# 3

# Stationary Currents

## 3.1 Fundamental Equations

The topic of stationary currents (also called direct currents) belongs within the subclass of stationary field phenomena. The properties of time-invariant electric currents, associated with free charges moving along closed conductor circuits, are analyzed in this chapter.

The fundamental laws governing stationary current problems are those in (PII.3) together with a constitutive relation concerning conductor media behavior. That is,

$$\begin{cases} \text{curl } \mathbf{E} = 0 \\ \text{div } \mathbf{J} = 0 \end{cases} \tag{3.1}$$

and

$$\mathbf{J} = \sigma \mathbf{E} \tag{3.2}$$

where $\sigma$ denotes conductor conductivity (units: S/m, siemens per meter).

The equation curl $\mathbf{E} = 0$ has been fully examined in Chapter 2. The properties of the electric field vector $\mathbf{E}$ we studied in electrostatics are exactly the same that you need to keep in mind throughout this new chapter.

However, here – contrary to electrostatics – because currents are allowed to exist ($\mathbf{J} \neq 0$), the electric field vector inside conductors is not zero, $\mathbf{E} = \mathbf{J}/\sigma$, and, consequently, conductors can no longer be considered equipotential bodies. Only in the limit case of perfect conductors ($\sigma \to \infty$) can you use the approximation $\mathbf{E} = 0$ and $V =$ constant.

## 3.2 Conductivity, Current Density, Electric Circuits

As far as conduction properties are concerned, material media can be coarsely split into two major categories, insulators and conductors. While insulators, like glass, mica, rubber, etc., are characterized by extremely low conductivity values in the range $10^{-8}$ to $10^{-17}$ S/m, conductors, like silver, copper, aluminum, etc., have extremely high conductivity values in the range $10^{6}$ to $10^{7}$ S/m. Typical conductivities (at 20 °C) for some common conductors

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**Table 3.1** Conductor conductivities

| Conductor | Conductivity (S/m) at 20 °C |
|---|---|
| Aluminum | $3.5 \times 10^7$ |
| Constantan | $2 \times 10^6$ |
| Copper | $5.6 \times 10^7$ |
| Gold | $4.1 \times 10^7$ |
| Graphite | $10^5$ |
| Iron | $10^6$ |
| Manganin | $2.3 \times 10^6$ |
| Silver | $6.1 \times 10^7$ |
| Tin | $9 \times 10^6$ |
| Water (sea water) | 5.6 |
| Water (tap water) | 0.01–0.1 |

are listed in Table 3.1. Note that the conductivity is a temperature-dependent parameter; conductivities decrease with increasing temperature in the case of metallic conductors.

Free charges inside a conducting medium can move under the influence of impressed electric fields. In the case of good conductors (metals), free charges are electrons and their movement occurs in the direction opposite to **E**; however, from a formal point of view, you can imagine that an equivalent flow of positive charges occurs parallel to **E** (Figure 3.1).

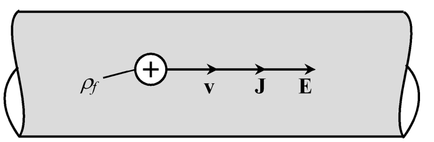


**Figure 3.1** Current flow inside a conductor driven by an electric field

Electric currents are free charges in movement. Hence, in order to provide a physical interpretation for the current density vector **J**, we can write

$$\mathbf{J} = \rho_f \mathbf{v} \tag{3.3}$$

where $\rho_f$ (C/m$^3$) represents the positive free charge per unit volume and **v** denotes the average value of the charge velocity parallel to the impressed **E** field.

For not very intense fields (linear media) the velocity **v**, resulting from random collision processes inside the medium atomic lattice, is proportional to **E**

$$\mathbf{v} = m\mathbf{E} \tag{3.4}$$

where $m$ is the so-called charge mobility. (Note: The electron mobility, for good conductors, can typically be found in the range $10^{-2}$ to $10^{-3}\,\mathrm{m^2\,V^{-1}\,s^{-1}}$.)

Finally, from (3.2)–(3.4), we obtain $\mathbf{J} = \sigma\mathbf{E} = \rho_f m\mathbf{E}$, from which you can see that $\sigma = \rho_f m$.

At this stage it is worth making an important remark. You have certainly heard before that electrical signals propagate at a velocity close to the speed of light ($3 \times 10^8$ m/s). Well, this is true. But do not confuse matters: such a velocity has absolutely nothing to do with the velocity that electrons move along conductors!

Considering **E**-field values that may typically occur inside good conductors ($\approx$100 V/m), you can check from (3.4) that the velocity of the electrons is merely about 1 m/s.

Let us now return to (3.1) and focus our attention on the equation div $\mathbf{J} = 0$.

Again using the Gauss theorem – see (2.9) from Chapter 2 – we obtain

$$\int_{S_V} \mathbf{J} \cdot \mathbf{n}_o \, dS = 0 \tag{3.5}$$

This means that, in the framework of stationary regimes ($\partial/\partial t = 0$), the flux of **J** across a closed surface bounding a given volume is always zero or, in other words, the number of **J**-field lines entering a given volume is equal to those leaving it (Figure 3.2).

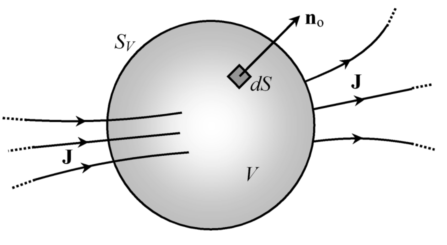


**Figure 3.2** The flux of **J** across a closed surface is zero for time-invariant regimes

An obvious consequence of this result is that the field lines of **J** cannot end or start anywhere. In general, any field vector whose *divergence* is zero must have its field lines closed.

An electric circuit is made of simple or multiple conductor connections forming closed loops so as to ensure that **J**-field lines are closed, otherwise one would end up with $\mathbf{J} = 0$ as in electrostatics.

The simplest electric circuit that can be imagined consists of a conductor loop immersed in a dielectric insulating medium. The conductor loop forms a closed tube where free charges can circulate. In most applications, given the huge discrepancy between conductor and insulator conductivities, leakage currents escaping the conductor loop (the circuit) can be considered absolutely negligible.

Although electric circuits are practically perfect tubes for **J**-field lines, you must be aware that the same is not true for the electric field **E**. In fact, **E** exists not only inside the circuit conductors but outside of them as well.

At the conductor/insulator interface (conductor side) you have a purely tangential component for the electric field, $\mathbf{E}_{\text{cond}} = \mathbf{E}_t = \mathbf{J}/\sigma$.

At the conductor/insulator interface (insulator side), although $\mathbf{J} = 0$, you will not get $\mathbf{E} = 0$. The electric field vector on the insulator side can be obtained by adding two orthogonal components

$$\mathbf{E}_{\text{insul}} = \mathbf{E}_{\text{n}} + \mathbf{E}_{\text{t}}$$

According to (2.17), the magnitude of the normal component, which is usually the dominant term, depends on the local surface charge density, $E_{\text{n}} = w/\varepsilon$.

As for the tangential component, it is a simple matter to show (using curl $\mathbf{E} = 0$) that it coincides with the one observed inside the conductor.

To prove the continuity of the tangential component of the electric field vector across the conductor/insulator interface, consider the illustration in Figure 3.3 where a closed rectangular infinitesimal path **s** is depicted.

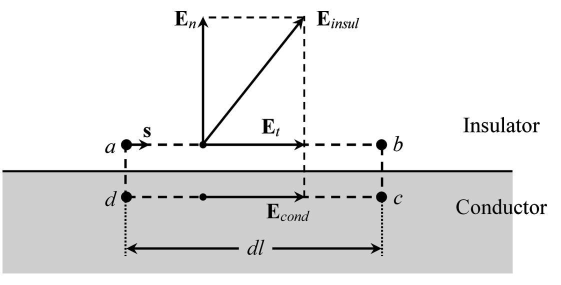


**Figure 3.3** The conductor's imperfection ($\sigma \neq \infty$) gives rise to a tangential component of the electric field vector which is continuous at the conductor/insulation interface

Then from

$$\oint_{\mathbf{s}} \mathbf{E} \cdot d\mathbf{s} = 0 = \int_{\overrightarrow{ab}} \mathbf{E}_{\text{insul}} \cdot d\mathbf{s} + \int_{\overrightarrow{cd}} \mathbf{E}_{\text{cond}} \cdot d\mathbf{s} = (E_{\text{t}} - E_{\text{cond}})\, dl$$

we conclude that $E_{\text{t}} = E_{\text{cond}} = J/\sigma$.

## 3.3 Current Intensity, Kirchhoff's Current Law

The notion of current intensity has already been introduced in Chapter 1.

Current intensity in a conductor, $I$, is just a simple measure of the flux of **J**-field lines through a conductor cross-section $S$ in a prespecified reference direction **n** (recall Figure 1.3(b))

$$I = \int_S \mathbf{J} \cdot \mathbf{n}\, dS \tag{3.6}$$

Not surprisingly, for stationary regimes ($\partial/\partial t = 0$), the current intensity through a conductor immersed in an insulating medium does not depend on the particular cross-section being

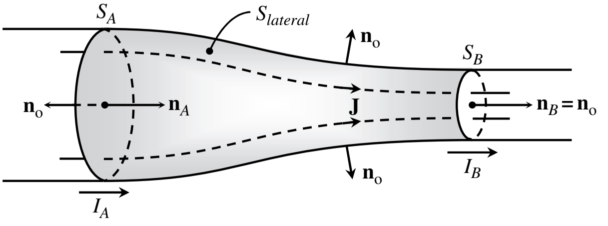


**Figure 3.4** Application of the Gauss theorem to show that the current intensity along a conductor remains unchanged, $I_A = I_B$, for time-invariant regimes

considered. Take the conductor volume bounded by $S_A$, $S_B$ and $S_{\text{lateral}}$ shown in Figure 3.4, and make use of the result in (3.5).

Then, taking into account that the external medium is an insulator ($\sigma \approx 0$), you get

$$0 = \int_{S_V} \mathbf{J} \cdot \mathbf{n}_o \, dS = \int_{S_A} \mathbf{J} \cdot \mathbf{n}_o \, dS + \int_{S_B} \mathbf{J} \cdot \mathbf{n}_o \, dS + \overbrace{\int_{S_{\text{lateral}}} \mathbf{J} \cdot \mathbf{n}_o \, dS}^{0}$$

$$= -\int_{S_A} \mathbf{J} \cdot \mathbf{n}_A \, dS + \int_{S_B} \mathbf{J} \cdot \mathbf{n}_B \, dS = -I_A + I_B = 0$$

Hence, you can see that $I_A$ across $S_A$ and $I_B$ across $S_B$ are identical, and for that reason you can drop the unnecessary subscript labels $A$ and $B$,

$$I_A = I_B = I$$

By the same token, Kirchhoff's current law (KCL) can be obtained. Consider a closed surface $S_V$ which is intersected by several current-carrying conductors – see Figure 3.5.

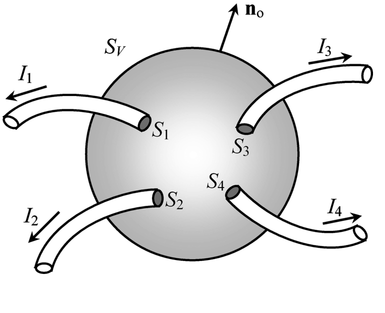


**Figure 3.5** Kirchhoff's current law, $\sum I_k = 0$

Then, from (3.5), you get

$$0 = \int_{S_V} \mathbf{J} \cdot \mathbf{n}_o \, dS = \int_{S_1} \mathbf{J} \cdot \mathbf{n}_o \, dS + \int_{S_2} \mathbf{J} \cdot \mathbf{n}_o \, dS + \int_{S_3} \mathbf{J} \cdot \mathbf{n}_o \, dS + \int_{S_4} \mathbf{J} \cdot \mathbf{n}_o \, dS = I_1 + I_2 + I_3 + I_4$$

or, more generally,

$$\sum_k I_k = 0 \tag{3.7}$$

As in Chapter 1, you should notice that KCL is not, strictly speaking, a general 'law'. Indeed, the result in (3.7) is only valid for stationary phenomena, $\partial/\partial t = 0$. For time-varying regimes (where div $\mathbf{J} \neq 0$) things are a little more complicated – see Chapter 6.

## 3.4 Resistor, Conductance, Resistance, Ohm's Law

To put it simply, a resistor is nothing but a piece of conducting material with two accessible terminals. When a voltage $U$ is applied between the resistor terminals a current of intensity $I$ will flow along the device – see Figure 3.6.

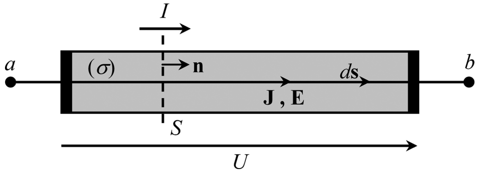


**Figure 3.6** Voltage and current in a resistor

If the conducting material behaves as a linear medium, or, put another way, if $\mathbf{J} = \sigma\mathbf{E}$, then $I$ and $U$ will be proportional:

$$I = GU \tag{3.8}$$

The proportionality constant $G$ is called conductance (units: S, siemens). This parameter depends not only on the geometrical configuration of the resistor, but also on the conductivity of the material of which the resistor is made. However, you can see from

$$G = \frac{I}{U} = \frac{\int_S \overbrace{\sigma\mathbf{E}}^{\mathbf{J}} \cdot \mathbf{n}\, dS}{\int_{\overrightarrow{ab}} \mathbf{E} \cdot d\mathbf{s}} \tag{3.9}$$

that $G$ does not depend on the intensity of the electric field in the device.

In many instances, it is often preferred to utilize the inverse of $G$ to describe the resistor's characteristics. The inverse of $G$ is called resistance, $R = 1/G$ (units: $\Omega$, ohm). By employing $R$, (3.8) translates into

$$U = RI \tag{3.10}$$

which you will certainly recognize as a statement of the familiar Ohm's law.

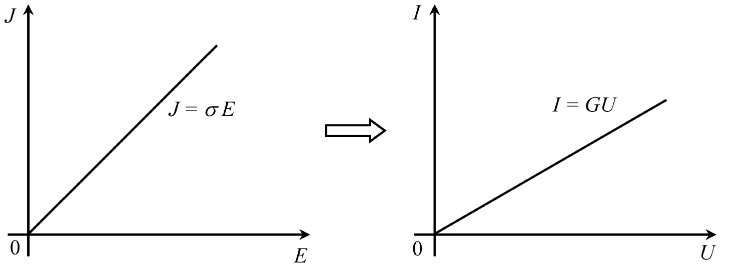


**Figure 3.7** The linear relationship $I(U)$ is just a consequence of the linear relationship $J(E)$

At this point two interesting remarks are in order.

The linear relationship between $I$ and $U$, (3.8), is just a consequence of the assumed linear relationship between **J** and **E**, (3.2) – see Figure 3.7.

Also, if you compare the definition of conductance in (3.9) to the one for the capacitance given in (2.21) you will see the striking analogy between them. This analogy provides you with a very effective means to easily compute $C$ from $G$, or $G$ from $C$, whenever you have a capacitor and a resistor with the same geometrical features.

As a simple example, consider a parallel-plate geometrical configuration (Figure 3.8), where in one case the medium sandwiched between the plates is an insulator with permittivity $\varepsilon$ (capacitor) and in the other case the medium is a conductor with conductivity $\sigma$ (resistor).

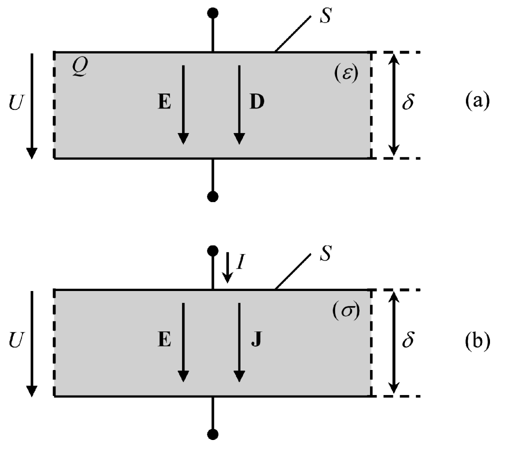


**Figure 3.8** Utilization of a parallel-plate structure for showing the analogy between electrostatics and stationary current problems. (a) Capacitor. (b) Resistor

As for the capacitance of the capacitor (see Chapter 2), you get

$$C = \frac{Q}{U} = \frac{DS}{E\delta} = \frac{\varepsilon S}{\delta}$$

Likewise, for the conductance of the resistor, you get

$$G = \frac{I}{U} = \frac{JS}{E\delta} = \frac{\sigma S}{\delta}$$

It then becomes obvious that

$$G = \frac{\sigma}{\varepsilon} C \tag{3.11}$$

## 3.5 Application Example (The Potentiometer)

Potentiometers are variable resistors that you may find in a large variety of electrical and electronic appliances. These devices are used principally as gain or volume controls, voltage dividers and current controls. Figure 3.9(a) shows the geometrical configuration of a common type of potentiometer and Figure 3.9(b) shows its corresponding equivalent electric circuit.

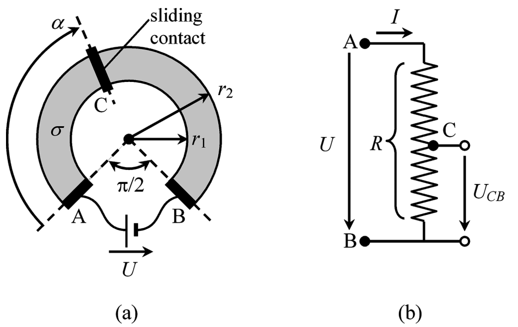


**Figure 3.9** Potentiometer. (a) Geometrical configuration. (b) Equivalent circuit representation

The resistor, which includes a sliding contact (terminal C), is made of a thin layer of conducting material of conductivity $\sigma$ with the shape of a circular crown of width $w = r_2 - r_1$ and thickness $t$ (perpendicular to the drawing plane).

A voltage $U$ is applied between the metallic terminals A and B.

Data: $r_2 = 9\,\text{mm},\ r_1 = 6\,\text{mm},\ t = 100\,\mu\text{m},\ \sigma = 10^3\,\text{S/m}.$

*Questions*

$Q_1$ Obtain the equation for the radial dependence of the current density $\mathbf{J}$. Using the latter find an expression for the current intensity $I$.

$Q_2$ Deduce an expression for the resistor resistance $R$ and compute its numerical value.

$Q_3$ Evaluate the voltage $U_{CB}$ as a function of both $\alpha$ and $U$.

$Q_4$ Show that if the width of the circular crown is small, then the device resistance can be approximately evaluated through $R = l/(\sigma S)$, where $l$ is the average length of the resistor and $S$ denotes its cross sectional area. Taking into account the problem data, estimate the relative error incurred by using such an approximation.

*Solutions*

$Q_1$ The electric field lines inside the resistor are circumferential arcs parallel to the circular crown walls. For geometrical reasons the intensity of **E** remains constant along each field line; however, when you jump from field line to field line (that is, when you change $r$) the field intensity must vary; it is weaker at the outer wall ($r = r_2$) but stronger at the inner wall ($r = r_1$). Hence you can write

$$\text{For } r_1 \le r \le r_2 : \mathbf{E} = E(r)\vec{e}_\phi$$

Integration of **E** along a circumferential arc starting at A and ending at B (infinitesimal path length $d\mathbf{s} = r\ d\phi\ \vec{e}_\phi$) yields the applied voltage $U$ between the potentiometer terminals

$$U = \int_{\overrightarrow{AB}} \mathbf{E} \cdot d\mathbf{s} = \int_{\phi=0}^{\phi=\frac{3}{2}\pi} E(r) r\ d\phi = \frac{3\pi r}{2} E(r) \rightarrow E(r) = \frac{2U}{3\pi r} \tag{3.12}$$

From (3.12) you can find the current density field,

$$\mathbf{J} = \sigma \mathbf{E} = \mathbf{J}(r) = \frac{2\sigma U}{3\pi r}\vec{e}_\phi$$

The current intensity $I$ is obtained by evaluating the flux of **J** through the rectangular cross-section $S$ of the resistor

$$I = \int_S \mathbf{J} \cdot \mathbf{n}\ dS, \text{ where } \mathbf{n} = \vec{e}_\phi \text{ and } dS = t\ dr$$

This gives

$$I = U \frac{2t\sigma}{3\pi} \int_{r_1}^{r_2} \frac{dr}{r} = U\left(\frac{2t\sigma}{3\pi} \ln \frac{r_2}{r_1}\right)$$

$Q_2$ The total resistance of the potentiometer is obtained from the above result through

$$R = \frac{U}{I} = \frac{3\pi}{2t\sigma \ln(r_2/r_1)} \tag{3.13}$$

Numerically, we obtain $R = 116.2\ \Omega$.

$Q_3$ The $U_{\mathrm{CB}}$ voltage is determined as in (3.12) by substituting C for A,

$$U_{CB} = \int_{\overrightarrow{CB}} \mathbf{E} \cdot d\mathbf{s} = \int_{\phi=\alpha}^{\phi=\frac{3}{2}\pi} E(r) r\ d\phi = \left(\frac{3\pi}{2} - \alpha\right)\frac{2U}{3\pi} = U\left(1 - \frac{\alpha}{3\pi/2}\right)$$

$Q_4$ Define the average length of the resistor as $l = r_{av}3\pi/2$, where $r_{av}$ is the average radius of the potentiometer, $r_{av} = (r_2 + r_1)/2$.
Take $r_2 = r_{av}(1+\delta)$ and $r_1 = r_{av}(1-\delta)$. By using (3.13) you find

$$R = \frac{3\pi}{2t\sigma \ln\left(\dfrac{1+\delta}{1-\delta}\right)}$$

Taking into account that, for small $\delta$,

$$\ln\left(\frac{1+\delta}{1-\delta}\right) \approx 2\delta = \frac{r_2 - r_1}{r_{av}}$$

you can obtain the approximation

$$R_{\text{approx}} = \frac{r_{av}3\pi/2}{\sigma\,(t(r_2 - r_1))} = \frac{l}{\sigma S}$$

thus $R_{\text{approx}} = 117.8\ \Omega$, giving an excess error of 1.4 %.

## 3.6 Application Example (The Wheatstone Bridge)

The Wheatstone bridge is a very simple circuit network which finds application in instrumentation and measurement. The circuit, represented in Figure 3.10 permits the experimental determination of an unknown resistance $R$ based on previous knowledge of $R_1$, $R_2$ and $R_3$ (one of them being a variable resistor).

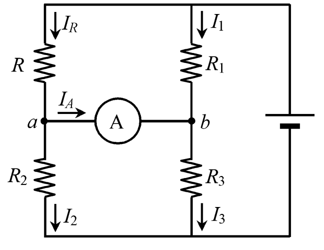


**Figure 3.10** The Wheatstone bridge

Assume that $R_3$ has been adjusted so as to ensure that the ammeter placed between $a$ and $b$ measures zero current – a balanced bridge ($I_A = 0$ and $U_{ab} = 0$).

*Questions*

$Q_1$ By using KVL and KCL, write the equations governing the circuit.

$Q_2$ Determine the relationship among $R$, $R_1$, $R_2$ and $R_3$ when the bridge is balanced.

*Solutions*

$Q_1$ $RI_R + U_{ab} - R_1I_1 = 0$; $R_3I_3 + U_{ab} - R_2I_2 = 0$.
$I_A + I_2 - I_R = 0$; $I_3 - I_A - I_1 = 0$.

$Q_2$ By making $U_{ab} = 0$ and $I_A = 0$, you find

$$R = \frac{R_1R_2}{R_3}$$

## 3.7 Joule Losses, Generator Applied Field

You should know that resistors get hot when submitted to currents. We have already mentioned that this effect – the Joule effect – is associated with electron random collisions inside the atomic lattice of the resistor medium. The energy dissipated by this process may vary from point to point inside the resistor. We will now see that dissipation is proportional to $E^2$, meaning that hot spots in a resistor are regions where $E$ has attained increased values.

Consider, as shown in Figure 3.11, that an infinitesimal volume $dV$ of the conductor contains an infinitesimal amount of free charge $dq = \rho_f dV$.

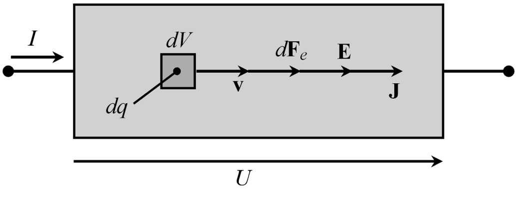


**Figure 3.11** Vectors involved in the analysis of Joule losses in a resistor

Under the influence of the impressed **E** field, the free charge $dq$ drifts with a velocity **v** driven by an elemental electric force $d\mathbf{F}_e = dq\mathbf{E}$.

The activity of the latter force produces an elemental power $dp$ which is dissipated in $dV$

$$dp = \mathbf{v} \cdot d\mathbf{F}_e = \rho_f \mathbf{v} \cdot \mathbf{E}\ dV = \hat{p}_J\ dV$$

Taking into account that $\mathbf{J} = \rho_f \mathbf{v}$ from (3.3), integration of the above result over the resistor's whole volume $V$ yields the total power losses (Joule losses), that is

$$P_J = \int_V \hat{p}_J\ dV; \quad \hat{p}_J = \mathbf{J} \cdot \mathbf{E} = \sigma E^2 \tag{3.14}$$

The local power losses density $\hat{p}_J$ (W/m$^3$) is thus shown to increase with $E^2$.

The preceding formulation for the power losses in a conductor not only permits the evaluation of the total losses in a resistor, but, further, has the additional advantage of allowing for a detailed perception of what is happening locally inside the resistor; for instance, where the hot spots are localized.

Sometimes this detailed knowledge is unimportant. Quite often you may only want to evaluate the resistor's total losses based on its voltage and current.

For illustrative purposes, let us revisit the resistor geometry previously shown in Figure 3.8(b). By making $J = I/S$ and $E = U/\delta$, we obtain from (3.14)

$$P_J = \int_V \frac{UI}{S\delta} dV = \frac{UI}{S\delta} \overbrace{\int_V dV}^{S\delta} = UI = GU^2 = RI^2 \tag{3.15}$$

Not surprisingly, we found $P_J = RI^2$, a result that you are certainly familiar with.

Let us pause for a moment to consider an apparently puzzling and paradoxical question. From the key equations (3.1), you have learnt that, on the one hand, the field lines of $\mathbf{E}$ are open (curl $\mathbf{E} = 0$) and, on the other hand, the field lines of $\mathbf{J}$ are closed (div $\mathbf{J} = 0$); in addition, from $\mathbf{J} = \sigma\mathbf{E}$, you can see that field lines of $\mathbf{E}$ and $\mathbf{J}$ should run parallel.

But how can all this happen? What is missing?

Clearly there is more to stationary currents than could be accounted for by (3.1) and (3.2).

In the framework of stationary phenomena, where do you think the energy necessary for driving the free charges in motion comes from? Where does the heat transferred to the conductor lattice by electron collision processes (Joule losses) come from?

The answer, as you might have already guessed, is: from generators (batteries, photovoltaic cells, electromechanical devices, and so on).

Therefore, the simplest electric circuit that can be imagined must include a generator and an external conductor, forming a closed loop for the circulation of currents – see Figure 3.12.

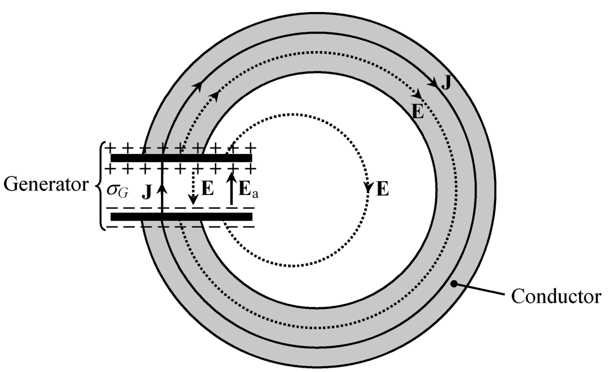


**Figure 3.12** A trivial electric circuit made of a generator and an external conductor loop. While the **J**-field lines are closed, the **E**-field lines are open

As shown in Figure 3.12, the relationship between **J** and **E** inside the generator cannot be the same as in (3.2); there, instead, you have to employ

$$\mathbf{J} = \sigma_G(\mathbf{E} + \mathbf{E}_a) \tag{3.16}$$

where $\sigma_G$ denotes the conductivity of the internal medium of the generator, and $\mathbf{E}_a$, the so-called applied electric field, represents from a macroscopic point of view the per-unit-charge internal force responsible for maintaining the separation of the electric charges residing at the positive and negative generator terminals (that would otherwise collapse together).

From (3.16) and Figure 3.12 you can easily observe that, when the generator is disconnected ($\mathbf{J} = 0$), the opposite fields $\mathbf{E}$ and $\mathbf{E}_a$ have the same magnitude; however, when the external conductor is connected you will get $|\mathbf{E}| < |\mathbf{E}_a|$.

## 3.8 Generator Electromotive Force, Power Balance

The actual electric circuit in Figure 3.12, containing a generator and an external load (a resistor), is symbolically represented in Figure 3.13.

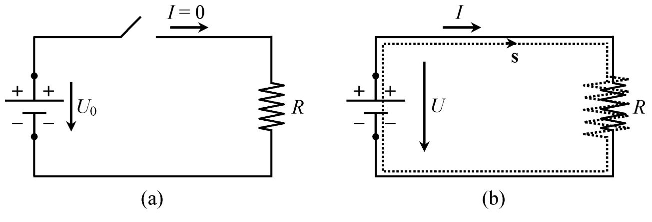


**Figure 3.13** Symbolic description of the circuit depicted in Figure 3.12. (a) With the switch open the generator voltage is given by its electromotive force. (b) With the switch closed the generator voltage is smaller than its electromotive force

When the load is disconnected ($\mathbf{J} = 0$) a voltage $U_0$ appears between the positive and negative terminals of the generator

$$U_0 = \int_{\overrightarrow{+-}} \mathbf{E} \cdot d\mathbf{s}$$

According to (3.16), when $\mathbf{J} = 0$, $\mathbf{E} = -\mathbf{E}_a$ inside the generator. Therefore, the above result can be rewritten as

$$U_0 = \int_{\overrightarrow{+-}} \mathbf{E} \cdot d\mathbf{s} = \int_{\substack{\overrightarrow{-+} \\ \text{generator}}} \mathbf{E}_a \cdot d\mathbf{s} \tag{3.17}$$

An intrinsic characteristic of the generator, since it only depends on the internal applied field $\mathbf{E}_a$, voltage $U_0$ is commonly known by the name of electromotive force (emf).

When the external resistor $R$ is connected across the generator terminals its voltage $U$ decreases compared to $U_0$.

In order to determine the resulting voltage $U$ let us evaluate the line integral of $(\mathbf{E}+\mathbf{E}_\mathrm{a})$ along the closed path $\mathbf{s}$ inside the circuit (Figure 3.13(b)). The line integration is performed by using two alternative processes:

$$\oint\limits_{\mathbf{s}}(\mathbf{E}+\mathbf{E}_\mathrm{a})\cdot d\mathbf{s}=\oint\limits_{\mathbf{s}}\mathbf{E}\cdot d\mathbf{s}+\oint\limits_{\mathbf{s}}\mathbf{E}_\mathrm{a}\cdot d\mathbf{s}=0+\int\limits_{\text{generator}}\mathbf{E}_\mathrm{a}\cdot d\mathbf{s}=U_0$$

and

$$\oint\limits_{\mathbf{s}}(\mathbf{E}+\mathbf{E}_\mathrm{a})\cdot d\mathbf{s}=\int\limits_{\text{generator}}(\mathbf{E}+\mathbf{E}_\mathrm{a})\cdot d\mathbf{s}+\int\limits_{\text{resistor}}(\mathbf{E}+\mathbf{E}_\mathrm{a})\cdot d\mathbf{s}$$

$$=\int\limits_{\text{generator}}\frac{1}{\sigma_G}\mathbf{J}\cdot d\mathbf{s}+\int\limits_{\text{resistor}}\mathbf{E}\cdot d\mathbf{s}=r_G I+U$$

from which we conclude $U_0=r_G I+U$ or, which is the same,

$$U=U_0-r_G I \tag{3.18}$$

The term $r_G I$ represents the internal voltage drop of the generator, where $r_G$ is its internal resistance (both are zero when ideal generators are considered, that is when $\sigma_G\to\infty$).

The relationship $U(I)$ in (3.18) describes the generator's behavior in terms of its intrinsic parameters $U_0$ and $r_G$. In the diagram shown in Figure 3.14 this relationship is represented by the straight line with negative slope.

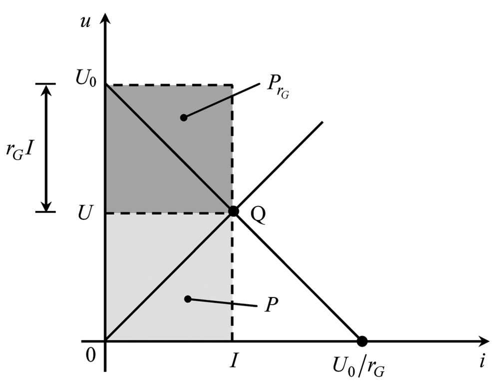


**Figure 3.14** Diagram for power balance analysis. The straight line with negative slope describes the generator's features. The straight line with positive slope characterizes the external resistor. Q is the operating point

On the other hand, the resistor characteristic is described by Ohm's law (3.10), $U=RI$, which in Figure 3.14 is represented by the straight line with positive slope.

The intersection of the two lines permits the identification of the circuit's operating point (point Q), from where $U$ and $I$ can simultaneously be obtained.

At last we turn our attention to power balance analysis. From $U_0 = r_G I + U$ we readily get

$$U_0 I = r_G I^2 + UI$$

The left-hand side of this equation represents the total power produced by the generator's applied field $P_G = U_0 I$. On the right-hand side, the first term represents dissipation losses internal to the generator, $Pr_G = r_G I^2$, whereas the second term represents the available power delivered to the load $P = UI$; hence

$$P_G = Pr_G + P \tag{3.19}$$

This power balance equation can be graphically interpreted by using Figure 3.14. While the area of the upper rectangle (with sides $I$ and $r_G I$) corresponds to $Pr_G$, the area of the lower rectangle (with sides $I$ and $U$) corresponds to $P$. Summing the two areas, we obtain $P_G$ (a rectangle whose sides are $I$ and $U_0$) as in (3.19).

## 3.9 Proposed Homework Problems

### *Problem 3.9.1*

In order to monitor and control the unavoidable stress phenomena occurring in some mechanical structures, they are usually provided with embedded resistor-type strain gauges. The simplest scheme used to detect resistance changes due to gauge deformation (tension or compression) utilizes the Wheatstone bridge already analyzed in Application Example 3.6. Assume, as shown in Figure 3.15, that the bridge is initially balanced (fixed resistors $R_1$, $R_2$ and $R_3$ are equal to $R$). Next, allow the embedded strain gauge resistor on the bridge's upper left arm to be subjected to a small variation $\Delta R$ on its resistance, ($\Delta R \ll R$).

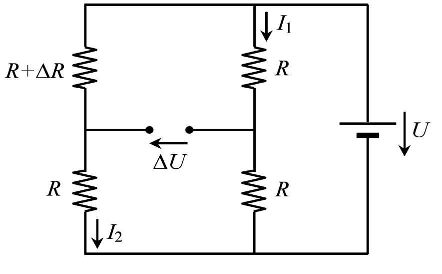


**Figure 3.15** Application of the Wheatstone bridge for the detection of strain gauge deformations

$Q_1$ Write the KVL equations governing the circuit.

$Q_2$ Find the relationship between the monitored voltage $\Delta U$ and $\Delta R$.

*Answers*

Q$_1$ $I_1 = \dfrac{U}{2R}$; $\quad I_2 = \dfrac{U}{2R + \Delta R}$; $\quad \Delta U = R(I_1 - I_2)$

Q$_2$ $\Delta U \approx \Delta R \dfrac{U}{4R}$

## *Problem 3.9.2*

For security reasons many types of electrical equipment ought to have a connection to ground. Ground electrodes are buried in the soil to provide a means for the flow of undesirable currents. Take the situation depicted in Figure 3.16 where a ground electrode of hemispherical shape is considered. Assume the metallic electrode is a perfect conductor. Choose for the potential $V(\infty) = 0$. Consider the following data: $I = 100\,\mathrm{A}$, $a = 10\,\mathrm{cm}$, $r_1 = 1\,\mathrm{m}$, $r_2 = 2\,\mathrm{m}$, $\sigma_{\mathrm{soil}} = 3.18 \times 10^{-2}\,\mathrm{S/m}$.

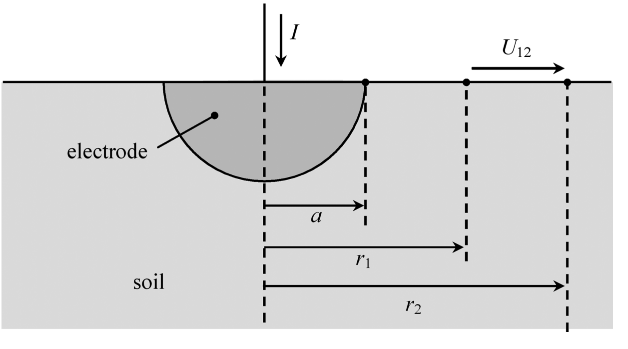


**Figure 3.16** Hemispherical ground electrode

Q$_1$ Obtain the equation for the radial dependence of the current density field $\mathbf{J}$ and potential function $V$ inside the soil.

Q$_2$ Compute the electrode potential $V_E$. Determine the boundary of the region around the electrode outside of which $V$ becomes smaller than $V_E/10$.

Q$_3$ Compute the electrode resistance

$$R_E = \frac{V_E - V(\infty)}{I}$$

Q$_4$ Evaluate the step voltage $U_{12}$.

Q$_5$ Find the power $P$ corresponding to the energy dissipated in the soil.

*Answers*

Q$_1$ $\mathbf{J}(r) = \dfrac{I}{2\pi r^2}\vec{e}_r$; $V(r) = \dfrac{I}{2\pi r \sigma_{\text{soil}}}$ (for $r \geq a$)

Q$_2$ $V_E = 5\,\text{kV}$; $V \leq V_E/10$ for $r \geq 1\,\text{m}$.

Q$_3$ $R_E = 50\ \Omega$.

Q$_4$ $U_{12} = 250\,\text{V}$.

Q$_5$ $P = 500\,\text{kW}$.

## *Problem 3.9.3*

Consider a two-wire transmission line like the one we dealt with in Chapter 2 (Application Example 2.8), where two identical cylindrical conductors, of length $l = 50\,\text{m}$, radius $r = 1\,\text{mm}$, run parallel separated by 4 mm. Assume that the surrounding dielectric medium is a perfect insulator. Line conductors, made of copper, have a conductivity $\sigma = 5.6 \times 10^7\,\text{S/m}$. The line is excited at the sending end by a voltage generator characterized by an electromotive force $U_0 = 50\,\text{V}$ and internal resistance $r_G = 1\,\Omega$. At the receiving end a resistor load $R_L = 50\ \Omega$ is placed (Figure 3.17).

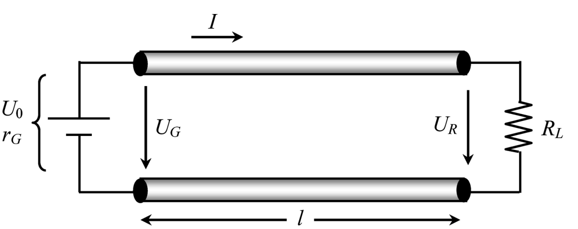


**Figure 3.17** A DC link employing a lossy two-conductor line

Q$_1$ In electrostatics we showed that proximity effects would give rise to non-uniform charge distributions over the conductor surfaces. Show, however, that as far as stationary currents are concerned, **J**-field lines are uniformly distributed inside the line conductors. (Note: Only for hight-frequency regimes the distribution of currents in the conductor cross-section becomes non-uniform due to skin effect phenomena – see Chapter 8.)

Q$_2$ Evaluate the resistance $R_{\text{cond}}$ of each line conductor.

Q$_3$ Evaluate $I$, $U_G$ and $U_R$.

Q$_4$ Determine the generator internal power losses $Pr_G$, as well as the transmission power losses $P_{\text{trans}}$. Compare the power produced by the generator applied field $P_G$ to the power delivered to the load $P$.

*Answers*

$Q_1$ Consider two neighboring parallel field lines $\mathbf{J}_1$ and $\mathbf{J}_2$ inside the cylindrical conductor. By applying the property

$$\oint_{\mathbf{S}} \mathbf{E} \cdot d\mathbf{s} = \frac{1}{\sigma} \oint_{\mathbf{S}} \mathbf{J} \cdot d\mathbf{s} = 0$$

you get $J_1 = J_2$.

$Q_2$

$$R_{\text{cond}} = \frac{l}{\sigma \pi r^2} = 284.2 \text{ m}\Omega$$

$Q_3$ $I = U_0/(r_G + 2R_{cond} + R_L) = 969.6\,\text{mA};\ U_G = 49.03\,\text{V};\ U_R = 48.48\,\text{V}.$

$Q_4$ $Pr_G = 0.94\,\text{W};\ P_{\text{trans}} = 0.53\,\text{W};\ P_G = U_0 I = 48.48\,\text{W};\ P = 47.01\,\text{W} = 97\% P_G.$

## *Problem 3.9.4*

Consider a coaxial cable which is terminated at its receiving end $(y = 0)$ by a resistor load $R_L = 1\,\text{k}\Omega$ whose power consumption is kept at $P_L = 250\,\text{W}$. The cable's longitudinal view and respective cross-section are shown in Figure 3.18. The length of the cable is $l = 10\,\text{km}$, and the remaining geometrical parameters are $r_1 = 1\,\text{mm},\ r_2 = 5\,\text{mm},\ r_3 = 5.1\,\text{mm}$.

The conductivity of the cable's internal and external conductors is $\sigma_{\text{cond}} = 31.67 \times 10^6\,\text{S/m}$. The dielectric medium is an imperfect insulator with conductivity $\sigma_{\text{insul}} = 5.123 \times 10^{-9}\,\text{S/m}$.

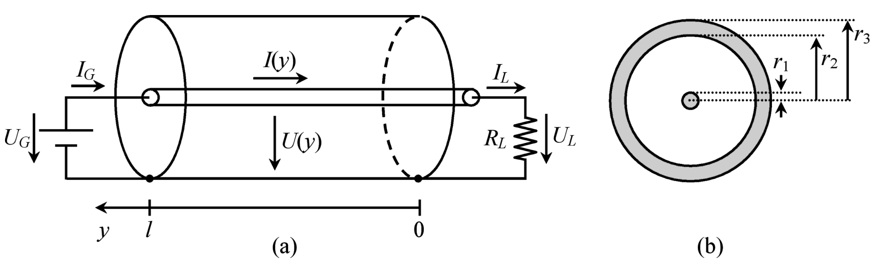


**Figure 3.18** A DC link employing a lossy coaxial cable. (a) General view. (b) Cable cross-section

$Q_1$ Determine the per-unit-length longitudinal cable resistance $R$ (including the internal and external conductors).

$Q_2$ Determine the per-unit-length transverse conductance $G$ of the dielectric medium.

$Q_3$ Consider the approximation that cable conductors are perfect (that is, cable voltage $U$ is constant along the longitudinal coordinate $y$).

Determine the generator voltage $U_G$ at the cable sending end ($y = l$).
Determine the evolution of the cable current intensity along $y$, $I(y)$, a consequence of the leakage currents crossing the imperfect dielectric.
(Hint: From div $\mathbf{J} = 0$, find $dI/dy$).
Obtain $I_G$ and determine the power dissipation in the insulating medium $P_{\text{insul}}$.

$Q_4$ Consider the approximation in which the dielectric medium is a perfect insulator (that is, cable current $I$ is constant along the longitudinal coordinate $y$).

Determine the generator current $I_G$ at the cable sending end ($y = l$).
Determine the evolution of the cable voltage along $y$, $U(y)$, a consequence of the voltage drop along the cable's imperfect conductors.
(Hint: From curl $\mathbf{E} = 0$, find $dU/dy$).
Obtain $U_G$ and determine the power dissipation in the cable conductors $P_{\text{cond}}$.

*Answers*

$Q_1$

$$R = \frac{1}{\pi\sigma_{\text{cond}}}\left(\frac{1}{r_1^2} + \frac{1}{r_3^2 - r_2^2}\right) = 20.0\,\text{m}\Omega/\text{m}$$

$Q_2$

$$G = \frac{2\pi\sigma_{\text{insul}}}{\ln(\text{r}_2/r_1)} = 20.0\,\text{nS/m}$$

$Q_3$

$$U_G = U = U_L = \sqrt{P_L R_L} = 500\,\text{V}$$

$$\frac{d}{dy}I(y) = GU \rightarrow I(y) = I_L + GUy; \quad I_L = U_L/R_L = 500\,\text{mA}$$

$$I_G = I_L + GUl = 600\,\text{mA}$$

$$P_{\text{insul}} = P_G - P_L = 300 - 250 = 50\,\text{W}$$

$Q_4$

$$I_G = I = I_L = \sqrt{P_L/R_L} = 500\,\text{mA}$$

$$\frac{d}{dy}U(y) = RI \rightarrow U(y) = U_L + RIy; \quad U_L = R_L I_L = 500\,\text{V}$$

$$U_G = U_L + RIl = 600\,\text{V}$$

$$P_{\text{cond}} = P_G - P_L = 300 - 250 = 50\,\text{W}$$

## *Problem 3.9.5*

Consider again the problem discussed in Problem 3.9.4. The approaches referred to in $Q_3$ and $Q_4$ are approximations because conductor voltage drops and dielectric leakage currents influence each other; in fact, here, we are dealing with a distributed coupled-phenomena problem. (Note: Distributed coupled phenomena will come to your attention in Part IV where the topic of electromagnetic wave propagation will be handled.)

$Q_1$ Determine the coupled differential equations describing the evolution of $U(y)$ and $I(y)$.

$Q_2$ Solve the equations.

$Q_3$ Make use of the boundary conditions at the receiving end of the cable to obtain the unknown integration constants.

$Q_4$ Obtain $U_G$ and $I_G$ at the generator terminals.

$Q_5$ Determine the total power losses and show how they break into conductor and insulator losses.

*Answers*

$Q_1$

$$\begin{cases} \dfrac{d}{dy} I(y) = GU(y) \\ \dfrac{d}{dy} U(y) = RI(y) \end{cases} \rightarrow \frac{d^2}{dy^2} \begin{Bmatrix} I(y) \\ U(y) \end{Bmatrix} - RG \begin{Bmatrix} I(y) \\ U(y) \end{Bmatrix} = 0$$

$Q_2$

$$\begin{cases} U(y) = U_1 \; e^{+y/D} + U_2 \; e^{-y/D} \\ I(y) = \dfrac{1}{R_0} \left( U_1 \; e^{+y/D} - U_2 \; e^{-y/D} \right) \end{cases}$$

where $D$ is the attenuation distance $D = 1/\sqrt{RG} = 50\,\text{km}$ and $R_0$ is the characteristic resistance $R_0 = \sqrt{R/G} = 1\,\text{k}\Omega$.

$Q_3$ Boundary conditions:

$$\begin{cases} U_{(y=0)} = U_L = 500\,\text{V} \\ I_{(y=0)} = I_L = 500\,\text{mA} \end{cases} \rightarrow U_1 = U_L = 500\,\text{V}; \;\; U_2 = 0$$

$Q_4$

$$U_G = U_L \; e^{+l/D} = 610.7\,\text{V}; \;\; I_G = \frac{U_L}{R_0} e^{+l/D} = 610.7\,\text{mA}$$

Q5
$$P_{\text{losses}} = P_G - P_L = 373 - 250 = 123\,\text{W}.$$

$$P_{\text{cond}} = \int_{y=0}^{y=l} R\, I^2(y)\; dy = 61.5\,\text{W}; \quad P_{\text{insul}} = \int_{y=0}^{y=l} GU^2(y)\; dy = 61.5\,\text{W}$$

## *Problem 3.9.6*

In electrostatics, the experimental determination of the partial capacitances among the conductors of a multiconductor system can be a very delicate subject. However, for homogeneous systems, an experimental procedure using stationary currents can be used indirectly to determine those capacitances. The electrolytic tank technique explores the existing analogy between capacitances and conductances as suggested by (3.11).

Consider a three-dimensional arrangement consisting of three metallic conductors in air ($\varepsilon_0$) as depicted in Figure 3.19(a). Next, assume that the same set of conductors is immersed into a tank filled with an electrolytic liquid of conductivity $\sigma$ (Figure 3.19(b)).The walls of the tank are made of an insulating material, and the size of the tank is very large compared to the overall conductor system dimensions.

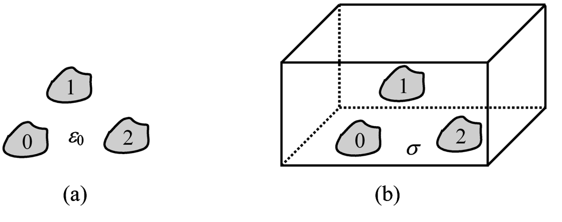


**Figure 3.19** A multiconductor system with three metallic bodies. (a) Immersed in air, $\varepsilon_0$. (b) Placed inside a tank filled with an electrolytic liquid of conductivity $\sigma$

From the point of view of stationary currents, the structure in Figure 3.19(b) can be replaced by the equivalent circuit in Figure 3.20, where $\hat{G}_{12}$, $\hat{G}_{10}$ and $\hat{G}_{20}$ represent the partial conductances of the system corresponding to the flow of currents through the electrolytic medium.

The following two experiments were conducted:

- Voltages $U_1 = 10\,\text{V}$ and $U_2 = 0$ were applied between accessible terminals. Ammeters used for current measurement gave the following readings: $I_1 = 0.6\,\text{A}$ and $I_2 = -0.2\,\text{A}$.
- Voltages $U_2 = 10\,\text{V}$ and $U_1 = 0$ were applied between accessible terminals. Ammeters used for current measurement produced the following readings: $I_1 = -0.2\,\text{A}$ and $I_2 = 0.5\,\text{A}$.

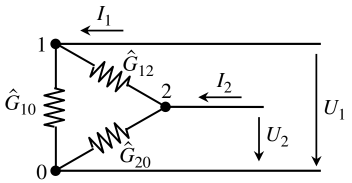


**Figure 3.20** Equivalent electric circuit made of partial conductances representing the arrangement depicted in Figure 3.19(b)

$Q_1$ Why must the size of the electrolytic tank be much larger than the conductor system dimensions?

$Q_2$ Write the equations governing the equivalent circuit in Figure 3.20.

$Q_3$ Determine the system partial conductances $\hat{G}_{12}$, $\hat{G}_{10}$ and $\hat{G}_{20}$.

$Q_4$ Knowing that the conductivity of the electrolyte is $\sigma = 17.68\,\text{mS/m}$, find the partial capacitances and the capacitance matrix $[C]$ that characterize the multiconductor electrostatic configuration in Figure 3.19(a).

*Answers*

$Q_1$ The configuration in Figure 3.19(a) is an unbounded system, so should be the one in Figure 3.19(b). This can be achieved (approximately) by placing the tank walls far away from the system conductors.

$Q_2$ $I_1 = \hat{G}_{12}(U_1 - U_2) + \hat{G}_{10}U_1;\ \ I_2 = \hat{G}_{12}(U_2 - U_1) + \hat{G}_{20}U_2.$

$Q_3$ First experiment:

$$\hat{G}_{12} = -\frac{I_2}{U_1} = 20\,\text{mS};\ \ \hat{G}_{10} = \frac{I_1 + I_2}{U_1} = 40\,\text{mS}$$

Second experiment:

$$\hat{G}_{12} = -\frac{I_1}{U_2} = 20\,\text{mS};\ \ \hat{G}_{20} = \frac{I_1 + I_2}{U_2} = 30\,\text{mS}$$

$Q_4$ From

$$\hat{C}_{jk} = \frac{\varepsilon_0}{\sigma}\hat{G}_{jk},\ \text{with}\ \frac{\varepsilon_0}{\sigma} = 0.5\,\text{ns}$$

we find $\hat{C}_{12} = 10\,\text{pF}$, $\hat{C}_{10} = 20\,\text{pF}$ and $\hat{C}_{20} = 15\,\text{pF}$.
From (2.53) we get the capacitance matrix

$$[C] = \begin{bmatrix} 30 & -10 \\ -10 & 25 \end{bmatrix}\text{pF}$$