4

FIRST- AND SECOND-ORDER CIRCUITS UNDER SINUSOIDAL AND STEP EXCITATIONS

4.1 INTRODUCTION

First-order circuits are very important in electrical and electronic engineering. Many higher order circuits can be reduced to a first-order circuit. Analyzing the behavior either in the time or in the frequency domains of a first-order circuit is unquestionably simpler than analyzing that of a higher order circuit. Essentially, first-order circuits have a single energy storage device. Such devices can be either a capacitor or an inductor. Examples of circuits that can be reduced to first-order circuits under certain conditions are electronic amplifiers, operational amplifiers, servomechanisms, electric motors, and other control networks.

Let us present an example of a first-order circuit.

Example 4.1 RL Series First-Order Circuit
Given a circuit that contains one resistor in series with an inductor, such as the one shown in Figure 4.1, we can calculate the output voltage to input voltage ratio as a function of frequency. Such ratio of voltages in the frequency domain is commonly referred to as $H(j\omega)$, where $H(j\omega)$ is called the circuit transfer function.

$$H(j\omega) = \mathbf{V}_{out}(j\omega)/\mathbf{V}_{in}(j\omega). \tag{4.1}$$

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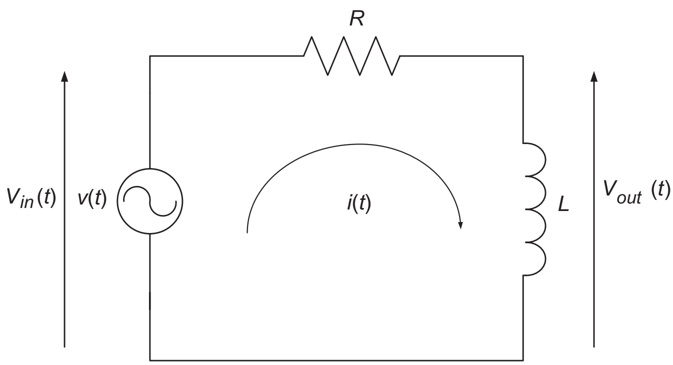


Figure 4.1 Circuit for Example 4.1, a first-order series RL circuit.

Previously reviewing the material from Chapter 3 on AC analysis we can easily calculate $H(j\omega)$ for the circuit in Figure 4.1.

The current in the RL circuit is calculated as follows:

$$\boldsymbol{I} = \boldsymbol{V}_{in}/(R + j\omega L), \tag{4.2}$$

where $\boldsymbol{I}$ and $\boldsymbol{V_{in}}$ are respectively the current and voltage phasors of the circuit.

Now the output voltage $\boldsymbol{V_{out}}$ is calculated by multiplying the circuit current times the impedance or reactance of inductor L. Thus, we obtain

$$\boldsymbol{V_{out}}(j\omega)/\boldsymbol{V_{in}}(j\omega) = j\omega L/(R + j\omega L). \tag{4.3}$$

And finally, our transfer function is

$$H(j\omega) = \mathbf{V_{out}}(j\omega)/\mathbf{V_{in}}(j\omega) = j\omega L/(R + j\omega L). \tag{4.4}$$

Furthermore, rationalizing the denominator, that is, multiplying numerator and denominator of Equation (4.4) by the complex conjugate of the denominator, $(R - j\omega\, L)$, we obtain

$$\boldsymbol{H}(j\omega) = \frac{\omega^2 L^2 + j\omega RL}{R^2 + (\omega L)^2} = \tag{4.5}$$

$$= \frac{\omega^2 L^2}{R^2 + (\omega L)^2} + j\frac{\omega RL}{R^2 + (\omega L)^2} \tag{4.6}$$

Equation (4.6) is of the form $a + jb$ where the terms a and jb are frequency dependent. Additionally term jb is of inductive nature.

We can also write the time domain circuit equation for the circuit of Figure 4.1; this leads to

$$v_{in}(t) = i(t)R + L\frac{di(t)}{dt}, \tag{4.7}$$

where $v_{in}(t)$ is the excitation, $i(t)$ is the current through the circuit, $i(t)R$ is the voltage drop across the resistor, and $L\ d\ i(t)/dt$ is the voltage drop across the inductor.

Equation (4.7) is the time domain first-order differential equation that describes the circuit on hand.

The highest derivative in a differential equation determines the *order* of the differential equation.

4.2 THE FIRST-ORDER *RC* LOW-PASS FILTER (LPF)

Let us investigate the *RC* circuit of Figure 4.2. This circuit is excited by a sinusoidal voltage waveform. Elements *R* and *C* are in series, the input voltage is applied to the two elements in series, the output is taken across the capacitor terminals.

4.2.1 Frequency Domain Analysis

Let us calculate the transfer function of this circuit:

$$\mathrm{H}(j\omega) = \mathbf{V}_{out}(j\omega)/\mathbf{V}_{in}(j\omega). \tag{4.8}$$

The impedance of the resistor and capacitor in series is

$$Z_{series}(j\omega) = R + 1/j\omega C. \tag{4.9}$$

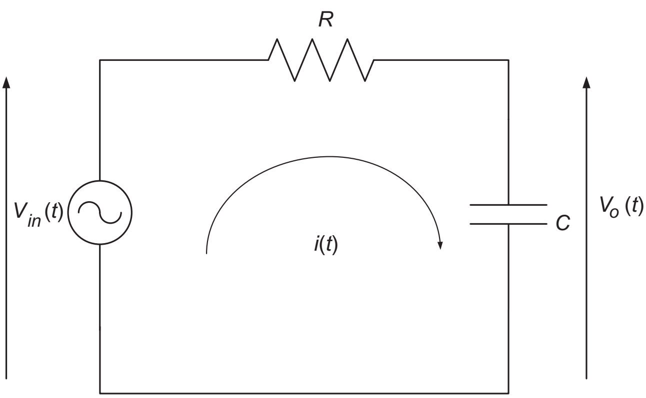


Figure 4.2 First-order *RC* low-pass filter.

Then the current through the circuit is

$$\mathbf{V}_{in}(j\omega)/Z_{series}(j\omega). \tag{4.10}$$

The above current times the impedance or reactive capacitance of the capacitor $1/j\omega C$ equals the output voltage $\boldsymbol{V_{out}}$ $(j\omega)$, thus we get

$$\mathrm{H}(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{1}{1 + j\omega RC}. \tag{4.11}$$

Remember that from Chapter 3, ω is the angular frequency, which equals $2\pi f$, where f is the frequency of the sinusoidal waveform in hertz.

Equation (4.11) describes the ratio of output voltage to input voltage of the *RC* circuit given by Figure 4.2, again referred to as the circuit transfer function in the frequency domain.

So let us now construct a table to plot the values of the transfer function $\mathrm{H}(j\omega)$ for a given R and a given C. The product

$$RC \tag{4.12}$$

is referred to as the circuit time constant. We will plot the transfer function of Equation (4.11) using two separate plots. One plot is for its magnitude, and the second plot for its phase, both are functions of frequency. Note that transfer function of Equation (4.11) is a complex quantity and as such, it has magnitude and phase.

4.2.2 Brief Introduction to Gain and the Decibel (dB)

It is common in circuit theory to refer to the output to input voltage ratio as the *gain* of the circuit. In the case of a first-order *RC* LPF circuit, such *gain* is always one* or less than one. Active circuits are those circuits containing operational amplifiers or transistors that usually are designed to have a gain larger than one. More on active circuits will be covered in Chapter 6.

When the *gain* is greater than *1*, it is referred to as *gain*, but when the gain is less than *1*, it is sometimes referred to as *attenuation*, or simply as *a less-than-one gain*. Passive first-order circuits have gains that are one or strictly less than 1.

When we talk about *gain* without any units associated to it, it is simply a ratio of voltages, thus it has no units because *gain* units are volts/volts. In circuit analysis, it is very common to define a new unit for *gain* and *attenuation* called the decibel (dB).

* Mathematically speaking, the gain of an *RC* circuit may be very close to *1*, but it is never exactly *1*. In practical terms the gain is *1* for frequencies one-tenth of the cutoff frequency and below.

The decibel is defined as follows for a ratio of voltages:

$$Gain\ in\ dB = 20log_{10}|V_{out}/V_{in}|. \tag{4.13}$$

It is important to mention that the argument of a decimal log must be a positive number. The log of a number less than or equal to zero is undefined.

So note that the *gain* (without units or in volts/volts), is also referred to, as the *linear gain* of a circuit. For example, given a circuit with a *linear gain* of *10*, by virtue of Equation (4.13) the *gain* in dB becomes:

$$20log_{10}(10/1) = 20\text{ dB}. \tag{4.14}$$

For a circuit whose *linear gain* is *0.1*, or an *attenuation* of *10*, the gain in dB becomes:

$$20log_{10}(1/10) = -20\text{ dB}. \tag{4.15}$$

So from Equations (4.14) and (4.15) larger than one *gains* have units of positive decibels, while *attenuations* are always given in negative decibels. Note that a linear gain of *1* is:

$$0\text{ dB} = 20log_{10}(1/1). \tag{4.16}$$

When we refer to a circuit with a *gain* of *1* or a *gain* of *zero dB*, we are talking about the exact same thing.

To summarize the relation between linear gain and logarithmic or gain in decibel we develop Table 4.1.

Positive linear gains are above 1 and negative linear gains are below 1. On the other hand, following Table 4.1, a –20 dB and a –40 dB gains can also be referred to as 20 dB and 40 dB attenuations, respectively.

Table 4.1 Relationship between linear gain or attenuation and gain in decibel

Linear Gain (V/V)	Gain (dB)
0.01	–40
0.1	–20
1	0
10	20
100	40
1,000	60
10,000	80
100,000	100
1,000,000	120

4.2.3 *RC* LPF Magnitude and Phase Bode Plots

Since *gains* may range from very small to very large values, plotting the gain in decibel is very advantageous. A very large range of *gains*, for example from 0.01 to 100,000 V/V, looks somewhat cumbersome when plotted in a linear scale. The gains are extremely compressed at low gain values and extremely expanded for large gain values.

To more evenly distribute the gain along the height of the *y*-axis the magnitude of the gain is plotted in decibel. Since the decibel is a logarithmic function, plotting gain in decibel is effectively *gain* in a logarithmic scale. Plotting the gain values in decibel (e.g., –40 dB for a *linear gain* of 0.01 and +100 dB for a *linear gain* of 100,000) allows the plot to have the same amount of vertical space allocated to display all the values of *gain*.

The horizontal axis of the magnitude Bode plot is frequency. Similarly to what is done to gain, frequency is plotted in logarithmic scale. So if we are interested in plotting the gain as a function of frequencies from 0.01 Hz to 100 kHz, the *x*-axis is scaled logarithmically. Doing this allows us to see the frequency range in a decompressed fashion. Plotting the frequency as a linear quantity would make the frequency axis very compressed at low frequencies and greatly expanded at high frequencies. So for all practical purposes, a Bode magnitude plot displays decibel linearly and frequency logarithmically; thus, it is a semi-log plot; but since the decibel is a logarithmic function, the magnitude plot is a log-log plot for gain in decibel versus frequency in hertz.

The phase portion of the Bode plot of a transfer function displays degrees in the vertical axis and logarithmic frequency along the *x*-axis. Degrees are always plotted in a linear scale, because the range of degrees is in general not very large as gains or frequencies are.

Example 4.2 Bode Plots of an *RC* LPF Transfer Function

Continuing with the circuit given in Figure 4.2 with transfer function given by Equation (4.11), the transfer function is repeated below for the reader's convenience:

$$H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{1}{1 + j\omega RC}. \tag{4.17}$$

We define a new term, the cutoff angular frequency ω_0 in radians per second and the *cutoff frequency* f_0 in hertz, as

$$\omega_0 = 1/RC, \tag{4.18}$$

where:

$$\omega_0 = 2\pi f_0. \tag{4.19}$$

Now let us assume that our filter has a cutoff frequency f_0 of 1 KHz, thus from combining Equations (4.18) and (4.19) we obtain

$$f_0 = 1/2\pi RC \tag{4.20}$$

and from Equation (4.20)

$$RC = 1/2\pi f_0 = 159.155\ \mu\text{s}. \tag{4.21}$$

From Equation (4.18) the term RC is known as the circuit *time constant* τ (Greek letter tau) and its units are ohms multiplied by farads, which lead to time in seconds in the SI system of units (see Chapter 1).

For our particular example,

$$\tau = 159.155\ \mu\text{s}. \tag{4.22}$$

Important Point

The time constant τ of the circuit, that is, the RC product, determines the circuit frequency behavior.

The cutoff frequency of the circuit f_0 is a very important characteristic of a circuit, as we will see shortly.

Now we can rewrite Equation (4.17) using ω_0 and this becomes

$$H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{1}{1 + j\omega/\omega_0} = \frac{1}{1 + jf/f_0}.$$

Since the cutoff frequency of the circuit is 1 kHz, we will pick to start plotting the Bode plots from frequencies much smaller than the cutoff frequency. In our case we will start at 1 Hz, somewhat arbitrarily we pick a high end frequency of 1 MHz.

Let us construct a table listing frequency on the leftmost column followed by the linear magnitude of our transfer function and a third and last column with the phase angle of our transfer function. Remember that the transfer function of interest given by Equation (4.11) is a complex number that has a magnitude and a phase.

The magnitude is

$$|\mathrm{H}(j\omega)| = \left|\frac{1}{1 + j\omega RC}\right| = \frac{1}{|1 + j\omega RC|} = \frac{1}{\sqrt{1 + (\omega RC)^2}} \tag{4.23}$$

and since $1/RC = \omega_0$ from Equation (4.18), simply becomes

$$|\mathrm{H}(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}. \tag{4.24}$$

Table 4.2 ***RC*** **LPF transfer function: magnitude and phase as a function of frequency**

Frequency (Hz)	Linear Gain (V/V)	Phase (Degrees)
1.000	0.999999500	–0.05729576
10.000	0.999950007	–0.57293870
100.000	0.995037481	–5.71059313
1,000.000	0.707117209	–44.99999998
10,000.000	0.099506625	–84.28940686
100,000.000	0.009999795	–89.42706130
1,000,000.000	0.001000029	–89.94270424

The phase of Equation (4.11) is

$$\angle \mathrm{H}(j\omega) = -\arctan(\omega/\omega_0). \tag{4.25}$$

So our Table 4.2 follows:

Table 4.2 lists frequencies in the leftmost column, the magnitude as a dimensionless number (volts/volts) in the center column and the phase in degrees in the rightmost column. In electronics the preferred way of displaying magnitude is in decibel. The output to input voltage ratio, which we will refer to as a *gain*, is usually expressed in decibels. A decibel was defined by Equation (4.13), which we repeat for the reader's convenience.

$$V_{out}/V_{in} \ in \ dB = 20log_{10}|V_{out}/V_{in}|. \tag{4.26}$$

We can easily verify using Equation (4.26) that for a gain ratio of 1 the gain in decibel equals 0 dB; for a gain ratio of 10, the gain equals 20 dB; a gain ratio of 100, the gain equals 40 dB; and a for gain ratio of 1000 equals 60 dB. So for every order of magnitude that the gain goes up, the gain in decibel goes up by 20 dB. On the other hand, for a gain of 0.1, the gain in decibel equals –20 dB, for 0.01 it equals –40 dB, for 0.001 it equals –60 dB, and so on.

Figure 4.3 depicts the magnitude and phase Bode plots of the *RC* LPF tabulated in Table 4.2.

Important Points

For a first-order RC LPF circuit, the gain is close to 0 dB at frequencies below one-tenth of the cutoff frequency f_0.

For a first-order RC LPF circuit the gain at the cutoff frequency f_0 *is –3.01 dB.*

For a first-order RC LPF, the circuit the gain drops at a rate of –20 dB per decade from its cutoff frequency. This is to say that the gain drops by 20 dB for a frequency 10 times f_0*, it drops another 20 dB for a frequency 100 times* f_0*, another 20 dB for a frequency 1000 times* f_0*, and so on. (Refer to Table 4.3.)*

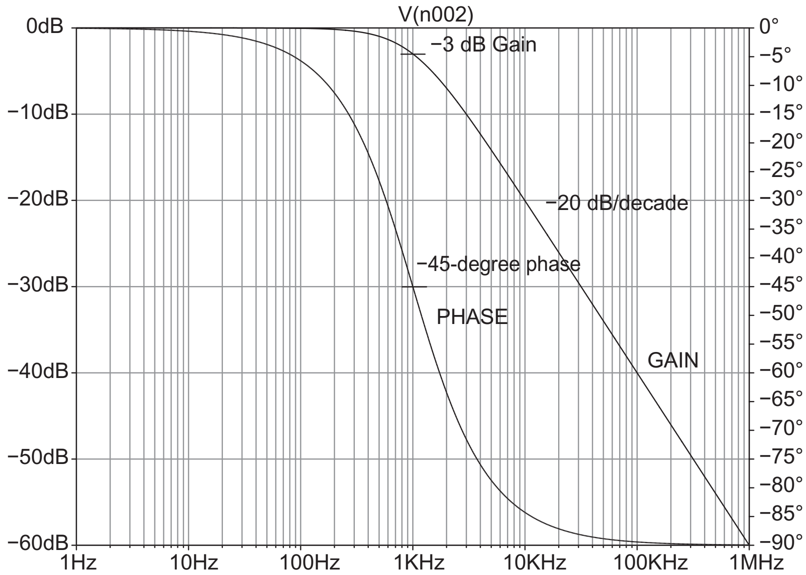


Figure 4.3 Exact magnitude and phase Bode plots of the first-order circuit of Example 4.2.

The frequency axis does not have a zero or origin of frequencies because log of zero is nonexistent. The lowest frequency can be represented with a value as small as we desire, but not with zero.

Frequency is represented logarithmically for both magnitude and phase plots.

The magnitude or gain in decibel is represented linearly. There is a 0 dB origin for the vertical axis because the scale is linear in decibels.

The phase in degrees is represented linearly on the vertical axis.

The phase of an RC LPF is approximately 0° at frequencies below 1/10 of f_0. *The phase of an RC LPF is approximately –90° at frequencies larger than 10* f_0. *At* f_0 *the phase equals –45°.*

Although the definition of the decibel may initially seem capricious, it is actually a better way that allows us to visualize the growth or the decay of the gain in a magnitude plot.

From the calculation in Table 4.2 we will build Table 4.3 that will contain the magnitude in decibels. Frequency, magnitude (linear gain), and phase are shown with a generous number of decimal places.

4.2.4 *RC* LPF Drawing a Bode Plot Using Just the Asymptotes

To draw the asymptotes of the Bode plots of our circuit (Fig. 4.2) we will normalize frequency. Instead of listing on the frequency axis the actual cutoff

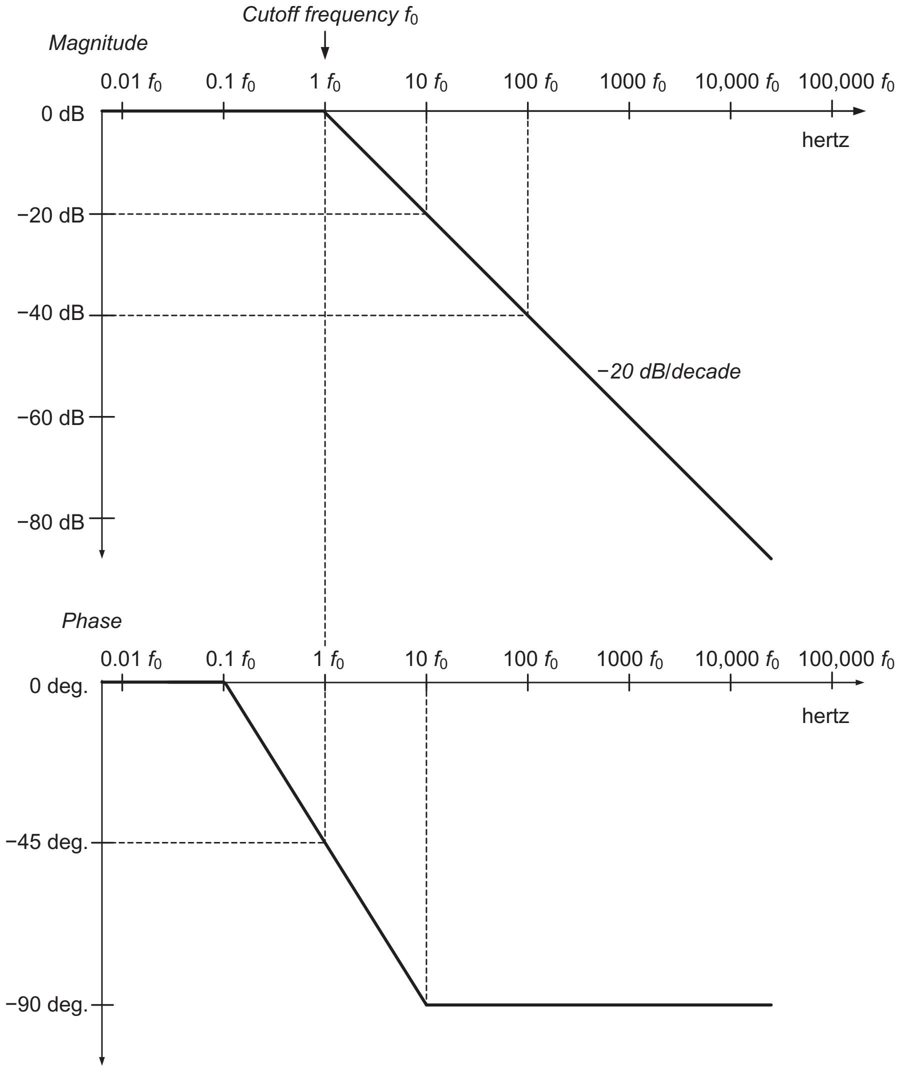


Figure 4.4 First-order *RC* LPF asymptotic Bode plots: magnitude and phase.

frequency value of 1 KHz, we will denote this frequency as f_0, so lower frequencies will be a tenth, a one-hundredth, and so on of f_0. Similarly, frequencies above f_0 are 10 times, 100 times, and so on of f_0. Figure 4.4 shows these normalized frequencies on the horizontal axis.

From Table 4.3 and Figure 4.3 we see that the magnitude asymptotically approaches 0 dB from the cutoff frequency f_0 to smaller frequencies. Also from the cutoff frequency to higher frequencies, the gain drops at a constant rate

Table 4.3 *RC* LPF transfer function: magnitude in *dB* and phase as a function of frequency

Frequency (Hz)	Normalized Frequency f/f_0	Linear Gain (V/V)	Gain in dB	Phase (Degrees)
1.000	$0.001\ f_0$	0.999999500	0.00	–0.05729576
10.000	$0.01\ f_0$	0.999950007	0.00	–0.57293870
100.000	$0.1\ f_0$	0.995037481	–0.04	–5.71059313
1,000.000	$1\ f_0$	0.707117209	–3.01	–44.99999998
10,000.000	$10\ f_0$	0.099506625	–20.04	–84.28940686
100,000.000	$100\ f_0$	0.009999795	–40.00	–89.42706130
1,000,000.000	$1,000\ f_0$	0.001000029	–60.00	–89.94270424

of –20 dB per decade. But at the cutoff frequency, the gain is approximately –3 dB. In linear terms, this means that the amplitude of the sinusoidal waveform that excites the *RC* circuit becomes attenuated to about 70% from its original value. Referring one more time to Table 4.3, we can see that at the cutoff frequency f_0 the output voltage magnitude is 0.707 of the original input, which has a magnitude of 1. That is, the output magnitude is approximately 70.7% of the input magnitude.

In a similar fashion, we can see that the phase is –45° at f_0. One more time looking at the phase in Table 4.3, we see that the phase at about one-tenth of f_0 is –5.7°. And for even lower frequencies, the phase asymptotically approaches 0°. On the other hand, at a frequency of 10 times f_0, the phase is approximately 5.7° below –90°.

So when Bode plots need to be drawn by hand, drawing its asymptotes is the preferred and quickest way of constructing magnitude and phase plots. This process not only saves a tremendous amount of number crunching but also makes the understanding of the plots more clear. The magnitude and phase plots are commonly referred to as the frequency response of the circuit. Figure 4.4 shows the asymptotic Bode plots for the circuit of Figure 4.2. Note that the gain is 0 dB flat from very low frequencies approximately up to the cutoff frequency f_0. The second magnitude asymptote simply decays from f_0 at a rate of –20 dB *per decade*. Once the magnitude asymptotes are drawn one can fill in by hand, the approximated gain curves. It is important to realize that the gain at f_0 is –3 dB and not 0 dB as Table 4.3 shows.

For the phase, we can also draw its plot using the phase asymptotes. The phase is a little bit more involved than the magnitude at least initially. Let us start with frequencies well below f_0, up to one-tenth of f_0, we draw a straight line at zero degrees from low frequencies all the way up to 1/10 of f_0. At a frequency of 10 times f_0 the frequency asymptote is a horizontal line at –90 degrees, starting at 10 f_0 continuing at –90° into higher frequencies. From Table 4.3, we know that at the cutoff frequency f_0, the phase is –45°. Looking at Figure 4.4, we now draw a straight line of a phase angle of 0 degrees at 1/10 of f_0 all the way to –90° at 10 f_0, such that this straight line passes through a –45° phase at f_0. In this way the phase asymptotes are drawn. Now one can draw by hand the approximate phase curves. Note that at 1/10 of f_0 and at 10 f_0 the phase is about 5.7° below zero degrees and 5.7° above –90°. Finally, it is important to note that the phase curve has an inflexion point at –45°, see Figure 4.3.

4.2.5 Interpretation of the *RC* LPF Bode Plots in the Time Domain

We will use the asymptotically drawn Bode plots to explain the meaning of the Bode magnitude and phase plots in terms of sinusoidal inputs applied to the *RC* LPF circuit. The same concepts can be extended to the actual (or exact) Bode plots that are tabulated in Table 4.3. So referring to Figure 4.4, the asymptotic Bode plots tell us the following:

First let us assume that a sinusoidal waveform of 1 V peak amplitude and a frequency of 0.01 f_0 is applied to the input of the first-order *RC* LPF. The output voltage waveform that will be observed across the capacitor terminals is for all practical purposes equal in magnitude to the input waveform and equal in frequency with no phase shift with respect to the input. That is, both inputs and outputs have the same magnitude: 1; and both input and output sinusoidal waveforms are in phase (phase = 0°). The lower the input waveform frequencies with respect to the cutoff frequency of the RC filter, the more accurate the preceding statement is. Refer to the numerical values of linear gain and gain in *dB* for frequencies much smaller than f_0 in Table 4.3.

Assume now that a sinusoidal input waveform of 1 V peak-amplitude and of a frequency f_0 (equal to the circuit cutoff frequency) is applied to the input of the RC circuit. The output voltage across the capacitor will be a sinusoidal waveform of peak amplitude 30.3% smaller than the input amplitude; however, it will still be of the same frequency, as the input waveform; but the output will be lagging the input by a *45* degree-phase. *Forty five degrees* is an eight of a full sinusoidal cycle.

Let us consider now that a sinusoidal input waveform of 1 V peak-amplitude and of a frequency 10 times larger than f_0, which is applied to the input of the *RC* circuit. The output voltage across the capacitor will be a sinusoidal waveform of peak amplitude 10 times smaller (20 dB) than the input amplitude; however, it will still be of the same frequency, as the input waveform; but the output will be lagging the input by about –84.3 degrees (almost –90 degrees). Clearly examining again the exact plots of Figure 4.3, the higher is the frequency of the input waveform with respect to the circuit cutoff frequency, the closer the output to input phase will be to –90 degrees. If the input waveform frequency is 100 times f_0 the output amplitude, while the input amplitude is always 1 V, the output waveform amplitude will be 100 times smaller (40 dB) than the input waveform amplitude. This behavior goes on and on, for every time the frequency goes up by a factor of 10 from the gain decays another 20 dB. (Refer again to Figs. 4.3 and 4.4.)

4.2.6 Why Do We Call This Circuit a LPF?

From the Bode plots just presented in Figures 4.3 and 4.4, it is clear to see that frequencies well below the cutoff frequency f_0 do not get attenuated, they just pass through the circuit with a *0 dB* gain (*linear gain of 1*), and a zero-degree phase shift. Frequencies well above the cutoff frequency become attenuated. We also see that the higher the frequency is above f_0, the higher the attenuation. The attenuation grows by 20 dB for every order of magnitude that the frequency grows above f_0. Alternatively, the gain decreases at a rate of 20 dB per decade of frequency.

In summary, the *RC* circuit just analyzed allows low frequency signals to go through the circuit without attenuation and without phase-shift, whereas the high frequencies become progressively attenuated as the input signal frequency goes up. At frequencies beyond 10 times the cutoff frequency, output signals exhibit a phase shift of approximately –90°. In reference to our *RC* LPF, which are low and which are high frequencies? The reference frequency is f_0 the filter cutoff frequency. One-tenth below f_0, the frequency is considered low. Ten times above f_0, a frequency is considered high.

4.2.7 Time Domain Analysis of the *RC* LPF

Now let us analyze the time domain equations of the low-pass *RC* circuit. Referring one more time to the circuit of Figure 4.2, it is possible to establish the differential equation for such circuit. Let us apply Kirchoff's voltage law (KVL) for the series of elements.

$$v_{in}(t) = i(t)R + v_o(t),$$

where $v_{in}(t)$ is the excitation or the circuit input voltage.

$i(t)$ is the current through the circuit, thus $i(t)\ R$ is the voltage drop across the resistor and $v_o\ (t)$ is the voltage across the capacitor:

We will define a unit-step function excitation $u(t)$ as follows:

$$u(t) = 1 \text{ for } t \geq 0$$

and

$$u(t) = 0 \text{ for } t < 0.$$

So applying the unit step function $u(t)$ *to* the input of *RC* LPF we have that

$$v_{in}(t) = u(t)$$

and substituting v_{in} with $u(t)$ into $v_{in}(t) = i(t)\ R + v_o(t)$, yields:

$$u(t) = i(t)R + v_o(t). \tag{4.27}$$

Since the current through the resistor and the capacitor have the same value,

$$i(t) = Cdv_o(t)/dt. \tag{4.28}$$

Plugging Equation (4.28) into Equation (4.27) leads to

$$u(t) = RC\frac{dv_o(t)}{dt} + v_o(t), \tag{4.29}$$

where Equation (4.29) is a first-order differential equation.

From calculus considerations to solve differential equations, the solution of Equation (4.29) has the form

$$v_o(t) = A_1 + A_2 e^{-t/\tau} \tag{4.30}$$

where A_1 and A_2 are two constants and τ is the circuit time constant RC. A_1 is the steady-state value of the output voltage since

$$\begin{gathered} for\ t \to \infty, \\ v_o(t) \to A_1. \end{gathered} \tag{4.31}$$

This means that A_1 is the final value of output voltage $v_o(t)$after the transient is over. We call this final value V_{final}.

Thus,

$$V_{\text{final}} = A_1. \tag{4.32}$$

Also note that the initial value of the output voltage $v_o(t)$is found by making $t = 0$. Thus, from Equation (4.30),

$$= V_{\text{initial}} = v_o(0) = A_1 + A_2. \tag{4.33}$$

Finally plugging Equations (4.32) and (4.33) in Equation (4.30) we obtain that

$$v_o(t) = V_{final} + (V_{initial} - V_{final})e^{(-t/\tau)}. \tag{4.34}$$

Equation (4.34) is a general solution for a first-order circuit or a circuit with a single-time constant. We will use Equation (4.34) several times throughout this text and the homework problems.

Example 4.3 Given the single-time constant circuit of Figure 4.2, assuming that the input voltage is a unit-step function $u\ (t - 1)$, $R = 1\ \text{M}\Omega$, $C = 1\ \mu\text{F}$, calculate the final value of the output voltage $v_o(t)$ across the capacitor. Plot the output voltage waveform for positive values of time. Figure 4.5 displays

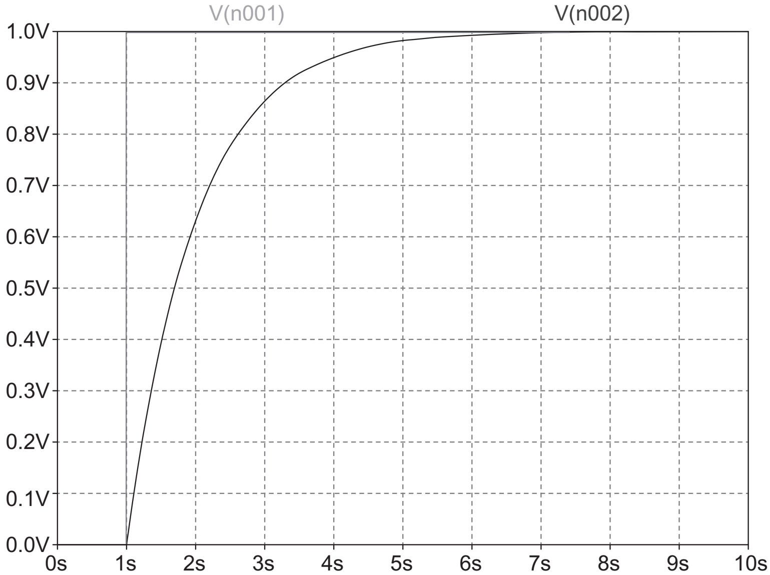


Figure 4.5 Unit step response of first-order *RC* LPF.

the input step function and the output as a function of time. Note that $u\,(t-1)$ is displaced or delayed from the origin of time by 1 second.

Now, since $R = 1$ MΩ and $C = 1$ μF, the time constant τ is 1 second. Now using Equation (4.34) and knowing that $V_{initial} = 0$ V, $V_{final} = 1$ V we obtain

$$v_o(t) = 1 - e^{-(t-1)/\tau}. \tag{4.35}$$

Equation (4.35) is plotted in Figure 4.5 from time 0 to 10 seconds. Note that at time $t = 2$ seconds the output voltage $v_o(t)$ has risen to

$$v_o(2) = 1 - e^{-(2-1)/\tau} = 0.6321. \tag{4.36}$$

Observing the waveform $v_o(t)$ in Figure 4.5, Equation (4.36) tells us that after one time constant the output reaches approximately 63% of its final value. After five time constants the output voltage across the capacitor reaches approximately 99% of its final value. Figure 4.6 displays several step input responses to circuits that have a range of time constants from 0.1, 0.5, 1, 2, 5, and 10 seconds. The shorter the time constant of the circuit, the faster the output voltage will approach its final value. So for our Figure 4.6, the circuit with $\tau = 0.1$ second has the fastest response of all the waveforms displayed. On the other hand, the circuit with $\tau = 10$ seconds has the slowest response. Notice that this waveform (10-second time constant) just reaches 63% of its final value after *one* time constant. The trajectory of the output waveform for a 10-second time constant is only shown for two time constants (20 seconds). If we had plotted the Figure 4.6 up to $t = 50$ seconds, the waveform would have reached 99% of 1 V.

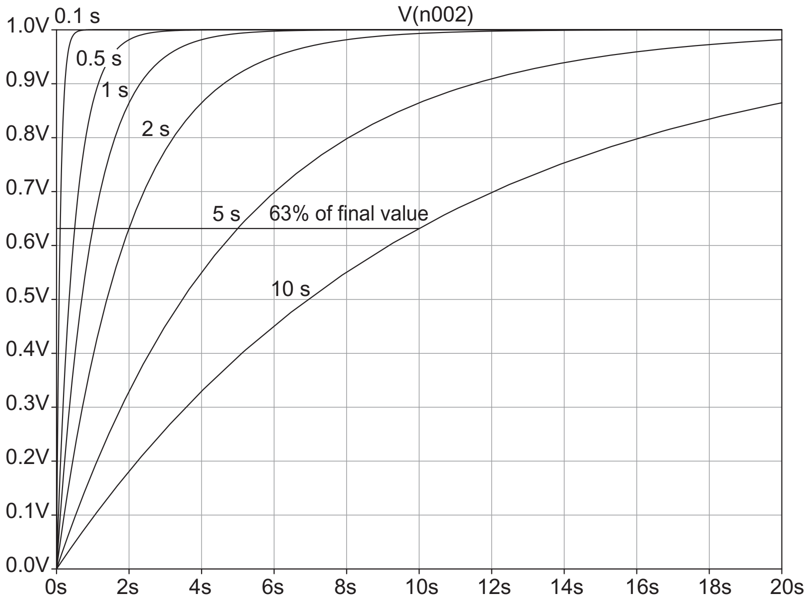


Figure 4.6 Unit step responses of *RC* LPF of time constants 0.1, 0.5, 1, 2, 5, and 10 seconds.

4.2.8 First-order *RC* LPF under Pulse and Square-Wave Excitation

We defined a unit step excitation $u(t)$ using Equation (4.29) in the previous section.

Let us combine to unit step functions such that the first one $u(t)$ is added, to a second $u(t)$ that is delayed by t_p seconds and inverted, so that:

$$u(\mathrm{t}) - u(t - t_p). \qquad (4.37)$$

Equation (4.37) is the expression of a single pulse of width t_p.

Applying such pulse to an *RC* LPF and its responses are shown in Figure 4.7 for a number of time constants. Note that for small time constants like 0.01 second and 0.05 second the output voltage waveform resembles the input more closely that the larger time constant curves. As the circuit time constant increases, the output waveform looks less exponential and more linear (2, 5, and 10-second time constants).

Now let us consider a square-wave input, as the one shown in Figure 4.8. Such waveform is a continuous train of pulses that swings between 0 V and 1 V with a 50% duty cycle. The waveform starts at 1 V at zero time for 1 second, at this time it drops very quickly to 0 V for another second. After this last second at 0 V, the earlier described process repeats itself indefinitely. Note that the period T of this pulse train is 2 seconds.

Let us apply such excitation to the input of a first-order *RC* LPF. We will look at the responses of several *RC* LPF with time constants equal to 0.01,

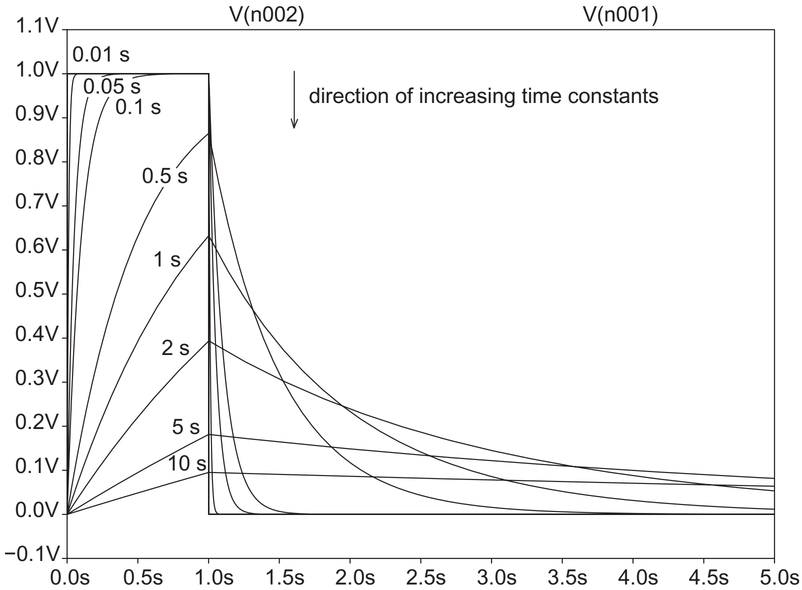


Figure 4.7 First-order *RC* LPF and pulse responses for various time constants.

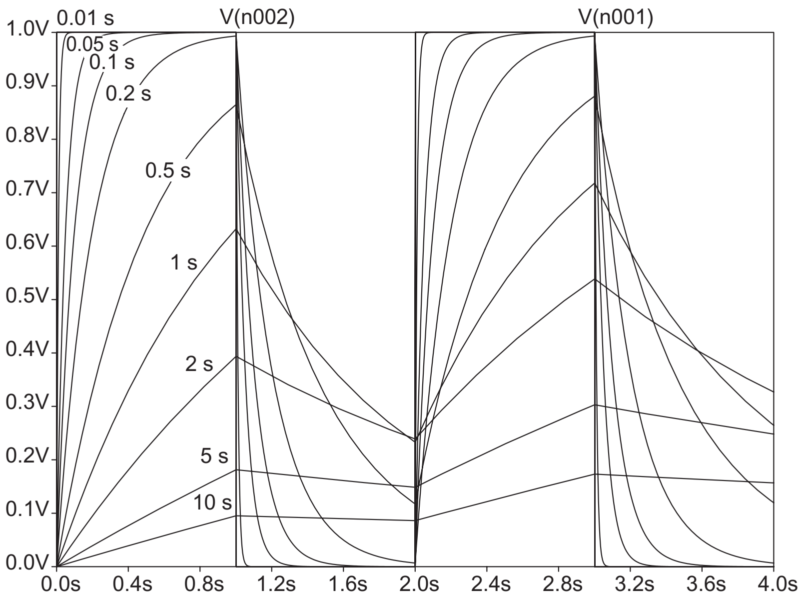


Figure 4.8 The outputs of all nine different time constants *RC* LPF circuits.

0.05, 0.1, 0.2, 0.5, 1, 2, 5, and 10 seconds. Note that the smallest time constant is 1/200th of the 2-second period of the excitation. The longest time constant is five times the period of the excitation.

Figure 4.8 shows 50% duty cycle square-wave driving *RC* LPF of nine different time constants, for two full excitation periods (i.e., 4 seconds).

The shorter is the time constant with respect to the excitation period, the faster the output of each *RC* circuit follows the 50% duty cycle square-wave input. For example, referring to Figure 4.8, the output of the circuit whose time constant is 0.01 seconds closely follows the square-wave input, some rounding is seen at the end of the rising and falling edges of the output response. Additionally, this behavior of reaching fairly quickly a steady state is reached virtually from the first excitation period. Now let us concentrate on the slowest time constant circuit of 10 seconds. Note that because this time constant is actually larger than the excitation period, it takes some time for the output of the 10-second time constant circuit to reach a steady-state value. Within this context, a steady-state value refers to the waveform moving up and down with time such that its average value settles down to a constant value, and it does not change significantly anymore. Figure 4.9 depicts the similar waveforms of those of Figure 4.8 display but for a much longer period of time, that is, 30 seconds. Note that Figure 4.9 only displays responses for nine different circuit of time constants equal to: 0.01, 0.05, 0.1, 0.2, 0.5, 1, 2, 5, and 10 seconds. Carefully observing the 10-second time constant circuit response, we see that during the first 20 seconds starting at zero time, the output little by little rises as

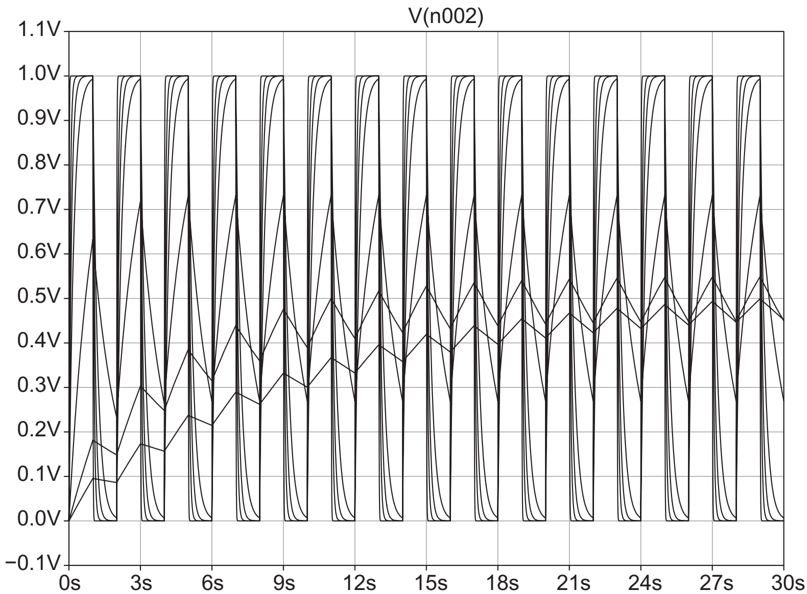


Figure 4.9 Square-wave responses of *RC* LPF of various time constants.

seconds go by. Then somewhere in the neighborhood of 30 seconds, the output settles around the mean value of the square-wave input. Since our input has a 50% duty cycle and swings between 0 and 1 V, its mean value is exactly 0.5 V. The average value to which the output voltage will settle is 0.5 V. Figure 4.9 does not quite show when the 10 seconds time constant circuit settles to 0.5 V because a few more seconds should have been plotted. Because the drawing becomes too busy and for a longer time, the author determined that 30-second was a better time frame to display. Looking at Figure 4.9 once more, the second slowest 5-second time constant response, second waveform from the time axis, more clearly reaches 0.5 V. On the other hand, note that those responses whose circuits have very fast time constant relative to the period of the square-wave excitation, simply follow relatively closely their input. Such responses will never settle to an average value of 0.5 V of the input waveform.

4.2.9 The *RC* LPF as an Integrator

When a first-order *RC* LPF has a large time constant in comparison with the time that it takes for the input signal to make an appreciable change, the voltage drop across the output capacitor is small compared to the drop across the resistor. Referring again to Figure 4.2, the current through the circuit is

$$i(t) = CdV_o(t)/dt \tag{4.38}$$

and since V_o is small compared to the voltage across resistor R then,

$$i(t) = V_{in}/R. \tag{4.39}$$

Combining Equations (4.38) and (4.39) we obtain

$$CdV_o(t)/dt = V_{in}/R, \tag{4.40}$$

which, after some algebraic manipulation and integration on both sides of the equal sign, it becomes

$$V_o = 1/RC\int V_{in}dt. \tag{4.41}$$

Equation (4.41) states that the output voltage of our *RC* LPF circuit is proportional to the integral of the input voltage. The constant of proportionality is $1/RC$.

Referring again to the responses of Figures 4.8 and 4.9, it is clear to note that the shorter the time constant of the circuit with respect to the period of the square-wave excitation, the output signal tends to follow the input waveform. This is noted for time constants of 0.01, 0.05, and 0.2 seconds. For longer time constants such as 5 and 10-second, the circuit behaves more like an integrator. Notice that the integral of a (constant) horizontal line is a ramp. Indeed waveforms responses for 5- and 10-second time constants like fairly linear ramps and not so much exponential as described by Equation (4.34).

Summary of Important Points about RC LPFs in the Frequency Domain and Integrators in the Time Domain

A first-order RC LPF circuit allows sinusoidal frequencies smaller than one order of magnitude of its cutoff frequency to go through the circuit with little attenuation and no significant change in phase with respect to the input sinusoidal.

Sinusoidal frequencies of one order of magnitude higher that the cutoff frequency of the circuit become attenuated by 20 dB.

Sinusoidal frequencies of at least one order of magnitude higher that the cutoff frequency of the circuit or higher, approach a –90-degree phase shift with respect to the sinusoidal input.

The same first-order RC circuit performs time integration of the signals that are at least one order of magnitude higher in frequency than the filter cutoff frequency.

A practical limitation of the integrator implemented with a first-order RC LPF circuit is that the integrated output signal is attenuated, while other lower frequency signals below the cutoff frequency pass through the filter practically unaltered. We will see how to overcome these problems using an operational amplifier in Chapter 5.

4.3 THE FIRST-ORDER *RC* HIGH-PASS FILTER (HPF)

Let us investigate the *RC* circuit of Figure 4.10. This circuit is excited by a sinusoidal voltage waveform. Elements *R* and *C* are in series, the input voltage

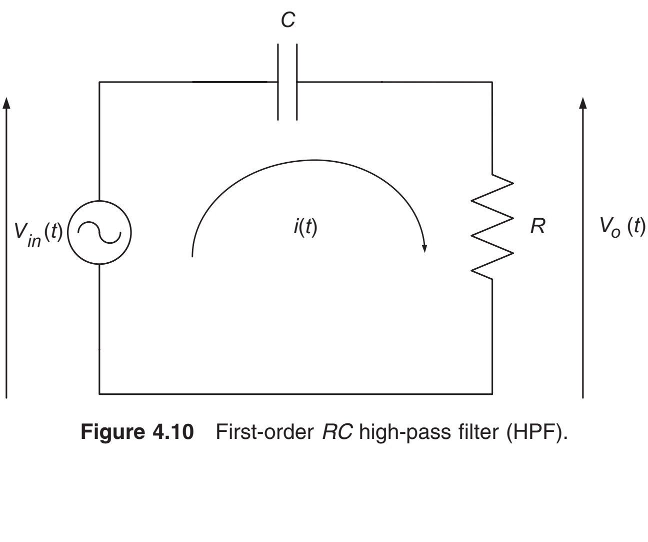


Figure 4.10 First-order *RC* high-pass filter (HPF).

is applied to the two elements in series, the output is taken across the resistor terminals.

4.3.1 *RC* HPF Frequency Domain Analysis

Let us calculate the transfer function of this circuit:

$$\mathrm{H}(j\omega) = \mathbf{V}_{out}(j\omega)/\mathbf{V}_{in}(j\omega). \tag{4.42}$$

The impedance of the resistor and capacitor in series is

$$Z_{series}(j\omega) = R + \frac{1}{j\omega C}. \tag{4.43}$$

Then, the current through the circuit is

$$\mathbf{V}_{in}(j\omega)/Z_{series}(j\omega). \tag{4.44}$$

The above current times the resistance equals the output voltage $V_{out}(j\omega)$, thus we get

$$\mathrm{H}(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{R}{R + 1/j\omega C} = \tag{4.45}$$

After some algebraic manipulations,

$$= \frac{j\omega RC}{1 + j\omega RC}. \tag{4.46}$$

Remember that from Chapter 2, ω is the angular frequency, which equals $2\pi f$, where f is the frequency of the sinusoidal waveform in hertz. We also define the angular cutoff frequency:

$$\omega_0 = 1/RC, \tag{4.47}$$

and the *RC* HPF cutoff frequency,

$$f_0 = 1/2\pi RC. \tag{4.48}$$

Equation (4.46) describes the ratio of output voltage to input voltage of the *RC* circuit given by Figure 4.10, again referred to as the circuit transfer function in the frequency domain. Using the definition for f_0 by Equation (4.48), we can rewrite the transfer function of the circuit as follows:

$$H(j\omega) = \frac{j\omega/\omega_0}{1 + j\omega/\omega_0} \tag{4.49}$$

$$|H(j\omega)| = \frac{\omega/\omega_0}{\sqrt{1 + (\omega/\omega_0)^2}} \tag{4.50}$$

$$\angle H(j\omega) = \pi/2 - arctan(\omega/\omega_0), \tag{4.51}$$

where Equation (4.50) is the linear magnitude of Equation (4.49). Equation (4.51) is the phase in radians of Equation (4.49).

So let us now construct a table to plot the values of the transfer function $H(j\omega)$ for a given *RC* HPF with a cutoff frequency f_0. Assuming a 1 kHz cutoff frequency f_0,

$$RC = 1/2\pi 1000 = 159.155\ \mu s. \tag{4.52}$$

We will plot the transfer function of Equation (4.49) using two separate plots. One plot is for its magnitude and the second plot for its phase, both as functions of frequency. Note that transfer function Equation (4.49) is a complex quantity and as such, it has magnitude and phase. Repeating the procedure that we previously used for the *RC* LPF we will tabulate the magnitude and phase values for the *RC* HPF. We show the outcome of such calculations in Table 4.4.

The gain and phase values of Table 4.4 have been calculated finding the magnitude and phase of the complex expression given by Equation (4.49).

Figure 4.11 depicts the exact magnitude and phase Bode plots for the *RC* HPF, generated using the values of Table 4.4.

4.3.2 Drawing an *RC* HPF Bode Plot Using Just the Asymptotes

To draw the asymptotes of the Bode plots of our first-order *RC* HPF (Fig. 4.10), we will normalize the frequency axis. Instead of listing on the frequency axis the actual cutoff frequency value of 1 kHz, we will denote this frequency

Table 4.4 ***RC* HPF transfer function: magnitude and phase as functions of frequency**

Frequency (Hz)	Normalized Frequency f/f_0	Linear Gain (V/V)	Gain in dB	Phase (Degrees)
1.000	0.001 f_0	0.001000253	–60.00	89.94270424
10.000	0.01 f_0	0.010002036	–40.00	89.42706130
100.000	0.1 f_0	0.099528982	–20.04	84.28940687
1,000.000	1 f_0	0.707296534	–3.01	45.00000002
10,000.000	10 f_0	0.995318596	–0.04	5.710593139
100,000.000	100 f_0	1.000233088	0.00	0.572938695
1,000,000.000	1,000 f_0	1.000282601	0.00	0.057295758

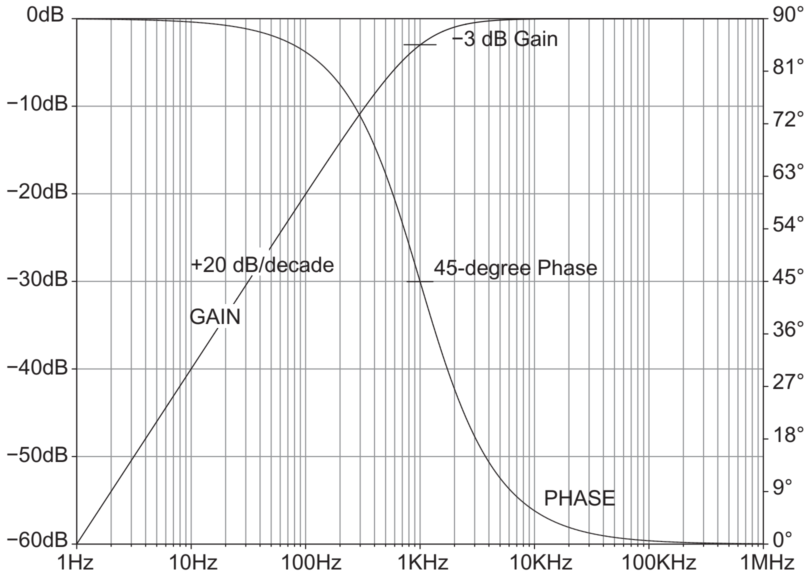


Figure 4.11 First-order *RC* HPF exact Bode plots.

as f_0, so lower frequencies will be a tenth, a one-hundredth, and so on of f_0. Similarly, frequencies above f_0 are 10 times, 100 times, and so on of f_0. Figure 4.11 shows these frequencies on the horizontal axis.

From Table 4.4 we see that the magnitude asymptotically approaches 0 dB from the cutoff frequency f_0 to higher frequencies. Also from the cutoff frequency to lower frequencies, the gain drops at a constant rate of 20 dB per decade. But at the cutoff frequency the gain is approximately –3 dB. In linear terms, this means that the amplitude of the sinusoidal waveform of frequency equal to f_0, the cutoff frequency of our circuit that excites the *RC* circuit, becomes attenuated down to about 70.7% from its original value. Referring one more time to Table 4.4 we can see that at the cutoff frequency f_0, the output voltage magnitude is 0.707 of the original input, which has a magnitude of 1. That is, the output magnitude is approximately 70.7% of the input magnitude.

In a similar fashion we can see that the phase is +45° at f_0. One more time looking at the phase in Table 4.4, we see that the phase at one-tenth of f_0 is about +84.3°. And for even lower frequencies, the phase asymptotically approaches +90°. On the other hand, at a frequency 10 times f_0, the phase is approximately 5.7° *more* than the high frequency value of the phase, which is 0°. For all frequencies higher than about 10 times f_0, the phase of the HPF is approximately 0 degrees (Figure 4.11).

We can also draw the HPF phase Bode plot using the phase asymptotes. Let us start with frequencies well below f_0, up to one-tenth of f_0, we draw a straight line at +90 degrees from low frequencies all the way up to 1/10 of f_0. At a frequency of 10 times f_0, the phase asymptote is a horizontal line at 0 degrees, continuing into higher frequencies. From Table 4.4, we know that at the cutoff frequency f_0, the phase is +45°. Looking at Figure 4.12, we now draw a straight line of a phase angle of 90° at 1/10 of f_0 all the way to 0° at 10 f_0,

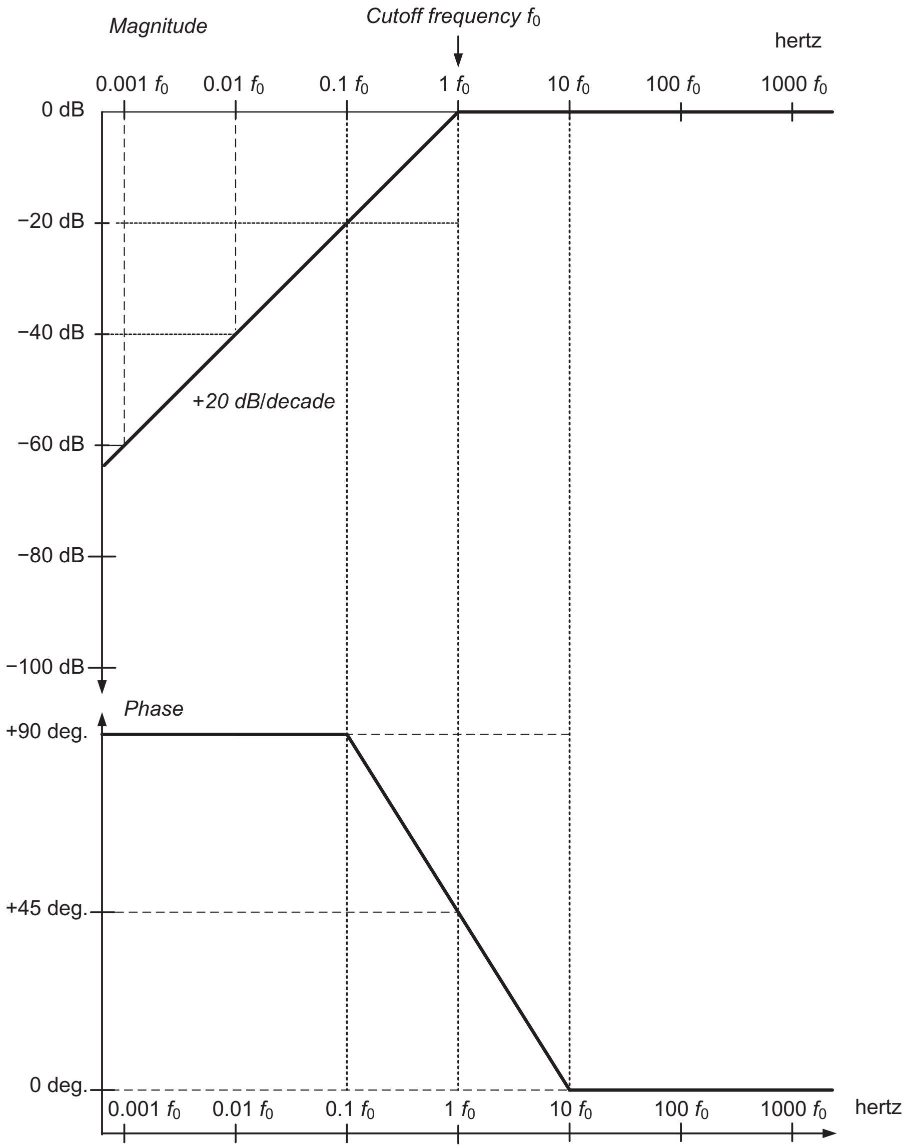


Figure 4.12 First-order *RC* HPF Bode plots asymptotes: magnitude and phase.

such that this straight line passes through a +45° phase at f_0. In this way, the phase asymptotes are drawn. Now one can draw by hand the approximate phase curves. Note that at 1/10 of f_0 and at 10 f_0, the phase is about 5.7° below +90° and about 5.7° above 0°, respectively. Finally, it is important to note that the phase curve has an inflexion point at 45°, see Figure 4.12. Figure 4.12 depicts the asymptotes of the magnitude and phase Bode plots for a first-order *RC* HPF.

Important Points about the First-Order RC HPF

For a first-order RC HPF the circuit gain at the cutoff frequency f_0 *is –3.01 dB.*

For a first-order RC HPF the gain is practically 0 dB at 10 f_0 *all the way to larger frequencies.*

From f_0 *down in frequency the gain starts decreasing at a 20 dB per decade rate. So that at 0.1* f_0 *the gain is 20 dB below 0 dB. At 0.01* f_0 *the gain is another 20 dB below, or 40 dB below the 0 dB gain line. At 0.001* f_0 *the gain is another 20 dB below the preceding decibel at the previous decade in frequency or 60 dB below the 0 dB gain line. This gain behavior continues to drop 20 dB as the frequency decreases by an order of decimal magnitude.*

The frequency axis does not have a zero or origin of frequencies because log of zero is nonexistent. The lowest frequency can be represented with a value as small as we desire, but not with zero.

Frequency is represented logarithmically for both magnitude and phase plots.

The magnitude or gain in dB is represented linearly. There is a 0 dB origin for the vertical axis because the scale is linear in dB.

The phase in degrees is represented linearly on the vertical axis.

The phase of an RC HPF is approximately +90° at frequencies below 1/10 of f_0*. The phase of an RC HPF is 0° at frequencies larger than 10* f_0*. At* f_0 *the phase equals +45°.*

4.3.3 Interpretation of the *RC* HPF Bode Plots in the Time Domain

We will use the asymptotically drawn Bode plots to explain the meaning of the Bode magnitude and phase plots in terms of sinusoidal inputs applied to the first-order *RC* HPF circuit (Fig. 4.12). We will explain this section at a faster pace because of the similarity that exists with first-order *RC* LPF, Section 4.2.

First let us assume that a sinusoidal waveform of 1 V peak amplitude and a frequency of 0.01 f_0 is applied to the input of the first-order *RC* HPF. The output voltage waveform that will be observed across the resistor terminals is 40 dB (a factor of 100) smaller than the 1 V input. At 0.1 f_0, the output waveform is 20 dB (factor of 10) smaller than the 1 V input. At the cutoff frequency f_0, the output magnitude is 3 dB below the 1 V input, meaning that the output is 70.7% of 1 V.

For the phase of the *RC* HPF, there is 90° phase shift for frequencies below 1/10th of f_0. A phase of +45° exists at the cutoff frequency. Finally, the phase becomes close to 0° (actually 5.7°) for 10 times f_0 and practically 0° at 100 f_0 frequencies and above.

4.3.4 Why Do We Call This Circuit an HPF?

From the Bode plots just presented in Figures 4.11 and 4.12 it is clear to see that frequencies below 1/10th of the cutoff frequency f_0 get attenuated. Frequencies above 10 times the cutoff frequency pass through the circuit with little or no attenuation. In summary, the *RC* circuit just analyzed allows high frequency signals to go through the circuit without attenuation, whereas the low frequencies become progressively attenuated as the input signal frequency goes below 1/10th of f_0. In summary, our first-order *RC* HPF greatly attenuates low frequencies and passes high frequencies without any significant attenuation. As usual, low frequencies are those that are smaller than 1/10th of f_0, and high frequencies are those that are larger than 10 times f_0. It is also interesting to notice that at frequencies equal to 10 f_0 and above, the region of frequency at which the gain is 0 dB, the phase shift is also 0°. The range of frequencies starting at 10 f_0 and going to larger frequencies is the pass-band frequency range of the filter. Within such range, signals pass through the filter unaltered in magnitude and in phase.

4.3.5 Time Domain Analysis of the *RC* HPF

Now let us analyze the time domain equations of the high-pass *RC* circuit. Referring one more time to the circuit of Figure 4.10, it is possible to establish the differential equation for such circuit. Simply applying KVL for the series of elements,

$$v_{in}(t) = v_{cap}(t) + i(t)R, \tag{4.53}$$

where $v_{in}(t)$ is the excitation or the circuit input voltage, $i(t)R$ equals the output voltage $v_o(t)$ and $v_{cap}(t)$ is the voltage across the capacitor of our *RC* HPF.

From Figure 4.10 we see that

$$v_{cap}(t) = v_{in}(t) - v_o(t). \tag{4.54}$$

Also,

$$i(t) = C\frac{d[v_{in}(t) - v_o(t)]}{dt} \tag{4.55}$$

because $[v_{in}(t) - v_o(t)]$ is the voltage across the capacitor.

Since the current times the resistor R is the output voltage $v_o(t)$, then,

$$v_o(t) = RCdv_{in}(t)/dt - RCdv_o(t)/dt. \tag{4.56}$$

Rearranging terms, Equation (4.56) becomes

$$\frac{dv_0(t)}{dt} + \frac{1}{RC}v_o(t) = \frac{dv_{in}(t)}{dt}, \tag{4.57}$$

where Equation (4.57) is a first-order differential equation. When input $v_{in}(t)$ is a step function $u(t)$, the solution is given by Equation (4.34), repeated here for the reader's convenience.

$$v_o(t) = V_{final} + (V_{initial} - V_{final})e^{(-t/\tau)}. \tag{4.58}$$

In particular for our *RC* HPF, Figure 4.10, we calculate the values of the initial and final values from circuit boundary considerations.

The initial value of the output waveform is 1 V, the magnitude of our step input excitation function $u(t)$. Why? Because upon impressing the 1-V pulse at the input of the circuit, assuming that the capacitor is initially discharged, the capacitor behaves like a short circuit to the 1-V edge. The final value of the output voltage after the transient behavior of the output is 0 V. Note that the capacitor has a *blocking* effect to the DC value of the step input. The output waveform will have no average value.

Now using Equation (4.54) and knowing that $V_{initial} = 1$ V, $V_{final} = 0$ V we obtain

$$v_o(t) = e^{-\frac{t}{\tau}}. \tag{4.59}$$

Equation (4.59) is plotted in Figure 4.13 with six different time constant values: 0.1, 0.5, 1, 2, 5, and 10 seconds from time 0 to 20 seconds. By observation of the response curve for time constant 10, note that its value is down to 36.8% from its original value of 1 V after 10 seconds from the origin of time.

Note that an *RC* HPF with a time constant of 10 seconds, at time $t = 10$ seconds, the output voltage $v_o(t)$ decays from its initial value of 1 V to

$$v_o(10) = e^{-\frac{10}{10}} = 0.368 \text{ V}. \tag{4.60}$$

The same is true for all other waveforms, as an example the response curve for 1-second time constant is down to 36.8% of its original value of 1 V after 1 second.

Back to Equation (4.58), we can verify that after five time constants, the response of the *RC* HPF will be down from its initial value of 1 V to 1% of 1 V.

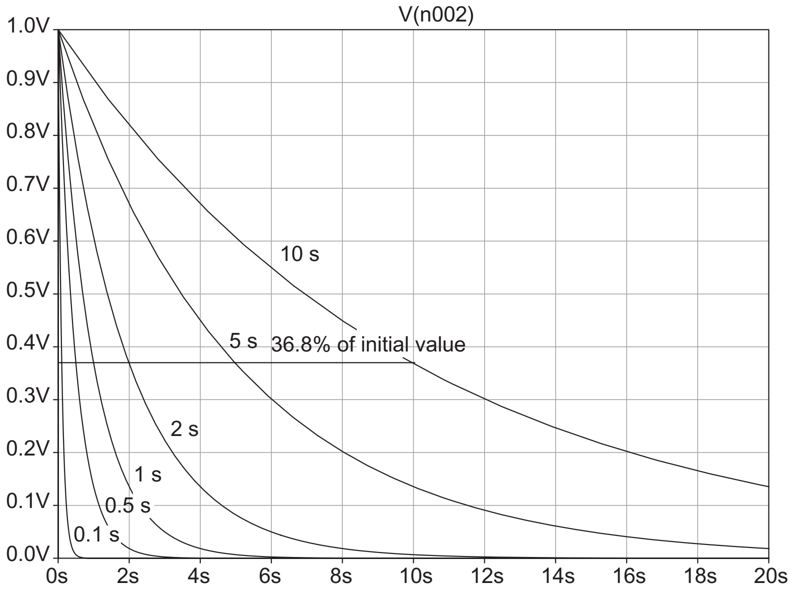


Figure 4.13 Unit step response of an *RC* HPF for six different time constants.

4.3.6 First-Order *RC* LPF under Pulse and Square-Wave Excitation

Let us apply a pulse of unity magnitude and 1-second duration to a first-order *RC* HPF with a 1-second time constant. The excitation and the corresponding response can be seen in Figure 4.14. The positive portion of the response is not new to us, since this is what we previously obtained in Figure 4.13 upon applying a unit-step. The difference in this example is that we are not applying a step, but a pulse. The 1-second pulse shown on the top of Figure 4.14 can be thought as the sum of a unit step at the origin, plus a 1-second delay inverted unit step. The equation for such step follows:

$$u(t) - u(t-1). \tag{4.61}$$

The positive portion of the response to such unit pulse is shown on the bottom section of Figure 4.14, and it is very much what we obtained for a unit step. This positive portion of the response is exponential and follows Equation (4.58). The difference in this case is that we are applying a pulse. The pulse cuts short the step at 1 second. So at such time, a negative 1-V step is applied to the to the circuit input. Note that the response of the circuit due to this negative 1-V step applied at *time* = 1 second, will also be exponential but will start at $t = 1$ second and 1 volt below the voltage of magnitude V_p in the first exponential in Figure 4.14. From that point on, the pulse over the exponential response from the negative portion continues to decay

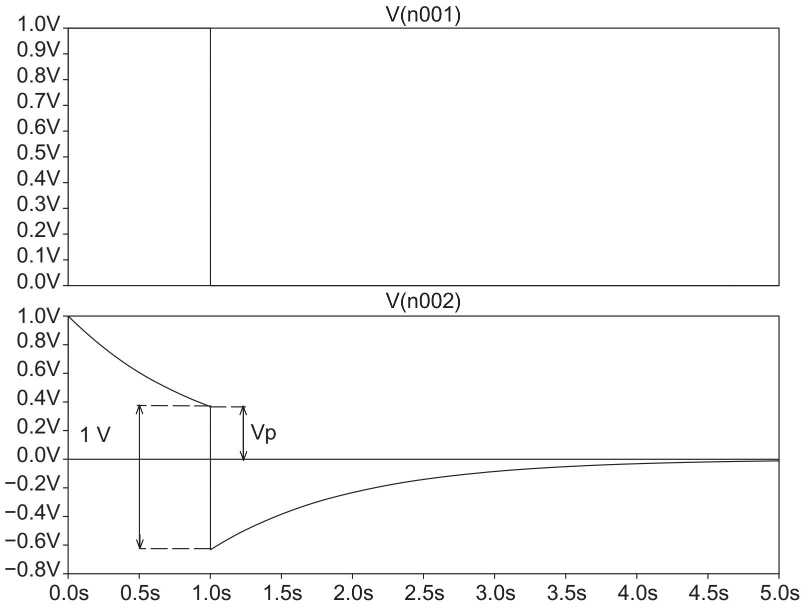


Figure 4.14 First-order *RC* HPF input pulse response.

exponentially (actually to increase exponentially) toward 0 V. After several time constants, the response reaches a zero value. Let us ask ourselves, what is the voltage V_p at time $t = 1$ second? Using Equation (4.58) and knowing that the circuit time constant is 1 second, we obtain

$$V_p = e^{-1/1} = +0.368 \text{ V}. \tag{4.62}$$

The voltage at which the negative portion of the exponential response begins at $t = 1$ second is:

$$0.368 - 1 = -0.632 \text{ V}. \tag{4.63}$$

It is important to observe that the average value of the complete response to the pulse, that is, the positive and the negative exponentials, have an average value of 0 V. In other words, the response has no *DC* component. Another way of saying this is that the area under the positive exponential equals the area above the negative exponential with respect to the time axis in both cases.

Now let us consider a square-wave input, as the one shown in Figure 4.15. Such waveform is a continuous train of pulses that swings between 0 V and 1 V with a 50% duty cycle. The waveform starts at 1 V at zero time for 1 second, at this time it drops very quickly to 0 V for another second. After this last second at 0 V, the earlier described process repeats itself indefinitely. Note that the period T of this square wave is 2 seconds.

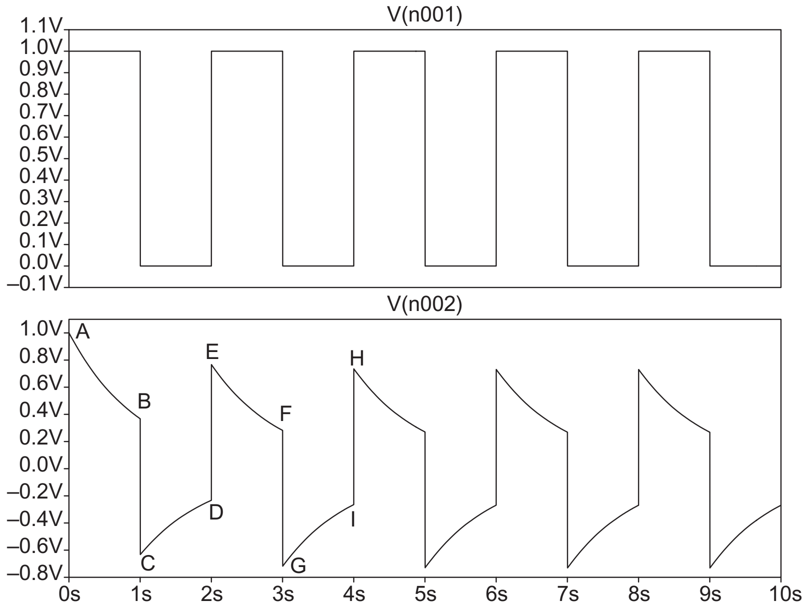


Figure 4.15 50% duty cycle square-wave driving *RC* HPF.

Let us apply such excitation to the input of a 1-second time constant first-order *RC* HPF. Note that during the first period of the square-wave input the response of the HPF settles down to a periodic response. The waveform indicated by points A, B, C, and D is a transient waveform; the second portion of the response, D, E, F, G, H, and I becomes the waveform that will be repeated over and over as long as the excitation is applied to the input. In our next example, Figure 4.16, the excitation input is the same as in the previous example, 0 V to 1 V swing, 50% duty cycle square wave, period $T = 2$ seconds. But now the time constant of the HPF is very small compared to the period of the excitation, that is, τ is 0.1 seconds. The RC circuit is much faster that the period of the excitation; this is the cause why the response attains steady-state value within the first period of the excitation.

Our next and final example of an *RC* HPF response to a square-wave input is applied to a circuit with a 100-second time constant. This is slower than the 2-second period of the excitation. Referring to Figure 4.17, it is clear to see that it takes in the order of 300 seconds (or three time constants) for the response to attain its steady state.

It is important and interesting to observe from Figures 4.15–4.17 that regardless of the *RC* circuit time constant, once the response attains a steady state, the average value or DC component of the response is zero. Let us remember that this occurs because of the DC blocking capacitor in the circuit. That is to say the output waveform has a zero average or zero DC value after the output transient behavior is over.

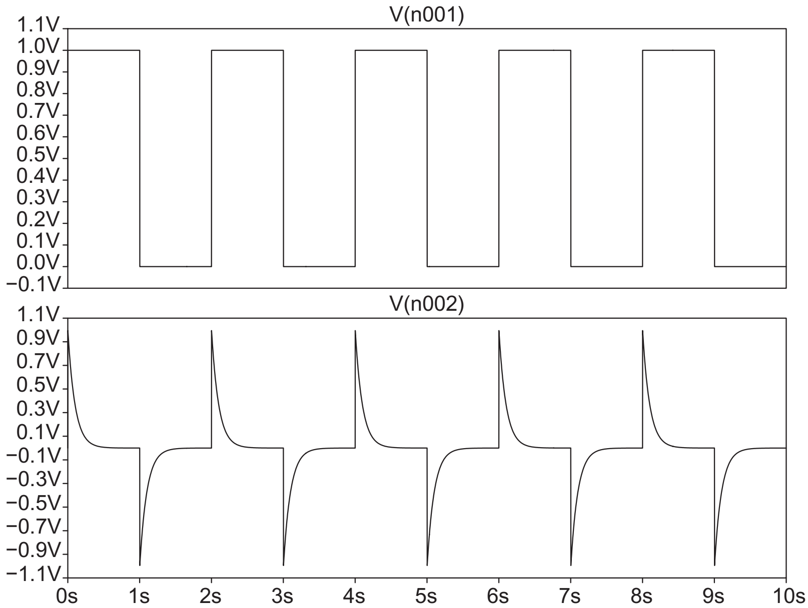


Figure 4.16 Square-wave excitation applied to a 0.1-second time-constant *RC* HPF.

4.3.7 The *RC* HPF as a Differentiator

From Equation (4.57), repeated here for the reader's convenience,

$$\frac{dv_0(t)}{dt}+\frac{1}{RC}v_o(t)=\frac{dv_{in}(t)}{dt}. \tag{4.64}$$

When the *RC* time constant and the output voltage are small, Equation (4.64) becomes

$$v_o(t)=RC\frac{dv_{in}(t)}{dt}. \tag{4.65}$$

Equation (4.65) shows that under the conditions previously stated, the output voltage is proportional to the derivative of the input voltage.

As an example of differentiation, let us look back at Figure 4.16, the *RC* HPF has a time constant of 0.1 second, which is smaller than the excitation 2-second period. Note that the circuit produces the derivative of the input waveform; the positive going transitions of the square-wave input become positive spikes, the negative going transitions become negative spikes upon the square wave becoming differentiated. Note that the constant levels of the square wave are zero, because the derivative of any constant is zero. As a counterexample of what is not a differentiator, refer this time to Figure 4.17,

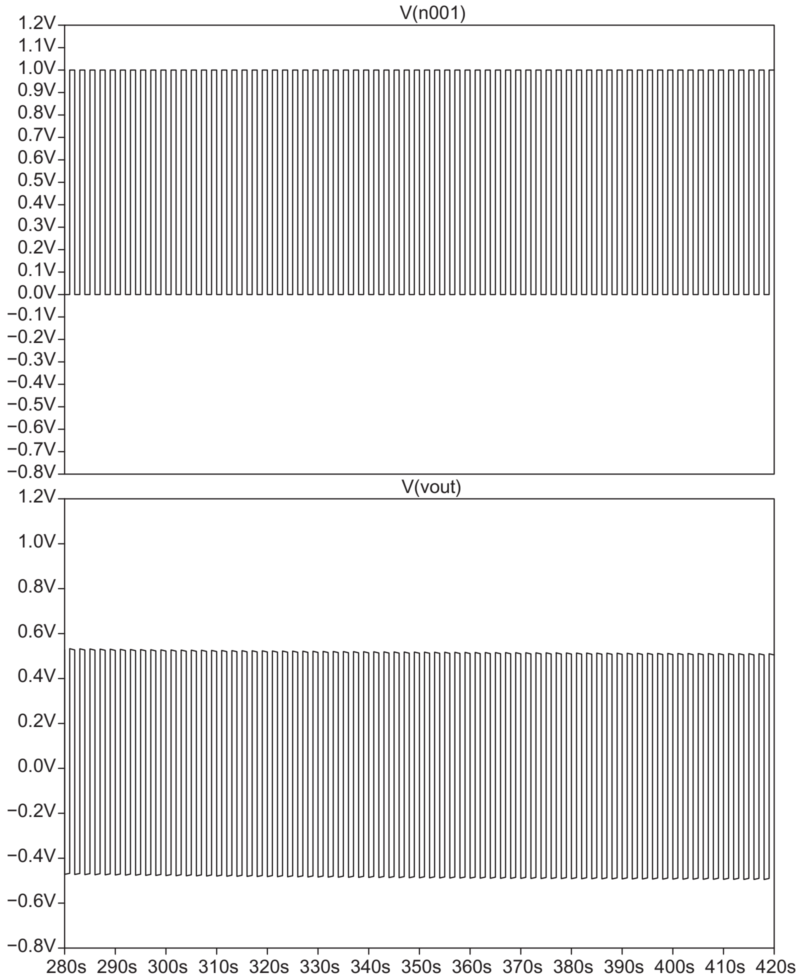


Figure 4.17 Second square-wave excitation applied to a 100-second time-constant *RC* HPF, in steady state.

the time constant of this *RC* HPF is 100 seconds (a large number), while the period of the excitation waveform is still 2 seconds as in Figures 4.15 and 4.16.

The plot shown presents the signals after its transient portion, in other words, in steady state condition. Note that the input square wave of 50% duty cycle, period of 2 seconds, swings between 0 V and 1 V. This input waveform contains a non-zero DC component of 0.5 V. The *RC* HPF allows the waveform to pass straight through with little attenuation, but notice that its DC component of 0.5 V has been removed by the filter. This is noticed by the fact that the output now swings between –0.5 V and +0.5 V, its peak to peak amplitude is still 1 V, no change with respect to the input. One more time referring to Figure 4.17, looking closely at the output waveform of the filter, we notice a slight slope on the top and the bottom of the output waveform positive and negative cycles. The reason for this is that the filter passes through high frequencies; however, it changes the phase of each frequency component by a positive 90° phase. Figure 4.17 displays the HPF response 280 seconds after the excitation was applied at the origin of time, 0 second.

Summary of Important Points about RC HPFs in the Frequency Domain and Differentiators in the Time Domain

A first-order RC HPF circuit allows sinusoidal waveforms of frequencies larger than one order of magnitude of its cutoff frequency to go through the circuit with little attenuation and with a 0° phase shift with respect to the sinusoidal input.

Sinusoidal waveforms whose frequencies are one order of magnitude lower than the cutoff frequency of the circuit are blocked by the RC HPF by being attenuated by 20 dB. Frequencies two orders of magnitude smaller than the cutoff are attenuated 40 dB. This goes on at a rate of 20 dB attenuation per decade. The phase of all frequencies at least one order of magnitude lower than f_0 *experience an approximate +90-degree phase shift.*

The same first-order RC circuit performs time differentiation of the signals that have frequencies at least one order of magnitude lower than than the filter cutoff frequency.

A practical limitation of the differentiator implemented with a first-order RC HPF circuit is that the differentiated output signal is attenuated, while other higher frequency signals above the cutoff frequency pass through the filter practically unaltered. We will see how to overcome these problems using an operational amplifier in Chapter 5.

4.4 SECOND-ORDER CIRCUITS

Second-order circuits are described by second-order ordinary differential equations with constant coefficients. Refer to Equation (4.66) to observe a second-order circuit differential equation:

$$a_0 \frac{d^2 f(t)}{dt^2} + a_1 \frac{df(t)}{dt} + a_2 f(t). \tag{4.66}$$

In Equation (4.66), $f(t)$ usually is $i(t)$ or $v(t)$, respectively current or voltage varying with respect to time. a_0, a_1, and a_2 are the constant coefficients, typically real numbers. t is time, the independent variable.

Equation (4.66) may be equated to zero or to a constant or to a function of time. Equation (4.66) equates the differential equation to zero, thus Equation (4.67):

$$a_0 \frac{d^2 f(t)}{dt^2} + a_1 \frac{df(t)}{dt} + a_2 f(t) = 0. \tag{4.67}$$

The differential equation of Equation (4.67) describes a second-order circuit without any external excitation. When Equation (4.66) is equated to a constant, it is usually when the second-order circuit is excited by a step. From now on, we will refer to a circuit described by a differential equation of the form given by Equation (4.66) simply as a second-order circuit. Second-order circuits have one inductor, one capacitor, and they may or may not have a resistor. When the second-order circuit does not have any resistors, it is said to be lossless.

4.5 SERIES RLC SECOND-ORDER CIRCUIT

We will analyze now a series *RLC* circuit, with a step input. Figure 4.18 depicts such a circuit.

From the circuit of Figure 4.18 we can derive the time domain equations. We obtain

$$L \frac{di(t)}{dt} + i(t)R + v_C = V_{step}. \tag{4.68}$$

In Equation (4.68), the first term on the left is the voltage drop on the inductor, the voltage drop on the resistor follows, and v_C is the drop across the capacitor. The differential equation, that is, Equation (4.68) is equated to V_{step}, assumes that a step input is applied to the circuit at time $t = 0$.

Since

$$i_C = C \frac{dv_C(t)}{dt}, \tag{4.69}$$

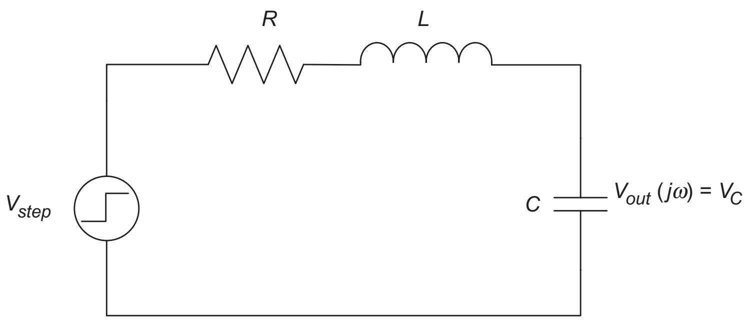


Figure 4.18 RLC series circuit with a step input excitation.

Then

$$v_C = \frac{1}{C}\int i_C(t)dt = \frac{1}{C}\int i(t)dt, \tag{4.70}$$

because $i_C(t)$ equals $i(t)$. We are analyzing a series circuit so the current through any one of its elements is the same current in the circuit.

We plug Equation (4.70) into Equation (4.68):

$$L\frac{di(t)}{dt} + i(t)R + \frac{1}{C}\int i(t)dt = V_{step}. \tag{4.71}$$

Differentiating Equation (4.71) and rearranging terms, we obtain

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = 0. \tag{4.72}$$

In Equation (4.72) $i(t)$ is the current through the capacitor which is the same as the current through the series circuit. When we solve, or find the solutions for differential Equation (4.72), we are finding the value of i as a function of time.

To find the solution of differential Equation (4.72), we will always end up with solutions that are functions of the following form:

$$i_1(t) = k_1 e^{s_1 t}. \tag{4.73}$$

$$i_2(t) = k_2 e^{s_2 t}. \tag{4.74}$$

Note that two solutions are found because it is a second-order system.

Equations (4.73) and (4.74) are solutions of Equation (4.72), and this means that if we plug each of the solutions into the differential equation, the solution will satisfy the mathematical operations of differential Equation (4.72). In Equations (4.73) and (4.74), k_1, s_1, k_2, and s_2 are constants, which can be real, imaginary, or complex. The differential equation solutions will determine three classic behaviors of second-order systems. These are

1. *Overdamped,*
2. *Critically damped, and*
3. *Underdamped.*

The reader is encouraged to plug Equation (4.73) into Equation (4.72) and validate the equation; similarly with Equation (4.74). So let us now plug a generic solution of the form of Equation (4.73) to our differential Equation (4.72):

$$\frac{d^2(ke^{st})}{dt^2} + \frac{R}{L}\frac{d(ke^{st})}{dt} + \frac{ke^{st}}{LC} = 0. \tag{4.75}$$

Computing the derivatives of Equation (4.75) we obtain

$$s^2ke^{st} + \frac{R}{L}ske^{st} + \frac{1}{LC}ke^{st} = 0. \tag{4.76}$$

Since e^{st} can never be zero for any finite time t, we can eliminate the instances of ke^{st} from Equation (4.76) and obtain

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0. \tag{4.77}$$

Equation (4.77) is called the *characteristic equation* of our differential equation.

Now, finding the roots for Equation (4.77) yields

$$s_1, s_2 = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{(R/L)^2 - 4\frac{1}{LC}}. \tag{4.78}$$

The roots of the characteristic equation are of three possible types:

1. *Both roots are real and different, or*
2. *Both roots are real and equal, or*
3. *Both roots are complex conjugates**

The solutions of differential Equation (4.72) have one of the following forms:

$$i(t) = k_1e^{s_1t} + k_2e^{s_2t}\text{: both roots are real and different: overdamped} \tag{4.79}$$

$$i(t) = (k_1 + k_2t)e^{\alpha t}\text{: both roots are real and identical: critically damped} \tag{4.80}$$

$$i(t) = (k_1 \cos\omega t + k_2 \sin\omega t)e^{-\alpha t}\text{: roots are complex conjugates: underdamped} \tag{4.81}$$

When the roots of the characteristic equation are complex conjugates, the roots have the following complex notation:

$$s_1, s_2 = -\alpha \pm j\omega. \tag{4.82}$$

* There is a fourth case when the roots are complex conjugate but pure imaginary. However, this is a special case of Equation (4.81).

In Equations (4.80) and (4.81),

$$\alpha = -\zeta\omega_n, \tag{4.83}$$

where ξ is defined as the damping factor, and ω_n is the natural or undamped frequency.

ω in Equation (4.84) is called the damped frequency, equal to

$$\omega = \omega_n\sqrt{1-\zeta^2}. \tag{4.84}$$

Based on Equation (4.72), repeated here for the reader's convenience,

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = 0. \tag{4.85}$$

We can find the relationship between the damping factor ξ, the damped frequency ω, and the undamped or natural frequency ω_n with the circuit *R, L,* and *C* components.

From Equation (4.72) R/L is defined as $2\xi\omega_n$. $1/LC$ is ω_n^2, that is, the square of the circuit natural frequency. The notation using ξ and ω_n is commonly used in control theory. Given those new defined parameters, we can rewrite Equation (4.77) as follows:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = s^2 + 2\zeta\omega_n s + \omega_n^2. \tag{4.86}$$

In reference to Equations (4.77) through (4.79), constants k_1 and k_2 are evaluated for a specific problem like ours, by the knowledge of the circuit *initial conditions*. Exponents s_1 and s_2 are the roots of the characteristic equation. Referring to Equation (4.80), α is the real part of the s_1 and s_2 roots of our system. And $\pm\ \omega$ is the imaginary part of the complex conjugate roots, also called the damped frequency. Referring once more to Figure 4.18 at time $t = 0$ when the voltage step is applied, the current in the circuit cannot change instantaneously, because the inductor is initially opposed to any current changes. Thus, $i(0+) = 0$. This means that the second and third voltage terms of Equation (4.68) are zero. The iR term is zero because $i(0+) = 0$ and v_C because the initial voltage across the capacitor is zero. Equation (4.68) is reduced to

$$\frac{di(0^+)}{dt} = \frac{V_{step}}{L}. \quad \text{ampère per second}$$

Example 4.4 Using the circuit of Figure 4.18, assume the following circuit components parameters:

$$R = 5\ \Omega, \mathrm{L} = 1\ \mathrm{H}, \mathrm{C} = \frac{1}{6}\mathrm{F} \tag{4.87}$$

And a step input of 1 V at t_0+.

Note: The large values of inductance and capacitance are simply used to simplify the arithmetic of the problem.

Referring to Equation (4.72), repeated here for the reader's convenience, thus,

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC} i(t) = 0. \tag{4.88}$$

Using the values provided by (4.87) in Equation (4.88) yields

$$\frac{d^2 i_C(t)}{dt^2} + 5\frac{di_C(t)}{dt} + 6i_C = 0. \tag{4.89}$$

The characteristic equation of Equation (4.89) is

$$s^2 + 5s + 6 = 0. \tag{4.90}$$

The roots of Equation (4.84) are

$$s_1 = -2;\ s_2 = -3. \tag{4.91}$$

Thus, the solution of Equation (4.88) is of the form

$$i_C(t) = k_1 e^{s1t} + k_2 e^{s2t}. \tag{4.92}$$

Using the roots of the characteristic equation, Equation (4.92) becomes

$$i_C(t) = k_1 e^{-2t} + k_2 e^{-3t}. \tag{4.93}$$

Let us determine constants k_1 and k_2 based on the problem initial conditions.

For $t = 0$, Equation (4.93) becomes:

$$0 = k_1 + k_2. \tag{4.94}$$

Now taking the derivative of Equation (4.92) yields

$$\frac{di(t)}{dt} = -2k_1 e^{-2t} - 3k_2 e^{-3t}. \tag{4.95}$$

$$\frac{di(t0^+)}{dt} = \frac{V}{L} = 1. \tag{4.96}$$

Because the current in the series circuit is zero, the inductor current cannot instantaneously change at $t = t_{0+}$. The voltages across the capacitor and resistor are zero.

As previously explained, using Equation (4.95) with the numerical values on-hand we obtain

$$1 = -2k_1 - 3k_2. \tag{4.97}$$

Solving the system of simultaneous Equations (4.94) and (4.97) yields

$$k_1 = 1; k_2 = -1. \tag{4.98}$$

Using the values of k_1 and k_2 from Equations (4.98) and (4.93), we obtain the complete current response:

$$i_C(t) = e^{-2t} - e^{-3t} \quad \textit{\textbf{<<< overdamped case}} \tag{4.99}$$

This example had a characteristic equation with two real and distinct roots; this is an overdamped-type response. In the next example we will study the response of the same second-order *RLC* circuit but with characteristic equation roots that are real and both are identical to each other. Since Example 4.4 was covered in great detail, the next two examples will be dealt without that many steps.

Example 4.5 Using the circuit of Figure 4.18, assume the following circuit component parameters:

$$R = 4\ \Omega, L = 1\ \text{H}, C = \frac{1}{4}\,\text{F} \tag{4.100}$$

and a 1-V step input. Derive an equation for the transient response of the circuit current, $i(t)$.

From Equation (4.88),

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = 0. \tag{4.101}$$

Equation (4.101) holds because we are dealing with the same series *RLC* circuit. Using the value given by Equation (4.100), Equation (4.101) becomes

$$\frac{d^2 i_C(t)}{dt^2} + 4\frac{di_C(t)}{dt} + 4i_C = 0. \tag{4.102}$$

From Equation (4.102) the characteristic function is

$$s^2 + 4s + 4 = 0. \tag{4.103}$$

The roots of Equation (4.103) are s_1, $s_2 = -2$; that is, -2 is a double root of characteristic Equation (4.103).

The solution will have the form of Equation (4.80), repeated below for the reader's convenience:

$$i_C(t) = (k_1 + k_2 t)e^{\alpha t}. \tag{4.104a}$$

Same as before for t_0^+, when the step is applied to the circuit, since the inductor will not allow an instantaneous current change, $i_C(t) = 0$ at the initial time t_{0+}.

Thus,

$$i_C(t_0^+) = 0 = k_1. \tag{4.104b}$$

Differentiating Equation (4.104a) after we substitute k_1 with 0, we obtain

$$\frac{di_C(t)}{dt} = \alpha k_2 t e^{\alpha t}. \tag{4.105}$$

Evaluating Equation (4.105) at time t_{0+}, yields

$$\frac{di_C(t_0^+)}{dt} = \alpha k_2 t e^{\alpha t} = 1. \tag{4.106}$$

Since $\alpha = -2$, then

$$k_2 = -\frac{1}{2}. \tag{4.107}$$

Using Equations (4.104) and (4.107), the solution is

$$i_C(t) = \frac{1}{2} t e^{-2t} \quad \textit{<<< critically damped case} \tag{4.108}$$

Example 4.6 This example will address the series RLC circuit, with a 1-V step input when the roots of the characteristic equation are complex conjugates: $R = 2\ \Omega$, $L = 1$ H, $C = 1/2$ F.

The second-order differential equation that describes such system is

$$\frac{d^2 i_C(t)}{dt} + 2\frac{di_C(t)}{dt} + 2i_C = 0. \tag{4.109}$$

The circuit characteristic equation is

$$s^2 + 2s + 2 = 0. \tag{4.110}$$

The roots of Equation (4.109) are

$$s_1, s_2 = -1 \pm j1. \tag{4.111}$$

The general solution of Equation (4.109) is of the form

$$\begin{aligned} i_C(t) &= k_1 e^{(-1+j1)t} + k_2 e^{(-1-j1)t} = \\ &= e^{-t}(k_1 e^{jt} + k_2 e^{-jt}) \end{aligned} \tag{4.112}$$

which is also of the general form previously shown by Equation (4.81):

$$i_C(t) = e^{-t}(k_3 \cos t + k_4 \sin t). \tag{4.113}$$

Note: The mathematical equivalence between Equations (4.112) and (4.113) is justified with Euler's identity; that is,

$$e^{\pm j\omega t} = \cos \omega t \pm j \sin \omega t. \tag{4.114}$$

Repeating Equation (4.81) for the reader's convenience,

$$i_C(t) = (k_1 \cos \omega t + k_2 \sin \omega t)e^{-\alpha t} \tag{4.115}$$

and now we equate Equations (4.113) and (4.115):

$$e^{-t}(k_3 \cos t + k_4 \sin t) = e^{-t}(k_1 e^{jt} + k_2 e^{-jt}). \tag{4.116}$$

From Euler's Equation (4.114) it can be shown that

$$\cos \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2} \tag{4.117}$$

and

$$\sin \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2j}. \tag{4.118}$$

Expanding the right-hand side term of Equation (4.116) using Euler's identities we obtain

$$e^{-t}(k_3 \cos t + k_4 \sin t) = e^{-t}[k_1(\cos t - j \sin t) + k_2(\cos t - j \sin t)]. \tag{4.119}$$

Rearranging terms on the right-hand side of Equation (4.119) and comparing them against the left-hand side of Equation (4.119) we obtain that

$$k_3 = k_1 + k_2 \quad \text{and} \quad k_4 = j(k_1 - k_2). \tag{4.120}$$

The initial conditions for this problem are exactly the same as what they were for Examples 4.4 and 4.5.

$$i_C(t0+) = 0 \quad \text{and} \quad \frac{di_C(0+)}{dt} = 1. \tag{4.121}$$

We evaluate the left-hand side term of Equation (4.119) at time $t = 0+$ and get

$$i_C(0+) = e^{-0}(k_3 \cos 0 + k_4 \sin 0) = 0 = k_3. \tag{4.122}$$

Now since k_3 is 0,

$$\frac{di_C(t)}{dt} = \frac{d}{dt}(e^{-t} k_4 \sin t). \tag{4.123}$$

Thus,

$$\frac{di_C(t)}{dt} = k_4(e^{-t} \cos t - e^{-t} \sin t). \tag{4.124}$$

Since the initial condition $\frac{di_C(0+)}{dt} = 1$ from Equation (4.121), we evaluate Equation (4.124) at time $t = 0+$

And this yields

$$\frac{di_C(0)}{dt} = k_4(e^{-0} \cos 0 + e^{-0} \sin 0) = 1 \tag{4.125}$$

$$k_4 = 1. \tag{4.126}$$

Now we are ready to find our particular solution for

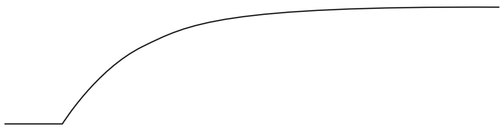
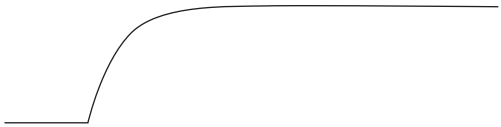
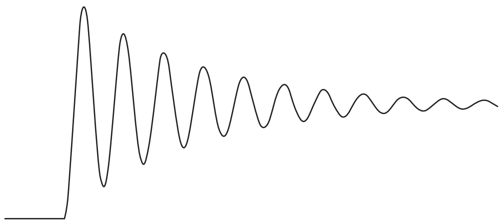
$$i_C(t) = e^{-t}(k_3 \cos t + k_4 \sin t). \tag{4.127}$$

Recall that $k_3 = 0$ and $k_4 = 1$. Thus,

$$i_C(t) = e^{-t} \sin t \quad \text{<<<} \textbf{\textit{ underdamped case}} \text{ (Table 4.5)} \tag{4.128}$$

Exercise for the reader: Technically speaking there is a fourth case, when the roots are pure imaginary and conjugate. Find the series *RLC* circuit voltage response across the capacitor due to a 1-V step voltage. Hint: Assume that the characteristic equation is $s^2 + 1 = 0$. Determine the values of all three circuit components for the given characteristic equation.

Table 4.5 Time domain step-input responses

Case	Type of Roots	Time Domain Response
Overdamped	Negative real and distinct	
Critically damped	Negative real and equal	
Underdamped	Complex conjugates with negative real parts	

4.6 SECOND-ORDER CIRCUIT IN SINUSOIDAL STEADY STATE: BODE PLOTS

In this section we will observe the behavior of a second-order circuit in the frequency domain. That is to say we will look at its magnitude in decibels and its phase in degrees.

The circuit of Figure 4.18 depicts a second-order *RLC* circuit. We are interested in the voltage across the capacitor. Let us apply an AC voltage source to the input of the series.

Now we can calculate the ratio of the output voltage over the input voltage of the circuit.

The total impedance seen by the AC source is the series of the R, L, and C circuit elements. That is,

$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}. \tag{4.129}$$

We are interested in the output voltage, which is the voltage across the capacitor. If we think of the R, L series as one impedance, say we call it Z_1, and we think of the capacitor as being impedance which we call Z_2, we have then

$$Z_1(j\omega) = R + j\omega L \tag{4.130}$$

and

$$Z_2(j\omega) = \frac{1}{j\omega C}. \tag{4.131}$$

The output voltage is calculated as if the impedances worked as resistor dividers.

Thus,

$$\frac{V_{out}}{V_{in}} = \frac{Z_2}{Z_1 + Z_2}. \tag{4.132}$$

However, it is important to understand that all the voltages and impedances in Equation (4.132) are complex numbers, because they are representing components operating at a the same sinusoidal frequency.

Plugging the values from Equations (4.130) and (4.131) into Equation (4.132), replacing the variable $j\omega$ with the operator s yields after doing some arithmetic:

$$\frac{V_{out}}{V_{in}} = \frac{1}{LC}\left(\frac{1}{s^2 + s\frac{R}{L} + \frac{1}{LC}}\right) \tag{4.133}$$

Note: The s operator is called the Laplace variable or operator. We simply used the operator as a substitute for the complex number $j\omega$. A whole entire course can be taken on the Laplace transforms and its applications. Certainly, this is not the book to read about Laplace transforms.

Equation (4.133) is also referred to as the circuit or system transfer function.

This is the transfer function that we will plot to understand the magnitude and the phase behavior with respect to frequency.

The denominator of Equation (4.133) is a second-order equation (nothing new here). This denominator can be factored as $(s - root_1).(s - root_2)$, where $root_1$ and $root_2$ are the denominator roots.

For the sake of simplicity and a clear presentation, we will assume the following numerical values for R, L, and C.

Assume that: $L = 1$ H, $C = 1$ F, and we plot 10 magnitude and 10 phase plots for the following values of R in ohms: 0.1, 0.3, 0.6, 0.9, 1, 2, 3, 4, 5, 10. Figure 4.19 is a computer generated Bode plot (magnitude and phase) for the transfer function given by Equation (4.133).

The purpose of this demonstration is to reveal the most important characteristics that a second-order system transfer function Bode plot has. Also compare those against the first-order Body plots at the beginning of this chapter.

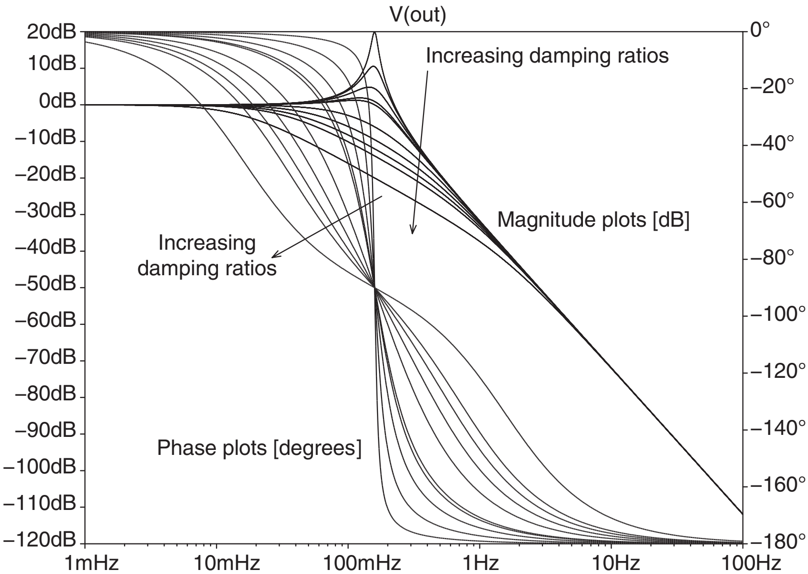


Figure 4.19 Magnitude and phase Bode plots of *RLC* circuit.

Figure 4.19 shows magnitude and phase plots on the same sheet. The unit of magnitude is the decibel, the unit of phase is the degree.

Magnitude characteristics:
10 different magnitude plots are shown for the 10 given values of R.

The natural frequency f_n, according to Equation (4.86), is $\omega_n^2 = 1/LC$, where $\omega_n = 2\pi f_n$, which for $L = 1$ H and $C = 1$ F, $f_n = 0.15924$ Hz. By inspection of Figure 4.19 we see that the magnitude peaks for small damping rations, and as the damping factor increases, the magnitude becomes less "peaky."

It is also of importance to mention that the magnitudes peak at the natural frequency of the circuit:

$$f_n = \frac{1}{2\pi\sqrt{LC}} \text{ [hertz].} \tag{4.134}$$

The negative slope of the magnitude plots are –40 dB per decade. Once the magnitude is at a frequency greater than or equal to 10 times the natural frequency, the slope is –40 dB/dec regardless of the damping ratio of the circuit.

Phase characterisitics:
The phase changes from 0 degrees to –180 degrees in approximately two decades of frequency. This statement is more accurate for lower damping ratios. The natural frequency is the crossover point for all phase plots. All phase plots will cross over at the f_n regardless of the value of the damping ratio. The phase crossover point for the second-order system is –90 degrees.

4.7 DRAWING THE SECOND-ORDER BODE PLOTS USING ASYMPTOTIC APPROXIMATIONS

The approximate methodology allows one to get very quickly approximate magnitude and phase plots. The Bode plots of a second-order system can be constructed as the composite plots of 2 first-order Bode plots.

In a generic way, assume that the natural frequency is f_n. Following the asymptotic magnitude plot of Figure 4.20, we see that for frequencies less than or equal to 1/10 f_n the magnitude plot is approximated by a 0 dB line.

From frequency f_n, we draw a line with a –40 dB/dec slope.

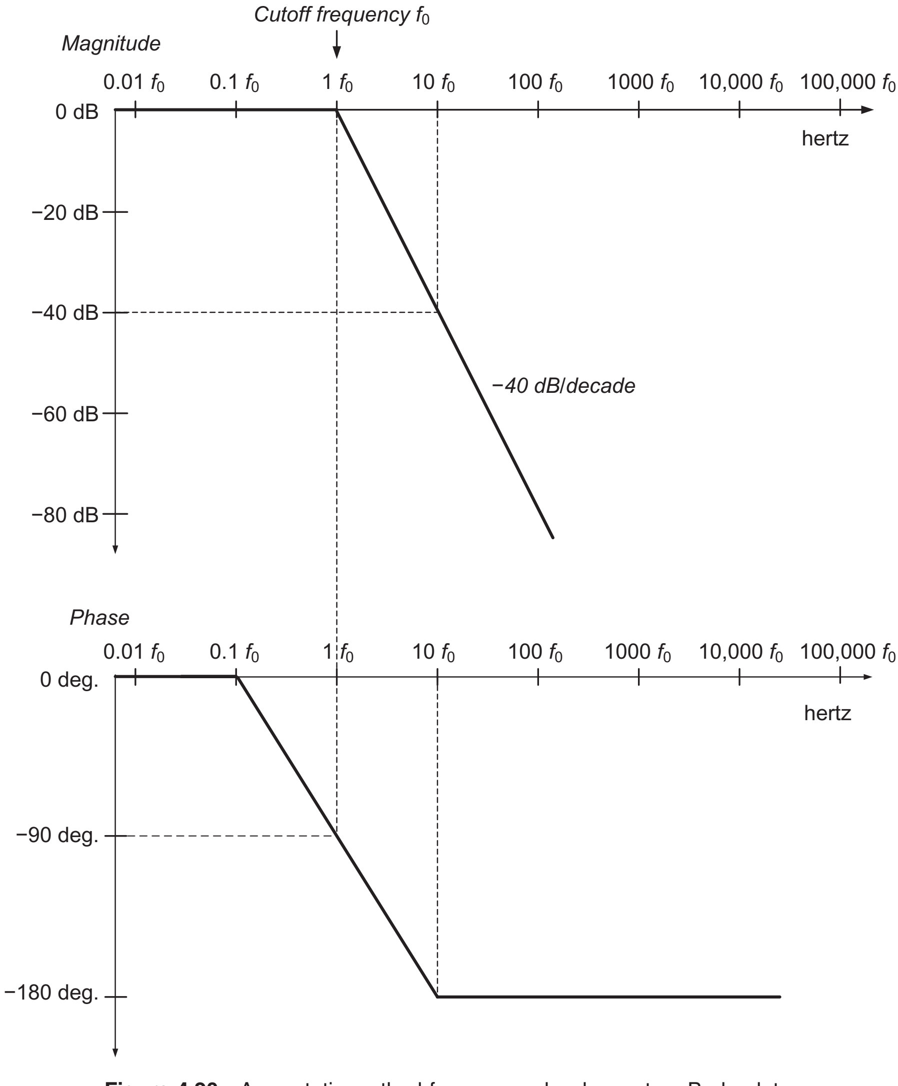


Figure 4.20 Asymptotic method for a second-order system Bode plots.

For the phase, we approximate it with a zero degree phase line at frequencies less than or equal to $1/10 \times f_n$. We then draw a horizontal phase line starting at a frequency greater or equal to $10 \times f_n$. Finally and once more referring to Figure 4.20, we draw a line for the phase from 0 degrees at 1/10 of f_n all the way through the –180 degree point at a frequency 10 times f_n.

4.8 SUMMARY

We looked at two of the most fundamental circuits in electrical and electronics engineering, the first-order *RC* LPF and HPF. They are first-order circuits because they have a single energy storage element, a capacitor. Their time domain equations are first-order differential equations. The circuits are fully characterized; that is, their time behavior as well as their frequency behavior are completely known by their *RC* time constant.

The *RC* LPF as its name states allows low frequencies to pass through it unaltered. The *RC* HPF allows high frequencies to pass through it, unaltered.

The behavior of an *RC* LPF can be that of an LPF or that of an integrator at frequencies well above the filter cutoff frequency. The behavior of an *RC* HPF can be that of an HPF or that of a differentiator at frequencies well below the filter cutoff frequency.

The *RC* LPF integrates when the frequency of the signal to be integrated is at least 10 times f_0 or more. The *RC* HPF differentiates when the frequency of the signal to be differentiated is at most 0.1 times f_0 or less. Recall that f_0 is for both, HPF and LPF, their cutoff frequency.

Second-order circuits, have one capacitor and one inductor in addition to some resistance. Those two energy-storing circuit elements are what cause the overshooting and undershooting of the second-order time response, of course depending on the damping ratio. The larger the damping ratio, the smoother the response and no overshoot/undershoot will be observed. The smaller the damping ratio, the larger the overshoots and undershoots will be. Overshooting and undershooting are phenomena not observed in first-order circuits. For overshooting and undershooting to occur, the circuit has to be a second-order system or higher.

FURTHER READING

1. M. E. Van Valkenburg, *Analog Filter Design*, HRW, New York, 1982.
2. M. E. Van Valkenburg, *Network Analysis*, 3rd ed., Prentice Hall, Englewood Cliffs, NJ, 2006.
3. Sergio Franco, *Design with Operational Amplifiers and Analog Integrated Circuits*, McGraw-Hill Book Company, New York, 1988.
4. Mahmood Nahvi and Joseph Edminister, *Electric Circuits*, 4th ed., Schaum's Outline Series, McGraw-Hill, New York, 2003.

PROBLEMS

4.1 Given an *RC* low-pass filter circuit, like the one shown in Figure 4.2, assume that $R = 1$ kΩ and $C = 1$ μF. (a) Determine the filter cutoff frequency, (b) determine the time constant of the circuit, and (c) draw the magnitude and phase asymptotic Bode plots of such filter for the following frequencies: 0.01 f_0, 0.1 f_0, 1 f_0, 10 f_0, 100 f_0, where f_0 refers to the cutoff or corner frequency. Make sure to use semi-log paper to draw the Bode plots.

4.2 For an *RC* low-pass filter with $R = 1$ kΩ and $C = 1$ μF, determine the steady-state output $v_{out}(j\omega)$ magnitude and phase when the a sinusoidal voltage $v_{in}(j\omega)$ is applied at the input. Tabulate magnitude and phase for the following frequencies: 0.01 f_0, 0.1 f_0, 1 f_0, 10 f_0, 100 f_0, where f_0 refers to the cutoff or corner frequency. Note: $v_{out}(j\omega)$ is the voltage across the capacitor.

4.3 For the circuit given in Figure 4.22, initially the capacitor is completely discharged. Determine the voltage that the capacitor will get charged up to, after the switch is closed instantaneously at time t_o and waiting for two circuit time constants.

4.4 Recall the current–voltage relationship of the voltage across a capacitor and the current flowing through it, is given by: $i_C(t) = CdV_C/dt$. (a) Calculate the voltage developed across an initially discharged 1 μF capacitor when a DC current source is applied as shown by Figure 4.21. (b) Justify your answer based on the capacitor current–voltage relationship.

4.5 Using the circuit depicted by Figure 4.23, (a) draw the current through the 10 nH inductor when the square wave shown is applied to the inductor for two complete periods; (b) determine the current numerical value at $t = 1$ μs; (c) determine the current numerical value at $t = 2$ μs.

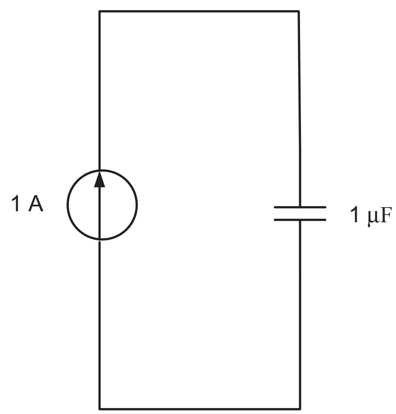


Figure 4.21 Circuit for Problem 4.4.

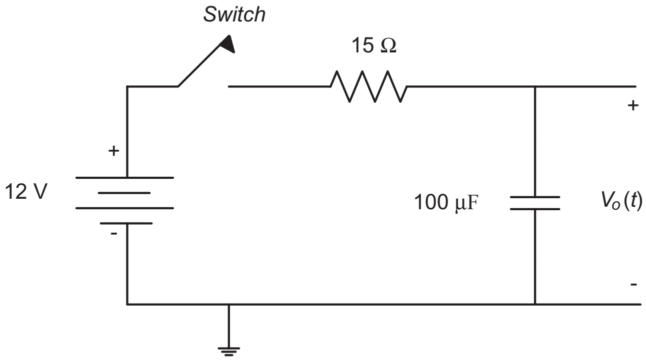


Figure 4.22 Circuit for Problem 4.3.

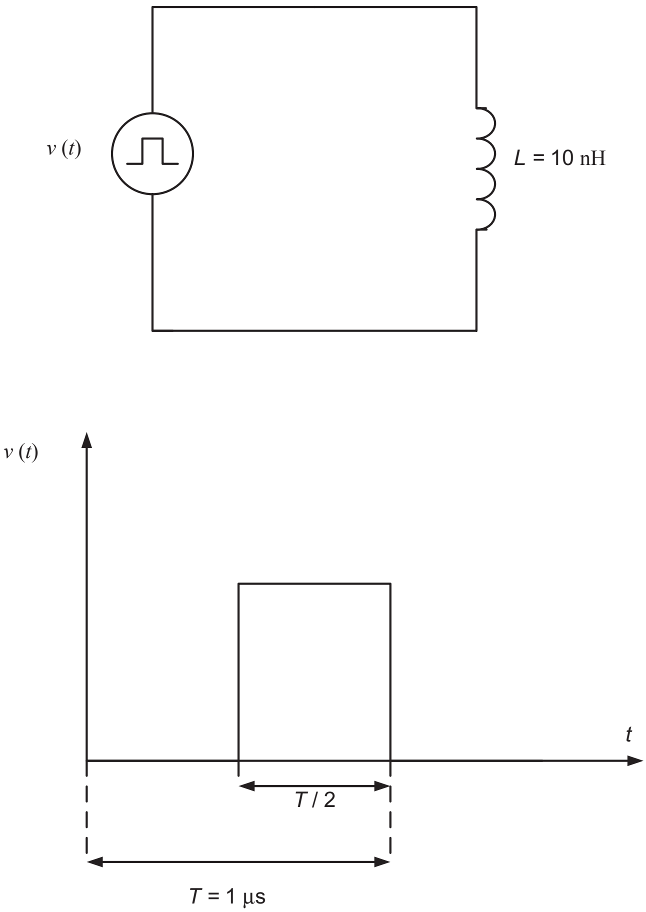


Figure 4.23 Circuit for Problem 4.5.

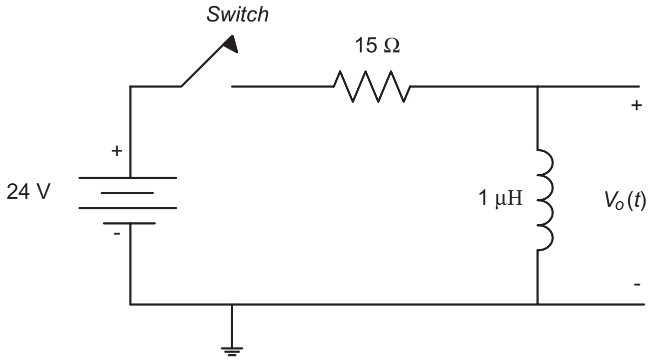


Figure 4.24 Circuit for Problem 4.6.

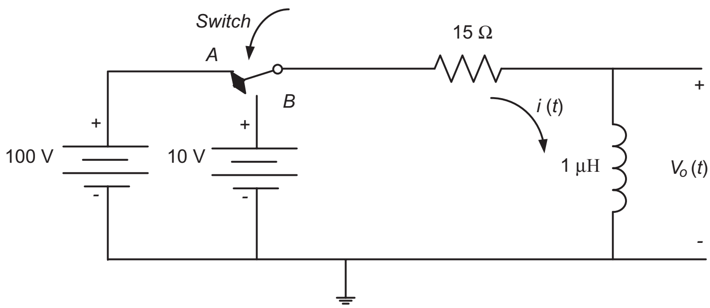


Figure 4.25 Circuit for Problem 4.7.

4.6 Using the circuit depicted by Figure 4.24, (a) determine the output voltage $V_o(t)$ equation as a function of time, when the switch is quickly closed, (b) draw the output voltage $v_o(t)$, (c) determine the output voltage after 10 μs from closing the switch, and (d) Determine the output voltage after one second from closing the switch.

4.7 The switch in Figure 4.25 has been closed for a very long time in position A. At time $t = 0$, the switch is quickly moved to position B. (a) Determine the equation of $i(t)$ for $t > 0$; (b) draw current $i(t)$ for $t > 0$.

4.8 The switch in Figure 4.26 has been closed for a very long time in position A. At time $t = 0$, the switch is quickly moved to position B. (a) Determine the equation of $i(t)$ for $t > 0$; (b) draw current $i(t)$ for $t > 0$.

4.9 Given the circuit of Figure 4.27, (a) determine the circuit time constant, (b) determine the circuit cutoff frequency f_0, and (c) construct the magnitude and phase Bode plots using the asymptotic method for the transfer function: $V_{out}(j\omega)/V_{in}(j\omega)$. Use as frequency range, 2 decades below cutoff frequency f_0 up to 2 decades above f_0.

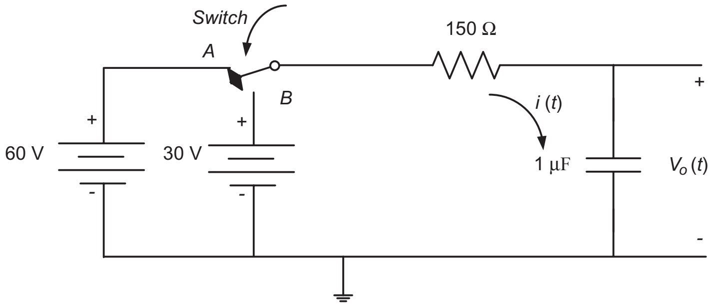


Figure 4.26 Circuit for Problem 4.8.

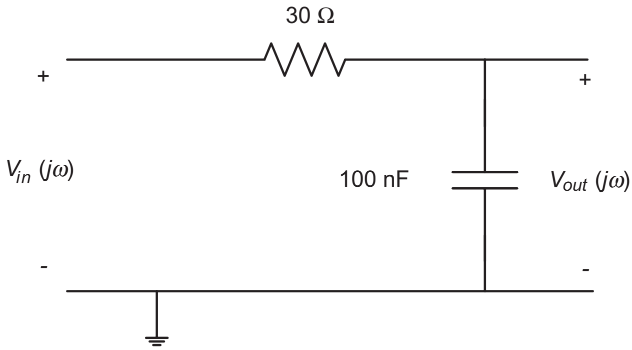


Figure 4.27 Circuit for Problem 4.9.

4.10 For the *RL* series circuit of Figure 4.28, (a) determine the transfer function of the circuit, that is, $V_{out}(j\omega)/V_{in}(j\omega)$; (b) determine the circuit cutoff frequency f_0; (c) determine and draw the asymptotic Bode plots for the magnitude and the phase of the transfer function; the frequency range used should be from 0.01 f_0 to 100 f_0. This is a total of four decades of frequency; (d) which type of filter this circuit represents?

4.11 The capacitor in the circuit of Figure 4.29 is charged up to 50 V DC when the switch is open. Upon closing the switch very quickly, determine the transient current as a function of time that will flow through the circuit. Note: The circuit that initially charged the capacitor is not shown.

4.12 For the circuit of Figure 4.30, (a) calculate the circuit transfer function as a function of $j\omega$; (b) calculate the cutoff or corner frequency of the circuit; (c) draw the asymptotic magnitude and phase Bode plots.

4.13 For the circuit of Figure 4.30, determine the current transient response for a step input voltage of 0 to 1.

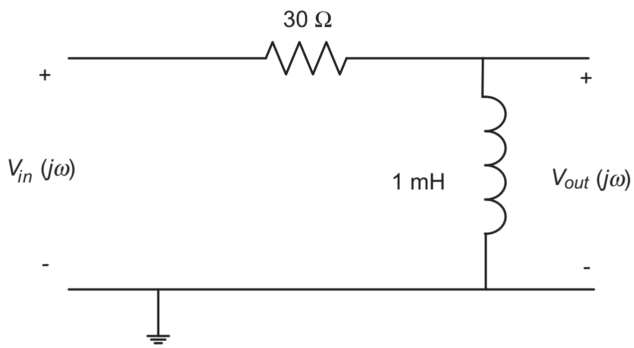


Figure 4.28 Circuit for Problem 4.10.

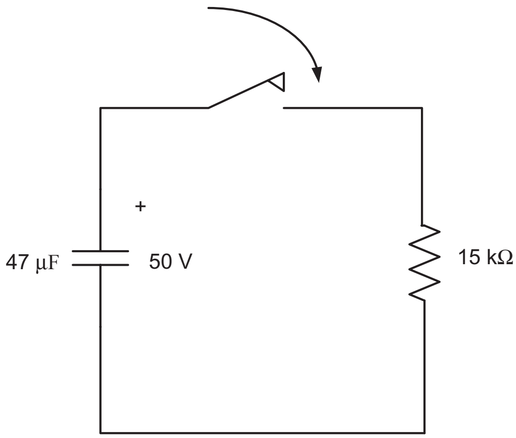


Figure 4.29 RC circuit for Problem 4.11.

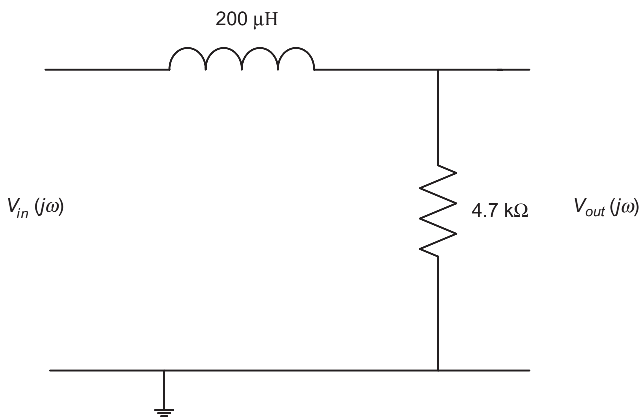


Figure 4.30 Circuit for Problems 4.12 and 4.13.

4.14 For an *RC* high-pass filter, such as the one shown in Figure 4.10, if $R = 100\ \Omega$ and $C = 0.159\ \mu F$, (a) derive the transfer function of the circuit $V_{out}(j\omega)/V_{in}(j\omega)$; (b) draw the magnitude and phase Bode plots from 2 frequency decades below the corner frequency up to 2 decades above the corner frequency.

4.15 Assume that you are given an *RC* low-pass filter, whose corner frequency is 10 kHz. Calculate the exact magnitude in decibels and phase in degrees at 100 Hz, 10 kHz, and 100 kHz.

4.16 For the filter shown in Figure 4.31, assume a 1-V step is applied to the input. (a) Derive a time domain equation of the current through the circuit, (b) calculate the circuit time constant, and (c) plot the current response for two time constants. Hint: Apply Thévenin to simplify the problem.

4.17 Refer to the circuit of Figure 4.32. Determine the time domain equation of the current as a function of time. Make sure that you find all the initial conditions of the circuit. Assume that the capacitor is initially discharged. Hint: Apply Thévenin to the left-hand side of the 47 μF capacitor to simplify the problem.

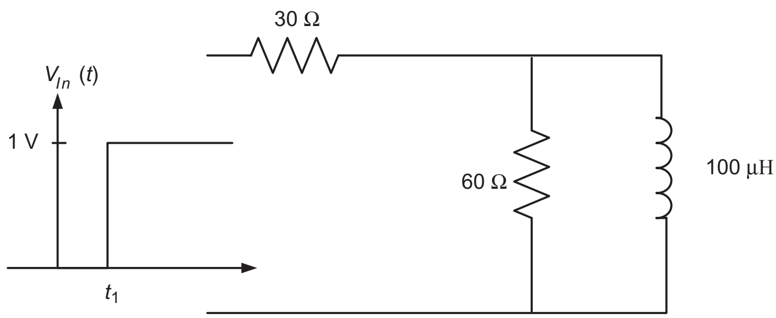


Figure 4.31 Circuit for Problem 4.16.

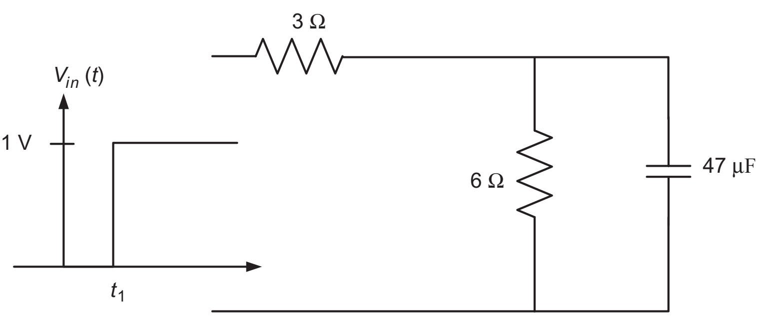


Figure 4.32 Circuit for Problem 4.17.

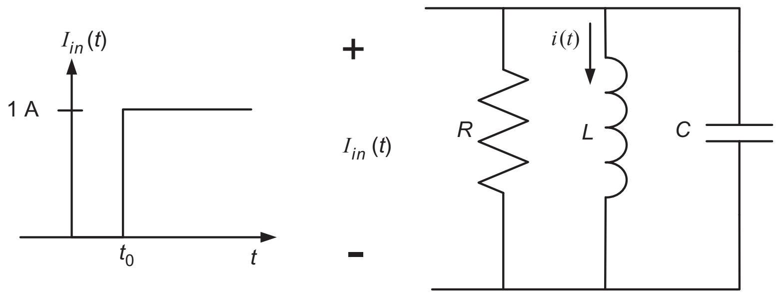


Figure 4.33 Circuit for Problem 4.19.

4.18 Refer to the second-order *RLC* series circuit of Figure 4.18. A 1-V step input voltage is applied at time t_0, all the circuit initial conditions are zero.

(a) State the time domain equation of the circuit; ensure that you show the equation as a second-order system equation. (b) In a general fashion, explain the consequences when the roots are

(i) negative real and different, (ii) negative real and equal, and (iii) when the root are complex conjugates.

4.19 Refer to the second-order *RLC* parallel circuit of Figure 4.33. Assume that a 1-A step input current is applied at time t_0, all the circuit initial conditions are zero.

(a) State the time domain equation of the circuit inductor current; ensure that you show the equation as a second-order system equation. (b) In a general fashion, explain the consequences when the roots are

(i) negative real and different, (ii) negative real and equal, and (iii) when the roots are complex conjugates.