2

ALTERNATING CURRENT CIRCUITS

2.1 AC VOLTAGE AND CURRENT SOURCES, ROOT MEAN SQUARE VALUES (RMS), AND POWER

When we plug a toaster into an electrical outlet in our kitchen, insert a slice of bread into a slot, we notice that the toaster starts to get hot very quickly. If we take a peek in the slot where the slice of bread is, we can see that the internal wires in the toaster become red hot. The toaster-heating elements are approximately 1 to 2 kW rated resistors, depending on the toaster make and model, rated to operate at the household AC supply voltage. This is a simple example of an alternating current (AC) voltage source, supplying an AC current to the toaster-heating elements in operation. Both of these waveforms, voltage and current, vary sinusoidally with respect to time. The AC current, being "*pushed*" by the AC voltage source, is the cause of heat being produced in the immediate vicinity of the toaster-heating element. The outlet on the kitchen wall is the point where we connect the appliance to the AC voltage source. The AC voltage source from the electric utility company is usually located in a remote site, far away from the home. In most households in the Unites States, the standard AC voltage is 120 V. The 120 V refers to the root mean square (RMS) value of the sinusoidal waveform, where RMS is defined mathematically by the following equation:

Electrical, Electronics, and Digital Hardware Essentials for Scientists and Engineers, First Edition. Ed Lipiansky.

$$f_{RMS} = \sqrt{\frac{1}{T}\int_0^T [f(t)]^2 dt}. \quad (2.1)$$

Equation (2.1) is the RMS value of waveform $f(t)$.

In Equation (2.1) T is the period of the waveform. The waveform $f(t)$ can be either a voltage or a current, and t is the time, the independent variable. RMS of a waveform is also referred to as the effective value of the waveform.

When $f(t)$ is a sinusoidal waveform such as $v(t) = V \sin(\omega t + \theta)$; V is the amplitude (or peak value) of the sinusoidal waveform in volts, ω is its angular frequency equal to $2\pi f$, where f equals the inverse of the sinusoid's period T or the sinusoid frequency, given in units of second^{-1} or hertz, and θ is the sinusoidal waveform phase shift. The units of the angular frequency ω are given in radians per second. Solving Equation (2.1) for a sinusoidal voltage, the RMS value of it is

$$V_{effective} = V_{RMS} = V/\sqrt{2} \cong 0.707 \text{ V}, \quad (2.2)$$

where V is the peak value or magnitude of the sinusoidal waveform. So the 120 V at the kitchen outlet is the RMS value of the sinusoidal waveform that the electric utility company provides to U.S. households. Also applying Equation (2.2) to a current waveform, $i(t) = I \sin(\omega t + \theta)$, we find that its RMS value is also

$$I_{effective} = I_{RMS} = 0.707\ I. \quad (2.3)$$

In Equation (2.3) I is the peak value or amplitude of the current waveform.

2.1.1 Ideal and Real AC Voltage Sources

An ideal AC voltage source is one that produces a sinusoidal voltage that varies with time. Most importantly, the amplitude and the RMS value of such voltage source does not vary based on how much current the load across the source terminals is drawing. This means that the internal resistance of an ideal AC voltage source is zero. So whether the voltage source supplies no current or very large currents, the voltage amplitude and RMS value remain constant. It is also true that the waveforms retain their sinusoidal shape and original frequency f and phase angle θ. On the other hand, a real AC voltage source amplitude does not remain constant with the level of current being supplied by the real AC voltage source. This concept is similar to that of ideal and real DC voltage sources. The real AC voltage source can be modeled as an ideal AC voltage source in series with its internal resistance, the real AC voltage source internal resistance is not zero, and it is a finite number as shown by Figure 2.1.

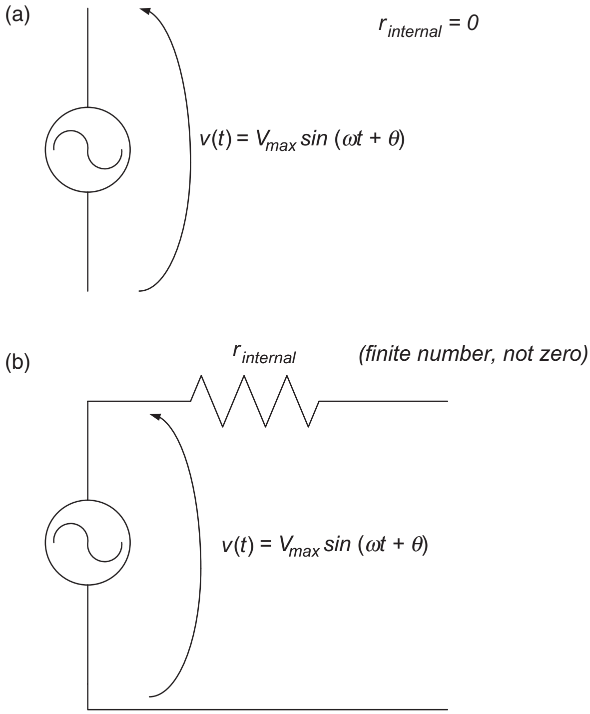

Figure 2.1 Representation of (a) ideal and (b) real AC voltage sources.

- *The internal resistance of an ideal AC voltage source is zero, which means that the source can supply an unlimited current to its load.*
- *In contrast, real AC voltage sources cannot provide infinite current when the source terminals are short-circuited.*
- *The internal resistance of a real AC voltage source is never zero and it is a finite number.*
- *The internal resistance of a real AC voltage source is always greater than 0 Ω.*
- *The internal resistance is an indicator of the current sourcing capability of the voltage source.*

Example 2.1 Ideal versus Real Voltage Sources

Let us assume that we have an ideal voltage source, this can be a DC or an AC source. An ideal voltage source has zero internal resistance. Thus, a load connected across the terminals of the voltage source can draw any amount of current dictated by the load value. For example, given an ideal 12-V DC source connected across a 10 Ω, 1 Ω, 0.1 Ω, 0.01 Ω, or any other resistor value (except for zero), the current is always determined by Ohm's law. A 10 Ω resistor draws 12 V/10 Ω = 1.2 A from the ideal 12-V source. The 1 Ω resistor draws 12 V/1 Ω = 12 A; a 0.1 Ω draws 12 V/0.1 Ω = 120 A and the 0.01 Ω resistor draws 12 V/0.01 Ω = 1200 A from the voltage source. Now what happens if the resistor placed across the ideal voltage source has a 0 Ω value? The current that the ideal voltage source would have to supply is infinite. So to be realistic with how much current an ideal voltage source can supply, it is fair to say that any amount of current desired can be provided by the source, but not an infinite current. Using circuit simulators, if we simulated a short-circuited voltage source with a 0-Ω internal resistor, it produces an indetermination.

For the ideal AC voltage source, there is conceptually no difference with respect to the ideal DC source. The key difference is that the AC source supplies a perfectly sinusoidal time varying waveform; which has a peak value or amplitude, a frequency, and a phase angle.

Sinusoidal Waveforms: A sinusoid, from basic trigonometry, is a periodic waveform that repeats itself with a period T; it is also positive half of the time, and it is negative the other half of the time. Figure 2.2 shows a sinusoidal voltage of frequency f, amplitude V, and phase θ. Figure 2.3 shows a sinusoidal source applied to a resistor.

When we discussed DC circuits we stated that $V = I\,R$ (Ohm's law), where I and V are DC values of current and voltage, respectively. For sinusoidal-varying waveforms $v(t)$ and $i(t)$, Ohm's law holds true as well:

Generalized Form of Ohm's Law

$$v(t) = i(t)R \tag{2.4}$$

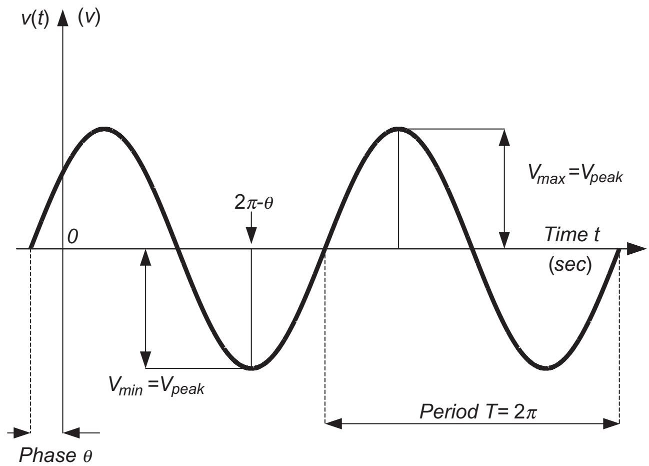

Figure 2.2 Sinusoidal voltage.

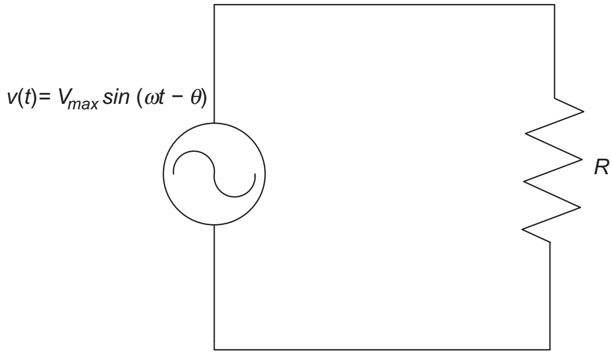

Figure 2.3 An AC voltage source applied across a resistor.

Example 2.2 Assume an ideal AC voltage source that generates a voltage equal to $v(t) = V_{peak} \sin(\omega t + \theta)$, $V_{peak} = 12$ V, $\omega = 2\pi\ 1$ rad/s, and $\theta = 0°$. Evaluate the current waveform obtained for the following resistive loads R_L: (a) 10 Ω; (b) 1 Ω; (c) 0.1 Ω, and (d) 0.01 Ω.

Answer to Example 2.2

From Equation (2.4), since $v(t) = i(t)\ R_L$, thus $i(t) = V_{peak}/R_L \sin(\omega t + \theta)$.
Thus, we obtain

(a) $i(t) = 1.2 \sin(2\pi t)$ for $R_L = 10\ \Omega$;
(b) $i(t) = 12 \sin(2\pi t)$ for $R_L = 1\ \Omega$;
(c) $i(t) = 120 \sin(2\pi t)$ for $R_L = 0.1\ \Omega$; and
(d) $i(t) = 1200 \sin(2\pi t)$ for $R_L = 0.01\ \Omega$.

All currents are given in amperes.

Equation (2.4) holds for all values of time such that $t \geq 0$. Moreover, Equation (2.4) is not limited to sinusoidal-varying waveforms but to any real-world time-varying currents and voltages that are functions of time. Finally, Equation (2.4) tells us that whatever the current as a function of time waveform is, the voltage developed across such resistor is proportional to the current waveform.

In particular, when $v(t) = V \sin(\omega t + \theta)$ and $i(t) = I \sin(\omega t + \theta)$,

$$V \sin(\omega t + \theta) = RI \sin(\omega t + \theta). \tag{2.5}$$

Figure 2.4 below depicts a plot of Equation (2.5). Note that both current and voltage waveforms are sinusoidal and proportional to each other. Resistance R is the constant of proportionality. It is also important to note that the angular frequency ω (or $2\pi f$) is the same for the current and voltage waveforms.

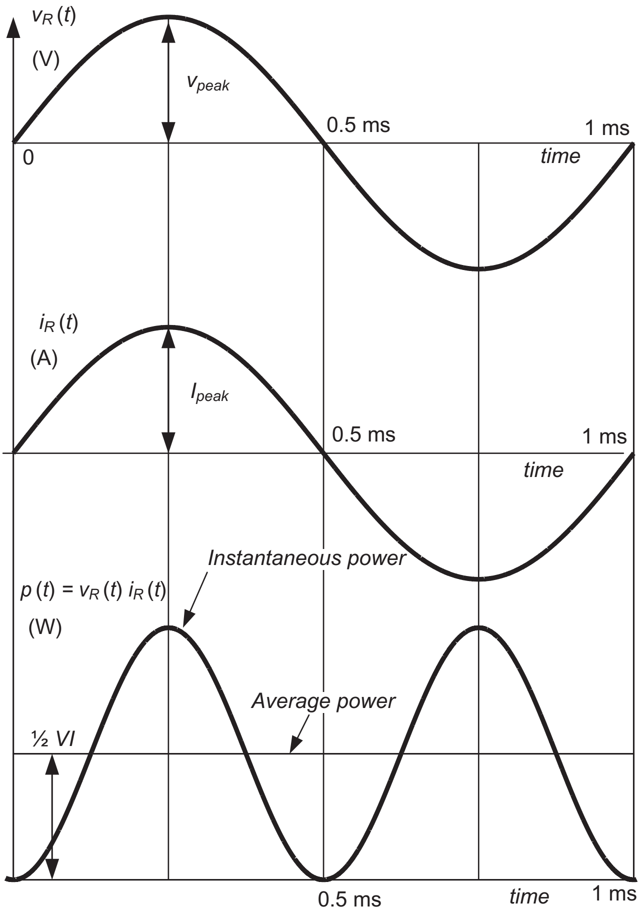

Figure 2.4 Resistor with sinusoidal voltage, current, and instantaneous power.

From Figure 2.4, we can see that both sine waves are exactly in-phase. This means that the voltage and current peak values occur at the same time, as well as their valleys (negative peaks), zero crossings, and so on.

Referring to the toaster example powered by a sinusoidal voltage source, we calculate that the instantaneous power consumed by the resistor is

$$p(t) = v(t)i(t). \tag{2.6}$$

In particular, when $v(t) = V\sin(\omega t + \theta)$ and $i(t) = V\sin(\omega t + \theta)$, then

$$p(t) = VI\sin^2(\omega t + \theta). \tag{2.7}$$

From the trigonometric equality,

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x). \tag{2.8}$$

Using Equation (2.8) in Equation (2.7), we obtain

$$p(t) = \frac{1}{2}[VI - \cos(2\omega t + \theta)]. \tag{2.9}$$

Equation (2.9) is the instantaneous power on resistor R. Refer to Figure 2.4 which depicts, from top to bottom, voltage across the resistor, current through the resistor, instantaneous power consumed by the resistor, and the average power on the resistor.

Equation (2.9) has two terms, a constant power term equal to

$$\frac{1}{2}VI. \tag{2.10}$$

The second term varies with twice the original frequency and is given by Equation (2.11):

$$-\frac{1}{2}\cos(2\omega t + \theta). \tag{2.11}$$

The average power consumed by the resistor is given by

$$P_{average} = \frac{1}{T}\int_0^T p(t)dt. \tag{2.12}$$

Integrating Equation (2.12), where $p(t)$ is given by Equation (2.9), yields

$$P_{average} = \frac{1}{2}VI. \tag{2.13}$$

V and I, respectively, are the peak values of voltage and current.

The resistor will dissipate an amount of heat that is the average value of its instantaneous power. Again looking at Equation (2.9), the average value of $p(t)$ is

$$P_{average} = \frac{1}{T}\int_0^T VI \sin^2(\omega t + \theta)\,dt. \tag{2.14}$$

And solving the integral of Equation (2.14) yields

$$P_{average} = \frac{1}{2} VI. \tag{2.15}$$

V and I are respectively the amplitude (or peak values) of voltage and current.

Integrating the term

$$-\frac{1}{2}\cos(2\omega t + \theta) \tag{2.16}$$

between 0 and period T, yields zero.

Now from Equations (2.2) and (2.3), we know that for sinusoidal waveforms,

$$V_{RMS} = V/\sqrt{2} \approx 0.707\ V \tag{2.17}$$

and

$$I_{RMS} = I/\sqrt{2} \approx 0.707\ I. \tag{2.18}$$

Substituting the V_{RMS} and I_{RMS} values in Equation (2.15), we obtain that

$$P_{average} = I_{RMS} V_{RMS}. \tag{2.19}$$

The average power dissipated by a resistor when a sinusoidal current flows through it, developing a sinusoidal voltage across it, is the product of the RMS (or effective) values of such current and voltage.

The *RMS* values of current and voltage on the resistor are thermally equivalent to DC values of same current and voltage. The following example explains.

Example 2.3 Power Calculations on a Resistor Powered by an AC Voltage
Given a 10 Ω resistor R, with an AC voltage source $v(t)$ applied across its terminals, where $v(t) = 25\sin(2\pi\ 60t)$, where $f = 60$ Hz, note that the phase θ, in this example has a value of zero. Note: The peak value of the sinusoidal waveform above is 25 V.

(a) Determine the value of the AC current developed through the resistor.
(b) Find the average AC power dissipated by the resistor finding the AC waveform corresponding RMS values.
(c) Find equivalent values of DC voltage and DC current that will produce the same power dissipation as the RMS values of the AC waveforms produce.

Solution to Example 2.3

(a) The current through the resistive circuit is

$$\begin{aligned} i(t) &= v(t)/R \\ &= 25/10\sin(120\pi t) \\ &= 2.5\sin(120\pi t)\text{ A.} \end{aligned}$$

(b) Using Equations (2.17) and (2.18), we find the RMS values of voltage and current waveforms are

$$V_{RMS} = V/\sqrt{2} \cong 17.68\text{ V.} \tag{2.20}$$

$$I_{RMS} = \text{I}/\sqrt{2} \cong 1.77\text{ A} \tag{2.21}$$

Thus, the power dissipated by the resistor equals

$$P_{dissipated} = V_{RMS}I_{RMS} = 17.68\text{ V }1.77\text{ A} = 31.29\text{ W.}$$

(c) Since $V_{RMS} = 17.68$ V and $I_{RMS} = 1.77$ A, DC values of 17.68 V and 1.77 A will produce the same power dissipation of

$$\begin{aligned} P_{dissipated} &= V_{RMS}I_{RMS} = V_{DC}I_{DC} \\ P_{dissipated} &= 17.68 \times 1.77 = 31.29\text{ W.} \end{aligned} \tag{2.22}$$

From a thermal perspective, the resistor sees no difference between the power produced by sinusoidal current and voltage or by equivalent DC values.

Example 2.4 Given a 1 Ω resistor and a 1 A DC current source, determine the peak value of an AC current source with a 1 Ω load, which produces the same power dissipation as the DC source. Hint: The resistor dissipates 1 W in DC and must also dissipate 1 W in AC.

Solution to Example 2.4

Given that the DC current value is 1 A, a sinusoidal AC current with an RMS current of 1 A will produce the same power dissipation as the DC value. Thus, the peak value of the sinusoidal current is $1 \times \sqrt{2} \approx 1.41$ A. Refer to Equation (2.18).

2.1.2 Ideal and Real AC Current Sources

An ideal AC current source is one that produces a current that varies in a sinusoidal fashion with respect to time. Most importantly, the amplitude of an ideal AC current source does not vary based on how much voltage gets developed across the current source, based on the load that it has across its terminals. So whether the current source supplies current to a short-circuit load or a very light load (resistor of high ohmic value), the current amplitude remains constant. The internal resistance of an ideal current source is infinitely large; this means that regardless of the load applied across its terminal, the current remains constant and the voltage is given by the current times the voltage across the load. The standby condition of a current source is obtained by short-circuiting the current source terminals. When ideal current source terminals are left *open-circuited*, the voltage developed across the current source approaches an infinitely large value. When we have a real current source and leave its terminals open-circuited, the voltage developed across the current source terminals is very large, and there is a great likelihood of damaging the current source. A real AC current source amplitude does not remain constant with the level of voltage being developed across the real AC current source. An ideal current source is depicted in Figure 2.5a. The real AC current source can be modeled as an ideal AC current source in parallel with its internal resistance, as shown by Figure 2.5b. Current source terminals, whether the source is real or ideal, must always be short-circuited when no load is

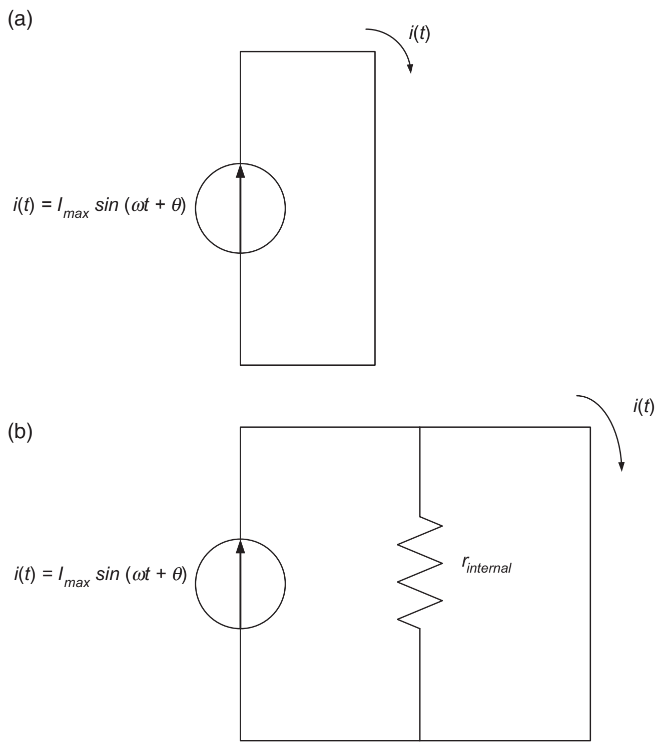

Figure 2.5 (a) Ideal and (b) real current source models in standby mode.

connected to its terminals (standby mode). Why? Because when the current source terminals are open-circuited, then the ideal and real current source voltage approaches a very large voltage value.

Just like with DC circuit voltage sources, an AC voltage source is in a standby mode when its terminals are in an open-circuit condition; its open-circuit voltage is read, but since there is no load applied across its terminals, no current is delivered by the voltage source.

An ideal or real AC current source in standby mode must have its terminals short-circuited, or a 0-Ω resistance across its terminals. A current source is in a benign state when its terminals are short-circuited. Figure 2.6a shows a basic voltage source with internal resistance and load resistance in series. Figure 2.6b depicts an ideal load line of an ideal 10-V voltage source with internal

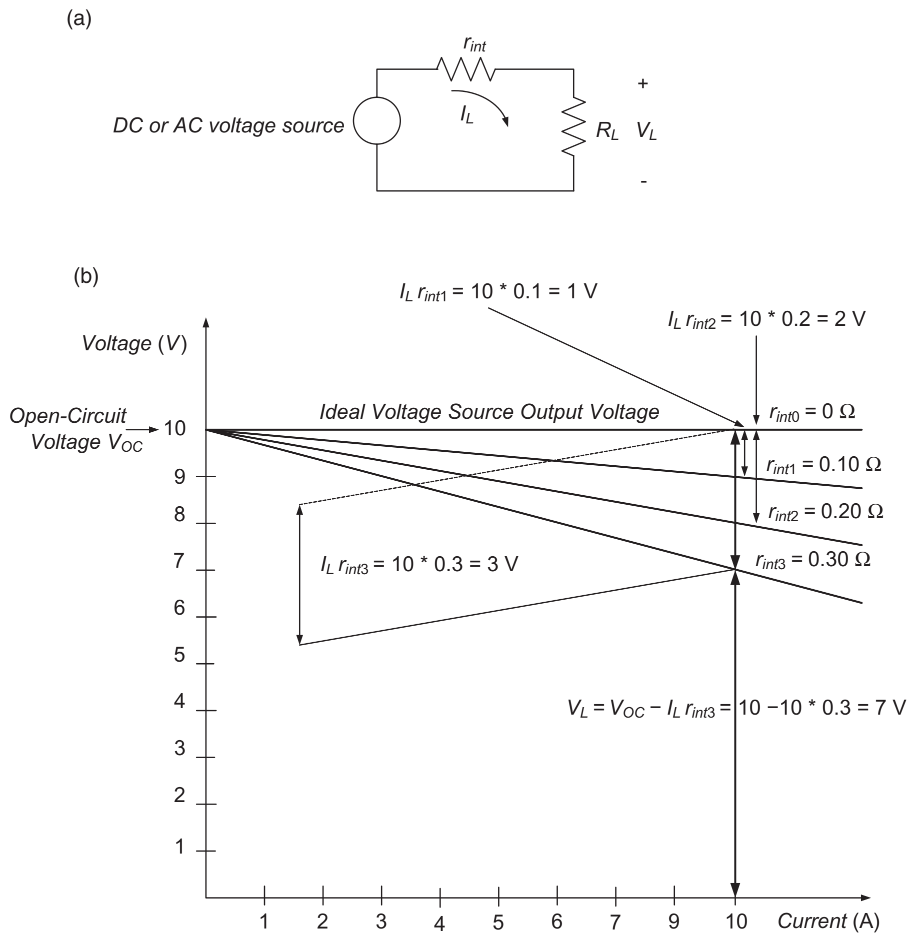

Figure 2.6 (a) Voltage source model with internal resistance; (b) different internal resistance voltage sources models under the same load condition.

resistance $r_{int0} = 0\ \Omega$, and three real voltage sources with internal resistances of $r_{int1} = 0.1\ \Omega$, $r_{int2} = 0.2\ \Omega$, and $r_{int3} = 0.3\ \Omega$. All four voltage sources have a 10-A current load. It is important to note that for equal current loading, the output voltage (load voltage V_L) of the source with the largest internal resistance (r_{int3}) is the lowest. The ideal source with zero internal resistance produces the highest possible voltage, which is 10 V. The load line equation is given from Kirchhoff's and Ohm's laws by

$$V_L = V_{oc} - r_{int} I_L. \tag{2.23}$$

In Equation (2.23) V_L is the voltage across the load (V_L). Refer again to Figure 2.6a.

The load voltages for each load line equation for r_{int0}, r_{int1}, r_{int2}, and r_{int3} for $V_{oc} = 10$ V and load current $I_L = 10$ A, respectively, are

$$V_L = V_{oc} - r_{int0} I_L = 10\text{ V}. \tag{2.24}$$

$$V_L = V_{oc} - r_{int1} I_L = 10 - 0.1 \times 10 = 9\text{ V}. \tag{2.25}$$

$$V_L = V_{oc} - r_{int2} I_L = 10 - 0.2 \times 10 = 8\text{ V}. \tag{2.26}$$

$$V_L = V_{oc} - r_{int3} I_L = 10 - 0.3 \times 10 = 7\text{ V}. \tag{2.27}$$

Exercise: For Equations (2.25), (2.26), and (2.27) determine the actual load resistance R_L at the given conditions.

For Equation (2.25), the output or load voltage is 9 V, and since the load current is 10 A, then $R_L = 9$ V/10 A = 0.9 Ω. Similarly for Equation (2.26), $R_L = 8$ V/10 A = 0.8 Ω and for Equation (2.27), $R_L = 7$ V/10 A = 0.7 Ω.

Figure 2.7 depicts an ideal load line of an ideal 10-A current source with internal resistance $r_{int0} \to \infty$ and three real current sources with internal resistances of $r_{int1} = 10\ \Omega$, $r_{int2} = 5\ \Omega$, and $r_{int3} = 3.333\ \Omega$. All four current sources have a load that causes the load voltage to be 10-V. It is important to note that for equal voltage at the load, the current of the source with the numerically smallest internal resistance (r_{int3}) produces the lowest load current. The goal is to obtain as much of the current source current to flow through the load. The ideal source with an infinite internal resistance produces the highest possible load current I_L, which is 10 A. For the example on hand a current source with a 10-Ω internal resistance and 10 V at the load produces 9 A through the load and 1 A through the internal resistance. A current source with a 5-Ω internal resistance and 10 V at the load produces 8 A through the load and 2 A through the internal resistance. Finally, a current source with a 3.3333-Ω internal resistance and 10 V at the load produces 7 A through the load and 3 A through the internal resistance. Refer to Equations 2.24 through 2.27.

The load line equation is given from Kirchhoff's and Ohm's laws by Equation (2.28):

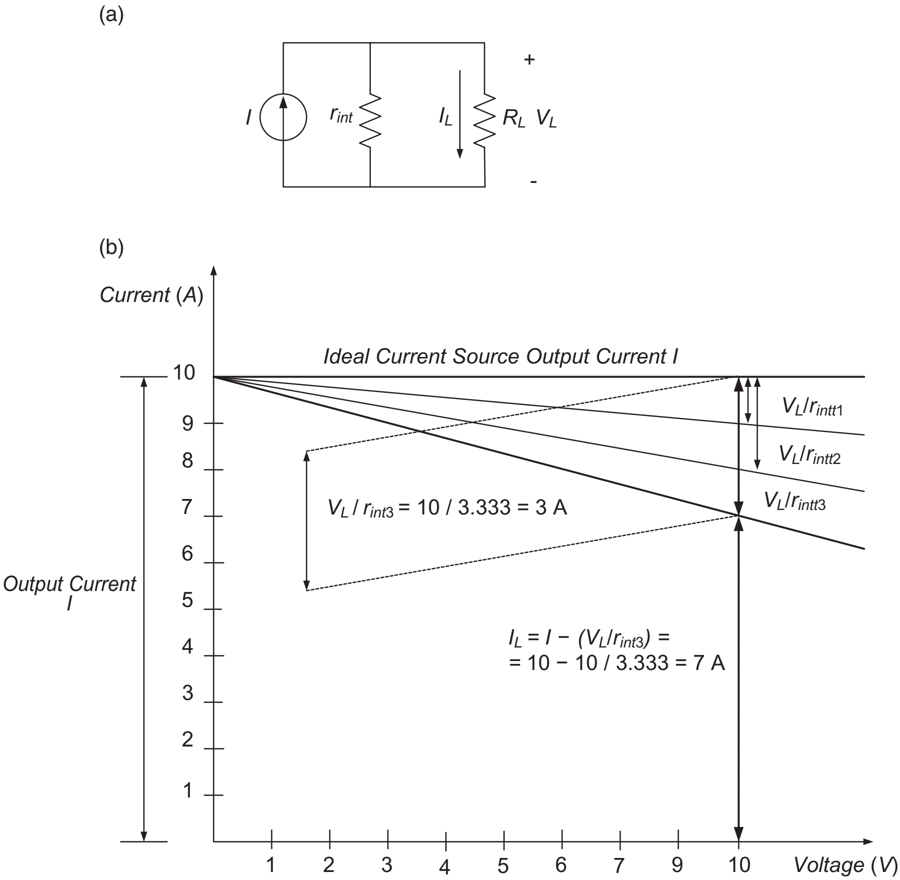

Figure 2.7 (a) Current source model with internal resistance; (b) different internal resistance current sources at the same load voltage.

$$I_L = I - \frac{V_L}{r_{int}}. \tag{2.28}$$

In Equation (2.28), I_L is the current through the load resistance R_L, I is the total current that the current source supplies, r_{int} is the current source internal resistance, and V_L is the load voltage or the voltage across R_L (R_L does not appear in Equation (2.28), refer to Figure 2.7a for the location of R_L). Thus, using Equation (2.28), the line load equations for r_{int0}, r_{int1}, r_{int2}, and r_{int3} for $I = 10$ A and load voltage $V_L = 10$ V, respectively, are

$$I_L = I = 10 \text{ A, because } r_{int0} \to \infty. \tag{2.29}$$

$$I_L = I - \frac{V_L}{r_{int1}} = 10 - (10/10) = 9 \text{ A}. \tag{2.30}$$

$$I_L = I - \frac{V_L}{r_{int2}} = 10 - (10/5) = 8 \text{ A}. \quad (2.31)$$

$$I_L = I - \frac{V_L}{r_{int3}} = 10 - (10/3.333) = 7 \text{ A}. \quad (2.32)$$

Independent current source I can be a *DC* current or an *AC* current source. When using a DC current source, I simply is the current DC value; when using an AC current source, I is typically the peak value of the sinusoidal current.

2.2 SINUSOIDAL STEADY STATE: TIME AND FREQUENCY DOMAINS

When sinusoidal voltage or current sources excite an *RLC* network, the sinusoidal voltage and current waveforms are of the same angular frequency ω in sinusoidal steady state. Sinusoidal steady state means that transient behavior is over. For the circuit given in Figure 2.8, which shows an AC voltage source in series with a resistor R, capacitor C, and inductor L, we can state the circuit equations using Kirchoff's voltage law (KVL) and Kirchoff's current law (KCL) directly in the time domain.

The time domain circuit equations for a resistor, capacitor, and inductor are summarized in Table 2.1 from previous sections of this chapter. The various scientists and engineers that developed basic circuit theory throughout most of the 19th century experimentally obtained such equations. It is important to state that the equations of Table 2.1 hold true regardless of the waveform that excites each element. For example, for a resistor, if its current $i_R(t)$ is a constant (DC), then its voltage $v_R(t)$ is a constant, since the voltage–current behavior is $v_R(t) = i_R(t)R$. Similarly, if the resistor current is a sinusoidal function of time, so will be the voltage across it. A second example for an inductor, if its current is a sinusoidal waveform with respect to time, the voltage developed across such inductor varies proportionally to the derivative of its current with respect to time. That is, $v_L(t) = Ldi(t)_L/dt$.

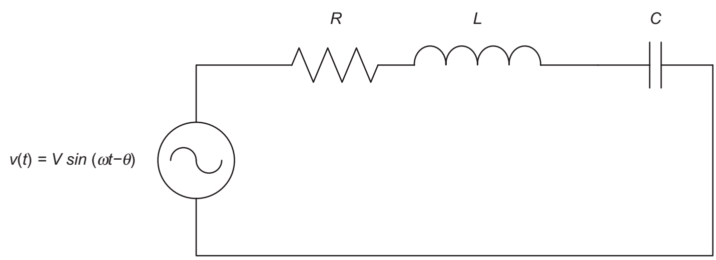

Figure 2.8 Series *RLC* circuit with sinusoidal voltage source.

Table 2.1 Voltage–current and current–voltage relationships for electric components (Universal time domain equations)

Circuit Element	Basic Voltage–Current Relationship	Basic Current–Voltage Relationship
R	$v_R(t) = i_R(t)\ R$	$i_R(t) = \dfrac{v_R(t)}{R}$
L	$v_L(t) = L\dfrac{di_L(t)}{dt}$	$i_L(t) = \dfrac{1}{L}\int_{-\infty}^{t} v_L(t)\,dt$
C	$v_C(t) = \dfrac{1}{C}\int_{-\infty}^{t} i_C(t)\,dt$	$i_C(t) = C\dfrac{dv_C(t)}{dt}$

2.2.1 Resistor under Sinusoidal Steady State

Based on the voltage–current relationships for R in Table 2.1, when

$$i_R(t) = I\sin(\omega t + \theta). \tag{2.33}$$

$$v_R(t) = i_R R = IR\sin(\omega t + \theta) = V\sin(\omega t + \theta). \tag{2.34}$$

V, *the peak voltage*, is defined as:

$$V = IR. \tag{2.35}$$

In Equation (2.35), I is the peak value of the current waveform and R is the resistor value.

Previously seen Figure 2.4 depicts the voltage and current waveform of a resistor with sinusoidal excitation. Figure 2.4 also shows the instantaneous power on the resistor and the average value of the power dissipated. Important facts to observe are that both voltage and current waveforms are exactly in phase; that is, they both have the same zero crossings, positive and negative peaks.

2.2.2 Inductor under Sinusoidal Steady State

Based on the voltage–current relationships for L in Table 2.1, when

$$i_L(t) = I\sin(\omega t + \theta). \tag{2.36}$$

$$v_L(t) = L di_L/dt = \omega LI\cos(\omega t + \theta) = V\cos(\omega t + \theta). \tag{2.37}$$

V *peak* is defined as

$$V = \omega LI, \tag{2.38}$$

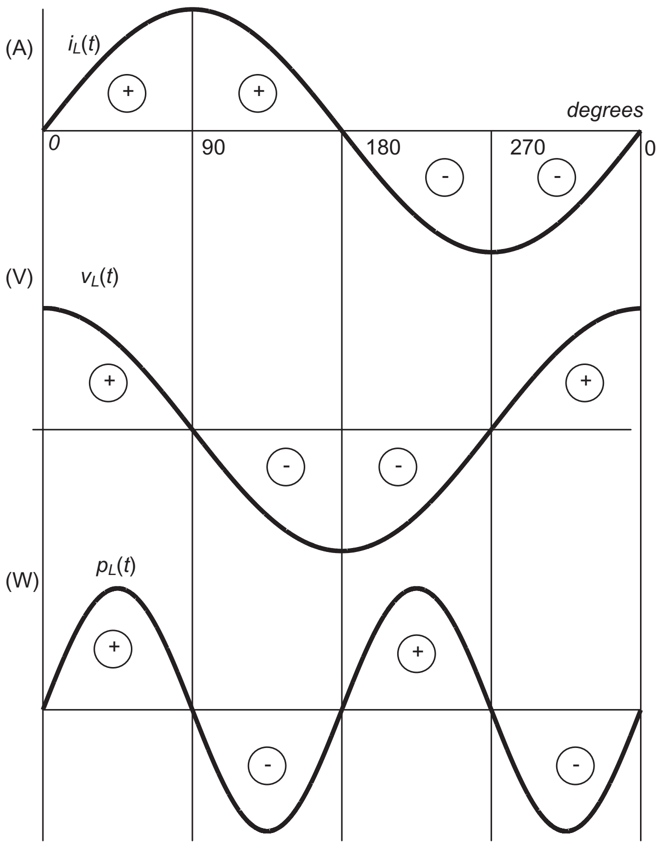

Figure 2.9 Inductor under sinusoidal voltage, current, and instantaneous power.

where ω is the angular frequency of the exciting current, I is the peak value of the current waveform, and L is the inductor value.

Figure 2.9 depicts the voltage and current waveform of an inductor with sinusoidal excitation. An important fact to observe is that the voltage waveform leads the current waveform by 90° (or π/2 radians). It is also interesting to note that the peak value of the voltage waveform (V) is a frequency-dependent term (recall that $\omega = 2\pi f$). We will discuss instantaneous power in the inductor shortly.

2.2.3 Capacitor under Sinusoidal Steady State

Based on the voltage–current relationships for C in Table 2.1, when

$$v_C(t) = V\sin(\omega t + \theta). \tag{2.39}$$

$$i_C(t) = C dv_C/dt = \omega CV\cos(\omega t + \theta) = I\cos(\omega t + \theta). \tag{2.40}$$

I, the peak current, is defined as:

$$I = \omega CV, \tag{2.41}$$

where ω is the waveform angular frequency, C is the capacitance value, V is the peak value of the voltage waveform; thus, I is the peak value of the current waveform through the capacitor.

Figure 2.10 depicts the current, voltage, and instantaneous power waveforms of a capacitor with sinusoidal excitation. An important fact to observe is that in a capacitor, the current waveform leads the voltage waveform by 90° (or $\pi/2$ rad). It is also interesting to note that the peak value of the current waveform (i_C) is an angular frequency-dependent term ($\omega = 2\pi f$). We will discuss instantaneous power in the capacitor shortly.

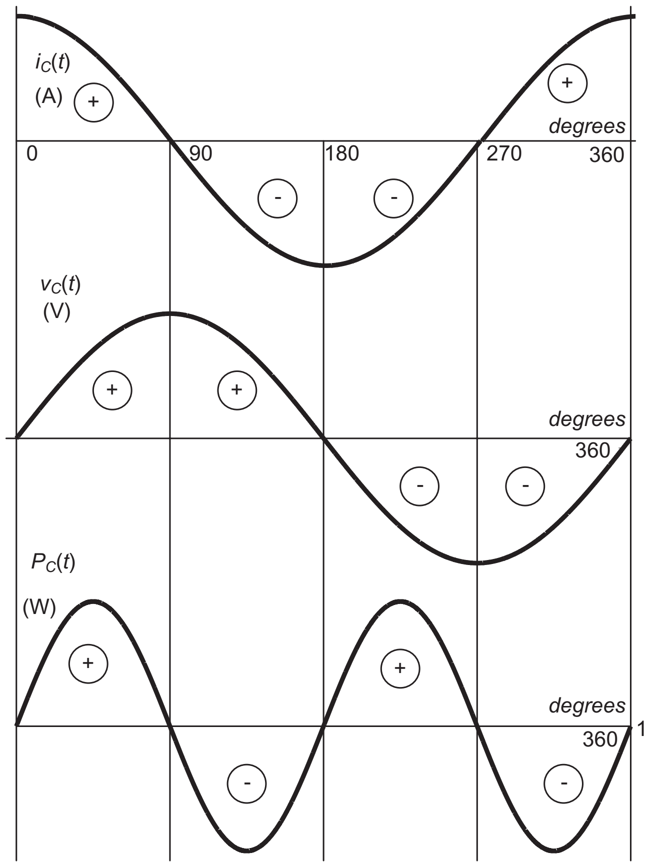

Figure 2.10 Capacitor under sinusoidal voltage, current, and instantaneous power.

Table 2.2 Time domain equations for *R*, *L*, and *C* with sinusoidal excitation

Electric Element	Voltage[a]	Current[a]	Voltage–Current Phase Relationship
Resistor	$v_R(t) = V\sin(\omega t + \theta)$	$i_R(t) = I\sin(\omega t + \theta)$	v_R and i_R are in-phase
Inductor	$v_L(t) = V\cos(\omega t + \theta)$	$i_L(t) = I\sin(\omega t + \theta)$	v_L leads i_L by 90°
Capacitor	$v_C(t) = V\sin(\omega t + \theta)$	$i_C(t) = I\cos(\omega t + \theta)$	i_C leads v_C by 90°

[a] All waveforms are referred to as sinusoidal, regardless whether they are expressed by a sine or a cosine function.

From Equations (2.33) through (2.41), Table 2.2 summarizes the results obtained.

Note: Figures 2.9 and 2.10 depict degrees in their horizontal axis; this is totally equivalent to display time, where 90° is 1/4 of a sinusoidal period, 180° is half-a-period, and so on.

When circuits operate in sinusoidal steady state, it is particularly useful to use complex numbers instead of manipulating time domain equations. When using time domain equations, differential equations need to be solved. When dealing with complex numbers, complex algebra manipulations are required instead of having to solve differential equations. This topic will be addressed further in the section about phasors.

2.2.4 Brief Complex Number Theory Facts

The purpose of this section is to provide a brief review on complex numbers and their basic operations.

Mathematically, "*i*" is the imaginary number unit; however, electrical engineers prefer to use "*j*" because the letter *i* is reserved for current.

Complex number theory begins with its fundamental assumption or definition:

$$j = \sqrt{-1} \tag{2.42}$$

A complex number z is a number of the form $a + jb$, where a is the real part of the complex number z, $\text{Re}\{z\} = a$ and b is the imaginary part of z, $\text{Im}\{z\} = b$.

Complex number:

$$z = a + jb \tag{2.43}$$

is said to be represented in *rectangular* form. Complex numbers can be represented on the complex plane. The horizontal axis of this plane is used to represent the real part of the complex number, and the vertical axis or the

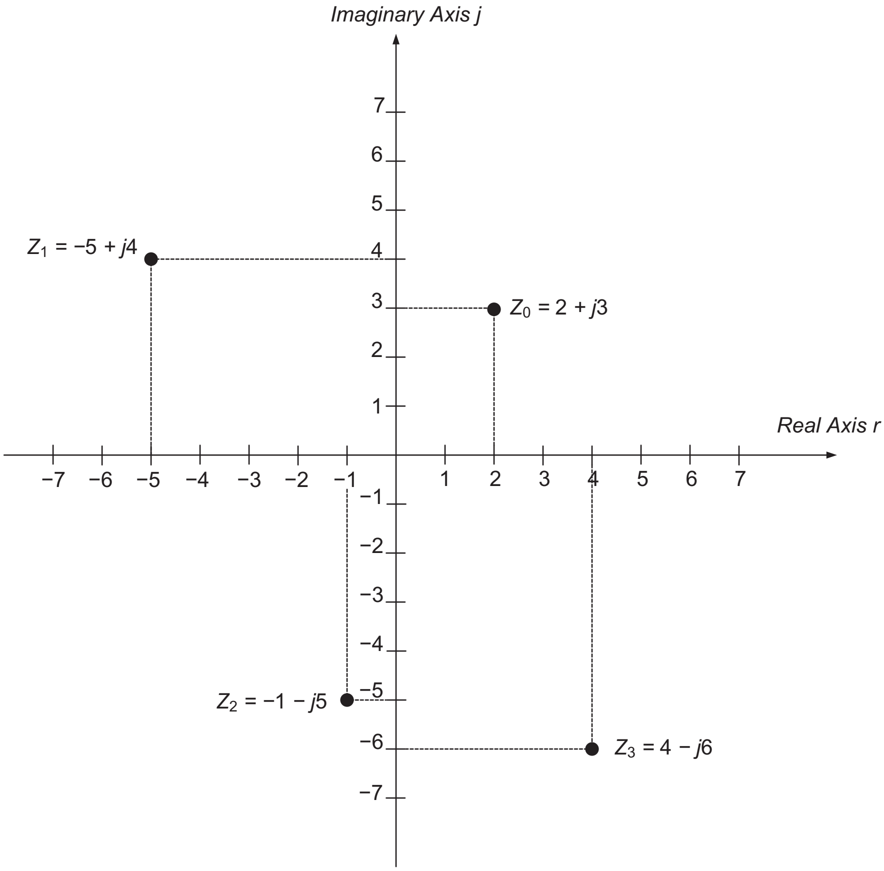

Figure 2.11 Complex plane showing complex numbers in rectangular form.

j-axis is used to represent the imaginary part of the complex number. Figure 2.11 shows the complex plane, and on the plane there are four examples of complex numbers.

In particular, a complex number with its zero real part is said to be a pure imaginary number. Conversely, a complex number with zero imaginary part is said to be a real number.

Examples of *pure imaginary numbers* in rectangular form are

$$0 + j3 = j3;$$

$$0 + j4.5 = j4.5;$$

$$0 + j1 = j;$$

$$0 + j\pi = j\pi.$$

Examples of *real numbers* in rectangular form are

$$+1 + j0 = 1;$$

$$\pi - j0 = \pi;$$

$$23.7 + j0 = 23.7;$$

$$1 + j0 = 1.$$

2.2.4.1 Complex Numbers in Polar Form Complex numbers can also be represented in *polar* form. Figure 2.12 shows a complex number with real part a, imaginary part b, and how it relates to its modulus or absolute value ρ (rho) and its phase angle θ (theta) with respect to the real axis.

From Figure 2.12 and trigonometric identities, one can see that the absolute value of the complex number is related to its rectangular component as follows:

$$\rho = \sqrt{a^2 + b^2}. \qquad (2.44)$$

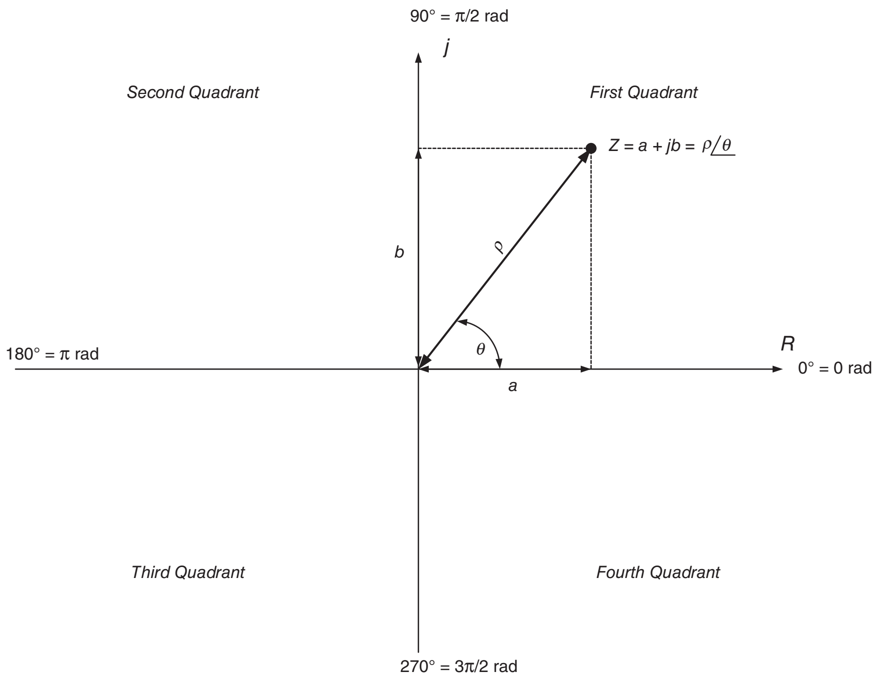

Figure 2.12 Complex numbers in rectangular and in polar forms.

The phase angle θ, also called the argument of z, is related to its *rectangular* components as follows:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right), \tag{2.45}$$

where a and b are respectively the real and imaginary part of complex number z.

The complex number in polar form is represented as follows:

$$z = \rho\angle\theta. \tag{2.46}$$

Figure 2.12 also depicts the four quadrants within the trigonometric circle:

Quadrant I encompasses angles in the range: $90° < \theta < 0°$
Quadrant II encompasses angles in the range: $180° < \theta < 90°$
Quadrant III encompasses angles in the range: $270° < \theta < 180°$
Quadrant IV encompasses angles in the range: $360° < \theta < 270°$

where 0°, 90°, 180°, 270°, and 360° angles are the boundaries between quadrants.

It is also important to note that the following convention is also accepted:

Negative angles whose angle absolute value is within the range: $90° < |\theta| < 0°$ are in Quadrant IV.

Example 2.5 Negative angles whose angle absolute value is within the range: $90° < |\theta| < 0°$.

–30°, –5°, and –75° are all examples of angles that reside in Quadrant IV.

Example 2.6 Negative angles whose angle absolute value is within the range: $180° < |\theta| < 90°$.

–110°, –145°, and –175° are all examples of angles that reside in Quadrant III.

Example 2.7 Negative angles whose angle absolute value is within the range: $270° < |\theta| < 180°$.

–190°, –205°, and –265° are all examples of angles that reside in Quadrant II.

Example 2.8 Negative angles whose angle absolute value is within the range: $270° < |\theta| < 360°$.

–280°, –300°, and –334° are all examples of angles that reside in Quadrant I.

Example 2.9 Convert the following complex numbers from *rectangular* form to *polar* form:

(a) $z_0 = 2 + j3$
(b) $z_1 = -5 + j4$
(c) $z_2 = -1 - j5$
(d) $z_3 = 4 - j6$

Applying Equations (2.43) through (2.45) for (a) through (d) we obtain

(a) $z_0 = 2 + j3 = (2^2 + 3^2)^{½} \angle\tan^{-1}(3/2) = 3.606\angle 56.31°$
(b) $z_1 = -5 + j4 = [(-5)^2 + 4^2]^{½} \angle\tan^{-1}[4/(-5)] = 6.403\angle 141.34°$
(c) $z_2 = -1 - j5 = [(-1)^2 + (-5)^2]^{½} \angle\tan^{-1}[(-5)/(-1)] = 5.099\angle 258.69°$
(d) $z_3 = 4 - j6 = [4^2 + (-6)^2]^{½} \angle\tan^{-1}[(-6)/4] = 7.211\angle -56.31°$

2.2.4.2 Complex Numbers in Euler's Form From Euler's identity,

$$z = \rho e^{j\theta} = \rho(\cos\theta + j\sin\theta), \tag{2.47}$$

where ρ is the modulus or amplitude of the complex number z and θ the angle that its module has with respect to the real axis; complex number z then is

$$z = \rho e^{j\theta} = \rho\angle\theta. \tag{2.48}$$

From Euler's equality, Equation (2.47), it can be seen by looking at the *rectangular* representation of a complex number, previously given by Equation (2.43), that

$$\text{Re}\{z\} = a = \rho\cos\theta \tag{2.49}$$

and

$$\text{Im}\{z\} = b = \rho\sin\theta \tag{2.50}$$

Equations (2.49) and (2.50) show a direct conversion of complex number from polar form into Euler form.

Example 2.10 Convert the following complex numbers from *polar* form to *Euler's* form:

$$z_1 = 3.606\angle 56.31°$$
$$z_2 = 6.403\angle 141.34°$$

$$z_3 = 5.099\angle 258.69°$$
$$z_4 = 7.211\angle -56.31°$$

The conversion from polar form is straightforward; it just uses the modulus and the phase angle in Euler's equation. Yielding

$$z_1 = 3.606\angle 56.31° = 3.606e^{j56.31°}$$
$$z_2 = 6.403\angle 141.34° = 6.403e^{j141.34°}$$
$$z_3 = 5.099\angle 258.69° = 5.099e^{j258.69°}$$
$$z_4 = 7.211\angle -56.31° = 7.211e^{-j56.31°}$$

2.2.4.3 Arithmetic Operations with Complex Numbers

2.2.4.3.1 Rectangular Form Addition/Subtraction

Given $\mathbf{z_1} = a + jb$ and $\mathbf{z_2} = \mathrm{c} + jd$, then $\mathbf{z_1} + \mathbf{z_2} = (a + c) + j(b + d)$. (2.51)

Given $\mathbf{z_1} = a + jb$ and $\mathbf{z_2} = c + jd$, then $\mathbf{z_1} - \mathbf{z_2} = (a - c) + j(b - d)$. (2.52)

From Equations (2.51) and (2.52), it can be seen that for addition or subtraction in *rectangular* form, real parts get added or subtracted, and imaginary parts get added or subtracted.

Example 2.11 Given complex numbers $\mathbf{z_0}$, $\mathbf{z_1}$, $\mathbf{z_2}$, and $\mathbf{z_3}$ in rectangular form, perform the following operations: (a) $\mathbf{z_0} + \mathbf{z_1}$; (b) $\mathbf{z_2} - \mathbf{z_3}$; (c) $\mathbf{z_1} + \mathbf{z_2} - \mathbf{z_3}$; and (d) $-\mathbf{z_0} - \mathbf{z_2}$.

$$\mathbf{z_0} = 2 + j3$$
$$\mathbf{z_1} = -5 + j4$$
$$\mathbf{z_2} = -1 - j5$$
$$\mathbf{z_3} = 4 - j6$$

Solutions to Example 2.11

(a) $\mathbf{z_0} + \mathbf{z_1} = (2 - 5) + j(3 + 4) = -3 + j7$

(b) $\mathbf{z_2} - \mathbf{z_3} = (-1 - j5) - (4 - j6) = -1 - 4 - j5 + j6 = -5 + j$

(c) $\mathbf{z_1} + \mathbf{z_2} - \mathbf{z_3} = -5 + j4 - 1 - j5 - (4 - j6) = -10 + j5$

(d) $-\mathbf{z_0} - \mathbf{z_2} = -(2 + j3) - (-1 - j5) = -1 + j2$

2.2.4.3.2 Polar and Euler's Forms Addition/Subtraction To add or subtract complex numbers in *polar* or *Euler*'s forms, it is convenient to convert the complex numbers to *rectangular* form, do the addition (or subtraction), and convert the results back to *polar* or *Euler*'s form.

2.2.4.3.3 Rectangular Form Multiplication

$$\text{Given } \mathbf{z_1} = a + jb \text{ and } \mathbf{z_2} = c + jd, \text{ then } \mathbf{z_1} \times \mathbf{z_2} = (a + jb) \times (c + jd). \tag{2.53}$$

Performing the term-by-term multiplication of both complex numbers in rectangular form and taking into account that $j^2 = -1$, leads to

$$\mathbf{z_1} \times \mathbf{z_2} = (ac - bd) + j(ad + bc) \tag{2.54}$$

Example 2.12 Multiplication of complex numbers given in rectangular form.
Given $\mathbf{z_1} = 8 + j6$, $\mathbf{z_2} = 2 - j1$; find the product $\mathbf{z_1} \times \mathbf{z_2}$ operating with both numbers in their given rectangular form.

Solution to Example 2.12

$$\begin{aligned}\mathbf{z_1} \times \mathbf{z_2} &= (8 + j6)(2 - j1) = 16 + (j6)(-j1) + j \times 6 - j1 \times 8 \\ &= 16 + 6 + j12 - j8 = 22 + j4\end{aligned}$$

2.2.4.3.4 Euler's and Polar Forms Multiplication Given: $\mathbf{z}_1 = \rho_1 e^{j\theta_1} = \rho_1 \angle \theta_1$ and $\mathbf{z}_2 = \rho_2 e^{j\theta_2} = \rho_2 \angle \theta_2$, respectively in Euler's form and polar form.

The product is obtained by multiplying ρ_1 and ρ_2, and by adding their respective phase angles, $\theta_1 + \theta_2$, so that the final product is

$$\mathbf{z_1} \times \mathbf{z_2} = (\rho_1 + \rho_2) e^{j(\theta 1 + \theta 2)}. \tag{2.55}$$

Equation (2.55) is in Euler's form and similarly in polar form:

$$\mathbf{z_1} \times \mathbf{z_2} = (\rho_1 + \rho_2) \angle (\theta 1 + \theta 2). \tag{2.56}$$

Example 2.13 Find the product of $\mathbf{z_1}$ and $\mathbf{z_2}$. $\mathbf{z_1} = 12\angle 25°$ and $\mathbf{z_2} = 3\angle 60$.

Solution to Example 2.13

Applying Equation (2.55), we calculate the desired product:

$$\mathbf{z_1} \times \mathbf{z_2} = (12\angle 25° \times (3\angle 60°) = 12.3\angle(25° + 60°) = 36\angle 85°.$$

2.2.4.3.5 Rectangular Form Division

$$\text{Given } \mathbf{z_1} = a + jb \text{ and } \mathbf{z_2} = c + jd, \text{ then } \mathbf{z_2}/\mathbf{z_1} = (c + jd)/(a + jb). \tag{2.57}$$

Multiplying the numerator and denominator by the complex conjugate of the denominator allows rationalizing the complex number. That is, it eliminates the imaginary part of the number of the denominator.

Since the denominator in the given case is

$$(a + jb),$$

its complex conjugate has the same real part but complementary imaginary part; that is,

$$\text{Complex Conjugate } (a + jb) = a - jb. \tag{2.58}$$

Since $z_1 = a + jb$, its complex conjugate is indicated as

$$\mathbf{z_1^*} = a - jb. \tag{2.59}$$

Then,

$$\mathbf{z_2}/\mathbf{z_1} = \frac{(c + jd)(a - jb)}{(a + jb)(a - jb)} \tag{2.60}$$

$$\frac{(c + jd)(a - jb)}{(a^2 + b^2)} = \frac{(ac + bd) + j(ad - bc)}{(a^2 + b^2)}. \tag{2.61}$$

Example 2.14 Division of complex numbers given in rectangular form.

Given $\mathbf{z_2} = 8 + j6$, and $\mathbf{z_1} = 2 - j1$, find the quotient $\mathbf{z_2}/\mathbf{z_1}$ using both numbers in rectangular form.

Solution to Example 2.14

$$\begin{aligned} \mathbf{z_2}/\mathbf{z_1} &= (8 + j6)/(2 - j1) = \frac{(8 + j6)(2 + j1)}{(2 - j1)(2 + j1)} = \frac{16 - 6 + j(12 + 8)}{(2^2 + 1^2)} \\ &= \frac{10 + j20}{5} = 2 + j4. \end{aligned} \tag{2.62}$$

2.2.4.3.6 Polar and Euler's Forms Division Given: $\mathbf{z_1} = \rho_1 e^{j\theta_1} = \rho_1\angle\theta_1$ and $\mathbf{z_2} = \rho_2 e^{j\theta_2} = \rho_2\angle\theta_2$, where we define $\mathbf{z_2}$ as the dividend and $\mathbf{z_1}$ as the divisor.

The quotient of $\mathbf{z_2}/\mathbf{z_1}$ is obtained by dividing the modulus of the dividend by the modulus of the divisor (ρ_2/ρ_1), and by subtracting the divisor phase angle θ_1 from the dividend phase angle θ_2 so that the final quotient is

$$\boldsymbol{z_2}/\boldsymbol{z_1} = (\rho_2/\rho_1)e^j(\theta_2 - \theta_1) \tag{2.63}$$

in Euler's form and similarly in polar form:

$$\boldsymbol{z_2} \times \boldsymbol{z_1} = (\rho_2/\rho_1)\angle(\theta_2 - \theta_1). \tag{2.64}$$

Example 2.15 Find the quotient of $\boldsymbol{z_2}/\boldsymbol{z_1}$, where $z_2 = 3\angle 60°$ and $\boldsymbol{z_1} = 12\angle 25°$.

Solution to Example 2.15

Applying Equation (2.64) we calculate the desired quotient:

$$\boldsymbol{z_2}/\boldsymbol{z_1} = (3\angle 60°)\times(12\angle 25°) = 3/12\angle(60° - 25°) = 0.25\angle 35°. \tag{2.65}$$

2.3 TIME DOMAIN EQUATIONS: FREQUENCY DOMAIN IMPEDANCE AND PHASORS

The basic equations describing the voltage–current relationships, where voltage and current are functions of time in resistors, capacitors, and inductors (see Table 2.1), are referred to as the time domain equations of those electric components. Those equations were experimentally determined. In the particular case that we need to deal with sinusoidal steady-state regime, current and voltage waveforms have a single frequency, and they vary sinusoidally with respect to time; it is possible to manipulate the waveform with phasors instead of the time domain differential or integral equations.

We will address phasors shortly, but the main advantage of using phasors, provided that the circuit is in sinusoidal steady state, is that the voltage and current calculations need not be in the time domain; consequently, no differential equations need to be solved. Phasors allow current and voltage calculations to be made with simple arithmetic equations. The catch is that such arithmetic is complex arithmetic; real and imaginary numbers are involved.

2.3.1 Phasors

A sinusoidal voltage or current waveform varying with respect to time, such as

$$v(t) = V\sin(\omega t + \theta)$$

can also be described with a phasor of amplitude or peak value V, rotating at a constant angular frequency ω, and θ its phase shift with respect to zero degrees. Figure 2.13 depicts a phasor rotating in counterclockwise direction generating as it rotates each ordinate or sine value of our sinusoid.

For Figure 2.13 above, the phase angle θ is assumed to be zero, which is the reason why the sine wave in the time domain begins at the origin of the time axis.

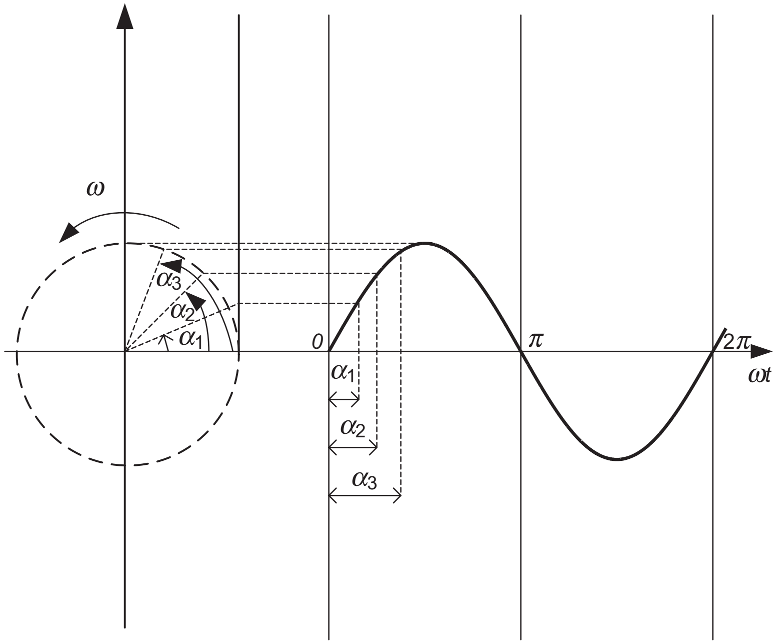

Figure 2.13 Phasor-generated sine wave with zero-phase angle.

Example 2.16 Determine the phasor of the following sinusoidal, time domain equation.

Given: $v(t) = V \sin(\omega t + \theta)$, where $V = 20$ V, $\omega = 60$ rad/s, and $\theta = 45°$.

Rewriting the sinusoidal waveform with the given numerical values results in

$$v(t) = 20\sin(60t - 45°).$$

We can represent the sinusoidal waveform with its generating phasor instead of using the time-varying sine function. The phasor is: $20\angle 45°$. Figure 2.14 depicts this phasor.

Example 2.17 Phasor of sinusoidal waveform $i(t) = 5\sin(60t + 30°)$.

$i(t)$ is a current waveform, of a 5 A peak amplitude, 60 rad/s angular frequency ω, and a 30° phase angle θ. The phasor is: $5\angle 30°$.

Figure 2.15 depicts such current phasor.

2.3.2 The Impedance Concept

The impedance of a circuit element, where a circuit element can be a resistor, an inductor or a capacitor, is defined as the ratio of its voltage phasor **V** divided by its current phasor **I**. So referring to the time domain equations for R, L, and C circuit elements (Table 2.1) with sinusoidal excitation, we will find their equivalent voltage and current phasor to determine what their impedance is.

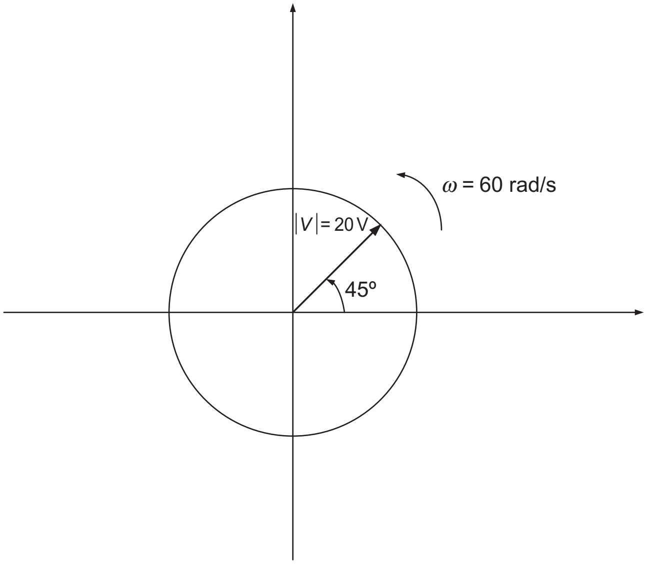

Figure 2.14 Phasor of sinusoidal waveform: $v(t) = 20\ \sin(60t - 45°)$.

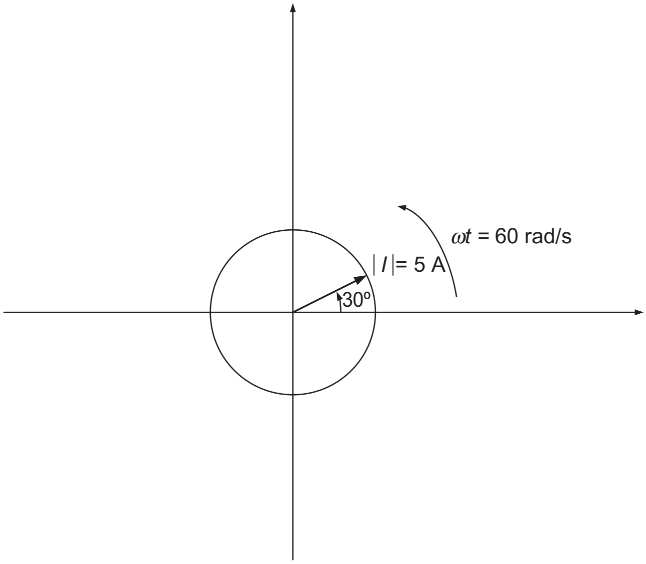

Figure 2.15 Current phasor for current waveform $i(t) = 5\ \sin(60t - 30°)$.

Before proceeding much further, it is important to emphasize that the impedance concept is only meaningful at one angular frequency and when the voltage and current waveforms are sinusoidal. To evaluate the phasors for each circuit element, we will make use of Table 2.2. For the reader's convenience, Table 2.2 is repeated here:

Table 2.2 Time domain equations for *R*, *L*, and *C* with sinusoidal excitation

Electric Element	Voltage[a]	Current[a]	Voltage–Current Phase Relationship
Resistor	$v_R(t) = V\sin(\omega t + \theta)$	$i_R(t) = I\sin(\omega t + \theta)$	v_R and i_R are in-phase
Inductor	$v_L(t) = V\cos(\omega t + \theta)$	$i_L(t) = I\sin(\omega t + \theta)$	v_L leads i_L by 90°
Capacitor	$v_C(t) = V\sin(\omega t + \theta)$	$i_C(t) = I\cos(\omega t + \theta)$	i_C leads v_C by 90°

[a] All waveforms are referred to as sinusoidal, regardless whether they are expressed by a sine or a cosine function.

2.3.3 Purely Resistive Impedance

For a resistor from Table 2.2 we have that both sinusoidal voltage and current are in phase, so the impedance of a *pure** resistor is a real number, expressed by Equation (2.66):

$$\mathbf{Z_R} = \mathbf{V_R}/\mathbf{I_R}. \tag{2.66}$$

In Equation (2.66), $\mathbf{V_R}$ is the voltage phasor that corresponds to the time-varying sinusoidal voltage developed across the resistor. $\mathbf{I_R}$ is the current phasor that corresponds to the time-varying current through the resistor.

That is, $v_R(t) = V\sin(\omega t + \theta)$. $\mathbf{I_R}$ is the current phasor of the sinusoidal time-varying waveform that flows through the resistor. $\mathbf{Z_R}$ denotes impedance and, in a general sense, **V**, **I,** and **Z** are complex numbers (actually referred to as phasors). However, because both voltage and current phasors are always in phase on a resistor, the actual impedance for a *pure* resistor is a real number. Often times, the impedance of a pure resistor is referred to as simply *R*, the resistance itself.

Example 2.18 Given a sinusoidal voltage and current equal to

$$v_R(t) = 120\sin(2\pi 60t - 45°) \tag{2.67}$$

and

$$i_R(t) = 20\sin(2\pi 60t - 45°), \tag{2.68}$$

* A *pure resistor*, also called an *ideal* resistor, means within this context, that the resistor exclusively has resistive properties and has no parasitic inductive or capacitive characteristics.

where $v_R = 120$ V is the voltage peak value, $\omega = 2\pi$ 60 rad/s, ($f = 60$ Hz), and phase angle θ is 45° for the resistor voltage waveform of Equation (2.67). Similarly for the current waveform, $i_R = 20$ A is the current peak value, $\omega = 2\pi$ 60 rad/s, ($f = 60$ Hz), and phase angle θ is 45° (Eq. 2.68).

Determine the voltage and current phasors on the resistor and the resistor value.

Solution to Example 2.18

From Equation (2.66) we can see that the voltage phasor corresponding to such time domain waveform is

$$\mathbf{V_R} = 120\angle 45° \tag{2.69}$$

and the current phasor is

$$\mathbf{I_R} = 20\angle 45°. \tag{2.70}$$

Thus, $\mathbf{Z_R} = 120/20 = 6\ \Omega$, a real number, which means that the impedance is purely resistive in this case.

Note that the resistive impedance turns out to be a real number after all. This will not happen with inductors and capacitors. In general, impedance phasors will always be of the complex form, with nonzero real and imaginary parts, when a circuit contains resistance, plus inductance and/or capacitance.

Graphical interpretation of phasors $\mathbf{V_R} = 120\angle 45°$ and $\mathbf{I_R} = 20\angle 45°$
Both phasors $\mathbf{V_R}$ and $\mathbf{I_R}$ rotate at a constant angular frequency $\omega = 2\pi$ 60 rad/s = 376.98 rad/s, and since both phasors are in phase, their phase difference is zero. Figure 2.16 is a representation of both phasors in the complex plane.

2.3.4 Inductive Impedance: Inductive Reactance

For an inductor, from Table 2.2, we have that the sinusoidal voltage across the inductor leads the sinusoidal current through the inductor by 90°. The impedance of a *pure** inductor is

$$\mathbf{Z_L} = \mathbf{V_L}/\mathbf{I_L}. \tag{2.71}$$

In Equation (2.71), $\mathbf{V_L}$ is the voltage phasor that corresponds to the time-varying sinusoidal voltage developed across the inductor. That is, $v_L(t) = V\cos(\omega t + \theta)$. $\mathbf{I_L}$ is the current phasor of the sinusoidal time-varying waveform that flows through the inductor. $\mathbf{Z_L}$ denotes impedance and in a general sense,

* A *pure inductor* also called an ideal inductor means, in this context, that the inductor exclusively has inductive properties and no parasitic resistive or capacitive characteristics.

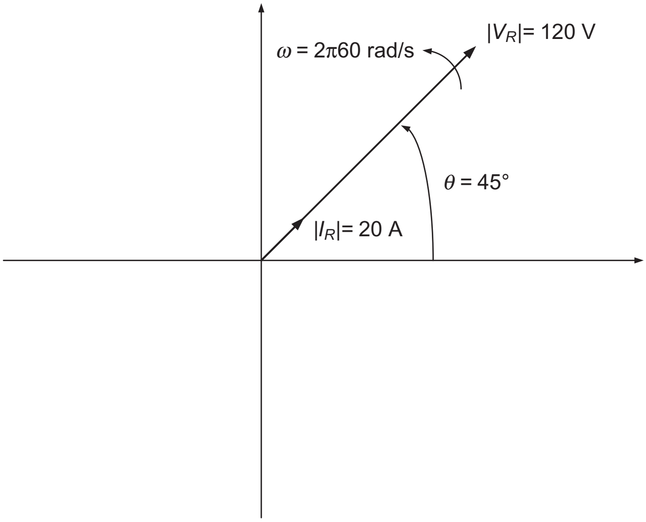

Figure 2.16 Resistor voltage and current phasors.

$\mathbf{V_L}$, $\mathbf{I_L}$, and $\mathbf{Z_L}$ are complex numbers (they are also referred to as phasors). However, because on a pure inductor its voltage phasor always leads the current phasor by 90°, the actual impedance of a *pure* inductor is a pure imaginary number (i.e., has zero real part).

Example 2.19 Given: a sinusoidal voltage and current equal to

$$v_L(t) = 45\cos(2\pi 100 + 20°) \tag{2.72}$$

and

$$i_L(t) = 5\sin(2\pi 100t + 20°). \tag{2.73}$$

From trigonometry we know that

$$\cos x \text{ leads } \sin x \text{ by } 90°. \tag{2.74}$$

Thus

$$v_L(t) \text{ leads } i_L(t) \text{ by } 90° \tag{2.75}$$

We can now proceed and determine that the respective phasors for $v_L(t)$ and $i_L(t)$ are

$$\boldsymbol{V_L} = 45\angle 110° \tag{2.76}$$

and

$$\boldsymbol{I_L} = 5\angle 20°, \tag{2.77}$$

where $|\boldsymbol{V_L}| = 45$ V, $\omega = 2\pi$ 100 rad/s, ($f = 100$ Hz), and phase angle θ is 110° for the inductor voltage waveform of Equation (2.61). Similarly for the current waveform, $\boldsymbol{I_L} = 5$ A, $\omega = 2\pi$ 100 rad/s, ($f = 100$ Hz), and the phase angle θ is 20°.

Using Equation (2.71) with Equations (2.76) and (2.77), we get that $\boldsymbol{Z_L} = \boldsymbol{V_L}/\boldsymbol{I_L} = 9\Omega\angle 90° = j9\Omega$, an imaginary number.

Example 2.20 Determine the voltage and current phasors and the impedance of the pure inductor from Example 2.19.

Equations (2.76) and (2.77) are repeated here for the reader's convenience:

$$\boldsymbol{V_L} = 45\angle 110°. \tag{2.78}$$

$$\boldsymbol{I_L} = 5\angle 20°. \tag{2.79}$$

From Equations (2.78) and (2.79), since we know that the impedance of an inductor is the ratio of voltage and current phasors, this leads to

$$\begin{aligned} \mathbf{Z}_L &= 45/5\angle(110° - 20°) \\ &= \mathbf{Z}_L = 9\ \Omega\angle 90° \end{aligned} \tag{2.80}$$

or simply

$$\mathbf{X}_L = 9\ \Omega\angle 90° \text{ (in polar form)} \tag{2.81}$$

or

$$\mathbf{X}_L = j9\ \Omega \text{ (in rectangular form)}, \tag{2.82}$$

where $\mathbf{X}_L$ is defined as the *reactive inductance* of the given inductor. The *reactive inductance*, Equation (2.81), represents a pure imaginary number as predicted earlier.

We can further look at equations for v_L and i_L from Table 2.2 since

$$v_L(t) = V\cos(\omega t + \theta) \tag{2.83}$$

and

$$i_L(t) = I\sin(\omega t + \theta), \tag{2.84}$$

also remembering from Table 2.2 that $v_L(t) = Ldi_L(t)/dt$. Using this equation into Equations (2.83) and (2.84) we obtain for

$$v_L(t) = Ldi_L(t)/dt = Ld[I\sin(\omega t + \theta)]/dt \tag{2.85}$$

$$v_L(t) = \omega LI\cos(\omega t + \theta), \tag{2.86}$$

where the term (ωLI) is the peak voltage V of Equation (2.86) for $v_L(t)$:

$$V = V_{peak} = \omega LI = |\mathbf{X_L}|, \tag{2.87}$$

where $|\mathbf{X_L}|$ is the absolute value of the inductive reactance given in Equation (2.71). The absolute value of the inductive reactance equals the absolute value of the inductor impedance, because its impedance real part is zero.

Again identifying the phasors for time domain Equations (2.85) and (2.86), we get that for an inductor,

$$\mathbf{V_L} = \mathbf{I_L X_L}. \tag{2.88}$$

Equation (2.88) is very important because it describes Ohm's law in phasor form or for an inductor when used in sinusoidal steady state. Note that all three terms, $\mathbf{V_L}$, $\mathbf{I_L}$, and $\mathbf{X_L}$, are complex numbers, and in a general sense they have magnitude and phase.

From Equations (2.78) and (2.79), the impedance of a pure inductor in rectangular form is

$$\mathbf{Z_L} = \mathbf{X_L} = \mathbf{V_L}/\mathbf{I_L} = j\omega L \quad (\text{for an inductor, } \mathbf{Z_L} = \mathbf{X_L}). \tag{2.89}$$

Graphical interpretation of inductor phasors $\mathbf{V_L} = 45\angle 110°$ and $\mathbf{I_L} = 5\angle 20°$
Both phasors $\mathbf{V_L}$ and $\mathbf{I_L}$ are rotating at a constant angular frequency $\omega = 2\pi\ 100$ rad/s $= 628.30$ rad/s; both phasors are separated by a fixed 90° phase difference, where $\mathbf{V_L}$ leads $\mathbf{I_L}$ by 90°. Figure 2.17 is a representation of both $\mathbf{V_L}$ and $\mathbf{I_L}$ inductor phasors in the complex plane. Note that the initial phase

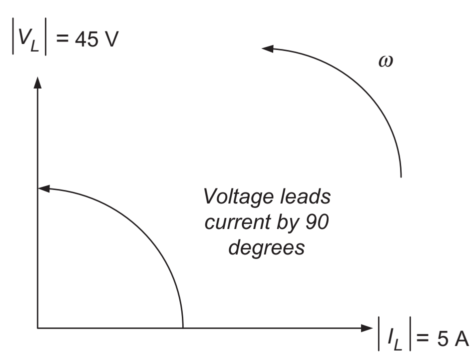

Figure 2.17 Inductor voltage and current phasors.

angle from both the inductor voltage and capacitor current were eliminated from Figure 2.17.

2.3.5 Purely Capacitive Impedance: Capacitive Reactance

For a capacitor from Table 2.2 we have that the sinusoidal current through the capacitor leads the sinusoidal voltage drop across the capacitor by 90°. The impedance of a *pure** capacitor is

$$\mathbf{Z_C} = \mathbf{V_C} / \mathbf{I_C}, \tag{2.90}$$

where $\mathbf{V_C}$ is the voltage phasor that corresponds to the time-varying sinusoidal voltage developed across the inductor, i.e., $v_C(t) = V \sin(\omega t + \theta)$. $\mathbf{I_C}$ is the current phasor of the sinusoidal time-varying waveform that flows through the capacitor. $\mathbf{Z_C}$ denotes impedance, and in a general sense, $\mathbf{V_C}$, $\mathbf{I_C}$, and $\mathbf{Z_C}$ are complex numbers (actually referred to as phasors).

However, because on a pure capacitor its current always leads the voltage phasor by 90°, the actual impedance of a *pure* capacitor is a pure imaginary number (has zero real part).

Example 2.21 Given a capacitor with a sinusoidal voltage and current equal to

$$v_C(t) = 14\sin(2\pi\ 2\text{ MHz}\ t + 45°) \tag{2.91}$$

and

$$i_C(t) = 2\cos(2\pi\ 2\text{ MHz}\ t + 45°). \tag{2.92}$$

Thus

$$v_C(t) = 14\sin(4\pi \times 10^6 t + 45°) \tag{2.93}$$

and

$$i_C(t) = 2\cos(4\pi \times 10^6 t + 45°) \tag{2.94}$$

We can see from (2.93) and (2.94) that the capacitor current leads the capacitor voltage by 90°. Please also refer to Table 2.2.

$$\textit{Capacitor phasor } I_C \textit{ leads capacitor phasor } V_C \textit{ by } 90° \tag{2.95}$$

We can now proceed and determine that the respective phasors for $v_C(t)$ and $i_C(t)$ are

$$\boldsymbol{V_C} = 14\angle 45° \tag{2.96}$$

* A *pure capacitor*, also called an ideal capacitor, means in this context, that the capacitor exclusively has capacitive properties and has no parasitic resistive or inductive characteristics.

and since $\boldsymbol{I_C}$ leads $\boldsymbol{V_C}$ by 90°

$$\boldsymbol{I_C} = 2\angle 135°, \tag{2.97}$$

where $\boldsymbol{V_C} = 14$ V peak voltage, $\omega = 12.57$ Mrad/s, ($f = 2$ MHz), and phase angle θ is 45° for the capacitor voltage waveform of Equation (2.91). Similarly for the current waveform, $\boldsymbol{I_C} = 2$ A peak current, $\omega = 12.57$ Mrad/s, ($f = 2$ MHz), and phase angle θ is 135°.

Determine the voltage and current phasors and the impedance of the pure capacitor.

Solution to Example 2.21

From Equation (2.91), we can see that the voltage phasor corresponding to such time domain waveform is

$$V_C = 14\angle 45°. \tag{2.98}$$

And from Equation (2.95), the current phasor is

$$I_C = 2\angle 135°. \tag{2.99}$$

From Equations (2.98) and (2.99), we know that since the impedance of a capacitor is the ratio of voltage and current phasors, this leads to

$$\begin{aligned} \mathbf{Z_C} &= 14/2\angle(45° - 135°) \\ &= \mathbf{Z}_C = 7\ \Omega\angle -\mathbf{9}0° \end{aligned} \tag{2.100}$$

or simply

$$\mathbf{X_C} = 7\ \Omega\angle -\mathbf{9}0°\ (\textit{in polar form}). \tag{2.101}$$

Or

$$\begin{aligned} \mathbf{X_C} &= -j7\ \Omega\ (\textit{in rectangular form}) \\ &= 1/j(7\ \Omega)\ (\textit{also in rectangular form, without rationalizing } j). \end{aligned} \tag{2.102}$$

In Equation (2.102), $\mathbf{X_C}$ is defined as the *reactive capacitance* of the given capacitor. The *reactive capacitance*, Equation (2.102), represents a pure imaginary number as predicted earlier.

We can further look at equations for v_C and i_C from Table 2.2, and since

$$v_C(t) = V\sin(\omega t + \theta) \tag{2.103}$$

and

$$i_C(t) = I\cos(\omega t + \theta), \tag{2.104}$$

also remember from Table 2.1 that $i_C(t) = Cdv_C(t)/dt$. Using this expression into Equations (2.103) and (2.104) we obtain for

$$v_C(t) = Cdv_C(t)/dt = Cd[V\sin(\omega t + \theta)]/dt \tag{2.105}$$

$$v_C(t) = \omega CV\cos(\omega t + \theta), \tag{2.106}$$

where the term $\omega\ C\ V$ is the peak voltage V of Equation (2.106):

$$V = V_{peak} = \omega CV. \tag{2.107}$$

Again identifying the phasors for time domain Equations (2.104) and (2.106), we get that for a capacitor,

$$\mathbf{V_C} = \mathbf{I_C X_C}. \tag{2.108}$$

Equation (2.108) is very important because it describes Ohm's law in phasor form, for a capacitor when used in sinusoidal steady state. Note that all three terms, $\mathbf{V_C}$, $\mathbf{I_C}$, and $\mathbf{X_C}$ are complex numbers that have magnitude and phase.

In general, for a pure capacitor, the reactive capacitance is given by

$$\mathbf{Z_C} = \mathbf{X_C} = \frac{1}{j\omega C} = -j\frac{1}{\omega C}. \tag{2.109}$$

Note that $|\mathbf{X_C}| = 1/\omega C$ is the absolute value of the capacitive reactance.

Graphical interpretation of inductor phasors $\mathbf{V_C} = 14\angle 45°$ and $\mathbf{I_C} = 2\angle 135°$
Both phasors $\mathbf{V_C}$ and $\mathbf{I_C}$ are rotating at a constant angular frequency $\omega = 2\pi$ 2 Mrad/s = 12.57 Mrad/s; both phasors are separated by a fixed 90° phase difference, where $\mathbf{V_C}$ lags $\mathbf{I_C}$ by 90°. Figure 2.18 is a representation of both $\mathbf{V_C}$ and $\mathbf{I_C}$ capacitor phasors in the complex plane.

2.3.6 R, L, and C Impedances Combinations

From the previous three sections we can summarize that the impedances for R, L, and C elements are

$$\mathrm{Z_R} = \mathrm{V_R}/\mathrm{I_R} = R \tag{2.110}$$

$$\mathrm{Z_L} = \mathrm{X_L} = \mathrm{V_L}/\mathrm{I_L} = j\omega L \tag{2.111}$$

$$\mathbf{Z_C} = \mathbf{X_C} = \frac{1}{j\omega C} = -j\frac{1}{\omega C}. \tag{2.112}$$

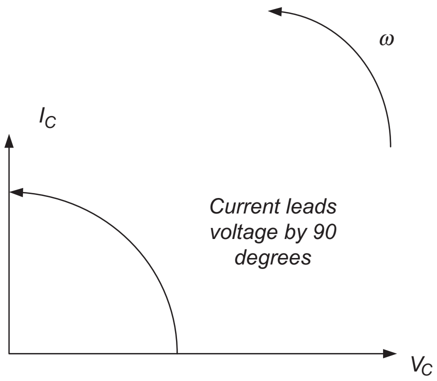

Figure 2.18 Capacitor voltage and current phasors.

Given any R, L, C circuit series and parallel combinations of impedance are handled similarly to how serial and parallel combinations of resistors are calculated. In DC circuit calculations with resistors, all the operations are done with real numbers. In AC, sinusoidal steady-state analysis impedances are in general complex numbers and in phasor form. Note: The concept of impedance and phasors is defined for the frequency domain. Impedance and phasors do not make any sense in the time domain.

Example 2.22 Compute the series equivalent impedance of $\mathbf{Z_R}$, $\mathbf{Z_C}$, and $\mathbf{Z_L}$.

$$\mathbf{Z}_{\mathbf{series\text{-}equivalent}} = \mathbf{Z_R} + \mathbf{Z_C} + \mathbf{Z_L}. \tag{2.113}$$

Given that $\mathbf{Z_R} = 10\ \Omega$, $\mathbf{Z_L} = j\,60\ \Omega$ and $\mathbf{Z_C} = -j30\ \Omega$, calculate the series equivalent impedance.

Since

$$\mathbf{Z}_{\mathbf{series\text{-}equivalent}} = \mathbf{Z_R} + \mathbf{Z_C} + \mathbf{Z_L}, \tag{2.114}$$

using the given values leads to

$$\begin{aligned} \mathbf{Z}_{\mathbf{series\text{-}equivalent}} &= 10\ \Omega - j30\ \Omega + j60\ \Omega \\ &= 10\ \Omega + j30\ \Omega \text{ (in rectangfular form)} \end{aligned} \tag{2.115}$$

and

$$\begin{aligned} &= (10^2 + 30^2)^{1/2} \angle{-\arctan(+30/10)} \\ &= 31.62\angle 71.57^\circ \text{ (in polar form).} \end{aligned} \tag{2.116}$$

Example 2.23 For the previous example, calculate the values of capacitance and inductance assuming that the angular frequency is 1 Mrad/s.

Solution to Example 2.23

From Equations (2.111) and (2.112) we know that

$$\mathbf{Z_L} = \mathbf{X_L} = \mathbf{V_L}/\mathbf{I_L} = j\omega L. \tag{2.117}$$

$$\mathbf{Z_C} = \mathbf{X_C} = \frac{1}{j\omega C} = -j\frac{1}{\omega C}. \tag{2.118}$$

Since $\omega = 1$ Mrad/s and since $|\mathbf{Z_L}| = |j\omega L| = \omega L$,

$$L = |\mathbf{Z_L}|/1\ \text{Mrad/s} = 60\ \Omega/1\ \text{Mrad/s} = 60\ \mu\text{H} = 60 \times 10^{-6}\ \text{H},$$

and for C, using Equation (2.118),

$$C = 1/\omega|\mathbf{Z_C}| = 1/1\ \text{Mrad/s} \times 30\ \Omega = 33.33\ \text{nF} = 33.33 \times 10^{-9}\ \text{F}.$$

Example 2.24 Parallel Equivalent Impedance

Given that $\mathbf{Z_R} = 10\ \Omega$, $\mathbf{Z_L} = j60\ \Omega$ and $\mathbf{Z_C} = -j30\ \Omega$, calculate the parallel equivalent impedance in rectangular, polar, and Euler's forms.

Similarly to what we did with resistors, we do it with impedances, but remembering that impedances are phasors, or complex numbers with magnitude and phase, then,

$$\frac{1}{\mathbf{Z}_{\textbf{parallel-equivalent}}} = \frac{1}{\mathbf{Z_R}} + \frac{1}{\mathbf{Z_L}} + \frac{1}{\mathbf{Z_C}}. \tag{2.119}$$

It is convenient to transform the given impedance into their polar forms, which are $\mathbf{Z_R} = 10\ \Omega\angle 0°$, $\mathbf{Z_L} = 60\Omega\angle{+90°}$, and $\mathbf{Z_C} = 30\Omega\angle{-90°}$.

Then using Equation (2.119) and using the impedances in polar form we obtain

$$\mathbf{1/Z}_{\textbf{parallel-equivalent}} = 1/10\ \Omega\angle 0° + 1/60\ \Omega\angle{-90°} + 1/30\ \Omega\angle{+90°}. \tag{2.120}$$

After converting each impedance term on the right-hand side of Equation (2.120) from polar form into rectangular form, adding the three of them and then obtaining the inverse of the addition, leads to

$$\mathbf{1/Z}_{\textbf{parallel-equivalent}} = 1/10 + 1/j60 - j30 \tag{2.121}$$

$$\mathbf{Z}_{\textbf{parallel-equivalent}} = 9.729 - j1.622, \tag{2.122}$$

where Equation (2.122) is the equivalent impedance in rectangular form.

$$\mathbf{Z}_{\text{parallel-equivalent}} = 9.863\angle{-9.465°} \quad (2.123)$$

And finally,

$$\mathbf{Z}_{\text{parallel-equivalent}} = 9.863e^{-j9.465°} \quad (2.124)$$

where Equation (2.122) is in rectangular form, Equation (2.123) is in polar form, and Equation (2.124) is in Euler's form.

2.4 POWER IN AC CIRCUITS

Circuits on sinusoidal steady state draw power from their sinusoidal power source. When *R, L, C* circuits are connected to a sinusoidal power source, some of the power drawn by the circuit is consumed by it; this is called real, true, active, or average power. *Active* power is measured in Watts (W). Active power is power that the load takes from the source to perform useful work. Active power gets converted into heat on the resistive part of the impedance. The capacitors and/or inductors present in the circuit cause the source to produce some additional power that is not consumed by the load. In the capacitor case, this power establishes the electric field on the capacitor itself; in the inductor case, this power establishes the magnetic field on the inductor. The power drawn from the power source by the capacitive and inductive elements does not produce any active power. This capacitive and inductive power is referred to as *reactive* power, and it is measured in *reactive volt-amperes* (VAR). The total power taken by a load from the AC power source is some combination of the total active power plus the total reactive power. This total power is referred to as the *apparent* power (S), measured in volt-amperes (VAs). So what is the relationship between apparent (S), active (P), and reactive (Q) powers?

We will answer this question soon, but first let us study the instantaneous power drawn by a resistor, a capacitor, and an inductor when they are fed by an AC source.

Let us go over active, reactive, and apparent power one more time. Active or real power is the easiest to understand. And it is the total energy absorbed by the resistive component of the load during each sinusoidal cycle. Energy is measured in units of power (W) multiplied by units of time, for example, watt-seconds or watt-hours. Real or active power is measured in watts.

The physical meaning of reactive power is not as easy or intuitive to understand. Reactive power, denoted by Q, refers to the maximum value of instantaneous power absorbed by the reactive component of the load. The instantaneous reactive power is alternatively positive and negative, twice per sinusoidal cycle. For an inductor, refer to Figure 2.9, and for a capacitor, refer to Figure 2.10. Note that the instantaneous power in a reactive element (i.e., either an inductor or a capacitor) is positive for the first quarter of the sinusoidal cycle, and then it becomes negative during the second quarter of the cycle, positive on the third quarter, and negative on the final quarter cycle. Refer to Figures 2.9 and 2.10. Positive instantaneous power means that the generator provides power to the reactive load; negative instantaneous power means to the load returns the power back to the source. Note that the average or active power consumed by a reactive element is zero on a cycle per cycle basis. Active and reactive powers are related, and the combination of both is referred to as apparent power measured in VAs. In the next several sections we will discuss instantaneous power in resistors, inductors, and capacitors. This will lead to active, reactive, and apparent powers and their relationship which is explained by means of the *triangle of powers*.

2.4.1 AC Instantaneous Power Drawn by a Resistor

From Table 2.1, when a sinusoidal current and voltage are produced on a resistor, we know that both waveforms are in phase. And from Equation (2.9), repeated here for the reader's convenience, the instantaneous power on the resistor is

$$p_R(t) = 1/2VI - 1/2VI\cos(2\omega t + \theta), \tag{2.125}$$

where V and I are respectively peak values of voltage and current.

We also have seen (from Eqs. 2.12 through 2.19) that the average power consumed by the resistor is evaluated as follows:

$$\mathrm{P}_{\text{average-resistor}} = 1/T\int_0^T [I_{RMS}V_{RMS} - V_{RMS}I_{RMS}\sin(2\omega t)]\,dt, \tag{2.126}$$

which leads to

$$\mathrm{P}_{\text{average}} = 1/2VI = I_{RMS}V_{RMS} \tag{2.127}$$

because the term $V_{RMS}I_{RMS}\sin 2\omega t$ average value is zero.

Earlier in this Chapter, Figure 2.4 shows the sinusoidal current, voltage on a resistor, the instantaneous power, and the average power consumed by the resistor. It is important and interesting to observe that the average power consumed by the resistor always flows from the AC power source into the resistor. Such average power is always positive.

2.4.2 AC Instantaneous Power Drawn by a Capacitor

The instantaneous power drawn by a capacitor is the product of its instantaneous voltage and current. From previous sections we know that the instantaneous voltage across the capacitor lags the instantaneous current waveform by 90°. That is to say,

$$V_C = -jX_C I_C \text{ in phasor form or the frequency domain} \tag{2.128}$$

and

$$I_C \cos(\omega t + \theta);\, V_C \sin(\omega t + \theta) \text{ in the time domain,} \tag{2.129}$$

where in both Equations (2.128) and (2.129), I_C and V_C are respectively the peak values of AC current and AC voltage on the capacitor. As usual, ω is the *angular frequency* and θ is an arbitrary phase angle. Note that θ shows up on both AC current and voltage. For simplicity and without loss of generality we will assume that θ is zero.

The product of its AC voltage and current gives the instantaneous power on the capacitor;

$$p_C(t) = v_C(t) \times i_C(t), \tag{2.130}$$

where we substitute the waveforms from Equation (2.129) into Equation (2.130) and obtain

$$p_C(t) = V_C \sin(\omega t) \times I_C \cos(\omega t) \tag{2.131}$$

$$= V_C I_C \sin(\omega t) \cos(\omega t). \tag{2.132}$$

In Equation (2.132), V_C and I_C are respectively the peak voltage and current values.

Using the following trigonometric identity in Equation (2.132),

$$\sin 2x = 2 \sin x \cos x \tag{2.133}$$

leads to

$$= 1/2 V_C I_C \sin 2\omega t \tag{2.134}$$

$$= V_{RMS} I_{RMS} \sin 2\omega t, \tag{2.135}$$

where $V_{RMS} = V_C/\sqrt{2}$ and $I_{RMS} = I_C/\sqrt{2}$ are the RMS values of voltage and current.

$$\mathrm{P}_{\text{average-capacitor}} = 1/T \int_0^T V_{RMS} I_{RMS} \sin 2\omega t \, dt \tag{2.136}$$

$$\mathrm{P}_{\text{average-capacitor}} = 0. \tag{2.137}$$

From Equation (2.137) we observe that the average power consumed by the capacitor during an AC period is zero. Referring to the double frequency instantaneous power waveform of Figure 2.10, it is possible to see that the integral of the instantaneous power waveforms between an integral number of cycles T is zero.

Again referring to Figure 2.10, it can be seen that the instantaneous power drawn by the capacitor is alternatively positive and negative every quarter of a period of the original voltage and current waveforms. When the instantaneous power is positive, it means that the source is providing instantaneous power to the capacitor; when the instantaneous power is negative, the capacitor is returning power to the source. This is what originates the *capacitive reactive* power in a capacitor, and it is sometimes called as the *entertaining power* between the source and the capacitor.

2.4.3 AC Instantaneous Power Drawn by an Inductor

The instantaneous power drawn by the inductor is the product of its instantaneous voltage and current. From previous sections we know that the instantaneous voltage across the inductor leads the instantaneous current waveform by 90°. That is to say,

$$V_L = jX_L I_L \text{ in phasor form or the frequency domain} \tag{2.138}$$

and

$$V_L \cos(\omega t + \theta);\ I_L \sin(\omega t + \theta) \text{ in the time domain,} \tag{2.139}$$

where in both equations above, V_L and I_L are respectively the peak values of AC voltage and AC current. As usual, ω is the *angular frequency* and θ is the phase angle.

Note that θ shows up on both AC voltage and current. For simplicity and without loss of generality we will assume that θ is zero.

The product of its AC voltage and current gives the instantaneous power on the inductor.

$$p_L(t) = v_L(t) \times i_L(t), \tag{2.140}$$

where we substitute the waveforms from 2.139 into Equation (2.140) and obtain

$$p_L(t) = V_L \cos\omega t \times I_L \sin\omega t \tag{2.141}$$

$$= V_L I_L \cos\omega t \sin\omega t \tag{2.142}$$

Using the following trigonometric identity in Equation (2.142),

$$\sin 2x = 2\sin x \cos x \tag{2.143}$$

leads to

$$= 1/2 V_L I_L \sin 2\omega t \tag{2.144}$$

$$= V_{RMS} I_{RMS} \sin 2\omega t, \tag{2.145}$$

where $V_{RMS} = V_L/\sqrt{2}$ and $I_{RMS} = I_L/\sqrt{2}$ are the RMS values of voltage and current:

$$\mathrm{P}_{\text{average-inductor}} = 1/T \int_0^T V_{RMS} I_{RMS} \sin 2\omega t \, dt. \tag{2.146}$$

Evaluating the integral

$$\mathrm{P}_{\text{average-inductor}} = 0. \tag{2.147}$$

From Equation (2.147) we can see that the average power consumed by the inductor during a sinusoidal AC period is zero. Refer to previously seen Figure 2.9. From this figure it can be seen that the instantaneous power drawn by the inductor is alternatively positive and negative every quarter of a period. A period refers to the voltage or current period. Both voltage and current waveforms on an inductor have the same frequency when the excitation is sinusoidal.

When the instantaneous power is positive, it means that the source is providing instantaneous power to the inductor; when the instantaneous power is negative, the inductor is returning power to the source. This is what originates the *inductive reactive* power in an inductor, and it is sometime called as the *entertaining power* between the source and the inductor.

2.4.3.1 AC Power Triangle Active, capacitive reactive, inductive reactive, and apparent powers are geometrically related by the power triangle. When an impedance **Z** has all three electric components (R, L, and C), the active power, dissipated on the resistive part of the impedance, is drawn horizontally and is denoted as P. The inductive reactive power is represented vertically and pointing toward the positive side of the complex plane. It is denoted as Q inductive and has a positive sign. The capacitive reactive power is represented vertically and pointing down toward the negative side of the complex plane. It is denoted as Q capacitive and has a negative sign. The active and net reactive power (*inductive reactive power or capacitive reactive power*) are related to each other by the Pythagorean relationship, from phasor analysis:

$$S^2 = P^2 + Q^2, \tag{2.148}$$

where S in VA is the apparent power; P is the active power in watts consumed by the resistive part of the impedance; Q is the net reactive power in VAR (reactive volt-ampere).

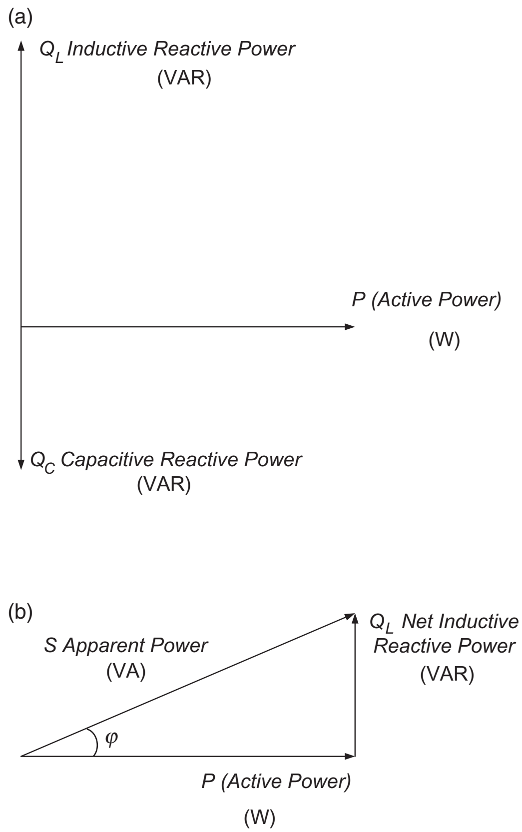

Figure 2.19 Power triangle with net inductive reactive power.

Figure 2.19 depicts active power with net inductive reactive power, and Figure 2.20 depicts active power with net capacitive reactive power.

From basic trigonometry it can be observed that

$$P = S\cos\varphi \tag{2.149}$$

and

$$Q = S\sin\varphi, \tag{2.150}$$

where φ is defined as the *power factor* for sinusoidal steady state.

Thus,

$$\text{Power Factor} = \text{PF} = \cos\varphi. \tag{2.151}$$

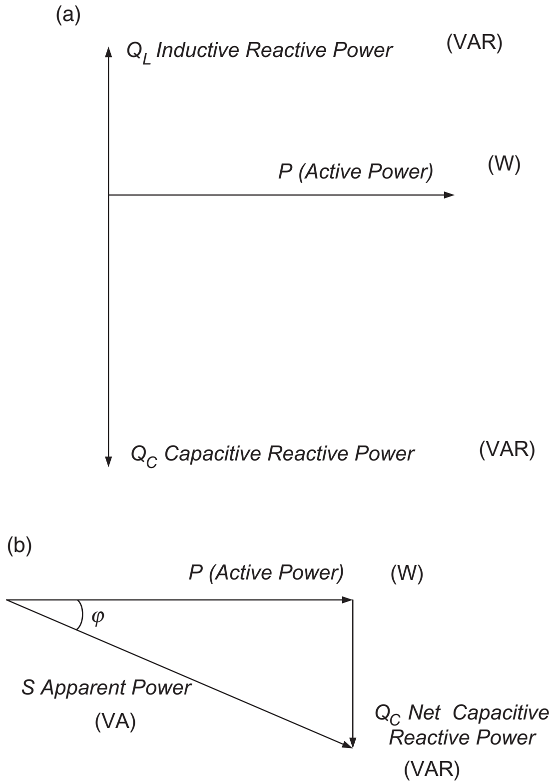

Figure 2.20 Power triangle with net capacitive reactive power.

For sinusoidal steady state, it can also be observed that

$$PF = P/S. \tag{2.152}$$

Power factor is an important figure of merit that electric utility companies observe closely. The utility company does not want its customer's electrical loads to demand too much reactive power. Why? Because the electric generators need to produce an excess power (reactive power) that does not end up as useful work developed at the load. Remember that reactive power is power that is supplied by the generator to the load and returned back from the load to the generator on a cyclical basis. When a capacitor's electric fields and inductor's magnetic fields are created, they cause for the existence of capacitive and inductive reactive power respectively. Ideally, the electric utility company wants that reactive power to be zero, or in other words, they want to see a very close to unity power factor (PF = 1). From Equation (2.151) for PF to be one, φ has to be zero. For inductive loads, the current through the inductor lags the voltages across it, and the power factor is said to be *lagging*. For capacitive loads, the current through the capacitor leads the voltage across it, and the power factor is said to be *leading*.

After all being said, why is it that important that the power factor of an electrical load be one or very close to one? The reasons are that if power factor is smaller or much smaller than one, the utility company electric generator has to generate excessive power that will not end up being used by the load as active power. Let us remember that the load produces useful work consuming active power. The power distribution wiring needs to be thicker if the power factor is smaller than one. The dimensioning of the power distribution wiring must be made based on the apparent power drawn by the load. This ensures that the AC power distribution wires to the load are appropriately sized.

Example 2.25 Determine the total apparent, active, and reactive power that a 2 Ω resistive load with a unity power factor draws from an AC 220 V_{RMS} voltage generator.

Solution to Example 2.25

Since

$$S^2 = P^2 + Q^2 \tag{2.153}$$

where

$$P = S\cos\varphi \tag{2.154}$$

and

$$Q = S\sin\varphi \tag{2.155}$$

where $\cos\varphi$ equals one, as stated by the problem assumption, thus, φ equals 0° and $\sin\varphi = 0$.

From Equations (2.154) and (2.155),

$$P = S \tag{2.156}$$

and

$$Q = 0 \tag{2.157}$$

where

$$P = S = V_{RMS}^2/R = 220^2/2 = 24.2\ \text{kW} = 24.2\ \text{kVA}$$

In this example the apparent power equals the active power dissipated by the load, and there is zero reactive power between the generator and the load.

Example 2.26 Determine the total apparent, active, and reactive powers that an impedance of an absolute value of 11 Ω, and an inductive power factor of 0.8. The impedance draws 20 A_{RMS} from a 220 V_{RMS} AC voltage generator. Also determine the impedance real and imaginary parts.

Solution to Example 2.26

Since apparent power

$$S = I_{RMS}V_{RMS} \tag{2.158}$$

$$S = 20\ \text{A}220\ \text{V} = 4400\ \text{VA}.$$

And since

$$S^2 = P^2 + Q^2 \tag{2.159}$$

where

$$P = S\cos\varphi \tag{2.160}$$

and

$$Q = S\sin\varphi \tag{2.161}$$

$$P = 4400 \times 0.8 = 3520\ \text{W}$$

and where $\cos\varphi$ equals 0.8, as stated by the problem assumption, thus, $\varphi = 36.87°$ and $\sin\varphi = 0.6$.

Then,

$$Q = 4400 \times 0.6 = 2640\ \text{VAR}.$$

$$R = |Z|\cos 36.87° = 8.8\ \Omega \tag{2.162}$$

and

$$X_L = |Z|\sin 36.87° = 6.6\ \Omega \tag{2.163}$$

where X_L is inductive.

2.5 DEPENDENT VOLTAGE AND CURRENT SOURCES

Dependent sources produce either a voltage or a current, where such voltage or current depends on either a voltage or a current on some other part of the circuit or network.

Dependent sources are widely used to model active circuits like operational amplifiers, transistor-based amplifiers, and transistors such as bipolars and MOSFETs.

There are four basic kinds of dependent sources; two dependent voltage sources and two dependent current sources. Within each type there are current- and voltage-controlled sources.

The four types of dependent sources are listed below:

1. Voltage-controlled dependent voltage source or VCVS
2. Current-controlled dependent voltage source or CCVS
3. Voltage-controlled dependent current source or VCCS
4. Current-controlled dependent current source or CCCS

2.5.1 Voltage-Controlled Voltage Source (VCVS)

The voltage-controlled voltage source is a dependent source that allows us to model a voltage amplifier. Without knowing yet about the internals of a voltage amplifier, we can define such a circuit element as a two-port device. One input port that receives and input voltage V_{in} and one output port that generates an output voltage which is a magnification of the input voltage V_{in} by some constant A, where A stands for *amplification factor* or simply its *amplification*.

Figure 2.21 depicts the symbol diagram of a VCVS which is very appropriate to model the behavior of a voltage amplifier, such as the one just described.

Figure 2.22 depicts the use of a VCVS in a circuit example. Note that $V_{out} = A\,V_{in}$; thus, A has to be dimensionless because $A = V_{out}/V_{in}$, and its units are then volts/volts.

Note that in the VCVS circuit example, the voltage source of value $V_{out} = A\,V_{in}$ produces an output voltage that depends on the value of input voltage V_{in},

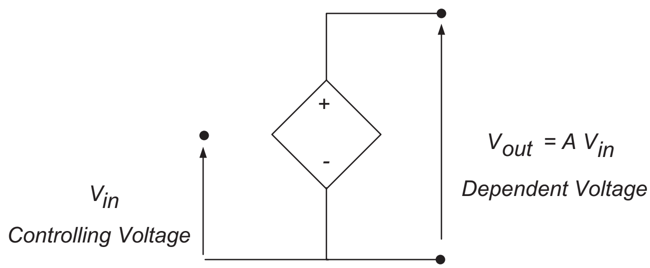

Figure 2.21 Voltage-controlled voltage source.

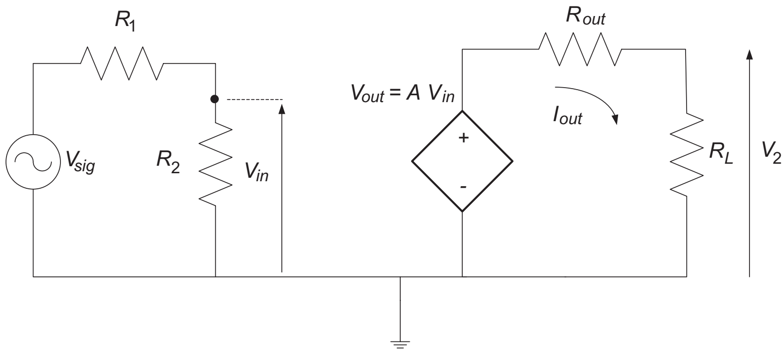

Figure 2.22 Use of a VCVS in a circuit example.

which is shown across resistor R_2. Just to wrap up this example, let us evaluate the overall output voltage of the complete circuit, V_2 as a function of V_{sig}.

From Figure 2.22, using KVL, we can write for the left-hand side loop that

$$V_{in} = \frac{R_2}{R_1 + R_2} V_{sig}. \tag{2.164}$$

Using KVL for the right-hand side loop we get

$$A V_{in} = I_{out}(R_{out} + R_L). \tag{2.165}$$

Combining Equation (2.164) with Equation (2.165), and since $V_2 = I_{out} \times R_L$, it yields

$$V_2 = \frac{A}{R_{out} + R_L} R_L \frac{R_2}{R_1 + R_2} V_{sig}. \tag{2.166}$$

2.5.2 Current-Controlled Voltage Source (CCVS)

A current-controlled voltage source is a dependent source that allows us to model a trans-resistance amplifier. Without knowing yet about the internals of a trans-resistance amplifier, we can define such a circuit element as a two-port device. One input port that receives and input current I_{in} and one output port that generates an output voltage which is a magnification of the input current I_{in} by some constant Γ (rho), where Γ stands for *trans-resistance amplification factor* or simply its *amplification* Γ. Note that the units of Γ are ohms.

Figure 2.23 depicts the symbol diagram of a CCVS which is very appropriate to model the behavior of a trans-resistance amplifier. Figure 2.24 depicts the use of a CCVS in a circuit example. Note that $V_{out} = \Gamma I_{in}$; thus, Γ is in ohms, because $\Gamma = V_{out}/I_{in}$ units are volts/ampere.

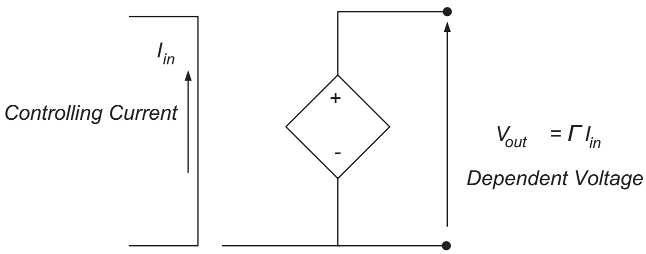

Figure 2.23 Current-controlled voltage source.

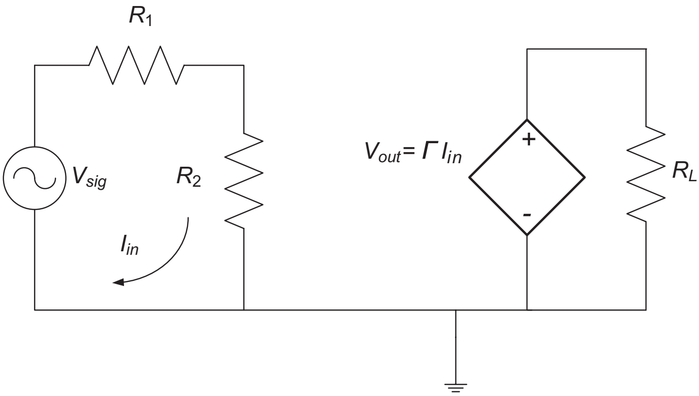

Figure 2.24 Use of a CCVS in a circuit example.

Note that in the CCVS circuit example of Figure 2.24, the voltage source of value $V_{out} = \Gamma\ I_{in}$ produces an output voltage that depends on the value of input current I_{in}, which flows in the circuit of R_1 and R_2 and excited by V_{sig}.

2.5.3 Voltage-Controlled Current Source (VCCS)

A voltage-controlled current source is a dependent source that allows us to model a trans-conductance amplifier. Without knowing yet about the internals of a trans-conductance amplifier, we can define such a circuit element as a two-port device. One input port that receives and input voltage V_{in} and one output port that generates an output current which is a magnification of the input voltage V_{in} by some constant G, where G stands for *trans-conductance amplification factor* or simply its *amplification* G, where G has conductance units (Ω^{-1}).

Figure 2.25 depicts the symbol diagram of a VCCS which is very appropriate to model the behavior of a trans-conductance amplifier, such as the one just described.

Figure 2.26 depicts the use of a VCCS in a circuit example. Note that I_{out} = G V_{in}.

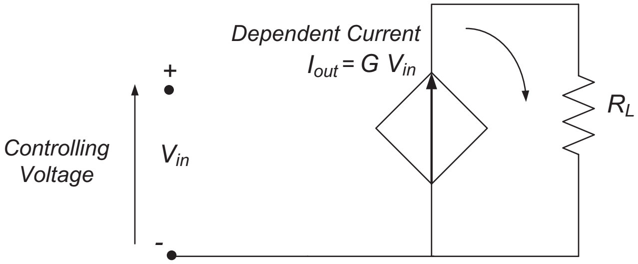

Figure 2.25 Voltage-controlled current source.

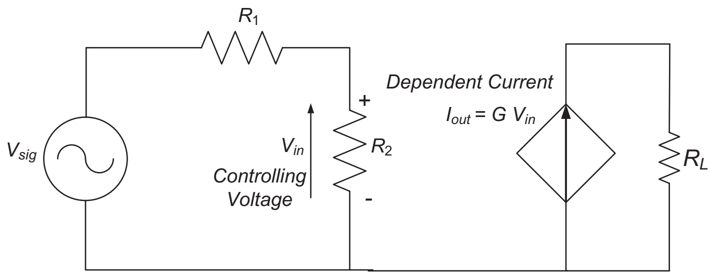

Figure 2.26 Use of a VCCS in a circuit example.

Note that in the VCCS circuit example the current source of value: $I_{out} = G\,V_{in}$ produces an output current that depends on the value of input voltage V_{in}, which is shown across resistor R_2.

2.5.4 Current-Controlled Current Source (CCCS)

A current-controlled current source is a dependent source that allows us to model a current amplifier. Without knowing yet about the internals of a current amplifier, we can define such a circuit element as a two-port device. One input port that receives and input current I_{in} and one output port that generates an output current which is a magnification of the input current I_{in} by some constant β, where β stands for *current amplification factor* or simply its *amplification* β. Note that β has no dimensions since it is obtained as the ratio of two currents. A current amplifier is also referred to as a buffer. We will see in later chapters that buffers can be implemented with transistors or with operational amplifiers.

Figure 2.27 depicts the symbol diagram of a CCCS which is very appropriate to model the behavior of a current amplifier, such as the one just described.

Figure 2.28 depicts the use of a CCCS in a circuit example.

Note that in the CCCS circuit example the voltage source of value $I_{out} = \beta I_{in}$ produces an output current that depends on the value of input current I_{in}, which is shown in the circuit of Figure 2.28.

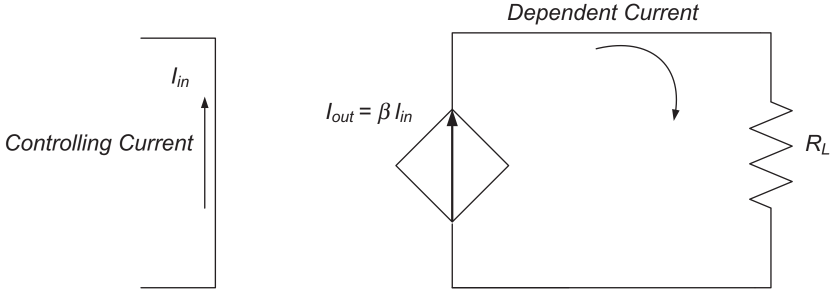

Figure 2.27 Current-controlled current source.

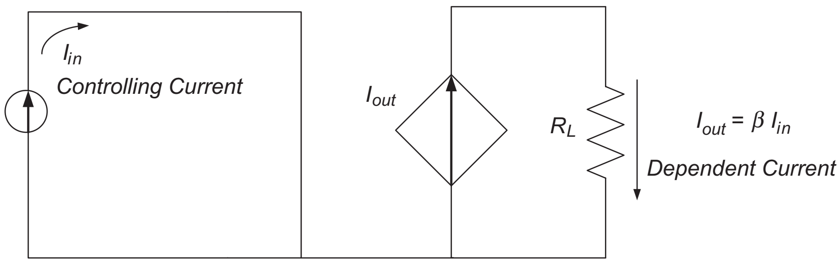

Figure 2.28 Use of a CCCS in a circuit example.

2.6 SUMMARY OF KEY POINTS

This chapter covers the fundamentals of AC circuits. It is important to understand the concept of effective value (RMS) of voltage and current and the role they play on R, L, and C elements and how they produce different kinds of AC power: active, reactive, and apparent. It is also of great interest to know how to manipulate circuit equations in the time domain as well as in the frequency domain. In the time domain, derivatives and integrals of current or voltage usually apply. In the frequency domain, phasors replace the tedious-to-deal-with differential equations. Phasor diagrams make AC circuit calculations easier. The catch is that this method works when the voltage and current frequencies are the same. "Dependent sources" is a topic of great interest to model electronic devices or active devices that have gain. More on this subject is covered in Chapters 5 and 6.

FURTHER READING

1. Charles Alexander and Matthew Sadiku, *Fundamentals of Electric Circuits*, 2nd ed., McGraw Hill, New York, 2004.
2. Charles Monier, *Electric Circuit Analysis*, Prentice Hall, Upper Saddle River, NJ, 2001.
3. David Bell, *Fundamentals of Electric Circuits*, 4th ed., Prentice Hall, Upper Saddle River, NJ, 1988.

PROBLEMS

2.1 A toaster is rated at 1 kW and for an AC voltage of 120 V at 60 Hz.

(a) Determine the resistance of the toaster, before its temperature increases. Assume that the resistance is at room temperature.

(b) Determine the RMS value of current flowing through the toaster when it is dissipating 1 kW.

(c) Determine the peak value of the current through the toaster when it is dissipating 1 kW.

(d) If the resistance of the toaster has +/–10% tolerance, calculate the minimum and maximum power that the toaster will consume under the two extremes of resistance values.

2.2 Evaluate the RMS value of the voltage waveform drawn in Figure 2.29. Assume that the peak amplitude of the waveform is 1 V, its period *T* is 1 msec, and 50% duty cycle.

2.3 Evaluate the average DC voltage waveform for the double rectified sine-wave waveform depicted in Figure 2.30. Analytically, the waveforms can be described as follows:

$$v(t) = \sin \omega t;\ \text{for: } \pi/2 \le t \le 0$$

$$v(t) = -\sin \omega t;\ \text{for: } \pi/2 \le t \le \pi$$

This alternatively can be expressed as

$$v(t) = |\sin \omega t|.$$

2.4 Calculate the RMS value of a 10 A DC current.

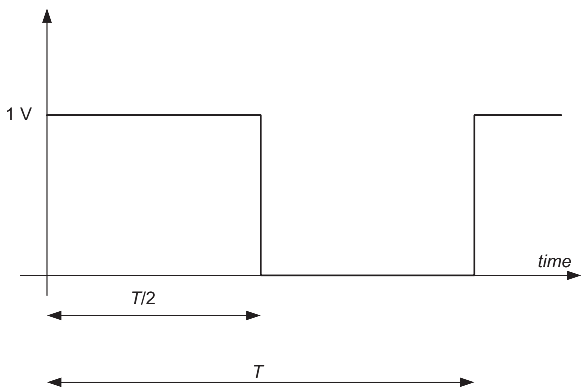

Figure 2.29 50% duty cycle square-wave voltage waveform for Problem 2.2.

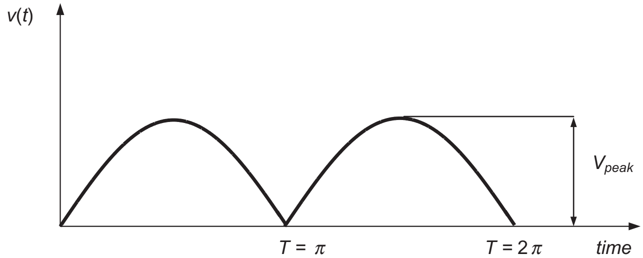

Figure 2.30 Double rectified sine-wave waveform for Problem 2.3.

2.5 Given an *RLC*-series circuit, where $R = 10\ \Omega$, $L = 320$ nH, $C = 100\ \mu$F, find the absolute value of the impedance of the circuit at the following frequencies:

(a) 1 Hz
(b) 10 Hz
(c) 100 Hz
(d) 1 kHz
(e) 10 kHz
(f) 100 kHz
(g) 1 MHz, and
(h) 10 MHz.

2.6 For an RLC-series circuit, where $R = 10\ \Omega$, $L = 320$ nH, $C = 100\ \mu$F, calculate the absolute value of inductive reactance and the capacitive reactance at the following frequencies:

(a) 1 Hz
(b) 10 Hz
(c) 100 Hz
(d) 1 kHz
(e) 10 kHz
(f) 100 kHz
(g) 1 MHz, and
(h) 10 MHz.

2.7 (a) For the circuit given in Problem 2.5, find the frequency at which the absolute value of the inductive reactance equals the absolute value of the capacitive reactance (i.e., resonance condition). (b) At this frequency find the peak value of current for a 1-V peak sinusoidal voltage at the resonant frequency.

2.8 The circuit of Problem 2.5 is said to be at its resonant frequency when the absolute value of its inductive reactance equals the absolute value of its capacitive reactance. The resonant frequency was calculated in Problem 2.7. Assuming a 1-V peak sinusoidal voltage, (a) find the value of current in the circuit at a frequency equal to 10 times the resonant frequency of the circuit, and (b) find the value of current in the circuit at a frequency equal to one-tenth of the resonant frequency of the circuit.

Draw conclusions from the numerical answers that you obtain for this problem.

2.9 An impedance of value $Z = (400 + j350)\ \Omega$ is connected to a sinusoidal voltage of 416 V RMS. (a) Compute the apparent, active, and reactive powers that the impedance absorbs from the AC generator. (b) Determine the power factor of the circuit.

2.10 Establish the time domain equations (i.e., differential equations) of an *RLC*-series circuit powered by a sinusoidal voltage source $v(t) = V_{peak} \sin(\omega t + \theta)$. Hint: The final equation is a second-order differential equation with constant coefficients.

2.11 Establish the time domain equations (i.e., differential equations) of a parallel *RLC* circuit powered by a sinusoidal current source $i(t) = I_{peak} \sin(\omega t + \theta)$.

Hint: The final equation is a second-order differential equation with constant coefficients.

2.12 Given a 10 Ω resistor in series with a 10 μF capacitor, and an AC voltage source of $V_{in} = 100\ \text{V}\ e^{j0}$, of a 1 kHz frequency, determine: (1) if the current through the circuit leads or lags the input voltage V_{in} across the *RC* series; (2) the phase angle between the input voltage and the circuit series current), and (3) the phase angle between the voltage source and the voltage across the capacitor.

2.13 Given an *RLC* series circuit, where $R = 100\ \Omega$, $L = 1\ \mu\text{H}$, and $C = 10\ \mu\text{F}$, determine the frequency at which the circuit goes into resonance.

2.14 Express the series impedance given in Problem 2.12 in complex notation. Hint: $Z(j\omega)$.

2.15 Given that $Z_1 = (30 + j25)\ \Omega$ and $Z_2 = (20 - j15)\ \Omega$, calculate: (1) the series combination of both impedances, and (2) the parallel combination of both impedances.

2.16 Given impedance $(Z_1 = 30 + j25)\ \Omega$, find the value of inductance of its inductive reactance at a frequency of 1 kHz.

2.17 Given impedance $(Z_2 = 20 - j15)\ \Omega$, find the value of the capacitive reactance of the impedance at a frequency of 1 kHz.

2.18 Given an *RLC* series circuit, where $R = 5\ \Omega$, the reactive reactance is $+j18\ \Omega$, and the capacitive reactance is $-j10\ \Omega$ connected to a sinusoidal

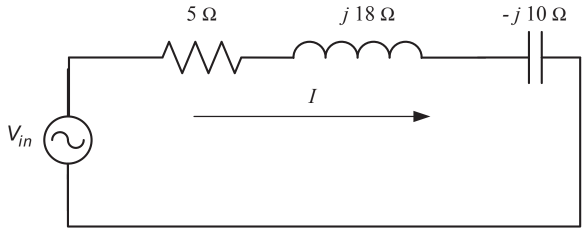

Figure 2.31 *RLC* series circuit for Problem 2.18.

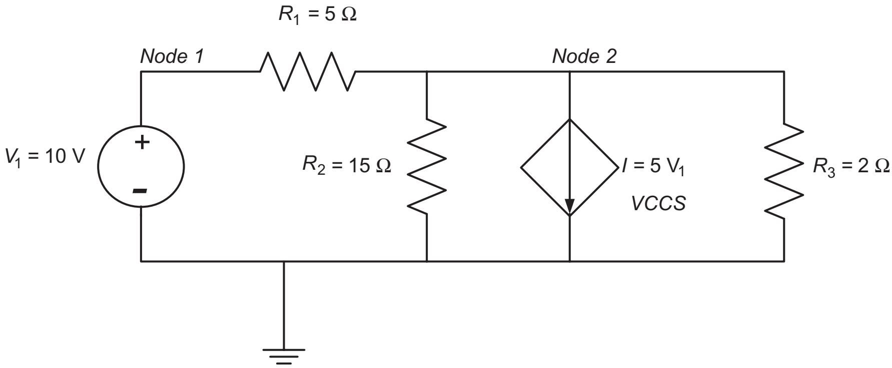

Figure 2.32 Circuit for Problem 2.20.

voltage source V_{in} of *RMS* value of 100 V, determine the circuit complete phasor diagram. The following phasors must be shown: (1) circuit current phasor, $\boldsymbol{I}$, (2) resistor voltage phasor, $\boldsymbol{V_R}$, (3) capacitive reactance voltage phasor, $\boldsymbol{V_C}$, and (4) inductive reactance voltage phasor, $\boldsymbol{V_L}$. Find and show on the phasor diagram all the numerical phase angle values between the current and the three voltages (Fig. 2.31).

2.19 Given an RLC series circuit with an impedance $Z = 100 - j45$ at 60 Hz, assume that the circuit is energized by a 240 V 60 Hz sinusoidal voltage generator. (1) Calculate the real, apparent, and reactive power of the circuit, and (2) calculate the circuit power factor.

2.20 Given the circuit of Figure 2.32, note that $I = 5\ V_1$, between node 2 and ground, is a voltage-controlled current source (VCCS), whose output current value is $I = 5\ V_1$, and the control voltage is $V_1 = 10$ V. Calculate the voltage at every node with respect to ground and the currents through every resistor, the independent voltage source, and the VCCS.

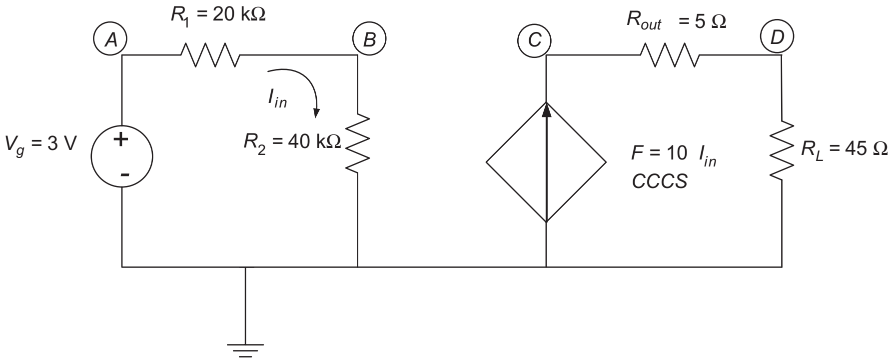

Figure 2.33 Circuit for Problem 2.21.

2.21 Given the circuit of Figure 2.33, note that the element between node C and ground is a current-controlled current source (CCCS), whose output current is 10 I_{in}, and the control current is I_{in}. Calculate the voltage at every node (A through D) with respect to ground and the currents through every resistor, the independent voltage source, and the VCCS dependent source.