Chapter 6

Parallel Wrists: Limb Types, Singularities and New Perspectives

This chapter reviews the criteria to identify the limb architectures suitable for parallel wrists (PWs) and the instantaneous kinematics of PWs. Special attention will be given to the use of the screw theory and of the displacement groups. The relationship between the limb architecture and the resulting PW mobility will be discussed. The main results reported in the literature about the singularity analysis of PWs will be summarized and the problem of characterizing the kinetostatic performances of PWs will be addressed. Eventually, innovative wrist architectures, recently proposed, will be presented.

6.1. Limb architectures and mobility analysis

PWs are parallel manipulators (PMs) with three degrees of freedom (DoF) where the end-effector motion can only be spherical. In PWs, the end-effector is connected to the frame through a number, *n*, of kinematic chains (limbs).

For a long time, there have been only two architectures of PWs diffusely studied in the literature: the 3-RRR wrist [GOS 89, GOS 94, GOS 95, ALI 94] and the S-3UPS wrist [INN 93, WOH 94]. The 3-RRR wrist (Figure 6.1) has three limbs of type RRR (R stands for revolute pair) with all

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the revolute pair axes converging toward a single point of the frame (the center of the end-effector spherical motion). The S-3UPS wrist (Figure 6.2) has three limbs of type UPS (U, P and S stand for universal joint, prismatic pair and spherical pair, respectively) and a fourth limb that is constituted by a spherical pair that directly connects the end-effector and the frame.

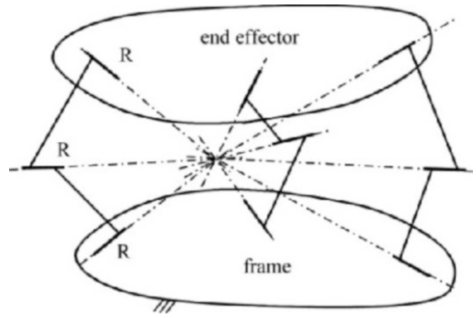

Figure 6.1. *The 3-RRR wrist*

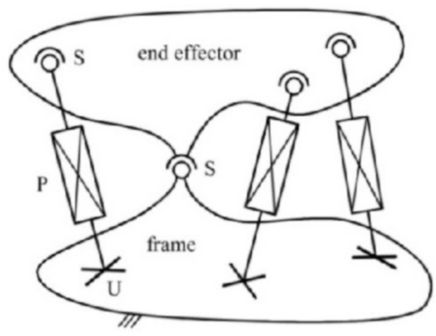

Figure 6.2. *The S-3UPS wrist*

Since 2000 [KAR 00], the interest in PWs has increased by following the increasing attention that the academic and industrial worlds paid to the PMs with lower mobility. This renewed interest in the PWs led to lot of new PW architectures appearing in the literature (see, for instance, [KAR 00, DIG 01a, DIG 01b, KAR 02, DIG 04a, KON 04a, KON 04b, KAR 04, KAR 06]). The analysis of this literature reveals that there are many techniques to generate limb architectures suitable for PWs. Such techniques can be collected into three sets: (1) the techniques that use the screw theory, (2) the techniques that use group theory and (3) other techniques less general than screw and group theories.

In the following sections, these approaches to the problem of identifying PW limbs will be presented and discussed.

6.1.1. *Use of the screw theory*

6.1.1.1. *Basic concepts*

By introducing a reference point O (pole), a straight line, (P,**u**), passing through a point P and with the direction of the unit vector **u**, and two scalar coefficients, named q (signed magnitude) and p (pitch), whose product, *k* (= q p), is always a finite real number, a screw can be geometrically defined as the following six-dimensional vector:

$$\hat{\mathbf{\$}} = \mathrm{q}\begin{bmatrix} \mathbf{u} \\ \mathrm{p}\mathbf{u} + (\mathbf{P} - \mathbf{O}) \times \mathbf{u} \end{bmatrix} \qquad [6.1]$$

The set of screws with the same pole, O, constitutes a vector space where the rules of the sum of two screws can be immediately deduced from the rules of the sum of two three-dimensional vectors.

According to the physical meaning given to q and to the line (P, **u**), a screw can represent either a rigid body motion (in this case, the screw is named twist) or a system of forces (in this case, the screw is named wrench).

If q is the signed magnitude of a rigid body angular velocity, $\boldsymbol{\omega}$, that is equal to q **u**, and the line (P, **u**) is the instantaneous screw axis of an infinitesimal motion, the screw (twist) will represent the instantaneous motion of a rigid body since it has the following kinematic meaning:

$$\hat{\mathbf{\$}}_{\mathrm{t}} = \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_{\mathrm{O}} \end{bmatrix} \qquad [6.2]$$

where $\mathbf{v}_O$ is the velocity that the pole O would have if it was fixed to the rigid body.

If q is the signed magnitude of the resultant force **F** (= q **u**) of a force system, and the line (P, **u**) is Poinsot's central axis of the same force system, the screw (wrench) will represent the force system since it has the following static meaning:

$$\hat{\$}_w = \begin{bmatrix} \mathbf{F} \\ \mathbf{M}_O \end{bmatrix} \qquad [6.3]$$

where $\mathbf{M}_O$ is the resultant moment about the pole O of the force system.

The reciprocal product, $\hat{\$}_t \circ \hat{\$}_w$, between a twist, $\hat{\$}_t$, of the set of the twists with pole O (motion space) and a wrench, $\hat{\$}_w$, of the set of the wrenches with pole O (wrench space) is defined as follows:

$$\hat{\$}_t \circ \hat{\$}_w = \mathbf{F} \cdot \mathbf{v}_O + \mathbf{M}_O \cdot \boldsymbol{\omega} \qquad [6.4]$$

The reciprocal product has the commutative and the distributive properties. It has the physical meaning of instantaneous virtual power that the system of forces, represented by $\hat{\$}_w$, introduces into a mechanical system that performs the instantaneous rigid motion represented by $\hat{\$}_t$.

A twist, $\hat{\$}_t$, is said to be reciprocal to a wrench, $\hat{\$}_w$, and vice versa if their reciprocal product, $\hat{\$}_t \circ \hat{\$}_w$, is equal to zero. Since, for an instantaneous motion compatible with the joint, the instantaneous virtual power introduced by the reaction forces of a passive frictionless joint is zero, it can be stated that, in a passive frictionless joint, the wrench that represents the reactions in the joint is reciprocal to the twist that represents the instantaneous relative motion between the links connected by the joint.

6.1.1.2. *Identification of PW limbs*

Let us consider a generic PM without actuators (i.e. all the joints are passive joints) where the end-effector is connected to the frame through n limbs (Figure 6.3), and call m_i the connectivity[1] of the *i*th limb (i = 1, …, n).

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1 The connectivity of a limb is the number of DoF that the end-effector would have, if it was connected to the frame only by that limb. If the limb is a serial kinematic chain, its connectivity will be equal to the number of joint variables of the limb. Such a number is also equal to the number of one DoF kinematic pairs, the limb would have, if all its multiple DoF pairs were transformed into a serial kinematic chain with one DoF pairs.

Hereafter, without losing generality, all the limbs will be assumed serial kinematic chains.

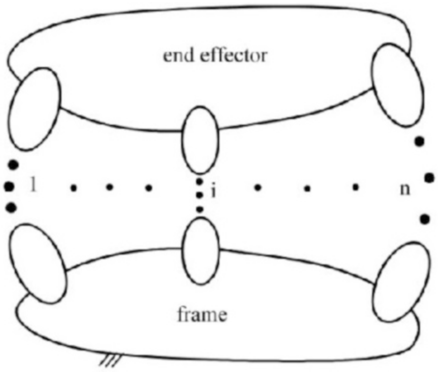

Figure 6.3. *Parallel manipulator with n limbs*

If only the *i*th limb connected the end-effector to the frame, the instantaneous virtual motion of the end-effector would be represented by a twist, $\hat{\$}_{t,i}$, that can be expressed as follows:

$$\hat{\$}_{t,i} = \sum_{j=1,m_i} \alpha_{ij} \hat{\$}_{t,ij} \quad [6.5]$$

where α_{ij} is the virtual rate of the *j*th joint variable of the *i*th limb; whereas $\hat{\$}_{t,ij}$ is the twist that represents the motion, the end-effector would have, if all the α_{ik} for $k = 1, \ldots, (j-1), (j+1), \ldots, m_i$ were equal to zero and α_{ij} was equal to 1. The explicit expressions of the $\hat{\$}_{t,ij}$ twists depend on the architecture and the configuration of the *i*th limb.

The force systems that the *i*th limb can exert on the end-effector are, of course, represented by wrenches, $\hat{\$}_{w,i}$, that are all reciprocal to $\hat{\$}_{t,i}$. The generic wrench $\hat{\$}_{w,i}$ can be represented through a base of six independent screws, $\hat{\$}_b$ for $b = 1, \ldots, 6$, as follows:

$$\hat{\$}_{w,i} = \sum_{b=1,6} \beta_{ib} \hat{\$}_b \quad [6.6]$$

where β_{ib} for $b = 1, \ldots, 6$ are scalar coefficients.

By using [6.5] and [6.6], the reciprocity of $\hat{\$}_{w,i}$ and $\hat{\$}_{t,i}$ yields the following analytic condition:

$$\sum_{j=1,m_i} \alpha_{ij} \left[\sum_{b=1,6} \beta_{ib} (\hat{\$}_b \circ \hat{\$}_{t,ij}) \right] = 0 \qquad [6.7]$$

Since the α_{ij} coefficients can take any arbitrary value, condition [6.7] can be satisfied if and only if the following m_i conditions are matched:

$$\sum_{b=1,6} \beta_{ib} (\hat{\$}_b \circ \hat{\$}_{t,ij}) = 0 \quad j = 1, \ldots, m_i \qquad [6.8]$$

If the ith limb architecture is known, condition [6.8] is a system of m_i linear equations in the six β_{ib} coefficients. Provided that singular limb configurations are not considered, system [6.8] concludes that, if the connectivity m_i is greater than or equal to six, all the β_{ib} coefficients must be zero (i.e. the ith limb does not apply forces[2] to the end-effector). Therefore, only limbs with connectivity m_i less than six can actually constrain the end-effector motion.

According to [6.8], if m_i is less than six, there will be $(6 - m_i)$ coefficient β_{ib} that can be freely chosen in expression [6.6]. Therefore, the set of all the $\hat{\$}_{w,i}$ wrenches constitutes a screw system of order $(6 - m_i)$, which can be analytically expressed as follows:

$$\hat{\$}_{w,i} = \sum_{k=1,(6-m_i)} \gamma_{ik} \hat{\$}_{ik} \qquad [6.9]$$

where γ_{ik} for $k = 1, \ldots, (6 - m_i)$ are free coefficients, and the $\hat{\$}_{ik}$ are a set of $(6 - m_i)$ independent screws obtained by introducing the solutions of system [6.8] into expression [6.6].

The force systems that all the limbs apply to the end-effector are represented by the following wrench system:

$$\hat{\$}_e = \sum_{i=1,n} \sum_{k=1,(6-m_i)} \gamma_{ik} \hat{\$}_{ik} \qquad [6.10]$$

2 External loads (gravity, inertia forces, etc.) applied to the limb's links are not considered, and all the joints are considered passive in the analyses that bring to identify a limb architecture.

The PM is a PW if and only if the wrench system [6.10] coincides with the third-order screw system, $\hat{\$}_{\mathbf{F}_P}$, that collects all the wrenches that represent a force, $\mathbf{F}_P$, with line of action passing through the center of the spherical motion, P. In order to make $\hat{\$}_e$ coincide with $\hat{\$}_{\mathbf{F}_P}$, the following relationship must hold:

$$\hat{\$}_{ik} = \sum_{s=1,3} \delta_{ik,s} \begin{bmatrix} \mathbf{u}_s \\ (\mathbf{P}-\mathbf{O})\times\mathbf{u}_s \end{bmatrix} \quad i = 1, \ldots, n; \quad k = 1, \ldots, (6 - m_i) \qquad [6.11]$$

where $\delta_{ik,s}$ for $s = 1, \ldots, 3$ are suitable coefficients, and $\mathbf{u}_s$ for $s = 1, 2, 3$ are a set of three mutually orthogonal unit vectors.

In addition to [6.11], if the PW is non-overconstrained[3], the following condition must hold too:

$$\sum_{i=1,n} (6 - m_i) = 3 \qquad [6.12]$$

Provided that the limbs (redundant limbs) with m_i equal to six or greater than six are not considered; relationship [6.12] reveals that the non-overconstrained PWs without redundant limbs can be collected into two sets: (1) the PWs with three limbs with connectivity five, and (2) the PWs with two limbs: one with connectivity four and the other with connectivity five.

Once the connectivity and the topology of the *i*th limb are chosen, the generic expressions of the $\hat{\$}_{ik}$ can be computed by solving system [6.8]. Such expressions will contain the geometric parameters and the joint variables of the limb. And, by imposing that the computed $\hat{\$}_{ik}$ expressions satisfy relationship [6.11], the geometric conditions (i.e. the manufacturing and mounting conditions) that a limb with the chosen topology must satisfy to be used as a PW limb are found.

Such a technique is general and can be used to answer the questions such as "which are the PW limbs with only prismatic and revolute pairs?". This

3 Parallel manipulators whose DoF number is greater than (equal to) the DoF number computed through the Grübler–Kutzbach rule are called overconstrained (non-overconstrained).

and other related problems have been investigated with this technique in [KON 04a].

6.1.2. *Use of the group theory*

6.1.2.1. *Basic concepts*

A group is a set, say {A}, with an associative binary operation, •, so defined that (1) $a_1 \bullet a_2$ always exists and is an element of {A} for any couple, a_1 and a_2, of elements of {A}, (2) the identity element, e, of the operation • is an element of {A} and (3) each $a \in$ {A} has an inverse element $a^{-1} \in$ {A}.

The set of rigid body displacements (motions), {D}, is a six-dimensional group where the associative binary operation, •, is the composition law of two displacements. The generic element of {D} or of one out of its subgroups can be analytically represented by the screw identifying the finite or infinitesimal motion belonging to the subgroup. The dimension of a displacement subgroup is the number of independent scalar parameters that, in the analytic expression of the generic element's screw, must be varied to generate all the screws of the subgroup.

In addition to the identity subgroup, {E}, that corresponds to the absence of motion, {D} contains 10 motion subgroups [HER 78, HER 99] with dimensions greater than zero and less than six:

(1) Subgroups of dimension 1:

(i) *Linear translation subgroup*, {T(**u**)}, that collects all the translations parallel to the unit vector **u**. As many {T(**u**)} as unit vectors, **u**, can be defined. A prismatic pair with sliding direction parallel to **u** physically generates the motions of {T(**u**)}.

(ii) Revolute *subgroup*, {R(P, **u**)}, that collects all the rotations around an axis (rotation axis) passing through point P and parallel to the unit vector **u**. As many {R(P, **u**)} as rotation axes, (P, **u**), can be defined. A revolute pair with rotation axis (P, **u**) physically generates the motions of {R(P, **u**)}.

(iii) *Helical subgroup*, {H(P, **u**, p)}, that collects all the helical motions with axis (P, **u**) and finite pitch p that is different from zero and constant. As many {H(P, **u**, p)} as sets of helix parameters, (P, **u**, p), can be defined. A helical pair (hereafter denoted with H) with helix parameters (P, **u**, p) physically generates the motions of {H(P, **u**, p)}.

(2) Subgroups of dimension 2:

(i) *Planar translation subgroup*, {T($\mathbf{u}_1$, $\mathbf{u}_2$)}, that collects all the translations parallel to a plane perpendicular to $\mathbf{u}_1 \times \mathbf{u}_2$ where $\mathbf{u}_1$ and $\mathbf{u}_2$ are two orthogonal unit vectors. As many {T($\mathbf{u}_1$, $\mathbf{u}_2$)} as unit vectors $\mathbf{u}_1 \times \mathbf{u}_2$ can be defined.

(ii) *Cylindrical subgroup*, {C(P, **u**)}, that collects all the motions obtained by combining a rotation around a rotation axis (P, **u**) and a translation parallel to the unit vector **u**. As many {C(P, **u**)} as (P, **u**) axes can be defined. A cylindrical pair (hereafter denoted with C) with axis (P, **u**) physically generates the motions of {C(P, **u**)}.

(3) Subgroups of dimension 3:

(i) *Spatial translation subgroup*, {T}, that collects all the spatial translations. Only one subgroup {T} can be defined.

(ii) *Spherical subgroup*, {S(P)}, that collects all the spherical motions with center P. As many {S(P)} as P points can be defined. A spherical pair (hereafter denoted with S) with center P physically generates the motions of {S(P)}.

(iii) *Planar subgroup*, {G($\mathbf{u}_1$, $\mathbf{u}_2$)}, that collects all the planar motions with motion plane perpendicular to $\mathbf{u}_1 \times \mathbf{u}_2$ where $\mathbf{u}_1$ and $\mathbf{u}_2$ are two orthogonal unit vectors. As many {G($\mathbf{u}_1$, $\mathbf{u}_2$)} as unit vectors $\mathbf{u}_1 \times \mathbf{u}_2$ can be defined.

(iv) *Pseudo-planar subgroup*, {Y($\mathbf{u}_1$, $\mathbf{u}_2$, p)}, that collects all the motions obtained by combining a planar translation belonging to {T($\mathbf{u}_1$, $\mathbf{u}_2$)} with a helical motion belonging to {H(P, $\mathbf{u}_1 \times \mathbf{u}_2$, p)}.

(4) Subgroups of dimension 4:

(i) *Schoenflies subgroup*, {X($\mathbf{u}_1$, $\mathbf{u}_2$)}, that collects all the motions obtained by combining a planar translation belonging to {T($\mathbf{u}_1$, $\mathbf{u}_2$)} with a cylindrical motion belonging to {C(P, $\mathbf{u}_1 \times \mathbf{u}_2$)}.

A kinematic chain is called a mechanical bond when it connects one rigid body to another so that the relative motion between the two bodies is constrained. A mechanical bond is called mechanical generator when all the allowed relative motions between the two bodies belong to only one of the 10 subgroups of {D}.

6.1.2.2. *Identification of PW limbs*

If the end-effector was connected to the frame only by the *i*th limb, the generic element, l_i, of the set, {L_i}, of the displacements, the end-effector can perform, could always be expressed as follows:

$$l_i = a_{i1} \bullet a_{i2} \bullet \cdots \bullet a_{i(m_i-1)} \bullet a_{im_i} \quad [6.13]$$

where a_{ij} for j = 1, …, m_i is a generic element of the set, $\{A_{ij}\}$, of the displacements that the *j*th joint[4] of the *i*th limb allows.

The existence of relationship [6.13] between the elements of $\{L_i\}$ and the elements of the m_i sets, $\{A_{ij}\}$, can be reminded by introducing the following notation:

$$\{L_i\} = \{A_{i1}\} \bullet \{A_{i2}\} \bullet \cdots \bullet \{A_{i(m_i-1)}\} \bullet \{A_{im_i}\} \quad [6.14]$$

The set, $\{M\}$, of the displacements that the end-effector of a PM with *n* limbs can perform is a connected subset of the intersection of all the $\{L_i\}$ sets, that is:

$$\{M\} \subset \bigcap_{i=1,n} \{L_i\} \quad [6.15]$$

If $\{M\}$ is a subset of a spherical subgroup, $\{S(P)\}$, then the PM is a PW. Therefore, in order that the *i*th limb can be a PW limb, the existence of a connected subset, $\{S_i\}$, of $\{L_i\}$, which is a subset of the spherical subgroup, $\{S(P)\}$, too, is necessary. By imposing this condition, PW limbs can be identified; whereas, by imposing that $\{M\}$ is a subset of $\{S(P)\}$, PW architectures can be identified.

This scheme has been used in [KAR 00, KAR 02, KAR 04, KAR 06] to identify PW limbs and PW architectures.

6.1.3. *Other approaches*

Many other approaches have been used to demonstrate that specific PM topologies could be used as PW architectures. In this section, some of them are presented.

An approach different from the previous approaches has been used in [DIG 01a, DIG 01b]. This approach uses the velocity loop equations. The use of the velocity loop equations consists of exploiting the kinematic properties of the *n* limbs for writing *n* times both the end-effector angular velocity, $\boldsymbol{\omega}$,

4 It has been assumed that the limb's joints are numbered from the frame to the end-effector.

and the velocity, $\mathbf{v}_P$, of the end-effector point, P, which is a candidate for being the center of the end-effector's spherical motion. By doing so, *n* expressions of the couple of vectors ($\boldsymbol{\omega}$, $\mathbf{v}_P$) are obtained where the *i*th expression, $i = 1, \ldots, n$, is a linear combination of the joint rates of the *i*th limb. The analysis of these ($\boldsymbol{\omega}$, $\mathbf{v}_P$) expressions is sufficient to determine the geometric conditions that each limb has to satisfy in order to make (1) all the *n* expressions compatible, and (2) the velocity $\mathbf{v}_P$ equal to zero. Since this approach deduces geometric conditions by analyzing the instantaneous end-effector motion, the characteristics of the finite end-effector motion are stated by demonstrating that those conditions are sufficient to warrant an infinite sequence of instantaneous motion of the same type provided that no singular configuration is encountered.

The use of the velocity loop equations could be used as a general technique to find and/or enumerate PW topologies, but its implementation for this aim is much more complex than the implementation of the methods presented in the previous subsection. On the other hand, it gives many more pieces of information, about the instantaneous behavior of a specific architecture, than other methods, and it is specially recommended to find all the PW singularities since it considers all the joint rates of the PW.

Di Gregorio [DIG 04a] demonstrated that, if three non-aligned points of a rigid body are constrained to move on concentric spheres, the rigid body will be constrained to perform only a spherical motion. This demonstration can be used for finding PW architectures by assembling limbs that constrain an end-effector point to lie on a sphere. In [DIG 04a], it was used to propose the 3-RRS wrist (Figure 6.4).

Once a PW limb has been identified, other PW limbs can be generated. Indeed, by connecting two adjacent links of a PW limb, for instance the links joined by the *j*th joint, through a kinematic chain that, together with the *j*th joint, yields a single loop with one DoF, the added kinematic chain does not modify the type of motion that the end-effector can perform. Moreover, the configuration of the one DoF loop built around the *j*th joint is uniquely determined by the *j*th joint variable, and vice versa, the relative pose between the links is uniquely determined by the configuration of the added kinematic chain. Thus, the added chain is able to keep the mobility constraint, between the two links, due to the *j*th joint and, if the *j*th joint is replaced by that chain, the resulting limb architecture is a new PW limb. This procedure has been used in [KON 04b].

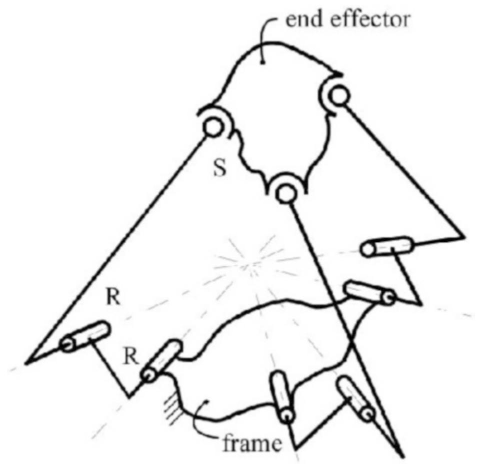

Figure 6.4. *The 3-RRS wrist*

Finally, new PW architectures can be obtained by adding limbs with connectivity 6 to already identified parallel or serial wrist architectures. The added limbs can contain actuated joints, the same as those already present in the original wrist architecture. Therefore, such a practice can be used to redistribute the actuators in a more convenient way. This principle can be seen as the one that generated the S-3UPS wrist (Figure 6.2) from a simple spherical pair that connects the end-effector to the frame.

6.1.4. *Conclusion*

Many architectures for PW limbs have been recently proposed by using either the screw theory or the group theory. Nevertheless, these are not the only tools that identify architectures suitable for being PW limbs. Moreover, once a limb architecture or a PW architecture has been identified, simple reasoning can be used to generate new limb architectures or new PW architectures.

6.2. Singularities and performance indices

Singularities are manipulator configurations where the relationship (input–output instantaneous relationship) between the rates of the actuated joint variables and the characteristic vectors[5] of the end-effector's instantaneous motion fails [GOS 90, MA 91, ZLA 95]. According to the input–output instantaneous relationship [GOS 90], they are of three types: (1) singularities of the inverse kinematic problem, (2) singularities of the direct kinematic problem and (3) singularities both of the inverse and of the direct kinematic problems.

Type 1 singularities occur when at least one of the input variable rates (actuated joint rates) is undetermined even though all the output variable rates (end-effector's motion characteristics $\{\boldsymbol{\omega}, \mathbf{v}_P\}$) are assigned. All the manipulator configurations where the end-effector reaches the border of the workspace are type 1 singularities, and finding type 1 singularities is one way to determine the workspace border. From a static point of view, in type 1 singularities, at least one component of output torque (force), applied to the end-effector, is equilibrated by the manipulator structure without applying any input torque (force) in the actuated joints.

Type 2 singularities occur when at least one component of end-effector's motion characteristics, $\{\boldsymbol{\omega}, \mathbf{v}_P\}$, is undetermined even though all the actuated joint rates are assigned. These singularities may be present only in PMs and fall inside the workspace. From a static point of view, in type 2 singularities, an (finite or infinitesimal) output torque (force), applied to the end-effector, needs at least one infinite input torque (force) in the actuated joints to be equilibrated. Since, long before the input torque (force) becomes infinite, the manipulator breaks down, type 2 singularities must be found during the design phase and avoided during operation.

This singularity classification has been extended in [ZLA 95] by taking into account the rates of the non-actuated joints.

In the literature [DIG 01a, DIG 01b, ZLA 01, DIG 02, ZLA 02, DIG 04b], the possibility of changing the type of motion the end-effector performs, in correspondence of particular type 2 singularities (constraint singularities) has been highlighted. Constraint singularities affect only PMs with lower mobility where the limbs' connectivity is greater than the manipulator's DoF. PWs are particular PMs with lower mobility. Therefore, PWs may have constraint singularities, that is configurations where the end-effector is no longer constrained to perform spherical motions.

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5 The characteristic vectors of the instantaneous motion of a rigid body are the rigid body's angular velocity, $\boldsymbol{\omega}$, and the velocity, $\mathbf{v}_P$, of a rigid body point, P.

This section reviews the main results reported in the literature about the singularity analysis of PWs and addresses the problem of characterizing the kinetostatic performances of PWs.

6.2.1. ***Singularity analysis of PWs***

The singularity analysis is the determination of all the singularities of a manipulator. In the configuration space of a manipulator (joint space or operational space), the geometric locus collecting all the points that identify manipulator singularities is named as the singularity locus.

According to the above-reported definitions, singularities are related to the input–output instantaneous relationship; thus, the implementation of the singularity analysis reduces itself to discuss such a relationship.

6.2.1.1. *Analytical determination of the singularity loci*

The input–output instantaneous relationship of a PM with lower mobility can be deduced by using the velocity loop equations [DIG 01a, DIG 01b, DIG 02]. Such a technique consists of two step: (1) the analytic calculus of a number of different expressions of the end-effector motion characteristics, $\{\boldsymbol{\omega}, \mathbf{v}_P\}$, equal to the limb number, say n (such a calculus considers each limb separately as it acted on the end-effector by itself); and (2) the elimination of all the rates of the non-actuated joint variables from the $6n$ scalar equations obtained in the previous step.

By doing so, the following input–output instantaneous relationship is obtained for a generic three DoF PM:

$$\begin{bmatrix} \mathbf{W} & \mathbf{V} \\ \mathbf{T} & \mathbf{U} \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_P \end{bmatrix} = \begin{bmatrix} \mathbf{H} \\ \mathbf{G} \end{bmatrix} \dot{\mathbf{q}} \qquad [6.16]$$

where $\dot{\mathbf{q}}$ is the time derivative of the three-dimensional vector, $\mathbf{q}$, which collects the three actuated joint variables, q_i for $i = 1, 2, 3$. $\mathbf{H}$, $\mathbf{G}$, $\mathbf{T}$, $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ are 3×3 matrices that depend on the mechanism configuration (i.e. on $\mathbf{q}$). Such matrices can be seen as triplets of three-dimensional column vectors defined by the following formulas ($[\cdot]^T$ stands for transpose of $[\cdot]$):

$$\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3];\ \mathbf{G} = [\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3];\ \mathbf{T} = [\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3]^T \qquad [6.17a]$$

$$\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]^T;\ \mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]^T;\ \mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3]^T \qquad [6.17b]$$

In a PW, if the center of the end-effector spherical motion is chosen as point P, the matrices **T** and **G** become null matrices, and relationship [6.16] becomes [DIG 01a]:

$$\begin{bmatrix} \mathbf{W} & \mathbf{V} \\ \mathbf{0}_{3\times3} & \mathbf{U} \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_P \end{bmatrix} = \begin{bmatrix} \mathbf{H} \\ \mathbf{0}_{3\times3} \end{bmatrix} \dot{\mathbf{q}} \qquad [6.18]$$

where $\mathbf{0}_{3\times3}$ is the 3 × 3 null matrix. The homogeneity of the equations of system [6.18] allows the normalization of the $\mathbf{u}_i$ and $\mathbf{w}_i$ vectors for $i = 1, 2, 3$ (see definition [6.17b] of matrices **U** and **W**). Therefore, without losing generality, it will be assumed that the $\mathbf{u}_i$ and $\mathbf{w}_i$ vectors are unit vectors.

The analysis of relationship [6.18] reveals that the end-effector is constrained to perform an infinitesimal (elementary) spherical motion if and only if matrix **U** is not singular. Indeed, only under this condition, the last three equations of system [6.18], that is:

$$\mathbf{U}\,\mathbf{v}_P = \mathbf{0} \qquad [6.19]$$

have the only solution

$$\mathbf{v}_P = \mathbf{0} \qquad [6.20]$$

that warranties the spherical motion.

Since matrix **U** is singular, if and only if its determinant is equal to zero, the values of **q** that satisfy the following singularity condition

$$\det(\mathbf{U}) = 0 \qquad [6.21]$$

identify PW configurations where the type of motion of the end-effector may be non-spherical. Thus, such configurations are constraint singularities. Equation [6.21] is the analytical expression of the singularity locus that collects all the joint space points[6] that correspond to constraint singularities of the PW.

If condition [6.21] is matched, system [6.19] has an infinite number of solutions for $\mathbf{v}_P$ (i.e. the velocity $\mathbf{v}_P$ is undetermined), even though the

6 The vector of the actuated-joint variables, **q**, is the position vector that locates the points of the joint space.

actuated joint rates, $\dot{\mathbf{q}}$, are assigned. According to the above-reported singularity classification, this kinematic condition makes constraint singularities belong to the set of the type 2 singularities[7].

By using the set theory, it can be demonstrated that constraint singularities may occur only in PM with lower mobility where all the limbs have a connectivity that is greater than the PM DoF (see [DIG 06] for details). Therefore, the PWs that have at least one limb with connectivity equal to three, like (see [GOS 95]) the 3-RRR (Figure 6.1) or the S-3UPS (Figure 6.2), have no constraint singularities.

Another criterion for avoiding constraint singularities is the use of PW architectures that have a constant and non-singular **U** matrix, like the 3-RRS wrist [DIG 04a] (Figure 6.4).

Out of constraint singularities, the input–output instantaneous relationship [6.18] reduces itself to the following relationship[8]:

$$\mathbf{W}\,\boldsymbol{\omega} = \mathbf{H}\,\dot{\mathbf{q}} \qquad [6.22]$$

The analysis of [6.22] brings to the conclusion that: (1) if and only if matrix **W** is singular, the end-effector's angular velocity, $\boldsymbol{\omega}$, is not determined, even though the actuated joint rates, $\dot{\mathbf{q}}$, are assigned (i.e. a type 2 singularity occurs), and (2) if and only if matrix **H** is singular, the actuated joint rates, $\dot{\mathbf{q}}$, are not determined, even though the end-effector's angular velocity, $\boldsymbol{\omega}$, is assigned (i.e. a type 1 singularity occurs).

Because of the indetermination of $\boldsymbol{\omega}$, the type 2 singularities identified by condition (1) are also named rotation singularities [DIG 01a, DIG 01b, DIG 02, DIG 04b].

From an analytic point of view, the singularity condition (1) yields:

$$\det(\mathbf{W}) = 0 \qquad [6.23]$$

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7 Because of the indetermination of $\mathbf{v}_P$, the constraint singularities of PWs are also named translation singularities [DIG 01a, DIG 01b, DIG 02, DIG 04b].

8 Relationship [6.22] had been erroneously considered the complete input–output instantaneous relationship before the presence of constraint singularities was highlighted in the literature.

whereas the singularity condition (2) yields:

$$\det(\mathbf{H}) = 0 \qquad [6.24]$$

Equations [6.23] and [6.24] are the analytical expressions of two singularity loci that collect all the joint space points that correspond, respectively, to the rotation singularities, and to the type 1 singularities.

6.2.1.2. *Geometric interpretation of the singularity conditions*

Definition [6.17b] of matrix **U** allows system [6.19] to be split into the following three scalar equations:

$$\mathbf{u}_i \cdot \mathbf{v}_P = 0 \quad i = 1, 2, 3 \qquad [6.25]$$

From a kinematic point of view, the *i*th equation [6.25] shows that the velocity $\mathbf{v}_P$ has no component parallel to the vector $\mathbf{u}_i$, which, in other words, means that the end-effector point P cannot translate along the direction of the vector $\mathbf{u}_i$. Since the $\mathbf{u}_i$ vectors for $i = 1, 2, 3$ identify three directions along which the translation of P is forbidden, if the $\mathbf{u}_i$ vectors are linearly independent, point P cannot perform any elementary translation (i.e. condition [6.20] holds), otherwise it can translate along directions that are orthogonal to all the $\mathbf{u}_i$ vectors. The $\mathbf{u}_i$ vectors are linearly dependent if and only if they are all parallel to a unique plane. This geometric condition is analytically expressed as follows:

$$\mathbf{u}_1 \cdot \mathbf{u}_2 \times \mathbf{u}_3 = 0 \qquad [6.26]$$

Because of definition [6.17b] of matrix **U**, condition [6.26] coincides with condition [6.21], and it is its geometric counterpart.

Out of the constraint singularities, the end-effector motion is spherical, and system [6.22] is the input–output instantaneous relationship to be considered. Such a system is not singular, if and only if it states a one-to-one relationship between $\boldsymbol{\omega}$ and $\dot{\mathbf{q}}$. Since system [6.22] is linear and homogeneous with respect to $\boldsymbol{\omega}$ and $\dot{\mathbf{q}}$, $\boldsymbol{\omega}$ is determined (i.e. type 2 singularities do not occur), if and only if the only solution of system [6.22] for $\boldsymbol{\omega}$ is the null vector when the actuated joints are locked (i.e. $\dot{\mathbf{q}}$ is equal to zero). On the other hand, $\dot{\mathbf{q}}$ is determined (i.e. type 1 singularities do not occur), if and only if the only solution of system [6.22] for $\dot{\mathbf{q}}$ is the null vector when the end-effector is locked (i.e. $\boldsymbol{\omega}$ is equal to zero).

When the actuated joints are locked, definition [6.17b] of matrix **W** allows system [6.22] to be split into the following three scalar equations:

$$\mathbf{w}_i \cdot \boldsymbol{\omega} = 0 \quad i = 1, 2, 3 \qquad [6.27]$$

From a kinematic point of view, the *i*th equation [6.27] shows that the end-effector's angular velocity $\boldsymbol{\omega}$ has no component parallel to the vector $\mathbf{w}_i$, which, in other words, means that the end-effector cannot rotate around an axis parallel to the vector $\mathbf{w}_i$ and passing through P. Since the $\mathbf{w}_i$ vectors for $i = 1, 2, 3$ identify three directions around which the end-effector rotation is forbidden, if the $\mathbf{w}_i$ vectors are linearly independent, the end-effector cannot perform any elementary rotation (i.e. $\boldsymbol{\omega}$ must be equal to the null vector), otherwise it can rotate around an axis passing through P and orthogonal to all the $\mathbf{w}_i$ vectors. The $\mathbf{w}_i$ vectors are linearly dependent if and only if they are all parallel to a unique plane. This geometric condition is analytically expressed as follows:

$$\mathbf{w}_1 \cdot \mathbf{w}_2 \times \mathbf{w}_3 = 0 \qquad [6.28]$$

Because of definition [6.17b] of matrix **W**, condition [6.28] coincides with condition [6.23], and it is its geometric counterpart.

By using definition [6.17a] of matrix **H**, system [6.22] can be written as follows ($\dot{\mathbf{q}} \equiv [\dot{q}_1, \dot{q}_2, \dot{q}_3]^T$):

$$\mathbf{W}\,\boldsymbol{\omega} = \sum_{i=1,3} \mathbf{h}_i \dot{q}_i \qquad [6.29]$$

Moreover, singularity condition [6.24], which identifies the type 1 singularities, can be rewritten as follows:

$$\mathbf{h}_1 \cdot \mathbf{h}_2 \times \mathbf{h}_3 = 0 \qquad [6.30]$$

Condition [6.30] shows that a type 1 singularity occurs if and only if the three $\mathbf{h}_i$ vectors, for $i = 1, 2, 3$, are all parallel to a unique plane. When this condition occurs, the direction that is normal to all the $\mathbf{h}_i$ vectors is given by the cross product of any couple of non-parallel $\mathbf{h}_i$ vectors, say $\mathbf{n} \equiv \mathbf{h}_1 \times \mathbf{h}_2$. Thus, the dot product of the vector equation [6.29] by $\mathbf{n}$ yields, after rearrangements:

$$\boldsymbol{\omega}^T \mathbf{b} = 0 \qquad [6.31]$$

where

$$\mathbf{b} = \mathbf{W}^{T}\,\mathbf{n} \qquad [6.32]$$

Equation [6.31] shows that, when condition [6.30] is satisfied, the end-effector's angular velocity cannot assume any direction since it must be orthogonal to the vector **b**. The existence of motion limitations on the end-effector identifies a manipulator configuration that is located at the borders of the workspace. Therefore, equation [6.30] is the analytic expression of the workspace borders in the manipulator's configuration space (joint space or operational space).

6.2.2. *Kinetostatic performances*

Every relationship of the instantaneous (first-order) kinematics has a static interpretation that can be determined through the virtual work principle. That is why the static interpretation of the input–output instantaneous relationship of a manipulator, sometimes, is referred to as kinetostatics [ANG 03].

6.2.2.1. *Statics of PWs*

By considering only the PW skeleton (i.e. the PW without generalized torques applied in the active joints), the virtual work principle yields the following relationship:

$$\mathbf{F} \cdot \mathbf{v}_P + \mathbf{M}_P \cdot \boldsymbol{\omega} = 0 \qquad [6.33]$$

where $\mathbf{F}$ and $\mathbf{M}_P$ are resultant force and resultant moment about the pole P, respectively, of an external force system that is applied to the end-effector and is balanced by the reactions that all the limbs[9] exert on the end-effector. $\mathbf{v}_P$ and $\boldsymbol{\omega}$ are any set of end-effector's motion characteristics compatible with the end-effector constraints (virtual motion characteristics).

Since the end-effector's constraints are analytically expressed by the instantaneous input–output relationship, the virtual motion characteristics are any set $\{\boldsymbol{\omega}, \mathbf{v}_P\}$ that satisfies relationship [6.18].

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9 No external load (gravity, inertia forces, etc.) is applied to the limbs' links, all the kinematic pairs are non-actuated (passive) and frictionless.

According to [6.18], $\boldsymbol{\omega}$ is practically free to assume any value, whereas $\mathbf{v}_P$ must follow the vector equation [6.19]. Therefore, equation [6.33] can be satisfied (i.e. the manipulator skeleton is in equilibrium) if and only if $\mathbf{F}$ and $\mathbf{M}_P$ have the following analytic expressions:

$$\mathbf{M}_P = 0 \qquad [6.34a]$$

$$\mathbf{F} = \sum_{i=1,3} \alpha_i \mathbf{u}_i \qquad [6.34b]$$

where α_i, for $i = 1, 2, 3$, are three coefficients whose values must be computed through [6.34b] once the value of $\mathbf{F}$ is assigned.

Expressions [6.34] reveal that the PW skeleton can equilibrate a force system that is equivalent to a unique force with line of action passing through the center, P, of the end-effector spherical motion. Moreover, the detailed static analysis of any real case (see, for instance, [DIG 04b]) reveals that the loads applied on the limbs' links are proportional to the α_i coefficients.

By solving [6.34b] with respect to the α_i coefficients, the following explicit expressions of these coefficients result in:

$$\alpha_i = \frac{\mathbf{F} \cdot \mathbf{u}_j \times \mathbf{u}_k}{\mathbf{u}_i \cdot \mathbf{u}_j \times \mathbf{u}_k} \qquad i, j, k \in \{1, 2, 3 \,|\, i \neq j,\ i \neq k, j \neq k\} \qquad [6.35]$$

Since the absolute value of the denominator of expression [6.35] coincides with the absolute value of det($\mathbf{U}$), the more det($\mathbf{U}$) is near to zero (i.e. the more a PW configuration is near to a constraint singularity), the greater the absolute values of the α_i coefficients are (i.e. the greater the loads on the limbs' links are).

By considering the PW with generalized torques applied in the active joints, the virtual work principle yields the following relationship:

$$\mathbf{F} \cdot \mathbf{v}_P + \mathbf{M}_P \cdot \boldsymbol{\omega} = \boldsymbol{\tau} \cdot \dot{\mathbf{q}} \qquad [6.36]$$

where $\boldsymbol{\tau}$ is a three-dimensional vector collecting the three generalized torques, τ_i for $i = 1, 2, 3$. $\dot{\mathbf{q}}$ is any vector of actuated joint rates compatible with the constraints (virtual rates of the actuated joint variables). $\mathbf{F}$, $\mathbf{M}_P$, $\mathbf{v}_P$ and $\boldsymbol{\omega}$ have the same meaning as they have in [6.33].

By choosing **F** according to [6.34b], the first term at the left-hand side of [6.36] is identically equal to zero. Moreover, out of constraint singularities, the input–output instantaneous relationship reduces itself to vector equation [6.29]. Thus, equation [6.36] is identically satisfied if and only if:

$$\boldsymbol{\tau} = \mathbf{H}^{\mathrm{T}}\,\mathbf{W}^{-\mathrm{T}}\,\mathbf{M}_{\mathrm{P}} \qquad [6.37a]$$

$$\mathbf{F} = \sum_{i=1,3} \alpha_i \mathbf{u}_i \qquad [6.37b]$$

Since the explicit expression of $\mathbf{W}^{-\mathrm{T}}$ is:

$$\mathbf{W}^{-\mathrm{T}} = \frac{1}{\mathbf{w}_1 \cdot \mathbf{w}_2 \times \mathbf{w}_3}\left[\mathbf{w}_2 \times \mathbf{w}_3, \quad \mathbf{w}_3 \times \mathbf{w}_1, \quad \mathbf{w}_1 \times \mathbf{w}_2\right]^{\mathrm{T}} \qquad [6.38]$$

the right-hand side of [6.37a] has a common factor that is the reciprocal of det(**W**). As a result, the more det(**W**) is near to zero (i.e. the more a PW configuration is near to a rotation singularity), the greater the values of the generalized torques, τ_i for i = 1, 2, 3, are.

6.2.2.2. *Performance indices*

Characterizing the kinetostatic performances is important in order to compare different architectures during the design of a new manipulator.

The above-reported static analysis highlights that the force transmission from the end-effector to the links and to the actuators depends on the manipulator configuration. Thus, the indices devised to measure these manipulator performances must be configuration dependent, whereas the overall performances of the architecture can be measured by means of suitable averages on the workspace of the configuration-dependent indices.

In the literature, many indices were proposed to measure the kinetostatic performances of serial manipulators (see [ANG 03] for references), and, successively, adapted to PMs [GOS 91, GOS 95, ZAN 97].

In order to understand the static meaning of the indices used for PWs, the concept of manipulability measure introduced in [YOS 85] will be briefly recalled.

By using relationship [6.37a], a manipulating ability (manipulability) ellipsoid that identifies all the $\mathbf{M}_P$ values corresponding to a unit generalized-torque vector, $\boldsymbol{\tau}$, can be defined as follows:

$$\mathbf{x}^T \mathbf{J} \mathbf{J}^T \mathbf{x} = 1 \qquad [6.39]$$

where

$$\mathbf{x} = \frac{\mathbf{M}_P}{|\boldsymbol{\tau}|} \qquad [6.40a]$$

$$\mathbf{J} = \mathbf{W}^{-1} \mathbf{H} \qquad [6.40b]$$

Since matrix **J** depends on the manipulator configuration, the shape of the ellipsoid [6.39] depends on the manipulator configuration. In particular, the lengths of the semi-axes are equal to the reciprocals of the absolute values of the **J** eigenvalues, the directions of the ellipsoid axes are given by the directions of the **J** eigenvectors, and the volume of the ellipsoid is proportional to the reciprocal of the product of all the **J** eigenvalues, which is equal to det(**J**).

The lengths of the semi-major and of the semi-minor axes of the manipulability ellipsoid give the maximum and the minimum values, respectively, of the ratio (mechanical advantage (MA)) between the magnitudes of $\mathbf{M}_P$ and $\boldsymbol{\tau}$. The manipulability ellipsoid highlights that, in general, the MA depends on the mechanism configuration and on the direction of $\mathbf{M}_P$. The manipulator configurations where the eigenvalues of the **J** matrix are all equal have an MA that does not depend on the direction of $\mathbf{M}_P$. Such configurations are named isotropic [ANG 03], and are characterized by the fact that their manipulability ellipsoid is a sphere. A PW with at least one isotropic configuration is named isotropic.

The distortion of the isotropic condition at a generic PW configuration can be measured by the following index (dexterity)[10]

$$\zeta(\mathbf{J}) = \frac{\lambda_{min}}{\lambda_{max}} \qquad [6.41]$$

10 Definition [6.41] is a particular case of the following more general definition of dexterity [GOS 95] as inverse of the condition number of **J**:

$$\zeta(\mathbf{J}) = \frac{1}{\|\mathbf{J}\|\|\mathbf{J}^{-1}\|} \qquad [6.42]$$

where $\|\cdot\|$ denotes a norm of its matrix argument.

where λ_{min} and λ_{max} are the minimum and the maximum absolute values of the eigenvalues of matrix **J**.

Moreover, at the parity of an ellipsoid shape (i.e. of dexterity), the greater the volume of the manipulability ellipsoid, the higher the MA values in any direction are. Such a kinetostatic property can be measured by the following "manipulability measure" [YOS 85]:

$$\mu(\mathbf{J}) = \sqrt{\det(\mathbf{J}\,\mathbf{J}^{\mathrm{T}})} \qquad [6.43]$$

Since the determinant of the product of two square matrices is equal to the product of the determinants of the two matrices, definition [6.43] states that the "manipulability measure", μ, is equal to the absolute value of det(**J**). And, definition [6.40b] makes it possible to state that det(**J**) is equal to the ratio between det(**H**) and det(**W**). As a result, definition [6.43] can be given in the following alternative way:

$$\mu(\mathbf{J}) = \frac{|\det(\mathbf{H})|}{|\det(\mathbf{W})|} \qquad [6.44]$$

The "manipulability measure", μ, ranges from zero to infinity and is inversely proportional to the ellipsoid volume. Definition [6.44] highlights that μ is equal to zero (i.e. the MA is equal to infinity) where det(**H**) is equal to zero (i.e. at the workspace borders (see singularity condition [6.24])), whereas it is equal to infinity (i.e. the MA is equal to zero) where det(**W**) is equal to zero (i.e. when a rotation singularity occurs (see singularity condition [6.23])). This result brings us to conclude that, from a static point of view, a PW must work far from the workspace borders and the rotation singularities. Finally, it is worth noting that, since the three $\mathbf{w}_i$ vectors (see definitions [6.17b]) are unit vectors, the PW configurations that are the farthest from rotation singularities are characterized by the fact that the absolute value of det(**W**) is equal to one (i.e. the three $\mathbf{w}_i$ vectors are mutually orthogonal (recall that det(**W**) is equal to $\mathbf{w}_1 \cdot \mathbf{w}_2 \times \mathbf{w}_3$).

Both the dexterity and the manipulability measure characterize the force transmission, from the actuators to the end-effector, at a given PW configuration. In order to characterize the kinetostatic properties of a PW architecture, a global dexterity, ζ_{av}, and a global manipulability measure, μ_{av}, can be defined as follows:

$$\zeta_{av} = \frac{\int_Q \zeta(\mathbf{J})\, dQ}{\int_Q dQ} \qquad [6.45]$$

$$\mu_{av} = \frac{\int_Q \mu(\mathbf{J})\, dQ}{\int_Q dQ} \qquad [6.46]$$

where Q denotes the workspace of the PW.

So far, the defined indices take into account only the force transmission between end-effector and actuators (i.e. relationship [6.37a]). The force transmission between the end-effector and the PW skeleton is ruled by relationship [6.37b], which brings to relationship [6.35] for the α_i coefficients. Relationship [6.35] can be alternatively written as follows:

$$\boldsymbol{\alpha} = \mathbf{U}^{-T}\,\mathbf{F} \qquad [6.47]$$

where

$$\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \alpha_3]^T \qquad [6.48]$$

If, as usual [DIG 04b], the α_i coefficients are signed magnitudes of loads applied to the limbs' links, the resting ability (restability) of the PW skeleton can be characterized through the following restability ellipsoid:

$$\mathbf{y}^T\,\mathbf{U}^{-1}\,\mathbf{U}^{-T}\,\mathbf{y} = 1 \qquad [6.49]$$

where

$$\mathbf{y} = \frac{\mathbf{F}}{|\boldsymbol{\alpha}|} \qquad [6.50]$$

Since matrix **U** depends on the PW-skeleton configuration, the shape of the ellipsoid [6.49] depends on the PW-skeleton configuration. In particular, the lengths of the semi-axes are equal to the absolute values of the **U** eigenvalues, the directions of the ellipsoid axes are given by the directions of the **U** eigenvectors and the volume of the ellipsoid is proportional to the product of all the **U** eigenvalues, which is equal to det(**U**).

The lengths of the semi-major and of the semi-minor axes of the restability ellipsoid give the maximum and the minimum values, respectively, of the ratio (passive mechanical advantage (PMA)) between the magnitudes of **F** and α. The restability ellipsoid highlights that, in general, the PMA depends on the PW-skeleton configuration and on the direction of **F**. The PW-skeleton configurations where the eigenvalues of the **U** matrix are all equal have a PMA that does not depend on the direction of **F**. Such configurations uniformly distribute the force **F** among the links of the PW skeleton, and are characterized by the fact that their restability ellipsoid is a sphere.

The distortion of the uniform distribution condition at a generic configuration can be measured by the following redistribution index:

$$\xi(\mathbf{U}) = \frac{\eta_{min}}{\eta_{max}} \qquad [6.51]$$

where η_{min} and η_{max} are the minimum and the maximum absolute values of the eigenvalues of matrix **U**.

Moreover, at the parity of the ellipsoid shape (i.e. of ξ value), the greater the volume of the restability ellipsoid is, the higher the PMA values in any direction are. Such a kinetostatic property can be measured by the following restability index:

$$\nu(\mathbf{U}) = \frac{1}{\sqrt{\det(\mathbf{U}\,\mathbf{U}^T)}} \qquad [6.52]$$

that can be alternatively defined as follows:

$$\nu(\mathbf{U}) = \frac{1}{|\det(\mathbf{U})|} \qquad [6.53]$$

Since the three $\mathbf{u}_i$ vectors (see definitions [6.17b]) are unit vectors, the absolute value of det(**U**) (remind that det(**U**) is equal to $\mathbf{u}_1 \cdot \mathbf{u}_2 \times \mathbf{u}_3$) ranges from 0 at a constraint singularity (i.e. where singularity condition [6.21] is satisfied) to one at a configuration where the three $\mathbf{u}_i$ vectors are mutually orthogonal. As a result, the ν index ranges from one to infinity and is inversely proportional to the ellipsoid volume. This result brings us to conclude that, from a static point of view, a PW skeleton must work far from constraint singularities. Finally, it is worth noting that, the PW configurations that are the farthest from constraint singularities are characterized by the fact that the absolute value of det(**U**) is equal to one.

In order to globally characterize the kinetostatic properties of a PW skeleton, a global redistribution index, ξ_{av}, and a global restability index, ν_{av}, can be defined as follows:

$$\xi_{av} = \frac{\int_{Q} \xi(\mathbf{U})\, dQ}{\int_{Q} dQ} \qquad [6.54]$$

$$\nu_{av} = \frac{\int_{Q} \nu(\mathbf{U})\, dQ}{\int_{Q} dQ} \qquad [6.55]$$

6.2.3. *Conclusion*

Instantaneous kinematics and statics are two sides of the same model: kinetostatics. Every relationship that is stated through the analysis of the instantaneous kinematics of a mechanism has a static meaning, too. Such a meaning can be deduced through the virtual work principle.

The input–output instantaneous relationship of a PM is the reference relationship for the introduction of the concept of singularity and for the classification of the PM singularities.

The input–output instantaneous relationship of a PW can be written in the general case, and the singularity conditions of PWs can be deduced both in analytical and geometrical form. The static meaning of the matrices that appear in the input–output instantaneous relationship is known together with the static interpretation of the singularity conditions.

Eventually, in the literature, kinetostatics has been exploited for proposing indices that characterize PWs’ performances. Here, such indices have been discussed and integrated by proposing new indices that characterize the static efficiency of the PW skeleton.

6.3. New perspectives

Underactuated manipulators are able to position their end-effector in a workspace whose dimensionality[11] is greater than the number of actuators. Such a feature can be obtained by reducing the practicable paths that the end-effector can perform to move between two any end-effector poses of the workspace. Making the manipulator instantaneous DoF lower than its configuration DoF is a necessary condition to obtain a reduction of practicable paths. Since non-holonomic constraints are able to introduce this difference between instantaneous and configuration DoF, their introduction, in a manipulator architecture, is a way to obtain underactuated manipulators.

The roller–sphere contact is a non-holonomic constraint that constrains the sphere to rotate around axes that lie on the plane located by the roller axis and the sphere center. Rotations around axes that lie on a fixed plane are sufficient to make a rigid body assume any orientation. This fact was exploited by Stammers *et al*. [STA 91, STA 92] to conceive an underactuated wrist where two actuated rollers in contact with a sphere controlled the orientation (i.e. three configuration DoF) of an end-effector fixed to the sphere.

Grosch *et al*. [GRO 10] highlighted that, in "ordinary" (i.e. non-underactuated) manipulators, the substitution of passive spherical (S) pairs with as many non-holonomic spherical (nS) pairs yields underactuated manipulators that practically have the same workspace of the ordinary manipulators that have generated them. In [DIG 11], this substitution of S pairs with nS pairs was used to obtain 10 novel types of underactuated PWs from S-3U$\underline{\text{P}}$S[12] wrists (Figure 6.2). The so-deduced underactuated PWs have the same workspace as the S-3U$\underline{\text{P}}$S "ordinary" wrists that generated them, even though their hardware is simpler, but, due to their reduced instantaneous DoF, they cannot perform tracking tasks.

11 In a manipulator, the workspace's dimensionality is the minimum number of geometric parameters necessary to locate the end-effector in the operational space. If the manipulator is not redundant, such a number will coincide with the configuration (or finite) degrees of freedom (DoF) [ANG 03] of the manipulator, which is the minimum number of geometric parameters necessary to uniquely identify the manipulator configuration [ORE 08]. It may be different from the instantaneous DoF, also called velocity DoF [ANG 03], of the same manipulator.

12 The underscore indicates an actuated pair.

The use of nS pairs was first proposed in [BEN 08]. Any S pair can be transformed into an nS pair (Figure 6.5) by introducing, in parallel with the S pair, a sphere–roller contact, whose sphere is a spherical shell, fixed to one of the two joined links and having the center coincident with the center of the S pair, whereas the roller is hinged on the other link (see [DIG 11, GRO 10] for details and constructive schemes). In Figure 6.5, the S pair is constituted by three revolute pairs in series, whose axes intersect one another at the same point; such a point is the center both of the S pair and of the nS pair. Because of the static friction, the sphere–roller contact forbids rotations around axes perpendicular to the plane (axes' plane (AP)), which the sphere center and the roller axis lie on. As a result, the relative motion of the two links joined by an nS pair can only be a rotation around an axis passing through the nS center and lying on the AP.

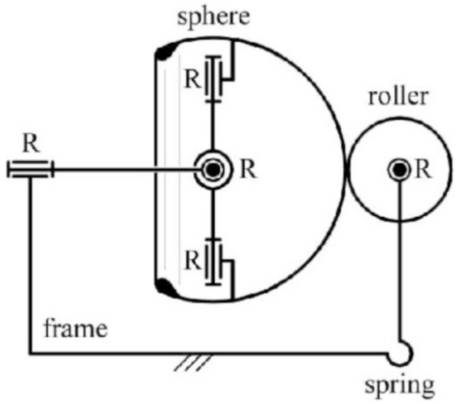

Figure 6.5. *A manufacturing scheme for passive nS pairs, proposed in [DIG 11] (R stands for revolute pair): the nS center is the sphere center, and the axes' plane (AP) of this nS pair is the plane containing the roller axis and the sphere center*

Three of the 10 underactuated PWs proposed in [DIG 11] contain only one nS pair: the (nS)-2SPU (Figure 6.6), the S-(nS)PU-SPU (Figure 6.7) and the S-(nS)PU-2SPU (Figure 6.8) underactuated PWs. These three PWs have been studied in depth [DIG 12a, DIG 12b, DIG 12c]. Such studies showed that their position analyses can be solved with simple formulas and that their kinetostatics is as simple as the kinetostatics of the S-3UPS that generated them. All these features brought us to conclude that they are a valid alternative to "ordinary" PWs when the wrist must not perform tracking tasks.

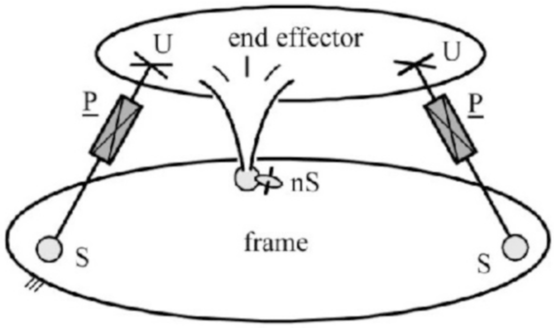

Figure 6.6. *Underactuated parallel wrist of type (nS)-2SPU*

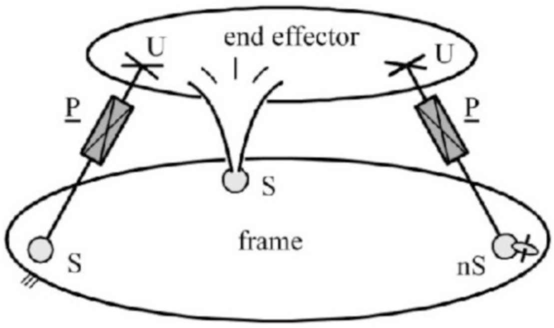

Figure 6.7. *Underactuated parallel wrist of type S-(nS)PU-SPU*

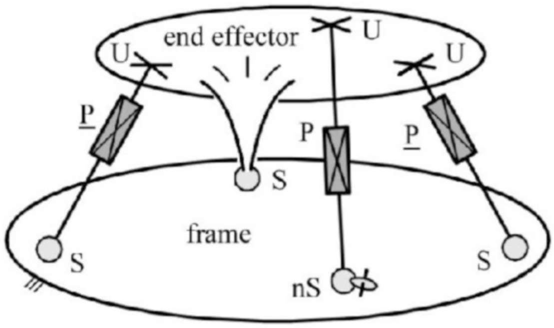

Figure 6.8. *Underactuated parallel wrist of type S-(nS)PU-2SPU*

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