Chapter 10

Bending Vibration of Rod Instrumentation and Measurements

10.1. Introduction

Of the experimental set-ups proposed in the technical literature for the evaluation of the Young's modulus, an apparatus using bending vibration is the easiest to realize. Theories involving the bending vibration of rods to obtain the Young's modulus (and also to obtain some non-diagonal compliance matrix coefficients for anisotropic materials), from the simplest theories (Bernoulli-Euler's) to the most elaborate (Timoshenko's and Mindlin's[1]), are presented elsewhere (see [CHE 10], Chapter 6).

This chapter places its emphasis on practical applications.

10.2. Realization of an elasticimeter

The main objective of an elasticimeter is essentially the measurement of the Young's modulus. We begin this section with a discussion concerning boundary conditions, namely: the free end; additional weight; clamping end and support.

Various boundary conditions were discussed in Chapter 3, from a practical point of view. Figure 10.1 presents the four possible boundary conditions and

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1 Mindlin-Timoshenko's theory is valid up to the second elastodynamic mode. Beyond this mode, the theory is not valid. However, for industrial applications, this is largely sufficient.

Mechanical Characterization of Materials and Wave Dispersion: Instrumentation and Experiment Interpretation
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commentaries about their feasibilities. The mechanical set-up adopted corresponds to free ends with excitation in the middle by an electromechanical shaker (System 3 in Figure 10.1). The advantages of this system are presented below.

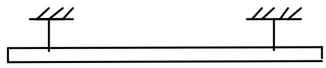
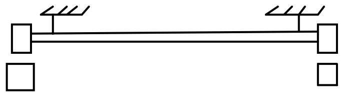
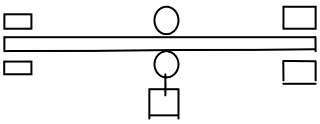
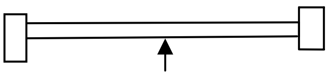
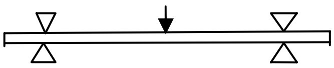
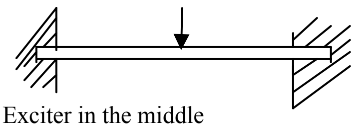
Sample mounting system	**Commentaries**
 1. Wire suspension	- free ends with glued thin metallic blades; - inductive exciter at one end; - contactless displacement transducers.
 2. Wire suspension	- additional weights at the free ends facilitate resonance amplitude visualization and measurements; - inductive exciter at one end; - contactless displacement transducer.
 3. Shaker in the middle	- free ends; - excitation by electromechanical shaker at the middle of the sample; - contactless displacement transducers; - pseudo-clamping between two cylinders.
 4. Exciter in the middle	- additional weights at free ends; - exciter in the middle of the rod; - contactless displacement transducers; - requires more power.
 5. Exciter in the middle	- support at both ends; - difficulty to practically realize these boundary conditions.
 6. Exciter in the middle	- clamping at both ends; - needs length correction which depends on clamping force.

Figure 10.1. *Various boundary conditions at both ends. System (3) is adopted in this chapter*

10.2.1. *Forced vibration of symmetrical free-free rod*

The forced vibration of a symmetrical rod, excited in the middle of the rod, requires less power from an electromechanical shaker compared to System 6 (Figure 10.1) using a double clamping, or System 4 (Figure 10.1) which has additional weights at both ends.

10.2.2. *Pseudo-clamping between two cylinders*

This system, presented in Chapter 3, has the advantage that a sample length correction is not necessary. Taking into account the symmetry of the system and the position of the exciter, the compressive clamping force is reduced to a minimum.

However, a disadvantage exists, when tests are carried out at high temperature and with "soft" materials such as rubber. In such cases, the sample can move by sliding between the two cylinder jaws.

10.2.3. *Free-free rod suspended by yarns*

System 1 (Figure 10.1) is particularly interesting if one wants to study bending wave dispersion in a large range of frequencies. If possible, the rod must be very long (see Chapter 3, Figure 3.14), and the contactless transducer can be replaced by a lightweight accelerometer, whilst a contactless inductive-type transducer is used as the exciter.

10.2.4. *Additional weights at free ends*

As shown in Systems 2 and 4 (Figure 10.1), the two set-ups require length correction. These systems are used when the objective of testing is to work at very low frequency, the additional weights working as a mechanical amplifier for oscillation amplitudes at the free ends.

10.2.5. *Mechanical system*

As for any other elasticimeter, the stand on which the sample holder and the exciter are fixed must be heavy enough to be a motionless base. The stand receives the electromechanical shaker whose moving axis is connected to the set (the frame and sample combined); see Figure 10.2. The sample clamping between two cylinders is described in Chapter 3, section 3.6 and Figure 3.9.

As the shaker's moving axis connecting the moving coil to the sample is long, it is necessary to protect the moving coil by a shaft bearing which is self-lubricated. This precaution avoids any parasitic motion of the sample in torsion and horizontal translation. The use of a translational ball bearing is excluded for the reason that, during alternative motion of the shaft, impacts between balls are a source of noise.

The main role of the shaft bearing is to ensure a vertical alternative motion of the sample holder without backlash.

In order to prevent damage of the moving coil when the electrodynamic shaker is submitted to an excessive voltage or a sudden electric transient signal, it is necessary to limit the stroke of the cylinder which is integral with the moving coil.

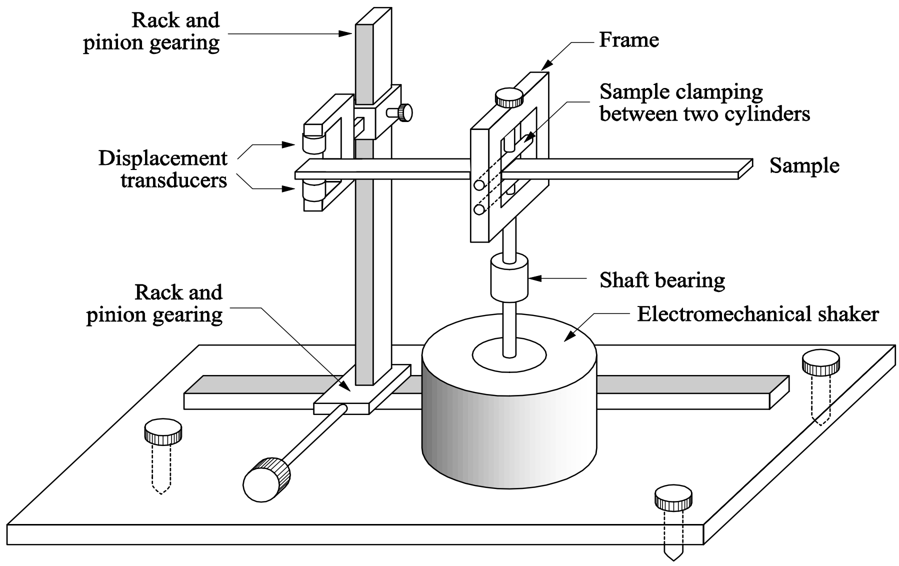


Figure 10.2. *General view of the mechanical set-up for bending tests. The shaft bearing is connected to a heavy cowl , not represented. The cowl serves to guide the vertical motion of the moving coil shaker*

The shaft bearing is connected to a cowl which also supports the shaker (and is not represented in Figure 10.2). When the cowl is fixed on the stand, the only visible parts of the elasticimeter are the sample holder, the transducers and the vertical column which is part of the vertical rack and pinion gearing.

10.2.6. *Electronic set-up*

If we only have to deal with elastic material characterization, then the electronic instrumentation required is simple. Figure 10.3 presents a schematic view. For details about the choice of transducers and exciters, see Chapters 4 and 5.

10.2.7. *Successive operations in sample mounting and measurements*

10.2.7.1. *Fixing the sample in the sample holder*

The excitation adopted in the middle of the sample requires a preliminary positioning of the sample in an exact place between the two cylinders. Successive operations are indicated as follows (Figure 10.4);

– loosen the bolt which applies compressive force to the sample;

– let the sample lie in equilibrium on the lower cylinder, and try to obtain the horizontal equilibrium of the sample whilst, at the same time, making sure that the sample is perpendicular to the cylinder axes. For this purpose, a simple additional system can be used (Figure 10.4). Two thick circular disks are placed at both sides of the sample on either side of the cylinders. The line joining the center of the disks serves as a reference axis in the adjustment of a sample axis.

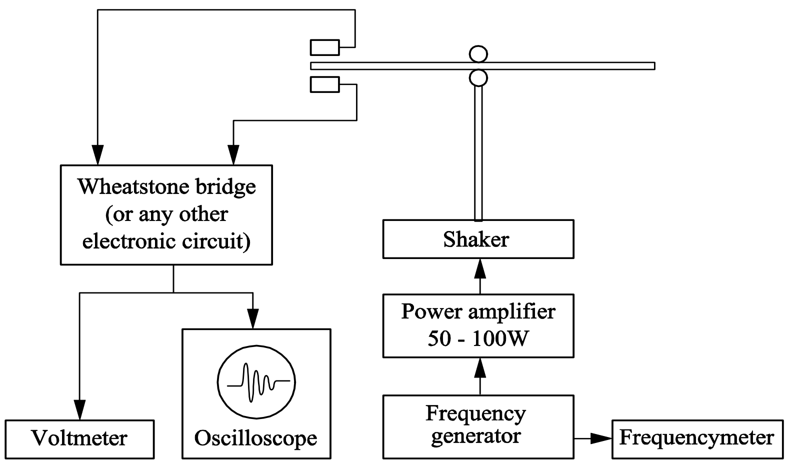


Figure 10.3. *Electronic instrumentation for estimation of Young's modulus by bending test*

Each disk can rotate around its central axis and has two pins. By rotation in one direction or the other and by moving the sample and adjustment of the two sample half lengths so as to obtain sample equilibrium on the lower cylinder. The bolt on the upper side of the frame is then tightened to apply a compressive force on the two cylinders. For this operation the frame integer with the two cylinders must be taken off the shaker axis, see Figure 10.4.

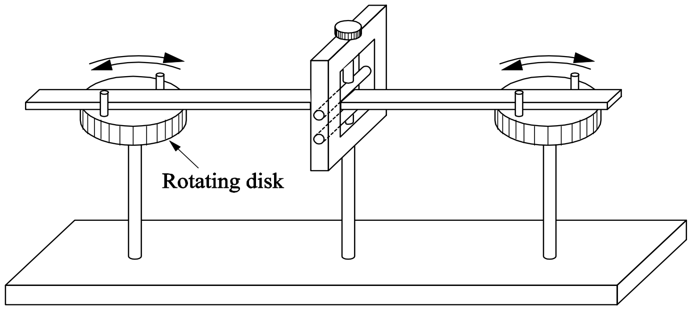


Figure 10.4. *Mechanical system to adjust the middle and perpendicularity of a sample with respect to the cylinder axes. The two circular disks serve to maintain the horizontal position of the sample during adjustment. The two pins on both disks contribute to maintain the sample in a horizontal position*

10.2.7.2. Transducer installation

An adjustment needs to be made to the distance between the sample and the transducers:

– let the sample vibrate at resonance frequency and adjust the output electric voltage of the power amplifier to ensure that the displacement amplitude of the sample is not too large and no contact occurs between the sample and the contactless transducers at resonance frequency;

– the Wheatstone bridge (or any other signal conditioner) equilibrium in gain and phase needs to be adjusted, the two transducers being at an equal distance with respect to the sample.

10.2.8. *Electronic set-up for measurement over a large frequency range*

The electronic set-up presented above requires additional instruments. Figure 10.5 shows three groups of electronic instruments: an electrodynamic exciter and related electronic equipment, and the two pairs of transducers with electronic equipment.

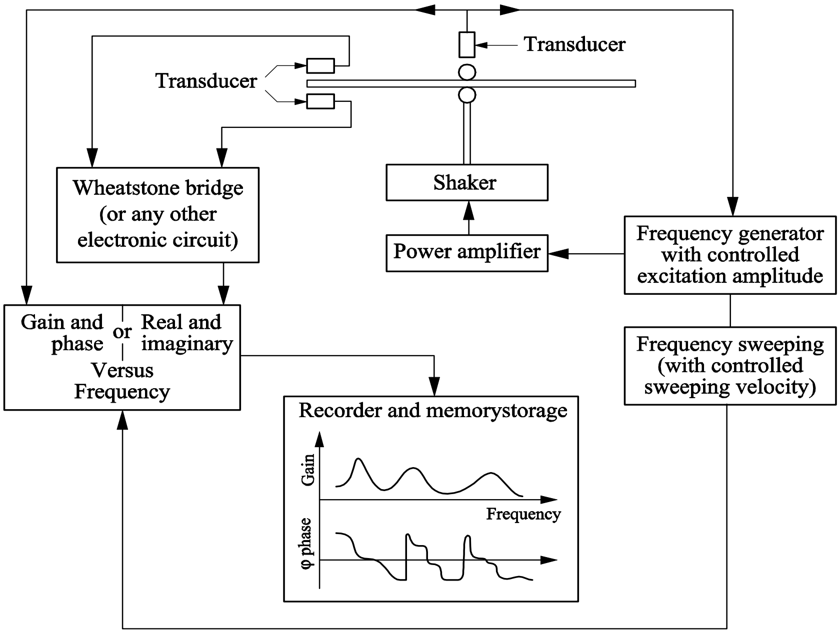


Figure 10.5. *Electronic set-up to obtain the transfer function (gain and phase, or real and imaginary parts, versus frequency). From these graphs, the variation of the complex Young's modulus versus frequency is obtained, after using computing programs*

10.2.8.1. *Power amplifier and electromechanical exciter*

A power amplifier with an available power output of the order of 100 Watts and electrodynamic shaker requiring the same power constitute a reasonable choice.

10.2.8.2. *Variable frequency generator*

The variable frequency generator allows the controlled displacement amplitude of the excitation signal applied to the middle of the sample. This amplitude is maintained at a constant level during the whole frequency sweeping.

The sweeping velocity is defined as the ratio of frequency interval to time interval ($\frac{\Delta f(frequency)}{\Delta t(time)}$) on a linear or logarithmic scale.

10.2.8.3. *Fourier analyzer with transfer function program and recorder*

The sweeping velocity is the most important parameter, and needs to be adjusted with caution and accuracy (see Chapter 6, Figure 6.16).

The lower the damping coefficient of a material, the more often a low sweeping velocity is chosen. If this precaution is not respected, the wrong damping and wrong dynamic curves are obtained. The reason for this is that, in the resonance region and with a varying frequency response, the amplitude oscillates before reaching its equilibrium. The transient regime is, in this case, predominant during the whole sweeping time covering the resonance zone. This phenomenon (to be avoided) is pronounced when the damping coefficient of the material is weak (tan $\leq 10^{-2}$).

10.3. How to conduct bending tests

10.3.1. *Preliminary remarks*

It is important to remember the remarks within the following sections when considering the Bernoulli-Euler and Timoshenko-Mindlin equations of motion.

10.3.1.1. *Inertia effect*

Bernoulli-Euler's equation resorts to elementary theory in which the inertia effect and shear effect are neglected. Consequently, for the first effect, the dimensionless coefficient is useful to evaluate its influence:

$$u_I = \frac{\mathrm{I(cross{-}section\ inertia\ with\ respect\ to\ neutral\ line)}}{\mathrm{SL^2\ (S\ cross{-}section\ area,\ \ L\ sample\ length)}}$$

where S is the cross-section area and 2L the sample length.

If $u_I \ll 1$ for a rectangular cross-section, with width b, thickness h, and length L:

$$u_I = \frac{bh^3}{12(bh)L^2} = \frac{h^2}{12L^2} \ll 1 \qquad [10.1]$$

This ratio is extensively adopted in discussions on Bernoulli's theory, Appendix 10.A.

10.3.1.2. *The shear effect*

The shear effect intervenes in Timoshenko's equation by the ratio:

$$\frac{E_z(\text{ Young's modulus in the direction z axis of the sample})}{G_{xz}(\text{ shear modulus in the plane xz})} \qquad [10.2]$$

This ratio is equal to $2(1+\nu) \approx 2.6$, ν being Poisson's number, for an isotropic material. This parameter enables the validity of Bernoulli's equation to be evaluated at a very low frequency which does not exceed the first eigenmode.

For composite materials, the ratio indicated in [10.2] might exceed 40(!) and the applicability of Bernoulli's equation is controversial, and further calculations are necessary to validate this hypothesis. The shear effect requires the preliminary estimation of the shear modulus G_{xz}. This is possible when using a trial and error method. At the beginning of the procedure neither Young's modulus E_z nor shear modulus G_{xz} are known, z being the rod axis.

10.3.1.3. *Combined shear and inertia effects*

The last term of equation [10.A.2] includes the following coefficient (see [CHE 10], Chapter 6, [6.38]):

$$u_2 = \left(\frac{E_z}{G_{xz}K}\right)(u_1) \qquad [10.3]$$

K takes into account the non-uniform shear in the cross-section. Even for high slenderness ($u_1 << 1$) in equation [10.A.2.] the shear coefficient can counterbalance u_1 in the expression of u_2. This remark should be taken into account for composite materials.

Appendix 10A presents useful equations of motion (Bernoulli-Euler's and Timoshenko-Mindlin's equations) as well as the principal parameters to be taken into account.

10.3.2. *Tests on isotropic materials*

Two tests using bending and torsional vibrations on the same sample permit elastic characterization.

Let us focus our attention on the bending test. The first idea is to adopt Bernoulli-Euler's equation (see Appendix 10.A, [10.A.1]). If validity conditions attached to this equation are satisfied, i.e.:

$$u_1 << 1 \quad \text{and} \quad [E_z/G_{xz}]K = 2(1+\nu)/K \qquad [10.4]$$

then, for a rectangular cross-section K≈ 0.8, the upper bound of [10.4] is:

$$\frac{E_z}{G_{xz}\,K} \leq 2.6\,/0.8 = 3.25 \quad \text{(valid only for isotropic materials)}$$

10.3.2.1. *First case: long sample*

If there is a possibility of choice of the half length, L with L>> b (width), then the coefficient u_l is small. It is reasonable to assume that Bernoulli's equation is applicable on the condition that the first eigenmode is adopted. Then, the corresponding resonance circular frequency (first mode) gives Young's modulus (see Appendix 10A):

$$E= \frac{48\pi^2}{\beta^4}\left[\frac{M_b}{b}\right]\left[\frac{L}{h}\right]^3 .\ f^2 \qquad [10.5]\ [10.A.6a]$$

where the weight of the sample is 2M, its length is 2L, with width b, thickness h, and where β is the first eigenvalue in [10.A.6b], valid for a clamped-free half sample.[2]

Measurement is reduced to the evaluation of resonance frequency, $f = f_r$.

It should be mentioned that the cube of thickness in equation [10.A.4] indicates that Young's modulus is strongly dependent on the accuracy obtained on the sample thickness h. For some artificial composite materials, the accuracy obtained for h rarely exceeds 0.1 mm. This constitutes a serious limitation in the accuracy of Young's modulus evaluation.

10.3.2.2. *Second case: short sample*

In Appendix 10A, the Young's modulus is deduced from Bernoulli's equation (presented above) being known, from which curves $\frac{\beta^4}{\beta_0^4}$ can be drawn with $\frac{E_z}{G_{xz}}$ as a parameter.

Figure 10.6 presents such curves for steel. Depending on the mode rank, the variation of the curve mentioned above is more or less pronounced. If such curves are available, we can see that Timoshenko's correction is necessary to obtain β.

2 Remember that the sample is free at both ends and excited in the middle, which is clamped.

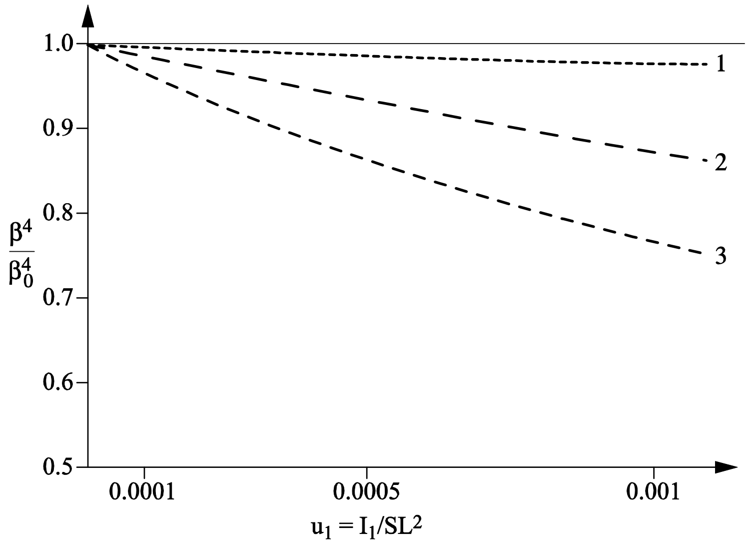


Figure 10.6. *Deviation of β (Timoshenko's equation) with respect to the value β_0 (Bernoulli's equation) versus the slenderness coefficient $u_1=I_1/SL^2$. The material is steel (E_z/G_{xz}= 2.6). The figures 1,2,3 on the curves represent the mode rank*

10.3.3. *Tests on a composite anisotropic material*

The diskrepancy of β obtained by Timoshenko's theory with respect to β_0 obtained by Bernoulli's theory is represented by $(\beta/\beta_0)^4$ versus u_1= I/ SL^2.

This ratio deviates strongly from unity for a higher eigenmode (Figure 10.7).

For a composite material it is possible to use previsional calculus (see [CHE 10]) to obtain the order of elastic moduli whose number depends on the symmetry degree of the material itself. For transverse isotropic materials (long fibers periodically distributed in the matrix) formulae giving elastic moduli versus the fiber volume percentage and the elastic moduli of fiber and matrix are available.

Studies concerning bending wave dispersion over a very large frequency range are not detailed in this chapter.

10.3.3.1. *Slenderness is low*

Figure 10.6 is drawn with dimensionless coordinates. If the slenderness is low, (of the order 10^{-4}), the calculation of Young's modulus using Bernoulli-Euler's equation is acceptable formula [10.A.5]. Eventually we can use the formula [10.A.7] as an improved one.

10.3.3.2. *Slenderness is high*

The curve in Figure 10.7 shows that the eigenvalue β departs from the eigenvalue β_0. In this case, it is necessary to use Timoshenko's equation, [10.A.7] The eigenvalue equation is more complicated and recourse to numerical computation of eigenvalues is necessary (see [CHE 10], Chapter 6, pp. 309). Let us recall that slenderness coefficient concerns inertia effect

10.3.3.3 *Experimental set-up for the study of bending wave dispersion*

The set-up presented in Figures 6.1 to 6.5 is not appropriate for measurements of higher resonance frequencies beyond 10,000 Hertz. [3]

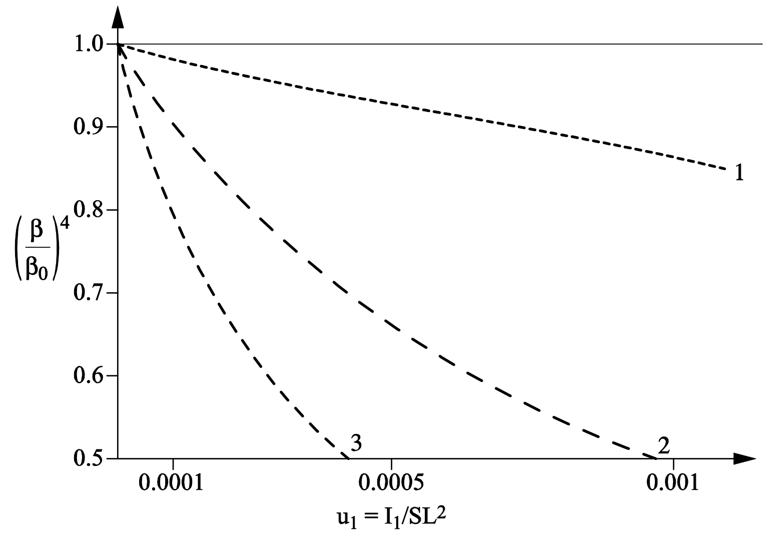


Figure 10.7. *Deviation of eigenvalue β (obtained from Timoshenko's equation) with respect to β_0 (from Bernoulli-Euler's equation) versus the slenderness coefficient $u_1 = I_1/SL^2$. Material: long graphite fiber-epoxy resin composite (E_z/G_{xz}=40). Capital figures 1;2;3 on the curves designate the mode rank*

3 See Chapter 9 [CHE 10] devoted to wave dispersions of the three kinds of wave dispersions. Details of experimental set-ups are given.

10.3.3.4. *Shear effect in bending tests of composite materials*

Shear effect plays a predominant role. Let us recall that it is characterized in Timoshenko's equation by the ratio $E_e/(G_{xz}$ or G_{yz}). This ratio can reach the value of 40 instead of 3 for isotropic materials. Before effecting the bending tests, it is necessary to obtain the scale of sizes of this ratio by using, if possible, previsional calculations for composite materials (see [CHE 10], Chapter 1). This ratio can be high, in this case recourse to Timoshenko's equation is necessary (see [CHE 10, Chapter 6, pp. 309).

10.4. Concluding remarks

10.4.1. *Apparatus*

The mechanical set-up is easy to fabricate. We have to choose the most convenient system. The sample holder system is the most important one to adopt to facilitate testing. The boundary conditions must be chosen with care by the experimenter.

10.4.2. *Samples and useful parameters*

The symmetry consideration for the sample might be helpful and sometimes length correction is not necessary (pseudo-clamping).

Sample parameters to consider include the inertia effect (slenderness) defined as the ratio of thickness to length, and the shear effect represented by the ratio of the axial Young's modulus E_Z to shear modulus (G_{xz} or G_{yz}), should be taken into account.

When one enlarges the frequency range towards a higher frequency, it is important to take into account the best equation of motion (Timoshenko's equation).

10.4.3. *Transducers*

If possible, contactless transducers should be used which do not create a mechanical influence, such as the influence of their own weight at the free ends or the eventual magnetic force (in the case of an inductance transducer with a permanent magnet as its core) which creates additional forces at the sample end which can influence the damping measurement as well as the resonance frequency of the sample.

10.4.4. *Electronic equipment*

When using the automatic sweeping velocity of the generator frequency, the experimenter must take care concerning the recording of the sample response (see Chapter 6) which strongly depends on the damping coefficient of the sample itself.

10.4.5. *Young's modulus evaluation*

The choice of the appropriate equation of motion depends on the range of working frequency on the geometric parameters of the sample and also on the ratio E_z/G_{xz}. Shear modulus plays an important role, particularly for composite materials.

Computer programs are needed to evaluate eigenvalues related to Timoshenko's equation and adopted boundary conditions (see [CHE 10], Chapter 6).

10.4.6. *Wave dispersion*

Systematic wave dispersion (say the wave velocity versus the frequency f or wave number k) study is presented in detail (for isotropic and/or anisotropic materials) in a very large range of frequency (or wave number) in Chapter 15 of this book. Special set-ups (which are different from the one in Figure 10.1) might give interesting information for researchers.

10.5 Bibliography

[CHE 10] CHEVALIER, Y and VINH, J T, (eds), *Mechanics of Viscoelastic Materials and Wave Dispersion*, ISTE, London and John Wiley and Sons, New York, 2010.

10.6. Appendix 10A. Useful formulae to evaluate the Young's modulus by bending vibration of rods[4]

The objective of this Appendix is to gather necessary equations and formulae to help the reader use them to interpret practical experiments.

10A.1. *Bernoulli-Euler's equation*

$$\rho S \frac{\partial^2 w}{\partial t^2} + E_z I \frac{\partial^4 w}{\partial z^4} = 0 \qquad [10A.1]$$

4 Equations of motion are presented without demonstrations; see [CHE 10], Chapter 6.

where ρ , S, E_z and I represent rod density, cross-section area, axial Young's modulus, and the cross-section area quadratic momentum, respectively; w(z,t) and t are the rod displacement, coordinate along rod axis z, and time, respectively.

When using equation [10.A.1], readers must recall that the inertia effect, as well as the shear effect, is neglected. Remember also that the inertia effect is a dynamic effect whose influence is pronounced at higher frequency. The shear effect takes place over the whole range of frequency.

10A.2. *Timoshenko-Mindlin's equation*

$$\left(\rho S\frac{\partial^2 w}{\partial t^2}+E_z I\frac{\partial^4 w}{\partial t^4}\right)+\frac{\rho^2 I}{KG_{xz}}\frac{\partial^4 w}{\partial t^4}-\left(\rho I+\frac{E_z\rho I}{KG_{xz}}\right)\frac{\partial^4 w}{\partial t^2\partial z^2}=0 \qquad [10.A.2]$$

where K is the correcting coefficient depending on the calculation method adopted (whether Timoshenko's static theory of shear, or Mindlin's dynamic asymptotic phase velocity)[5] and G_{xz} is the shear modulus in the plane (xz).

10A.3. *Boundary conditions and wave number equation*

10A.3.1. *Clamped-free rod*

a) Bernoulli-Euler's theory:

$$1+\cosh\beta.\cos\beta=0 \qquad [10A.3]$$

b) Timoshenko-Mindlin's theory: the wave number equation is more complex (remember: eigenvalues β and β_0 concern Bernoulli's and Timoshenko's equations, respectively). [CHE 10] chapter 6.

10A.4. *Important parameters in rod bending vibration*

10A.4.1. *Inertia effect and slenderness factor for a rectangular cross-section*

$$u_1=\frac{\text{I (cross−section area moment with respect to neutral line)}}{SL^2}=\frac{h^2}{12L^2} \qquad [10.A.4]$$

5 Coefficients in equation [10.A.2] differ only by two different considerations. Timoshenko suggested to use shear distribution in the thickness and calculate the coefficient in the neutral plane in static bending. Mindlin proposed a coefficient based on frequency consideration when frequency goes to infinity, the bending wave becomes a Rayleigh's wave whose velocity is known.

10A.4.2. *Shear factor*

$$\text{Ratio}: \frac{Young's\ modulus\ E_z}{Shear\ modulus\ in\ plane (xz)\ G_{xz}} \begin{cases} = 2(1+\nu) \approx 2.6 \text{ (isotropic material)} \\ \cong 10 \text{ to } 50 \text{ (anisotropic material)} \end{cases}$$

The numerical value of this ratio is obtained by previsional calculations applied to composite materials concerning the two moduli E_z and G_{xz}.

10A.5. *Expression of wave number*

The expression of the wave number is deduced from [10.A.1] or [10.A.2] depending on the bending theory adopted (either Bernoulli or Timoshenko).

10A.6. *Young's modulus (Bernoulli's theory)*

$$E_z = \frac{\rho S \omega^2 L^4}{I \beta^4} \qquad [10.A.5]$$

For a rectangular cross-section: $E_z = \frac{48\pi^2}{\beta^4} \left(\frac{M}{b}\right)\left(\frac{L}{h}\right)^3 f^2$ [10.A.6a]

where S=bh (area), $I=bh^3/12$ (inertia moment of the cross-section), ρ=density, M is the rod mass, and β the eigenvalue solution of equation [10.A.3].

$$\beta \in (1.8751, 4.69409, 7.875, 10.996, \text{etc.}) \qquad [10.A.6b]$$

The eigenvalues β different order is related to the set of boundary conditions (say clamping at one side: the middle and free end at the other side) of the adopted mechanical experimental set-up, as shown in Figure 10.1.

10A.7. *Young's modulus (Timoshenko-Mindlin's equation)*

$$E_z = \frac{\omega^2\left(\rho S - \frac{\rho I \beta^2}{L^2}\right)}{\left[\frac{I\beta^4}{L^4} + \frac{\rho I \omega^2 \beta^2}{G_{xz} K \omega^2 \frac{\beta^2}{L^2}}\right]} \qquad [10.A.7]$$

β is given by a more complex eigenvalue equation (see footnote 1 above) and is evaluated numerically by a computer program, based on a trial and error method.