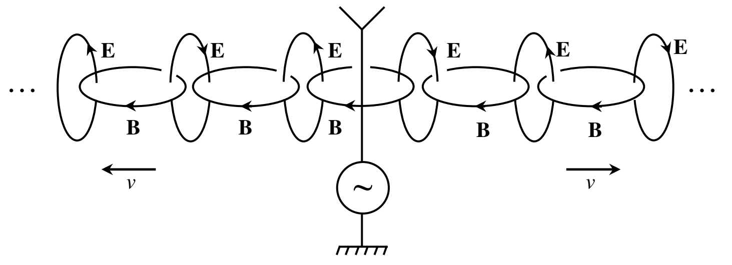


Figure 8.2 A pictorial sketch of the self-sustainable electromagnetic field radiated by an antenna, and propagating through space with velocity $v = 1/\sqrt{\mu_0 \varepsilon_0}$

8.2 Poynting Theorem, Poynting Vector, Power Flow

Based on intuitive energy balance considerations, the Poynting theorem has already been introduced in the framework of quasi-stationary regimes (Chapter 7). Here, we are going to show you a proof of such a theorem based directly on Maxwell's equations. In addition, a new interpretation for the concept of instantaneous power $p(t)$ is provided.

Take $\operatorname{curl} \mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t$ and multiply (inner product) both sides of the equation by $\mathbf{E}$. Then you get

$$\mathbf{E} \cdot \operatorname{curl} \mathbf{H} = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \mathbf{E} \cdot \mathbf{J} + \frac{\partial}{\partial t}\left(\frac{\mathbf{E} \cdot \mathbf{D}}{2}\right) \tag{8.7a}$$

From (2.26) in Chapter 2, you will recognize that $(\mathbf{E} \cdot \mathbf{D})/2$ represents the per-unit-volume electric energy stored in the electric field, $\hat{w}_e$. Also, from (3.14) in Chapter 3, you will recognize that $(\mathbf{E} \cdot \mathbf{J})$ represents the per-unit-volume power losses associated with the Joule effect, $\hat{p}_J$. Hence, we can rewrite (8.7a) as

$$\mathbf{E} \cdot \operatorname{curl} \mathbf{H} = \hat{p}_J + \frac{\partial \hat{w}_e}{\partial t} \tag{8.7b}$$

Similarly, take $\operatorname{curl} \mathbf{E} = -\partial \mathbf{B}/\partial t$ and multiply (inner product) both sides of the equation by $\mathbf{H}$. Then you get

$$\mathbf{H} \cdot \operatorname{curl} \mathbf{E} = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{\mathbf{H} \cdot \mathbf{B}}{2}\right) \tag{8.8a}$$

From (4.36) in Chapter 4, you will recognize that $(\mathbf{H} \cdot \mathbf{B})/2$ represents the per-unit-volume magnetic energy stored in the magnetic field, $\hat{w}_m$. Hence, we can rewrite (8.8a) as

$$\mathbf{H} \cdot \operatorname{curl} \mathbf{E} = -\frac{\partial \hat{w}_m}{\partial t} \tag{8.8b}$$

By subtracting the equations in (8.7b) and (8.8b), we obtain

$$\mathbf{E} \cdot \operatorname{curl} \mathbf{H} - \mathbf{H} \cdot \operatorname{curl} \mathbf{E} = \hat{p}_J + \frac{\partial \hat{w}_m}{\partial t} + \frac{\partial \hat{w}_e}{\partial t} \tag{8.9}$$

But, from vector calculus we have $\mathbf{E} \cdot \operatorname{curl} \mathbf{H} - \mathbf{H} \cdot \operatorname{curl} \mathbf{E} = -\operatorname{div}(\mathbf{E} \times \mathbf{H})$. The vector entity

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{8.10}$$

named the Poynting vector (units: $\mathrm{W/m^2}$, watt per square meter), is to be physically interpreted as the per-unit-area power flow carried by the electromagnetic field.

Therefore, the Poynting theorem, in local form, can be established as

$$-\operatorname{div} \mathbf{S} = \hat{p}_J + \frac{\partial \hat{w}_m}{\partial t} + \frac{\partial \hat{w}_e}{\partial t} \tag{8.11}$$

If this result is integrated over a volume V, bounded by a closed surface S_V, we obtain, using the already familiar Gauss theorem,

$$\underbrace{\int_{S_V} \mathbf{S} \cdot \mathbf{n}_\mathrm{i} \, dS}_{p(t)} = p_J(t) + \frac{d}{dt}\left(W_m(t) + W_e(t)\right) \tag{8.12}$$

where $\mathbf{n}_\mathrm{i}$ is the inward unit normal to S_V.

Let us emphasize the following aspects:

- The result in (8.12) is a restatement of the result in (7.20).
- The result in (8.12) is not valid when hysteresis phenomena are present.
- The instantaneous power $p(t)$ in play in a given volume of space is to be interpreted as the inward flux of the Poynting vector across the volume border.
- The Poynting vector is the carrier of electromagnetic energy; it is ultimately responsible for the flow of power in any electromagnetic system.
- Electromagnetic energy is not carried by wires. Wires are simply used to guide electromagnetic waves, but (apart from wire losses) the energy flow is essentially external to the wires.
- At any chosen point of an electromagnetic wave, the power flow density is totally determined, in both in magnitude and direction, by $\mathbf{S}$ – see Figure 8.3.

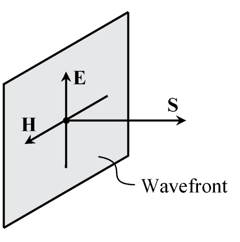


Figure 8.3 Wavefront. The Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ determines the flow of electromagnetic energy

Well before the introduction of the Poynting theorem we utilized the instantaneous power definition $p = ui$. Now, we know that such a definition is not general; it pertains to slow time-varying field phenomena. Here, we are going to show that for stationary and quasi-stationary phenomena we have

$$\int_{S_V} \mathbf{S} \cdot \mathbf{n}_\mathrm{i}\, dS = p(t) = ui \tag{8.13}$$

For that purpose, consider the general situation depicted in Figure 8.4(a) where a generator is connected to a load using a pair of wires. Define S_V as the global surface enclosing the load, and define S_T as the transversal plane surface intersecting the two wires (which are assumed to be perfect conductors). On the transverse plane, the field lines of $\mathbf{E}$ are circumferential arcs starting and ending at the wires, and the lines of $\mathbf{H}$ are non-coaxial circumferences around the wires – see Figure 8.4(b). The two families of curves intersect perpendicularly, originating Poynting vectors oriented along z, $\mathbf{S} = \mathbf{E} \times \mathbf{H} = S\vec{e}_z$.

For stationary or quasi-stationary regimes, at any point belonging to S_V, we have $\mathbf{E} = -\mathrm{grad}\, V$ and $\mathrm{curl}\, \mathbf{H} = \mathbf{J}$.

Hence, the Poynting vector is given by $\mathbf{S} = \mathbf{E} \times \mathbf{H} = -\mathrm{grad}\, V \times \mathbf{H}$.

Taking into account the vector identity

$$\mathrm{curl}\,(V\mathbf{H}) = V\,\mathrm{curl}\,\mathbf{H} + \mathrm{grad}\, V \times \mathbf{H}$$

we get $\mathbf{S} = V\mathbf{J} - \mathrm{curl}(V\mathbf{H})$. The inward flux of $\mathbf{S}$ across S_V gives the instantaneous power that flows from the generator to the load

$$p = \int_{S_V} \mathbf{S} \cdot \mathbf{n}_\mathrm{i}\, dS = \int_{S_V} V\,\mathbf{J} \cdot \mathbf{n}_\mathrm{i}\, dS + \int_{S_V} \mathrm{curl}\,\underbrace{(V\mathbf{H})}_{\mathbf{K}} \cdot \mathbf{n}_\mathrm{o}\, dS$$

where $\mathbf{n}_\mathrm{o}$ is the outward unit normal.

The last term on the right is zero. In fact, from the Gauss theorem, we have

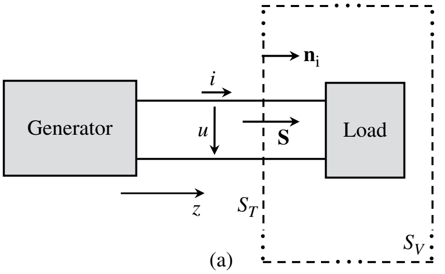
$$\int_{S_V} \mathrm{curl}\,\mathbf{K} \cdot \mathbf{n}_\mathrm{o}\, dS = \int_V \mathrm{div}\,\mathrm{curl}\,\mathbf{K}\, dV = 0$$

since the operator div curl $\equiv 0$. Therefore, we have

$$p = \int_{S_V} V\,\mathbf{J} \cdot \mathbf{n}_\mathrm{i}\, dS \tag{8.14}$$

The flux of $V\mathbf{J}$ across S_V in (8.14) is zero everywhere except at the regions S_1 and S_2 where the current-carrying wires intersect the transverse plane. Hence, we finally find

$$p = \int_{S_V} V\,\mathbf{J} \cdot \mathbf{n}_\mathrm{i}\, dS = \int_{S_T} V\,\mathbf{J} \cdot \mathbf{n}_\mathrm{i}\, dS = V_1 \underbrace{\int_{S_1} \mathbf{J} \cdot \mathbf{n}_\mathrm{i}\, dS}_{+i} + V_2 \underbrace{\int_{S_2} \mathbf{J} \cdot \mathbf{n}_\mathrm{i}\, dS}_{-i} = \underbrace{(V_1 - V_2)}_{u}\, i = ui$$



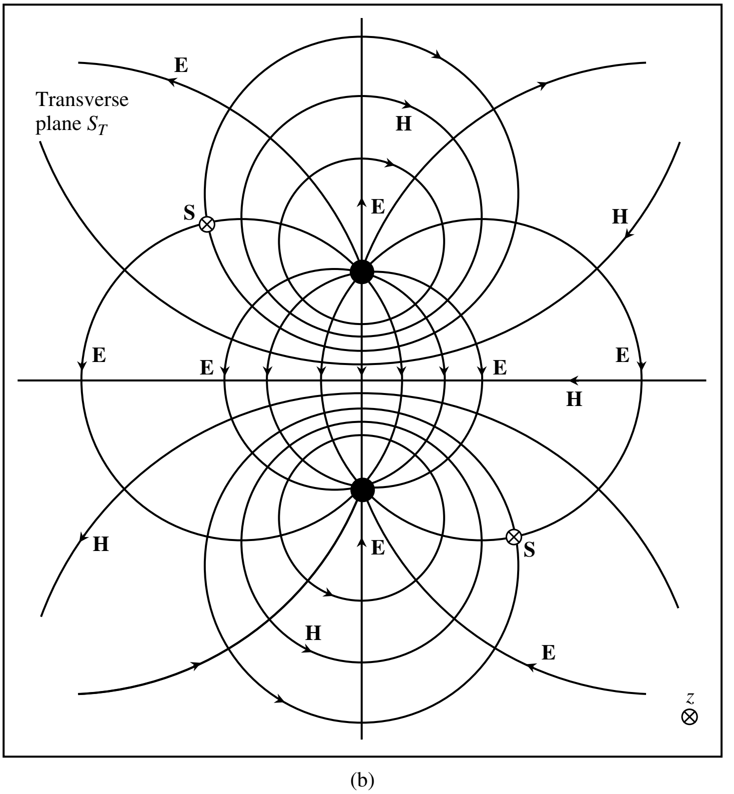


Figure 8.4 (a) The instantaneous power transmitted to the load $p = ui$ can be interpreted as the Poynting vector flow across the transverse plane S_T. (b) Electromagnetic field lines in the transverse plane

Note that, for low- or high-frequency regimes, in a transmission system made of two parallel perfect conductors, both electric and magnetic field lines belong to transverse planes. On these planes, because $\mathbf{B} \perp \mathbf{n}_S$ and $\mathbf{D} \perp \mathbf{n}_S$, you have

$$\oint_{\mathbf{S}} \mathbf{E} \cdot d\mathrm{s} = -\int_{S_s} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}_S \, dS = 0 \rightarrow \operatorname{curl} \mathbf{E} = 0 \rightarrow \mathbf{E} = -\operatorname{grad} V$$

$$\oint_{\mathbf{s}} \mathbf{H} \cdot d\mathbf{s} = \int_{S_S} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{n}_S \, dS = \int_{S_S} \mathbf{J} \cdot \mathbf{n}_S \, dS \rightarrow \text{curl}\, \mathbf{H} = \mathbf{J}$$

That is, you encounter the same conditions we talked about when stationary regimes were dealt with, and, consequently, you can still use

$$p = \int_{S_T} \mathbf{S} \cdot \mathbf{n} \, dS = ui$$

where S_T is the transverse plane surface where u and i are defined.

8.3 Time-Harmonic Fields, Field Polarization, RMS Field Values

In Chapter 7 we defined complex amplitudes (phasors) of sinusoidal time-varying scalar quantities. Now we do the same with vector fields. To start with, let us consider the simplest case of a unidirectional electric field vector, for example $\mathbf{E}(x, y, z, t) = E_x \cos(\omega t + \alpha_x) \vec{e}_x$, where $E_x = E_x(x, y, z)$ and $\alpha_x = \alpha_x(x, y, z)$. At a given point P in space, such a field is said to be linearly polarized along x because the tip of vector $\mathbf{E}$ defines a straight line aligned with the x axis. The above field can be written in complex form as

$$\mathbf{E}(x, y, z, t) = \Re \left(\overline{\mathbf{E}}(x, y, z) \, e^{j\omega t} \right), \text{with } \overline{\mathbf{E}} = E_x \, e^{j\alpha_x} \, \vec{e}_x \tag{8.15}$$

where $\overline{\mathbf{E}}$ is the time-independent phasor associated with the harmonic field $\mathbf{E}(x, y, x, t)$.

Let us complicate matters a little by assuming that $\mathbf{E}$ has two components of the same frequency:

$$\mathbf{E}(x, y, z, t) = E_x \cos(\omega t + \alpha_x) \, \vec{e}_x + E_y \cos(\omega t + \alpha_y) \, \vec{e}_y \tag{8.16a}$$

At a given point P in space, if $\alpha_x \neq \alpha_y$ the tip of vector $\mathbf{E}$ will move with time and describe an ellipse in the xy plane. In this case the field is said to be elliptically polarized.

In the particular case $E_x = E_y$ and $\alpha_x - \alpha_y = \pm\pi/2$, the ellipse degenerates into a circumference and the field is said to be circularly polarized.

The complex representation of $\mathbf{E}(x, y, z, t)$ in (8.16a) is given by

$$\mathbf{E}(x, y, z, t) = \Re \left(\overline{\mathbf{E}}(x, y, z) \, e^{j\omega t} \right), \text{with } \overline{\mathbf{E}} = E_x \, e^{j\alpha_x} \, \vec{e}_x + E_y \, e^{j\alpha_y} \, \vec{e}_y \tag{8.16b}$$

where, again, $\overline{\mathbf{E}}$ denotes the complex amplitude (phasor) of the harmonic field $\mathbf{E}(x, y, z, t)$.

How do you generalize the foregoing results when the harmonic field $\mathbf{E}$ is described by three components with $\alpha_x \neq \alpha_y \neq \alpha_z$?

$$\mathbf{E}(x, y, z, t) = E_x \cos(\omega t + \alpha_x) \, \vec{e}_x + E_y \cos(\omega t + \alpha_y) \, \vec{e}_y + E_z \cos(\omega t + \alpha_z) \, \vec{e}_z \tag{8.17a}$$

How does the tip of vector $\mathbf{E}$ move in space, at a given point P, as time elapses?

Most likely, your intuition will tell you that the tip of $\mathbf{E}$ will describe a 3D ellipsoid.

Well, that is not true. Intuition has misled you!

To help you find the correct answer let us rewrite (8.17a) using the phasor-domain representation

$$\mathbf{E}(x, y, z, t) = \Re\left(\overline{\mathbf{E}}(x, y, z)\, e^{j\omega t}\right), \text{with } \overline{\mathbf{E}} = \underbrace{E_x\, e^{j\alpha_x}}_{\overline{E}_x}\, \vec{e}_x + \underbrace{E_y\, e^{j\alpha_y}}_{\overline{E}_y}\, \vec{e}_y + \underbrace{E_z\, e^{j\alpha_z}}_{\overline{E}_z}\, \vec{e}_z$$

Breaking down $\overline{\mathbf{E}}$ into its real and imaginary parts we obtain

$$\overline{\mathbf{E}} = \left(\sum_{k=x,y,z} E_k \cos\alpha_k\, \vec{e}_k\right) + j\left(\sum_{k=x,y,z} E_k \sin\alpha_k\, \vec{e}_k\right) = E_1\, \vec{e}_1 + jE_2\, \vec{e}_2$$

where the field magnitudes E_1 and E_2 are time independent.

From $\mathbf{E} = \Re\left(\overline{\mathbf{E}}\, e^{j\omega t}\right)$ we find

$$\mathbf{E} = E_1 \cos(\omega t)\, \vec{e}_1 - E_2 \sin(\omega t)\, \vec{e}_2 \tag{8.17b}$$

Again, as shown in Figure 8.5, we find an elliptically polarized field in the plane defined by the unit vectors $\vec{e}_1$ and $\vec{e}_2$ (which, in general, are not mutually orthogonal).

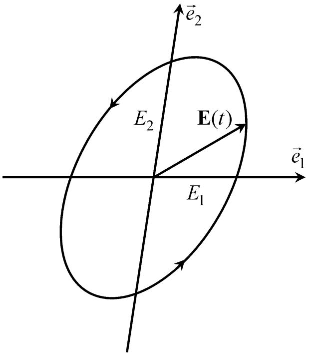


Figure 8.5 The elliptically polarized electric field vector can be constructed from two linearly polarized fields directed along $\vec{e}_1$ and $\vec{e}_2$

To conclude this section we must address the important topic of the evaluation of rms values for time-harmonic vector fields.

For sinusoidal time-varying scalar quantities we showed, in Chapter 7, that

$$U_{rms} = \sqrt{(u^2(t))_{av}} = \frac{\overline{U}\,\overline{U}^*}{2} \tag{8.18}$$

yielding $U_{rms} = U/\sqrt{2}$.

For vector fields, we have a definition similar to the one given in (8.18), that is

$$E_{rms} = \sqrt{(E^2(t))_{av}} = \sqrt{(\mathbf{E}(t) \cdot \mathbf{E}(t))_{av}} = \sqrt{\frac{E_x^2 + E_y^2 + E_z^2}{2}}$$

In addition, noting that

$$E_x^2 + E_y^2 + E_z^2 = \overline{E}_x \overline{E}_x^* + \overline{E}_y \overline{E}_y^* + \overline{E}_z \overline{E}_z^* = \overline{\mathbf{E}} \cdot \overline{\mathbf{E}}^*$$

we can also write, using complex amplitudes,

$$E_{rms} = \sqrt{\frac{\overline{\mathbf{E}} \cdot \overline{\mathbf{E}}^*}{2}} \tag{8.19}$$

A mistake that you should avoid is to write $E_{rms} = E_{\max}/\sqrt{2}$. For time-harmonic vector fields, the rms value is not obtained from the maximum value $E_{\max}$ by dividing it by $\sqrt{2}$ (as happens with scalar quantities) – that is only true for the especial case of linearly polarized fields.

Remember, for example, that in the case of circular polarization, the field intensity remains constant with time and, consequently, you get $E_{rms} = E_{\max}$.

In short, depending on the field's polarization state, you can have $E_{\max}/\sqrt{2} \le E_{rms} \le E_{\max}$.

8.4 Phasor-Domain Maxwell's Equations, Material Media Constitutive Relations

The correspondence between operative rules in the time domain and in the phasor domain summarized in Table 7.1 also applies to sinusoidal (ω) time-varying vector fields. Therefore, as far as Maxwell's equations are concerned, we can write

Time domain		Phasor domain	
$\operatorname{curl}\mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\leftrightarrow$	$\operatorname{curl}\overline{\mathbf{E}} = -j\omega\overline{\mathbf{B}}$	(8.20)
$\operatorname{curl}\mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\leftrightarrow$	$\operatorname{curl}\overline{\mathbf{H}} = \overline{\mathbf{J}} + j\omega\overline{\mathbf{D}}$	(8.21)
$\operatorname{div}\mathbf{B} = 0$	$\leftrightarrow$	$\operatorname{div}\overline{\mathbf{B}} = 0$	(8.22)
$\operatorname{div}\mathbf{D} = \rho$	$\leftrightarrow$	$\operatorname{div}\overline{\mathbf{D}} = \overline{\rho}$	(8.23)

In addition, the material media constitutive relations, analyzed in Chapters 2, 3 and 4, translate in the phasor domain as

$$\mathbf{D} = \varepsilon\,\mathbf{E} \quad \leftrightarrow \quad \overline{\mathbf{D}} = \varepsilon\,\overline{\mathbf{E}} \tag{8.24}$$

$$\mathbf{J} = \sigma\,\mathbf{E} \quad \leftrightarrow \quad \overline{\mathbf{J}} = \sigma\,\overline{\mathbf{E}} \tag{8.25}$$

$$\mathbf{B} = \mu\,\mathbf{H} \quad \leftrightarrow \quad \overline{\mathbf{B}} = \mu\,\overline{\mathbf{H}} \tag{8.26}$$

Rigorously speaking, the time-domain equations in (8.24) and (8.26) are valid for stationary and quasi-stationary regimes. The corresponding phasor-domain equations on the right can be further generalized for rapid time-varying fields, considering the possibility that losses may eventually be present in the linear medium (note that linearity is a prerequisite).

In order to take into account dielectric polarization losses, you can introduce a complex permittivity, $\overline{\varepsilon} = \varepsilon' - j\varepsilon''$. Likewise, magnetic losses can be accounted for by making use of a complex permeability, $\overline{\mu} = \mu' - j\mu''$. Hence, (8.24) and (8.26) can be generalized through

$$\overline{\mathbf{D}} = \overline{\varepsilon}\,\overline{\mathbf{E}} \quad \text{and} \quad \overline{\mathbf{B}} = \overline{\mu}\,\overline{\mathbf{H}} \tag{8.27}$$

The time-domain interpretation of the equations in (8.27) is that, for sinusoidal regimes, the existence of material losses makes **D** lag **E**, and **B** lag **H**. The parametric curves $D = D(E)$ and $B = B(H)$ are ellipses where the operating point moves counterclockwise as time goes on – see Figure 8.6. The areas circumscribed by the ellipses denote the energy loss per period, per unit volume of the material.

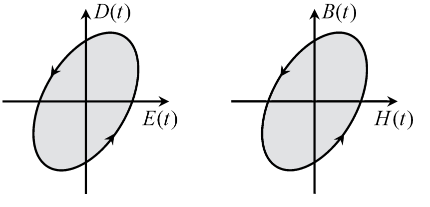


Figure 8.6 For sinusoidal regimes the parametric curves $D(E)$ and $B(H)$ change from straight lines to ellipses, when dielectric and magnetic losses are present, respectively

8.5 Application Example (Uniform Plane Waves)

A uniform plane wave is one where fields **E** and **H** belong to planes perpendicular to the propagation direction, the fields in those planes being invariant from point to point. A monochromatic wave is one where fields are sinusoidal with time with a given frequency ω.

Uniform plane waves are a physical abstraction, but their analysis is very important because any type of wave can be synthesized via a summation of uniform plane waves (Fourier space transforms).

Consider the propagation of a wave produced by a light source. Assume that such a wave can be approximately described by a uniform monochromatic plane wave, propagating along the positive z axis in free space, where the electric field is given by

$$\mathbf{E}(z, t) = E_0 \cos(\omega t - \theta(z))\,\vec{e}_x$$

where $E_0 = 20\,\mathrm{V/m}$, $\omega = 2\pi f$, $f = 474\,\mathrm{THz}$.

Questions

Q_1 Write the complex amplitude corresponding to $\mathbf{E}(z, t)$.

Q_2 Write the phasor-domain equation corresponding to the wave equation in (8.6), and determine the function $\theta(z)$. Assume $\theta(0) = 0$.

Q_3 Determine the operating wavelength λ of the light source.

Q_4 Find the complex amplitude, as well as the time-domain representation, of the magnetic field **H**.

Q_5 Determine the Poynting vector and evaluate its time-averaged value.

Solutions

Q_1 $\overline{\mathbf{E}} = \overline{E}(z)\,\vec{e}_x$, where $\overline{E}(z) = E_0\,e^{-j\theta(z)}$.

Q_2

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \rightarrow \frac{d^2\overline{E}(z)}{dz^2} + \left(\frac{\omega}{v}\right)^2 \overline{E}(z) = 0$$

where the wave velocity is $v = c = 1/\sqrt{\mu_0\varepsilon_0} = 3\times 10^8\,\text{m/s}$.

Noting that

$$\frac{d^2\overline{E}(z)}{dz^2} = -E_0\,e^{-j\theta(z)}\left[\left(\frac{d\theta}{dz}\right)^2 + j\frac{d^2\theta}{dz^2}\right]$$

we obtain

$$\left(\frac{d\theta}{dz}\right)^2 + j\frac{d^2\theta}{dz^2} = \left(\frac{\omega}{v}\right)^2 + j0 \rightarrow \left\{\begin{aligned} &\frac{d^2\theta}{dz^2} = 0 \\ &\left(\frac{d\theta}{dz}\right)^2 = \left(\frac{\omega}{v}\right)^2 \end{aligned}\right\} \rightarrow \theta(z) = \pm\beta z$$

where β , the so-called phase constant, is given by

$$\beta = \frac{\omega}{v} \tag{8.28}$$

The plus or minus sign in $\theta(z) = \pm\beta z$ means that two solutions are available. The plus sign, which we will adopt here, corresponds to a wave propagating along the positive z axis (check the answer to question Q_5 showing that such an option indeed leads to an energy flow along the positive z axis). The choice $\theta(z) = -\beta z$ would describe a propagating wave along the negative z axis.

Q_3 Taking into account the result $\theta(z) = \beta z$, the expression for the electric field wave is

$$\mathbf{E}(z,t) = E_0\cos(\omega t - \beta z)\,\vec{e}_x$$

If at a given moment $t = t_0$ you take a snapshot of the electric field wave, you will find that $\mathbf{E}(z,\,t_0)$ is periodic along z. The space period, or wavelength λ, is such that $\beta\lambda = 2\pi$. Hence you have

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi v}{\omega} = \frac{v}{f} \tag{8.29}$$

Numerically you find $\lambda = 0.633\,\mu\text{m}$ (red light).

Q_4 From

$$\begin{cases} \operatorname{curl} \overline{\mathbf{E}} = -j\omega\mu_0 \overline{\mathbf{H}} \\ \operatorname{curl} \overline{\mathbf{E}} = \dfrac{d\overline{E}(z)}{dz}\,\vec{e}_y \end{cases}$$

we see that $\mathbf{H}$ is oriented along the y axis, $\overline{\mathbf{H}} = \overline{H}(z)\,\vec{e}_y$,

$$\overline{H}(z) = \frac{j}{\omega\mu_0}\frac{d\overline{E}(z)}{dz} = \frac{\overline{E}(z)}{R_w}$$

where R_w, the so-called characteristic wave resistance of free space, is

$$R_w = \frac{\omega\mu_0}{\beta} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi = 377\,\Omega \tag{8.30}$$

$\mathbf{H}(z, t) = \Re\left(\overline{\mathbf{H}}\,e^{j\omega t}\right) = H_0\cos(\omega t - \beta z)\,\vec{e}_y$, with $H_0 = E_0/R_w = 53\,\text{mA/m}$.

Q_5 $\mathbf{S} = \mathbf{E} \times \mathbf{H} = S\,\vec{e}_z$.

$$S(z, t) = \frac{E_0^2}{R_w}\cos^2(\omega t - \beta z),\ (S)_{av} = \frac{E_0^2}{2R_w} = 530\,\text{mW/m}^2$$

8.6 Complex Poynting Vector

The complex Poynting theorem, valid for time-harmonic regimes, has already been dealt with in (7.26) in Chapter 7,

$$\overline{P} = P_J + j2\omega\left((W_m)_{av} - (W_e)_{av}\right) \tag{8.31}$$

where the complex power $\overline{P}$ is obtained as $\overline{P} = \overline{U}\,\overline{I}^*/2$ for quasi-stationary regimes.

As mentioned earlier, the validity of the complex Poynting theorem is not limited to quasi-stationary regimes. It also holds in the analysis of rapid time-varying fields.

In fact, the theorem can be logically deduced from the phasor-domain Maxwell's equations in (8.20) to (8.23), following a procedure very similar to the one that led us to the Poynting theorem in Section 8.2. There is no point in again repeating the derivation steps. However, a salient aspect that must be emphasized is the general interpretation that should be given to the complex power $\overline{P}$.

When deducing the complex Poynting theorem from the phasor-domain Maxwell's equations, the complex power appears as

$$\overline{P} = \int_{S_V}\left(\frac{\overline{\mathbf{E}} \times \overline{\mathbf{H}}^*}{2}\right) \cdot \mathbf{n}_\text{i}\,dS \tag{8.32}$$

where S_V is a closed surface bounding a volume V where electromagnetic phenomena occur (energy dissipation, electric energy storage, magnetic energy storage). The complex vector under the integration symbol is the complex Poynting vector $\overline{\mathbf{S}}$ (units: W/m^2)

$$\overline{\mathbf{S}} = \frac{\overline{\mathbf{E}} \times \overline{\mathbf{H}}^*}{2} \tag{8.33}$$

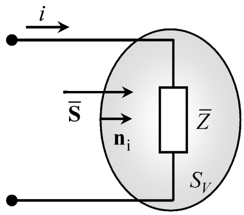


Figure 8.7 The inward flux of the complex Poynting vector across a closed surface containing a linear device allows for the evaluation of the device's impedance

The inward flux of $\overline{\mathbf{S}}$ across a closed surface containing a linear device permits the evaluation of its impedance (Figure 8.7),

$$\overline{P} = \int_{S_V} \left(\frac{\overline{\mathbf{E}} \times \overline{\mathbf{H}}^*}{2} \right) \cdot \mathbf{n}_i \, dS = \overline{Z} I_{rms}^2 \rightarrow \overline{Z} = \frac{\overline{P}}{I_{rms}^2} \tag{8.34}$$

In Part III we showed that the instantaneous power $p(t)$ and the complex power $\overline{P}$ were related through $(p(t))_{av} = \Re\left(\overline{P}\right)$. Likewise, for the Poynting vector we also have

$$(\mathbf{S}(t))_{av} = \Re\left(\overline{\mathbf{S}}\right)$$

Electromagnetic power losses are included in (8.31) in the form of conductor losses associated with the Joule effect P_J

$$P_J = \int_V \sigma E_{rms}^2 \, dV$$

However, the complex Poynting theorem can be easily extended to account for two additional frequency-dependent loss mechanisms that may be present, namely the dielectric polarization losses and the magnetization losses. For linear media, where the constitutive relations (8.27) apply, you have

$$P_{Polarization} = \int_V \omega \varepsilon'' E_{rms}^2 \, dV$$

$$P_{Magnetization} = \int_V \omega \mu'' H_{rms}^2 \, dV$$

Consequently the most general form for the complex Poynting theorem will read as

$$\overline{P} = \underbrace{\left(P_{Joule} + P_{Polarization} + P_{Magnetization}\right)}_{P_{loss}} + j2\omega\left((W_m)_{av} - (W_e)_{av}\right)$$

8.7 Application Example (Skin Effect)

Consider a cylindrical conductor of radius r_0 and indefinite length l. In Chapter 3 dedicated to stationary currents we concluded, as a result of curl $\mathbf{E} = 0$, that the $\mathbf{J}$-field lines had constant intensity at any point inside the current-carrying conductor, that is $J = I/(\pi r_0^2)$, where I denotes the current intensity. The per-unit-length DC resistance of the conductor was then evaluated as $R_{DC} = 1/(\sigma \pi r_0^2)$, where σ is the conductivity. For the case of rapid time-varying currents, the current density is no longer uniformly distributed inside the conductor – most of it tends to flow near to the conductor's surface, the so-called skin-effect phenomenon.

Consider a time-varying sinusoidal current $i(t) = I\cos(\omega t + \alpha_i)$, and assume, as shown in Figure 8.8, that conduction currents are longitudinally oriented. According to the geometry of the problem, a particular field solution to Maxwell's equations must be invariant under rotation operations around z and translation operations along z; in other words, all vector fields are independent of ϕ and z. All vector fields will, however, depend on the radial coordinate. In particular, we will have $\mathbf{J} = J(r)\,\vec{e}_z$ for the current density.

Take the permittivity and permeability of the conductor as ε_0 and μ_0, respectively.

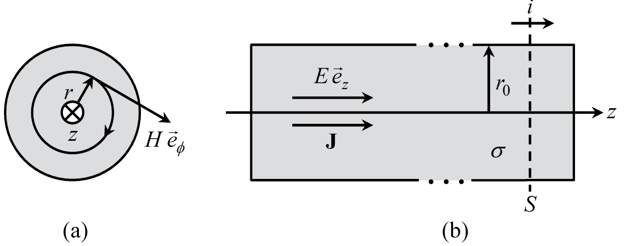


Figure 8.8 Skin-effect phenomena in a cylindrical conductor of radius r_0. (a) Conductor transverse cross-section showing the internal magnetic field $H_\phi(r)$. (b) Conductor longitudinal cross-section showing the internal longitudinal electric field $E_z(r)$

Questions

Q_1 Show that the displacement current density inside the conductor is negligibly small for the case of good conductors.

Q_2 By using a cylindrical coordinate reference frame, write the phasor-domain Maxwell's *curl* equations for the problem.

Q_3 Obtain a solution for $\overline{\mathbf{H}}$ and $\overline{\mathbf{E}}$.

Q_4 Determine the solution for $\overline{\mathbf{J}}$ and particularize it for very-low frequencies.

Q_5 Find the complex Poynting vector $\overline{\mathbf{S}}(r_0)$ at any point belonging to the conductor surface.

Q_6 Determine the per-unit-length frequency-dependent impedance $\overline{Z}(\omega)$ of the conductor.

Q_7 Particularize the results concerning $\overline{Z}(\omega)$ for low- and high-frequency regimes.

Solutions

Q_1 Conduction current density: $\overline{\mathbf{J}} = \sigma\,\overline{\mathbf{E}}$. Displacement current density: $j\omega\overline{\mathbf{D}} = j\omega\varepsilon_0\overline{\mathbf{E}}$. Their absolute ratio is

$$\eta = \left|\frac{j\omega\overline{\mathbf{D}}}{\sigma\overline{\mathbf{E}}}\right| = \frac{\omega\varepsilon_0}{\sigma} \ll 1$$

For ordinary good conductors (where $\sigma \approx 10^7\,\mathrm{S/m}$) the ratio η approaches unity for frequencies above $10^{17}\,\mathrm{Hz}$ (X-ray band). This means that for typical applications up to the gigahertz range, the displacement current can be neglected inside good conductors.

Q_2

$$\operatorname{curl}\overline{\mathbf{E}} = \operatorname{curl}\left(\overline{E}_z(r)\,\vec{e}_z\right) = -j\omega\mu_0\overline{\mathbf{H}} \to \frac{d}{dr}\left(\overline{E}_z(r)\right)\vec{e}_\phi = j\omega\mu_0\overline{\mathbf{H}}$$

From this, you can see that the magnetic field is purely azimuthal, $\overline{\mathbf{H}} = \overline{H}_\phi(r)\,\vec{e}_\phi$, its field lines being coaxial circumferences.

$$\operatorname{curl}\overline{\mathbf{H}} = \operatorname{curl}\left(\overline{H}_\phi(r)\,\vec{e}_\phi\right) = \sigma\overline{\mathbf{E}} \to \frac{1}{r}\frac{d\left(r\overline{H}_\phi(r)\right)}{dr}\vec{e}_z = \sigma\overline{E}_z(r)\,\vec{e}_z$$

To abbreviate the notation let us put $\overline{E} = \overline{E}_z(r)$ and $\overline{H} = \overline{H}_\phi(r)$.
In conclusion, the governing equations for the problem are

$$\frac{d\overline{E}}{dr} = j\omega\mu_0\overline{H} \tag{8.35}$$

$$\frac{d\overline{H}}{dr} + \frac{\overline{H}}{r} = \sigma\overline{E} \tag{8.36}$$

Q_3 Take the derivative of (8.36) with respect to r:

$$\frac{d^2\overline{H}}{dr^2} + \frac{1}{r}\frac{d\overline{H}}{dr} - \frac{\overline{H}}{r^2} = \sigma\frac{d\overline{E}}{dr}$$

Making use of (8.35), you find

$$\frac{d^2\overline{H}}{dr^2} + \frac{1}{r}\frac{d\overline{H}}{dr} + \left(\overline{p}^2 - \frac{1}{r}\right)\overline{H} = 0, \text{ with } \overline{p} = \sqrt{-j\omega\mu_0\sigma} = \frac{1-j}{\delta} \tag{8.37}$$

where δ is the so-called penetration depth

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} \tag{8.38}$$

The differential equation in (8.37) is a particular case of the more general Bessel equation[1]

$$\frac{d^2\overline{H}}{dr^2} + \frac{1}{r}\frac{d\overline{H}}{dr} + \left(\overline{p}^2 - \frac{n^2}{r^2}\right)\overline{H} = 0, \text{ with } n \geq 0$$

[1] For details on Bessel functions refer to G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1995.

whose solution is of the type $\overline{H} = H_1 J_n(\overline{p}r) + H_2 Y_n(\overline{p}r)$, and where J_n is the Bessel function of the first kind of order n, and Y_n is the Neumann function of order n. H_1 and H_2 are constants to be determined using the boundary conditions of the problem.

Since in our case we have $n = 1$, we write for the magnetic field solution

$$\overline{H}(r) = H_1 J_1(\overline{p}r) + H_2 Y_1(\overline{p}r) \tag{8.39}$$

Application of Ampère's law to our problem yields the following boundary conditions:

$$\overline{H}(r_0) = \frac{\overline{I}}{2\pi r_0} \text{ and } \overline{H}(0) = 0 \tag{8.40}$$

Taking into account that $J_1(0) = 0$, and that $Y_1(0) \to \infty$, we readily conclude from (8.39) and (8.40) that

$$H_2 = 0 \text{ and } H_1 = \frac{\overline{I}}{2\pi r_0 J_1(\overline{p}r_0)}$$

Substituting the above results into (8.39), we finally obtain the magnetic field inside the conductor

$$\overline{H}(r) = \frac{\overline{I}}{2\pi r_0} \frac{J_1(\overline{p}r)}{J_1(\overline{p}r_0)}, \text{ for } 0 \le r \le r_0 \tag{8.41}$$

As for the electric field, we have from (8.35), $\overline{E}(r) = j\omega\mu_0 \int \overline{H}(r)dr$, and noting that $\int J_1(\xi)d\xi = -J_0(\xi)$, we obtain

$$\overline{E}(r) = \frac{\overline{p}\overline{I}}{2\pi\sigma r_0} \frac{J_0(\overline{p}r)}{J_1(\overline{p}r_0)}, \text{ for } 0 \le r \le r_0 \tag{8.42}$$

Q_4 From $\overline{J} = \sigma\overline{E}$ we obtain for the current density

$$\overline{J}(r) = \frac{\overline{p}\overline{I}}{2\pi r_0} \frac{J_0(\overline{p}r)}{J_1(\overline{p}r_0)}, \text{ for } 0 \le r \le r_0 \tag{8.43}$$

For the case of very low frequencies, that is $|\overline{p}r| \ll 1$, or, put another way, when the conductor radius is much smaller than the penetration depth (8.38), $r_0 \ll \delta$, we may substitute the Bessel functions J_0 and J_1 in (8.43) by their Taylor expansions, retaining only the leading terms, $J_0(\overline{p}r) \simeq 1$ and $J_1(\overline{p}r_0) \simeq \overline{p}r_0/2$, and yielding

$$\overline{J} \simeq \frac{\overline{I}}{\pi r_0^2} = \frac{\overline{I}}{S} \tag{8.44}$$

This result is no surprise. It shows that for very low frequencies the current density is uniformly distributed over the conductor cross-section S.

Q_5 Particularizing (8.41) and (8.42) for $r = r_0$, recalling that $\overline{I}\,\overline{I}^* = 2I_{rms}^2$, and noting that $\vec{e}_z \times \vec{e}_\phi = -\vec{e}_r$, we obtain for the complex Poynting vector

$$\overline{\mathbf{S}}(r_0) = \frac{\overline{\mathbf{E}}(r_0) \times \overline{\mathbf{H}}^*(r_0)}{2} = -\overline{S}(r_0)\vec{e}_r, \text{ where } \overline{S}(r_0) = \frac{\overline{p}I_{rms}^2}{\sigma(2\pi r_0)^2} \frac{J_0(\overline{p}r_0)}{J_1(\overline{p}r_0)} \tag{8.45}$$

Q_6 Now we compute the complex power corresponding to the inward flux of $\overline{\mathbf{S}}(r_0)$ across the conductor's cylindrical surface of radius r_0 and unit length

$$\overline{P} = \int_{S_V} \overline{\mathbf{S}}(r_0) \cdot (-\vec{e}_r)\, dS = 2\pi r_0 \overline{S}(r_0) = \left(\frac{\overline{p}}{2\pi r_0 \sigma} \frac{J_0(\overline{p} r_0)}{J_1(\overline{p} r_0)} \right) I_{rms}^2 \tag{8.46}$$

According to (8.34), the term in the large parentheses is to be interpreted as the per-unit-length impedance of the conductor, hence

$$\overline{Z}(\omega) = R(\omega) + jX(\omega) = \frac{\overline{p}}{2\pi r_0 \sigma} \frac{J_0(\overline{p} r_0)}{J_1(\overline{p} r_0)} = R_{DC} \overline{p} r_0 \frac{J_0(\overline{p} r_0)}{2J_1(\overline{p} r_0)} \tag{8.47}$$

where $R_{DC} = 1/(\sigma\pi r_0^2)$ is the per-unit-length DC resistance of the conductor ($\omega = 0$).

Q_7 Here we analyze the limit cases of the low- and high-frequency regimes.

For low frequencies, $|\overline{p} r_0| \ll 1$, $r_0 \ll \delta$, we may substitute the Bessel functions J_0 and J_1 in (8.47) by their Taylor expansions, retaining only the first two terms

$$J_0(\overline{p} r_0) \simeq 1 - \left(\frac{\overline{p} r_0}{2} \right)^2, \quad J_1(\overline{p} r_0) \simeq \frac{\overline{p} r_0}{2} \left[1 - 2\left(\frac{\overline{p} r_0}{4} \right)^2 \right]$$

On doing this, and recalling that $\overline{p}^2 = -j\omega\mu_0\sigma$, we find

$$\overline{Z}(\omega) = R_{DC} + j\omega R_{DC} \frac{\mu_0 \sigma r_0^2}{8} = R_{DC} + j\omega \underbrace{\left(\frac{\mu_0}{8\pi} \right)}_{L_{\text{inner}}} \tag{8.48}$$

While the real part of the impedance coincides with the R_{DC} contribution, the imaginary part, which is associated with the magnetic energy stored inside the conductor itself, is characterized by a per-unit-length internal inductance $L_{\text{inner}} = \mu_0/(8\pi)$. This result is not new – go to Chapter 4 and check Problem 4.15.1.

For high-frequencies, $|\overline{p} r_0| \gg 1$, $\delta \ll r_0$, we may substitute the Bessel functions J_0 and J_1 in (8.47) for their asymptotic behavior,

$$\lim_{\overline{p} r_0 \to \infty} \frac{J_0(\overline{p} r_0)}{J_1(\overline{p} r_0)} = j$$

from which we get

$$\overline{Z}(\omega) = j \frac{\overline{p}}{2\pi r_0 \sigma} = R_\delta + jR_\delta, \text{ with } R_\delta(\omega) = \frac{1}{(2\pi r_0 \delta)\sigma} \propto \sqrt{\omega} \tag{8.49a}$$

where use was made of

$$\overline{p} = \frac{1-j}{\delta}, \text{ with } \delta = \sqrt{\frac{2}{\omega \mu_0 \sigma}}$$

For high-frequency regimes the real and imaginary parts of the impedance tend to be equal and to increase proportionally to the square root of the frequency. For the physical interpretation of the per-unit-length resistance R_δ introduced in (8.49a), you can think of the conductor current as being concentrated near to the conductor's peripheral surface and flowing across a thin circular crown of thickness equal to the penetration depth δ. From this point of view, the conductor's effective area utilized by the current is $S_\delta = 2\pi r_0 \delta$.

The result in (8.49a) can be rewritten in order to reveal the high-frequency conductor's internal inductance

$$\overline{Z}(\omega) = R_\delta + j\omega L_{\text{inner}}, \text{ with } L_{\text{inner}}(\omega) = \frac{1}{2\pi r_0}\sqrt{\frac{\mu_0}{2\omega\sigma}} \propto \frac{1}{\sqrt{\omega}} \tag{8.49b}$$

As the frequency increases, the magnetic field is expelled from inside the conductor, becoming confined to a peripheral layer whose thickness gradually decreases. In the limiting case $\omega \to \infty$, the magnetic energy stored in the conductor goes to zero and, consequently, $L_{\text{inner}} \to 0$.

The general result in (8.47) is illustrated in Figure 8.9, where a graph of the typical evolution of the conductor's impedance against the frequency is shown.

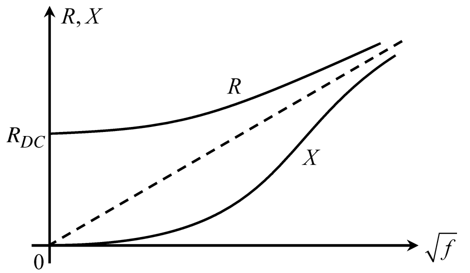


Figure 8.9 Graph of the conductor's internal resistance and reactance against frequency

8.8 Proposed Homework Problems

Problem 8.8.1

Consider a coaxial cable characterized by inner and outer radii r_1 and r_2. Assume that the cable conductors and the insulation medium are perfect. Denote by $u(t)$ and $i(t)$ the cable voltage and current at a given cross-section, $z = \text{constant}$.

Q_1 Write the equations for the electric and magnetic fields inside the cable, $r_1 \le r \le r_2$.

Q_2 Determine the Poynting vector $\mathbf{S}$.

Q_3 Determine the power p transmitted by the cable.

Answers

Q_1

$$\mathbf{E}(t) = \frac{u(t)}{r\ln(r_2/r_1)}\,\vec{e}_r,\;\; \mathbf{H}(t) = \frac{i(t)}{2\pi r}\,\vec{e}_\phi$$

Q_2

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \frac{ui}{2\pi r^2\ln(r_2/r_1)}\,\vec{e}_z$$

Q_3

$$p = \int\limits_{S_z} \mathbf{S}\cdot\vec{e}_z\,dS = \frac{ui}{2\pi\ln(r_2/r_1)}\int\limits_{r_1}^{r_2}\frac{1}{r^2}\,2\pi r\,dr = ui$$

Problem 8.8.2

A uniform monochromatic plane wave propagating along the positive z axis is characterized, in free space, by the following electric field:

$$\mathbf{E}(z,t) = E_0\cos(\omega t - \beta z)\,\vec{e}_x + E_0\cos(\omega t - \beta z - \pi/2)\,\vec{e}_y$$

Q_1 Write $\overline{\mathbf{E}}(z)$ and determine $\overline{\mathbf{H}}(z)$ and $\mathbf{H}(z,t)$.

Q_2 Classify both fields as to their type of polarization and determine the corresponding rms values.

Q_3 Determine the complex Poynting vector.

Q_4 Assuming $E_0 = 5\,\mathrm{V/m}$, find the time-averaged value of $\mathbf{S}$.

Answers

Q_1 $\overline{\mathbf{E}}(z) = E_0\,e^{-j\beta z}\,\vec{e}_x + E_0\,e^{-j(\beta z+\pi/2)}\,\vec{e}_y.$

$$\overline{\mathbf{H}}(z) = \frac{j}{\omega\mu_0}\operatorname{curl}\overline{\mathbf{E}}(z) = H_0\,e^{-j(\beta z-\pi/2)}\,\vec{e}_x + H_0\,e^{-j\beta z}\,\vec{e}_y$$

where

$$H_0 = \frac{E_0}{R_w} \text{ and } R_w = \frac{\omega\mu_0}{\beta} = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$\mathbf{H}(z,t) = \Re\left(\overline{\mathbf{H}}(z)\,e^{j\omega t}\right) = H_0\cos(\omega t - \beta z + \pi/2)\,\vec{e}_x + H_0\cos(\omega t - \beta z)\,\vec{e}_y.$$

Q_2 Both **E** and **H** are circularly polarized fields. The tip of both vectors rotates in the xy plane with angular velocity ω, **E** and **H** remaining orthogonal to each other.

$$E_{rms} = E_0, \ H_{rms} = H_0.$$

Q_3

$$\overline{\mathbf{S}} = \frac{\overline{\mathbf{E}} \times \overline{\mathbf{H}}^*}{2} = E_0 H_0 \vec{e}_z$$

Q_4

$$(\mathbf{S})_{av} = \Re\left(\overline{\mathbf{S}}\right) = E_0 H_0 \vec{e}_z, E_0 H_0 = 66.3\,\mathrm{mW/m^2}.$$

Problem 8.8.3

An electric Hertz dipole of length $l = 50\,\mathrm{m}$ oriented along the z axis is excited by a sinusoidal current $i(t) = I\cos\omega t$, with $I = 100\,\mathrm{A}$ and frequency $f = 60\,\mathrm{kHz}$ (LF band/radio beacon applications). In a spherical coordinate system (r, θ, ϕ), the radiated electromagnetic field in free space (μ_0, ε_0) far away from the antenna is given by

$$\mathbf{E} = E_r(r,\theta,t)\,\vec{e}_r + E_\theta(r,\theta,t)\,\vec{e}_\theta, \ \mathbf{H} = H_\phi(r,\theta,t)\,\vec{e}_\phi$$

where

$$E_r = \frac{IR_w l}{2\pi r^2}\cos\theta\cos(\omega t - \beta r), \ E_\theta = \frac{IR_w l}{2\lambda r}\sin\theta\cos(\omega t - \beta r + \pi/2), \ H_\phi = \frac{E_\theta}{R_w}$$

where $R_w = \sqrt{\mu_0/\varepsilon_0}$, $\beta = \omega\sqrt{\mu_0\varepsilon_0} = \omega/c = 2\pi/\lambda$, with $\lambda = c/f = 5\,\mathrm{km}$ denoting the wavelength.

Q_1 What is the type of polarization of the radiated fields?

Q_2 Show that there is no radiation along the dipole axis, $\theta = 0$.

Q_3 Determine the complex amplitudes of **E** and **H** in the dipole equatorial plane ($\theta = \pi/2$), at a distance $r = 100\,\mathrm{km}$. Obtain the corresponding complex Poynting vector.

Q_4 Evaluate the time-averaged power P radiated by the antenna.

Q_5 Determine the dipole radiation resistance defined as $R_R = P/I_{rms}^2$.

Answers

Q_1 The electric field is elliptically polarized in meridian planes, that is planes defined by $\vec{e}_r$ and $\vec{e}_\theta$. The magnetic field is linearly polarized along $\vec{e}_\phi$.

Q_2 For $\theta = 0$ you have $E_\theta = H_\phi = 0$. Since $\mathbf{H} = 0$, the Poynting vector is also zero, $\mathbf{S} = 0$, which implies the absence of radiated power along z, that is along the antenna axis.

Q_3 For $\theta = \pi/2$:

$$\begin{cases} E_\theta = \dfrac{IR_w l}{2\lambda r}\cos(\omega t - \beta r + \pi/2) \\ E_r = 0 \\ H_\phi = \dfrac{Il}{2\lambda r}\cos(\omega t - \beta r + \pi/2) \end{cases}$$

$$\overline{\mathbf{E}} = j\frac{IR_w l}{2\lambda r}e^{-j\beta r}\,\vec{e}_\theta = 1.885\,e^{j\pi/2}\,\vec{e}_\theta \text{ mV/m}$$

$$\overline{\mathbf{H}} = j\frac{Il}{2\lambda r}e^{-j\beta r}\,\vec{e}_\phi = 5\,e^{j\pi/2}\,\vec{e}_\phi \text{ µA/m}$$

Noting that $\vec{e}_\theta \times \vec{e}_\phi = \vec{e}_r$, you find

$$\overline{\mathbf{S}} = \frac{\overline{\mathbf{E}} \times \overline{\mathbf{H}}^*}{2} = 4.71\,\vec{e}_r \text{ nW/m}^2$$

Q_4 The complex amplitudes of **E** and **H**, which depend on r and θ, are

$$\overline{\mathbf{E}}(r,\theta) = \frac{IR_w l}{2\pi r^2}\cos\theta\, e^{-j\beta r}\,\vec{e}_r + j\frac{IR_w l}{2\lambda r}\sin\theta\, e^{-j\beta r}\,\vec{e}_\theta,\ \ \overline{\mathbf{H}}(r,\theta) = j\frac{Il}{2\lambda r}\sin\theta\, e^{-j\beta r}\,\vec{e}_\phi$$

The corresponding complex Poynting vector has two components, one directed along $\vec{e}_\theta$ and the other along $\vec{e}_r$. The component $\overline{S}_\theta$ not only is of little importance for $r \gg \lambda$, but, above all, does not contribute to the radiation of energy. The radial component $\overline{S}_r$ is given by

$$\overline{S}_r = \frac{R_w}{8}\left(\frac{Il}{\lambda r}\sin\theta\right)^2$$

The power radiated outward by the antenna is obtained by integrating $\overline{\mathbf{S}}$ across a spherical surface S_V of radius $r \gg \lambda$, centered at the antenna feeding point:

$$\overline{P} = P = \int_{S_V} \overline{\mathbf{S}} \cdot \vec{e}_r\, dS = \int_{S_V} \overline{S}_r\, dS$$

where $dS = r^2 \sin\theta\, d\theta\, d\phi$, with $\theta \in [0,\ \pi]$ and $\phi \in [0,\ 2\pi]$. Therefore we have

$$P = \frac{R_w}{8}\left(\frac{Il}{\lambda}\right)^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta\, d\theta = R_w \frac{\pi}{3}\left(\frac{Il}{\lambda}\right)^2 = 395\,\text{W}$$

Q_5

$$R_R = R_w \frac{2\pi}{3}\left(\frac{l}{\lambda}\right)^2 = 80\pi^2\left(\frac{l}{\lambda}\right)^2\ \Omega = 79\,\text{m}\Omega$$

Problem 8.8.4

As far as quasi-stationary regimes are concerned, a capacitor is a circuit component whose characteristic parameter is its capacitance. For high-frequency regimes, things are much more complicated. At certain frequencies a capacitor may behave as an inductor!

Consider a parallel-plate capacitor consisting of two circular metallic electrodes of radius r_0 separated by a perfect insulating dielectric medium of very small thickness h – see Figure 8.10. The capacitor current is a time-varying sinusoidal current $i(t) = I\cos(\omega t + \alpha_i)$.

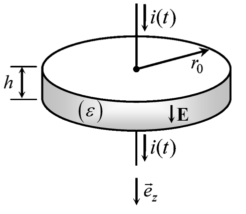


Figure 8.10 A parallel-plate disk-capacitor structure. For high-frequency regimes the device may exhibit an inductor-like behavior

Given the small thickness of the insulation medium, assume that all fields are independent of z. Assume, further, that the electric field **E** between the capacitor plates is vertically oriented along z. According to the geometry of the problem, a particular field solution to Maxwell's equations must be invariant under rotation operations around the z axis.

Take the permittivity and permeability of the insulation medium as ε and μ_0 respectively.

Questions

Q_1 By using a cylindrical coordinate reference frame, write the phasor-domain Maxwell's equations for the problem and obtain a solution for $\overline{\mathbf{H}}$ and $\overline{\mathbf{E}}$.

Q_2 Find the complex Poynting vector $\overline{\mathbf{S}}$ at any point belonging to the boundary cylindrical surface of radius $r = r_0$.

Q_3 Evaluate the complex power $\overline{P}$ corresponding to the inward flux of $\overline{\mathbf{S}}$ across the boundary cylindrical surface of radius r_0 and height h.

Q_4 Determine the frequency-dependent impedance $\overline{Z}(\omega)$ of the capacitor and discuss the resonance conditions of the device. Particularize $\overline{Z}(\omega)$ for very low- and very high-frequency regimes.

Q_5 Take $\varepsilon = 4\varepsilon_0$ and $r_0 = 15.92\,\text{mm}$. Find the first five resonance frequencies of the device.

(Hint: Follow the same line of thought employed in Application Example 8.7, substituting $\varepsilon\,\partial\mathbf{E}/\partial t$ for $\sigma\mathbf{E}$.)

Answers

Q_1

$$\overline{\mathbf{E}} = \overline{E}(r)\,\vec{e}_z, \quad \overline{\mathbf{H}} = \overline{H}(r)\,\vec{e}_\phi.$$

$$\frac{d\overline{E}}{dr} = j\omega\mu_0\overline{H}$$

$$\frac{d\overline{H}}{dr} + \frac{\overline{H}}{r} = j\omega\varepsilon\overline{E}$$

$$\frac{d^2\overline{E}}{dr^2} + \frac{1}{r}\frac{d\overline{E}}{dr} + \underbrace{\left(\omega\sqrt{\mu_0\varepsilon}\right)}_{\beta}{}^2\overline{E} = 0$$

$$\overline{E}(r) = E_1\,J_0(\beta r) + E_2\,Y_0(\beta r), \text{ where } \begin{cases} E_1 = \dfrac{\beta\overline{I}}{j\omega\varepsilon 2\pi r_0 J_1(\beta r_0)} \\ E_2 = 0 \end{cases}$$

$$\overline{E}(r) = -j\frac{\overline{I}}{2\pi r_0}\sqrt{\frac{\mu_0}{\varepsilon}}\frac{J_0(\beta r)}{J_1(\beta r_0)}$$

$$\overline{H}(r) = \frac{1}{j\omega\mu_0}\frac{d\overline{E}(r)}{dr} = \frac{\overline{I}}{2\pi r_0}\frac{J_1(\beta r)}{J_1(\beta r_0)}$$

Q_2

$$\overline{\mathbf{S}}(r_0) = \frac{\overline{\mathbf{E}}(r_0)\times\overline{\mathbf{H}}^*(r_0)}{2} = -\overline{S}(r_0)\vec{e}_r, \text{ where } \overline{S}(r_0) = \frac{I_{rms}^2}{j(2\pi r_0)^2}\sqrt{\frac{\mu_0}{\varepsilon}}\frac{J_0(\beta r_0)}{J_1(\beta r_0)}$$

Q_3

$$\overline{P} = \int_{S_V}\overline{\mathbf{S}}(r_0)\cdot(-\vec{e}_r)\,dS = 2\pi r_0 h\overline{S}(r_0) = \left(\frac{h}{j2\pi r_0}\sqrt{\frac{\mu_0}{\varepsilon}}\frac{J_0(\beta r_0)}{J_1(\beta r_0)}\right)I_{rms}^2$$

Q_4 According to (8.34), the term above, in the large parentheses, is to be interpreted as the impedance of the capacitor, which can be rewritten as

$$\overline{Z}(\omega) = jX(\omega) = \frac{1}{j\omega C_0}\beta r_0\frac{J_0(\beta r_0)}{2J_1(\beta r_0)} \tag{8.51}$$

where C_0 is the electrostatic capacitance, $C_0 = \left(\varepsilon\pi r_0^2\right)/h$, and β depends on the frequency, $\beta = \omega\sqrt{\mu_0\varepsilon}$.

From (8.51) you can see that the capacitor reactance $X(\omega)$ changes sign whenever the Bessel functions J_0 and J_1 go through zero. The first zeros of J_0 occur for $\beta r_0 \approx 2.405, 5.520, 8.654, \ldots$. The first zeros of J_1 occur for $\beta r_0 \approx 0$, 3.832, 7.016, 10.174, ...

In the range $0 < \beta r_0 < 2.405$ the parallel-plate device behaves as a capacitor, $X(\omega) < 0$. For $\beta r_0 = 2.405$ it behaves as a short circuit (resonance). In the range $2.405 < \beta r_0 < 3.832$

the device behaves as an inductor, $X(\omega) > 0$. For $\beta r_0 = 3.832$ it behaves as an open circuit (resonance). In the range $3.832 < \beta r_0 < 5.520$ the device again behaves as a capacitor, $X(\omega) < 0$. And so on.

This alternate behavior between capacitive and inductive character corresponds, respectively, to the dominance of electric energy or the dominance of magnetic energy stored in the insulation medium. (Note: Magnetic energy storage in the dielectric is associated with the magnetic field produced by displacement currents.)

Resonance situations occur for frequencies such that $(W_m)_{av} = (W_e)_{av}$.
The general result in (8.51) is illustrated in Figure 8.11, where a graph of the typical evolution of the capacitor reactance is plotted against βr_0.

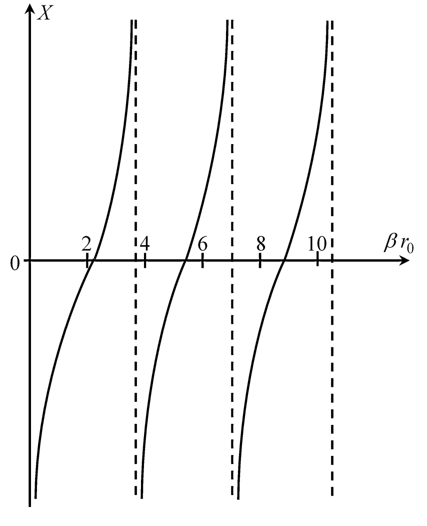


Figure 8.11 Graph of the reactance X of the disk capacitor against βr_0, where $\beta = \omega\sqrt{\mu_0 \varepsilon}$

For very low-frequency regimes, $\beta r_0 \to 0$, you have

$$\beta r_0 \frac{J_0(\beta r_0)}{2J_1(\beta r_0)} \to 1$$

hence

$$\overline{Z}(\omega) \approx \frac{1}{j\omega C_0}$$

For very high-frequency regimes, $\beta r_0 \gg 1$, you must consider the asymptotic behavior of the Bessel functions, that is

$$\frac{J_0(\beta r_0)}{J_1(\beta r_0)} \to \frac{1}{\tan(\beta r_0 - \pi/4)}$$

hence

$$\overline{Z}(\omega) \approx \frac{1}{j\omega C_0} \frac{\beta r_0/2}{\tan(\beta r_0 - \pi/4)}$$

Q_5 $f_1 = 4.81\,\text{GHz}$; $f_2 = 7.66\,\text{GHz}$; $f_3 = 11.04\,\text{GHz}$: $f_4 = 14.03\,\text{GHz}$: $f_5 = 17.31\,\text{GHz}$.

Problem 8.8.5

Consider a copper wire ($\mu = \mu_0$, $\sigma = 5.7 \times 10^7\,\text{S/m}$) of radius $r_0 = 1\,\text{mm}$ and length $l = 1\,\text{m}$. The wire carries a time-varying current given by $i(t) = I_0 + I_1 \cos(\omega_1 t)$, with $\omega_1 = 2\pi f_1$, $f_1 = 1\,\text{MHz}$.

Q_1 Making use of skin-effect results (see Application Example 8.7) determine the wire's resistance and inductance parameters for the DC component and for the AC component.

Q_2 Using the superposition principle, determine the longitudinal voltage drop $u(t)$ along the wire length.

Q_3 Show that $u(t)$ and $i(t)$ cannot, in any way, be related by an equation of the usual type $u = Ri + L\,di/dt$.

Answers

Q_1 From (8.48) you have

$$\omega = 0 \begin{cases} R_0 = \dfrac{l}{\sigma\pi r_0^2} = 5.6\,\text{m}\Omega \\ L_0 = \dfrac{\mu_0 l}{8\pi} = 50\,\text{nH} \end{cases}$$

For the AC component, the skin-effect penetration depth, (8.38), is

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} = 67\,\mu\text{m}$$

Since $\delta \ll r_0$, you can use the high-frequency results established in (8.49):

$$\omega = \omega_1 \begin{cases} R_1 = \dfrac{l}{\sigma 2\pi r_0 \delta} = R_0 \dfrac{r_0}{2\delta} = 41.7\,\text{m}\Omega \\ L_1 = \dfrac{R_1}{\omega_1} = 6.64\,\text{nH} \end{cases}$$

Q_2 For the DC component: $u_0 = R_0 I_0$.

For the AC component: $\overline{U}_1 = \sqrt{2} R_1 I_1 e^{j\pi/4} \rightarrow u_1(t) = \sqrt{2} R_1 I_1 \cos(\omega_1 t + \pi/4)$.

Hence

$$u(t) = R_0 I_0 + \sqrt{2} R_1 I_1 \cos(\omega_1 t + \pi/4) \tag{8.52}$$

Q_3 Using $i(t) = I_0 + I_1 \cos(\omega_1 t)$ in an equation like $u = Ri + L\,di/dt$ would yield

$$u(t) = RI_0 + RI_1 \cos(\omega_1 t) - \omega_1 L I_1 \sin(\omega_1 t)$$

or, which is the same,

$$u(t) = RI_0 + \sqrt{R^2 + (\omega_1 L)^2}\ I_1 \cos\left(\omega_1 t + \arctan\frac{\omega_1 L}{R}\right) \tag{8.53}$$

In order to ensure that (8.53) would coincide with the correct result in (8.52), it would be required that $R = R_0 = R_1 = \omega_1 L$, which is an impossible condition!

This clearly shows that, when analyzing skin-effect phenomena in the time domain, utilization of equations of the type $u = Ri + L\,di/dt$ is, in general, a mistake.