Chapter 7

The Frequency Hilbert Transform and Detection of Hidden Non-linearities in Frequency Responses

7.1. Introduction

The frequency Hilbert transform is a signal processing method which has proved useful at the post processing stage. We will take a practical look at it in this chapter.

When conducting experiments, one records frequency responses from a sample during dynamic tests. There are inevitably some hidden non-linearities in the whole chain of the experimental set-up: in the mechanical system including the sample and the holder, and in the electronic chain of apparatus as well.

Often, non-linear transducer responses are due to the relationship between measured physical parameters and electrical signals (see Chapter 6). The linear domain of the transducer is not respected in some cases, when the sample is submitted to high excitation levels.

Damping measurements of viscoelastic materials are often in themselves deduced from the damping measurements of the whole structure in which the sample constitutes only one part (see Chapter 8).

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The experimental problem seems, at first sight, to be an intractable one which can only be solved by the sagacity of experimenters themselves!

What we present here is certainly not a miracle remedy. Instead, it constitutes one tool amongst others to help experimenters detect hidden non-linearities which are localized in some parts of the sample frequency responses. A new method of detection and linearization is proposed.

7.1.1. *The non-causality principle of the physical system response*

In Chapter 6, devoted to signal processing, the coherence function is a candidate for non-linearity detection. Its use is largely adopted in experimental structural dynamics. Unfortunately, this function only statistically correlates the output signal to the input signal. It is not certain that it constitutes a good and efficient tool to evaluate non-linearities as a priority. There are so many possible symptoms and non-linear behavior of the physical system would be just one of them, ascertained among other causes.

In this respect, the frequency Hilbert transform, which is based on a fundamental mathematical principle governing all physical systems, i.e. governing causality, is an efficient tool to detect non-causality, when a linear transfer function of a sample is used. In other words, it detects the imperfections of the system, including hidden non-linearities.

The time response, i.e. the transient response, deduced from the experimental frequency transfer function of the physical system, allows experimenters to eventually ascertain that a system is non-causal and closely related to any non-linearities of the system.

7.1.2. *Complementary information provided by the frequency Hilbert transform*

In textbooks and scientific papers devoted to viscoelasticity and signal processing, Ferry [FER71], Goyder [GOY 80] and Gross [GRO 53], among others, have shown that, in the frequency domain, if one part of the transfer function is known, the other part, which is not measured, can be supplied by the Hilbert transform.

To take an example, in the viscoelastic behavior of a material in the ultrasonic high frequency range, the wave velocity curve versus frequency is sometimes obtained indirectly via the Hilbert transform.

7.2. Mathematical expression of the Hilbert transform

The frequency Hilbert transform[1] is derived from Cauchy's integral formula involving the function of a complex variable. Mathematically speaking, the French mathematician Cauchy, in the first part of the 19th Century, was the first scholar to study the function of a complex variable, z, being defined as:

$$z = \omega + j\sigma \qquad [7.1]$$

where ω is the circular frequency and σ the damping coefficient.

The two main ideas to be retained for a special class of function of complex variable, z, are presented below.

The analytical (or holomorphous) function in the domain D (see Appendix 7A) is defined in a closed domain D delimited by a contour C:

– if it has definite derivative sin D; and

– if it is single valued in D.

The consequence is that, if the function derivative has only one limit, the Cauchy-Riemann condition is satisfied (see Appendix 7A, equation [7.A.4].

7.2.1. *Choice of contour C*

If the function f(z) has complex poles designated by stars in Figure 7.1. and a real pole on the ω real axis, a contour C constituted by a semi-circle $\mathcal{C}$ completed by a diameter 2R along the ω axis and a small semi-circle of radius γ so that inside the contour there are no singular points such as a simple pole, the function f(z) satisfies the following integral:

$$F(z) = \frac{1}{2j\pi} \oint \frac{f(\alpha)}{(\alpha - z)} d\alpha \qquad [7.2]$$

1 There are two kinds of Hilbert transform: the time Hilbert transform, using an integral with time variable which is considered to be a complex quantity; see [GOL 70] and [OPE 75]. The frequency Hilbert transform uses an integral with a complex frequency variable and is presented above.

If the system is stable, damping σ must be negative. The integral in [7.2] is evaluated along the contour and after calculation, the following limits are obtained:

R →∞ for the large semi-circle $\mathcal{C}$

γ → 0 for the small semi-circle γ

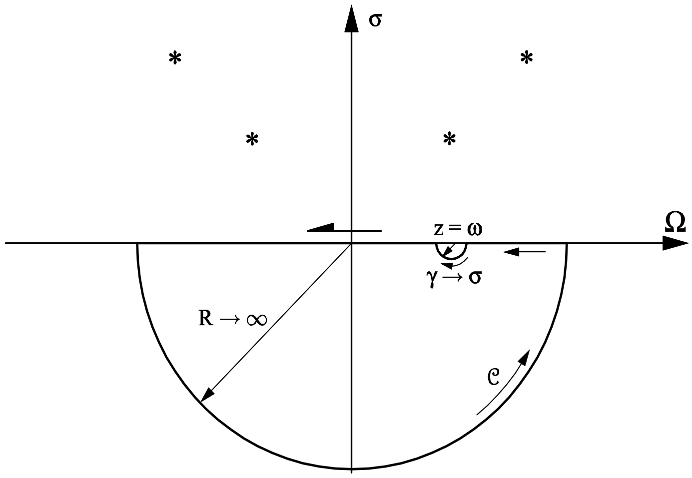


Figure 7.1. *Bromwich-Wagner contour in the complex plane $\alpha = \Omega + j\sigma$ so that the function $f(\alpha)/(\alpha - z)$ is analytic in the domain D defined inside the contour. Stars designate negative damping coefficients. They correspond to the instability of the system and must be excluded*

7.2.2. *Evaluation of integral [7.2.]*

Symbolically, the contour integral in equation [7.2] is decomposed into four integrals (see Figure 7.1):

$$\oint_{C} = \int_{\mathcal{C}} + \int_{+\infty}^{\omega+\epsilon} + \int_{\gamma} + \int_{\omega-\epsilon}^{-\infty} \qquad [7.3]$$

Jordan's lemma says that f(z) = G(z) → 0 when z→ ∞. The boundedness of the transfer function is a physical property of the system G(z) → o. In some circumstances it is not true, as we shall see in this chapter when the inertance transfer function is examined. The third integral evaluated along γ is non-zero. For this purpose, a change of variable is effected:

$$\alpha = \omega + r\,e^{j\theta} \; ; \; d\alpha = r\,.e^{j\theta}.\,jd\theta \qquad [7.4]$$

Bringing [7.4] into [7.A.11] integrating along the contour γ we obtain:

$$\int_{\gamma} = j\pi\, G(\omega) \qquad [7.5]$$

We are able, at this stage, to express [7.A.1] in the following form, where PV designates the principal value of the integral:

$$G(\omega) = \frac{1}{2j\pi}\oint f(\alpha)\frac{d\alpha}{(\alpha - z)} = \frac{1}{2j\pi}\left[-\mathrm{PV}\int_{-\infty}^{+\infty} G(u)du/(u-\omega) + j\pi G(\omega)\right] \qquad [7.6]$$

Inside the bracket, in the second member the integral corresponding in [7.3] to the second and fourth integrals which operate on the real function G(Ω), on the real axis, Ω is the circular frequency variable.

Bringing the non-integral term into the first member:

$$G(\omega) = -\frac{1}{j\pi}\,\mathrm{PV}\int_{-\infty}^{+\infty}\frac{G(u)du}{u-\omega} \qquad [7.7]$$

7.2.3. *Hilbert transform expressed in real and imaginary parts*

Let us express G in Cartesian coordinates:

$$G(\omega) = \mathrm{Re}\, G(\omega) + j\, \mathrm{Im}\, G(\omega) \qquad [7.8]$$

Bringing [7.8] into [7.7] we obtain the two following relationships:

$$\mathrm{Re}\, G(\omega) = \frac{-1}{\pi}\,\mathrm{PV}\int_{-\infty}^{+\infty}\frac{ImG(\omega)}{(u-\omega)}du \qquad [7.9]$$

$$\mathrm{Im}G(\omega) = \frac{1}{\pi}\,\mathrm{PV}\int_{-\infty}^{+\infty}\frac{ReG(\omega)}{(u-\omega)}du \qquad [7.10]$$

[7.9] and [7.10] relate the real and imaginary parts of the transfer function of a linear and stable system.[2]

2 Readers unfamiliar with the function of complex variables can find another demonstration of the Hilbert transform, using decomposition of real function into the sum of impulses, in Appendix 7B.

7.2.4. *Physical interpretation of Hilbert transform*

In this section, various considerations concerning stability, parity, causality and linearity will be raised. These ideas have, of course, been presented and exploited since 1925 in numerous scientific and technical domains. A specialist in viscoelasticity, Ferry [FER 71] presented formulae to deduce one of the parts of the frequency response from the other without mentioning Cauchy's integral. Kramer [KRA 27], as early as 1927, raised this same possibility in the study of dielectrics, and Kronig [KRO 27], raised it with regard to the dispersion of X-rays.

The massive introduction of computers in around 1980 and advances in digital signal processing made a drastic change and now make it possible and easy to perform the calculation of the integrals presented in this section.

7.2.4.1. *Analytic function*

If the system is stable and has a minimum phase (no poles inside the contour of Figure [7.1]), then equations [7.9] and [7.10] tell us that we need not have the complete portrayal of $G(\omega)$ in both real and imaginary parts. The complete characteristics of the system are included either in Re $G(\omega)$ or Im $G(\omega)$. We can deduce one from the other. This property is the consequence of the analytic function $G(\omega)$ in the complex plane.

We know that the real part of the transfer function is an even function and that the imaginary part of the same function is odd.

$$G(\omega) = \text{Even function}(\omega) + j\ \text{Odd function}(\omega) \quad [7.11]$$

If we change the circular frequency ω into $-\omega$, we have:

$$G(-\omega) = \text{Even function}(-\omega) - j\ \text{Odd function}(-\omega) \quad [7.12]$$

Adding these two equations member to member:

$$\text{Re } G(\omega) = \text{Even function} = \frac{1}{2}[G(\omega)+G(-\omega)]$$

This construction is possible by separation of the real and imaginary parts, if the complex function $G(z)$ is analytic in the specified domain D.

7.2.4.2. *Sufficiency and causality*

To discuss causality, we must use the time function which is the inverse Fourier transform of the transfer function. This is the impulse response of the system:

$$g(t) = \int_{-\infty}^{+\infty} G(\omega)e^{j\omega}df \quad \text{where } \omega = 2\pi f$$

$$g(t) = \int_{-\infty}^{+\infty} ReG(\omega)\cos\omega t df - \int_{-\infty}^{+\infty} ImG(\omega)\sin\ \omega t df \qquad [7.13]$$

If the system is initially at rest, g(t) must be zero for negative time $t < 0$. Taking the parity of the real and imaginary parts of $G(\omega)$ into account:

$$0 = \int_{-\infty}^{0-\epsilon} ReG(\omega)\cos(\omega t)\, df - \int_{-\infty}^{0-\varepsilon} ImG(\omega)\sin(\omega t)\, df \ ; \ t < 0 \qquad [7.14]$$

The integrals in the second member of [7.14] must be equal and equation [7.13] is rewritten as:

$$g(t) = 2\int_{0}^{+\infty} ReG(\omega)\cos\omega t\,.df \qquad [7.15]$$

$$\text{or } g(t) = -2\int_{0}^{+\infty} ImG(\omega) sin\omega t\,.df \qquad [7.16]$$

[7.15] and [7.16] show that either the real or imaginary part of $G(\omega)$ suffices to evaluate impulse response.

7.2.4.3. *Linearity test*

The most interesting property of the Hilbert transform is that it constitutes a straightforward method to test the linearity of the system itself.

Theorem I: if the system is linear and stable, its transfer function G(ω) coincides with its Hilbert transform. This is the consequence of equation [7.7].

7.2.4.4. *Comparison between the Hilbert transform and coherence function*

The coherence function is widely used in signal processing (see Chapter 6). As we mentioned in the introduction, it portrays the statistical parenthood between the input and output signals. Non-linearities are only one possibility among other eventualities and no mathematical demonstration can be presented in this respect.

The Hilbert transform is based on the analyticity of the transfer function. The aforementioned theorem is a consequence of this property.

7.2.4.5. *Does the Hilbert transform (equation [7.7]) constitute a sufficient condition for system linearity?*

If equation [7.7] is validated, is the system linear or not? The answer is that it is not possible to assert that equation [7.7] constitutes a sufficient condition.

7.2.4.6. *Is the Hilbert transform a tool to prove whether a system is causal or not?*

Yes. Equation [7.7] gives rise to either a linear system or a non-linear system as well. However, it must be remarked that while the necessary condition has been found, the sufficient condition has not yet been either found or demonstrated. The following remark might be a great help: Methods to exploit a linear system are extensively presented and used in all domains of science i.e. the Fourier transform with one frequency variable, linear convolution to relate input to output signals, Boltzmann's superposition theorem, etc. The Hilbert transform presented above uses one complex frequency variable. What we have presented above is adapted to a linear system and might be considered as a linear filter tools privileging the linear part of the signal. That constitutes a remark and not necessary a demonstration.

Hilbert transforms exist, however, for transfer functions with many complex frequency variables (not used in this book). They are suited to examining non-linear component responses. Used concurrently with appropriate methods to define a signal with linear and non-linear components, for example the Volterra functional series, they constitute a good framework to analyze the whole signal. One uses linear functions (with one time variable) and also non-linear functions (with many time variables). These tools are more adapted to study non-linear systems.

In spite of the lack of sufficient conditions mentioned above, the Hilbert transform presented in this chapter is a new and efficient tool for rheologists to detect non-linearities in sample dynamic responses.

7.3. Kramer-Kronig's relationships

This relationship is well known in publications devoted to viscoelasticity [FER 71]. It is useful when a transfer function is not known in totality. From equations [7.8] and [7.9], and by exploiting the parity property of Re G(ω) and Im G(ω):

$$\mathrm{Re}\, G(\omega) = -\frac{1}{\pi}\, \mathrm{PV}\left[\int_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty}\right] \tag{7.17}$$

$$\mathrm{Re}\, G(\omega) = -\frac{1}{\pi}\, \mathrm{PV}\left[\int_{0}^{+\infty} \frac{ImG(u)}{(u+\omega)}\, du + \int_{0}^{+\infty} \frac{Im\, G(u)}{(u-\omega)}\, du\right]$$

$$\mathrm{ReG}(\omega) = \frac{-1}{\pi} PV \int_0^{+\infty} [\frac{2u}{u^2-\omega^2}] \, ImG(u) \text{ du with variable u=ω} \quad [7.18]$$

In a similar manner, ImG(ω) can be expressed as:

$$\mathrm{ImG}\ (u{=}\omega) = \frac{1}{\pi} \mathrm{PV}\ [- \int_0^{+\infty} \frac{\mathrm{ReG(u)du}}{(u+\omega)} + \int_0^{+\infty} \frac{\mathrm{ReG(u)du}}{(u-\omega)}\ ]$$

$$\mathrm{Im\ G}\ (u = \omega) = \frac{2\omega}{\pi} \mathrm{PV} \int_0^{+\infty} [1/\ (u^2\text{-}\omega^2)]\ \mathrm{Re\ G(u)\ du} \quad [7.19]$$

Equations [7.18] and [7.19] are largely exploited in studies of viscoelasticity and also in electricity [FRO 58].

7.4. Causal signal and Fourier transform

We present here the expression of a causal signal and its Fourier transform. This will serve to us to establish the computer code needed to numerically obtain the Hilbert transform

A signal is causal if, for t < 0, the signal is zero. This signal can physically be considered as the product of a continuous signal x(t) and the unit step Heaviside function u(t):

$$\left.\begin{array}{l} x_{causal}(t) = u(t).x(t) \quad [7.20] \\ u(t) = 0 \quad \text{for } t < 0 \\ u(t) = 1 \quad \text{for } t > 0 \quad [7.21] \end{array}\right\}$$

Let us apply the Fourier transform (symbol $\mathfrak{F}$) to [7.20]. Using "FT" to designate the Fourier transform, and the symbol * to designate a linear convolution:

$$X_c(f) = U(f) * X(f) \quad [7.22]$$

$$X_c(f) = \mathfrak{F}[x_c(t)] ; \quad U(f) = \mathfrak{F}[u(t)] \quad [7.23]$$

$$U(f) = \mathrm{PF}\ (1/2j\ \pi f) + \delta(f)/2 \quad [7.24]$$

PF is a pseudo-function, δ is the Dirac impulse function.

Equality [7.23] becomes

$$X_C(f) = [\mathrm{PF}\ (\frac{1}{2j\pi f}) + \frac{1}{2}\delta(f)] * X(f) \quad [7.25]$$

We recognize the expression of the Hilbert transform in [7.25]. If x(t) is the impulse response of the system, X(f) is the transfer function G(f). [7.25] demonstrates the causal characteristics of the Hilbert transform without recourse to a function with a complex frequency variable.

7.5. Hilbert transform of a truncated transfer function

7.5.1. *How to write Hilbert transform computer code*

In the framework of digital signal processing, it is not theoretically possible to evaluate unbounded integrals (for example, for the Fourier transform). Special precautions are necessary to reduce the discrepancies between the theoretical definition (equation [7.25]) and bounded integrals used in signal processing by using appropriate methods, such as time windows or correction at high frequencies, for the transfer functions to satisfy asymptotic behavior. Such methods are used to overcome inevitable discontinuities in signal processing when one has to deal with bounded samples of a signal, etc.

7.5.2. *Hilbert transform of a truncated transfer function*

The Hilbert transform is an integral whose bounds are -∞ to +∞. This means that the *eventual non-causality detection is not local but concerns the whole frequency range*. This constitutes the first difficulty in writing a computer code.

In practice, this frequency interval is finite and limited to upper and lower bounds for circular frequencies ω_U and ω_L:

$$\omega_L < \omega < \omega_U \qquad [7.26]$$

The Hilbert transform ($\mathscr{H}$) is then broken down into three integrals:

$$\mathscr{H}[G(\omega)] = \int_0^{\omega_L-\varepsilon} + \int_{\omega_L-\varepsilon}^{\omega_U+\varepsilon} + \int_{\omega_U+\varepsilon}^{+\infty} \qquad [7.27]$$

The second integral will be effectively computed in [7.27] but the first and third integrals cannot be evaluated, the transfer function being unknown in the following intervals:

$$0 < \omega < \omega_L \; ; \quad \omega_U < \omega < +\infty \qquad [7.28]$$

The truncating effect, due to the experimental limitation, consequently introduces errors in computation. Two cases can be envisaged:

– the existence of possible resonances in the frequency intervals indicated in [7.28]; errors in computation may be important. To improve the calculation, new transfer functions in a larger frequency interval should be recorded and additional resonance zones included in the transfer function (Figure 7.2);

– no resonance zones are observed outside the frequency interval [7.26]; this case gives rise to correcting terms which are calculated with the help of asymptotic expansions of G(ω) (Figure 7.2).

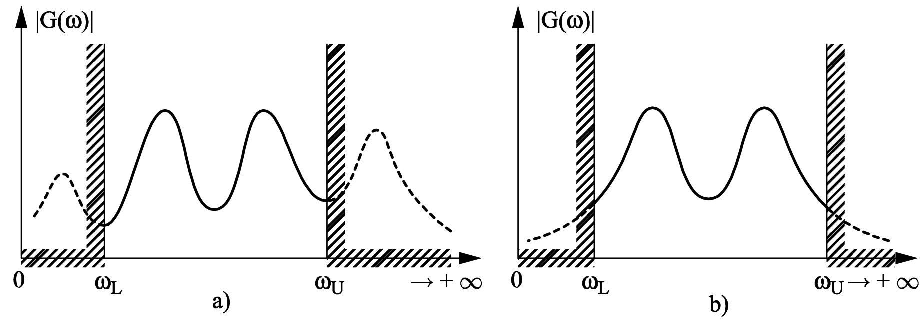


Figure 7.2. *Truncated transfer function. (a) In hatched intervals, there exists resonance zones and large errors in correcting terms; (b) no resonance zones in hatched intervals: possibility to evaluate additional correcting terms*

Two correction methods are presented below. Calculation of the correcting terms is based on the closed form expression of the transfer function using the decomposition of the dynamic response of the system into the sum of modal parameters. The system is supposed to be linear. Such expressions might admit low damping in the Simon's approach.

7.5.3. *Simon's correction [SIM 83]*

In his thesis, Simon [SIM 83] suggested that modal parameters (resonance frequencies and damping coefficients) be evaluated by:

$$G(\omega) = \sum_{r=1}^{N} \left(\frac{j\omega A_r^*}{[\,\omega^2 - \omega_r^2(1+j\delta_r)]} \right) \qquad [7.29]$$

where N is the system degree of freedom, A^*_r the complex number, ω_r the resonance frequency, and δ_r structural damping[3].

For the sake of simplicity, only one mode is examined below:

$$G(\omega) = j\omega\,(\text{Re}\,A_r + j\,\text{Im}A_r) / [(\omega^2 - \omega_r^2(1+j\delta_r)] \qquad [7.30]$$

A generalization to a multimode system will be made in the next stage.

7.5.3.1. *Real part of the Hilbert transform*

When truncation is effected at a frequency ω which is not in the neighborhood of the resonance zone and if the damping is weak, we can write the denominator of [7.30] as:

$$\omega^2 - \omega_r^2(1+j\delta_r) \cong \omega^2 - \omega_r^2 \qquad [7.31]$$

The denominator of [7.30] is then real and we can write, for the real part of the HT (Hilbert transform):

$$\text{Re}\,H(\omega) = -\frac{2}{\pi}\,\text{PV}\left[\int_0^{\omega_L} \frac{u.\text{Im}[G(u)]du}{u^2-\omega^2} + \int_{\omega_L}^{\omega_U} \frac{u.\text{Im}(G(\omega)]du}{u^2-\omega^2} + \int_{\omega_U}^{+\infty} \frac{u.Im(G(u)]du}{u^2-\omega^2}\right] \qquad [7.32]$$

Let us designate $B^R(\omega)$ and $C^R(\omega)$ as the correcting terms corresponding to the first and third integrals in [7.32], respectively:

$$B^R(\omega) = -\frac{2}{\pi}\text{PV}\int_0^{\omega_L} \frac{u.\text{Im}[G(u)]du}{u^2-\omega^2} \qquad [7.33]$$

$$C^R(\omega) = -\frac{2}{\pi}\,\text{PV}\int_{\omega_U}^{+\infty} \frac{u.\text{Im}[G(u)]du}{u^2-\omega^2} \qquad [7.34]$$

[7.33] and [7.34] are easily calculated by introducing [7.30] and [7.31] and using the Naeperian (natural) logarithm:

3 Structural damping is commonly adopted in structural dynamics. However, it creates mathematical difficulties which will be discussed in the next section.

$$B^R(\omega) = -\frac{2}{\pi}\sum_{r=1}^{N} \int_0^{\omega_L} \frac{u^2.ReA_r}{(u^2-\omega^2)(u^2-\omega_r^2)}\,du \qquad [7.35]$$

$$C^R(\omega) = -\frac{2}{\pi}\sum_{r=1}^{N} \int_{\omega_U}^{+\infty} \frac{u^2.ReA_r\,du}{(u^2-\omega_r^2)(u^2-\omega^2)} \qquad [7.36]$$

Adding the two correcting terms:

$$B^R(\omega) + C^R(\omega) = \frac{ReA_r}{\pi(\omega^2-\omega_r^2)}\left[\omega_r \,\mathrm{Ln}\,\frac{(\omega_u+\omega_r)(\omega_r-\omega_L)}{(\omega_u-\omega_r)(\omega_r+\omega_L)} + Ln\,\frac{(\omega+\omega_U)(\omega_r-\omega)}{(\omega-\omega_L)(\omega_r+\omega)}\right] \qquad [7.37]$$

7.5.3.2. *Imaginary part of Hilbert transform*

The two correcting terms are obtained in a similar manner:

$$B^R(\omega) + C^R(\omega) = \sum_{r=1}^{N} \int_0^{\omega_U} \frac{u^2.ImA_r du}{\pi(u^2-\omega_r^2)(u^2-\omega^2)} + \sum_{r=1}^{N} \int_{\omega_U}^{+\infty} \frac{u^2.Im\,A_r du}{(u^2-\omega_r^2)(u^2-\omega^2)} \qquad [7.38]$$

$$B^R(\omega) + C^R(w) = \sum_{r=1}^{N} \frac{ImA_r}{\pi(\omega^2-\omega_r^2)} \left\{ \omega_r Ln\left[\frac{(\omega_u+\omega_r)(\omega_r-\omega_L)}{(\omega_u-\omega_r)(\omega_r+\omega_L)}\right] + \omega Ln\left[\frac{(\omega+\omega_L)(\omega_u-\omega)}{(\omega-\omega_L)(\omega_u+\omega)}\right] \right\} \qquad [7.39]$$

When many modes are present in the interval, summation is to be made in [7.38] and [7.39], with subscript r varying from 1 to N.

7.5.4. *Fei's correction*

Fei, in his thesis [FEI 83], suggested a correction method using an upper bound truncation and asymptotic behavior of the transfer function at higher frequency. The method is valid for viscous damping and gives rise to correcting terms which depend on the nature of the transfer function itself (receptance, mobility, inertance[4]).

4 Receptance = displacement/force; mobility = velocity/force; inertance = acceleration/force. In dynamic tests, one can use G(ω) = displacement/displacement (or another parameter of the same nature).

7.5.4.1. *Receptance correction*

Remember that receptance, by definition, is the ratio of displacement to force:

$$G(\omega)=\sum_{r=1}^{N}\frac{A_r}{\omega^2-\omega_r^2-j\omega\omega_r Z_r} \qquad [7.40]$$

where A_r is the real modal constant, ω_r the eigencircular frequency, and Z_r the modal damping.

If damping is supposed to be weak and the truncation upper frequency different from the resonance frequency ω_r then [7.40] can be simplified to:

$$G(\omega)=\sum_{r=1}^{N}\frac{A_r}{\omega^2}\,; \quad \omega>\omega_r \qquad [7.41]$$

7.5.4.1.1. Correction terms for imaginary part of Hilbert transform

$$\mathrm{Im}[H(\omega)]=\mathscr{H}\,[\mathrm{Re}\,G(\omega)]=\frac{2\omega}{\pi}\int_0^{\omega_U}\frac{\mathrm{Re}(G(u)du}{u^2-\omega^2}+\frac{2\omega}{\pi}\int_{\omega_U}^{\infty}\frac{\mathrm{Re}[G(u)]du}{u^2-\omega^2} \qquad [7.42]$$

Taking into account the approximation in [7.41]:

$$C^{\mathrm{I}}(\omega)=\frac{2\omega}{\pi}\int_0^{\infty}\frac{\sum_{r=1}^{N}A_r du}{u^2(u^2-\omega^2)} \qquad [7.43]$$

$$C^{\mathrm{I}}(\omega)=\frac{2}{\pi\omega}\left(\sum_{r=1}^{N}A_r\right)\left[\frac{1}{2\omega}\mathrm{Ln}\frac{\omega_u+\omega}{\omega_u-\omega}-\frac{1}{\omega_u}\right] \qquad [7.44]$$

In [7.44], we can write:

$$\mathrm{Re}\,G(\omega)=\sum_{r=1}^{N}\frac{A_r}{\omega_U^2}$$

$$\text{Then:}\sum_{r=1}^{N}A_r=\omega_u^2\mathrm{ReG}(\omega) \qquad [7.45]$$

Bringing [7.45] into [7.44]:

$$C^{\mathrm{I}}(\omega)=\frac{-2\omega_U^2}{\pi\omega}\mathrm{ReG}(\omega)\left[\frac{1}{2\omega}\mathrm{Ln}\frac{\omega_U+\omega}{\omega_U-\omega}-\frac{1}{\omega_U}\right] \qquad [7.46]$$

7.5.4.2. *Mobility correction*

By definition, mobility is the ratio of velocity to force. The transfer function is:

$$G(\omega) = \sum_{r=1}^{N} A_r \frac{j\omega}{[(\omega^2-\omega_r^2)-j2\omega\omega_r Z_r]} \; [\qquad [7.47]$$

with $\omega_r < \omega_U < \omega$

The following approximation is adopted:

$$G(\omega) \cong \left(\sum_{r=1}^{N} A_r\right)\frac{j}{\omega} = A\,\frac{j}{\omega} \qquad [7.48]$$

with $A=\sum_{r=1}^{N} A_r$

7.5.4.2.1. Real part of Hilbert transform

$$C^R(\omega) = -\frac{2}{\pi}\int_{\omega_u}^{+\infty}\frac{A\,du}{(u^2-\omega^2)}$$

$$C^R(\omega) = \frac{A}{\pi\omega}\mathrm{Ln}\left[\frac{\omega_u+\omega}{\omega_u-\omega_r}\right] \; [\qquad [7.49]$$

As approximation [7.48] is adopted

$$\mathrm{Im}\,G(\omega) \cong \sum_{r=1}^{N}\frac{A_r}{\omega_u} = \frac{A}{\omega_u} \qquad [7.50]$$

Then, $\sum A_r = A = \omega_u \,\mathrm{Im}\,G(\omega_u)$

$$C^R(\omega) = \frac{\omega_u}{\pi\omega}\mathrm{Im}G(\omega_u)\,.\,\mathrm{Ln}\left(\frac{\omega_u+\omega}{\omega_u-\omega}\right) \qquad [7.51]$$

with $\omega \neq 0$.

7.5.4.3. *Inertance correction*

Inertance is defined as the ratio of acceleration to force.

$$G(\omega) = -\omega^2 \sum_{r=1}^{N} A_r \frac{\omega^2-\omega_r^2+2jZ_r\omega_r\omega}{(\omega^2-\omega_r^2)^2+4Z_r^2\omega^2\omega_r^2} \qquad [7.52]$$

Notice that

$$\text{Lim } G(\omega) = G(\infty) = -\sum_{r=1}^{N} A_r \neq 0 \qquad [7.53]$$

where $\omega \to \infty$ ∞.

This equation shows that this asymptotic behavior must be accounted for in calculating the correction term, as we shall see below.

7.5.4.3.1. Truncation correction

If $\omega > \omega_u$:

$$G(\omega) \cong -\sum_{r=1}^{N} A_r = \text{A} \qquad [7.54]$$

for $\omega > \omega_u > \omega_r$ (resonance).

A correction is made for the imaginary part of the Hilbert transform:

$$C^{I}(\omega) = -A\frac{j}{\pi}\int_{\omega_u}^{+\infty}\left(\frac{1}{u-\omega_U} - \frac{1}{u+\omega_U}\right)du \qquad [7.55]$$

where ω = u.

Integration of this equation is easy:

$$C_I(\omega) = A\,\frac{j}{\pi}\text{Ln}\left(\frac{\omega_{u}+\omega}{\omega_u-\omega}\right) \qquad [7.56]$$

7.5.4.3.2. Correction due to asymptotic behavior of the inertance

$$G(\omega) = \frac{1}{j\pi}\,\text{PV}\int_{-\infty}^{+\infty}\frac{G(u)du}{(\omega-u)} + \frac{1}{j\pi}\oint\frac{G(u)du}{(\omega-u)} \qquad [7.57]$$

The first term corresponds to the line integral along the real axis. The second term corresponds to integration along a circle of radius R which tends to infinity by Jordan's Lemma.

Let us change Cartesian variables into polar complex variables:

$$\lim \oint_C \frac{G(u)du}{(\omega-u)}\,du = \int_0^{-\pi}\frac{-ARe^{-j\sigma}jd\sigma}{Re^{j\sigma}} = jA\pi \qquad [7.58]$$

R→ ∞

$$G(\omega) = \frac{1}{j\pi} \mathrm{PV} \int_{-\infty}^{+\infty} \frac{G(u)du}{(\omega-u)du} + \mathrm{A} \quad [7.59]$$

where frequency ω tends to zero, and PV designates the principal value.

$$\frac{1}{j\pi} \mathrm{PV} \int_{-\infty}^{+\infty} \frac{G(u)du}{\omega-u} = G(0) - A \quad [7.60]$$

The Hilbert transform takes the value:

$$H(\omega = 0) = \frac{1}{j\pi} \mathrm{PV} \int_{-\omega_u}^{+\omega_u} \frac{G(u)du}{\omega-u} \quad [7.61]$$

Then

$$A = G(0) - H(0) = -H(0) \quad [7.62]$$

Adding the two corrections presented in [7.56] and [7.62]:

$$C^{l}(\omega) = -\left[H(0) + j\frac{H(0)}{\pi} \mathrm{Ln}\left(\frac{\omega+\omega_u}{\omega_u-\omega}\right) \right] \quad [7.63]$$

Table 7.1 summarizes the main characteristics of the two correcting methods.

	Receptance	Mobility	Inertance
Simon's correction	Only mobility is considered. Modal parameters are to be evaluated first. Weak structural damping		
Fei's correction	One correction term for upper correction	One correction term for upper correction	One correction term for upper correction. Correction due to the behavior of inertance at infinite frequency

Table 7.1. *Comparison between Simon and Fei's correction methods*

7.6. Impulse response of a system. Non-causality due to measurement defects

Hilbert transform, as we have seen, expresses causal properties of a physical system. It is surprising, *a priori*, that HT can detect in many cases non-causality in the system response which can be either linear or non-linear. The detected non-causality of the system response is due to the defects attributed to the measurement techniques themselves, being either mechanical or electrical.

7.6.1. *Transfer function of a linear system*

Let x(t), y(t) be input and output signals. Let capital letters designate the Fourier transforms:

$$\mathfrak{F}\,[x(t)] = X(f)$$

$$\mathfrak{F}\,[y(t)] = Y(f) \qquad [7.64]$$

The transfer function is defined, as the star designates the conjugate quantity:

$$TF(f) = \frac{Y(f)}{X(f)} = \frac{Y(f).X^*(f)}{X(f)X^*(f)} \qquad [7.65]$$

$$TF(f) = \frac{S_{yx}(f)}{S_{xx}(f)} = G(f) \qquad [7.66]$$

We know that linear system obeys Boltzmann's superposition principle:

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \qquad [7.67]$$

Applying the Fourier transform to [7.66], we obtain:

$$\frac{Y(f)}{X(f)} = H(f) = G(f) \qquad [7.68]$$

Taking the inverse Fourier transform of [7.67] we obtain the impulse response:

$$h(t) = \mathfrak{F}^{-1}[H(f)] = \mathfrak{F}^{-1}[G(f)] \qquad [7.69]$$

h(t) must satisfy the causality principle applicable to any realizable system.

$$h(t) \equiv 0 \qquad \text{for} \quad t < 0 \qquad [7.70]$$

$$h(t) \neq 0 \qquad \text{for} \quad t > 0 \qquad [7.71]$$

Application of inverse Fourier's transform to [7.71] hypothesizes that the system is linear. Unfortunately, the system submitted to a dynamic test might have a non-linear behavior. The non-linearities are hidden and the output signal y(t) is a sum of linear and non-linear time responses:

$$\underset{\text{total}}{y(t)} = \underset{\text{linear}}{y_l(t)} + \underset{\text{all non-linear terms}}{y_{NL}(t)} \qquad [7.72]$$

In [7.64] and [7.65], y(t) and Y(f) is the Fourier transform concerning the global response of the system. The Fourier transform is applied to non-linear time responses as well.

If there is no attempt to extract linear term y_l (t) from the total response, there is an error in measurement. For simplicity, let us hypothesize that the second order time response is described by[5]:

$$y_{NL}(t) \approx y_2(t) = \int_{-\infty}^{+\infty} [\iint_{-\infty}^{+\infty} h_2(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2] e^{-j\omega t} dt \qquad [7.73]$$

It is the second order Volterra functionals which extend Boltzmann's principle to a second order non-linear system.

Applying a one-dimensional Fourier transform to [7.73]:

$$\mathfrak{F}_1 [y_2(t)] = \int_{-\infty}^{+\infty} [\iint_{-\infty}^{+\infty} h_2(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2] e^{-j\omega t} dt$$

$$Y_2^{(1)}(f) = \int_{-\infty}^{+\infty} H_2(f_1, f - f_1) X(f_1) X(f - f_1)\, df_1 \qquad [7.74]$$

The superscript between brackets with number 1 designates the one-dimensional Fourier transform[6].

5 Volterra functional series makes it possible to express the non-linear responses as a sum of functionals of various orders under the form of multiple integrals with many time variables.
6 For the demonstration see Bendat [BEN 90].

The inverse Fourier transform from [7.67] is:

$$h(t) = h_1(t) + \mathcal{F}^{-1}\left[\, Y_2^{(1)}(f) \,/\, X(f)\,\right] + \text{other terms} \qquad [7.75]$$

total linear non-linear

In the second member, extraneous terms are the errors in computation including non-causality of the total impulse response.

It is consequently interesting to compute the impulse response from the transfer function. If for negative time $t < 0$ and the impulse response is non-zero, we shall try to exploit this characteristic in the following sections.

7.7. Summary of principal result in sections 7.5 and 7.6

– Hilbert transform (HT) derived from Cauchy's integrals makes it possible to evaluate the real part of HT of transfer function $G(\omega)$ from the imaginary part of $G(\omega)$ and imaginary part of HT applied to the real part of $G(\omega)$.

– Comparison between Hilbert transform ($\mathcal{H}$) and the transfer function is interesting:

$\mathrm{Re}[\mathcal{H}\{\, \mathrm{Im}\, G(\omega)\,\}]$ and $\mathrm{Re}\, G(\omega)$

$\mathrm{Im}[\mathcal{H}\,\{\, \mathrm{Re}\, G(\omega)\}]$ and $\mathrm{Im}\, G(\omega)$

The comparison makes it possible to detect some hidden non-linearities of the system or defects of measurement set-up.

– Inverse Fourier transform of Hilbert transform:

$$\mathcal{F}^{-1}[\, \mathcal{H}\,[G(\omega)\,]\ \text{gives}\ h_e(t) \qquad [7.76]$$

which is to be compared with the first order impulse response of the system:

$$\mathcal{F}^{-1}[\, G(\omega)\,] = h(t) \qquad [7.77]$$

If [7.76] and [7.77] are identical, we can ascertain that the system is linear.

If $h_e(t) \neq h(t)$, and both contain a non-causal part, in the next sections we shall present some methods to improve transfer function by modifying impulse responses $h_s(t)$ and $h(t)$ to linearize the system or to improve the measurements.

7.8. Causalized Hilbert transform[7]

If in the time domain we set to zero the non-causal part of h(t), we create a new system that is physically significant, the modified impulse response $h_c(t)$ is "causalized".

Let g(t) be the inverse Fourier transform of the transfer function:

$$g(t) = \mathcal{F}^{-1}[G(f)] \tag{7.78}$$

g(t) includes a non-causal part:

$$g(t) \neq 0 \text{ for } t < 0 \tag{7.79}$$

We proceed to a "*causalization*" of g(t) by multiplying with unit step function (which is identically zero for negative time):

$$G_c(t) = g(t) \,.\, U(t) = h_c(t) \tag{7.80}$$

Taking now the direct Fourier transform of $h_c(t)$:

$$\mathcal{F}[\,h_c(t)\,] = H_c(f) \tag{7.81}$$

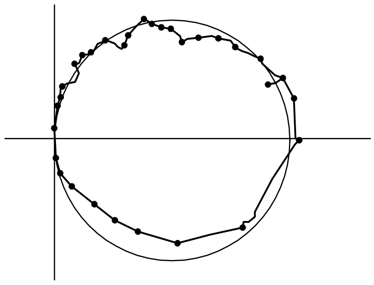
$H_c(f)$ possesses analyticity properties in the frequency complex domain $H_c(\omega)$.

To illustrate the interest of $h_c(t)$ and $H_c(f)$, let us examine Figure 7.3 concerning a one-degree-of-freedom system with a back-lash. The Nyquist plots present respectively the frequency response G(f), the extended Hilbert transform and the causalized Hilbert transform. We notice that the Nyquist (Argand) plot of the last one almost approaches a circle; see Appendix 7B.

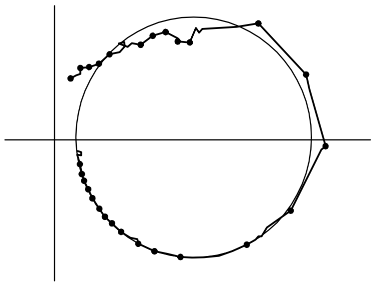
What we have presented above might present a practical interest in viscoelasticity with hidden non-linearities. We shall return to this problem when dealing with the practical problem of viscoelasticity in Chapter 8.

7 We suggest this vocabulary to designate a mathematical method based on Hilbert transform to suppress the non-causal part of the linear component.

7.9. Some practical aspects of Hilbert transform computation



a) Transfer function G(f) . . .



b) Causalised Hilbert tr. $H_c(f)$ . . .

Figure 7.3. *One-degree-of-freedom system. Non-linear system with backlash: (a) measured transfer function G(f); (b) causalized Hilbert transform*

Succinctly we will present some practical aspects concerning this transform in the framework of signal processing. In the preceding sections we have presented the computation of Hilbert transform in the frequency domain. The inverse Fourier transform applied to transfer function $G(\omega)$ enables us to obtain the impulse response h(t). It gives rise to classical programs which can be found in any Fourier analyzer or computer codes.

However, examination of h(t) for negative time ($t < 0$) is particularly interesting. As we have presented above, suppressing the non-causal part of h(t) is the way to obtain causalized impulse response $h_c(t)$. This operation does not consist of setting to zero for t< 0. In signal processing, it introduces *sharp discontinuities* in h(t) and there are problems when we return to the frequency domain by direct Fourier transform. This problem is very well known when we have to deal with a signal record sample with discontinuities at both ends. A time window is then necessary.

7.9.1. *Hilbert time window [FEI 83]*

Without going into detail, we will mention the main ideas and successive steps to solve practical problems of signal processing. The bibliography at the end of this chapter presents principal contributions in this field by researchers in our laboratory in France and those who have been working with Tomlinson's team in England since 1981.

7.9.1.1. *Ideal time window*

For the moment let us examine the continuous time signal corresponding to what we have presented above.

Suppressing the non-causal part of h(t) signifies that we multiply the original signal with the function:

$$2U(t) - 1 = f(t) \quad [7.82]$$

$$h_i(t) = h(t)\,[2U(t) - 1] \quad [7.83]$$

U(t) designates the Heaviside step function:

$$U(t) = 1 \qquad \text{for } t > 0$$

$$U(t) = 0 \qquad \text{for } t < 0 \quad [7.84]$$

The Fourier transform of [7.82] is:

$$H_i(f) = G(f) * \mathrm{Pf}\,\frac{1}{j\pi f} \quad [7.85a]$$

$$\text{and } \mathrm{Pf}\left(\frac{1}{j\pi f}\right) = \mathfrak{F}\,[2U(t) - 1] \quad [7.85b]$$

$*$ designates a convolution, Pf designates a pseudo-function, f frequency, G(f) the transfer function of the system.

7.9.1.2. *Hilbert time window in the framework of digital signal processing*

Practically, the signal is digitized and decomposed into a collection of samples. Figure 7.5 shows, in the time domain, the representation of the ideal Hilbert time window.

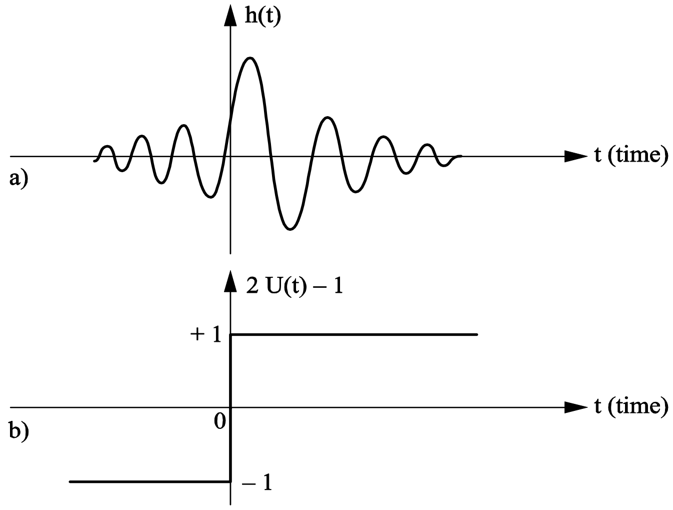


Figure 7.4. *(a) Impulse function h(t) with non-causal part (t*< 0); *(b) ideal Hilbert time window for continuous signal*

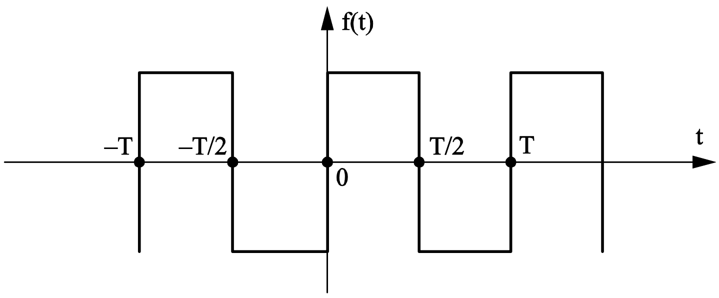


Figure 7.5. *In digital signal processing, the Hilbert time window is rendered artificially periodic by considering a collection of samples. Each sample has a length T*

Figure 7.5 shows that signal presented in [7.4b] is transformed into periodic square signal:

$$f(t) = \begin{cases} 1 \quad \text{with} \quad kT \le t \ \le kT + \dfrac{T}{2} \\ 0 \quad \text{with} \quad t = kT, (k+1)t/2 \\ -1 \ \text{with} \quad kT\text{-}T/2 \le t \le kT \end{cases} \qquad [7.86]$$

The Fourier transform of f(t) is shown to be:

$$F(k) = -j \cot \frac{\pi k}{1024} \text{ (with k odd)} \qquad [7.87]$$

where k is the discretized frequency variable. 1024 represents the maximum value of points adopted for calculation in the binary system 2^n with integer n.

Clearly this function F(k) is different from:

$F(k) = \frac{1}{j\pi k}$ in equation [7.85] representing the Fourier transform of [7.85]

7.9.1.3. *Optimized Hilbert time window and circular convolution problem*

There are additional problems to be solved, i.e.:

– the replacement of ideal time window (equation [7.83]) with an optimized time window which avoids the two sharp discontinuities at the time origin and at the end of a sample;[8]

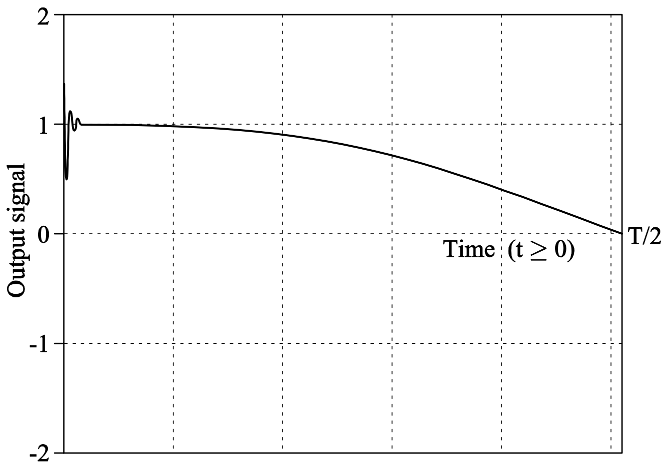


Figure 7.6. *Improved version of Hilbert time window which has no discontinuities at the time origin and at the end of the sample*

– The circular convolution in the frequency domain is due to the fact that the Fourier transform is applied to a succession of finite length samples[9].

8 Sample here designates a portion of time recording of a response in signal processing.
9 In signal processing we call it discrete Fourier transform (DFT).

It can be shown [FEI 86] that the optimized time window is represented in Figure 7.6. Only one half of the optimized time window is represented for time $t > 0$. A mirror symmetry is adopted to complete the figure in the time interval with respect to the time origin:

$$T/2 \leq t \leq +T/2$$

7.9.1.4. *The circular convolution*

This is due to the use of the fast Fourier transform applied to a succession of finite length sample. Relation [7.85a] indicates that in the frequency domain evaluating the Fourier transform is equivalent to a convolution. However, the ideal Hilbert time window is to be replaced by the optimized time window presented in Figure 7.7 and whose Fourier transform is indicated in equation [7.87].

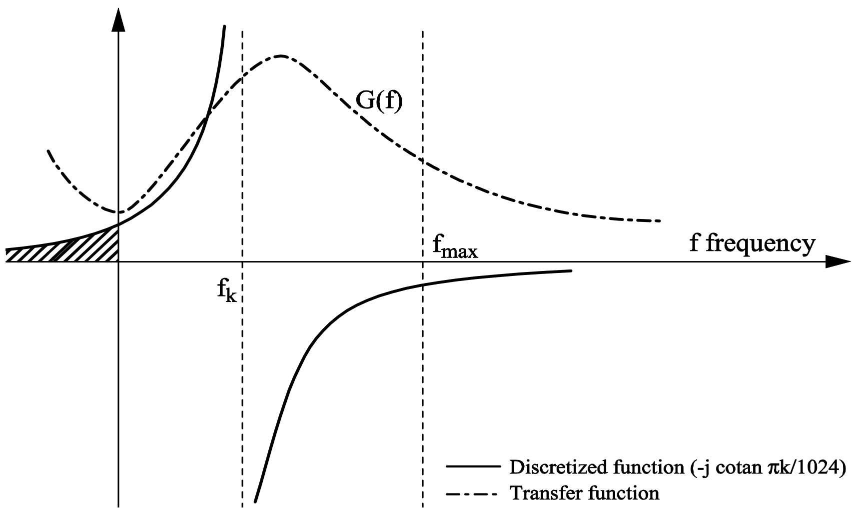


Figure 7.7. *Fourier transform of the optimized Hilbert time window adopted in [7.88]*

Instead of [7.87] we adopt:

$$H_{opt}(f) = G(f) *(-j \operatorname{cotan}\frac{\pi k}{1024}) \qquad [7.88]$$

In the second member, the bracket represents the Fourier transform of the optimized time window represented in Figure 7.7.

7.10. Conclusion

Causality is a principle which characterizes real physical systems. Mathematically, Cauchy's integral in the complex domain allows this principle to be exploited. The Hilbert transform for frequency has been used to complete missing (or wrong) information in electric and/or ultrasonic measurement.

In the domain of the characterization of materials by dynamic methods, this tool has proved to be efficient in detecting hidden non-linearities and, as such, it merits the attention of specialists in this field.

Signal processing invades all scientific domains. In this respect, dynamic tests on samples to evaluate dynamic moduli (or stiffness coefficients) are just some examples amongst others. Recent results obtained in computer programming to obtain Hilbert transforms of digitized signals in the frequency domain are proving to be efficient when the non-linearities are hidden and spoil the interpretation of dynamic functions such as the transfer functions. One of the properties of this transform is that it does not require knowledge of the nature of the non-linearities themselves.

7.11. Bibliography

[BEN 90] BENDAT J.S., *Non Linear System – Analysis and Identification from Random Data*, Wiley, New York, USA, 1990.

[FEI 83] FEI B.J., Digital Hilbert transform extended to non–linear system: Use of Volterra series, thesis ISMCM, Saint Ouen, France, in French, 1986.

[FER 71] FERRY J.D., *Viscoelastic Properties of High Polymers*, John Wiley, USA, 1971.

[FRO 58] FROLICH, *Theory of Dielectrics*, Clarendon Press, Oxford, England, 1958.

[GOL 70] GOLD B., OPPENHEIM A.V., RADER C.M., "Theory and implementation of the discrete Hilbert transform", *Symposium Computer Processing in Communications*, Vol. 19, Polytechnic Press, New York, 1970.

[GOY 80] GOYDER H.G.D., "Some theory and applications between the real and imaginary parts of a frequency response function provided by Hilbert transform", *Pro. Conf. Advances in Structural Dynamics*, I.S.V.R., Southampton, 1980.

[GRO 53] GROSS B., *Mathematical Structure of the Theories of Viscoelasticity*, Hermann, Paris, 1953.

[GUI 63] GUILLEMIN E.A., *Theory of Linear Physical System*, John Wiley & Sons, New York, U.S.A. pp. 89-97, 1963.

[HAO 84] HAOUI A., Hilbert transform theory and applications to non-linear system, PhD dissertation. ISMCM Department of structure, Saint Ouen, France, 1984.

[KLO 71] KLOSTERMAN A., On the experimental determination and use of modal representations of dynamic characteristics, PhD Dissertation, University of Cincinnnati, 1970.

[KRA 27] KRAMERS H.A. *Atti. Cong. DEI. Fisici. Como*., Italy, 1927.

[KRO 26] KRONIG R. de.L., "On the Theory of the Dispersion of X Rays", *Journ. of Optical Soc. Americ*., Vol. 12, p.547, 1926.

[OPE 75] OPPENHEIM A.V., SCHAFER R. W., *Digital Signal Processing*, Prentice Hall pp. 363-364, 1975.

[RAB 74] RABINER L.R., SCHAFER R.W. "On the Behavior of Minimax FIR Digital Hilbert Transformers", *Bell System Journal*, Vol. 53, N°2, pp. 361-388, February 1974.

[SDR 75] S.D.R.C. Modal Analysis User Manual - Modal Parameter Extraction, Cincinnati, Ohio, USA, 1975.

[SIM 83] SIMON M., Development in the modal analysis of linear and non linear structures, PhD, thesis in the Faculty of Sciences, University of Manchester, 1983.

[VIN 86] VINH T., *An Introduction to the Theory of the Hilbert Transform*: *Courses in Non Linear System Behavior*, Heriott-Watt University, Scotland, 1986.

7.12. Appendix 7A. Line integral of complex function and Cauchy's integral

Let us take the following modified frequency complex variable[10]

$$z = \omega + j\sigma = x + j\,y \qquad [7.A.1]$$

where ω is the circular frequency and σ the damping coefficient.

7A.1. *Analyticity of a function f(z) of complex variable z*

A function f(z) is analytic if it has definite derivatives in a region D and if it has a single value in D.

The integral along a contour C is broken down into:

$$\int_C f(z)dz = \int_C (udx - vdy) + j \int_C (vdx + udy) \qquad [7.A.2]$$

10 Readers will notice that there is another way to define the complex variable $z = \sigma + j\omega$, commonly used in electricity and physics, with $p = j\omega$ used as Heaviside's notation. Physically, damping must be negative for a stable system. It depends on the method of writing the exponent of exponentials, with or without $j = \sqrt{-1}$.

Real and imaginary parts of the function f(z) are:

$$f(z) = u + j\,v \qquad [7.A.3]$$

If the function derivative has only one limit, in [7.A.2], we must take the Cauchy Riemann condition:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad [7.A.4]$$

If f(z) is analytic in D, the integral is independent of the path.

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1) \qquad [7.A.5]$$

The integral calculated along a closed contour C is zero:

$$f(z)dz = 0 \qquad [7.A.6]$$

This is Cauchy's integral formula.

7A.2. *Expression of Cauchy's integral of the function f(z)/(z-α)*

This function has a pole at z = α. Let us choose two contours surrounding this pole. The domain delimited by contour C and C_ε is consequently analytic, inside the domain (Figure 7A.1).

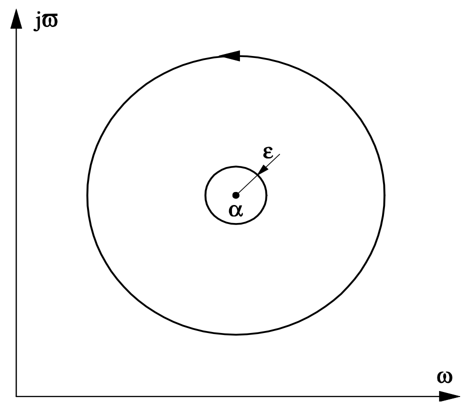


Figure 7A.1. *In a complex frequency plane, two contours C and C_ε are drawn with a pole α at the center of a circle C_e whose radius is ε*

C_ε is excluded. As the line integration does not depend on the path

$$\frac{f(z)dz}{(z-\alpha)} = \oint \frac{f(z)dz}{(z-\alpha)} \forall \varepsilon > 0 \qquad [7.A.7]$$

$$\oint \frac{f(z)dz}{(z-\alpha)} = \oint \frac{f(z)dz}{(z-\alpha)} \qquad for\ \forall\varepsilon$$

$along\ C \qquad along\ \ C_\varepsilon$

For the contour C_ε, let us change the complex variable:

$$z = \alpha + \varepsilon e^{j\vartheta};\ dz = j\varepsilon e^{j\theta} d\vartheta \qquad [7.A.8]$$

When $\varepsilon \to 0$

$$\oint \frac{f(z)dz}{(z-\propto)} = lim_{\varepsilon\to 0} \int_0^{2\pi} f(\alpha + \varepsilon e^{j\theta}) j d\theta = 2\pi j f(\alpha)$$

$$C_\varepsilon \qquad [7.A.9]$$

Then, by interchanging z and α:

$$f(z) = \frac{1}{2j\pi} \oint_C \frac{f(\alpha)d\alpha}{(\alpha - z)} \qquad [7.A.10]$$

This is Cauchy's integral for the analytic function f(z).

7.13. Appendix 7B. Hilbert transform obtained directly by Guillemin's method

Guillemin [GUI 63] did not use a function of a complex variable to find the Hilbert transform.

Let the function of the complex variable s be:

$$Z(s) = \frac{1}{\pi(s+\alpha)} \text{ with } s = \sigma + j\omega \qquad [7.B.1]$$

which has simple pole in the complex plane s.

Let $s = j\omega$

$$Z(s) = \frac{\alpha}{\pi(\alpha^2+\omega^2)} - j\frac{j\omega}{\pi(\alpha^{2+}\omega^{2)}} = \Re e\,(Z) + j\Im m(Z) \qquad [7.B.2]$$

If $\alpha = 0$ the pole is on the j axis. The real part in [7.B.2] is apparently zero. In reality this real part is a bell shaped curve. Integrating under the curve, Figure 7B.1, we obtain:

$$\int_{-\infty}^{+\infty} R(\omega)d\omega = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\alpha d\omega}{\alpha^2+\omega^2} = \frac{1}{\pi}tan^{-1}\alpha_{-\infty}^{+\infty} = 1 \qquad [7.B.3]$$

when $\propto \rightarrow 0$

$\Re e\,[Z(\omega)]$ becomes an unit impulse $\delta(\omega)$. Evaluating Z(s) for s = jω yields:

$$Z(j\omega) = \delta\,(\omega) + \frac{j}{\pi\omega} \qquad [7.B.4]$$

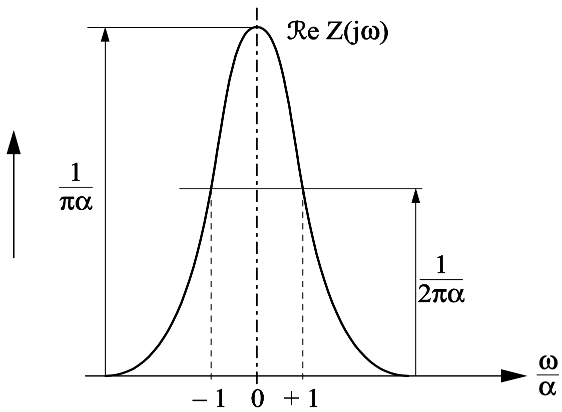


Figure 7B.1. *Integration under the bell shaped curve, equation [7.B.3]*

Similarly if s = $\mp j\omega_0$

$$Z(s) = \frac{1}{\pi}\left(\frac{1}{s+j\omega} + \frac{1}{s-j\omega}\right) \qquad [7.B.5]$$

We can build an arbitrary function by using a collection of impulses, Figure 7B.2.

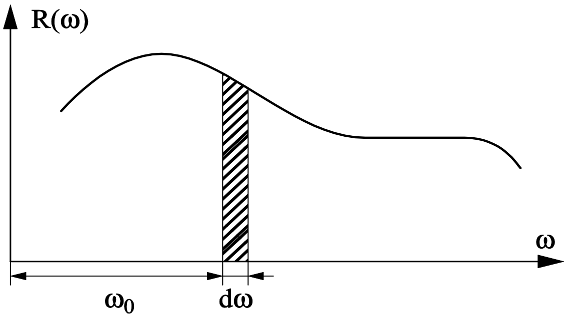


Figure 7B.2. *Integration to obtain the area under the curve*

The pulse area of an element is R(ω_0) dω , where R(ω_0) designates the real part. Using [7.B.5]:

$$dZ(s) = \frac{R(\omega_0)d\omega}{\pi}\left(\frac{1}{s+j\omega_0}+\frac{1}{s-j\omega_0}\right) = \frac{2sR(\omega_0)d\omega}{\pi(s^2+\omega_0^2)} \qquad [7.B.6]$$

Consequently, we show that the real and imaginary parts of the complex function [7.B.4] constitute a conjugate potential. Thus the conjugate of a unit impulse located at $\omega = \omega_0$ is equal to $\frac{-1}{\pi(\omega - \omega_0)}$

Let us take an analytic function

$$G(j\omega) = g_1(\omega) + jg_2(\omega) \qquad [7.B.7]$$

If integration is effected on the element $\frac{g_{1(u)du}}{(\omega-u)}$, the imaginary part is obtained:

$$g_2(\omega) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{g_1(u)du}{(\omega-u)} \qquad [7.B.8]$$

Using Cauchy-Riemann equation:

$$\frac{dg_1}{d\sigma} = \frac{dg_2}{d\omega} \;;\; \frac{dg_1}{d\omega} = -\frac{dg_2}{d\sigma} \qquad [7.B.9]$$

$$g_1(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{g_2(u)du}{(\omega-u)} \qquad [7.B.10]$$

[7.B.8] and [7.B.9] match equations [7.9] and [7.10] exactly.